

## TRANSITION FRONTS FOR INHOMOGENEOUS FISHER-KPP REACTIONS AND NON-LOCAL DIFFUSION

TAU SHEAN LIM AND ANDREJ ZLATOŠ

ABSTRACT. We prove existence of and construct transition fronts for a class of reaction-diffusion equations with spatially inhomogeneous Fisher-KPP type reactions and non-local diffusion. Our approach is based on finding these solutions as perturbations of appropriate solutions to the linearization of the PDE at zero. Our work extends a method introduced by one of us to study such questions in the case of classical diffusion.

### 1. INTRODUCTION AND MAIN RESULTS

In this paper we study the existence of transition fronts for a class of reaction-diffusion equations with inhomogeneous Kolmogorov-Petrovskii-Piskunov (KPP) type non-linearities (also called Fisher-KPP [13, 15]) and non-local diffusion. We consider the PDE

$$(1.1) \quad u_t = Hu + f(x, u),$$

with the *non-local diffusion operator*

$$(Hu)(x, t) := (J * u)(x, t) - u(x, t) = \int_{\mathbb{R}} J(y)[u(x - y, t) - u(x, t)]dy.$$

The kernel  $J \in C^1(\mathbb{R})$  satisfies on  $\mathbb{R}$ :

- (J1)  $J \geq 0$  is even and non-increasing on  $\mathbb{R}^+$ ;
- (J2)  $\text{supp } J = [-\delta, \delta]$  and  $\int_{-\delta}^{\delta} J(y)dy = 1$  (here  $\delta > 0$  need not be small).

The *inhomogeneous KPP reaction function*  $f \in C^2(\mathbb{R} \times [0, 1])$  satisfies on  $\mathbb{R} \times [0, 1]$ :

- (F1)  $f \geq 0$  and  $f(x, 0) = f(x, 1) = 0$ ;
- (F2) there is  $\theta_1 \in (0, 1)$  such that  $f_u(x, u) \leq 0$  when  $u \in [\theta_1, 1]$ ;
- (F3)  $a(x)g(u) \leq f(x, u) \leq a(x)u$ , with  $a(x) := f_u(x, 0)$  and some  $g$  as below.

The function  $g \in C^1([0, 1])$  in (F3) satisfies on  $[0, 1]$ :

- (G1)  $g \geq 0$  and  $g(0) = g(1) = 0$ ;
- (G2)  $g'(0) = 1$ ,  $g'$  is decreasing,  $g'(1) \geq -1$ , and  $\int_0^1 u^{-2}(u - g(u))du < \infty$ .

We denote  $a_- := \inf_{x \in \mathbb{R}} a(x)$  and  $a_+ := \sup_{x \in \mathbb{R}} a(x)$ , and also require

$$(1.2) \quad a_- > 0.$$

We have  $a_+ < \infty$  by  $f \in C^1$ , and  $f$  is of KPP type because  $f(x, u) \leq f_u(x, 0)u$ .

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A (*right moving*) *transition front* for (1.1) is any solution  $0 \leq u \leq 1$  on  $\mathbb{R} \times \mathbb{R}$  such that

$$(1.3) \quad \lim_{x \rightarrow -\infty} u(x, t) = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} u(x, t) = 0$$

for each  $t \in \mathbb{R}$ , and  $u$  has a *bounded width*. The latter means that for each  $\varepsilon > 0$ ,

$$(1.4) \quad \sup_{t \in \mathbb{R}} L_{u, \varepsilon}(t) := \sup_{t \in \mathbb{R}} \text{diam}\{x \in \mathbb{R} : \varepsilon \leq u(x, t) \leq 1 - \varepsilon\} < \infty.$$

This notion of transition fronts is the 1-dimensional case of the definition by Berestycki-Hamel, which was stated for equations with classical diffusion (i.e.,  $\partial_{xx}$  in place of  $H$ ) in [5]. It is a generalization of the notion of *traveling fronts* for homogeneous media and *pulsating fronts* for periodic media. The former are solutions of (1.1) (or its classical diffusion counterpart) with  $f(x, u) = f(u)$ , which are of the form  $u(x, t) = U(x - ct)$  for some speed  $c \in \mathbb{R}$  and profile  $U : \mathbb{R} \rightarrow (0, 1)$  such that  $\lim_{s \rightarrow -\infty} U(s) = 1$  and  $\lim_{s \rightarrow \infty} U(s) = 0$ . The latter are solutions of (1.1) with  $x$ -periodic  $f$ , which are of the form  $u(x, t) = U(x - ct, x)$ , with  $U$  periodic and the above limits uniform in the second argument.

Traveling and pulsating fronts in the presence of classical diffusion have been extensively studied, starting with the works of Fisher [13] and Kolmogorov-Petrovskii-Piskunov [15]. Instead of surveying the vast literature, let us refer to the review articles by Berestycki [3] and Xin [25], and mention specifically that in the homogeneous/periodic KPP case, there exists a traveling/pulsating front precisely when the speed  $c \geq c_f$ , where the number  $c_f > 0$  is the *minimal front speed* for  $f$  (in the homogeneous case  $c_f = 2\sqrt{f'(0)}$ ).

The corresponding results for the non-local diffusion equation (1.1) are considerably more recent. For instance, in [2, 7, 9, 11, 12], existence, uniqueness, and other properties of traveling fronts are proved for various kernels  $J$  and various types of homogeneous reactions  $f$  (KPP, monostable, ignition, and bistable). The case of periodic KPP reactions was also addressed by Coville, Dávila, and Martínez in [10], where pulsating fronts were proved to exist precisely when the speed  $c \geq c_{J, f}$  (for homogeneous reactions this was proved in [11]). In fact, [10] applies in several spatial dimensions, where it proves that for each unit vector  $e$  there again exists a pulsating front in direction  $e$  with speed  $c$  precisely when  $c \geq c_{J, f, e}$ . We mention that traveling fronts for equations with non-local diffusion represented by the fractional Laplacian and homogeneous ignition reactions [19], as well as with classical diffusion and non-local homogeneous KPP reactions [6], were also studied recently.

In these studies, both for classical and non-local diffusion, it has been of crucial help that the traveling front ansatz  $u(t, x) = U(x - ct)$  turns the PDE (1.1) into an ODE. The pulsating front ansatz  $u(t, x) = U(x - ct, x)$  ( $U$  periodic in the second argument) similarly yields a degenerate elliptic PDE. For general (non-periodic) inhomogeneous reactions, on the other hand, no such simplification is available. Because of this, the question of existence and properties of transition fronts for (1.1) with classical diffusion and general inhomogeneous reactions has been addressed only recently in, among other works, [17, 18, 20–22, 24, 26, 27]. The present paper is, to the best of our knowledge, the first study of the analogous non-local diffusion problem.

Our main result is existence of transition fronts for (1.1) with KPP reactions whose  $a(x) = f_u(x, 0)$  is sufficiently close to a constant (while  $f$  itself need not be close to a homogeneous reaction). We prove this by extending to this model a

method introduced by one of us in [26] for the classical diffusion case. The idea here is to exploit the close relationship between (1.1) and its linearization at  $u = 0$ ,

$$(1.5) \quad v_t = H v + a(x)v.$$

We will therefore first study the simpler case of *front-like solutions* of (1.5) of the form

$$(1.6) \quad v_\lambda(x, t) = e^{\lambda t} \phi_\lambda(x).$$

Here  $\phi_\lambda > 0$  is a generalized eigenfunction of the operator  $H + a(x)$ , satisfying

$$(1.7) \quad H\phi_\lambda + a(x)\phi_\lambda = \lambda\phi_\lambda$$

on  $\mathbb{R}$ , which grows exponentially to  $\infty$  as  $x \rightarrow -\infty$  and decays exponentially to 0 as  $x \rightarrow \infty$ .

In the case of classical diffusion, Sturm-Liouville theory assures existence of (a unique up to a multiple) such  $\phi_\lambda$  if and only if  $\lambda > \sup \sigma(\partial_{xx} + a(x))$  (with  $\sigma(\mathcal{L})$  the spectrum of  $\mathcal{L}$ ). We will prove that for (1.7) such  $\phi_\lambda$  exists for each  $\lambda > a_+$ . Note that  $H$  is a negative operator on  $L^2(\mathbb{R})$ , so  $a(x) \leq a_+$  shows  $a_+ \geq \sup \sigma(H + a(x))$ . In fact,  $-2I \leq H \leq 0$ , with  $I$  the identity operator, since  $\|J * \phi\|_2 \leq \|J\|_1 \|\phi\|_2 = \|\phi\|_2$  by Young’s inequality.

Also note that if  $a$  is constant, then  $\sup \sigma(H + a) = a$  and for each  $\lambda > a$  there is  $p_\lambda > 0$  such that  $\phi_\lambda(x) = e^{-p_\lambda x}$  solves (1.7). This  $p_\lambda$  is unique and given by  $\int_{\mathbb{R}} J(y)e^{p_\lambda y} dy = 1 + \lambda - a$ . In this case the solution (1.6) can also be written as  $v_\lambda(x, t) = e^{-p_\lambda(x-ct)}$ , with speed  $c = \lambda p_\lambda^{-1}$ . In the general inhomogeneous case, however, fronts for (1.1) and (1.5) typically do not have specific speeds, so one cannot anymore “parametrize” fronts via their speeds  $c$ . Instead, one can use the “energies”  $\lambda$  for this purpose.

Next we note that by (F3), solutions of (1.5) are super-solutions of (1.1). The main result of [26] is showing that in the case of classical diffusion, for each  $\lambda \in (\sup \sigma(\partial_{xx} + a(x)), 2a_-)$  there is a function  $h_\lambda : [0, \infty) \rightarrow [0, 1)$  such that  $w_\lambda := h_\lambda(v_\lambda) \leq v_\lambda$  is a sub-solution of (1.1), and then finding a transition front  $u_\lambda$  for (1.1) between  $w_\lambda$  and  $\min\{v_\lambda, 1\}$ . This  $h_\lambda$  satisfies

$$(1.8) \quad h_\lambda(0) = 0, \quad h'_\lambda(0) = 1, \quad \lim_{v \rightarrow \infty} h_\lambda(v) = 1, \quad \text{and} \quad h''_\lambda < 0 \quad \text{on} \quad (0, \infty),$$

which also means that  $h_\lambda$  is increasing and  $h_\lambda(v) \leq v$  on  $[0, \infty)$ . From  $\lim_{v \rightarrow 0} v^{-1} h_\lambda(v) = 1$ ,  $\lim_{x \rightarrow \infty} v_\lambda(x, t) = 0$  for each  $t \in \mathbb{R}$ , and  $w_\lambda \leq u_\lambda \leq v_\lambda$  it follows that

$$(1.9) \quad \lim_{x \rightarrow \infty} \frac{u_\lambda(x, t)}{v_\lambda(x, t)} = 1$$

for each  $t \in \mathbb{R}$ . We note that the bound  $\lambda < 2a_-$  is not just a technical limitation; it is sharp for constant  $a$ , and there are also examples of KPP  $f$  with  $\sup \sigma(\partial_{xx} + a(x)) > 2a_-$  for which no transition fronts exist at all [20].

In the present paper we show that this approach can be extended to the non-local diffusion equation (1.1). To do so, we need to overcome three new difficulties. First, we are not aware of a version of the Sturm-Liouville theory for operators  $H + a(x)$  and have to prove the necessary result below (Lemma 2.1). Second, due to the non-locality of  $H$ , we need to obtain very good estimates on the oscillation of the generalized eigenfunctions  $\phi_\lambda$  (Lemma 3.2) in order to apply the (local in nature) method of finding sub-solutions from [26]. And third, (1.1) lacks the regularizing

effects of its classical diffusion counterpart. In fact, the fundamental solution of  $u_t = Hu$  is

$$\Gamma(x - x_0, t) := e^{-t}\delta_0(x - x_0) + e^{-t}(e^{t\hat{J}} - 1)(x - x_0),$$

where  $\delta_0$  is the delta function at 0 (see [1, Lemma 1.6]). We overcome this lack of parabolic regularity theory for (1.1) by showing that while the regularity of solutions of the PDE does not improve with time, for at least some solutions it does not worsen arbitrarily either (Lemma 4.1). Our main result is as follows.

**Theorem 1.1.** *Assume that  $J, f, g$  satisfy the hypotheses (J), (F), (G) above and (1.2).*

(i) *If  $\lambda > a_+$ , then (1.7) has a continuous solution  $\phi_\lambda > 0$  with*

$$(1.10) \quad \lim_{x \rightarrow -\infty} \phi_\lambda(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} \phi_\lambda(x) = 0.$$

(In fact, by Lemma 2.1,  $\phi_\lambda$  grows and decays at least exponentially as  $x \rightarrow -\infty$  and  $x \rightarrow \infty$ .) Thus  $v_\lambda$  from (1.6) is a super-solution of (1.1).

(ii) *There are  $\lambda_0 = \lambda_0(J, a_-) > 0$  (which is non-decreasing in  $a_-$ ) and a function  $h_g : [0, \infty) \mapsto [0, 1)$  satisfying (1.8) such that if  $a_+ < a_- + \lambda_0$ , then for each  $\lambda \in (a_+, a_- + \lambda_0)$  the function  $w_\lambda := h_g(v_\lambda)$  is a sub-solution of (1.1).*

(iii) *If  $\lambda \in (a_+, a_- + \lambda_0)$ , then there is a transition front  $u_\lambda$  for (1.1) satisfying*

$$(1.11) \quad w_\lambda \leq u_\lambda \leq \min\{v_\lambda, 1\}.$$

*Remarks.* 1. Obviously (1.9) holds again.

2. Here  $h_g$  depends on  $g$  only, so not on  $\lambda \in (a_+, a_- + \lambda_0)$ .

3. In the case of classical diffusion [26] obtains  $\lambda_0 = a_-$ , which is sharp. An explicit expression for our  $\lambda_0$  can be found from the formulas in Sections 3 and 6, but we do not know what the sharp value is in this case.

4. At the end of Section 4 we obtain an explicit upper bound on the  $\varepsilon$ -width of  $u_\lambda$  (defined in (1.4)) for  $\lambda \in (a_+, a_- + \lambda_0)$ . This bound depends only on  $\varepsilon, J, g$  and on an upper bound for  $a_-$  and  $(\lambda - a_+)^{-1}$ .

5. As can be easily seen from the proof, the theorem extends to time-dependent  $f$  such that  $f_u(t, x, 0)$  is time independent and (F) holds for each  $t \in \mathbb{R}$ .

6. If  $a_+ < \inf_n \lambda_n \leq \sup_n \lambda_n < a_- + \lambda_0$  and  $b_n > 0$  are such that  $\sum_n b_n < \infty$ , then as in [26], the result holds with  $\phi_\lambda$  and  $v_\lambda$  replaced by  $\sum_n b_n \phi_{\lambda_n}$  and  $\sum_n b_n v_{\lambda_n}$ . The corresponding fronts are a combination of a countable number of the “pure” fronts from (iii). Their existence is new even in the cases of homogeneous and periodic reactions (and non-local diffusion).

We prove the three parts of Theorem 1.1 in the next three sections, postponing the proofs of two crucial estimates needed for the construction of sub-solutions until Sections 5 and 6.

## 2. PROOF OF THEOREM 1.1(i) (CONSTRUCTION OF A SUPER-SOLUTION)

Recall that  $v_\lambda$  from (1.6) is a super-solution of (1.1) when  $\phi_\lambda$  solves (1.7). We thus only need to prove the following result.

**Lemma 2.1.** *If  $\lambda > a_+$ , then there is a continuous solution  $\phi_\lambda > 0$  of (1.7) and  $L = L(J, \lambda - a_-, \lambda - a_+) > 0$  such that  $\phi_\lambda(x) \geq 2\phi_\lambda(y)$  whenever  $y \geq x + L$ .*

To prove this, we will need an appropriate regularity estimate.

**Lemma 2.2.** *Assume that  $\lambda > a_+$  and  $\phi > 0$  is continuous on  $\mathbb{R}$  and solves (1.7) on  $[b, \infty)$ . There are  $C = C(J, \lambda - a_-) > 0$  and  $m = m(J, \lambda - a_-, \lambda - a_+) > 0$  such that the following hold.*

(i) *If  $x \geq b + \delta$ , then*

$$(2.1) \quad |(J * \phi)'(x)| \leq C(J * \phi)(x).$$

*In particular, for all  $x, y \in [b + \delta, \infty)$ ,*

$$(2.2) \quad (J * \phi)(y) \leq e^{C|x-y|}(J * \phi)(x).$$

(ii) *If  $\lim_{x \rightarrow \infty} \phi(x) = 0$  and  $y \geq x \geq b + \delta$ , then*

$$(2.3) \quad (J * \phi)(y) \leq \frac{C}{m} e^{-m(y-x)}(J * \phi)(x).$$

*Remark.* (2.2) and (1.7) imply  $\phi(y) \leq (1 + \lambda - a_-)e^{C|x-y|}\phi(x)$  for  $x, y \in [b + \delta, \infty)$  (note that  $1 + \lambda - a_+ \geq 1$ ). This is a special case of the main result in [8].

*Proof.* (i) Let us rewrite (1.7) for  $x \geq b$  as

$$(2.4) \quad (J * \phi)(x) = (1 + \lambda - a(x))\phi(x).$$

Since  $\text{supp } J' \subseteq [-\delta, \delta]$  and  $J * J > 0$  on  $(-2\delta, 2\delta)$ , we have

$$C_J := \frac{\|J'\|_\infty}{\inf_{x \in [-\delta, \delta]}(J * J)(x)} > 0$$

and  $|J'(x)| \leq C_J(J * J)(x)$  for  $x \in \mathbb{R}$ . Hence, by  $\phi, J \geq 0$  and (2.4), we have for  $x \geq b + \delta$ ,

$$|(J * \phi)'(x)| \leq (|J'| * \phi)(x) \leq C_J(J * J * \phi)(x) \leq C_J(1 + \lambda - a_-)(J * \phi)(x)$$

(we need  $x \geq b + \delta$  in the last inequality). This is (2.1) when we take  $C := C_J(1 + \lambda - a_-)$ .

(ii) We first claim that there is  $m = m(J, \lambda - a_-, \lambda - a_+) > 0$  such that for  $x \geq b + \delta$ ,

$$(2.5) \quad (J * \phi)(x) \geq m \int_x^\infty (J * \phi)(\tau) d\tau.$$

Let us assume this is the case. Then  $[e^{mx} \int_x^\infty (J * \phi)(\tau) d\tau]' \leq 0$ , so for  $y \geq x \geq b + \delta$ ,

$$(2.6) \quad \int_y^\infty (J * \phi)(\tau) d\tau \leq e^{-m(y-x)} \int_x^\infty (J * \phi)(\tau) d\tau.$$

Hence, by this, (2.1), and (2.5),

$$\begin{aligned} (J * \phi)(y) &= - \int_y^\infty (J * \phi)'(\tau) d\tau \leq C \int_y^\infty (J * \phi)(\tau) d\tau \\ &\leq C e^{-m(y-x)} \int_x^\infty (J * \phi)(\tau) d\tau \leq \frac{C}{m} e^{-m(y-x)}(J * \phi)(x). \end{aligned}$$

It remains to prove (2.5). We have  $\lim_{x \rightarrow \infty} (J * \phi)(x) = 0$  by the hypothesis. Hence for each  $\varepsilon > 0$ , there is  $R_0$  such that  $\sup_{y > R_0} (J * \phi)(y) < \varepsilon$ . Let  $R := \max\{R_0, b + \delta, \frac{1}{\varepsilon}\}$ . Then,

$$(2.7) \quad \int_x^R (J * \phi)(\tau) d\tau = \int_x^{x+\delta} (J * \phi)(\tau) d\tau + \int_{x+\delta}^R (J * \phi)(\tau) d\tau = I + II.$$

By (2.2),  $I \leq \delta e^{C\delta}(J * \phi)(x)$ . On the other hand, by (2.4) and (J2),

$$\begin{aligned} II &\leq \int_x^{R+\delta} \phi(\tau) d\tau = \int_x^{R+\delta} \frac{(J * \phi)(\tau)}{1 + \lambda - a(\tau)} d\tau \\ &\leq \frac{1}{1 + \lambda - a_+} \int_x^R (J * \phi)(\tau) d\tau + \frac{\varepsilon\delta}{1 + \lambda - a_+}. \end{aligned}$$

The estimates for  $I$  and  $II$  and (2.7) now yield

$$\delta e^{C\delta}(J * \phi)(x) \geq \frac{\lambda - a_+}{1 + \lambda - a_+} \int_x^R (J * \phi)(\tau) d\tau - \frac{\varepsilon\delta}{1 + \lambda - a_+},$$

and (2.5) follows by letting  $\varepsilon \rightarrow 0$ , with  $m := \delta^{-1}e^{-C\delta}(\lambda - a_+)(1 + \lambda - a_+)^{-1}$ .  $\square$

*Proof of Lemma 2.1.* Obviously,  $\lambda \notin \sigma(H + a(x))$  by  $\lambda > a_+$ . Let  $0 \neq \eta \leq 0$  be continuous and compactly supported and let  $\varphi := (H + a(x) - \lambda)^{-1}\eta \in L^2(\mathbb{R})$ . Since  $J \in L^2(\mathbb{R})$  as well,  $J * \varphi$  is uniformly continuous and  $\lim_{|x| \rightarrow \infty} (J * \varphi)(x) = 0$ . Since also

$$\varphi = \frac{J * \varphi - \eta}{1 + \lambda - a},$$

$\varphi$  is continuous and  $\lim_{|x| \rightarrow \infty} \varphi(x) = 0$ .

Furthermore,  $\varphi > 0$ . Indeed, otherwise  $\varphi$  achieves a non-positive minimum, and since  $\varphi \not\equiv 0$ , the set of global minima of  $\varphi$  has a boundary point  $x_0$ . From the properties of  $J$  it now follows that  $(H\varphi)(x_0) > 0$ . But then  $\eta(x_0) = (H\varphi)(x_0) + (a(x_0) - \lambda)\varphi(x_0) > 0$  by  $\lambda > a_+$ , contradicting  $\eta \leq 0$ . Thus  $\varphi > 0$ , and Lemma 2.2 applies to  $\varphi$ .

Let us choose  $\eta$  with  $\text{supp } \eta = [-1, 0]$ , define  $\eta_j(x) := \eta(x + j)$ ,  $\varphi_j := (H + a(x) - \lambda)^{-1}\eta_j$ , and  $\phi_j := \varphi(0)^{-1}\varphi_j$  (recall that  $\varphi(0) > 0$ ). Then  $\phi_j$  solves (2.4) on  $[-j, \infty)$ , so Lemma 2.2(i) gives for  $x \geq -j + \delta$ ,

$$|[\log([1 + \lambda - a(x)]\phi_j(x))]'| = |[\log(J * \phi_j)]'(x)| \leq C.$$

Since also  $\log \phi_j(0) = 0$ , there is a locally uniform limit  $\Phi > 0$  for some sub-sequence of  $\{(1 + \lambda - a(x))\phi_j(x)\}_j$ . Let  $\phi(x) := (1 + \lambda - a(x))^{-1}\Phi(x)$ , which is positive and continuous. We have  $\phi_j \rightarrow \phi$  locally uniformly because  $1 + \lambda - a_+ \geq 1$ ; hence  $\phi$  solves (1.7) on  $\mathbb{R}$ .

By Lemma 2.2(ii) and (2.4), for any  $j$  and  $y \geq x \geq -j + \delta$  we have

$$\phi_j(y) \leq (J * \phi_j)(y) \leq \frac{C}{m}(1 + \lambda - a_-)e^{-m(y-x)}\phi_j(x).$$

Hence Lemma 2.1 holds with  $\phi_\lambda := \phi$  and  $L := \max\{\frac{1}{m} \log \frac{2C(1+\lambda-a_-)}{m}, \delta\}$ .  $\square$

### 3. PROOF OF THEOREM 1.1(II) (CONSTRUCTION OF A SUB-SOLUTION)

We now turn to the construction of sub-solutions of (1.1), extending the method from [26]. The function  $h_g$  will be taken from a family of functions  $\{h_{g,\alpha}\}_{\alpha \in (0,1)}$  satisfying (1.8), which have been constructed in [26] (we note that our  $h_{g,\alpha}$  equals  $h_{g,\alpha^2}$  from [26]).

It was proved in [23] that under the hypotheses (G) and for each  $\alpha \in (0, 1)$ , the homogeneous PDE  $u_t = u_{xx} + g(u)$  (with classical diffusion) has a (unique) *traveling front* solution  $u(x, t) = U_{g,\alpha}(x - c_\alpha t) \in (0, 1)$  (with  $c_\alpha := \alpha + \alpha^{-1}$ ) which

satisfies  $\lim_{s \rightarrow \infty} e^{\alpha s} U_{g,\alpha}(s) = 1$ . The pair  $(U_{g,\alpha}, c_\alpha)$  here solves the traveling front boundary value problem

$$(3.1) \quad U''_{g,\alpha} + c_\alpha U'_{g,\alpha} + g(U_{g,\alpha}) = 0, \quad \lim_{s \rightarrow -\infty} U_{g,\alpha}(s) = 1, \quad \lim_{s \rightarrow \infty} U_{g,\alpha}(s) = 0,$$

whose solutions are (up to translation in  $s$ ) precisely  $\{(U_{g,\alpha}, c_\alpha)\}_{\alpha \in (0,1]}$ . They satisfy  $U'_{g,\alpha} < 0$  on  $\mathbb{R}$ , and the critical front  $U_{g,1}$  (which we will not use) satisfies  $\lim_{s \rightarrow \infty} s^{-1} e^s U_{g,1}(s) = 1$ .

The linearization  $v_t = v_{xx} + v$  of  $u_t = u_{xx} + g(u)$  at  $u = 0$  has corresponding traveling front solutions  $v(x, t) = e^{-\alpha(x - c_\alpha t)}$ , and  $h_{g,\alpha}$  is chosen to be the function which takes  $e^{-\alpha s}$  to  $U_{g,\alpha}(s)$  for  $\alpha \in (0, 1)$ . That is,

$$(3.2) \quad h_{g,\alpha}(v) := \begin{cases} U_{g,\alpha}(-\alpha^{-1} \log v), & v > 0, \\ 0, & v = 0. \end{cases}$$

Notice that (3.1) yields

$$(3.3) \quad \alpha^2 v^2 h''_{g,\alpha}(v) - v h'_{g,\alpha}(v) + g(h_{g,\alpha}(v)) = 0,$$

and (1.8) follows from the definition of  $h_{g,\alpha}$ , with  $h'_{g,\alpha}(0) = 1$  due to  $\lim_{s \rightarrow \infty} e^{\alpha s} U_{g,\alpha}(s) = 1$ , and  $h''_{g,\alpha} < 0$  proved in [26] (also in Lemma 5.1 below).

It turns out that the same  $h_{g,\alpha}$  can be used for our non-local diffusion problem (1.1). To do that, we will need the following two lemmas, whose proofs we postpone until after the proof of Theorem 1.1.

**Lemma 3.1.** *Let  $g$  satisfy (G) and for  $\alpha \in (0, 1)$  let  $\beta := 2 + \alpha^{-2}$  and  $h_{g,\alpha}$  be from (3.2). Then  $\rho_{g,\alpha}(x) := -h''_{g,\alpha}(e^{-x}) > 0$  satisfies  $|\rho'_{g,\alpha}(x)| \leq \beta \rho_{g,\alpha}(x)$  for  $x \in \mathbb{R}$  and, in particular,  $\rho_{g,\alpha}(y) \leq e^{\beta|x-y|} \rho_{g,\alpha}(x)$  for  $x, y \in \mathbb{R}$ .*

**Lemma 3.2.** *Let  $\phi_\lambda > 0$  satisfy (1.7) with  $\lambda > a_+$  and (1.10). For each  $s > 0$  there is  $\gamma_s = \gamma_s(J) > 0$  with  $\lim_{s \searrow 0} \gamma_s = 0$  and such that if  $|x - y| \leq \delta$  (with  $\delta$  from (J2)), then*

$$(3.4) \quad |\phi_\lambda(x) - \phi_\lambda(y)| \leq \gamma_{\lambda - a_-} \phi_\lambda(y).$$

*Remark.* Lemma 3.2 is an improvement of the remark after Lemma 2.2.

Let  $h_g := h_{g,3/4}$ , with  $h_{g,\alpha}$  from (3.2). We will suppress the subscripts  $g, \lambda$  in what follows, denoting  $w = w_\lambda = h_g(v_\lambda) = h(v)$ . Then by (1.6) and (1.7),

$$w_t - Hw = h'(v)Hv + a(x)vh'(v) - \int_{-\delta}^\delta J(y)[w(x - y, t) - w(x, t)]dy.$$

By Taylor’s theorem for  $h(v)$  we have

$$w(x - y, t) - w(x, t) = h'(v(x, t))[v(x - y, t) - v(x, t)] + \frac{1}{2} h''(\zeta_{x,y,t})[v(x - y, t) - v(x, t)]^2,$$

where  $\zeta_{x,y,t}$  is some number between  $v(x - y, t)$  and  $v(x, t)$ . This and the definition of  $Hv$  yield

$$(3.5) \quad w_t - Hw = a(x)vh'(v) - \frac{1}{2} \int_{-\delta}^\delta h''(\zeta_{x,y,t})J(y)[v(x - y, t) - v(x, t)]^2 dy.$$

Since  $\zeta_{x,y,t}$  is between  $v(x - y, t)$  and  $v(x, t)$  (and  $|y| \leq \delta$ ), Lemma 3.2 implies

$$|\log \zeta_{x,y,t} - \log v(x, t)| \leq \log(1 + \gamma_{\lambda - a_-}).$$

Lemma 3.1 with  $\beta = 2 + (3/4)^{-2} < 4$  now gives

$$(3.6) \quad \begin{aligned} -h''(\zeta_{x,y,t}) &= \rho(-\log \zeta_{x,y,t}) \leq e^{4\log(1+\gamma_{\lambda-a_-})} \rho(-\log v(x,t)) \\ &= -(1 + \gamma_{\lambda-a_-})^4 h''(v(x,t)). \end{aligned}$$

On the other hand, by Lemma 3.2,

$$(3.7) \quad \int_{-\delta}^{\delta} J(y)[v(x-y,t) - v(x,t)]^2 dy \leq \gamma_{\lambda-a_-}^2 v(x,t).$$

Using (3.6), (3.7), and  $h'' < 0$ , we obtain from (3.5),

$$(3.8) \quad w_t - Hw \leq a(x)v h'(v) - \frac{1}{2} \gamma_{\lambda-a_-}^2 (1 + \gamma_{\lambda-a_-})^4 v^2 h''(v).$$

Since  $\lim_{s \searrow 0} \gamma_s = 0$  by Lemma 3.2, there exists (non-decreasing in  $a_-$ )  $\lambda_0 = \lambda_0(J, a_-)$  such that

$$\gamma_s^2 (1 + \gamma_s)^4 \leq a_-$$

for all  $s \in (0, \lambda_0)$ . If now  $a_+ < a_- + \lambda_0$  and  $\lambda \in (a_+, a_- + \lambda_0)$ , then we have

$$\frac{1}{2} \gamma_{\lambda-a_-}^2 (1 + \gamma_{\lambda-a_-})^4 \leq \left(\frac{3}{4}\right)^2 a(x).$$

Thus (3.8),  $h'' < 0$ , (3.3) for  $h = h_g = h_{g,3/4}$ , and (F3) yield

$$w_t - Hw \leq a(x)[v h'(v) - (3/4)^2 v^2 h''(v)] = a(x)g(w) \leq f(x, w).$$

So  $w = w_\lambda = h_g(v_\lambda)$  is a sub-solution of (1.1).

#### 4. PROOF OF THEOREM 1.1(III) (CONSTRUCTION OF A TRANSITION FRONT)

For reaction-diffusion equations with classical diffusion, there is a simple and standard way to construct a transition front for (1.1) between the super-solution  $v_\lambda$  and the sub-solution  $w_\lambda = h_g(v_\lambda) \leq v_\lambda$  from the last two sections. One lets  $u_n : \mathbb{R} \times (-n, \infty) \rightarrow [0, 1]$  be the solution of the Cauchy problem with initial datum  $u_n(x, -n)$  between  $w_\lambda(x, -n)$  and  $\min\{v_\lambda(x, -n), 1\}$ , and recovers a transition front  $u_\lambda : \mathbb{R}^2 \rightarrow [0, 1]$  as a locally uniform limit along a sub-sequence of  $\{u_n\}_{n \geq 1}$ , using parabolic regularity results and the Arzelà-Ascoli theorem.

Such regularization results are not available for the non-local diffusion operator  $H$ , as was discussed in the introduction. Nevertheless,  $H$  does not (qualitatively) worsen the regularity of the solutions of (1.1), so one might hope that if the initial datum is sufficiently regular (in our case, Lipschitz or Hölder continuous would suffice), then this regularity will persist indefinitely for bounded solutions. In fact, a simple argument from [16] (where the homogeneous case was treated) shows that if  $\sup_{(x,u) \in \mathbb{R} \times [0,1]} f_u(x, u) < 1$ , then Lipschitz initial data give rise to uniformly-in-time (and  $n$ ) Lipschitz solutions. We do not assume such a bound here, and thus will have to prove a similar result for the sequence of solutions  $u_n$  in a different way.

We consider the Cauchy problem

$$(4.1) \quad \begin{cases} u_t = Hu + f(x, u) & \text{on } \mathbb{R} \times (-n, \infty), \\ u(x, -n) = w(x, -n) \quad (= h(v(x, -n))) & \text{on } \mathbb{R}, \end{cases}$$

where we dropped the subscripts  $n, g, \lambda$ . The proof of existence and uniqueness of a bounded continuous classical solution to this problem with bounded continuous initial data is standard and identical to the homogeneous case (see, e.g., [16]). The

proofs of the maximum and comparison principles for (1.1) are also standard. These imply, in particular,

$$(4.2) \quad w \leq u \leq \min\{v, 1\}.$$

We then obtain the following bound on  $u$  from (4.1).

**Lemma 4.1.** *There is  $\bar{C} = \bar{C}(J, \lambda - a_-, \|f\|_{C^2}, g)$  such that the solution of (4.1) satisfies*

$$(4.3) \quad \eta(t) := \sup_{0 < |y-x| \leq \delta} \frac{|u(y, t) - u(x, t)|}{|y-x|u(x, t)} \leq \bar{C}$$

for any  $t \geq -n$ .

*Remark.* In particular,  $u_x(\cdot, t)$  exists almost everywhere for each  $t \geq -n$ , and  $|u_x| \leq \bar{C}u$ .

*Proof.* From (2.1), (2.4),  $\|a\|_{C^1} \leq \|f\|_{C^2}$ , and  $1 + \lambda - a_+ \geq 1$  we have

$$(4.4) \quad |\phi'_\lambda(x)| = \left| \frac{(J * \phi_\lambda)'(x) + a'(x)\phi_\lambda(x)}{1 + \lambda - a(x)} \right| \leq [C(1 + \lambda - a_-) + \|f\|_{C^2}] \phi_\lambda(x) =: C_1 \phi_\lambda(x).$$

Since  $v(x, t) = e^{\lambda t} \phi_\lambda(x)$ ,  $|v_x| \leq C_1 v$ . From concavity of  $h$  we have  $vh'(v) \leq h(v)$ . Thus

$$|w_x| = h'(v)|v_x| \leq C_1 h'(v)v \leq C_1 h(v) = C_1 w,$$

so  $\eta(-n) \leq C_1 e^{C_1 \delta}$  (a bound which is independent of  $n$ ).

The comparison principle for (1.1) shows  $w \leq u \leq \tilde{v} := \min\{v, 1\}$  on  $\mathbb{R} \times [-n, \infty)$ . Concavity of  $h$  then yields  $\tilde{v} \leq h(v)h(1)^{-1} = wh(1)^{-1} \leq uh(1)^{-1}$ . Since from (4.4) we have  $\tilde{v}(x, t) \leq e^{2C_1 \delta} \tilde{v}(y, t)$  for  $|x - y| \leq 2\delta$  (with  $\delta$  from (J2)), it follows that

$$(4.5) \quad u(y, t) \leq \tilde{C}u(x, t)$$

for  $|x - y| \leq 2\delta$  and  $\tilde{C} := e^{2C_1 \delta} h(1)^{-1}$ .

Now let  $u^s(x, t) := u(x + s, t)$ ,  $q^s := \frac{1}{s}(u^s - u)$ , and  $z^s := q^s/u$ . The lemma will follow if we show  $|z^s(x, t)| \leq \tilde{C}$  for some  $\tilde{C} = \tilde{C}(J, \lambda - a_-, \|f\|_{C^2}, h) < \infty$  and all  $x \in \mathbb{R}$ ,  $t \geq -n$ , and  $0 < |s| \leq \delta$  (recall that  $h = h_g$  depends only on  $g$ ). We have

$$(4.6) \quad q_t^s - Hq^s = \frac{f(x + s, u^s) - f(x, u^s)}{s} + \frac{f(x, u^s) - f(x, u)}{s}.$$

By (4.1) and (4.6),

$$(4.7) \quad z_t^s = \alpha(x, t) + \beta(x, t)z^s,$$

with

$$(4.8) \quad \alpha(x, t) = \frac{J * q^s}{u} + \frac{f(x + s, u^s) - f(x, u^s)}{su},$$

$$(4.9) \quad \beta(x, t) = -\frac{J * u}{u} + \frac{f(x, u^s) - f(x, u)}{u^s - u} - \frac{f(x, u)}{u}.$$

Recall  $0 < |s| \leq \delta$ . We have  $J * q^s = \frac{1}{s}(J^{-s} - J) * u$ , so (4.5) implies  $|J * q^s| \leq 3\delta \|J'\|_\infty \tilde{C}u$ . Since also  $f_x(\cdot, 0) \equiv 0$ , we obtain  $|f(x + s, u^s) - f(x, u^s)| \leq \|f\|_{C^2} |s| u^s$ , and (4.5) now gives

$$(4.10) \quad |\alpha(x, t)| \leq \tilde{C} (3\delta \|J'\|_\infty + \|f\|_{C^2}) =: M.$$

From (4.5) we obtain

$$(4.11) \quad -\frac{J * u}{u} \leq -\frac{1}{\tilde{C}},$$

as well as

$$(4.12) \quad \left| \frac{f(x, u^s) - f(x, u)}{u^s - u} - \frac{f(x, u)}{u} \right| \leq \frac{1}{2\tilde{C}}$$

whenever  $u \leq \theta_0 := (2\tilde{C}^2 \|f\|_{C^2})^{-1}$  (then also  $u^s \leq (2\tilde{C} \|f\|_{C^2})^{-1}$ ). Thus

$$(4.13) \quad \beta(x, t) \leq -\frac{1}{2\tilde{C}}$$

when  $u \leq \theta_0$ .

We now fix any  $x \in \mathbb{R}$  and regard (4.7) as an ODE in  $t$ . If  $t_x := \inf\{t \geq -n : u(x, t) > \theta_0\}$ , then (4.13) holds for all  $t \in (-n, t_x)$ . Next define, with  $\theta_1$  from (F2),

$$T := \frac{1}{\lambda} \log \frac{\tilde{C}h^{-1}(\theta_1)}{\theta_0}.$$

From (4.2),  $h' > 0$ ,  $u(x, t_x) \geq \theta_0$ , and (4.5) we obtain for  $|r| \leq \delta$  and  $t \geq t_x + T$ ,

$$\begin{aligned} u^r(x, t) &\geq h(v^r(x, t)) \geq h(e^{\lambda T} v^r(x, t_x)) \geq h(e^{\lambda T} u^r(x, t_x)) \geq h(e^{\lambda T} \tilde{C}^{-1} \theta_0) \\ &= h(h^{-1}(\theta_1)) = \theta_1. \end{aligned}$$

So (F2) implies

$$\frac{f(x, u^s) - f(x, u)}{u^s - u} \leq 0$$

for  $t \geq t_x + T$ , and then (4.9) and (4.11) show (4.13) for  $t \geq t_x + T$ . Finally, for  $t \in [t_x, t_x + T)$ ,

$$(4.14) \quad \beta(x, t) \leq \|f\|_{C^1}.$$

From (4.10) and (4.13) for  $t \in (-n, t_x)$  we obtain  $z(x, t) \leq \max\{\eta(-n), 2\tilde{C}M\}$  for  $t \leq t_x$  (recall that  $\eta(-n)$  is bounded uniformly in  $n$ ), and then (4.14) for  $t \in [t_x, t_x + T)$  and (4.13) for  $[t_x + T, \infty)$  yield

$$|z^s(x, t)| \leq \left( \max\{\eta(-n), 2\tilde{C}M\} + \frac{M}{\|f\|_{C^1}} \right) e^{\|f\|_{C^1} T} - \frac{M}{\|f\|_{C^1}} =: \bar{C}$$

for all  $t \geq -n$ , and  $x \in \mathbb{R}$  and  $0 < |s| \leq \delta$ . This proves (4.3). □

*Remark.* The Harnack-type bound (4.5) played a crucial role in the above proof. We note that without it, one can still prove that  $\eta(t)$  is locally bounded if it is finite initially. Indeed, the absolute value of the right-hand side of (4.6) is bounded by  $\|f\|_{C^1}(1 + |q^s|)$ , so the comparison principle shows (with initial time  $t_0$ )

$$\|q^s(\cdot, t)\|_\infty \leq [1 + \eta(t_0)]e^{\|f\|_{C^1}(t-t_0)} - 1$$

for each  $s \neq 0$ . Hence,  $\eta(t)$  satisfies the same bound.

Let  $u_n$  be the (unique) solution of (4.1). The constant  $\bar{C}$  from Lemma 4.1 is a uniform-in- $n$  bound on  $|(u_n)_x|$  because  $0 \leq u_n \leq 1$ . Since  $Hu + f(x, u)$  is also uniformly bounded in  $0 \leq u \leq 1$ , we find that  $|(u_n)_t| \leq 2 + \|f\|_{C^1}$ . Since

$$\frac{\partial}{\partial t}[Hu + f(x, u)] = Hu_t + f_u(x, u)u_t$$

by the dominated convergence theorem, we have  $|(u_n)_{tt}| \leq (2 + \|f\|_{C^1})^2$ . Thus we see that  $u_n$  and  $(u_n)_t$  converge, along a sub-sequence, locally uniformly to  $u_\lambda$  and  $(u_\lambda)_t$  for some  $u_\lambda : \mathbb{R}^2 \rightarrow [0, 1]$ . Then obviously  $u$  solves (1.1), and (1.11) holds by (4.2) for each  $u_n$ .

From (1.11) we obtain (1.3), so it remains to show (1.4). If  $L$  is from Lemma 2.1 for  $\phi_\lambda$  from (1.6), then the lemma and (1.11) yield

$$\sup_{t \in \mathbb{R}} L_{u,\varepsilon}(t) \leq L[\log_2(\varepsilon^{-1}h_g^{-1}(1 - \varepsilon))],$$

which gives (1.4). So  $u$  is a transition front, and the proof of Theorem 1.1 is finished.

5. PROOF OF LEMMA 3.1 (ESTIMATE ON THE THIRD DERIVATIVE OF  $h_{g,\alpha}$ )

We will again drop the subscript  $g, \alpha$  in  $\rho_{g,\alpha}$ ,  $h_{g,\alpha}$ , and  $U_{g,\alpha}$ . From (3.1), (3.2), and  $c_\alpha = \alpha + \alpha^{-1}$  we have

$$(5.1) \quad \rho(x) = \alpha^{-3}e^{2x} [U'(\alpha^{-1}x) + \alpha g(U(\alpha^{-1}x))] = \alpha^{-3}e^{2x}\eta(\alpha^{-1}x),$$

with  $\eta = \eta_{g,\alpha}$  given by

$$(5.2) \quad \eta := U' + \alpha g(U).$$

By differentiating we obtain

$$(5.3) \quad \rho'(x) = 2\rho(x) + \alpha^{-4}e^{2x}\eta'(\alpha^{-1}x).$$

Thus Lemma 3.1 will follow if we show  $|\eta'| \leq \alpha^{-1}\eta$ . Using (3.1) and  $c_\alpha = \alpha + \alpha^{-1}$ , we obtain

$$(5.4) \quad \eta' = -\alpha^{-1}\eta - \alpha U'(1 - g'(U)).$$

Since  $U' < 0 \leq 1 - g'(U)$ , the latter by (G2), it suffices to prove  $-\alpha U'(1 - g'(U)) \leq 2\alpha^{-1}\eta$ . By (5.2), this is equivalent to

$$(5.5) \quad -U' \leq \frac{2\alpha}{2 + \alpha^2(1 - g'(U))}g(U).$$

Since  $0 \leq 1 - g'(U) \leq 2$  by (G2), this (and hence Lemma 3.1) will be proved once we prove the following lemma.

**Lemma 5.1.** For  $r_\alpha : (-\infty, 1] \rightarrow \mathbb{R}$ , given by

$$r_\alpha(v) := \begin{cases} \frac{\alpha}{1 + \alpha^2(1 - v)}, & v \in [0, 1], \\ \frac{\alpha}{1 + \alpha^2}, & v < 0, \end{cases}$$

we have  $-U' \leq r_\alpha(g'(U))g(U)$ .

*Remark.* This is an improvement of Lemma 3.1 in [26], which shows that  $-U' \leq \alpha g(U)$  (and thus  $\eta > 0$  and  $h'' < 0$ ).

*Proof of Lemma 5.1.* We will in fact prove the stronger estimate  $-U' \leq q(g'(U))g(U)$ , where  $q : (-\infty, 1] \rightarrow \mathbb{R}$  is given by (recall that  $c_\alpha = \alpha + \alpha^{-1} \geq 2$ )

$$(5.6) \quad q(v) := \begin{cases} \frac{2}{c_\alpha + \sqrt{c_\alpha^2 - 4v}}, & v \in [0, 1], \\ \frac{1}{c_\alpha}, & v < 0. \end{cases}$$

It is easy to check that  $q \leq r_\alpha$  on  $(-\infty, 1]$ . Also,  $q > 0$  is continuous and non-decreasing, and for  $v \in [0, 1]$  we have

$$(5.7) \quad vq(v)^2 - c_\alpha q(v) + 1 = 0.$$

Since  $g'$  and  $U$  are decreasing,  $g'(U(x))$  is increasing in  $x$  with limits  $g'(1) < 0$  and  $g'(0) = 1$  as  $x \rightarrow \pm\infty$ . Let  $x_0 \in \mathbb{R}$  be the unique number such that  $g'(U(x_0)) = 0$ , and let us prove

$$(5.8) \quad -U'(x) \leq q(g'(U(x)))g(U(x))$$

separately for  $x \geq x_0$  and  $x < x_0$ .

First fix any  $x \geq x_0$ . Then  $g'(U(x)) \in [0, 1]$ . (5.7) shows that  $s := q(g'(U(x)))$  satisfies

$$(5.9) \quad g'(U(x))s^2 - c_\alpha s + 1 = 0.$$

Define the region  $D_x \subseteq \mathbb{R}^2$  by

$$D_x := \{(u, v) : u \in (U(x), 1) \text{ and } v \in (-sg(u), 0)\}.$$

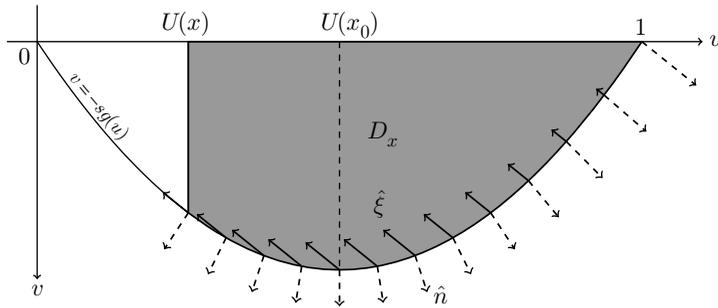


FIGURE 1. The region  $D_x$  in the case  $x \geq x_0$  (so that  $U(x) \leq U(x_0)$ ).

Consider the curve  $\{(U(y), V(y))\}$ , with  $V := U'$ . By problem (3.1),  $(U', V') = (V, -c_\alpha V - g(U))$ . Notice that the vector  $\hat{\xi} := (v, -c_\alpha v - g(u))$  is pointing inside  $D_x$  when  $u \in (U(x), 1)$  and  $v = -sg(u)$ . Indeed, the vector

$$\hat{n} := (-sg'(u), -1)$$

is an outer normal to  $D_x$ , and  $v = -sg(u)$  gives

$$\hat{\xi} = g(u)(-s, c_\alpha s - 1).$$

Since  $g > 0$  and  $g'$  is decreasing on  $(0, 1)$ ,  $u \in (U(x), 1)$  and (5.9) now yield

$$\hat{n} \cdot \hat{\xi} = g(u)[g'(u)s^2 - c_\alpha s + 1] < g(u)[g'(U(x))s^2 - c_\alpha s + 1] = 0.$$

As a consequence, if  $(U(y_0), V(y_0)) \in D_x$  for some  $y_0 < x$ , then  $(U(y), V(y)) \in D_x$  for all  $y \in [y_0, x)$ . Or equivalently, if  $(U(y_0), V(y_0)) \notin D_x$  for some  $y_0 < x$ , then  $(U(y), V(y)) \notin D_x$  for all  $y \leq y_0$ . In this latter case we have

$$(5.10) \quad V(y) < -sg(U(y))$$

for all  $y \leq y_0$ . From (3.1), (5.10), (5.9), and  $g'(U(x)) > 0$  it follows that

$$V'(y) = -c_\alpha V(y) - g(U(y)) > (c_\alpha s - 1)g(U(y)) = g'(U(x))s^2 g(U(y)) > 0$$

for all  $y \leq y_0$ . But then  $U'(y_0) = \int_{-\infty}^{y_0} V'(y)dy > 0$ , a contradiction.

Thus we must have  $(U(y_0), V(y_0)) \in D_x$  for all  $y_0 < x$ , which yields  $V(x) \geq -sg(U(x))$  by continuity. This is precisely (5.8), proving the lemma for  $x > x_0$ .

We actually proved  $-U'(y_0) \leq q(g'(U(x)))g(U(y_0))$  whenever  $y_0 \leq x$  and  $x \geq x_0$ . Taking  $x := x_0$  and renaming  $y_0$  to  $x (\leq x_0)$ , this becomes  $-U'(x) \leq q(0)g(U(x))$  for  $x \leq x_0$ . But this is again (5.8) because for  $x \leq x_0$  we have  $g'(U(x)) \leq 0$ , so  $q(g'(U(x))) = q(0)$ .  $\square$

6. PROOF OF LEMMA 3.2 (IMPROVED HARNACK-TYPE ESTIMATE FOR  $\phi_\lambda$ )

Let us drop the subscript  $\lambda$  in  $\phi_\lambda$ . Define

$$(6.1) \quad \kappa(x) := H \left[ \frac{|x|}{2} \right] = \frac{1}{2} \int_{-\delta}^\delta J(y)(|x-y| - |x|)dy,$$

which is continuous, even (because  $J$  is), and supported in  $[-\delta, \delta]$ . We also have

$$(6.2) \quad 0 \leq \kappa \leq \frac{\delta^2}{2} J.$$

To show this, observe that  $\kappa = H[x_+]$ , where  $x_+ := \max\{x, 0\}$ . So for  $x \in [-\delta, 0]$ ,

$$\kappa(x) = \int_{-\delta}^x J(y)(x-y)dy \in \left[ 0, \int_{-\delta}^x J(x)(x-y)dy \right] \subseteq \left[ 0, \frac{\delta^2}{2} J(x) \right]$$

because  $J$  is even and non-decreasing on  $\mathbb{R}^-$ . Since  $\kappa$  is also even and vanishes outside  $[-\delta, \delta]$ , (6.2) follows.

We will first prove an estimate as in the lemma for the function

$$(6.3) \quad \psi := \|\kappa\|_{L^1}^{-1}(\kappa * \phi)$$

and then show that  $\phi\psi^{-1}$  is close to 1 when  $\lambda - a_- > 0$  is small. The motivation for introducing the function  $\psi$  is the fact that

$$(6.4) \quad (\kappa * \varphi)'' = H\varphi$$

for any continuous function  $\varphi$ , showing that

$$(6.5) \quad \psi'' = \|\kappa\|_{L^1}^{-1} H\phi = \|\kappa\|_{L^1}^{-1}(\lambda - a(x))\phi$$

(which is small when  $\lambda - a_-$  is small).

Identity (6.4) should hold because for  $m := \frac{1}{2}|x|$  we have  $m'' = \delta_0$  (the delta function at 0) in the sense of distributions, so formally  $\kappa'' = H[m''] = H\delta_0 = J - \delta_0$ . To prove (6.4), let  $0 \leq \eta \leq 1$  be a smooth bump function around  $x$  with  $\eta = 1$  on  $[x - 2\delta, x + 2\delta]$ , and  $\eta = 0$  outside  $[x - 4\delta, x + 4\delta]$ . If  $\tilde{\varphi} := \varphi\eta$ , then  $\kappa * \varphi = \kappa * \tilde{\varphi}$  and  $H\varphi = H\tilde{\varphi}$  on  $[x - \delta, x + \delta]$ . We have

$$\kappa * \tilde{\varphi} = (J * m - m) * \tilde{\varphi} = J * m * \tilde{\varphi} - m * \tilde{\varphi}$$

because  $\tilde{\varphi}$  and  $J$  are compactly supported. Since

$$\int_{\mathbb{R}} \int_{\mathbb{R}} m(x-y)\tilde{\varphi}(y)\theta''(x)dydx = - \int_{\mathbb{R}} \tilde{\varphi}(y) \int_{\mathbb{R}} m'(x-y)\theta'(x)dx dy = \int_{\mathbb{R}} \tilde{\varphi}(y)\theta(y)dy$$

for any  $\theta \in C_0^\infty(\mathbb{R})$ , we see that  $(m * \tilde{\varphi})'' = \tilde{\varphi}$  in the distributional sense. Similarly, we have  $(J * m * \tilde{\varphi})'' = J * \tilde{\varphi}$ , and both equalities hold pointwise because the right-hand sides are continuous functions. Thus  $(\kappa * \tilde{\varphi})'' = H\tilde{\varphi}$ , so  $(\kappa * \varphi)''(x) = (H\varphi)(x)$ . This holds for any  $x \in \mathbb{R}$ , yielding (6.4).

The properties of  $\phi$  and (6.2) show  $\psi > 0$  and  $\lim_{x \rightarrow \infty} \psi(x) = 0$ . Then (6.5) and  $\lambda > a_+$  show  $\psi' < 0$ . We also claim the following.

**Lemma 6.1.** *There is  $m_s = m_s(J)$  such that  $\lim_{s \searrow 0} m_s = 0$  and  $|\psi'(x)| \leq m_{\lambda-a_-} \psi(x)$ . In particular,  $e^{-m_{\lambda-a_-} \delta} \psi(x) \leq \psi(x-y) \leq e^{m_{\lambda-a_-} \delta} \psi(x)$  whenever  $|y| \leq \delta$ .*

*Proof.* With  $C = C_J(1 + \lambda - a_-)$  from Lemma 2.2, and from the remark following it, we have

$$(6.6) \quad \frac{e^{-C\delta}}{1 + \lambda - a_-} \phi(x) \leq \psi(x) \leq (1 + \lambda - a_-) e^{C\delta} \phi(x).$$

Then (6.5) and (6.6) give

$$\psi''(x) \leq \frac{e^{C\delta}(\lambda - a_-)(1 + \lambda - a_-)}{\|\kappa\|_{L^1}} \psi(x),$$

which then implies

$$-\psi'(x) \leq \sqrt{\frac{e^{C\delta}(\lambda - a_-)(1 + \lambda - a_-)}{\|\kappa\|_{L^1}}} \psi(x).$$

To see the latter, let  $\mu$  be the constant on the right-hand side of the above inequality. Recall that  $\psi'' \leq \mu^2 \psi$  and  $\psi, \psi'' > 0 > \psi'$ . Thus  $Q := -\psi'/\psi > 0$  satisfies  $Q' \geq Q^2 - \mu^2$ . So if  $Q(x_0) > \mu$  for some  $x_0 \in \mathbb{R}$ , then  $Q' > 0$  on  $(x_0, \infty)$ . Together with  $Q' \geq Q^2 - \mu^2$  this shows that  $Q$  must blow up at some  $x_1 \in (x_0, \infty)$ , a contradiction. Thus  $Q \in (0, \mu]$ , as claimed.

So we can let  $m_{\lambda-a_-}$  be this  $\mu$ , and  $\lim_{s \searrow 0} m_s = 0$  is obvious.  $\square$

**Lemma 6.2.** *There are  $l_s = l_s(J) < L_s = L_s(J)$  such that  $\lim_{s \searrow 0} l_s = \lim_{s \searrow 0} L_s = 1$  and  $l_{\lambda-a_-} \phi(x) \leq \psi(x) \leq L_{\lambda-a_-} \phi(x)$ .*

*Proof.* Let  $M_s := e^{-m_s \delta}$ , with  $m_s$  from Lemma 6.1, and define

$$(6.7) \quad \mu := \inf_{x \in \mathbb{R}} \frac{\psi(x)}{\phi(x)}, \quad \nu := \sup_{x \in \mathbb{R}} \frac{\psi(x)}{\phi(x)}.$$

We have  $0 < \mu \leq \nu < \infty$  by (6.6). Given any  $\varepsilon > 0$ , let  $x_0$  be such that

$$(6.8) \quad (1 - \varepsilon)\nu \leq \frac{\psi(x_0)}{\phi(x_0)} \leq \nu.$$

If  $|y| \leq \delta$ , then by Lemma 6.1,

$$(6.9) \quad \frac{\phi(x_0 - y)}{\phi(x_0)} = \frac{\phi(x_0 - y)\psi(x_0 - y)\psi(x_0)}{\psi(x_0 - y)\psi(x_0)\phi(x_0)} \geq (1 - \varepsilon)M_{\lambda-a_-}.$$

Thus we find that

$$\begin{aligned} \int_{-\delta}^{\delta} J(y)[\phi(x_0 - y) - \phi(x_0)]_+ dy &= H\phi(x_0) + \int_{-\delta}^{\delta} J(y)[\phi(x_0 - y) - \phi(x_0)]_- dy \\ &\leq (\lambda - a_-)\phi(x_0) + [1 - (1 - \varepsilon)M_{\lambda-a_-}]\phi(x_0). \end{aligned}$$

So by the definition of  $\psi$  and (6.2),

$$\begin{aligned} \psi(x_0) &= \phi(x_0) + \frac{1}{\|\kappa\|_{L^1}} \int_{-\delta}^{\delta} \kappa(y)[\phi(x_0 - y) - \phi(x_0)]dy \\ &\leq \phi(x_0) + \frac{\delta^2}{2\|\kappa\|_{L^1}} \int_{-\delta}^{\delta} J(y)[\phi(x_0 - y) - \phi(x_0)]_+ dy \\ &\leq \phi(x_0) + \frac{\delta^2}{2\|\kappa\|_{L^1}} [\lambda - a_- + 1 - (1 - \varepsilon)M_{\lambda-a_-}] \phi(x_0). \end{aligned}$$

Hence (6.8) shows

$$(1 - \varepsilon)\nu \leq 1 + \frac{\delta^2}{2\|\kappa\|_{L^1}} [\lambda - a_- + 1 - (1 - \varepsilon)M_{\lambda-a_-}].$$

Taking  $\varepsilon \rightarrow 0$  yields

$$\nu \leq 1 + \frac{\delta^2}{2\|\kappa\|_{L^1}} [\lambda - a_- + 1 - M_{\lambda-a_-}] =: L_{\lambda-a_-} = L_{\lambda-a_-}(J),$$

and  $\lim_{s \searrow 0} L_s = 1$  follows from the same for  $M_s$ , which is due to Lemma 6.1.

A similar argument, using  $\psi(x_0)\phi(x_0)^{-1} \leq (1 + \varepsilon)\mu$  to show  $\phi(x_0 - y)\phi(x_0)^{-1} \leq (1 + \varepsilon)M_{\lambda-a_-}^{-1}$  for  $|y| \leq \delta$  and then

$$\begin{aligned} - \int_{-\delta}^{\delta} J(y)[\phi(x_0 - y) - \phi(x_0)]_- dy &= H\phi(x_0) - \int_{-\delta}^{\delta} J(y)[\phi(x_0 - y) - \phi(x_0)]_+ dy \\ &\geq (\lambda - a_+) \phi(x_0) - [(1 + \varepsilon)M_{\lambda-a_-}^{-1} - 1] \phi(x_0), \end{aligned}$$

shows (recall that  $\lambda > a_+$ )

$$\mu \geq 1 - \frac{\delta^2}{2\|\kappa\|_{L^1}} [M_{\lambda-a_-}^{-1} - 1] =: l_{\lambda-a_-} = l_{\lambda-a_-}(J).$$

Again,  $\lim_{s \searrow 0} l_s = 1$  is immediate. □

To prove Lemma 3.2, it suffices to show  $\phi(x - y) \leq C_{\lambda-a_-} \phi(x)$  whenever  $|y| \leq \delta$ , where  $C_s = C_s(J)$  and  $\lim_{s \searrow 0} C_s = 1$  (then  $\gamma_s := C_s - 1$ ). By Lemmas 6.2 and 6.1,

$$\phi(x - y) \leq l_{\lambda-a_-}^{-1} \psi(x - y) \leq l_{\lambda-a_-}^{-1} e^{m\lambda-a_- \delta} \psi(x) \leq l_{\lambda-a_-}^{-1} e^{m\lambda-a_- \delta} L_{\lambda-a_-} \phi(x).$$

Hence we set  $C_s := l_s^{-1} L_s e^{m_s \delta}$ , and the proof is finished.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706