DOMAINS OF DISCONTINUITY
FOR ALMOST-FUCHSIAN GROUPS

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ABSTRACT. An almost-Fuchsian group $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3)$ is a quasi-Fuchsian group such that the quotient hyperbolic manifold $\mathbb{H}^3/\Gamma$ contains a closed incompressible minimal surface with principal curvatures contained in $(-1, 1)$. We show that the domain of discontinuity of an almost-Fuchsian group contains many balls of a fixed spherical radius $R > 0$ in $\mathbb{C} \cup \{\infty\} = \partial_\infty(\mathbb{H}^3)$. This yields a necessary condition for a quasi-Fuchsian group to be almost-Fuchsian which involves only conformal geometry. As an application, we prove that there are no doubly-degenerate geometric limits of almost-Fuchsian groups.

1. INTRODUCTION

The systematic study of closed minimal surfaces in hyperbolic 3-manifolds began with the work of Uhlenbeck in the early 1980s [Uhl83]. There, she identified a class of quasi-Fuchsian hyperbolic 3-manifolds, the almost-Fuchsian manifolds, which contain a unique closed, incompressible minimal surface which has principal curvatures in $(-1, 1)$. The structure of almost-Fuchsian manifolds has been studied considerably by a number of authors [GHW10], [HW13]. In particular, the invariants arising from quasi-conformal Kleinian group theory (e.g. Hausdorff dimension of limit sets, distance between conformal boundary components) are controlled by the principal curvatures of the unique minimal surface.

Given an almost-Fuchsian manifold M, this paper further explores the relationship between the geometry of the unique minimal surface and the conformal surface at infinity. As a result we will show that there are no doubly-degenerate geometric limits of almost-Fuchsian groups. This will be achieved through a careful study of the hyperbolic Gauss map from the minimal surface, which serves to communicate information from the minimal surface to the conformal surface at infinity.

We briefly summarize the strategy: if Γ is the holonomy group of an almost-Fuchsian manifold $M = \mathbb{H}^3/\Gamma$, consider the Γ-invariant minimal disk $\widetilde{\Sigma} \subset \mathbb{H}^3$ which projects to the unique closed, minimal surface in M. We locate disks $D_i \subset \widetilde{\Sigma}$ which are very close to being totally geodesic; the hyperbolic Gauss map from these disks is nicely behaved and in particular satisfies a Koebe-type theorem [AG85], namely its image contains disks of bounded radius. Epstein [Eps86] has studied the hyperbolic Gauss map extensively. In particular, we apply his work to show that the images of the D_i under the hyperbolic Gauss map are contained in the domain

of discontinuity for Γ. These images form barriers for the limit set of Γ. Since the limit set of a doubly-degenerate group is equal to $\partial_\infty(\mathbb{H}^3)$, these barriers form an obstruction to Γ approaching a doubly-degenerate group.

If Σ is a surface immersed in some hyperbolic 3-manifold M, the immersion induces a conformal structure σ on Σ which underlies the induced Riemannian metric g. Provided the immersion is minimal, Hopf [Hop54] showed that the second fundamental form

$$B = B_{11}dx^2 + 2B_{12}dxdy + B_{22}dy^2$$

appears as the real part of

$$\alpha = (B_{11} - iB_{12})\ dz^2,$$

which is a holomorphic quadratic differential on (Σ, σ). The norm $\|\alpha\|_g$ measures how much Σ bends inside of M. In §3, we prove a Harnack inequality for $\|\alpha\|_g$ satisfying some bound $\|\alpha\|_g \leq K$. First we show that the norm of a bounded holomorphic quadratic differential on the hyperbolic plane satisfies a Harnack inequality. Then, we prove the induced metric g is uniformly comparable to the hyperbolic metric in that conformal class. Therefore, the growth of the principal curvatures of a minimal immersion is bounded; the surface cannot bend too much, too quickly. The disks D_i mentioned in the previous paragraph are obtained by taking balls around the zeros of α; the Harnack inequality ensures we may pick balls of a uniform radius.

In §4, we begin the study of the hyperbolic Gauss map: given an oriented surface $\widetilde{\Sigma} \subset \mathbb{H}^3$ in hyperbolic 3-space with oriented unit normal field N, the pair of hyperbolic Gauss maps $\mathcal{G}^\pm : \widetilde{\Sigma} \to \partial_\infty(\mathbb{H}^3)$ are defined by recording the endpoint of the geodesic ray in the direction of $\pm N$. We show that the images of the disks $D_i \subset \widetilde{\Sigma}$ under the hyperbolic Gauss map contain disks of a bounded radius in $\partial_\infty(\mathbb{H}^3)$. To achieve this, we utilize the generalization of the Koebe $\frac{1}{4}$-theorem to quasi-conformal maps due to Astala and Gehring [AG85].

In §5 the above technical results are used to prove that the domain of discontinuity of an almost-Fuchsian group Γ contains a disk (in fact many) of fixed radius in $\mathbb{C}$. As we mentioned above, this is an application of work of Epstein who showed [Eps86] that the hyperbolic Gauss map from the minimal Γ-invariant disk $\widetilde{\Sigma}$ is a (quasi-conformal) diffeomorphism onto the domain of discontinuity for Γ. In particular, we obtain definite regions R_i of $\partial_\infty(\mathbb{H}^3)$ into which the limit set of Γ cannot penetrate. Lastly in §5.1 we prove the promised theorem:

Theorem 1.1. *There are no doubly-degenerate geometric limits of almost-Fuchsian groups.*

There is a technical issue in the proof of the above theorem; in order to estimate the size of the regions R_i into which the limit set cannot penetrate, we must conjugate the group Γ by some element of $\mathrm{Isom}(\mathbb{H}^3)$ to put the surface $\widetilde{\Sigma}$ into a normalized position. It is possible that given a sequence of almost-Fuchsian groups Γ_n, the elements I_n by which we conjugate leave every compact set of $\mathrm{Isom}(\mathbb{H}^3)$. Since the action of $\mathrm{Isom}(\mathbb{H}^3)$ on $\partial_\infty(\mathbb{H}^3)$ is not isometric, this could destroy any control we had gained on the size of the R_i. Proving that we may always conjugate the group into a normalized position by an isometry with some bounded translation distance relies strongly on the geometry of the minimal surface.

1.1. **Historical context.** In [Uhl83], Uhlenbeck mentioned that "if the area minimizing surfaces in M are isolated, we expect a large number of minimal surfaces in

quasi-Fuchsian manifolds near M." The M she refers to is the doubly-degenerate manifold which arises as a cyclic cover of a closed manifold fibering over the circle. It seems likely that any doubly-degenerate hyperbolic 3-manifold M contains infinitely many distinct incompressible (stable) minimal surfaces which are homotopy equivalent to M. Examples of this phenomenon are provided by the doubly-degenerate manifolds M which arise as cyclic covers of closed hyperbolic 3-manifolds fibering over the circle. In this case, there is an infinite cyclic group of isometries of M. The general existence theory ([FHS83], [SU82] and [SY79]) yields an incompressible minimal surface Σ in the closed manifold which M covers. Σ lifts to an incompressible minimal surface in M. The translates of Σ by the infinite cyclic group of deck transformations yield infinitely many distinct incompressible minimal surfaces in M. Theorem 1.1 supports a general philosophy that the number of closed incompressible (stable) minimal surfaces in a quasi-Fuchsian manifold should serve as a measure of how far that manifold is from being Fuchsian.

In [HW13], the authors show that if $M = \mathbb{H}^3/\Gamma$ is an almost-Fuchsian manifold and λ is the maximum positive principal curvature of the unique closed minimal surface in M, then the Hausdorff dimension of the limit set of Γ is at most $1 + \lambda^2$. The question remains whether there exists a sequence of almost-Fuchsian groups Γ_n such that the Hausdorff dimension of the limit set approaches 2. Theorem 1.1 rules out the most naive way in which this might occur.

With these more technical comments in place, we comment on the main achievement of the paper. The primary advance in this paper is a soft qualitative necessary condition for a quasi-Fuchsian group Γ to be almost-Fuchsian. The recipe is as follows: look at the domain of discontinuity. For some small threshold $R > 0$, which we calibrate in the paper, if the domain of discontinuity does not contain many balls of radius larger than R as measured in the spherical metric on $\partial_\infty(\mathbb{H}^3)$, then Γ is not almost-Fuchsian. Since almost-Fuchsian groups are defined directly using the hyperbolic geometry of the quasi-Fuchsian manifold, it is quite interesting that there is a necessary condition which can be stated solely in terms of the complex geometry of Möbius transformations.

1.2. **Plan of the paper.** The preliminary section §2 assembles all of the definitions and basic facts we require from the theory of Kleinian groups, minimal surfaces and introduces the hyperbolic Gauss map. Then in §3 we prove the main technical propositions controlling the growth of the principal curvatures of a minimal surface in $\mathbb{H}^3$. In §4 we prove that under the hyperbolic Gauss map nearly flat regions in the minimal surface are sent to disks of controlled radius in the image of the Gauss map. Finally, in §5 we prove that the domain of discontinuity of an almost-Fuchsian group contains disks of a fixed radius. As an application we show that there are no doubly-degenerate geometric limits of almost-Fuchsian groups.

2. Preliminaries

In all that follows, Σ denotes a closed, connected, oriented, smooth surface of genus greater than 1.

2.1. **Kleinian groups and hyperbolic geometry.** The 3-dimensional hyperbolic space $\mathbb{H}^3$ is the unique, 3-dimensional complete, 1-connected Riemannian manifold with constant negative sectional curvature -1. We will primarily work in the upper

half-space model for hyperbolic space which consists of the smooth manifold

$$\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ | x_3 > 0\}$$

together with the Riemannian metric

$$dh_3^2 = \frac{\delta_{ij} dx^i dx^j}{x_3^2}.$$

For $\mathbb{H}^3$, we use coordinates $(z, t) \in \mathbb{H}^3$ with $z = x + iy \in \mathbb{C}$ and $t > 0$. A Kleinian group is a discrete (torsion-free) subgroup $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3) \simeq \mathrm{PSL}(2, \mathbb{C})$ of orientation-preserving isometries of hyperbolic 3-space. Given a Kleinian group Γ, the action on $\mathbb{H}^3$ extends to an action on the conformal boundary $\partial_\infty(\mathbb{H}^3) \simeq \mathbb{C} \cup \{\infty\}$ by Möbius transformations. This action divides $\partial_\infty(\mathbb{H}^3)$ into two disjoint subsets: $\Lambda(\Gamma)$ and $\Omega(\Gamma)$. The *limit set* $\Lambda(\Gamma)$ is defined to be the smallest non-empty, Γ-invariant closed subset of $\partial_\infty(\mathbb{H}^3)$. The *domain of discontinuity* $\partial_\infty(\mathbb{H}^3) \backslash \Lambda(\Gamma) = \Omega(\Gamma)$ is the largest open set on which Γ acts properly discontinuously. The quotient $M = \mathbb{H}^3/\Gamma$ is a complete hyperbolic 3-manifold with *holonomy* group Γ.

Given a sequence of Kleinian groups Γ_n, we say that Γ_n converges *geometrically* to the group Γ if,

(1) For all $\gamma \in \Gamma$ there exists $\gamma_n \in \Gamma_n$ such that $\gamma_n \to \gamma$ in $\mathrm{Isom}(\mathbb{H}^3)$.
(2) If $\gamma_n \in \Gamma_n$ and $\{\gamma_{n_j}\} \subset \{\gamma_n\}$ is a subsequence such that $\gamma_{n_j} \to \gamma$, then $\gamma \in \Gamma$.

We denote geometric convergence by $\Gamma_n \to \Gamma$. It is well known [CEG87] that the geometric convergence of Kleinian groups is equivalent to the base-framed, Gromov-Hausdorff convergence of the associated quotient manifolds.

Equipping $\partial_\infty(\mathbb{H}^3) = \mathbb{C} \cup \{\infty\}$ with the spherical metric $d_{\mathbb{S}^2} = \frac{|dz|^2}{(1+|z|^2)^2}$, a sequence of closed subsets $A_n \subset \mathbb{C} \cup \{\infty\}$ converges to $A \subset \mathbb{C} \cup \{\infty\}$ in the *Hausdorff* topology if $A_n \to A$ with respect to the distance

$$d(A_n, A) = \inf \left\{ r : A_n \subset \bigcup_{x \in A} B_{\mathbb{S}^2}(x, r) \text{ and } A \subset \bigcup_{x \in A_n} B_{\mathbb{S}^2}(x, r) \right\}.$$

If a sequence of Kleinian groups Γ_n converges geometrically to Γ, it is a simple exercise to verify that the limit set of Γ satisfies $\Lambda(\Gamma) \subset \lim_{n \to \infty} \Lambda(\Gamma_n)$ where the limit is taken with respect to the Hausdorff topology.

Now we restrict to the case that there is an isomorphism $\rho : \pi_1(\Sigma, p_0) \to \Gamma$. The representation ρ is *quasi-Fuchsian* if and only if $\Omega(\Gamma)$ consists of precisely two invariant, connected, simply-connected components. The quotient $\Omega(\Gamma)/\Gamma = X^+ \cup \overline{X^-}$ is a disjoint union of two marked Riemann surfaces $(X^+, \overline{X^-})$, each diffeomorphic to Σ, where the bar over X^- denotes the surface with the opposite orientation. The marking, which is a choice of homotopy equivalence $f^\pm : \Sigma \to X^\pm$, is determined by the requirement that $f^\pm_* = \rho$. Conversely, we have the Bers' simultaneous uniformization theorem [Ber60],

Theorem 2.1. *Given an ordered pair of marked, closed Riemann surfaces $(X^+, \overline{X^-})$ each diffeomorphic to Σ, there exists an isomorphism $\rho : \pi_1(\Sigma, p) \to \Gamma$, unique up to conjugation in $\mathrm{PSL}(2, \mathbb{C})$, such that $\Omega(\Gamma)/\Gamma = X^+ \cup \overline{X^-}$.*

It is a striking feature of the diversity of hyperbolic 3-manifolds that there exist isomorphisms $\rho : \pi_1(\Sigma, p) \to \Gamma$ onto Kleinian groups which are not quasi-Fuchsian. A Kleinian surface group $\Gamma \simeq \pi_1(\Sigma, p)$ is called *doubly degenerate* if the domain

of discontinuity for Γ is empty; $\Omega(\Gamma) = \emptyset$. The first explicit examples of doubly-degenerate groups were free groups of rank 2 discovered by Jørgensen [Jor77].

Due to the resolution of the ending lamination conjecture, the isometry classification of doubly-degenerate manifolds is now well understood. Nonetheless, their fine scale geometry is extremely intricate and interesting. Bonahon showed [Bon86] that they are diffeomorphic to $\Sigma \times \mathbb{R}$; then the ending lamination theorem (for surface groups) [Min10], [BCM12] in conjunction with Thurston's double limit theorem [Thu] can be applied to see that every doubly-degenerate group is a geometric limit of quasi-Fuchsian groups.

2.2. Minimal surfaces and almost-Fuchsian manifolds.
Let (M, h) be a 3-dimensional Riemannian manifold. Given an immersion $f : \Sigma \to M$, the area of f is the area of the Riemannian manifold (Σ, g) where $g = f^*(h)$. Suppose $f(\Sigma)$ is two-sided in M and let ν be a globally defined unit normal vector field. Then given X and Y tangent vectors to $f(\Sigma)$, we have the second fundamental form B defined by

$$B(X, Y) = h(\nabla_X \nu, Y)$$

where ∇ is the Levi-Cevita connection of h. The second fundamental form B defines a symmetric, contravariant 2-tensor on Σ. By the spectral theorem for self-adjoint operators, B is diagonalizable with eigenvalues λ_1, λ_2 whose product $\lambda_1 \lambda_2$ is a smooth function on Σ. They are called the *principal curvatures* of the immersion and $\|B\|_g^2 = \lambda_1^2 + \lambda_2^2$.

Contracting with the metric g yields the mean curvature of the immersion,

$$H = g^{ij} B_{ij},$$

where repeated upper and lower indices are to be summed over. The immersion $f : \Sigma \to M$ is *minimal* if $H = \lambda_1 + \lambda_2 = 0$. Equivalently, f is a critical point for the area functional defined on the space of immersions of Σ into M. A minimal surface is said to be *least area* if $f : \Sigma \to M$ has least area among all maps homotopic to f; least area maps are minimal.

Now suppose M is a hyperbolic 3-manifold. If $f : \Sigma \to M$ is an immersion as above, the Gauss equation of the immersion is

$$(2.1) \qquad\qquad K_g = -1 + \lambda_1 \lambda_2$$

where K_g is the sectional curvature of the metric g.

If f is a minimal immersion, it follows that

$$(2.2) \qquad\qquad K_g = -1 - \lambda^2$$

where we have packaged the principal curvatures into a single smooth function $\lambda^2 = \lambda_1^2 = \lambda_2^2$ since they satisfy $\lambda_1 = -\lambda_2$. Then

$$(2.3) \qquad\qquad \frac{1}{2}\|B\|_g^2 = \lambda^2$$

and

$$(2.4) \qquad\qquad K_g = -1 - \frac{1}{2}\|B\|_g^2.$$

We now present the principal object of study.

Definition 2.2. A quasi-Fuchsian 3-manifold M is almost-Fuchsian if there exists a minimal immersion $f : \Sigma \to M$ with $\|B\|_g^2(x) < 2$ for all $x \in \Sigma$.

The following classification theorem is due to Uhlenbeck [Uhl83].

Theorem 2.3. *Suppose M is almost-Fuchsian and homotopy equivalent to Σ. Then,*

- *$f : \Sigma \to M$ such that $\|B\|_g^2(x) < 2$ is the only closed minimal surface of any kind in M.*
- *$f : \Sigma \to M$ is incompressible, i.e., it is a π_1-injective, smooth embedding.*
- *The normal exponential map from $f(\Sigma)$ is a diffeomorphism onto M. In these coordinates the metric on $\Sigma \times \mathbb{R}$ can be expressed as*

  $dt^2 + g(\cosh{(t)}\ \mathbb{I}(\cdot) + \sinh{(t)}\ \mathbb{S}(\cdot), \cosh{(t)}\ \mathbb{I}(\cdot) + \sinh{(t)}\ \mathbb{S}(\cdot)).$

  Here $\mathbb{I}$ is the identity operator and $\mathbb{S}(X) = \nabla_X(\nu)$ is the shape operator associated to the second fundamental form.

The following theorem, first observed by Hopf [Hop54], will be of great importance to us.

Theorem 2.4. *Let $f : \Sigma \to M$ be a minimal immersion and M a constant curvature Riemannian 3-manifold. Let z be a local holomorphic coordinate for the Riemann surface structure induced on Σ. If B is the second fundamental form of the immersion, then the expression*

$$\alpha = (B_{11} - iB_{12})dz^2$$

is an invariantly defined holomorphic quadratic differential on Σ with $\mathfrak{Re}(\alpha) = B$.

In terms of the quadratic differential α, the almost-Fuchsian condition translates to the condition that $\lambda^2 = \|\alpha\|_g^2 < 1$. Uhlenbeck's Theorem 2.3 along with an implicit function theorem argument (see [Uhl83]) shows that the space of isotopy classes of almost-Fuchsian manifolds is parameterized via an open neighborhood of the zero section of the bundle of holomorphic quadratic differential over $\mathcal{T}(\Sigma)$ the Teichmuller space of Σ. Recall the Teichmuller space $\mathcal{T}(\Sigma)$ is the space of isotopy classes of complex structures on Σ compatible with the orientation. The bundle of holomorphic quadratic differentials can be identified with $T^*\mathcal{T}(\Sigma)$ the cotangent bundle of $\mathcal{T}(\Sigma)$.

We will denote the space (of isotopy classes) of almost-Fuchsian manifolds by $\mathcal{AF}$, and say that $(g, \alpha) \in \mathcal{AF}$ is an almost-Fuchsian manifold with data (g, α) if there exists an almost-Fuchsian manifold such that the unique minimal surface has induced metric g and second fundamental form $\mathfrak{Re}(\alpha)$. Throughout the paper, we are concerned with specific manifolds and not just manifolds up to isotopy, so $(g, \alpha) \in \mathcal{AF}$ is understood to be a representative of the isotopy class.

Given $(g, \alpha) \in \mathcal{AF}$ with holonomy group Γ, we call a lift of the unique, incompressible minimal surface in $\mathbb{H}^3/\Gamma$ to $\mathbb{H}^3$ an almost-Fuchsian disk with data (g, α). An almost-Fuchsian disk $\widetilde{\Sigma} \subset \mathbb{H}^3$ is *normalized* if

(1) The point $p = (0, 0, 1) \in \widetilde{\Sigma}$ and the oriented unit normal to $\widetilde{\Sigma}$ at p is $(0, 0, -1)$.
(2) The principal curvatures vanish at p: $\alpha(p) = 0$.

If $(g, \alpha) \in \mathcal{AF}$ with holonomy group Γ, we can always select $I \in \mathrm{Isom}(\mathbb{H}^3)$ such that the $I\Gamma I^{-1}$-invariant almost-Fuchsian disk is normalized. Such a choice of I is unique up to the action of the circle group $U(1) < \mathrm{Isom}(\mathbb{H}^3)$ of rotations about the z-axis in the upper half-space model of hyperbolic space.

2.3. **Hyperbolic Gauss map.** Details on the material in this section may be found in the paper of Epstein [Eps86]. Let $S \subset \mathbb{H}^3$ be an oriented, embedded surface. Let N be a global unit normal vector field on S such that if $\{X, Y\} \subset TS$ is an oriented basis of the tangent space of S, then $\{X, Y, N\}$ extends to an oriented basis of $T\mathbb{H}^3$. Given $p \in S$, let $\gamma_p(t)$ be the unit speed geodesic ray with initial point $\gamma_p(0) = p$ and initial velocity $\frac{d\gamma_p(t)}{dt}|_{t=0} = N$.

Definition 2.5. The forward hyperbolic Gauss map associated to S is the map $\mathcal{G}_S^+ : S \to \partial_\infty(\mathbb{H}^3)$ defined by

$$\mathcal{G}_S^+(p) = \lim_{t \to +\infty} \gamma_p(t).$$

If we use $-N$ in the definition we shall call the associated map $\mathcal{G}_S^-$ the backwards hyperbolic Gauss map.

We quickly recall the results we shall utilize from [Eps84], [Eps86]. Let $f : \mathbb{D} \to \mathbb{H}^3$ be an immersion of the unit disk $\mathbb{D}$ such that the principal curvatures are always contained in $(-1 + \varepsilon, 1 - \varepsilon)$ for some $\varepsilon > 0$. Epstein proves,

Theorem 2.6 ([Eps84], [Eps86]). *Let $S = f(\mathbb{D})$. The immersion $f : \mathbb{D} \to \mathbb{H}^3$ satisfies:*

(1) *f is a proper embedding.*
(2) *f extends to an embedding $\overline{f} : \overline{\mathbb{D}} \to \mathbb{H}^3 \cup \partial_\infty(\mathbb{H}^3)$. So $\partial_\infty(S) = \overline{f}(\partial\mathbb{D} \setminus \mathbb{D})$ is a Jordan curve.*
(3) *Each hyperbolic Gauss map $\mathcal{G}_S^\pm : S \to \partial_\infty(\mathbb{H}^3)$ is a quasi-conformal diffeomorphism onto a component of $\partial_\infty(\mathbb{H}^3) \setminus \partial_\infty(S)$. Note that the Jordan curve theorem implies $\partial_\infty(\mathbb{H}^3) \setminus \partial_\infty(S)$ consists of two components.*
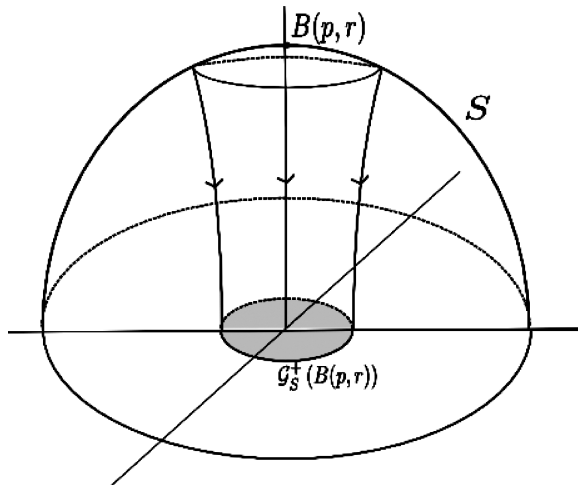

FIGURE 1. The image of a disk under the forward hyperbolic Gauss map from a totally geodesic plane in $\mathbb{H}^3$.

Recall that given a homeomorphism $f : D_1 \to D_2$ between domains $D_1, D_2 \subset \mathbb{C}$, we say that f is a *quasi-conformal* homeomorphism if:

- $f \in H^1_{loc}(D_1)$, that is, both f and its distributional derivatives f_z, $f_{\bar{z}}$ are locally square-integrable on D_1.
- There exists $\mu \in L^\infty(D_1)$ with $\|\mu\|_{L^\infty} < 1$ such that $f_{\bar{z}} = \mu f_z$ in the sense of distributions.

Remark. The quasiconformal maps we meet will be smooth, so the references to distributions above can be safely ignored.

Defining the *dilatation* of f to be the quantity

$$(2.5) \qquad K = \frac{1 + \|\mu\|_{L^\infty}}{1 - \|\mu\|_{L^\infty}},$$

f is said to be K-quasi-conformal. Observe that a 1-quasi-conformal map is conformal.

From Theorem 2.6 it immediately follows that if $\widetilde{\Sigma} \subset \mathbb{H}^3$ is an almost-Fuchsian disk invariant under an almost-Fuchsian group Γ, each hyperbolic Gauss map

$$\mathcal{G}^{\pm}_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \partial_\infty(\mathbb{H}^3)$$

is a (quasi-conformal) diffeomorphism onto one component of the domain of discontinuity Ω of Γ. Furthermore, $\pi_1(\Sigma, p_0) \simeq \Gamma$ implies $\partial_\infty(\widetilde{\Sigma})$ identifies homeomorphically with the limit set $\Lambda(\Gamma)$.

3. Technical estimates

This section contains some technical results which give us control on the growth of the principal curvature function for a minimal surface immersed in a hyperbolic 3-manifold.

We begin with a basic result establishing a bound on the growth of the L^∞-norm of a bounded, holomorphic quadratic differential $\alpha = f(z)\, dz^2$ on $\mathbb{H}^2$. We denote the canonical bundle of holomorphic 1-forms on $\mathbb{H}^2$ by $K_{\mathbb{H}^2}$. In this section we will utilize the Poincaré disk model of the hyperbolic plane consisting of the unit disk $\mathbb{D} \subset \mathbb{C}$ with the metric

$$\frac{4|dz|^2}{(1 - |z|^2)^2}.$$

Proposition 3.1. *Let $\alpha \in H^0\left(\mathbb{H}^2, K^2_{\mathbb{H}^2}\right)$ be a holomorphic quadratic differential on the hyperbolic plane and assume there exists a $C > 0$ such that $\sup_{z \in \mathbb{H}^2} \|\alpha\|_{\mathbb{H}^2}(z) \leq C$. Assume there is $z_0 \in \mathbb{H}^2$ such that $\alpha(z_0) = 0$. Then for all $\varepsilon > 0$, there exists $r = r(\varepsilon, C) > 0$ such that*

$$\|\alpha\|_{\mathbb{H}^2}(x) < \varepsilon$$

for all $x \in B_{\mathbb{H}^2}(z_0, r)$ where $B_{\mathbb{H}^2}(z_0, r)$ is the ball of hyperbolic radius r centered at z_0.

Proof. Without loss of generality, assume $z_0 = 0 \in \mathbb{D}$. Writing $\alpha(z) = f(z)\, dz^2$, the condition

$$(3.1) \qquad \sup_{z \in \mathbb{H}^2} \|\alpha\|_{\mathbb{H}^2}(z) \leq C$$

becomes

$$\tag{3.2} \frac{(1-|z|^2)^2}{4}|f(z)| \le C$$

for all $z \in \mathbb{D}$. Define an auxiliary holomorphic function defined on $\mathbb{D}$ by

$$\tag{3.3} g(z) = C'f\left(\frac{1}{2}z\right)$$

with

$$\tag{3.4} C' = \frac{\left(1-\frac{1}{4}\right)^2}{4C}.$$

Utilizing (3.2),

$$\tag{3.5} |g(z)| = \frac{\left(1-\frac{1}{4}\right)^2}{4C}\left|f\left(\frac{1}{2}z\right)\right| \le \frac{\left(1-\frac{1}{4}\right)^2}{\left(1-\frac{|z|^2}{4}\right)^2} \le 1$$

which proves that

$$|g(z)| \le 1.$$

By construction $g(0) = 0$, hence the Schwarz lemma implies

$$\tag{3.6} |g(z)| \le |z|.$$

Thus,

$$\tag{3.7} \|\alpha\|_{\mathbb{H}^2}\left(\frac{1}{2}z\right) = \frac{(1-\frac{1}{4}|z|^2)^2}{4}\left|f\left(\frac{1}{2}z\right)\right| \le \frac{(1-\frac{1}{4}|z|^2)^2}{4C'}|z| < \varepsilon$$

provided $|z| < 4C'\varepsilon$. Thus, on the Euclidean disk of radius $2C'\varepsilon$ the estimate stated in the theorem holds. Translating to hyperbolic radius we obtain

$$\tag{3.8} r = \log\left(\frac{1+2C'\varepsilon}{1-2C'\varepsilon}\right).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Our next task is to apply the bound from Proposition 3.1 to the case of the norm of a quadratic differential arising from a minimal surface in an almost-Fuchsian manifold. Fortunately, the next lemma shows that the induced metric on the surface is uniformly comparable to the hyperbolic metric. Recall that given a Riemannian metric g on Σ, the Koebe-Poincaré uniformization theorem provides a unique hyperbolic metric in the conformal class of g.

Lemma 3.2. *Let $(g,\alpha) \in \mathcal{AF}$ and suppose h is the unique hyperbolic metric in the conformal class of g with $g = e^{2u}h$. Then,*

$$\frac{-\ln(2)}{2} < u \le 0.$$

Proof. Since $g = e^{2u}h$ and the sectional curvature of h is equal to -1, we have the standard equation,

$$K_g = e^{-2u}(-\Delta_h u - 1),$$

where K_g is the sectional curvature of g. A further application of the Gauss equation (2.1) reveals,

$$\tag{3.9} -1 - \|\alpha\|_g^2 = e^{-2u}(-\Delta_h u - 1).$$

Since $(g, \alpha) \in \mathcal{AF}$, the inequality $-1 - \|\alpha\|_g^2 > -2$ holds. Let $p \in \Sigma$ be a local minimum for u. Then,

$$-2 < e^{-2u}(-\Delta_h u - 1) \leq -e^{-2u}$$

since $-\Delta_h u(p) \leq 0$. Thus, $\frac{-\ln(2)}{2} < u$.

The inequality $u \leq 0$ is proved in [Uhl83]. This completes the proof. $\qquad\square$

Now we establish uniform estimates on the norm of α near a zero for any $(g, \alpha) \in \mathcal{AF}$. This will be crucial for our applications. In lieu of the previous lemma, the following proposition is a direct consequence of Proposition 3.1.

Proposition 3.3. *Let $(g, \alpha) \in \mathcal{AF}$ and $p \in \Sigma$ such that $\alpha(p) = 0$. Then for all $\varepsilon > 0$, there exists $r(\varepsilon) > 0$ such that*

$$\|\alpha\|_g(x) < \varepsilon$$

for all $x \in B_g(p, r)$, where $B_g(p, r)$ is the ball of radius r centered at p as measured in the metric g.

Remark. The above estimate is obvious for each (g, α) by the continuity of $\|\alpha\|_g$. The key point is the uniformity, namely that r depends only on $\varepsilon > 0$ and not on (g, α).

Remark. Note that α is a holomorphic quadratic differential on a closed Riemann surface of genus $g > 1$. Thus, they are holomorphic sections of the square of the canonical bundle K^2 which has degree $4g - 4$. By standard Riemann surface theory, α has $4g - 4$ zeros counting multiplicity.

Remark. The above proposition is equally true if we work in the metric universal cover $\widetilde{\Sigma}$ of Σ. This will be important in our applications.

Proof of Proposition 3.3. Let $g = e^{2u}h$ where h is the unique hyperbolic metric in the conformal class of g. Then $\|\alpha\|_g = e^{-2u}\|\alpha\|_h$. Combined with Lemma 3.2 this implies $\|\alpha\|_g < 2\|\alpha\|_h$. Let $p \in \Sigma$ be such that $\alpha(p) = 0$. Since $\|\alpha\|_g < 1$, Proposition 3.1 implies that for all $\varepsilon > 0$ we can find $r' > 0$ such that $\|\alpha\|_h(x) < \frac{\varepsilon}{2}$ for all $x \in B_h(p, r')$. Now, since $u > \frac{-\ln(2)}{2}$ we observe $B_g(p, r) \subset B_h(p, r')$ where $r = \frac{r'}{\sqrt{2}}$. It follows that

$$\|\alpha\|_g(x) < 2\|\alpha\|_h(x) < \varepsilon$$

for all $x \in B_g(p, r) \subset B_h(p, r')$. $\qquad\square$

By the Gauss equation, on $B_g(p, r)$ the sectional curvature K_g of g satisfies $-1 - \varepsilon^2 \leq K_g \leq -1$. Thus the disks $B_g(p, r)$ are very close to being totally geodesic. These are the nearly geodesic regions referred to in the introduction.

4. Thick regions in the Gauss map image

In this section we show that the nearly geodesic regions previously obtained are mapped, via the hyperbolic Gauss map, to regions in $\partial_\infty(\mathbb{H}^3)$ which have a uniformly bounded thickness. This will rely on the generalization of the Koebe $\frac{1}{4}$-theorem due to Astala and Gehring [AG85].

Let (g, α) be the data for a normalized almost-Fuchsian disk $\widetilde{\Sigma} \subset \mathbb{H}^3$. By the uniformization theorem, there exists a conformal diffeomorphism

$$(4.1) \qquad\qquad\qquad\qquad \phi : \mathbb{D} \to (\widetilde{\Sigma}, g)$$

satisfying $f(0) = p$. Since ϕ is conformal,

$$\phi^* g = e^{2u} h \tag{4.2}$$

for some $u \in C^\infty(\Sigma)$ where h is the hyperbolic metric on $\mathbb{D}$. If dV_g is the volume element arising from the metric g, then

$$\phi^* dV_g = e^{2u} dV_h. \tag{4.3}$$

Convention. The following **notation** is in place for the rest of this section: $\widetilde{\Sigma} \subset \mathbb{H}^3$ is a normalized almost-Fuchsian disk with data (g, α). Recall, this means $(0, 0, 1) = p \in \widetilde{\Sigma}$ is such that $\alpha(p) = 0$. Choose an $\varepsilon > 0$; then Proposition 3.3 yields $r > 0$ such that the norm of α is less than ε on $B_g(p, r)$. Whenever r or ε appear in statements below, it is these fixed quantities to which we refer.

Proposition 4.1. *Let $\widetilde{\Sigma}$ be a normalized almost-Fuchsian disk with data $(g, \alpha) \in \mathcal{AF}$. Let*

$$\phi : \mathbb{D} \to (\widetilde{\Sigma}, g) \tag{4.4}$$

be a uniformization such that $\phi(0) = p$. Then there exists $r_1 = r_1(r) > 0$ such that

$$B_{\mathbb{C}}(0, r_1) \subset \phi^{-1}(B_g(p, r)). \tag{4.5}$$

Furthermore,

$$J_\phi(z) > 2 \tag{4.6}$$

for all $z \in \mathbb{D}$. Here $J_\phi(z)$ is the Jacobian of ϕ at z calculated with respect to the Euclidean metric on $\mathbb{D}$. Also, $B_{\mathbb{C}}(0, r_1)$ is the Euclidean ball centered at 0 of radius r_1.

Proof. First, any metric g such that $(g, \alpha) \in \mathcal{AF}$ is uniformly comparable to the hyperbolic metric by Lemma 3.2. This implies that there exists a fixed annulus $\mathcal{A} \subset \mathbb{D}$ centered at the origin which contains the boundary of $\phi^{-1}(B_g(p, r))$ for every $(g, \alpha) \in \mathcal{AF}$. On a fixed compact subset of $\mathbb{D}$ (which is taken to be the closed ball bounded by the outer ring of the annulus $\mathcal{A}$), the hyperbolic metric is uniformly comparable to the Euclidean metric. Thus there exists $r_1 > 0$ independent of (g, α) such that

$$B_{\mathbb{C}}(0, r_1) \subset \phi^{-1}(B_g(p, r)). \tag{4.7}$$

We note that $B_{\mathbb{C}}(0, r_1)$ can be taken to be the ball bounded by the inner ring of the annulus $\mathcal{A}$.

Writing $\phi^*(g) = e^{2u} h$, then by (4.3), the Jacobian of ϕ (with respect to the Euclidean metric on $\mathbb{D}$) is given by

$$J_\phi(z) = \frac{4 e^{2u(z)}}{(1 - |z|^2)^2}. \tag{4.8}$$

Applying the estimate in Lemma 3.2 reveals

$$J_\phi(z) > 2 \tag{4.9}$$

for all $z \in \mathbb{D} = \mathbb{H}^2$. This completes the proof. $\qquad\square$

The following proposition which controls the distortion of the hyperbolic Gauss map is due to Epstein [Eps86].

Proposition 4.2. *Let $\widetilde{\Sigma}$ be a normalized almost-Fuchsian disk with data (g, α). Then:*

  (1) *The hyperbolic Gauss map*

$$(4.10) \qquad\qquad \mathcal{G}_{\widetilde{\Sigma}}^+ : B_g(p, r) \to \mathbb{C}$$

  is quasi-conformal with dilatation $K \leq \left(\frac{1+\varepsilon}{1-\varepsilon} \right)^{\frac{1}{2}}$.

  (2) *There exists a universal constant $C > 0$ such that*

$$(4.11) \qquad\qquad J_{\mathcal{G}_{\widetilde{\Sigma}}^+}(x) > C(1 - \varepsilon^2)$$

  for all $x \in B_g(p, r)$ where the Jacobian of $\mathcal{G}^+$ is computed with respect to the Euclidean metric on $\mathbb{C}$.

Proof. These facts can be found on pages $120 - 121$ of [Eps86]. Epstein uses the spherical metric on $\mathbb{C}$, but in a bounded neighborhood of zero the spherical metric and Euclidean metric are uniformly comparable. Thus, the C we obtain in (4.11) is some multiple of that obtained by Epstein. $\qquad\square$

Next, we introduce the generalization of Koebe's $\frac{1}{4}$-theorem due to Astala and Gehring [AG85]. Let $U, V \subset \mathbb{C}$ be open domains and

$$(4.12) \qquad\qquad f : U \to V$$

a K-quasi-conformal mapping with Jacobian J_f. Then $\log J_f$ is locally integrable and the quantity

$$(4.13) \qquad\qquad (\log J_f)_B = \frac{1}{|B|} \int_B \log J_f \ dx,$$

with $B \subset U$ a ball, is well defined. For each $x \in U$, define

$$(4.14) \qquad\qquad B(x) = B_{\mathbb{C}}(x, d_{\mathbb{C}}(x, \partial U))$$

to be the largest open ball centered at x which remains in U. Lastly, define

$$(4.15) \qquad\qquad a_f(x) = \exp \left(\frac{1}{2} (\log J_f)_{B(x)} \right).$$

The promised generalization follows.

Theorem 4.3 ([AG85]). *Suppose U and V are open domains in $\mathbb{C}$ and $f : U \to V$ is K-quasi-conformal. Then there exists a constant $C = C(K)$ such that*

$$(4.16) \qquad\qquad \frac{1}{C} \frac{d_{\mathbb{C}}(f(x), \partial V)}{d_{\mathbb{C}}(x, \partial U)} \leq a_f(x) \leq C \frac{d_{\mathbb{C}}(f(x), \partial V)}{d_{\mathbb{C}}(x, \partial U)}.$$

Given $\widetilde{\Sigma} \subset \mathbb{H}^3$ a normalized almost-Fuchsian disk with data (g, α), fix a uniformization

$$(4.17) \qquad\qquad \phi : \mathbb{D} \to (\widetilde{\Sigma}, g)$$

such that $\phi(0) = p$ as in Proposition 4.1. Consider the composition

$$(4.18) \qquad\qquad \Phi := \mathcal{G}_{\widetilde{\Sigma}}^+ \circ \phi : \mathbb{D} \to \mathbb{C}$$

where we have identified $\partial_\infty(\mathbb{H}^3)$ with $\mathbb{C} \cup \{\infty\}$. Our strategy is to apply Theorem 4.3 to the function Φ. We collect the necessary properties of Φ below.

Proposition 4.4. *The map $\Phi : \mathbb{D} \to \mathbb{C}$ above satisfies:*

(1) $\Phi(0) = 0$.

(2) *The restriction of Φ to the sub-disk $B_{\mathbb{C}}(0, r_1)$ from Proposition 4.1 is K-quasi-conformal with K independent of (g, α).*

(3) *There exists $\beta(r_1) = \beta > 0$ such that $J_{\Phi}(z) > \beta$ for all $z \in B_{\mathbb{C}}(0, r_1)$ with β independent of (g, α).*

Proof. First,

$$\Phi(0) = 0$$

is a direct consequence of the fact that $\widetilde{\Sigma}$ is normalized. Next, Φ is a composition of a conformal map ϕ with the hyperbolic Gauss map $\mathcal{G}^+_{\widetilde{\Sigma}}$. Thus, the restriction of Φ to $B_{\mathbb{C}}(0, r_1)$ is K-quasi-conformal if $\mathcal{G}^+_{\widetilde{\Sigma}}$ is K-quasi-conformal on $B_g(p, r)$. This is proved in Proposition 4.2 which verifies (2). Lastly, the Jacobian is multiplicative:

$$(4.19) \qquad\qquad J_{\Phi}(z) = J_{\mathcal{G}^+_{\widetilde{\Sigma}}}(\phi(z)) \, J_{\phi}(z).$$

Combining Propositions 4.1 and 4.2, the product on the right is bounded below by $\beta = 2C(1 - \varepsilon^2)$. This proves (3) and the proof is complete. $\square$

Proposition 4.4 and Theorem 4.3 combine to show that the image of the Gauss map contains disks of a definite size.

Proposition 4.5. *Let $\widetilde{\Sigma}$ be a normalized almost-Fuchsian disk. Then there exists $R > 0$ such that $B_{\mathbb{C}}(0, R) \subset \mathcal{G}^+_{\widetilde{\Sigma}}(\widetilde{\Sigma})$ where $B_{\mathbb{C}}(0, R)$ is the Euclidean disk of radius R centered at $0 \in \mathbb{C}$.*

Proof. Consider the map

$$(4.20) \qquad\qquad \Phi := \mathcal{G}^+_{\widetilde{\Sigma}} \circ \phi : B_{\mathbb{C}}(0, r_1) \to \mathbb{C}.$$

By Proposition 4.4(3), there exists $\beta > 0$ such that

$$(4.21) \qquad\qquad a_{\Phi}(0) \geq \exp\left(\frac{1}{2} \log \beta\right).$$

Noting that Φ is K-quasi-conformal by Proposition 4.4(2), Theorem 4.3 yields

$$(4.22) \qquad \sqrt{\beta} \leq \frac{C d_{\mathbb{C}}\big(0, \partial(\Phi(B_{\mathbb{C}}(0, r_1)))\big)}{d_{\mathbb{C}}\big(0, \partial(B_{\mathbb{C}}(0, r_1))\big)} = \frac{C d_{\mathbb{C}}\big(0, \partial(\Phi(B_{\mathbb{C}}(0, r_1)))\big)}{r_1}$$

since $d_{\mathbb{C}}(0, \partial(B_{\mathbb{C}}(0, r_1))) = r_1$. Above, we have taken $U = B_{\mathbb{C}}(0, r_1)$ and $V = \Phi(U)$. Letting $R = \frac{r_1 \sqrt{\beta}}{C}$ completes the proof. $\square$

5. MAIN RESULTS

In this section, we present the main results concerning the domain of discontinuity of an almost-Fuchsian group. An almost-Fuchsian group Γ is normalized if the Γ-invariant almost-Fuchsian disk is normalized.

Theorem 5.1. *Let Γ be a normalized almost-Fuchsian group. Then the domain of discontinuity Ω of Γ contains $B_{\mathbb{C}}(0, R) \subset \mathbb{C}$ for some $R > 0$.*

Proof. Let $\widetilde{\Sigma}$ be the normalized Γ-invariant almost-Fuchsian disk. By Theorem 2.6, the forward hyperbolic Gauss map $\mathcal{G}^+_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \partial_{\infty}(\mathbb{H}^3)$ is a (quasi-conformal) diffeomorphism onto one connected component Ω^+ of the domain of discontinuity. By Proposition 4.5, $B_{\mathbb{C}}(0, R) \subset \mathcal{G}^+_{\widetilde{\Sigma}}(\widetilde{\Sigma}) = \Omega^+$ for some $R > 0$. $\square$
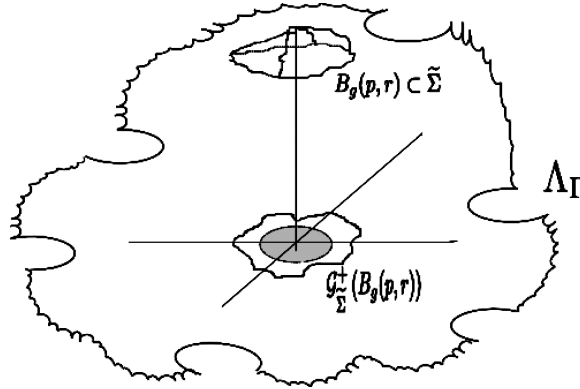
FIGURE 2. The disk contained inside the image of the hyperbolic Gauss map lying in the domain of discontinuity for some almost-Fuchsian group Γ with limit set Λ_Γ.

Note that given an almost-Fuchsian group Γ, conjugating Γ by a rotation so that $0 \in \Omega$ followed by a translation with arbitrarily large translation distance in the direction of the positive t-axis, we obtain an almost-Fuchsian group whose domain of discontinuity contains a disk around zero of arbitrarily large radius. Of course, the resulting almost-Fuchsian group is not normalized. This is the essential point of the above theorem.

We can actually do better than the previous theorem. Because the almost-Fuchsian disk $\widetilde{\Sigma}$ is the universal cover of a closed minimal surface whose non-negative principal curvature equals the norm of a holomorphic quadratic differential, the principal curvatures of $\widetilde{\Sigma}$ are zero at a countably infinite set of points. Thus, the arguments given above can be applied one by one to each such point. Furthermore, we may use the opposite pointing normal to obtain the same results for the opposing domain of discontinuity. With this in mind, we can refine the above theorem in the following way.

Theorem 5.2. *Let Γ be a normalized almost-Fuchsian group and $\widetilde{\Sigma}$ the almost-Fuchsian Γ-invariant disk with data $(g, \alpha) \in \mathcal{AF}$. For a fixed compact set $E \subset \mathbb{H}^3$ containing $p = (0,0,1)$, consider any set of points $\{p_i\} \subset \widetilde{\Sigma}$ contained in $E \cap \widetilde{\Sigma}$ such that $\|\alpha\|_g(p_i) = 0$ for each i. Then there exists an $R'(E) > 0$ such that $\bigcup_i B_{\mathbb{S}^2}\left(\mathcal{G}_{\widetilde{\Sigma}}^{\pm}(p_i), R'\right) \subset \Omega$ where Ω is the domain of discontinuity of Γ.*

Remark. By Theorem 2.6, $\widetilde{\Sigma}$ is properly embedded so that $E \cap \widetilde{\Sigma}$ is compact. As any set of zeros $\{p_i\}$ is necessarily discrete (they are zeros of a holomorphic quadratic differential), any compact set of zeros is finite.

Proof. First, we claim there exists a positive integer $K(E)$ such that the cardinality of the set of zeros of α contained in $\widetilde{\Sigma} \cap E$ is always less than K. As observed in the remark above, $\widetilde{\Sigma} \cap E$ is compact. Because the metric on the minimal surface is uniformly comparable to the hyperbolic metric, there exists a uniform constant $C > 0$ such that the area of $\widetilde{\Sigma} \cap E$ is bounded above by C. By the Gauss equation

the area of the closed minimal surface which $\widetilde{\Sigma}$ covers is at least $\pi(2g-2)$ where g is the genus, thus a fundamental domain for the action of the almost-Fuchsian group has area at least $\pi(2g-2)$. Then there are at most $\frac{C}{\pi(2g-2)}$ fundamental domains which lie in $\widetilde{\Sigma} \cap E$. Since each fundamental domain contains at most $4g-4$ zeros of α, we take $K(E)$ equal to the nearest integer greater than $(4g-4)\frac{C}{\pi(2g-2)} = \frac{2C}{\pi}$.

For any set of zeros $\{p_i\}_{i=1}^n \subset \widetilde{\Sigma} \cap E$, each member of a collection of normalizing elements $\{I_i\}_{i=1}^n \subset \mathrm{Isom}(\mathbb{H}^3)$ (i.e. I_i satisfies $I_i(p_i) = p = (0,0,1)$) lies in a fixed compact set in $\mathrm{Isom}(\mathbb{H}^3)$. This follows from the fact that each I_i will be a composition of a hyperbolic element of uniformly bounded translation distance (specifically bounded by the distance from p to ∂E) and a rotation. Fixing the $R > 0$ of Theorem 5.1, we set R' equal to the minimum of the spherical radii of the disks $I(B_{\mathbb{C}}(0,R))$ where $I \in \mathrm{Isom}(\mathbb{H}^3)$ ranges over the finite set of all words of length at most n in the I_i and their inverses. Remember that we first established that there exists a universal constant such that $n < K(E)$ and so this is a bounded list over all almost-Fuchsian disks. Each I_i is a conformal transformation so indeed $I(B_{\mathbb{C}}(0,R))$ is a disk. Then, the argument in Theorem 5.1 implies that $\bigcup_i B_{\mathbb{S}^2}\left(\mathcal{G}_{\widetilde{\Sigma}}^+(p_i), R'\right) \subset \Omega$.

Repeating the same argument with the opposite pointing normal proves that there exists $R'' > 0$ such that $\bigcup_i B_{\mathbb{S}^2}\left(\mathcal{G}_{\widetilde{\Sigma}}^-(p_i), R''\right) \subset \Omega$. Letting $R' = \min\{R', R''\}$ completes the proof. □

5.1. Geometric limits of almost-Fuchsian groups.
In this section, we prove that a geometrically convergent sequence of almost-Fuchsian groups cannot limit to a doubly-degenerate group. The mantra is that the disks guaranteed to exist in each domain of discontinuity form barriers into which the limit set cannot penetrate.

Theorem 5.3. *Suppose Γ_n is a sequence of almost-Fuchsian groups and $\Gamma_n \to \Gamma$ geometrically. Then Γ is not doubly-degenerate.*

Proof. Recall that the limit set of a doubly-degenerate group Γ is equal to $\partial_\infty(\mathbb{H}^3)$. Suppose Γ_n is a sequence of almost-Fuchsian groups converging geometrically to Γ. By Theorem 5.1 there exists an $R > 0$ and a sequence of transformations $I_n \in \mathrm{Isom}(\mathbb{H}^3)$ such that

$$(5.1) \qquad \Lambda(I_n \Gamma_n I_n^{-1}) \cap B_{\mathbb{C}}(0, R) = \emptyset.$$

We shall refer to any such sequence I_n as a normalizing sequence.

First suppose that the I_n remain in a compact subset of $\mathrm{Isom}(\mathbb{H}^3)$. Choose a subsequence such that $I_n \to I$ where we have relabeled the indices. Then $I_n \Gamma_n I_n^{-1} \to I\Gamma I^{-1}$ geometrically. Since $\Lambda(I\Gamma I^{-1}) = I\Lambda(\Gamma)$, the limit set of $I\Gamma I^{-1}$ is also equal to $\partial_\infty(\mathbb{H}^3)$. We conclude that

$$\partial_\infty(\mathbb{H}^3) = \Lambda(I\Gamma I^{-1}) \subset \lim_{n\to\infty} \Lambda(I_n \Gamma_n I_n^{-1})$$

and so

$$\lim_{n\to\infty} \Lambda(I_n \Gamma_n I_n^{-1}) = \partial_\infty(\mathbb{H}^3)$$

in the Hausdorff topology on closed subsets of $\partial_\infty(\mathbb{H}^3)$. In particular, since the spherical and Euclidean metrics are comparable on compact subsets in the Euclidean topology (they nearly agree around $0 \in \mathbb{C}$), given any $R > 0$ there exists an $N \in \mathbb{N}$ such that

$$\Lambda(I_n \Gamma_n I_n^{-1}) \cap B_{\mathbb{C}}(0, R) \neq \emptyset$$

for every $n \geq N$. This contradicts equation (5.1).

Now suppose that no normalizing sequence I_n remains in a compact subset of $\mathrm{Isom}(\mathbb{H}^3)$. We work towards a contradiction. Let $\widetilde{\Sigma_n}$ be the unique Γ_n-invariant almost-Fuchsian disk. We select a sequence of points $p_n \in \widetilde{\Sigma_n}$ satisfying the following properties:

(1) Each $p_n \in \widetilde{\Sigma_n}$ is such that the principal curvatures of $\widetilde{\Sigma_n}$ vanish at p_n.
(2) There exists $x \in \mathbb{C} \subset \partial_\infty(\mathbb{H}^3)$ such that, after perhaps passing to a subsequence, $p_n \to x$ where the convergence is in the Euclidean topology on the closed upper half-space $\mathbb{H}^3 \cup \partial_\infty(\mathbb{H}^3)$.

A (sub-)sequence can be found satisfying both conditions since we have assumed every sequence p_n of vanishing principal curvature leaves every compact subset of $\mathbb{H}^3$. Next, conjugate each group Γ_n by a parabolic isometry, represented by a Möbius transformation

$$J_n = z + a_n,$$

which maps p_n to some point $(0, 0, p_n) \in \mathbb{H}^3$ where we have abused notation in calling the z-coordinate by the same name. Note that assumption (2) on the sequence p_n implies $|a_n| \leq K$ for some $K > 0$. As the next step, conjugate each group further by an elliptic isometry to make the downward pointing unit normal vector to $J_n(p_n)$ directed at $0 \in \partial_\infty(\mathbb{H}^3)$. Further abusing notation, we label the new sequence of groups as Γ_n and denote the unique Γ_n-invariant almost-Fuchsian disks by $\widetilde{\Sigma_n}$.

It remains true that $\Lambda(\Gamma_n) \to \partial_\infty(\mathbb{H}^3)$ since the parabolic elements were chosen from a compact set of isometries and elliptic elements act as isometries of the spherical metric from which the Hausdorff topology was induced.

Given the above prerequisites, a $\widetilde{\Sigma_n}$-normalizing sequence I_n takes the form

$$(5.2) \qquad I_n = \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n^{-1} \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{Isom}^+(\mathbb{H}^3)$$

for some $\lambda_n \to \infty$.

Since $I_n(\widetilde{\Sigma_n})$ is a sequence of normalized almost-Fuchsian disks, by Theorem 5.2 there exists an $R' > 0$ such that

$$(5.3) \qquad \Lambda(I_n \Gamma_n I_n^{-1}) \cap B_{\mathbb{S}^2}(\infty, R') = \emptyset.$$

Since $\lambda_n \to \infty$, there exists an $N \in \mathbb{N}$ such that

$$(5.4) \qquad \partial_\infty(\mathbb{H}^3) \setminus I_n^{-1}(B_{\mathbb{S}^2}(\infty, R')) = (\mathbb{C} \cup \{\infty\}) \setminus I_n^{-1}(B_{\mathbb{S}^2}(\infty, R'))$$

$$(5.5) \qquad\qquad\qquad\qquad\qquad\qquad \subset B_{\mathbb{S}^2}(0, R')$$

for all $n > N$. Applying I_n^{-1} to equation (5.3) we obtain that,

$$(5.6) \qquad \Lambda(\Gamma_n) \cap I_n^{-1} B_{\mathbb{S}^2}(\infty, R') = \emptyset.$$

Note that the above is true because I_n^{-1} is injective, so it maps non-intersecting sets to non-intersectings sets. By line (5.4) this implies that for all $n > N$

$$(5.7) \qquad \Lambda(\Gamma_n) \subset B_{\mathbb{S}^2}(0, R'),$$

but this is impossible since $\Lambda(\Gamma_n) \to \partial_\infty(\mathbb{H}^3)$. This contradiction implies that we may always find a normalizing sequence which remains in a compact subset of $\mathrm{Isom}(\mathbb{H}^3)$. But, we have already shown this case above, hence Γ_n cannot geometrically limit to a doubly-degenerate group. This completes the proof. $\qquad \square$

Acknowledgments

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