NORMALIZATION AND SOLVABILITY OF VECTOR FIELDS
NEAR TRAPPED ORBITS

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Abstract. We study the solvability and normalization, in the real analytic
and smooth categories, of a class of vector fields in a neighborhood of an
invariant torus. The vector fields are supposed to satisfy Siegel type conditions.

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Introduction

This paper deals with the solvability and normalization of a class of vector fields
in a neighborhood of an invariant torus. For a nonsingular (complex) vector field $L$,
a necessary and sufficient condition for the local solvability of the equation $Lu = F$
is the Nirenberg-Treves Condition (P) (see [11] and [16]). The Condition (P) is also
sufficient for the solvability in a tubular neighborhood of a nonrelatively compact
orbit of $L$ (see [6] and [7]). However, the solvability in a neighborhood of a relatively
compact orbit is not well understood. In fact there are locally solvable vector fields
that are not solvable in any neighborhood of a trapped orbit (see Section 2.) The
solvability and normalization of planar vector fields are considered in [4], [8], [9]
and [10].

Received by the editors December 1, 2014 and, in revised form, April 30, 2015.
2010 Mathematics Subject Classification. Primary 35F05, 34K17; Secondary 35A01, 35A24,
35F35.
Key words and phrases. Vector field, normalization, solvability, Siegel condition, invariant
torus.
Let
\[ X = \sum_{k=1}^{m} (\omega_k + g_k(t, x)) \frac{\partial}{\partial t_k} \quad \text{and} \quad Y = \sum_{j=1}^{n} (\lambda_j x_j + f_j(t, x)) \frac{\partial}{\partial x_j}, \]

where \( \omega_k \in \mathbb{R}, \lambda_j \in \mathbb{R}, \) \( g_k \) and \( f_j \) are smooth \( \mathbb{R} \)-valued functions defined for \( t \in \mathbb{T}^m, \) \( x \in \mathbb{R}^n. \) Here \( \mathbb{T}^m := \mathbb{R}^m / (2\pi \mathbb{Z})^m \) is the \( m \)-torus. We assume that \( g_k = O(|x|) \) and \( f_j = O(|x|^2) \) for \( |x| < r \) for some \( r > 0. \) The main questions addressed here are the normalization and the solvability of the real vector field \( X + Y \) and of the complex vector field \( X + iY \) in a neighborhood of the torus \( \mathbb{T}^m_0 = \mathbb{T}^m \times \{0\} \) in \( \mathbb{T}^m \times \mathbb{R}^n. \) These vector fields have orbits that are trapped in the torus \( \mathbb{T}^m_0. \)

The real vector field \( X + Y \) is studied under the assumption that the pair \( (\omega, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n \) satisfies the following Siegel type condition: there exist \( C > 0 \) and \( \mu > 0 \) such that
\[ |i(\omega, K) + \langle \lambda, J \rangle| - \epsilon \lambda_j | \geq \frac{C}{(|K| + |J|)^\mu} \]
for every \( K \in \mathbb{Z}^m, \) \( J \in \mathbb{N}^n, \epsilon = 0 \) or \( 1, j = 1, \cdots, n \) with \( |K| + |J| > 0. \) The following results are obtained.

**Theorem 1 (Normalization).** Suppose that the functions \( g_k \) and \( f_j \) are of class \( C^a \) with \( a = \infty \) (smooth) or \( a = \varpi \) (real analytic). Then there exists a \( C^a \)-diffeomorphism \( \Phi \) in a neighborhood of \( \mathbb{T}^m_0 \) such that
\[ \Phi_*(X + Y) = \sum_{k=1}^{m} \omega_k \frac{\partial}{\partial t_k} + \sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j}. \]

It should be noted that this result is related to normalization of dynamical systems near singular points or near invariant manifolds. Among the many works in this direction, we would like to mention the following papers: [1], [3], [13], [14] and [15].

**Theorem 2 (Solvability).** Suppose that the functions \( g_k \) and \( f_j \) are as in Theorem 1. Then for every function \( F \) of class \( C^a \) in a neighborhood of \( \mathbb{T}^m_0 \) that satisfies
\[ \int_{\mathbb{T}^m} F(t, 0) dt = 0, \]
equation \( (X + Y)u = F \) has a \( C^a \) solution in a neighborhood of the torus \( \mathbb{T}^m_0. \)

For the complex vector field \( X + iY, \) we assume that \( X \) and \( Y \) commute \( ([X, Y] = 0) \) and that \( (\omega, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n \) satisfies the Siegel type condition: there exist \( C > 0 \) and \( \mu > 0 \) such that
\[ |\langle \omega, K \rangle + \langle \lambda, J \rangle| | \geq \frac{C}{(|K| + |J|)^\mu} \]
for every \( K \in \mathbb{Z}^m \) and \( J \in \mathbb{N}^n \) with \( |K| + |J| > 0. \) Under these two conditions, we prove the following results.

**Theorem 3 (Real analytic normalization).** Suppose that the functions \( g_k \) and \( f_j \) are real analytic. Then there exists a real analytic diffeomorphism \( \Phi \) in a neighborhood of \( \mathbb{T}^m_0 \) such that
\[ \Phi_*(X + iY) = \sum_{k=1}^{m} \omega_k \frac{\partial}{\partial t_k} + i \sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j}. \]
**Theorem 4** (Real analytic solvability). Suppose that the functions \( g_k \) and \( f_j \) are real analytic. For any given real analytic function \( F \) in a neighborhood of \( T_0^m \) satisfying \( \int_{T_0^m} F(t,0) dt = 0 \), the equation \((X + iy)u = F \) has a real analytic solution \( u \) in a neighborhood of \( T_0^m \).

In the \( C^\infty \)-category, the complex vector field \( X + iy \) is not always normalizable (see Proposition 6.1 for a nonnormalizable vector field). However, when all the \( \lambda_j \)'s have the same sign, \( X + iy \) is smoothly normalizable.

**Theorem 5** (Smooth normalization). Suppose that the functions \( g_k \) and \( f_j \) are of class \( C^\infty \) and that all the coefficients \( \lambda_1, \ldots, \lambda_n \) have the same sign. Then there exists a \( C^\infty \) diffeomorphism \( \Phi \) in a neighborhood of \( T_0^m \) such that

\[
\Phi_*(X + iy) = \sum_{k=1}^m \omega_k \frac{\partial}{\partial t_k} + i \sum_{j=1}^n \lambda_j x_j \frac{\partial}{\partial x_j}.
\]

The solvability of \( X + iy \) in the smooth case is more delicate.

**Theorem 6** (Smooth solvability). Suppose that \( X + iy \) is of class \( C^\infty \) and normalizable. For any given \( C^\infty \) function \( F \) in a neighborhood of \( T_0^m \) that satisfies \( \int_{T_0^m} F(t,0) dt = 0 \), and for any given \( q \in \mathbb{N} \), the equation \((X + iy)u = F \) has a solution \( u_q \) of class \( C^q \) in a neighborhood of \( T_0^m \).

Although, for any given \( q \), the equation \((X + iy)u = F \) has a solution of class \( C^q \), in general it does not have solutions of class \( C^\infty \) (see Theorem 8.4).

The organization of this paper is as follows. In Section 1, we set the necessary notation, introduce the class of vector fields and recall Condition (P). In Section 2, we give an example of a locally solvable vector field \( N \) that is not solvable (not even in the sense of distributions) in any neighborhood of the torus \( T_0^m \). Section 3 deals with the normalization of the real vector field \( X + Y \). Sections 4 and 5 complete the normalization theorem for \( X + Y \), with the formal normalization in Section 4 and the solvability of the associated homological equation in Section 5. Section 6 deals with the normalization of the complex vector field \( X + iy \) and with the construction (Proposition 6.1) of a nonnormalizable vector field with commuting real and imaginary parts. The solvability of the equations \((X + Y)u = F \) is discussed in Section 7. The equation \((X + iy)u = F \) is considered in Section 8, and a \( C^\infty \) function \( F \) is constructed for which the equation \((X + iy)u = F \) has no \( C^\infty \) solution (Theorem 8.4). Sections 9 and 10 contain technical results that complete the proof of the solvability of \( X + iy \).

1. **A class of vector fields**

We introduce here the class of vector fields and some notation that will be used throughout.

We denote by \( T^m \) the torus \( \mathbb{R}^m/(2\pi \mathbb{Z})^m \) and by \( B(r) \) the ball with center 0 and radius \( r \) in \( \mathbb{R}^m \). Set \( T_0^m = T^m \times \{0\} \subset T^m \times \mathbb{R}^n \). For \( \omega = (\omega_1, \ldots, \omega_m) \in \mathbb{R}^m \) with \( \omega_k \neq 0 \ (k = 1, \ldots, m) \) and \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \) with \( \lambda_j \neq 0 \ (j = 1, \ldots, n) \),
consider the vector fields $X$ and $Y$ defined in an open neighborhood of $\mathbb{T}_0^m \subset \mathbb{T}^m \times \mathbb{R}^n$ by

\begin{align}
(1.1) & \quad X = \sum_{k=1}^{m} (\omega_k + g_k(t, x)) \frac{\partial}{\partial t_k}, \\
(1.2) & \quad Y = \sum_{j=1}^{n} (\lambda_j x_j + f_j(t, x)) \frac{\partial}{\partial x_j},
\end{align}

where $g_k, f_j \in C^a(\mathbb{T}^m \times B(r); \mathbb{R})$ for some $r > 0$ and with $a = \infty$ (smooth case) or $a = \sigma$ (real analytic case), and such that

\begin{align}
(1.3) & \quad g_k(t, x) = O(|x|) \text{ for } k = 1, \cdots, m, \quad f_j(t, x) = O(|x|^2) \text{ for } j = 1, \cdots, n.
\end{align}

For brevity we write

\begin{align}
(1.4) & \quad X = (\omega + g(t, x))\partial_t \quad \text{and} \quad Y = (\Lambda x + f(t, x))\partial_x
\end{align}

where $g = (g_1, \cdots, g_m)$, $f = (f_1, \cdots, f_n)$, $\Lambda$ is the diagonal matrix $\Lambda := \text{diag}(\lambda_1, \cdots, \lambda_n)$, $\partial_t = \left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_m}\right)^T$, and $\partial_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)^T$. Depending on the context, we will view $X$ and $Y$ as either differential operators acting on functions defined in $\mathbb{T}^m \times \mathbb{R}^n$ or as mappings. The vector field $X$ is a mapping from $\mathbb{T}^m \times \mathbb{R}^n$ into $\mathbb{T}^m \times \mathbb{R}^n$ given by $(\omega + g, 0)$, and $Y$ is a mapping from $\mathbb{T}^m \times \mathbb{R}^n$ into $\mathbb{T}^m \times \mathbb{R}^n$ given by $(0, \Lambda x + f)$.

The main objective of this paper is to understand the equations

\begin{align}
(1.5) & \quad (X + Y)u = F \quad \text{(real case)} \quad \text{and} \quad (1.6) & \quad (X + iY)u = F \quad \text{(complex case)}
\end{align}

in a full neighborhood of the central torus $\mathbb{T}_0^m$, where $F(t, x)$ is a given $C^\infty$ or $C^\sigma$ function in a neighborhood of $\mathbb{T}_0^m$. It follows at once (by using Fourier series) that a necessary condition for equations (1.5) and (1.6) to have solutions is that $F$ satisfies

\begin{align}
(1.7) & \quad \int_{\mathbb{T}^m} F(t, 0)dt = 0.
\end{align}

We will assume throughout that condition (1.7) holds whenever considering such equations in a full neighborhood of $\mathbb{T}_0^m$.

Note that since the frequency vector $\omega$ is nonzero, $X$ is nonsingular along $\mathbb{T}_0^m$ and, consequently, the real vector field $X + Y$ is locally solvable at each point $(t_0, 0) \in \mathbb{T}_0^m$. That is, there exist open sets $V \subset \mathbb{T}^m, U \subset \mathbb{R}^n$ with $(t_0, 0) \in V \times U$ such that for every $F \in C^a(V \times U)$ ($a = \infty$ or $a = \sigma$), equation (1.5) has a $C^a$ solution in a neighborhood of $(t_0, 0)$.

For the complex vector field $X + iY$, the local solvability is a more complicated problem and there are nonsingular such vector fields that are not locally solvable. A necessary and sufficient condition for local solvability is given by the Nirenberg-Treves Condition (P) (see [2] or [16]). Briefly, a linear differential operator $P(D)$ with principal symbol $p$ satisfies Condition (P) in a neighborhood of a point $y_0$, if the function $\text{Im}(p)$ does not change sign on any null bicharacteristic of $\text{Re}(p)$. In particular, if $P$ is the vector field

\[ P = \frac{\partial}{\partial s} + i \sum_{j=1}^{n} b_j(s, y) \frac{\partial}{\partial y_j} \]
defined near $0 \in \mathbb{R}^{n+1}$, where the functions $b_j$ are $\mathbb{R}$-valued, then $P$ satisfies Condition (P) if and only if for every fixed $y$ near 0 and $\xi \in \mathbb{R}^n$, the function $s \mapsto \sum b_j(s, y) \xi_j$ does not change sign. Hence, any vector field defined in a neighborhood of $\mathbb{T}_0^m$ by

$$
\sum_{k=1}^m a_k(t) \frac{\partial}{\partial t_k} + i \sum_{j=1}^n b_j(x) \frac{\partial}{\partial x_j},
$$

with the $a_k$’s and $b_j$’s real valued and $\sum |a_k| > 0$, satisfies Condition (P).

The real vector field $X + Y$ defines a foliation near $\mathbb{T}_0^m$. Since $Y = 0$ when $x = 0$ and $X$ is tangent to $\mathbb{T}_0^m$, then for every $(t_0, 0) \in \mathbb{T}_0^m$ the integral curve $\Gamma_{(t_0,0)}$ of $X + Y$ through $(t_0,0)$ is entirely contained in the torus $\mathbb{T}_0^m$. In general, $\Gamma_{(t_0,0)}$ is not a closed curve unless there exist $K \in \mathbb{Z}^m$ and $\sigma \in \mathbb{R}$ such that $\omega = \sigma K$.

The complex vector field $X + iY$ does not generate a foliation in $\mathbb{T}_0^m \times B(r)$. However, there is the notion of Sussman orbits (see [2] or [16]). These are the equivalence classes obtained by identifying two points in $\mathbb{R}^m$. Let $\omega \in \mathbb{T}_0^m$ and let $X$ be tangent to $\mathbb{T}_0^m \times B(r)$ via the integral curve $\Gamma_{(t_0,0)}$ of $X + iY$. The Sussman orbits of $X + iY$ are one dimensional or two dimensional submanifolds. In particular, each orbit $\Gamma_{(t_0,0)}$ of $X$ is a one-dimensional Sussman orbit of $X + iY$ trapped in the torus $\mathbb{T}_0^m$.

Throughout the paper, we will use the following notation: for $K = (k_1, \cdots, k_m) \in \mathbb{Z}^m$ and $J = (j_1, \cdots, j_n) \in \mathbb{N}^n$, where $\mathbb{Z}$ is the set of integers and $\mathbb{N}$ the set of nonnegative integers, and for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ we set

$$
\langle \omega, K \rangle = \omega_1 k_1 + \cdots + \omega_m k_m; \quad \langle \lambda, J \rangle = \lambda_1 j_1 + \cdots + \lambda_n j_n;
$$

$$
|K| = |k_1| + \cdots + |k_m|; \quad |J| = j_1 + \cdots + j_n;
$$

$$
x^J = x_1^{j_1} \cdots x_n^{j_n}; \quad J! = j_1! \cdots j_n!.
$$

2. A NONSOLVABLE VECTOR FIELD

We give here an example of a locally solvable vector field that is not solvable in any neighborhood of the torus $\mathbb{T}_0^m$.

Let $\omega = (\omega_1, \cdots, \omega_m) \in \mathbb{R}^m$ with $\omega_k \neq 0$ for $k = 1, \cdots, m$ and let $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$ with $a_j \neq 0$ for $j = 1, \cdots, n$. Consider the vector field $N_{\omega,a}$ defined in $\mathbb{T}_0^m \times \mathbb{R}^n$ by

$$
N_{\omega,a} := \sum_{k=1}^m \omega_k \frac{\partial}{\partial k} + i \sum_{j=1}^n a_j \left( y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right).
$$

**Theorem 2.1.** The vector field $N_{\omega,a}$ defined by (2.1) has the following properties:

(i) It is locally solvable at each point $(t,0) \in \mathbb{T}_0^m$.

(ii) There exists $f \in C^\infty(\mathbb{T}_0^m \times \mathbb{R}^{2n})$ vanishing on $\mathbb{T}_0^m$ such that the equation $N_{\omega,a}u = f$ has no distribution solution in any neighborhood of $\mathbb{T}_0^m$.

(iii) The range of the operator

$$
N_{\omega,a} : C^\infty(\mathbb{T}_0^m \times B(r) \times B(r)) \to C^\infty(\mathbb{T}_0^m \times B(r) \times B(r))
$$

has infinite codimension.
Proof. (i) Since the real vector field $\text{Re}(N_{\omega,a})$ is constant and acts only on $\mathbb{T}^m$ and $\text{Im}(N_{\omega,a})$ depends on the $(x, y)$ variables and acts on $\mathbb{R}^n \times \mathbb{R}^n$, it then follows that $N_{\omega,a}$ satisfies the Nirenberg-Treves Condition (P) at each point $(t, 0) \in \mathbb{T}_y^n$ and it is therefore locally solvable.

(ii) This is an application of Theorem 1 of [17]. The vector field with real linear coefficients

$$\text{Im}(N_{\omega,a}) = \sum_{j=1}^n a_j \left( y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right)$$

is not solvable in any neighborhood of singular point 0. Indeed, with respect to the polar coordinates $(\rho, \theta)$ in the $(x_j, y_j)$-plane ($j = 1, \cdots, n$), the expression of $\text{Im}(N_{\omega,a})$ becomes

$$\text{Im}(N_{\omega,a}) = \sum_{j=1}^n a_j \frac{\partial}{\partial \theta_j}.$$

Now if $f(x, y)$ is any function that is independent on $t$ and depends only on the radii $\rho_1, \cdots, \rho_n$, ($f = f(\rho_1, \cdots, \rho_n)$), and $f(0,0) = 0$, then for a distribution $u \in \mathcal{D}'(\mathbb{T}^m \times B(t) \times B(r))$ to satisfy the equation $N_{\omega,a} u = f$, it is necessary (by using Fourier series in the $t$-variables) that the equation

$$\text{Im}(N_{\omega,a}) u = \sum_{j=1}^n a_j \frac{\partial u}{\partial \theta_j} = f(\rho_1, \cdots, \rho_n)$$

have a solution $v \in \mathcal{D}'(B(r) \times B(r))$. The last equation does not have a distribution solution if $f \neq 0$.

(iii) This is a consequence of the proof of (ii) since $f(x, y)$ can be taken as an arbitrary $C^\infty$ function that depends only on the radii $(x_1^2 + y_1^2)$.

\[\square\]

Remark 2.1. If a given function $f(t, \rho_1, \cdots, \rho_n, \theta_1, \cdots, \theta_n)$ satisfies the condition

$$\int_{\mathbb{T}^m} \int_{\mathbb{T}^n} f(t, \rho, \theta) d\rho dt = 0$$

for any $\rho = (\rho_1, \cdots, \rho_n)$ and if the vectors $\omega$ and $a$ satisfy Siegel conditions (see Sections 3 and 6), then equation $N_{\omega,a} u = f$ is solvable. It can be shown, by using Fourier series in the $t$ and $\theta$ variables, that the series

$$u(t, \rho, \theta) = \sum_{K \in \mathbb{Z}^m, L \in \mathbb{Z}^n} \frac{f_{KL}(\rho)}{i(\omega, K) - (a, L)} e^{i(K, t)} e^{i(L, \theta)},$$

where $f_{KL}(\rho)$ is the $KL$-th Fourier coefficient of $f$, defines such a solution.

3. Normal form: The real case

We prove that if the pair $(\omega, \lambda)$ satisfies a Siegel type condition, then the real vector field $X + Y$ can be transformed into a simple model vector field $M_{\omega,\lambda}$ by a diffeomorphism.

For given $\omega = (\omega_1, \cdots, \omega_m) \in \mathbb{R}^m$ and $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, the Siegel type condition $(S_\mathbb{R})$ is the following:

$(S_\mathbb{R})$: There exists $C > 0$ and $\mu > 0$ such that

$$|i(\omega, K) + \lambda, J - \epsilon \lambda_j| \geq \frac{C}{(|K| + |J|)^\mu}$$

for every $K \in \mathbb{Z}^m$, $J \in \mathbb{N}^n$, $j = 1, \cdots, n$, $\epsilon = 0$ or $1$ and $|K| + |J| > 0$. 

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This condition has appeared in earlier normalization results (see [1] or [3]).
Let \( g(t, x) \in C^a(\mathbb{T}^m \times B(r), \mathbb{R}^m) \) and \( f(t, x) \in C^a(\mathbb{T}^m \times B(r), \mathbb{R}^n) \), with \( a = \infty \) or \( a = \omega \), be such that \( g = O(|x|) \) and \( f = O(|x|^2) \). Let \( X \) and \( Y \) be the vector fields given in (1.4). Then we have the following theorem.

**Theorem 3.1.** Suppose that \((\omega, \lambda)\) satisfies \((S_\mathbb{R})\). Then there exists a diffeomorphism

\[
\Phi(t, x) = (t + \phi(t, x), x + \psi(t, x))
\]

of class \( C^a \) in a neighborhood of \( \mathbb{T}^m_0 \), with \( \phi(t, 0) = 0 \) and \( \psi(t, 0) = 0 \), and such that

\[
\Phi_*(X + Y) = M_{\omega, \lambda}
\]

where

\[
M_{\omega, \lambda} = \omega \partial_t + \Lambda x \partial_x = \sum_{k=1}^m \omega_k \frac{\partial}{\partial t_k} + \sum_{j=1}^n \lambda_j x_j \frac{\partial}{\partial x_j}.
\]

It should be noted that in the real analytic category, this result is contained in [3]. We will therefore prove the theorem in the \( C^\infty \) category. We devote the rest of this section to the proof of Theorem 3.1.

To a function \( \alpha(t, x) \in C^\infty(\mathbb{T}^m \times B(r), \mathbb{R}^p) \) we associate its \( x \)-Taylor series

\[
\tilde{\alpha}(t, x) = \sum_{k=0}^{\infty} \sum_{J \in \mathbb{N}^n, |J|=k} \frac{1}{J!} \frac{\partial^k \alpha}{\partial x^J}(t, 0) x^J.
\]

Denote by \( \tilde{X} \) and \( \tilde{Y} \) the formal \( x \)-Taylor expansions of the vector fields \( X \) and \( Y \):

\[
\tilde{X} = (\omega + \tilde{g}(t, x)) \partial_t = \left( \omega + \sum_k \sum_{|J|=k} \frac{1}{J!} \frac{\partial^k g}{\partial x^J}(t, 0) x^J \right) \partial_t,
\]

\[
\tilde{Y} = (\Lambda x + \tilde{f}(t, x)) \partial_x = \left( \Lambda x + \sum_k \sum_{|J|=k} \frac{1}{J!} \frac{\partial^k f}{\partial x^J}(t, 0) x^J \right) \partial_x.
\]

For each \( k \in \mathbb{N} \), we denote by \( \mathcal{H}_k(\mathbb{R}^s) \) the space of homogeneous polynomials in \( x \in \mathbb{R}^n \) of degree \( k \) and with coefficients in \( C^\infty(\mathbb{T}^m, \mathbb{R}^s) \):

\[
\mathcal{H}_k(\mathbb{R}^s) := \left\{ p(\theta, x) = \sum_{J \in \mathbb{N}^n, |J|=k} p_J(\theta)x^J; \ p_J \in C^\infty(\mathbb{T}^m, \mathbb{R}^s) \right\}.
\]

The following proposition establishes the formal normalization of \( X + Y \). The proof of this proposition is given in Section 4.

**Proposition 3.1.** Suppose that \((\omega, \lambda)\) satisfies \((S_\mathbb{R})\). Then for every \( k \in \mathbb{N} \), \( k > 0 \), there exist \( \phi_k \in \mathcal{H}_k(\mathbb{R}^m) \), \( \psi_k \in \mathcal{H}_k(\mathbb{R}^n) \), with \( \psi_1 = 0 \), such that the formal series on \( \mathbb{T}^m \times \mathbb{R}^n \) given by

\[
\hat{\Phi}(\theta, y) = \left( \theta + \sum_{k=1}^\infty \phi_k(\theta, y), \ y + \sum_{k=2}^\infty \psi_k(\theta, y) \right)
\]

is a formal diffeomorphism along \( \mathbb{T}^m_0 \) that transforms \( M_{\omega, \lambda} = \omega \partial_\theta + \Lambda y \partial_y \) into \( \tilde{X} + \tilde{Y} \). That is,

\[
\hat{\Phi}_* M_{\omega, \lambda} = \tilde{X} + \tilde{Y}.
\]
A $C^\infty$ function $\alpha(t, x)$ defined in a neighborhood of the torus $T_0^n$ is said to be flat along $T_0^n$ if
\[ \frac{\partial^{|J|+|L|}\alpha}{\partial x^J \partial t^L}(t, 0) = 0 \quad \forall J \in \mathbb{N}^n, \forall L \in \mathbb{N}^m. \]
Similarly, a vector field $W = A(t, x)\partial_t + B(t, x)\partial_x$ is said to be flat along $T_0^n$ if each component of $A$ and $B$ is flat along $T_0^n$.

The following lemma reduces the normalization of $X + Y$ to $M_{\omega, \lambda}$ up to flat vector fields.

**Lemma 3.1.** There exists a $C^\infty$ diffeomorphism
\[ \Psi(t, x) = (t + \alpha(t, x), x + \beta(t, x)) \]
defined in a neighborhood of $T_0^n$ with $\alpha = O(|x|)$, $\beta = O(|x|^2)$ such that
\[ \Psi_*(X + Y) = M_{\omega, \lambda} + W \]
where $W$ is a vector field flat along $T_0^n$.

**Proof.** Consider the formal diffeomorphism $\tilde{\Phi}(\theta, y)$ of Proposition 3.1 given by (3.5) with
\[ \phi_k(\theta, y) = \sum_{|J|=k} \phi_J(\theta)y^J \quad \text{and} \quad \psi_k(\theta, y) = \sum_{|J|=k} \psi_J(\theta)y^J \]
where $\phi_J \in C^\infty(T^n, \mathbb{R}^m)$ and $\psi_J \in C^\infty(T^n, \mathbb{R}^n)$. By using the Whitney Extension Theorem (see [18]), we can find functions $\phi(\theta, y) \in C^\infty(T^n \times \mathbb{R}^n, \mathbb{R}^m)$ and $\psi(\theta, y) \in C^\infty(T^n \times \mathbb{R}^n, \mathbb{R}^n)$ such that
\[ \frac{1}{J!} \frac{\partial^{|J|}\phi}{\partial y^J}(\theta, 0) = \phi_J(\theta) \quad \text{and} \quad \frac{1}{J!} \frac{\partial^{|J|}\psi}{\partial y^J}(\theta, 0) = \psi_J(\theta) \quad \forall J \in \mathbb{N}^n. \]
The map
\[ \Phi(\theta, y) = (\theta + \phi(\theta, y), y + \psi(\theta, y)) \]
is therefore a $C^\infty$ diffeomorphism in a neighborhood of $T_0^n$ such that
\[ \Phi_* M_{\omega, \lambda} = X + Y + \tilde{W}, \]
where $\tilde{W}$ is a vector field flat along $T_0^n$. The diffeomorphism $\Psi = \Phi^{-1}$ satisfies (3.7). \(\square\)

The proof of Theorem 3.1 is therefore reduced to removing the flat vector field $W$ appearing in (3.7). That is, we need to find a diffeomorphism $\Phi$ such that
\[ \Phi_* (M_{\omega, \lambda} + W) = M_{\omega, \lambda}. \]
We are going to construct such diffeomorphism $\Phi$ in a way that it coincides with the identity to infinite order along $T_0^n$:
\[ \Phi(t, x) = (t + \phi(t, x), x + \psi(t, x)) \]
with $\phi$ and $\psi$ flat along $T_0^n$. For this we use the periodic version of the homotopy method. This method was used in other normalization problems (see [5], [12], and [15] for example) and more recently by Stolovitch for the normalization in the Gevrey category of vector fields with isolated singular points ([15]). In our periodic context, we have the following proposition in which $[A, B]$ stands for the Lie bracket of the vector fields $A$ and $B$. 

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Proposition 3.2. For $\sigma \in [0, 1]$, consider the family of vector fields

\begin{equation}
M_{\omega, \lambda}^\sigma = M_{\omega, \lambda} + \sigma W
\end{equation}

with $W$ a flat vector field along $\mathbb{T}_0^m$. Then the (homological) equation

\begin{equation}
[M_{\omega, \lambda}^\sigma, Z^\sigma] = W
\end{equation}

has a solution $Z^\sigma$ for every $\sigma \in [0, 1]$, where the $C^\infty$ vector field $Z^\sigma$ is flat along $\mathbb{T}_0^m$.

The proof of this proposition is given in Section 5.

With this proposition, we can finish the proof of Theorem 3.1 as follows. Let $Z^\sigma(t, x)$ be the flat vector field solution of (3.13). Let $\Psi((t, x), \sigma)$ be the associated flow:

\begin{equation}
\frac{d\Psi}{d\sigma} = Z^\sigma(\Psi), \quad \Psi((t, x), 0) = (t, x).
\end{equation}

Then,

\begin{equation}
\Psi((t, x), \sigma) = (t + \phi(t, x, \sigma), x + \psi(t, x, \sigma)),
\end{equation}

with $\phi$ and $\psi$ flat along $\mathbb{T}_0^m$. This follows at once from the initial value problem (3.14) and the flatness of $Z^\sigma$ (see [5], for example). We need only to verify that for each $\sigma$ fixed in $[0, 1]$, we have

\begin{equation}
\Psi_* (M_{\omega, \lambda}) = M_{\omega, \lambda}^\sigma = M_{\omega, \lambda} + \sigma W.
\end{equation}

Relation (3.16) means that

\begin{equation}
D_{(t, x)} \left( \frac{d\Psi}{d\sigma} \right) M_{\omega, \lambda}(t, x) = (M_{\omega, \lambda} + \sigma W) (\Psi((t, x), \sigma)) .
\end{equation}

If we differentiate (3.17) with respect to $\sigma$ we get

\begin{equation}
D_{(t, x)} \left( \frac{d\Psi}{d\sigma} \right) M_{\omega, \lambda}(t, x) = W(\Psi) + D_{(t, x)} (M_{\omega, \lambda} + \sigma W) \frac{d\Psi}{d\sigma}.
\end{equation}

On the other hand, it follows from (3.14) that

\begin{equation}
D_{(t, x)} \left( \frac{d\Psi}{d\sigma} \right) = D_{(t, x)} Z^\sigma(\Psi) = (D_{(t, x)} Z^\sigma)(\Psi) D_{(t, x)} \Psi .
\end{equation}

Hence, after combining (3.19), (3.18), and (3.14), we find that

\begin{equation}
(D_{(t, x)} Z^\sigma)(\Psi) D_{(t, x)} \Psi M_{\omega, \lambda} = W(\Psi) + D_{(t, x)} (M_{\omega, \lambda} + \sigma W) (\Psi) Z^\sigma(\Psi) .
\end{equation}

This is precisely equation (3.13). The proof of Theorem 3.1 is complete.

4. Proof of Proposition 3.1: Formal normalization

In the space of formal power series

\begin{equation}
\hat{\alpha}(\theta, y) = \sum_{k=0}^{\infty} \phi_k(\theta, y) = \sum_{k=0, |J|=k}^{\infty} \phi_J(\theta) y^J ,
\end{equation}

with $\phi_J \in \mathcal{H}_k(\mathbb{R}^p)$, and for $N \in \mathbb{N}$, consider the truncation and projection operators defined by

\begin{equation}
T_N \hat{\alpha}(\theta, y) = \sum_{k=0}^{N} \phi_k(\theta, y) \quad \text{and} \quad P_N \hat{\alpha}(\theta, y) = \Phi_N(\theta, y) = \sum_{|J|=N} \phi_J(\theta) y^J .
\end{equation}

The following lemma will be used.
Lemma 4.1. Let \( A(\theta, y) \in C^\infty(\mathbb{T}^m \times B(r), \mathbb{R}^p) \) with \( A(\theta, 0) = 0 \) and let
\[
\hat{A}(\theta, y) = \sum_{k=1}^{\infty} A_k(\theta, y), \quad A_k \in \mathcal{H}_k(\mathbb{R}^p),
\]
be its \( y \)-Taylor series. Then, for the formal power series \( \hat{\Phi} \) given by (3.5) with \( \psi_1 = 0 \), we have
\[
\hat{A}
\left(\hat{\Phi}(\theta, y)\right) = \sum_{N=1}^{\infty} F_N(\theta, y), \quad F_N \in \mathcal{H}_N(\mathbb{R}^p),
\]
where \( F_N \) depends on the coefficients \( A_1, \ldots, A_N \) of \( \hat{A} \) and on the coefficients \( \phi_1, \ldots, \phi_{N-1} \) and \( \psi_2, \ldots, \psi_{N-1} \) of \( \hat{\Phi} \).

Proof. For a given \( \theta \in \mathbb{T}^m \), the Taylor polynomial of degree \( N \) of the function \( A(\theta + u, v) \) about \( (\theta, 0) \) is
\[
P_N(u, v) = \sum_{k=1}^{N} \left( \sum_{|H|=|L|=k} \frac{\partial^k A}{\partial^{H} \partial^L y} \right) (\theta, 0) \frac{u^H v^L}{H! L!}
\]
where \( H \in \mathbb{N}^m \) and \( L \in \mathbb{N}^n \). Hence, the truncated series of order \( N \) of \( \hat{A}(\hat{\Phi}) \) is
\[
\mathcal{T}_N \left( \hat{A}(\theta + \hat{\phi}, y + \hat{\psi}) \right) = \sum_{|H|=|L| \leq N, |L| > 0} \frac{\partial^{H+L} A}{\partial^{H} \partial^L y} (\theta, 0) \mathcal{T}_N \left[ \mathcal{T}_N(\hat{\phi})^H \mathcal{T}_N(y + \hat{\psi})^L \right] H! L!.
\]
Since \( \psi_1 = 0 \) and \( |L| \geq 1 \) (because \( A(\theta, 0) = 0 \)), then the only possible terms that remain in \( \mathcal{T}_N(\hat{\phi})^H \mathcal{T}_N(y + \hat{\psi})^L \) after truncation of order \( N \) are \( \phi_1, \ldots, \phi_{N-1} \) and \( \psi_2, \ldots, \psi_{N-1} \).

We continue with the proof of Proposition 3.1. Let
\[
X + Y = \omega \partial_t + \Lambda x \partial_x + g(t, x) \partial_t + f(t, x) \partial_x
\]
(4.4)
\[
= M_{\omega, \lambda} + g(t, x) \partial_t + f(t, x) \partial_x.
\]
For \( \hat{\Phi} \) as in (3.5) with \( \psi_1 = 0 \), equation (3.6) is equivalent to
\[
\begin{cases}
M_{\omega, \lambda}(\theta + \hat{\phi}) = \omega + \hat{g}(\theta + \hat{\phi}, y + \hat{\psi}), \\
M_{\omega, \lambda}(y + \hat{\psi}) = \Lambda(y + \hat{\psi}) + \hat{f}(\theta + \hat{\phi}, y + \hat{\psi}).
\end{cases}
\]
(4.5)
Since \( M_{\omega, \lambda} \theta = \omega \) and \( M_{\omega, \lambda} y = \Lambda y \), system (4.5) is equivalent to
\[
\begin{cases}
M_{\omega, \lambda} \hat{\phi} = \hat{g}(\theta + \hat{\phi}, y + \hat{\psi}), \\
M_{\omega, \lambda} \hat{\psi} = \Lambda \hat{\psi} + \hat{f}(\theta + \hat{\phi}, y + \hat{\psi}).
\end{cases}
\]
(4.6)
Note also that since \( M_{\omega, \lambda} J^J = (\lambda, J) y^J \), then after using the projection operators \( \mathcal{P}_k \) defined in (4.1), we find that for each \( k \in \mathbb{N}, k \geq 1 \), the functions \( \phi_k \in \mathcal{H}_k(\mathbb{R}^m) \) and \( \psi_k \in \mathcal{H}_k(\mathbb{R}^n) \) need to satisfy the system
\[
\begin{cases}
M_{\omega, \lambda} \phi_k(\theta, y) = \mathcal{P}_k \hat{g}(\theta + \hat{\phi}, y + \hat{\psi}), \\
M_{\omega, \lambda} \psi_k(\theta, y) - \Lambda \psi_k(\theta, y) = \mathcal{P}_k \hat{f}(\theta + \hat{\phi}, y + \hat{\psi}).
\end{cases}
\]
(4.7)
It follows from Lemma 4.1 and from the hypotheses on the functions $f$ and $g$ that

$$\mathcal{P}_k \hat{g}(\theta + \tilde{\phi}, y + \tilde{\psi}) \quad \text{and} \quad \mathcal{P}_k \hat{f}(\theta + \tilde{\phi}, y + \tilde{\psi})$$

depend only on $\phi_1, \ldots, \phi_{k-1}$ and $\psi_1, \ldots, \psi_{k-1}$ for $k \geq 2$.

For $k = 1$, the second equation in (4.7) is trivially satisfied by $\psi_1 = 0$ since $f(t, x) = O(|x|^2)$, and therefore the first equation reduces to

$$M_{\omega,\lambda} \phi_1(\theta, y) = \sum_{|J|=1} \frac{\partial g}{\partial y_J}(\theta, 0)y^J. \tag{4.8}$$

Set $\phi_1(\theta, y) = \sum_{|J|=1} \phi_J(\theta)y^J$. It follows from (4.8) that $\phi_J(\theta)$ needs to satisfy

$$\omega \partial_\theta \phi_J(\theta) + \langle \lambda, J \rangle \phi_J(\theta) = \frac{\partial g}{\partial y_J}(\theta, 0). \tag{4.9}$$

We can solve this equation by using Fourier series:

$$\frac{\partial g}{\partial y_J}(\theta, 0) = \sum_{K \in \mathbb{Z}^m} g_{JK} e^{i\langle K, \theta \rangle}, \quad \phi_J(\theta) = \sum_{K \in \mathbb{Z}^m} \phi_{JK} e^{i\langle K, \theta \rangle}. \tag{4.10}$$

We find that

$$\phi_{JK} = \frac{g_{JK}}{i\langle \omega, K \rangle + \langle \lambda, J \rangle}. \tag{4.11}$$

Since $\frac{\partial g}{\partial y_J}(\theta, 0) \in C^\infty(\mathbb{T}^m, \mathbb{R}^m)$, then its sequence of Fourier coefficients $\{g_{JK}\}_K$ decays rapidly as $|K| \to \infty$ and so does the sequence $\{\phi_{JK}\}_K$ given by (4.11) (thanks to $(\omega, \lambda)$ satisfying the $(S_\omega)$). Hence, the series in (4.10) defines $\phi_J \in C^\infty(\mathbb{T}^m, \mathbb{R}^m)$, and system (4.7) is solved when $k = 1$.

Now we continue the proof by induction. Suppose that

$$\phi_l(\theta, y) = \sum_{J \in \mathbb{N}^n, |J| = l} \phi_J(\theta)y^J, \quad \psi_l(\theta, y) = \sum_{J \in \mathbb{N}^n, |J| = l} \psi_J(\theta)y^J, \tag{4.12}$$

with $\phi_J$ and $\psi_J$ of class $C^\infty$ on the torus $\mathbb{T}^m$, have been determined and satisfy system (4.7) for $l = 1, \ldots, N - 1$. Then system (4.7) for $k = N$ becomes

$$\begin{align*}
M_{\omega,\lambda} \phi_N(\theta, y) &= A_N(\theta, y) = \sum_{J \in \mathbb{N}^n, |J| = N} A_J(\theta)y^J, \\
M_{\omega,\lambda} \psi_N(\theta, y) - \Lambda \psi_N(\theta, y) &= B_N(\theta, y) = \sum_{J \in \mathbb{N}^n, |J| = N} A_J(\theta)y^J, \tag{4.13}
\end{align*}$$

where $A_J(\theta) \in C^\infty(\mathbb{T}^m, \mathbb{R}^m)$, $B_J(\theta) \in C^\infty(\mathbb{T}^m, \mathbb{R}^m)$ depend only on the already known functions $\phi_L(\theta), \psi_L(\theta)$ with $|L| \leq N - 1$ and on the derivatives up to order $N$ of the functions $g$ and $f$. Let

$$\phi_N(\theta, y) = \sum_{|J| = N} \phi_J(\theta)y^J \quad \text{and} \quad \psi_N(\theta, y) = \sum_{|J| = N} \psi_J(\theta)y^J. \tag{4.14}$$
Equations (4.13) reduce then to the systems

\begin{equation}
\begin{aligned}
\omega \partial_\theta \phi_J(\theta) + & \langle \lambda, J \rangle \phi_J(\theta) = A_J(\theta), \\
\omega \partial_\theta \psi_J(\theta) - & (\langle \lambda, J \rangle I_n - \Lambda) \psi_J(\theta) = B_J(\theta),
\end{aligned}
\end{equation}

for each $J \in \mathbb{N}^n$ with $|J| = N$ ($I_n$ is the $n \times n$ identity matrix). Again, Fourier series gives the $K$-th Fourier coefficients of $\phi_J$ and $\psi_J$ as

\begin{equation}
\begin{aligned}
\phi_{JK} &= \frac{A_{JK}}{\iota (\omega, K) + \langle \lambda, J \rangle} \quad \text{and} \\
\psi_{JK} &= ((\iota (\omega, K) + \langle \lambda, J \rangle) I_n - \Lambda)^{-1} B_{JK},
\end{aligned}
\end{equation}

where $A_{JK}$ and $B_{JK}$ are, respectively, the $K$-th Fourier coefficients of $A_J(\theta)$ and $B_J(\theta)$. Condition $(S_\sigma)$ guarantees that $\phi_{JK}$ and $\psi_{JK}$ decay rapidly and $\phi_J$ and $\psi_J$ are $C^\infty$. This completes the proof of Proposition 3.1.

5. Proof of Proposition 3.2: The homological equation

We adapt the ideas contained in [3], [12], and [15] to our problem. First, we prove the case when the vector field $M_{\omega, \lambda}$ contracts onto the torus $\mathbb{T}_0^n$. That is, we consider the case

\begin{equation}
\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n < 0.
\end{equation}

Since the problem is local in the $x$-variable, we can assume that $W = 0$ for $|x| > r$. Indeed, this can be achieved by multiplying the vector field $W$ by a cutoff function $\chi(x) \in C^\infty(\mathbb{R}^n)$ with $\chi(x) = 1$ for $|x| \leq r/2$, $\chi(x) = 0$ for $|x| \geq r$, and $0 \leq \chi \leq 1$. Let $\rho^2 = |x|^2 = x_1^2 + \cdots + x_n^2$.

It follows from (5.1) and the flatness of $W$ that we can assume (after replacing $r$ by a smaller value if necessary) that

\begin{equation}
\lambda_1 \rho^2 \leq M_{\omega, \lambda} \rho^2 + \sigma W \rho^2 \leq \lambda_n \rho^2.
\end{equation}

We write

\begin{equation}
W = U(t, x) \partial_t + V(t, x) \partial_x,
\end{equation}

where the functions $U \in C^\infty(\mathbb{T}^m \times \mathbb{R}^n, \mathbb{R}^m)$ and $V \in C^\infty(\mathbb{T}^m \times \mathbb{R}^n, \mathbb{R}^n)$ are flat along $\mathbb{T}_0^n$ and with compact support in $\mathbb{T}^m \times B(r)$. For $p = (t, x) \in \mathbb{T}^m \times B(r)$, let $\gamma_{\sigma}(p, u) = (\alpha_{\sigma}(p, u), \beta_{\sigma}(p, u))$ be the flow of the vector field $M_{\omega, \lambda} + \sigma W$:

\begin{equation}
\begin{cases}
\frac{d\alpha_{\sigma}}{du} = \omega + \sigma U(\alpha_{\sigma}, \beta_{\sigma}), & \alpha_{\sigma}(p, 0) = t, \\
\frac{d\beta_{\sigma}}{du} = \Lambda \beta_{\sigma} + \sigma V(\alpha_{\sigma}, \beta_{\sigma}), & \beta_{\sigma}(p, 0) = x.
\end{cases}
\end{equation}

It follows from (5.1) that for every $(t, x) \in \mathbb{T}^m \times B(r/2)$ we have $\alpha_{\sigma}((t, x), u) \in \mathbb{T}^m$ and $\beta_{\sigma}((t, x), u) \in B(r)$ $\forall u \geq 0$, $\sigma \in [0, 1]$.

In fact, it follows from (5.2), that there exist constants $C > 0$ and $a > 0$ such that

\begin{equation}
|\beta_{\sigma}((t, x), u)| \leq Ce^{-au}|x| \quad \forall u \geq 0.
\end{equation}
Consider the linear differential equation

\[ \frac{dH}{dt} = [D(t,x)(M_{\omega,\lambda} + \sigma W)(\gamma_\sigma(p,u))] H + W(\gamma_\sigma(p,u)) \]

(5.6) with \( p = (t,x) \) and \( \sigma \) parameters. A solution \( H(p,\sigma,u) \) can be found in the form

\[ H(p,\sigma,u) = -F(p,\sigma,u) \int_0^\infty F(p,\sigma,s)^{-1} W(\gamma_\sigma(p,s))ds \]

(5.7) where \( F(p,\sigma,u) \) is the fundamental matrix of the associated homogeneous equation:

\[ \frac{dF}{du} = [D(t,x)(M_{\omega,\lambda} + \sigma W)(\gamma_\sigma(p,u))] F, \quad F(p,\sigma,0) = I. \]

Define the vector field \( Z^\sigma \) along the curve \( \gamma_\sigma \) by

\[ Z^\sigma(\gamma_\sigma(t,x,u)) := H(t,x,\sigma,u). \]

(5.9) In particular

\[ Z^\sigma(t,x) = H(t,x,\sigma,0) = -\int_0^\infty F(t,x,\sigma,s)^{-1} W(\gamma_\sigma(t,x,s))ds. \]

We need to verify that \( Z^\sigma \) satisfies equation (3.13) and that it is flat along \( T_0^m \). From the definition of \( Z^\sigma \) in (5.10), we get, after differentiating with respect to \( u \) and using (5.6), that

\[ D(t,x)Z^\sigma(\gamma_\sigma) \frac{d\gamma_\sigma}{du} = [D(t,x)(M_{\omega,\lambda} + \sigma W)(\gamma_\sigma)] H + W(\gamma_\sigma). \]

(5.11) The use of (5.4) and (5.9) allows us to rewrite (5.11) as

\[ D(t,x)Z^\sigma(\gamma_\sigma)(M_{\omega,\lambda} + \sigma W)(\gamma_\sigma) = D(t,x)(M_{\omega,\lambda} + \sigma W)(\gamma_\sigma) Z^\sigma(\gamma_\sigma) + W(\gamma_\sigma). \]

(5.12) This is precisely (3.13) evaluated at the point \( \gamma_\sigma(t,x,u) \).

To prove that \( Z^\sigma \) is flat along \( T_0^m \) we need to estimate the matrix \( F^{-1} \) that appears in the integral in (5.10). The differential equation for \( F^{-1} \) can be deduced from (5.8):

\[ \frac{dF^{-1}}{du} = -F^{-1} [D(t,x)(M_{\omega,\lambda} + \sigma W)(\gamma_\sigma(p,u))] , \quad F^{-1}(p,\sigma,0) = I. \]

Since \( W \) is flat along \( T_0^m \), we can find constants \( B > 0 \) and \( b > 0 \) such that

\[ |F(t,x,\sigma,u)^{-1}| \leq Be^{bu} \quad \forall u \geq 0 , \forall (t,x) \in T^m \times B(r) \quad \text{and} \quad \sigma \in [0, 1]. \]

(5.13) Recall that since \( W \) is flat along \( T_0^m \), then for every \( N \in \mathbb{N} \), we can find \( C_N > 0 \) such that \( |W(t,x)| \leq C_N |x|^N \) for every \( (t,x) \in T^m \times B(r) \). It follows then from (5.5), (5.10) and (5.13) that

\[ |Z^\sigma(t,x)| \leq C_N BC \int_0^\infty e^{(b-aN)N} ds |x|^N \leq \frac{C_N BC}{aN - b} |x|^N \]

(5.14) for \( N > b/a \). Since \( N \) is arbitrary, then \( Z^\sigma \) is flat along \( T_0^m \). This proves the proposition when all the \( \lambda_j \)'s are negative. The same proof works also when all the \( \lambda_j \)'s are positive by considering the integral curves for \( u \leq 0 \).

Now we consider the general case when the \( \lambda_j \)'s have different signs. Suppose that

\[ \lambda_1 \leq \cdots \leq \lambda_p < 0 < \lambda_{p+1} \leq \cdots \leq \lambda_n. \]

(5.15)
If \( n = p + q \), \( \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q \), then we can write \( x \in \mathbb{R}^n \) as \( x = (x', x'') \) where \( x' \in \mathbb{R}^p \) and \( x'' \in \mathbb{R}^q \). We need the following lemma about the decomposition of flat functions (there are various versions of such decompositions (see [12] and [15])).

Lemma 5.1. Let \( f(t, x', x'') \) be a \( C^\infty \) function defined in a neighborhood of \( \mathbb{T}_0^m = \mathbb{T}^m \times \{0\} \times \{0\} \) in \( \mathbb{T}^m \times \mathbb{R}^p \times \mathbb{R}^q \) such that \( f \) is flat along \( \mathbb{T}_0^m \). Then there exist \( C^\infty \) functions \( g \) and \( h \) defined in a neighborhood of \( \mathbb{T}_0^m \) in \( \mathbb{T}^m \times \mathbb{R}^p \times \mathbb{R}^q \) such that \( g \) is flat along \( x' = 0 \), \( h \) is flat along \( x'' = 0 \), and for every \( (t, x', x'') \) in a neighborhood of \( \mathbb{T}_0^m \).

Proof. We use the Whitney Extension Theorem ([13]) to define \( h(t, x', x'') \) as a \( C^\infty \) function such that

\[
\frac{\partial^{\mid J_1 \mid + \mid J_2 \mid} h}{\partial x'^{J_1} \partial x''^{J_2}} (t, x', 0) = 0,
\]

\[
\frac{\partial^{\mid J_1 \mid + \mid J_2 \mid} h}{\partial x'^{J_1} \partial x''^{J_2}} (t, 0, x'') = \frac{\partial^{\mid J_1 \mid + \mid J_2 \mid} f}{\partial x'^{J_1} \partial x''^{J_2}} (t, 0, x''),
\]

for every \( J_1 \in \mathbb{N}^p \) and \( J_2 \in \mathbb{N}^q \). Thus such a function \( h \) is flat along \( x'' = 0 \), and the function \( g = f - h \) is flat along \( x' = 0 \). \( \Box \)

For \( p = (t, x', x'') \) let \( \gamma_\sigma(p, u) = (\alpha_\sigma(p, u), \beta'_\sigma(p, u), \beta''_\sigma(p, u)) \) be the integral curve of \( M_{\omega, \lambda} + \sigma W \) through the point \( p \):

\[
\begin{align*}
\frac{d\alpha_\sigma}{du} &= \omega + \sigma U(\alpha_\sigma, \beta'_\sigma, \beta''_\sigma), & \alpha_\sigma(p, 0) = t, \\
\frac{d\beta'_\sigma}{du} &= \Lambda' \beta'_\sigma + \sigma V'(\alpha_\sigma, \beta'_\sigma, \beta''_\sigma), & \beta'_\sigma(p, 0) = x', \\
\frac{d\beta''_\sigma}{du} &= \Lambda'' \beta''_\sigma + \sigma V''(\alpha_\sigma, \beta'_\sigma, \beta''_\sigma), & \beta''_\sigma(p, 0) = x'',
\end{align*}
\]

(5.17)

where \( W = (U, V', V'') \), \( \Lambda' = \text{diag}(\lambda_1, \cdots, \lambda_p) \), and \( \Lambda'' = \text{diag}(\lambda_{p+1}, \cdots, \lambda_n) \).

Since \( M_{\omega, \lambda} + \sigma W \) is hyperbolic transversally to the torus \( \mathbb{T}_0^m \), we can define its stable and unstable invariant manifolds along \( \mathbb{T}_0^m \) as:

\[
\mathcal{M}_{\sigma}^{st} = \left\{ (t, x', x'') : \lim_{u \to \infty} \beta'_\sigma = 0 \quad \text{and} \quad \lim_{u \to -\infty} \beta''_\sigma = 0 \right\},
\]

(5.18)

\[
\mathcal{M}_{\sigma}^{un} = \left\{ (t, x', x'') : \lim_{u \to -\infty} \beta'_\sigma = 0 \quad \text{and} \quad \lim_{u \to \infty} \beta''_\sigma = 0 \right\}.
\]

(5.19)

Note that

\[
\mathcal{M}_{0}^{st} = E^{st} = \mathbb{T}^m \times \mathbb{R}^p \times \{0\} \quad \text{and} \quad \mathcal{M}_{0}^{un} = E^{un} = \mathbb{T}^m \times \{0\} \times \mathbb{R}^q.
\]

Since \( W \) is flat, the stable and unstable invariant manifolds \( \mathcal{M}_{\sigma}^{st} \) and \( \mathcal{M}_{\sigma}^{un} \) are tangent to infinite order with \( E^{st} \) and \( E^{un} \) at each point of \( \mathbb{T}_0^m \). We can therefore find a change of coordinates

\[
\chi(t, x', x'') = (t + \phi_\sigma(t, x', x''), x' + \psi'_\sigma(t, x', x''), x'' + \psi''_\sigma(t, x', x'')) \quad \text{with} \quad \phi_\sigma, \psi'_\sigma \text{ and } \psi''_\sigma \text{ flat along } \mathbb{T}_0^m,
\]

(5.20)

such that

\[
\chi_*(M_{\omega, \lambda} + \sigma W) = M_{\omega, \lambda} + \sigma \tilde{W},
\]

(5.21)
where \( \tilde{W} \) is flat along \( \mathbb{T}_0^m \), and moreover, the stable and unstable invariant manifolds of \( M_{\omega, \lambda} + \sigma \tilde{W} \) are, respectively, \( E^{st} \) and \( E^{un} \). Now, we can use Lemma 5.1 to decompose the flat vector field \( \tilde{W} \) as
\[
(5.22) \quad \tilde{W} = \tilde{W}^{st} + \tilde{W}^{un}
\]
with \( \tilde{W}^{st} \) flat along \( E^{st} \) and \( \tilde{W}^{un} \) flat along \( E^{un} \).

The proof given above to produce the vector field \( Z^\sigma \) when all \( \lambda_j \)'s are negative can be repeated to produce a vector field \( Z^{\sigma}_{st} \), flat along \( E^{st} \) and such that
\[
(5.23) \quad \left[ M_{\omega, \lambda} + \sigma \tilde{W}^{st}, Z^{\sigma}_{st} \right] = \tilde{W}^{st}.
\]

The corresponding flow \( \Psi^{st}_\sigma \) of \( Z^{\sigma}_{st} \) coincides with the identity to infinite order along \( E^{st} \) and satisfies
\[
(5.24) \quad \left( \Psi^{st}_\sigma \right)_* M_{\omega, \lambda} = M_{\omega, \lambda} + \sigma \tilde{W}^{st}.
\]

We repeat the construction for the vector field \( \tilde{W}^{un} \) to obtain a vector field \( Z^{\sigma}_{un} \) flat along \( E^{un} \), and such that
\[
(5.25) \quad \left[ M_{\omega, \lambda} + \sigma \tilde{W}^{st} + \sigma \tilde{W}^{un}, Z^{\sigma}_{un} \right] = \tilde{W}^{un}.
\]

The corresponding flow \( \Psi^{un}_\sigma \) of \( Z^{\sigma}_{un} \) coincides with the identity to infinite order along \( E^{un} \) and satisfies
\[
(5.26) \quad \left( \Psi^{un}_\sigma \right)_* \left( M_{\omega, \lambda} + \sigma \tilde{W}^{st} \right) = M_{\omega, \lambda} + \sigma \tilde{W}^{st} + \sigma \tilde{W}^{un}.
\]

Hence, the diffeomorphism \( \Psi^{un}_\sigma \circ \Psi^{st}_\sigma \) coincides with the identity to infinite order along \( \mathbb{T}_0^m \) and satisfies
\[
(5.27) \quad \left( \Psi^{un}_\sigma \circ \Psi^{st}_\sigma \right)_* M_{\omega, \lambda} = M_{\omega, \lambda} + \sigma \tilde{W}.
\]
This completes the proof of Proposition 3.2.

6. Normal form: The complex case

We consider the complex vector field
\[
X + iY = (\omega + g(t, x)) \partial_t + i(\Lambda x + f(t, x)) \partial_x
\]
and seek a diffeomorphism that transforms it into the vector field
\[
(6.1) \quad L_{\omega, \lambda} = \omega \partial_t + i\Lambda x \partial_x.
\]
It is clear, since the vector fields \( \omega \partial_t \) and \( \Lambda x \partial_x \) commute, that a necessary condition for such a transformation to exist is that the vector fields \( X \) and \( Y \) must commute.

We will show that this condition is also sufficient in the real analytic category provided that \( (\omega, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n \) satisfy a Siegel type condition. However, these two conditions alone are not sufficient in the \( C^\infty \) category unless all the \( \lambda_j \)'s have the same sign.

For given \( \omega = (\omega_1, \cdots, \omega_m) \in \mathbb{R}^m \) and \( \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \), the Siegel type condition \((\mathcal{S}_C)\) is the following:

\((\mathcal{S}_C)\): There exists \( C > 0 \) and \( \mu > 0 \) such that
\[
|\langle \omega, K \rangle + \langle \lambda, J \rangle| \geq \frac{C}{(|K| + |J|)^\mu}
\]
for every \( K \in \mathbb{Z}^m, J \in \mathbb{N}^n \), with \( |K| + |J| > 0 \).
We have the following normalization result.

**Theorem 6.1.** Suppose that \((\omega, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n\) satisfies \((S_C)\). Let \(g(t, x) \in C^\infty(\mathbb{T}^m \times B(r), \mathbb{R}^m)\), \(f(t, x) \in C^\infty(\mathbb{T}^m \times B(r), \mathbb{R}^n)\) with \(g = O(|x|)\), \(f = O(|x|^2)\) and such that the vector fields \(X = (\omega + g)\partial_t\) and \(Y = (\lambda x + f)\partial_x\) satisfy
\[
[X, Y] = 0.
\]
Then there exists a \(C^\infty\)-diffeomorphism
\[
\Phi(t, x) = (t + \phi(t, x), x + \psi(t, x))
\]
deﬁned in a neighborhood of \(\mathbb{T}_0^m\), with \(\phi(t, 0) = 0\) and \(\psi(t, 0) = 0\), such that
\[
(\Phi^* \omega, \lambda) = \Phi \omega, \lambda
\]
where \(L_{\omega, \lambda}\) is given in (6.1).

The proof uses the following lemma.

**Lemma 6.1.** Let \(Y\) be as in Theorem 6.1. If \(u(t, x)\) is a \(C^\infty\) function in a neighborhood of \(\mathbb{T}_0^m\) and satisfies \(Yu = 0\), then \(u\) is independent on the \(x\)-variable.

**Proof.** Note that since \(f = O(|x|^2)\), then for \(J \in \mathbb{N}^n\), we have
\[
Y x^J = \langle \lambda, J \rangle x^J + O(|x||J|^{1+1}).
\]
A function \(u(t, x) \in C^\infty(\mathbb{T}^m \times B(r))\) can be expanded as
\[
u(t, x) = \sum_{k=0}^\infty \sum_{|J|=k} u_J(t)x^J = u_0(t) + \sum_{|J|=1} u_J(t)x^J + O(|x|^2).
\]
If \(Yu = 0\), then
\[
0 = Yu = \sum_{|J|=1} \langle \lambda, J \rangle u_J(t)x^J + O(|x|^2).
\]
Consequently for \(J \in \mathbb{N}^n\) with \(|J| = 1\), we have \(u_J(t) = 0\) since \(\langle \lambda, J \rangle \neq 0\) (condition \((S_C)\)). By induction we can prove that for every \(j \in \mathbb{N}^n\) with \(|J| \geq 1\), we have \(u_J(t) = 0\).

**Proof of Theorem 6.1.** Condition (6.2) implies that
\[
XF(t, x) = 0 \quad \text{and} \quad Yg(t, x) = 0.
\]
It follows from Lemma 6.1 that the function \(g\) is independent on the \(x\)-variable and since \(g = O(|x|)\), then \(g = 0\) and consequently
\[
X = \omega \partial_t.
\]
With this expression for \(X\), we can solve equation \(XF = 0\) by using Fourier series. Let
\[
f(t, x) = \sum_{K \in \mathbb{Z}^m} f_K(x)e^{i(K, t)}
\]
be the Fourier series of \(f\). Then,
\[
XF(t, x) = \sum_{K \in \mathbb{Z}^m} i\langle \omega, K \rangle f_K(x)e^{i(K, t)} = 0.
\]
It follows then from \((S_C)\) that \(\langle \omega, K \rangle \neq 0\) for \(K \neq 0\). Thus, \(f_K(x) = 0\) for \(K \neq 0\). Hence, \(f(t, x)\) is independent on the \(t\)-variable:
\[
Y = (\Lambda x + f(x)) \partial_x.
\]
The hyperbolic vector field $Y$ is equivalent to its linear part $\lambda x \partial_x$ in a neighborhood of $0 \in \mathbb{R}^n$ since $\lambda$ satisfies the Siegel condition (see [1] for example). Therefore, there exists a $C^\infty$-diffeomorphism $\Psi(x) = x + \beta(x)$ in a neighborhood of $0 \in \mathbb{R}^n$ with $\beta(0) = 0$ and $D\beta(0) = 0$ such that $\Psi_* Y = \lambda x \partial_x$. The $C^\infty$-diffeomorphism
\[
\Phi(t, x) = (t, \Psi(x))
\]
defined in a neighborhood of $\mathbb{T}_0^n$ satisfies (6.3).

Remark 6.1. It follows from our proof that what is needed for the normalization is the real analyticity in the $x$-variable. Hence, the normalization result in Theorem 6.1 extends to vector fields that are $C^\infty$ in the $t$-variable and $C^\infty$ in the $x$-variable.

The normalization of $X + iY$ into $L_{\omega, \lambda}$ is not always possible in the $C^\infty$ category (see Proposition 6.1 below) unless $Y$ or $-Y$ is contracting onto the torus $\mathbb{T}_0^n$. More precisely, we have the following theorem.

Theorem 6.2. Suppose that $(\omega, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n$ satisfies (6.1) and that $\lambda_1, \ldots, \lambda_n$ have the same sign. Let $g(t, x) \in C^\infty(\mathbb{T}_m \times B(r), \mathbb{R}^m)$, $f(t, x) \in C^\infty(\mathbb{T}_m \times B(r), \mathbb{R}^n)$ with $g = O(|x|)$ and $f = O(|x|^2)$ and such that the vector fields $X = (\omega + g)\partial_t$ and $Y = (\lambda x + f)\partial_x$ commute. Then, there exists a $C^\infty$-diffeomorphism
\[
\Phi(t, x) = (t + \phi(t, x), x + \psi(t, x))
\]
defined in a neighborhood of $\mathbb{T}_0^m$, with $\phi(t, 0) = 0$ and $\psi(t, 0) = 0$, such that
\[
(6.7) \quad \Phi_* (X + iY) = L_{\omega, \lambda}
\]
where $L_{\omega, \lambda}$ is given in (6.1).

Proof. Note that since all the $\lambda_j$'s have the same sign and since $f = O(|x|^2)$, then for every fixed point $t$ in $\mathbb{T}_m$, every integral curve of $Y$ contains 0 in its closure. Hence, if $u(t, x)$ is any continuous solution of the equation $Yu = 0$, then it is independent on the x-variable. With this in hand, the rest of the proof continues as that of Theorem 6.1. \qed

Now, we give an example of a $C^\infty$ vector field that cannot be normalized to $L_{\omega, \lambda}$. Let $(\omega, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n$ with $n \geq 2$ and such that $\lambda_1 < 0$ and $\lambda_2 > 0$. Let $\beta(x)$ be the function defined in $\mathbb{R}^n$ by
\[
\beta(x_1, x_2, \ldots, x_n) = x_1^{\lambda_1} x_2^{-\lambda_2} \text{ if } x_1 > 0, x_2 > 0 \text{ and } \beta(x) = 0 \text{ elsewhere}.
\]
Note that $\lambda x \partial_x \beta = 0$. Let $\alpha \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that
\[
\alpha(s) = 0 \text{ for } s \leq 0 \text{ and } \alpha(s) > 0 \text{ for } s > 0.
\]
Define $g \in C^\infty(\mathbb{T}_m \times \mathbb{R}^n, \mathbb{R}^m)$, flat along $\mathbb{T}_0^n$, by
\[
g_1(t, x) = \alpha(\beta(x)) \quad \text{and} \quad g_j(t, x) = 0 \text{ for } j = 2, \ldots, m.
\]
Consider the vector field $T$ defined on $\mathbb{T}_m \times \mathbb{R}^n$ by
\[
(6.8) \quad T := (\omega + g(t, x))\partial_t + i\lambda x \partial_x = L_{\omega, \lambda} + g_1(x) \partial_{\ell_1}.
\]
We have the following proposition.

Proposition 6.1. The vector field $T$ given by (6.8) has the following properties.
\begin{enumerate}
\item [(i)] $[Re(T), Im(T)] = 0$.
\item [(ii)] There is no diffeomorphism $\Phi$ of a neighborhood of $\mathbb{T}_0^n$ such that $\Phi_* T = L_{\omega, \lambda}$.
\end{enumerate}
Proof. Property (i) follows directly from $\Lambda x \partial_x \beta = 0$. We prove (ii) by contradiction. Suppose that there exists a diffeomorphism $\Phi$ that transforms $T$ to $L_{\omega, \lambda}$ in a neighborhood of $T^m_0$. Then we can assume, after a linear change in the $t$-variables, that

$$
\Phi(t,x) = (t + \phi(t,x), \psi(t,x))
$$

(where $\phi$ and $\psi$ are $2\pi$-periodic in the $t$-variables). The condition $\Phi^* T = L_{\omega, \lambda}$ implies that for $j = 1, \cdots, m$ we have

$$
\sum_{k=1}^m (\omega_k + g_k(x)) \frac{\partial (t_j + \phi_j(t,x))}{\partial t_k} + i \sum_{l=1}^n \lambda_l x_l \frac{\partial \phi_j(t,x)}{\partial x_l} = \omega_j.
$$

In particular, for $j = 1$ the function $\phi_1$ must satisfy

$$
\sum_{k=1}^m (\omega_k + g_k(x)) \frac{\partial (\phi_1(t,x))}{\partial t_k} = -g_1(x).
$$

But this equation cannot have periodic solutions unless $g_1 = 0$, which is a contradiction. □

7. SOLVABILITY OF THE MODEL VECTOR FIELD $M_{\omega, \lambda}$

We study the solvability of the equation

$$
M_{\omega, \lambda} u(t,x) = f(t,x)
$$

in a neighborhood of the torus $T^m_0$ in $T^m \times \mathbb{R}^n$, where $M_{\omega, \lambda}$ is the model vector field of Section 3: $M_{\omega, \lambda} = \omega \partial_t + \Lambda x \partial_x$. It should be noted right away that a necessary condition for equation (7.1) to have a solution is that the right side $f$ must satisfy

$$
\int_{T^m} f(t,0) dt = 0.
$$

We will assume throughout, when considering equation (7.1), that the right side satisfies (7.2). In the real analytic category we have the following theorem.

**Theorem 7.1.** Assume that $(\omega, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n$ satisfies $(S_R)$. Let $f(t,x) \in C^\infty(T^m \times B(r))$ satisfy (7.2). Then equation (7.1) has a unique solution $u \in C^\infty(T^m \times B(r_1))$ for some $0 < r_1 < r$.

**Proof.** Consider the Fourier-Taylor expansion of $f$:

$$
f(t,x) = \sum_{J \in \mathbb{N}^n} f_{JK} e^{i\langle K,t \rangle} x^J
$$

(with $f_{00} = 0$). The unique formal series solution of equation (7.1) is

$$
\hat{u}(t,x) = \sum_{J \in \mathbb{N}^n} f_{JK} e^{i\langle K,t \rangle} x^J.
$$

We need to verify that the series $\hat{u}$ defines a real analytic function in a neighborhood of $T^m_0$.

We can assume that the holomorphic extension $\tilde{f}$ of $f$ is defined in the region

$$
D_{r_1} = \{(t + it', x + ix') \in \mathbb{C}^m \times \mathbb{C}^n, \max\{|t'|, |x|, |x'|\} \leq r_1\}
$$
for some $0 < r_1 < r$ and that $\tilde{f}$ is $2\pi$-periodic in the $t$-variables. In the space of such functions, define the norm

$$||\tilde{f}||_r = \sup_{(t,\bar{x}) \in D_{r_1}} |\tilde{f}(t,\bar{x})|.$$  

Then from Lemma 2.1 of [3], we obtain the following estimates for the coefficients $f_{JK}$:

$$(7.5) \quad |f_{JK}| \leq ||\tilde{f}||_r r_1^{-|J|} e^{-r_1|K|}.$$  

Finally, it follows from (7.5), (7.4), condition $(S_\pi)$, and Lemma 2.2 of [3] that there exists a constant $C = C(r_1)$ such that

$$|\check{u}(t,\bar{x})| \leq C ||\tilde{f}||_r.$$  

Therefore, $\check{u}(t,\bar{x})$ defines a holomorphic function in $D_{r_1}$, and its restriction $u(t, x)$ is the real analytic solution of (7.1) in $\mathbb{T}^m \times B(r_1)$. □

For the $C^\infty$-solvability, we have the following result.

**Theorem 7.2.** Assume that $(\omega, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n$ satisfies $(S_\pi)$. Suppose $f(t, x) \in C^\infty(\mathbb{T}^m \times B(r))$ satisfies (7.2). Then, equation (7.1) has a unique solution $u \in C^\infty(\mathbb{T}^m \times B(r_1))$ for some $0 < r_1 < r$.

The proof uses the following lemmas about formal solvability and solvability for flat functions.

**Lemma 7.1.** Let

$$\check{u}(t, x) = \sum_{J \in \mathbb{N}^m} v_J(t) x^J$$  

be a series with $v_J \in C^\infty(\mathbb{T}^m)$ and $v_0$ satisfying

$$\int_{\mathbb{T}^m} v_0(t) dt = 0.$$  

Then, the equation $M_{\omega, \lambda} \check{u} = \check{v}$ has a series solution

$$\check{u}(t, x) = \sum_{J \in \mathbb{N}^m} u_J(t) x^J$$  

with $u_J \in C^\infty(\mathbb{T}^m)$.

**Proof.** A series $\check{u} = \sum u_J(t) x^J$ solves $M_{\omega, \lambda} \check{u} = \check{v}$ if and only if for every $J \in \mathbb{N}^n$, the $C^\infty$ function $u_J(t)$ satisfies

$$\omega \partial_t u_J(t) + (\lambda, J) u_J(t) = v_J(t).$$  

By using Fourier series $v_J(t) = \sum_{K \in \mathbb{Z}^m} v_{JK} e^{i(K,t)}$, $u_J(t) = \sum_{K \in \mathbb{Z}^m} u_{JK} e^{i(K,t)}$, we get the coefficients $u_{JK}$ as

$$u_{JK} = \frac{v_{JK}}{i(\omega, K) + (\lambda, J)}.$$  

Since the function $v_J$ is $C^\infty$, then its Fourier coefficients $v_{JK}$ decay rapidly as $|K| \to \infty$. The coefficients $u_{JK}$ are also of rapid decay since $(\omega, \lambda)$ satisfies $(S_\pi)$. Hence, $u_J(t) \in C^\infty(\mathbb{T}^m)$.

**Lemma 7.2.** Let $f \in C^\infty(\mathbb{T}^m \times B(r))$ be flat along $\mathbb{T}^m_0$. Then equation (7.1) has a solution $u \in C^\infty(\mathbb{T}^m \times B(r_1))$ with $u$ flat along $\mathbb{T}^m_0$.
Proof. Let 
\[ \Phi_s(t, x) = (\omega s + t, e^{\lambda s} x) \]
be the flow of \( M_{\omega, \lambda} \). If all the \( \lambda_j \)'s are negative, then we define the solution \( u \) as

(7.8) \[ u(t, x) = -\int_0^\infty f(\Phi_s(t, x)) ds . \]

It follows at once from the flatness of \( f \) and the hypothesis \( \lambda_j < 0 \) for all \( j \) that \( u \) is flat along \( x = 0 \). That \( u \) solves (7.1) follows immediately from its definition in (7.8). If all the \( \lambda_j \)'s are positive, we can define \( u \) by a similar integral but this time integrating from \(-\infty \) to \( 0 \) instead of \( 0 \) to \( \infty \).

When the \( \lambda_j \)'s have different signs, we can assume that

\[ \lambda_1 \leq \cdots \leq \lambda_p < 0 < \lambda_{p+1} \leq \cdots \leq \lambda_n . \]

Set \( n = p + q, \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q \) and write \( x \in \mathbb{R}^n \) as \( x = (x', x'') \) with \( x' \in \mathbb{R}^p \) and \( x'' \in \mathbb{R}^q \). Since the given function \( f(t, x', x'') \) is flat along the torus \( \{ x' = 0, x'' = 0 \} \), then it can be written (Lemma 5.1) as

\[ f(t, x', x'') = g(t, x', x'') + h(t, x', x'') \]

with \( g \) flat along \( x' = 0 \) and \( h \) flat along \( x'' = 0 \). As in the previous case, the functions

(7.9) \[ v(t, x', x'') = -\int_0^\infty g(\Phi_s(t, x)) ds \quad \text{and} \quad w(t, x', x'') = \int_0^\infty h(\Phi_s(t, x)) ds \]

satisfy the equations \( M_{\omega, \lambda} v = g \) and \( M_{\omega, \lambda} w = h \). Therefore \( u = v + w \) is the sought-for flat solution of (7.1).

Proof of Theorem 7.1. Let \( f \in C^\infty(\mathbb{T}^m \times B(r)) \) and let \( \hat{f} \) be its \( x \)-Taylor expansion:

(7.10) \[ \hat{f}(t, x) = \sum_{J \in \mathbb{N}^n} \frac{1}{J!} \frac{\partial^{|J|} f}{\partial x^J} (t, 0) x^J . \]

The equation \( M_{\omega, \lambda} \hat{u} = \hat{f} \) has a formal series solution (Lemma 7.1):

(7.11) \[ \hat{u}(t, x) = \sum_{J \in \mathbb{N}^n} u_J(t) x^J, \quad u_J(t) \in C^\infty(\mathbb{T}^m) . \]

We use the Whitney Extension Theorem [18] to find a function \( v(t, x) \in C^\infty(\mathbb{T}^m \times B(r)) \) such that

(7.12) \[ \frac{\partial^{|J|} v}{\partial x^J} (t, 0) = J! u_J(t), \quad \forall J \in \mathbb{N}^n . \]

The function

\[ g(t, x) = f(t, x) - M_{\omega, \lambda} v(t, x) \]

is therefore flat along \( \mathbb{T}_{0}^m \) and, consequently (Lemma 7.2) there exists a \( C^\infty \) function \( w \) flat along \( \mathbb{T}_{0}^m \) such that \( M_{\omega, \lambda} w = g \). Hence, \( M_{\omega, \lambda} (v + w) = f \). \( \square \)

Remark 7.1. The vector field \( M_{\omega, \lambda} \) is not hypoelliptic in a neighborhood of \( \mathbb{T}_{0}^m \). For instance the function

\[ u(t, x) = e^{i(t_1 - (\omega_1/\lambda_1) \ln |x_1|)} \]

is only a bounded solution of \( M_{\omega, \lambda} u = 0 \).
8. Solvability of the model vector field $L_{\omega,\lambda}$

We consider the solvability, in a neighborhood of $\mathbb{T}_0^m$, of the equation
\begin{equation}
L_{\omega,\lambda}u(t, x) = f(t, x),
\end{equation}
where $L_{\omega,\lambda} = \omega \partial_t + iAx \partial_x$ is the model vector field of Section 6. We will assume throughout that the function $f$ satisfies the necessary condition
\begin{equation}
\int_{\mathbb{T}^m} f(t, 0) = 0.
\end{equation}
In the real analytic category we have the following theorem.

**Theorem 8.1.** Suppose that $(\omega, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n$ satisfies condition $(S_C)$ (see Section 6) and $f \in C^{\infty}(\mathbb{T}^m \times B(r))$ satisfies (8.2). Then equation (8.1) has a unique solution $u \in C^{\infty}(\mathbb{T}^m \times B(r_1))$ for some $0 < r_1 < r$.

**Proof.** Consider the Fourier-Taylor series expansion of $f$:
\[ f(t, x) = \sum_{J \in \mathbb{N}^n \, K \in \mathbb{Z}^m} f_{JK} e^{i(K,t)} x^J, \]
with $f_{00} = 0$. Then the series
\[ u(t, x) = \sum_{J \in \mathbb{N}^n \, K \in \mathbb{Z}^m} \frac{f_{JK}}{i(\langle \omega, K \rangle + \langle \lambda, J \rangle)} e^{i(K,t)} x^J \]
is the sought-for real analytic solution. The uniform convergence of the series defining $u$ follows from condition $(S_C)$ by using arguments similar to those used in the proof of Theorem 7.1.

The solvability of equation (8.1) is more complicated when $f$ is only a $C^\infty$ function. In general there is no $C^\infty$ solution $u$ (Theorem 8.4) but only solutions of class $C^q$ for any $q \in \mathbb{N}$. More precisely, we have the following theorem.

**Theorem 8.2.** Suppose that $(\omega, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n$ satisfies condition $(S_C)$ and $f \in C^{\infty}(\mathbb{T}^m \times B(r))$ satisfies (8.2). Then, for any $l \in \mathbb{N}$, equation (8.1) has a solution $u \in C^l(\mathbb{T}^m \times B(r_1))$ for some $0 < r_1 < r$.

This theorem is a consequence of Theorem 8.3 (below). To state the theorem we need to introduce the following definitions.

Let $a = (a_1, \cdots, a_k)$ and $b = (b_1, \cdots, b_{k'})$ with $a_j, b_l \in \{\lambda_1, \cdots, \lambda_n\}$ for $j = 1, \cdots, k$ and $l = 1, \cdots, k'$. For $J \in \mathbb{N}^{k'}$ and $K \in \mathbb{Z}^m$, and $a, b$ as above let
\begin{equation}
\Lambda(a) = \text{diag}(a), \quad \tau_b(J, K) = \langle b, J \rangle + \langle \omega, K \rangle.
\end{equation}
Define the operator $P_{k,a,\tau_b(J,K)}$ on $\mathbb{T}^m \times \mathbb{R}^k$ by
\begin{equation}
P_{k,a,\tau_b(J,K)} := \omega \partial_t + i\Lambda(a)x \partial_x + i\tau_b(J, K).
\end{equation}
We will say that an operator $P$ defined on $\mathbb{T}^m \times \mathbb{R}^n$ is $\nu$-solvable if for every $\nu \in \mathbb{N}$, there exists $q \in \mathbb{N}$ such that whenever $f \in C^q(\mathbb{T}^m \times B(r))$ (and satisfies a compatibility condition), the equation $Pu = f$ has a solution $u \in C^\nu(\mathbb{T}^m \times B(r_1))$ for some $0 < r_1 < r$.

**Theorem 8.3.** For every $k \in \mathbb{N}, a, b, J, K$, as above, the operator $P_{k,a,\tau_b(J,K)}$ is $\nu$-solvable.
The proof of this theorem is given in Sections 9 and 10. It is clear that Theorem
8.2 is a particular case since \( L_{\omega, \lambda} = P_{n, \lambda, \gamma_0(0,0)} \).

The next theorem states that, in general, equation (8.1) does not have \( C^\infty \)
solutions.

**Theorem 8.4.** There exists \( f \in C^\infty(\mathbb{T}^m \times B(r)) \) satisfying (8.2) for which equation
(8.1) does not have a \( C^\infty \) solution in any neighborhood of the torus \( \mathbb{T}^m_0 \).

**Proof.** The idea of the proof is contained in [8] and [9], where planar vector fields
are treated. Let \( \{c_j\} \) be a sequence of points in \( \mathbb{C} \) such that the series \( \sum c_j z^j \) has a
radius of convergence equal to 0. Let \( M(z) \in C^\infty(\mathbb{C}) \) be such that its Taylor series
at \( z = 0 \) coincides with the series \( \sum c_j z^j \). Thus,

\[
\frac{\partial^j M}{\partial z^j}(0) = j! c_j \quad \text{and} \quad \frac{\partial^{j+k} M}{\partial z^j \partial \bar{z}^k}(0) = 0, \quad \forall j, k \in \mathbb{N}, k \geq 1.
\]

Note that the function \( \frac{\partial M}{\partial \bar{z}} \) is flat at \( 0 \in \mathbb{C} \). Consider the vector field \( L_1 \) on \( \mathbb{T}^1 \times \mathbb{R} \)
given by

\[
L_1 = \omega_1 \frac{\partial}{\partial t_1} + i \lambda_1 x_1 \frac{\partial}{\partial x_1}.
\]

Suppose that \( \omega_1 \lambda_1 > 0 \). Then the function

\[
z(t_1, x_1) = |x_1|^{\omega_1 / \lambda_1} e^{-it_1}
\]

is a first integral of \( L_1 \) on \( \mathbb{T}^1 \times \mathbb{R}^n \). (If \( \omega_1 \lambda_1 < 0 \), replace the function \( z \) by \( 1/z \)).

Define the function \( f(t, x) \) in \( \mathbb{T}^m \times \mathbb{R}^n \) by

\[
f(t_1, \cdots , t_m, x_1, \cdots , x_n) = 2i \omega_1 \bar{z}(t_1, x_1) \frac{\partial M}{\partial \bar{z}}(z(t_1, x_1)).
\]

Note that \( f \) is independent on the variables \( t' = (t_2, \cdots , t_m) \), \( x' = (x_2, \cdots , x_n) \),
and \( f \in C^\infty(\mathbb{T}^m \times \mathbb{R}^n) \) is flat along \( x_1 = 0 \). Now we show that for such a function \( f \) equation (8.1) does not have a \( C^\infty \) solution.

By contradiction, suppose that \( u(t_1, t', x_1, x') \) is a \( C^\infty \) solution in a neighborhood
of \( \mathbb{T}^m_0 \). Then its restriction to \( x' = 0 \) satisfies

\[
\left( \omega \frac{\partial}{\partial t_1} + \lambda_1 x_1 \frac{\partial}{\partial x_1} \right) u(t_1, t', x_1, 0) = f(t_1, x_1).
\]

It follows, after using Fourier series in the \( t' \)-variables, that the \( C^\infty \) function

\[
v(t_1, x_1) = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^{m-1}} u(t_1, t', x_1, 0) dt'
\]

satisfies the equation

\[
L_1 v(t_1, x_1) = f(t_1, x_1)
\]
in a neighborhood of \( \mathbb{T}^1 \) in \( \mathbb{T} \times \mathbb{R} \). Now, the pushforward of equation (8.11) in the
region \( x_1 > 0 \) is the CR equation

\[
2i \omega_1 \bar{z} \frac{\partial \bar{v}}{\partial \bar{z}} = 2i \omega_1 \bar{z} \frac{\partial M}{\partial \bar{z}}
\]
in \( \mathbb{C} \setminus \{0\} \). Hence, the general solution of (8.12) near \( 0 \in \mathbb{C} \) is

\[
\bar{v}(z) = M(z) + H(z),
\]
with $H$ holomorphic in a neighborhood of $0 \in \mathbb{C}$. We can assume that $\bar{v}(0) = 0$. Since the Taylor series of $M$ is divergent and that of $H$ is convergent, then there exist $j \geq 1$ such that $d_j = c_j + (H^{(j)}(0)/j!) \neq 0$. Let $j_0 \geq 1$ be the smallest such $j$. Then

\begin{equation}
(8.14) \quad \bar{v}(z) = d_{j_0} z^{j_0} + O(|z|^{j_0+1}).
\end{equation}

Therefore,

\begin{equation}
(8.15) \quad v(t_1, x_1) = \bar{v}(z(t_1, x_1)) = d_{j_0} x_1^j e^{i j_0 t_1} + o(x_1^{j_0/d_{j_0}/\lambda_1}).
\end{equation}

Since the pair $(\omega, \lambda)$ satisfies $(S_C)$, then $\omega_1 j_0 \lambda_1 = \not \in \mathbb{N}$ and $v(t_1, x_1)$, given in (8.15), cannot be of class $C^\infty$. This is a contradiction. \hfill \Box

9. Some Lemmas

The following lemmas will be used in the next section to prove Theorem 8.3. To alleviate the notation, we will simply write $\tau$ for the real number $\tau_0(J, K)$ introduced in Section 8. We will assume throughout that $(\omega, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n$ satisfy $(S_C)$. For a function $f \in C^l(B(r))$ and $k \leq l$, $r_1 < r$, $||f||_{k, r_1}$ will denote the norm

$$||f||_{k, r_1} = \sup_{|J|\leq k} \sup_{|x|\leq r_1} |\frac{\partial^{|J|} f}{\partial x^J}(x)|.$$

**Lemma 9.1.** Let $a = \lambda_j$ for some $j = 1, \cdots, n$. Then, for every $l \in \mathbb{N}$, $N \in \mathbb{N}$, and $f \in C^l(B(r))$, where $B(r) = (-r, r) \subset \mathbb{R}$, the equation

\begin{equation}
(9.1) \quad \left(ax \frac{d}{dx} + \tau\right) u(x) = x^N f(x)
\end{equation}

has a solution $u \in C^l(B(r_1))$ with $0 < r_1 < r$ satisfying the following property. For every $k \leq l$ there exists $C_k > 0$ such that

\begin{equation}
(9.2) \quad \left|\frac{d^k u}{dx^k}(x)\right| \leq C_k ||f||_{k-1, r_1} |x|^{N-k}(N + |J| + |K|)^{k+1}
\end{equation}

(when $k = 0$, the norm $||f||_{k-1, r_1}$ is replaced by $||f||_{0, r_1}$).

**Proof.** We can assume $a > 0$. Let $r_1 < r$. For $0 < x < r_1$, we can define a solution $u$ of (9.1) by

$$u(x) = \frac{1}{a} x^{-\tau/a} v(x)$$

where

$$v(x) = \begin{cases} 
\frac{1}{\Gamma(N+\tau/a)} \int_0^x s^{N+(\tau/a)-1} f(s) \, ds & \text{if } aN + \tau > 0; \\
-\frac{1}{\Gamma(N+\tau/a)} \int_x^{r_1} s^{N+(\tau/a)-1} f(s) \, ds & \text{if } aN + \tau < 0.
\end{cases}
$$

We have then

$$|v(x)| \leq \frac{a}{|aN + \tau|} ||f||_{0, r_1} x^{N+\tau/a}.$$

Consequently,

$$|u(x)| \leq \frac{a}{|aN + \tau|} ||f||_{0, r_1} x^N.$$
Moreover, since \((\omega, \lambda)\) satisfies \((S_C)\), we have
\[
|aN + \tau| = |aN + \langle \lambda, J \rangle + \langle \omega, K \rangle| \geq \frac{C}{(N + |J| + |K|)^{\mu}},
\]
and estimate (9.2) follows for \(k = 0\).

Now, we verify (9.2) by induction on \(k\). Suppose that (9.2) holds for \(k = 0, \ldots, m - 1\) with \(m - 1 < l\). We have from (9.1) that
\[
a u'(x) = x^{N-1} - \tau x^{-1} u(x).
\]

By using the Leibniz formula we get
\[
au^{(m)}(x) = \sum_{s=0}^{m-1} \binom{m-1}{s} \left( \frac{(N-1)!}{(N+s-m)!} f^{(s)}(x) x^{N+s-m} + (-1)^{m-s} \tau u^{(s)}(x) x^{-m+s} \right).
\]

Therefore, there exists \(C_m > 0\) such that
\[
|au^{(m)}(x)| \leq C_m \sum_{s=0}^{m-1} \left( ||f||_{s,r} x^{N-(m-s)} + ||\tau||_{s} x^{-(m-s)} \right).
\]

Since
\[
|\tau| = |\tau_b(J, K)| \leq \max(|\omega|, |\lambda|) (|J| + |K|),
\]
the induction hypothesis gives
\[
|au^{(m)}(x)| \leq C_m \sum_{s=0}^{m-1} \left( ||f||_{s,r} x^{s} + C'' ||f||_{s-1,r} (N + |K| + |J|)^{\mu+s} \right) x^{N-m},
\]
and the estimates (9.2) follows for \(k = m\). A similar argument can be used when \(x < 0\).

Next we will develop estimates of the derivatives in spherical coordinates. Consider the spherical coordinates \((\rho, \theta_1, \ldots, \theta_{n-1})\) in \(\mathbb{R}^n\). That is, if \(x \in \mathbb{R}^n\), then
\[
\begin{aligned}
x_1 &= \rho \cos \theta_1, \\
x_2 &= \rho \sin \theta_1 \cos \theta_2, \\
& \vdots \\
x_{n-1} &= \rho \sin \theta_1 \cdots \sin \theta_{n-2} \cos \theta_{n-1}, \\
x_n &= \rho \sin \theta_1 \cdots \sin \theta_{n-2} \sin \theta_{n-1}.
\end{aligned}
\]

Clearly,
\[
(9.3) \quad \rho^2 = x_1^2 + \cdots + x_n^2 \quad \text{and} \quad \cot \theta_j = \frac{x_j}{\sqrt{x_1^2 + \cdots + x_{j+1}^2}} \quad \text{for} \ j = 1, \ldots, n-1.
\]

**Lemma 9.2.** For every \(m \in \mathbb{N}\) there exist \(C > 0\) and \(l \in \mathbb{N}\) such that for every \(J \in \mathbb{N}^n\) with \(|J| \leq m\), we have
\[
(9.4) \quad \left| \frac{\partial^{|J|}\rho}{\partial x^J} \right| \leq \frac{C}{\rho^{|J|-1}} \quad \text{and} \quad \left| \frac{\partial^{|J|}\theta_j}{\partial x^J} \right| \leq \frac{C}{|x_1 \cdots x_n|} \quad \text{for} \ j = 1, \ldots, n-1.
\]

**Proof.** It follows from the expression of \(\rho\) in (9.3) that
\[
\frac{\partial^{|J|}\rho}{\partial x^J} = \frac{P_J(x)}{\rho^{|J|-1}}
\]
where \( P_J(x) \) is a homogeneous polynomial in \( x \) of degree \(|J|\) and the estimate for the derivatives of \( \rho \) follow. For \( \theta_j \), we have

\[
\begin{align*}
\frac{\partial \theta_j}{\partial x_k} &= 0 & \text{if } k < j, \\
\frac{\partial \theta_j}{\partial x_j} &= -\sqrt{\frac{x_n^2 + \cdots + x_{j+1}^2}{x_n^2 + \cdots + x_j^2}}, \\
\frac{\partial \theta_j}{\partial x_k} &= \frac{x_j x_k}{(x_n^2 + \cdots + x_{j+1}^2)^{1/2}(x_n^2 + \cdots + x_j^2)^{1/2}} & \text{if } k > j.
\end{align*}
\]

The estimates for \(|J| = 1\) are clear where we take \( l = 1 \). The estimates when \(|J| > 1\) follow by induction from (9.5).

**Lemma 9.3.** Let \( a = (a_1, \ldots, a_k) \) with \( a_j \in \{\lambda_1, \ldots, \lambda_n\} \) be such that all the real numbers \( a_j \) have the same sign. Let \( \Lambda(a) \) and \( \tau = \tau_b(J,K) \) be as in (8.3). Then for every \( \nu \in \mathbb{N} \), there exists \( N \in \mathbb{N} \) and \( q \in \mathbb{N} \) such that for every \( f \in C^\nu(B(r)) \), equation

\[
(\Lambda(a)x \partial_x + \tau) u = (x_1 \cdots x_k)^{2N} f(x)
\]

has a solution \( u \in C^\nu(B(r_1)) \) (for some 0 < \( r_1 < r \)). Moreover, \( u = O((x_1 \cdots x_k)^N) \), and there exists \( C \nu > 0 \) such that

\[
|\frac{\partial^L u(x)}{\partial x^L}| \leq C \nu ||f||_{L^1,B(r)} (N + |K| + |J|)^{2N+\mu}
\]

for every \( L \in \mathbb{N}^k \) with \(|L| \leq \nu \) and \( x \in B(r_1) \).

**Proof.** We can assume that all the \( a_j \)'s are positive. Let \( \epsilon = (\epsilon_1, \ldots, \epsilon_k) \) with each \( \epsilon_j = 1 \) or \(-1\). Define \( \mathbb{R}_\epsilon^k \) as the region of \( \mathbb{R}^k \) that contains all points \( x \in \mathbb{R}^k \) with \( \epsilon_jx_j > 0 \) for \( j = 1, \ldots, k \). \( \mathbb{R}_\epsilon^k \) is the connected component of \( \mathbb{R}^k \setminus \{x_1 \cdots x_k = 0\} \) that contains the point \( \epsilon \). Set \( B_\epsilon(r) = B(r) \cap \mathbb{R}_\epsilon^k \). Consider the change of coordinates in \( \mathbb{R}_\epsilon^k \) given by

\[
y_j = (\epsilon_j x_j)^{1/a_j}, \quad j = 1, \ldots, k.
\]

With respect to these coordinates, equation (9.6) becomes

\[
\left( \sum_{j=1}^k y_j \frac{\partial}{\partial y_j} + \tau \right) v(y) = \prod_{j=1}^k y_j^{a_j N} g(y)
\]

with \( g(y) = \prod_{j=1}^k y_j^{a_j} f(\epsilon_1 y_1^{a_1}, \ldots, \epsilon_k y_k^{a_k}) \). Now we use the spherical coordinates to solve (9.9).

Let \( y = \rho \Psi(\theta) \), with \( \Psi(\theta) \in \mathbb{S}^{k-1} \), be the spherical coordinates as in (9.3). Equation (9.9) transforms into the equation

\[
\left( \frac{\partial}{\partial \rho} + \tau \right) w(\rho, \theta) = \rho^{AN} \prod_{j=1}^k \psi_j(\theta)^{a_j N} \tilde{g}(\rho, \theta)
\]
with \( A = a_1 + \cdots + a_k \). Note for any given \( q \in \mathbb{N} \), and for \( N \) large enough, that the function \( \tilde{g}(\rho, \theta) \) is of class \( C^q \). In this equation, we view \( \theta \) as a parameter. It follows (Lemma 9.1) that we can find a solution of the form

\[
(9.11) \quad w(\rho, \theta) = \rho^{AN} \prod_{j=1}^{k} \psi_j(\theta)^{a_j} \tilde{w}(\rho, \theta)
\]

that satisfies estimates (9.2). Going back to the original \( x \)-coordinates, this means that we have constructed a solution \( u_\varepsilon(x) = w(\rho(x), \theta(x)) \) of equation (9.6) in \( B_\varepsilon(r_1) \). Moreover, it follows from estimates (9.2), satisfied by \( w \), and from Lemma 9.2 that \( u_\varepsilon \) satisfies estimates (9.7) provided that \( N \) is large enough. Since \( u_\varepsilon \) vanishes to order larger than \( N \) on each hyperplane \( x_j = 0 \), then the functions \( u_\varepsilon \) glue together to produce the sought-for solution \( u \) in the full ball \( B(r_1) \).

**Lemma 9.4.** Let \( a = (a_1, \ldots, a_k) \) with \( a_j \in \{\lambda_1, \ldots, \lambda_n\} \). Let \( \Lambda(a) \) and \( \tau = \tau_0(J,K) \) be as in (8.3). Then for every \( \nu \in \mathbb{N} \), there exists \( N \in \mathbb{N} \) and \( q \in \mathbb{N} \) such that for every \( f \in C^q(B(r)) \), equation (9.6) has a solution \( u \in C^\nu(B(r_1)) \) (for some \( 0 < r_1 < \tau \)) such that \( u = O((x_1 \cdots x_k)^N) \) satisfies (9.7).

**Proof.** We can assume that \( a_j > 0 \) for \( j = 1, \ldots, k_1 \) with \( k_1 < k \) and \( a_j < 0 \) for \( j = k_1 + 1, \ldots, k \), and that \( k_2 = k - k_1 \leq k_1 \). Set \( \mathbb{R}^k = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \) and write \( x \in \mathbb{R}^k \) as \( x = (y, z) \) with \( y \in \mathbb{R}^{k_1} \) and \( z \in \mathbb{R}^{k_2} \). Let \( b_j = -a_{k_1+j} \) for \( j = 1, \ldots, k_2 \).

Equation (9.6) becomes

\[
(9.12) \quad \left( \sum_{j=1}^{k_1} a_j y_j \frac{\partial}{\partial y_j} - \sum_{j=1}^{k_2} b_j z_j \frac{\partial}{\partial z_j} + \tau \right) u = (y_1 \cdots y_{k_1} z_1 \cdots z_{k_2})^{2N} f(y, z).
\]

As in the proof of Lemma 9.3, we start by considering equation (9.12) in wedges of \( \mathbb{R}^k \). Let \( B_{\varepsilon_i}(r) = B(r) \cap \mathbb{R}^{k_i}_\varepsilon \) (\( i = 1, 2 \)), where \( \mathbb{R}^{k_i}_\varepsilon \) is the connected component of \( \mathbb{R}^{k_i} \setminus \{x_1 \cdots x_{k_i} = 0\} \) that contains the point \( \varepsilon_i \) with \( \varepsilon_i^j = 1 \) or \(-1\).

Consider the change of coordinates in \( \mathbb{R}^{k_1}_\varepsilon \times \mathbb{R}^{k_2}_\varepsilon \) given by

\[
\tilde{y}_j = (\varepsilon_1^j y_j)^{1/a_j}, \quad \text{for} \quad j = 1, \ldots, k_1; \quad \tilde{z}_j = (\varepsilon_2^j z_j)^{1/b_j}, \quad \text{for} \quad j = 1, \ldots, k_2.
\]

With respect to these coordinates equation (9.12) becomes

\[
(9.13) \quad \left( \sum_{j=1}^{k_1} \tilde{y}_j \frac{\partial}{\partial \tilde{y}_j} - \sum_{j=1}^{k_2} \tilde{z}_j \frac{\partial}{\partial \tilde{z}_j} + \tau \right) v(\tilde{y}, \tilde{z}) = \left( \tilde{y}_1^{a_1} \cdots \tilde{y}_{k_1}^{a_{k_1}} \tilde{z}_1^{b_1} \cdots \tilde{z}_{k_2}^{b_{k_2}} \right)^{2N} g(\tilde{y}, \tilde{z}).
\]

The use of the spherical coordinates \( \tilde{y} = \rho_1 \Phi(\theta_1) \) and \( \tilde{z} = \rho_2 \Psi(\theta_2) \) in \( \mathbb{R}^{k_1} \) and \( \mathbb{R}^{k_2} \) respectively leads to the equation

\[
(9.14) \quad \left( \rho_1 \frac{\partial}{\partial \rho_1} - \rho_2 \frac{\partial}{\partial \rho_2} + \tau \right) w = \rho_1^{AN} \rho_2^{BN} \left( \Phi(\theta_1)^a \Psi(\theta_2)^b \right)^N h(\rho_1, \rho_2, \theta_1, \theta_2)
\]

where \( A = a_1 + \cdots + a_{k_1}, \quad B = b_1 + \cdots + b_{k_2}, \quad \Phi(\theta_1)^a = \Phi_1(\theta_1)^{a_1} \cdots \Phi_{k_1}(\theta_1)^{a_{k_1}}, \quad \Psi(\theta_2)^b = \Psi_1(\theta_2)^{b_1} \cdots \Psi_{k_2}(\theta_2)^{b_{k_2}}, \)

and where the function

\[
h(\rho_1, \rho_2, \theta_1, \theta_2) = \rho_1^{AN} \rho_2^{BN} \left( \Phi(\theta_1)^a \Psi(\theta_2)^b \right)^N g(\rho, \theta)
\]
can be made up of class $C^q$ for any given $q$ provided that $N$ is large enough. We can solve equation (9.14) in the region $0 < \rho_1 < r_1$, $0 < \rho_2 < r_1$ by using the new variables

$$ s = \ln \rho_1 \quad \sigma = \rho_1 \rho_2 \quad \text{with} \quad s < \ln r_1 < 0 \quad 0 \leq \sigma < r_1 e^s. $$

With respect to the $(s, \sigma)$-variables, equation (9.14) transforms into

$$ (\frac{\partial}{\partial s} + \tau) T = \sigma^{BN} e^{(A-B)Ns} \Phi(\theta^1)^a \Phi(\theta^2)^b \tilde{h}(s, \sigma, \theta) $$

with a solution given by

$$ T(s, \sigma, \theta) = \begin{cases} 
 e^{-\tau s} \int_{-\infty}^{s} e^{\xi(\tau + (A-B)N)} \tilde{h}(\xi, \sigma, \theta) d\xi & \text{if} \quad \tau + (A-B)N > 0; \\
 e^{-\tau s} \int_{s}^{\ln r_1} e^{\xi(\tau + (A-B)N)} \tilde{h}(\xi, \sigma, \theta) d\xi & \text{if} \quad \tau + (A-B)N < 0.
\end{cases} $$

It follows that

$$ |T(s, \sigma, \theta)| \leq \frac{e^{s(A-B)N}}{\tau + (A-B)N} ||\tilde{h}||_0 \leq e^{s(A-B)N} ||\tilde{h}||_0 (N + |J| + |K|)^{\mu}. $$

Similar estimates hold for the derivatives of $T$ up to any preassigned order $\nu$, provided that $N$ is large enough.

Going back to the original $x$-variables, we have constructed a solution $u_{e_{1,e}1}(x)$ of equation (9.6) in $B_{e_1}(r_1) \times B_{e_2}(r_1)$. Estimates (9.7) follow from estimates for the function $T$ and from Lemma 9.2. Finally, since $u_{e_{1,e}2}$ has the form

$$ \rho_1^{AN} \rho_2^{BN} (\Phi(\theta^1)^a \Phi(\theta^2)^b)^N \tilde{u}(\rho, \theta) $$

in the spherical coordinates, then it vanishes to order $N$ on each hyperplane $x_j = 0$ of $\mathbb{R}^k$. This allows us to glue together all the $u_{e_{1,e}2}$ along the hyperplanes to get the desired solution $u$ in $B(r_1)$.

**10. Proof of Theorem 8.3**

Let $a$ and $\tau = \tau_0(J, K)$ be as defined in Section 8. We prove first that the operator $P_{1,a,\tau}$ is $\nu$-solvable. Let $\nu \in \mathbb{N}$ and $f \in C^q(\mathbb{T}^m \times B(r))$ with $B(r)$ the $r$-ball in $\mathbb{R}$. We assume that $f$ satisfies condition (8.2). Let $N \in \mathbb{N}$ with $N < q$ and consider the $x$-Taylor expansion of order $N$:

$$ f(t, x) = \sum_{j=1}^{N} f_j(t) x^j + x^N g(t, x), $$

with each $f_j \in C^{q-j}(\mathbb{T}^m)$ and $g \in C^{q-N}(\mathbb{T}^m \times B(r))$. For each $j = 1, \cdots, N$, the equation

$$ \omega \partial_t v_j(t) + i(ja + \tau)v_j(t) = f_j(t) $$

has a solution $v_j(t) \in C^\nu(\mathbb{T}^m)$. Indeed, the solution $v_j$ is given by

$$ v_j(t) = -i \sum_{K \in \mathbb{Z}^m} \frac{f_jK}{\{\omega, K\} + ja + \tau} e^{i\langle K, t \rangle}, $$
where \( f_{jk} \) is the \( K \)-th Fourier coefficient of \( f_j \), and since \((\omega, \lambda)\) satisfies \((\mathcal{S}_\lambda)\), then \( v_j \in C^\nu(T^m) \) provided that \( q > \nu + N + \mu \). We have then
\[
P_{1,a,\tau} (v_j(t)x^j) = f_j(t)x^j.
\]

Consequently,
\[
f(t, x) = x^Ng(t, x) + P_{1,a,\tau} \left( \sum_{j=0}^{N} v_j(t)x^j \right).
\]

Now we solve equation \( P_{1,a,\tau} v = x^Ng \) by using Fourier series:
\[
x^Ng(t, x) = \sum_{L \in \mathbb{Z}^m} x^N g_L(x)e^{i(L,t)}, \quad v(t, x) = \sum_{L \in \mathbb{Z}^m} v_L(x)e^{i(L,t)}.
\]

Each \( v_L(x) \) needs to satisfy the equation
\[
i \left( ax \frac{\partial}{\partial x} + \tau' \right) v_L(x) = x^Ng_L(x),
\]
where \( \tau' = \tau + (\omega, K) \). Equation (10.4) has a solution (Lemma 9.1) \( v_L \) that satisfies estimates (9.2):
\[
\left| v_L^{(l)}(x) \right| \leq C_l ||g_L||_{l-1,r_1} |x|^{N-l}(|L| + |K| + |J|)^{l+\mu}.
\]

Since, \( g \in C^{q-N}(T^m \times B(r)) \), then \( ||g_L||_{l-1,r_1} \) decays faster than \( |L|^{-(q-N-l)} \) as \( |L| \to \infty \). It follows that the series \( \sum v_L(x)e^{i(L,t)} \) defines a \( C^\nu \) function in \( T^m \times B(r_1) \). Therefore,
\[
f(t, x) = P_{1,a,\tau} \left( \sum_{j=0}^{N} v_j(t)x^j + \sum_{L \in \mathbb{Z}^m} v_L(x)e^{i(L,t)} \right),
\]
and the proposition is proved for \( k = 1 \).

We continue the proof by induction on the number \( k \) of the \( x \)-variables. Suppose that for every \( k \leq l - 1 \) (with \( l \leq n \)), the operator \( P_{k,a,\tau} \) is \( \nu \)-solvable for any parameters \( a \) and \( \tau \). Let \( f \in C^\nu(T^m \times B(r)) \) satisfy (8.2), with \( B(r) \) the \( r \)-ball in \( \mathbb{R}^l \). We write \( x \in \mathbb{R}^l \) as \( x = (x', x_l) \) with \( x' = (x_1, \ldots, x_{l-1}) \in \mathbb{R}^{l-1} \) and \( x_l \in \mathbb{R} \).

For \( N < q \), we use the \( x_l \)-Taylor expansion of \( f \) of order \( N \):
\[
f(t, x', x_l) = \sum_{j=1}^{N} f_j(t, x')x_l^j + x_l^Ng(t, x', x_l).
\]

Since solving the equation
\[
P_{l,a,\tau} u_j(t, x', x_l) = f_j(t, x')x_l^j
\]
can be reduced, via \( u_j(t, x', x_l) = v_j(t, x')x_l^j \), to solving the equation
\[
P_{l-1,a',\tau'} v_j(t, x') = f_j(t, x')
\]
with \( a' = (a_1, \ldots, a_{l-1}) \) and \( \tau' = \tau + ja_l \), then it follows from the induction hypothesis that equation (10.6) has a solution \( u_j \in C^\nu(T^m \times B(r_1)) \). Hence,
\[
f(t, x', x_l) = x_l^Ng(t, x', x_l) + P_{l,a,\tau} \left( \sum_{j=1}^{N} u_j(t, x', x_l) \right).
\]
Now consider the equation
\begin{equation}
(10.9) \quad P_{l,a,\tau} u(t, x', x_l) = x_l^N g(t, x', x_l).
\end{equation}
Set \( u(t, x', x_l) = v(t, x', x_l)x_l^N \). The equation for \( v \) is
\begin{equation}
(10.10) \quad (P_{l,a,\tau} + iNa_l) v(t, x', x_l) = g(t, x', x_l).
\end{equation}
We have
\begin{equation}
(10.11) \quad P_{l,a,\tau} + iNa_l = P_{l,a,\tau'} \quad \text{with} \quad \tau' = \tau + Na_l.
\end{equation}
We write \( x' \in \mathbb{R}^{l-1} \) as \( x' = (x'', x_{l-1}) \) with \( x'' \in \mathbb{R}^{l-2} \) and \( x_{l-1} \in \mathbb{R} \). Use the \( x_{l-1} \)-Taylor expansion of order \( N \) for \( g \):
\begin{equation}
(10.12) \quad g(t, x'', x_{l-1}, x_l) = \sum_{j=1}^{N} g_j(t, x'', x_l)x_{l-1}^j + x_{l-1}^N h(t, x'', x_{l-1}, x_l).
\end{equation}
We can therefore find
\begin{equation}
(10.13) \quad w^N(t, x'', x_{l-1}, x_l) = \sum_{j=1}^{N} w_j(t, x'', x_l)x_{l-1}^j \in C^\nu(\mathbb{T}^m \times B(r_1))
\end{equation}
such that
\begin{equation}
(10.14) \quad g(t, x'', x_{l-1}, x_l) = P_{l,a,\tau'} w^N + x_{l-1}^N h(t, x'', x_{l-1}, x_l)
\end{equation}
or equivalently
\begin{equation}
(10.15) \quad x_l^N g(t, x'', x_{l-1}, x_l) = P_{l,a,\tau'} (x_l^N w^N) + (x_l x_{l-1})^N h(t, x'', x_{l-1}, x_l).
\end{equation}
It is clear that this process can be continued until all the \( x_j \) variables are exhausted. Hence, there exists \( v(t, x) \in C^\nu(\mathbb{T}^m \times B(r_1)) \) such that
\begin{equation}
(10.16) \quad f(t, x) = (x_1, \cdots x_l)^N m(t, x) + P_{l,a,\tau} v(t, x).
\end{equation}
Finally, consider the equation
\begin{equation}
(10.17) \quad P_{l,a,\tau} w(t, x) = (x_1, \cdots, x_l)^N m(t, x).
\end{equation}
The use of Fourier series
\begin{equation}
(10.18) \quad m(t, x) = \sum_{K \in \mathbb{Z}^m} m_K(x)e^{i(K,t)}, \quad v(t, x) = \sum_{K \in \mathbb{Z}^m} v_K(x)e^{i(K,t)}
\end{equation}
leads to the following equation for the \( K \)-th coefficient \( v_K(x) \):
\begin{equation}
(10.19) \quad (\Lambda(a)x \partial_x + \tau') v_K(x) = -i(x_1, \cdots, x_l)^Nm_K(x),
\end{equation}
where \( \tau' = \tau + \langle \omega, K \rangle \). It follows from Lemma 9.4 that equation (10.19) has a solution \( v_K \) satisfying estimates (9.7). Moreover, if \( N < q \) is large enough, the series for \( v \) in (10.18) is of class \( C^\nu \) and satisfies equation (10.17). Hence \( f = P_{l,a,\tau}(w + v) \), and the proof of Theorem 8.3 is complete.
REFERENCES


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