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VARIATIONAL EQUALITIES OF ENTROPY IN NONUNIFORMLY HYPERBOLIC SYSTEMS

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ABSTRACT. In this paper we prove that for a nonuniformly hyperbolic system $(f, \widetilde{\Lambda})$ and for every nonempty, compact and connected subset K with the same hyperbolic rate in the space $\mathcal{M}_{inv}(\widetilde{\Lambda}, f)$ of invariant measures on $\widetilde{\Lambda}$, the metric entropy and the topological entropy of basin G_K are related by the variational equality

$$\inf\{h_\mu(f) \mid \mu \in K\} = h_{\mathrm{top}}(f, G_K).$$

In particular, for every invariant (usually nonergodic) measure $\mu \in \mathcal{M}_{inv}(\widetilde{\Lambda}, f)$, we have

$$h_\mu(f) = h_{\mathrm{top}}(f, G_\mu).$$

We also verify that $\mathcal{M}_{inv}(\widetilde{\Lambda}, f)$ contains an open domain in the space of ergodic measures for diffeomorphisms with some hyperbolicity. As an application, the historical behavior is shown to occur robustly with a full positive entropy for diffeomorphisms beyond uniform hyperbolicity.

1. INTRODUCTION

It is well-known that the evolution of dynamical systems is very sensitive to initial data [27, 41]. So the statistical qualitative method plays an important role in the study of dynamics. In this context, the most common quantities are metric entropy and topological entropy, related by the Variational Principle stating that the supremum of metric entropy is explicitly the topological entropy. In this paper we establish a new variational relation for nonuniformly hyperbolic systems which is not revealed by the classical Variational Principle.

Let (M, d) be a compact metric space and $f : M \to M$ be a continuous map. Given an invariant subset $\Gamma \subset M$, denote by $\mathcal{M}(\Gamma)$ the set consisting of all Borel probability measures on Γ, by $\mathcal{M}_{inv}(\Gamma, f) \subset \mathcal{M}(\Gamma)$ the subset consisting of f-invariant probability measures, and by $\mathcal{M}_{erg}(\Gamma, f) \subset \mathcal{M}_{inv}(\Gamma, f)$ the subset consisting of f-invariant ergodic probability measures. Clearly, if Γ is compact, then $\mathcal{M}(\Gamma)$ and $\mathcal{M}_{inv}(\Gamma, f)$ are both compact spaces in the weak*-topology of measures. Given $x \in M$, define the n-ordered empirical measure of x by

$$\mathcal{E}_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)},$$

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where δ_y is the Dirac mass at $y \in M$. The limit point set $V(x)$ of $\{\mathcal{E}_n(x)\}$ is always a compact connected subset of $\mathcal{M}_{inv}(M, f)$. Given a compact connected subset $K \subset \mathcal{M}_{inv}(M, f)$, we denote by G_K its basin consisting of points x with $V(x) = K$. In particular, for $\mu \in \mathcal{M}_{inv}(M, f)$, the basin of μ is defined as $G_\mu = G_{\{\mu\}}$. By the Birkhoff Ergodic Theorem, $\mu(G_\mu) = 1$ when μ is ergodic. However, this is somewhat a special case. For nonergodic μ, by the Ergodic Decomposition Theorem, G_μ has measure 0 and thus is "thin" with respect to measure. In addition, when f is uniformly hyperbolic or nonuniformly hyperbolic, G_μ is of first category ([13, 24, 39]), hence also "thin" with respect to topology.

In the physically observable sense (especially for the dissipative case), an invariant measure μ is called a Sinai-Ruelle-Bowen (SRB) measure if the basin G_μ admits positive volume. The measures of "thick" volume possess good dynamical characterization: the metric entropy $h_\mu(f)$ coincides with the sum of positive exponents iff μ admits absolutely continuous conditional measures along the unstable manifolds; see Pesin [33] and Ledrappier, Young [22].

If μ is ergodic, the result of Bowen [7] states that

$$h_{\text{top}}(f, G_\mu) = h_\mu(f). \tag{1}$$

Observe that G_μ is not compact in general, and that the topological entropy $h_{\text{top}}(f, G_\mu)$ adapts the definition given in [7], which we quote in Section 3. When f is mixing and uniformly hyperbolic (which implies the uniform specification property), applying [34] leads also to

$$h_{\text{top}}(f, G_\mu) = h_\mu(f), \quad \forall\, \mu \in \mathcal{M}_{inv}(M, f).$$

This implies that G_μ is "thick" with respect to topological entropy. In the setting of nonuniform hyperbolicity, the information of invariant measure can be shown to be well approximated by nearby measures [21, 23, 44], and Liang, Sun and Tian [24] further established that $G_\mu \neq \emptyset$. In the present paper we will show the "thickness" of G_μ with respect to entropy.

Let M be a compact connected boundary-less Riemannian manifold and $f : M \to M$ a $C^{1+\alpha}$ diffeomorphism. We use Df_x to denote the tangent map of f at $x \in M$. We say that $x \in M$ is a Lyapunov regular point of f if there exist numbers $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{\phi(x)}(x)$ and a decomposition on the tangent space

$$T_x M = E_1(x) \oplus \cdots \oplus E_{\phi(x)}(x)$$

such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|(D_x f^n)u\| = \lambda_j(x)$$

for every $0 \neq u \in E_j(x)$ and every $1 \leq j \leq \phi(x)$. The numbers $\lambda_i(x)$ and the spaces $E_i(x)$ are called the Lyapunov exponents and the eigenspaces of f at the point x, respectively. By Oseledets theorem [31], all Lyapunov regular points of f form a Borel set with total measure. For a regular point $x \in M$ we define

$$\lambda^+(x) = \min\{\lambda_i(x) \mid \lambda_i(x) > 0,\, 1 \leq i \leq \phi(x)\}$$

and

$$\lambda^-(x) = \min\{-\lambda_i(x) \mid \lambda_i(x) < 0,\, 1 \leq i \leq \phi(x)\},$$

where we set $\min \emptyset = \max \emptyset = 0$. Taking an ergodic invariant measure μ, by the ergodicity for μ-almost all $x \in M$ we can obtain uniform exponents $\lambda_i(x) = \lambda_i(\mu)$ for $1 \leq i \leq \phi(\mu)$. In this case define $\lambda^+(\mu) = \lambda^+(x)$ and $\lambda^-(\mu) = \lambda^-(x)$. We say an ergodic measure μ is hyperbolic if $\lambda^+(\mu)$ and $\lambda^-(\mu)$ are both nonzero.

Given $\beta_1, \beta_2 \gg \epsilon > 0$ and $k \in \mathbb{N}$, the hyperbolic block $\Lambda_k = \Lambda_k(\beta_1, \beta_2; \epsilon)$ with index k consists of all points $x \in M$ for which there is a splitting $T_{\mathrm{orb}(x)}M = E^s \oplus E^u$ over the orbit of x with the invariance property $Df^t(E^s_x) = E^s_{f^t x}$ and $Df^t(E^u_x) = E^u_{f^t x}$, and satisfying:

- $\|Df^n|E^s_{f^t x}\| \le e^{\epsilon k} e^{-(\beta_1-\epsilon)n} e^{\epsilon|t|}, \quad \forall\, t \in \mathbb{Z}, n \ge 1;$
- $\|Df^{-n}|E^u_{f^t x}\| \le e^{\epsilon k} e^{-(\beta_2-\epsilon)n} e^{\epsilon|t|}, \quad \forall\, t \in \mathbb{Z}, n \ge 1;$
- $\tan\angle(E^s_{f^t x}, E^u_{f^t x}) \ge e^{-\epsilon k} e^{-\epsilon|t|}, \quad \forall\, t \in \mathbb{Z}.$

Definition. $\Lambda(\beta_1, \beta_2; \epsilon) = \bigcup_{k=1}^{+\infty} \Lambda_k(\beta_1, \beta_2; \epsilon)$ is a nonuniformly hyperbolic set (NUH set for short).

Note that $\Lambda(\beta_1, \beta_2; \epsilon)$ is an f-invariant set (but usually not compact) and every $x \in \Lambda(\beta_1, \beta_2; \epsilon)$ satisfies the following pointwise hyperbolicity:

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|Df^n_x \mid_{E^s_x}\| \quad \le \quad -\beta_1 + \epsilon < 0,$$

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|Df^{-n}_x \mid_{E^u_x}\| \quad \le \quad -\beta_2 + \epsilon < 0.$$

On the other hand, if an ergodic hyperbolic measure ω admits $\lambda^-(\omega) \ge \beta_1$ and $\lambda^+(\omega) \ge \beta_2$, then $\omega \in \mathcal{M}_{inv}(\Lambda(\beta_1, \beta_2; \epsilon), f)$ (see [2]). From now on we fix such a measure ω and denote by $\omega \mid_{\Lambda_l}$ the conditional measure of ω on Λ_l. Set $\widetilde{\Lambda}_l = \mathrm{supp}(\omega \mid_{\Lambda_l})$ and $\widetilde{\Lambda} = \bigcup_{l\ge1} \widetilde{\Lambda}_l$. Clearly, $f^{\pm}(\widetilde{\Lambda}_l) \subset \widetilde{\Lambda}_{l+1}$, and the sub-bundles E^s_x, E^u_x depend continuously on $x \in \widetilde{\Lambda}_l$. Moreover, $\widetilde{\Lambda}$ is f-invariant with ω-full measure.[1] Intuitively those sets $\widetilde{\Lambda}$ are considered as hyperbolic "cells" which build up the hyperbolic part of diffeomorphisms.

Let $\{\eta_l\}_{l=1}^{\infty}$ be a decreasing sequence which approaches zero. As in [30] we say that a probability measure $\mu \in \mathcal{M}_{inv}(M, f)$ has *hyperbolic rate* $\{\eta_l\}$ with respect to the NUH set $\Lambda = \bigcup_{l\ge1} \Lambda_l$ if $\mu(\Lambda_l) \ge 1 - \eta_l$ for all $l \ge 1$.

Theorem 1.1. *Let $\eta = \{\eta_l\}$ be a sequence decreasing to zero and $\mathcal{M}(\widetilde{\Lambda}, \eta) \subset \mathcal{M}_{inv}(\widetilde{\Lambda}, f)$ the set of measures with hyperbolic rate η. Given any nonempty compact connected set $K \subset \mathcal{M}(\widetilde{\Lambda}, \eta)$, we have*

$$\inf\{h_\mu(f) \mid \mu \in K\} = h_{\mathrm{top}}(f, G_K).$$

For finitely many measures $\mu_1, \cdots, \mu_n \in \mathcal{M}_{inv}(\widetilde{\Lambda}, f)$, if we take a polyhedron

$$K \quad = \quad \{a_1\mu_1 + \cdots + a_n\mu_n \mid a_1 + \cdots + a_n = 1,\, a_i \ge 0,\, 1 \le i \le n\},$$

then K admits the hyperbolic rate $\{1 - \min_{1\le i\le n} \mu_i(\Lambda_l)\}_{l=1}^{\infty}$. In particular, letting $\mu \in \mathcal{M}_{inv}(\widetilde{\Lambda}, f)$ and $K = \{\mu\}$ in Theorem 1.1, we obtain the following.

Corollary 1.2. *For every $\mu \in \mathcal{M}_{inv}(\widetilde{\Lambda}, f)$, we have*

$$h_\mu(f) = h_{\mathrm{top}}(f, G_\mu).$$

[1]Here $\widetilde{\Lambda}$ is obtained by taking the support for each hyperbolic block Λ_l, so even for an ergodic measure with Lyapunov exponents away from $[-\beta_1, \beta_2]$, it is not necessary to have positive measure for $\widetilde{\Lambda}$. We give more details on $\widetilde{\Lambda}$ in section 5.

Observe that $\mathcal{M}_{inv}(M, f^{-1}) = \mathcal{M}_{inv}(M, f)$ for any homeomorphism f. Given $K \subset \mathcal{M}_{inv}(M, f)$, we denote G_μ^{-1} to be the basin of K with respect to f^{-1}. For each pair $(K_1, K_2) \in \mathcal{M}_{inv}(M, f) \times \mathcal{M}_{inv}(M, f)$, define $H(K_1, K_2) = G_{K_1} \cap G_{K_2}^{-1}$. If $K_1 = K_2$, we call $H(K_1, K_1)$ the homoclinic set of K_1; if $K_1 \neq K_2$, we call $H(K_1, K_2)$ the heteroclinic set of (K_1, K_2).

Theorem 1.3. *Let $\eta = \{\eta_l\}$ be a sequence decreasing to zero and $\mathcal{M}(\widetilde{\Lambda}, \eta) \subset \mathcal{M}_{inv}(\widetilde{\Lambda}, f)$ the set of measures with hyperbolic rate η. Given any nonempty compact connected sets $K_1, K_2 \subset \mathcal{M}(\widetilde{\Lambda}, \eta)$, we have*

- $h_{\text{top}}(f, H(K_1, K_2)) = \inf\{h_\mu(f) \mid \mu \in K_1\}$;
- $h_{\text{top}}(f^{-1}, H(K_1, K_2)) = \inf\{h_\mu(f) \mid \mu \in K_2\}$.

Letting $K_1 = \{\mu_1\}, K_2 = \{\mu_2\}$, by Theorem 1.3 we obtain the following.

Corollary 1.4. *Given two measures $\mu_1, \mu_2 \in \mathcal{M}_{inv}(\widetilde{\Lambda}, f)$, we have*

- $h_{\text{top}}(f, H(\mu_1, \mu_2)) = h_{\mu_1}(f)$;
- $h_{\text{top}}(f^{-1}, H(\mu_1, \mu_2)) = h_{\mu_2}(f)$.

Remark 1.5. When Γ is a compact invariant subset of M, by Proposition 1 of [7] it holds that $h_{\text{top}}(f, \Gamma) = h_{\text{top}}(f^{-1}, \Gamma)$. However, Theorem 1.3 and Corollary 1.4 show that for a noncompact invariant subset Γ, it may have $h_{\text{top}}(f, \Gamma) \neq h_{\text{top}}(f^{-1}, \Gamma)$.

Contrary to the notion of generic points, a point $x \in M$ is said to have historical behavior if there is a continuous function $\varphi \in C^0(M)$ satisfying

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) < \limsup_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)).$$

This terminology was introduced by Ruelle [38]. If the limit does not exist, the "partial averages" of $1/n \sum_{i=0}^{n-1} \varphi(f^i(x))$ keep giving new information when n increases. There are physical systems which are believed to have historical behavior [29] and Ruelle [38], and Takens [42] conjectured that historical behavior can occur persistently in the space of diffeomorphisms.

Denote by $\widehat{M}$ the set of all points with historical behavior. By the Variational Principle (together with (1)),

$$h_{\text{top}}(f, M \setminus \widehat{M}) = \sup_{\mu \in \mathcal{M}_{inv}(M,f)} h_\mu(f) = h_{\text{top}}(f).$$

Here we show that the topological entropy of $\widehat{M}$ can be positive and even large, possibly with full topological entropy in a robust manner beyond uniform hyperbolicity.

Theorem 1.6. *Let $M = \mathbb{T}^n$ $(n \geq 3)$ and $\mathcal{H}$ the set of Axiom A systems. There is a C^1 open subset $\mathcal{U}$ of $\text{Diff}^1(M) \setminus \overline{\mathcal{H}}$ such that for any $f \in \mathcal{U}$, $\mathcal{M}_{erg}(\widetilde{\Lambda}, f)$ contains a nonempty open set of $\mathcal{M}_{erg}(M, f)$ and*

$$h_{\text{top}}(f, \widehat{M}) = h_{\text{top}}(f) > 0.$$

2. Dynamics of nonuniformly hyperbolic systems

We start with some preliminary concepts and propositions of the nonuniformly hyperbolic theory [2, 20, 35].

2.1. **Lyapunov metric.** Assume $\Lambda(\beta_1, \beta_2; \epsilon) = \bigcup_{k\geq 1} \Lambda_k(\beta_1, \beta_2; \epsilon)$ is a nonempty Pesin set. Let $\beta_1' = \beta_1 - 2\epsilon$, $\beta_2' = \beta_2 - 2\epsilon$. Note that $\epsilon \ll \beta_1, \beta_2$; then $\beta_1' > 0, \beta_2' > 0$. For $x \in \Lambda(\beta_1, \beta_2; \epsilon)$, we define

$$\begin{aligned} \|v_s\|_s &= \sum_{n=1}^{+\infty} e^{-\beta_1' n} \|D_x f^n(v_s)\|, \ \forall\, v_s \in E_x^s, \\ \|v_u\|_u &= \sum_{n=1}^{+\infty} e^{-\beta_2' n} \|D_x f^{-n}(v_u)\|, \ \forall\, v_u \in E_x^u, \\ \|v\|' &= \max(\|v_s\|_s, \|v_u\|_u) \text{ where } v = v_s + v_u. \end{aligned}$$

We call the norm $\|\cdot\|'$ a Lyapunov metric. With the Lyapunov metric, $f : \Lambda \to \Lambda$ is uniformly hyperbolic:

$$\|Df\mid_{E_x^s}\|' \leq e^{-\beta_1'}, \ \|Df^{-1}\mid_{E_x^u}\|' \leq e^{-\beta_2'}.$$

This metric is in general not equivalent to the Riemannian metric. The following estimates are known:

$$\frac{1}{\sqrt{2}}\|v\|_x \leq \|v\|_x' \leq \frac{2}{1-e^{-\epsilon}} e^{\epsilon k}\|v\|_x, \ \forall\, v \in T_xM, \ x \in \Lambda_k.$$

In the local coordinate chart, a coordinate change $C_\varepsilon : M \to GL(m, \mathbb{R})$ is called a Lyapunov change of coordinates if for each regular point $x \in M$ and $u, v \in T_xM$, they satisfy

$$\langle u, v\rangle_x = \langle C_\varepsilon u, C_\varepsilon u\rangle_x'.$$

By any Lyapunov change of coordinates C_ϵ sends the orthogonal decomposition $\mathbb{R}^{\dim E^s} \oplus \mathbb{R}^{\dim E^u}$ to the decomposition $E_x^s \oplus E_x^u$ of T_xM. Additionally, denote $A_\epsilon(x) = C_\epsilon(f(x))^{-1} Df_x C_\epsilon(x)$. Then

$$A_\epsilon(x) = \begin{pmatrix} A_\epsilon^s(x) & 0 \\ 0 & A_\epsilon^u(x) \end{pmatrix},$$

$$\|A_\epsilon^s(x)\| \leq e^{-\beta_1'}, \quad \|A_\epsilon^u(x)^{-1}\| \leq e^{-\beta_2'}.$$

2.2. **Lyapunov neighborhood.** Fix a point $x \in \Lambda(\beta_1, \beta_2; \epsilon)$. By taking charts about $x, f(x)$ we can assume without loss of generality that $x \in \mathbb{R}^{\dim M}, f(x) \in \mathbb{R}^{\dim M}$. For a sufficiently small neighborhood U of x, we can trivialize the tangent bundle over U by identifying $T_UM \equiv U \times \mathbb{R}^{\dim M}$. For any point $y \in U$ and tangent vector $v \in T_yM$, we can then use the identification $T_UM = U \times \mathbb{R}^{\dim M}$ to translate the vector v to a corresponding vector $\bar{v} \in T_xM$. We then define $\|v\|_y'' = \|\bar{v}\|_x'$, where $\|\cdot\|'$ indicates the Lyapunov metric. This defines a new norm $\|\cdot\|''$ (which agrees with $\|\cdot\|'$ on the fiber T_xM). Similarly, we can define $\|\cdot\|_z''$ on T_zM (for any z in a sufficiently small neighborhood of fx or $f^{-1}x$). We write $\bar{v}$ as v whenever there is no confusion. We can define a new splitting $T_yM = E_y^{s\prime} \oplus E_y^{u\prime}, y \in U$, by translating the splitting $T_xM = E_x^s \oplus E_x^u$ (and similarly for $T_zM = E_z^{s\prime} \oplus E_z^{u\prime}$).

There exist $\beta_1'' = \beta_1 - 3\epsilon > 0, \beta_2'' = \beta_2 - 3\epsilon > 0$ and $\epsilon_0 > 0$ such that if we set $\epsilon_k = \epsilon_0 e^{-\epsilon' k}$, then for any $y \in B(x, \epsilon_k)$ in an ϵ_k neighborhood of $x \in \Lambda_k$, we have a

splitting $T_yM = E^{s\prime}_y \oplus E^{u\prime}_y$ with hyperbolic behavior:

(i) $\|D_yf(v)\|''_{fy} \le e^{-\beta''_1}\|v\|''$ for every $v \in E^{s\prime}_y$;

(ii) $\|D_yf^{-1}(w)\|''_{f^{-1}y} \le e^{-\beta''_2}\|w\|''$ for every $w \in E^{u\prime}_y$.

The constants ϵ_0, ϵ' here and afterwards depend on various global properties of f, e.g., the Hölder constants, the size of the local trivialization; see p. 73 in [35].

Definition 2.1. Let $\Psi_x = \exp_x \circ C_\epsilon(x)$. We define the Lyapunov neighborhood $\Pi = \Pi(x, a\epsilon_k)$ of $x \in \Lambda_k$ (with size $a\epsilon_k$, $0 < a < 1$) to be the Ψ_x-image of the tangent rectangle $(-a\epsilon_k,\, a\epsilon_k)\mathbb{R}^{\dim E^s} \oplus (-a\epsilon_k,\, a\epsilon_k)\mathbb{R}^{\dim E^u}$.

In the Lyapunov neighborhoods, Df appears uniformly hyperbolic in the Lyapunov metric. More precisely, one can extend the definition of $C_\epsilon(x)$ to the Lyapunov neighborhood $\Pi(x, a\epsilon_k)$ such that for any $y \in \Pi(x, a\epsilon_k)$,

$$A_\epsilon(y) := C_\epsilon(f(y))^{-1} Df_y C_\epsilon(y) = \begin{pmatrix} A^s_\epsilon(y) & 0 \\ 0 & A^u_\epsilon(y) \end{pmatrix},$$

$$\|A^s_\epsilon(y)\| \le e^{-\beta''_1}, \quad \|A^u_\epsilon(y)^{-1}\| \le e^{-\beta''_2}.$$

Given $x \in \Lambda_k$, we say that the set $H^u \subset \Pi(x, a\epsilon_k)$ is an admissible (u, γ_0, k)-manifold near x if $H^u = \Psi_x(\text{graph } \psi)$ for some γ_0-Lipschitz function ψ from $(-a\epsilon_k,\, a\epsilon_k)\mathbb{R}^{\dim E^u}$ to $(-a\epsilon_k,\, a\epsilon_k)\mathbb{R}^{\dim E^s}$ with $\|\psi\| \le a\epsilon_k/4$. Similarly, we can also define an admissible (s, γ_0, k)-manifold near x. Through each point $y \in \Pi(x, a\epsilon_k)$, we can take (u, γ_0, k)-admissible manifold $H^u(y) \subset \Pi(x, a\epsilon_k)$ and (s, γ_0, k)-admissible manifold $H^s(y) \subset \Pi(x, a\epsilon_k)$. Fixing γ_0 small enough, we can assume that

(i) $\|D_zf(v)\|''_{fz} \le e^{-\beta''_1+\epsilon}\|v\|''$ for every $v \in T_zH^s(y), z \in H^s(y)$;

(ii) $\|D_zf^{-1}(w)\|''_{f^{-1}z} \le e^{-\beta''_2+\epsilon}\|w\|''$ for every $w \in T_zH^u(y), z \in H^u(y)$.

For any regular point $x \in \Lambda$, define $\kappa(x) = \min\{i \in \mathbb{Z} \mid x \in \Lambda_i\}$. Using the local hyperbolicity above, we can see that each connected component of $f(H^u(y)) \cap \Pi(fx, a\epsilon_{\kappa(fx)})$ is an admissible $(u, \gamma_0, \kappa(fx))$-manifold; each connected component of $f^{-1}(H^s(y)) \cap \Pi(f^{-1}x, a\epsilon_{\kappa(f^{-1}x)})$ is an admissible $(s, \gamma_0, \kappa(f^{-1}x))$-manifold.

2.3. **Shadowing lemma.** In this subsection, we state a shadowing property for $C^{1+\alpha}$ nonuniformly hyperbolic systems, which is needed in the sequel.

Let $(\delta_k)_{k=1}^\infty$ be a sequence of positive real numbers. Let $(x_n)_{n=-\infty}^\infty$ be a sequence of points in $\Lambda = \Lambda(\beta_1, \beta_2; \epsilon)$ for which there exists a sequence $(s_n)_{n=-\infty}^{+\infty}$ of positive integers satisfying:

(a) $x_n \in \Lambda_{s_n}$, $\forall n \in \mathbb{Z}$;

(b) $|s_n - s_{n-1}| \le 1, \forall\, n \in \mathbb{Z}$;

(c) $d(f(x_n), x_{n+1}) \le \delta_{s_n}$, $\forall\, n \in \mathbb{Z}$.

Then we call $(x_n)_{n=-\infty}^{+\infty}$ a $(\delta_k)_{k=1}^\infty$ pseudo-orbit. Given $c > 0$, a point $x \in M$ is a c-shadowing point for the $(\delta_k)_{k=1}^\infty$ pseudo-orbit $(x_n)_{n=-\infty}^{+\infty}$ if $d(f^n(x), x_n) \le c\epsilon_{s_n}$, $\forall\, n \in \mathbb{Z}$.

Lemma 2.2 ([18,20]). *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism, with a nonempty NUH set $\Lambda = \Lambda(\beta_1, \beta_2; \epsilon)$ and fixed parameters, $\beta_1, \beta_2 \gg \epsilon > 0$. For $c > 0$ there exists a sequence $(\delta_k)_{k=1}^{\infty}$ such that for any $(\delta_k)_{k=1}^{\infty}$ pseudo-orbit there exists a unique c-shadowing point.*

3. Entropy theory

3.1. **Entropy for noncompact sets.** In our settings the basins are usually noncompact. [7] gave a definition of topological entropy for noncompact spaces. We state the definition in a slightly different way, but they are in fact equivalent by Dai-Jiang [12] when the whole space M is compact. Given $x \in M$, $n \in \mathbb{N}$, $\varepsilon > 0$, define the n-step dynamical ball

$$B_n(x, \varepsilon) = \{y \in M \mid d(f^i x, f^i y) < \varepsilon, \quad \text{for all } \ i \in [0, n-1]\}.$$

Let $E \subset M$ and $\mathcal{C}_n(E, \varepsilon)$ be the set of all finite or countable covers of E by the sets of form $B_m(x, \varepsilon)$ with $m \geq n$. Denote $\mathcal{Y}(E; t, n, \varepsilon) = \inf\{\sum_{B_m(x,\varepsilon)\in A} e^{-tm} \mid A \in \mathcal{C}_n(E, \varepsilon)\}$ and $\mathcal{Y}(E; t, \varepsilon) = \lim_{n\to\infty} \mathcal{Y}(E; t, n, \varepsilon)$. Define

$$h_{\text{top}}(E; \varepsilon) = \inf\{t \mid \mathcal{Y}(E; t, \varepsilon) = 0\} = \sup\{t \mid \mathcal{Y}(E; t, \varepsilon) = \infty\},$$

and the topological entropy of E is $h_{\text{top}}(E, f) = \lim_{\varepsilon\to 0} h_{\text{top}}(E; \varepsilon)$.

The following formula from [34, Theorem 4.1(3)] is a subcase of Bowen's variational principle and it remains true in the general topological setting.

Proposition 3.1. *Let $K \subset \mathcal{M}_{inv}(M, f)$ be nonempty, compact and connected. Then*

$$h_{\text{top}}(f, G_K) \leq \inf\{h_\mu(f) \mid \mu \in K\}. \tag{2}$$

By the above proposition, to prove Theorem 1.1, it suffices to show the following theorem.

Theorem 3.2. *Let $\eta = \{\eta_n\}$ be a sequence decreasing to zero and $\mathcal{M}(\widetilde{\Lambda}, \eta) \subset \mathcal{M}_{inv}(\widetilde{\Lambda}, f)$ be the set of measures with hyperbolic rate η. Given any nonempty compact and connected set $K \subset \mathcal{M}(\widetilde{\Lambda}, \eta)$, we have*

$$h_{\text{top}}(f, G_K) \geq \inf\{h_\mu(f) \mid \mu \in K\}. \tag{3}$$

Remark 3.3. In [34], C. E. Pfister and W. G. Sullivan proved (3) with the almost product property plus uniform separation. Note that if the almost product property holds for $\widetilde{\Lambda}$, so does the closure of $\widetilde{\Lambda}$ by taking suitable functions in the definition. In section 5, for some robustly transitive nonpartially hyperbolic diffeomorphisms we will construct $\widetilde{\Lambda}$ whose closure coincides with the whole manifold M. However, in the absence of hyperbolicity outside Λ, we do not have shadowing and the almost product property for the whole space M. Moreover, the function $m : \mathbb{R}^+ \to \mathbb{N}$ in the definition of the almost product property is independent of $x \in M$, but in NUH set Λ the hyperbolic index $\kappa(x)$ depends on x.

3.2. **Entropy structure.** We start this section by giving some elements from the theory of entropy structures developed by Boyle-Downarowicz [8]. Let $\mathcal{H} = (h_k)$ and $\mathcal{H}' = (h'_k)$ be two increasing sequences of functions on a compact domain $\mathcal{D}$.

We say $\mathcal{H}'$ *uniformly dominates* $\mathcal{H}$, denoted by $\mathcal{H}' \geq \mathcal{H}$, if for every index k and every $\gamma > 0$ there exists an index k' such that

$$h'_{k'} \geq h_k - \gamma.$$

We say $\mathcal{H}$ and $\mathcal{H}'$ are *uniformly equivalent* if both $\mathcal{H} \geq \mathcal{H}'$ and $\mathcal{H}' \geq \mathcal{H}$. Obviously, uniform equivalence is an equivalence relation. An increasing sequence $\alpha_1 \leq \alpha_2 \leq \cdots$ of partitions of M is called *essential* (for f) if

$$\begin{aligned} &(1)\ \operatorname{diam}(\alpha_k) \to 0 \text{ as } k \to +\infty, \\ &(2)\ \mu(\partial\alpha_k) = 0 \text{ for every } \ \mu \in \mathcal{M}_{inv}(M, f). \end{aligned}$$

Here $\partial\alpha_k$ denotes the union of the boundaries of elements in the partition α_k. Note that essential sequences of partitions may not exist (e.g., for the identity map on the unit interval). However, for any finite entropy system (f, M) it follows from the work of Lindenstrauss and Weiss [25], [26] that the product $f \times R$ with R an irrational rotation has essential sequences of partitions. Noting that the rotation does not contribute entropy for every invariant measure, we can always assume (f, M) has an essential sequence. By an *entropy structure* of a finite topological entropy dynamical system (f, M) we mean an increasing sequence $\mathcal{H} = (h_k)$ of functions defined on $\mathcal{M}_{inv}(M, f)$ which is uniformly equivalent to $(h_\mu(f, \alpha_k) \mid_{\mu \in \mathcal{M}_{inv}(M,f)})$.

For each ergodic measure ν, we use Katok's definition of metric entropy (see [21]). For $\varepsilon, \delta > 0$, let $N_n(\varepsilon, \delta)$ be the minimal number of ε-dynamical balls $B_n(x, \varepsilon)$ whose union covers a set of ν-measure of at least $1 - \delta$. We define

$$h_\nu^{Kat}(f, \varepsilon \mid \delta) = \limsup_{n \to \infty} \frac{\log N_n(\varepsilon, \delta)}{n}.$$

It follows by Theorem 1.1 of [21] that

$$h_\nu(f) = \lim_{\varepsilon \to 0} h_\nu^{Kat}(f, \varepsilon \mid \delta).$$

Recall that $\mathcal{M}_{erg}(M, f)$ denotes the set of all ergodic f-invariant measures supported on M. Given invariant measure μ, assume $\mu = \int_{\mathcal{M}_{erg}(M,f)} d\tau(\nu)$ is the ergodic decomposition of μ. Then by the Jacobs Theorem

$$h_\mu(f) = \int_{\mathcal{M}_{erg}(M,f)} h_\nu(f) d\tau(\nu).$$

Define

$$h_\mu^{Kat}(f, \varepsilon \mid \delta) \triangleq \int_{\mathcal{M}_{erg}(M,f)} h_\nu^{Kat}(f, \varepsilon \mid \delta) d\tau(\nu).$$

By the Monotone Convergence Theorem, we have

$$h_\mu(f) = \int_{\mathcal{M}_{erg}(M,f)} \lim_{\varepsilon \to 0} h_\nu^{Kat}(f, \varepsilon \mid \delta) d\tau(\nu) = \lim_{\varepsilon \to 0} h_\mu^{Kat}(f, \varepsilon \mid \delta).$$

Next we recall the notion of entropy introduced by Newhouse [30]. Given $\mu \in \mathcal{M}_{inv}(M,f)$, let $F \subset M$ be a measurable set. Define

(1) $H(n,\rho \mid x,F,\varepsilon) = \log\max\{\sharp E \mid E \text{ is an } (n,\rho)\text{-separated set in } F \cap B_n(x,\varepsilon)\}$;

(2) $H(n,\rho \mid F,\varepsilon) = \sup_{x\in F} H(n,\rho \mid x,F,\varepsilon)$;

(3) $h(\rho \mid F,\varepsilon) = \limsup_{n\to+\infty} \frac{1}{n} H(n,\rho \mid F,\varepsilon)$;

(4) $h(F,\varepsilon) = \lim_{\rho\to 0} h(\rho \mid F,\varepsilon)$;

(5) $h^{New}_{\mathrm{loc}}(\mu,\varepsilon) = \liminf_{\sigma\to 1}\{h(F,\varepsilon) \mid \mu(F) > \sigma\}$;

(6) $h^{New}(\mu,\varepsilon) = h_\mu(f) - h^{New}_{\mathrm{loc}}(\mu,\varepsilon)$.

Let $\{\theta_k\}_{k=1}^\infty$ be a decreasing sequence which approaches zero. One can verify that $(h^{New}(\mu,\theta_k) \mid_{\mu\in\mathcal{M}_{inv}(M,f)})_{k=1}^\infty$ is in fact an increasing sequence of functions defined on $\mathcal{M}_{inv}(M,f)$. Furthermore,

$$\lim_{\theta_k\to 0} h^{New}(\mu,\theta_k) = h_\mu(f) \quad \text{for any } \mu \in \mathcal{M}(f).$$

Combining with Katok's definition of entropy, we consider an increasing sequence of functions on $\mathcal{M}_{inv}(M,f)$ given by $(h^{Kat}_\mu(f,\theta_k \mid \delta) \mid_{\mu\in\mathcal{M}_{inv}(M,f)})$.

Theorem 3.4. *Both $(h^{Kat}_\mu(f,\theta_k \mid \delta) \mid_{\mu\in\mathcal{M}_{inv}(M,f)})$ and $(h^{New}(\mu,\theta_k) \mid_{\mu\in\mathcal{M}_{inv}(M,f)})$ are entropy structures; hence they are uniformly equivalent.*

Proof. This theorem is a part of Theorem 7.0.1 in [14]. □

Let $\eta = \{\eta_n\}_{n=1}^\infty$ be a sequence decreasing to zero. $\mathcal{M}(\widetilde{\Lambda},\eta)$ is the subset of $\mathcal{M}_{inv}(M,f)$ with respect to the hyperbolic rate η.

For $\delta,\varepsilon > 0$ and any $\Upsilon \subset \mathcal{M}(M)$, define

$$h^{Kat}_{\Upsilon,\mathrm{loc}}(f,\varepsilon \mid \delta) = \max_{\mu\in\Upsilon}\{h_\mu(f) - h^{Kat}_\mu(f,\varepsilon \mid \delta)\}.$$

Lemma 3.5. $\lim_{\theta_k\to 0} h^{Kat}_{\mathcal{M}(\widetilde{\Lambda},\eta),\mathrm{loc}}(f,\theta_k \mid \delta) = 0$.

Proof. The lemma is an adaptation of a formula contained in [30, p. 226], which reads as

$$\lim_{\varepsilon\to 0} \sup_{\mu\in\mathcal{M}(\widetilde{\Lambda},\eta)} h^{New}_{\mathrm{loc}}(\mu,\varepsilon) = 0. \tag{4}$$

By Theorem 3.4, (4) is equivalent to $\lim_{\theta_k\to 0} h^{Kat}_{\mathcal{M}(\widetilde{\Lambda},\eta),\mathrm{loc}}(f,\theta_k \mid \delta) = 0$. □

Remark 3.6. In [30], Lemma 3.5 was used to prove upper semi-continuity of metric entropy on $\mathcal{M}(\widetilde{\Lambda},\eta)$. However, the upper semi-continuity is broadly not true even if the underlying system is nonuniformly hyperbolic. For example, T. Downarowicz and S. E. Newhouse [15] established surface diffeomorphisms whose local entropy of arbitrary pre-assigned scale is always larger than a positive constant. Exactly, they constructed a compact subset E of $\mathcal{M}_{inv}(\Lambda,f)$ such that there exist a periodic

measure in E and a positive real number ρ_0 such that for each $\mu \in E$ and each $k > 0$,

$$\limsup_{\nu \in E, \nu \to \mu} h_\nu(f) - h_k(\nu) > \rho_0,$$

which implies infinity of symbolic extension entropy and also the absence of upper semi-continuity of metric entropy, thus no uniform separation in [34].

4. Entropy of basins

In this section, we will verify Theorem 3.2 and thus complete the proof of Theorem 1.1 by Proposition 3.1.

Assume $\{\varphi_i\}_{i=1}^{\infty}$ is the dense subset of $C^0(M)$ giving the weak* topology, that is,

$$D(\mu,\, \nu) = \sum_{i=1}^{\infty} \frac{|\int \varphi_i d\mu - \int \varphi_i d\nu|}{2^{i+1}\|\varphi_i\|}$$

for $\mu, \nu \in \mathcal{M}(M)$. It is easy to check the affine property of D, i.e., for any $\mu,\, m_1,\, m_2 \in \mathcal{M}(M)$ and $0 \le \theta \le 1$,

$$D(\mu,\, \theta m_1 + (1-\theta) m_2) \le \theta D(\mu, m_1) + (1-\theta) D(\mu, m_2).$$

In addition, $D(\mu, \nu) \le 1$ for any $\mu, \nu \in \mathcal{M}(M)$. For any integer $k \ge 1$ and $\varphi_1, \cdots, \varphi_k$, there exists $b_k > 0$ such that

$$d(\varphi_j(x), \varphi_j(y)) < \frac{1}{k}\|\varphi_j\| \text{ for any } d(x,y) < b_k,\ 1 \le j \le k. \tag{5}$$

Now fix $\varepsilon, \delta > 0$.

For any nonempty closed connected set $K \subset \mathcal{M}(\widetilde{\Lambda}, \eta)$, there exists a sequence of closed balls U_n in $\mathcal{M}_{inv}(M, f)$ with radius ζ_n in the metric D with the weak* topology such that the following hold:

$$(i) \quad U_n \cap U_{n+1} \cap K \neq \emptyset;$$

$$(ii) \quad \bigcap_{N \ge 1} \bigcup_{n \ge N}^{\infty} U_n = K;$$

$$(iii) \quad \lim_{n \to +\infty} \zeta_n = 0.$$

By (i), we take $\nu_k \in U_k \cap K$. Given $\gamma > 0$, using Lemma 3.5, we can find an $\varepsilon > 0$ such that

$$h^{Kat}_{\mathcal{M}(\widetilde{\Lambda},\eta),\mathrm{loc}}(f, \varepsilon \mid \delta) < \gamma.$$

For any integer $k \ge 1$, we can take a finite convex combination of ergodic probability measures with rational coefficients

$$\mu_k = \sum_{j=1}^{p_k} a_{k,j}\, m_{k,j}$$

such that

$$D(\nu_k, \mu_k) < \frac{1}{k},\ \ m_{k,j}(\widetilde{\Lambda}) = 1 \ \text{ and } \ |h^{Kat}_{\nu_k}(f, \varepsilon \mid \delta) - h^{Kat}_{\mu_k}(f, \varepsilon \mid \delta)| < \frac{1}{k}. \tag{6}$$

For each k, we can find l_k such that $m_{k,j}(\widetilde{\Lambda}_{l_k}) > 1 - \delta$ for all $1 \le j \le p_k$. Recall that ϵ_{l_k} is the scale of a Lyapunov neighborhood associated with the hyperbolic block Λ_{l_k}. For any $x \in \Lambda_{l_k}$, Df exhibits uniform hyperbolicity in $B(x, \epsilon_{l_k})$. For

$c = \frac{\varepsilon}{8\epsilon_0}$, by Lemma 2.2 there is a sequence of numbers $(\delta_l)_{l=1}^{\infty}$ such that each $(\delta_l)_{l=1}^{\infty}$ pseudo-orbit can be c-shadowed by a real orbit of f. Let ξ_k be a finite partition of M with $\operatorname{diam}\xi_k < \min\{\frac{b_k(1-e^{-\epsilon})}{4\sqrt{2}e^{(k+1)\epsilon}}, \epsilon_{l_k}, \delta_{l_k}\}$ and $\xi_k > \{\widetilde{\Lambda}_{l_k}, M \setminus \widetilde{\Lambda}_{l_k}\}$. Given $t \in \mathbb{N}$, consider the set

$$\begin{aligned}\Lambda^t(m_{k,j}) \quad = \quad & \big\{x \in \widetilde{\Lambda}_{l_k} \mid f^q(x) \in \xi_k(x) \text{ for some } q \in [t, [(1+\frac{1}{k})t] \\ & \text{and } D(\mathcal{E}_n(x), m_{k,j}) < \frac{1}{k} \ \text{ for all } n \geq t\big\},\end{aligned}$$

where $\xi_k(x)$ denotes the element in the partition containing the point x. By ergodicity of $m_{k,j}$, it holds that

$$m_{k,j}(\Lambda^t(m_{k,j})) \to m_{k,j}(\widetilde{\Lambda}_{l_k}) \ \text{ as } t \to +\infty.$$

So, we can take t_k such that

$$m_{k,j}(\Lambda^t(m_{k,j})) > 1 - \delta$$

for all $t \geq t_k$ and $1 \leq j \leq p_k$.

Let $E_t(k,j) \subset \Lambda^t(m_{k,j})$ be a (t,ε)-separated set of maximal cardinality. Then $\Lambda^t(m_{k,j}) \subset \bigcup_{x \in E_t(k,j)} B_t(x,\varepsilon)$, and by the definition of Katok's entropy there exist infinitely many t satisfying

$$\sharp\, E_t(k,j) \geq e^{t(h^{Kat}_{m_{k,j}}(f,\varepsilon|\delta) - \frac{1}{k})}.$$

For each $q \in [t, [(1+\frac{1}{k})t]$, let

$$V_q = \{x \in E_t(k,j) \mid f^q(x) \in \xi_k(x)\}$$

and let $n = n(k,j)$ be the value of q which maximizes $\sharp\, V_q$. Obviously,

$$t \geq \frac{n}{1+\frac{1}{k}} \geq n(1-\frac{1}{k}). \tag{7}$$

Since $e^{\frac{t}{k}} > \frac{t}{k}$, we deduce that

$$\sharp\, V_n \geq \frac{\sharp\, E_t(k,j)}{\frac{t}{k}} \geq e^{t(h^{Kat}_{m_{k,j}}(f,\varepsilon|\delta) - \frac{3}{k})}.$$

Consider the element $A_n(m_{k,j}) \in \xi_k$ for which $\sharp\,(V_n \cap A_n(m_{k,j}))$ is maximal. It follows that

$$\sharp\,(V_n \cap A_n(m_{k,j})) \geq \frac{1}{\sharp\,\xi_k}\sharp\, V_n \geq \frac{1}{\sharp\,\xi_k} e^{t(h^{Kat}_{m_{k,j}}(f,\varepsilon|\delta) - \frac{3}{k})}.$$

Thus taking t large enough so that $e^{\frac{t}{k}} > \sharp\,\xi_k$, we have by inequality (7) that

$$\sharp\,(V_n \cap A_n(m_{k,j})) \geq e^{t(h^{Kat}_{m_{k,j}}(f,\varepsilon|\delta) - \frac{4}{k})} \geq e^{n(1-\frac{1}{k})(h^{Kat}_{m_{k,j}}(f,\varepsilon|\delta) - \frac{4}{k})}. \tag{8}$$

Notice that $A_{n(k,j)}(m_{k,j})$ is contained in an open subset $U(k,j)$ of some Lyapunov neighborhood with $\operatorname{diam}(U(k,j)) < 2\operatorname{diam}(\xi_k)$. By the ergodicity of ω, for any two measures m_{k_1,j_1}, m_{k_2,j_2} we can find $y = y(k_1,j_1;k_2,j_2) \in U(k_1,j_1) \cap \widetilde{\Lambda}_{l_{k_1}}$ satisfying that for some $s = s(m_{k_1,j_1}, m_{k_2,j_2})$ one has

$$f^s(y) \in U(k_2,j_2) \cap \widetilde{\Lambda}_{l_{k_2}}.$$

Observe that $a_{k,j}$ is rational. Letting $C_{k,j} = \frac{a_{k,j}}{n(k,j)}$, we can choose an integer N_k large enough so that $N_k C_{k,j}$ are integers and

$$N_k \geq k \sum_{1\leq r_1,r_2\leq k+1, 1\leq j_i\leq p_{r_i}, i=1,2} s(m_{r_1,j_1}, m_{r_2,j_2}).$$

Arbitrarily take $x(k,j) \in A_n(m_{k,j}) \cap V_{n(k,j)}$. Denote sequences

$$\begin{aligned} X_k &= \sum_{j=1}^{p_k-1} s(m_{k,j}, m_{k,j+1}) + s(m_{k,p_k}, m_{k,1}), \\ Y_k &= \sum_{j=1}^{p_k} N_k n(k,j) C_{k,j} + X_k = N_k + X_k. \end{aligned}$$

So,

$$\frac{N_k}{Y_k} \geq \frac{1}{1+\frac{1}{k}} \geq 1 - \frac{1}{k}. \tag{9}$$

We further choose a strictly increasing sequence $\{T_k\}$ with $T_k \in \mathbb{N}$,

$$Y_{k+1} \leq \frac{1}{k+1} \sum_{r=1}^{k} Y_r T_r, \tag{10}$$

$$\sum_{r=1}^{k} (Y_r T_r + s(m_{r,1}, m_{r+1,1})) \leq \frac{1}{k+1} Y_{k+1} T_{k+1}. \tag{11}$$

In order to obtain shadowing points z with our desired property $\mathcal{E}_n(z) \to \mu$ as $n \to +\infty$, we first construct pseudo-orbits with satisfactory property in the measure theoretic sense. For simplicity, for $x \in M$, define the segments of orbits by

$$\begin{aligned} L_{k,j}(x) &\triangleq (x, f(x), \cdots, f^{n(k,j)-1}(x)), \ 1 \leq j \leq p_k, \\ \widehat{L}_{k_1,j_1;k_2,j_2}(x) &\triangleq (x, f(x) \cdots, f^{s(m_{k_1,j_1}, m_{k_2,j_2})-1}(x)), \ 1 \leq j_i \leq p_{k_i}, i = 1, 2. \end{aligned}$$

Consider now the pseudo-orbit

$$\begin{aligned} &O(\overline{x}(1,1;1,1), \cdots, \overline{x}(1,1;1,N_1C_{1,1}), \cdots, \overline{x}(1,p_1;1,1), \cdots, \overline{x}(1,p_1;1,N_1C_{1,p_1}); \\ &\cdots; \\ &\overline{x}(1,1;T_1,1), \cdots, \overline{x}(1,1;T_1,N_1C_{1,1}), \cdots, \overline{x}(1,p_1;T_1,1), \cdots, \overline{x}(1,p_1;T_1,N_1C_{1,p_1}); \\ &\vdots \\ &\overline{x}(k,1;1,1), \cdots, \overline{x}(k,1;1,N_kC_{k,1}), \cdots, \overline{x}(k,p_k;1,1), \cdots, \overline{x}(k,p_k;1,N_kC_{k,p_k}); \\ &\cdots; \\ &\overline{x}(k,1;T_k,1), \cdots, \overline{x}(k,1;T_k,N_kC_{k,1}), \cdots, \overline{x}(k,p_k;T_k,1), \cdots, \overline{x}(k,p_k;T_k,N_kC_{k,p_k}); \\ &\vdots \\ &\cdots) \end{aligned}$$

where $\overline{x}(k,j;i,t)$ is defined by

$$\begin{cases} L_{k,j}(x(k,j;i,t)), \quad 1 \le i \le T_k, 1 \le t < N_k C_{k,j}, 1 \le j < p_k; \\ L_{k,j}(x(k,j;i,t))\widehat{L}_{k,j,k,j+1}(y(k,j;k,j+1)), \quad 1 \le i \le T_k, t = N_k C_{k,j}, 1 \le j < p_k; \\ L_{k,j}(x(k,j;i,t))\widehat{L}_{k,p_k,k,1}(y(k,p_k;k,1)), \quad 1 \le i < T_k, t = N_k C_{k,j}, j = p_k; \\ L_{k,j}(x(k,j;i,t))\widehat{L}_{k,p_k,k+1,1}(y(k,p_k;k+1,1)), \quad i = T_k, t = N_k C_{k,j}, j = p_k, \end{cases}$$

and $x(k,j;i,t) \in V_{n(k,j)} \cap A_{n(k,j)}(m_{k,j})$.

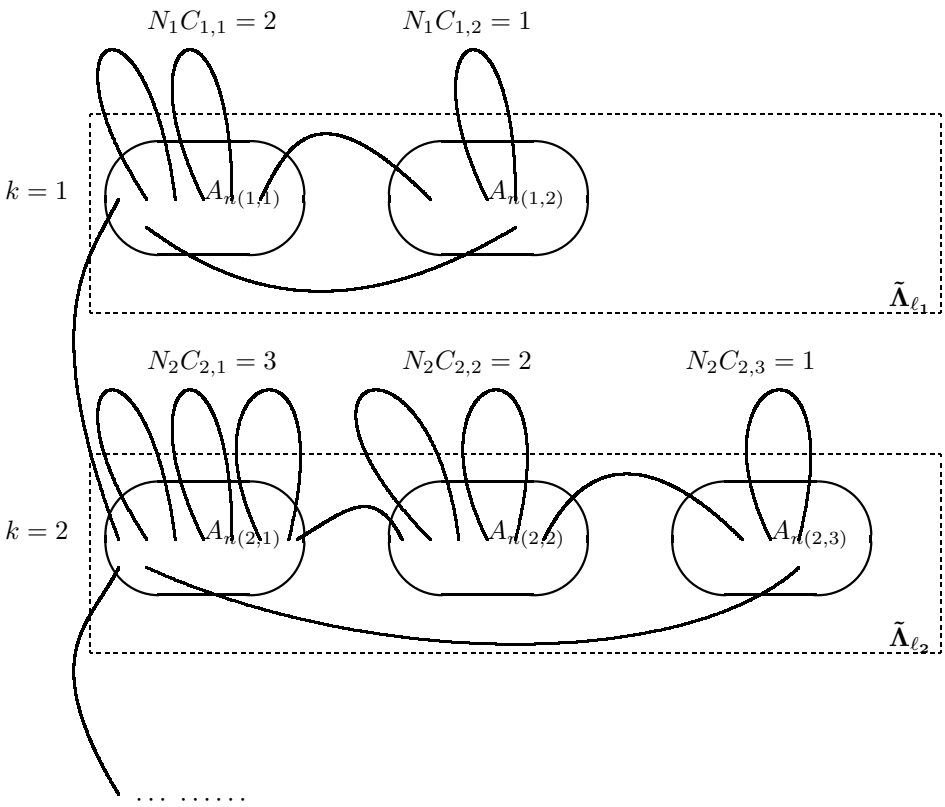

FIGURE 1. Quasi-orbits

For $k \ge 1$, $1 \le i \le T_k$, $1 \le j \le p_k$, $t \ge 1$, let $M_1 = 0$,

$$\begin{aligned} M_k &= M_{k,1} = \sum_{r=1}^{k-1}(T_r Y_r + s(m_{r,1}, m_{r+1,1})), \\ M_{k,i} &= M_{k,i,1} = M_k + (i-1)Y_k, \\ M_{k,i,j} &= M_{k,i,j,1} = M_{k,i} + \sum_{q=1}^{j-1}(N_k\, n(k,q)C_{k,q} + s(m_{k,q}, m_{k,q+1})), \\ M_{k,i,j,t} &= M_{k,i,j} + (t-1)n(k,j). \end{aligned}$$

By Lemma 2.2, there exists a shadowing point z of O such that

$$d(f^{M_{k,i,j,t}+q}(z), f^q(x(k,j;i,t))) < c\epsilon_0 e^{-\epsilon l_k} < \frac{\varepsilon}{4\epsilon_0}\epsilon_0 e^{-\epsilon l_k} \le \frac{\varepsilon}{4},$$

for $0 \le q \le n(k,j)-1$, $1 \le i \le T_k$, $1 \le t \le N_k C_{k,j}$, $1 \le j \le p_k$. In fact, z can be considered as a map with variables $x(k,j;i,t) \in V_{n(k,j)} \cap A_{n(k,j)}(m_{k,j})$. We denote by $\mathcal{J}$ the set of all shadowing points z obtained in the above procedure.

Lemma 4.1. $\overline{\mathcal{J}} \subset G_K$.

Proof. We begin by estimating $d(f^{M_{k,i,j,t}+q}(z), f^q(x(k,j;i,t)))$ for $0 \le q \le n(k,j)-1$. Recall that a pseudo-orbit and its shadowing points are in the same Lyapunov neighborhoods $\Pi(x(k,j;i,t), a\epsilon_k)$. And recall that we have required $\operatorname{diam}\xi_k < \frac{b_k(1-e^{-\epsilon})}{4\sqrt{2}e^{(k+1)\epsilon}}$, which implies that for every two adjacent orbit segments $x(k,j;i_1,t_1)$ and $x(k,j;i_2,t_2)$ the ending point of the front orbit segment and the beginning point of the segment following are $\frac{b_k(1-e^{-\epsilon})}{4\sqrt{2}e^{(k+1)\epsilon}}$ close to each other. Let y be the unique intersection point of admissible manifolds $H^s(f^{M_{k,i,j,t}}(z))$ and $H^u(x(k,j;i,t))$. We denote d'' to be the distance induced by $\|\cdot\|''$ in the local Lyapunov neighborhoods. By the hyperbolicity of Df in the Lyapunov coordinates,[2] we obtain

$$\begin{aligned}
& d(f^{M_{k,i,j,t}+q}(z), f^q(x(k,j;i,t))) \\
\le\ & d(f^{M_{k,i,j,t}+q}(z), f^q(y)) + d(f^q(y), f^q(x(k,j;i,t))) \\
\le\ & \sqrt{2}d''(f^{M_{k,i,j,t}+q}(z), f^q(y)) + \sqrt{2}d''(f^q(y), f^q(x(k,j;i,t))) \\
\le\ & \sqrt{2}e^{-(\beta_1''-\epsilon)q}d''(f^{M_{k,i,j,t}}(z), y) \\
& +\sqrt{2}e^{-(\beta_2''-\epsilon)(n(k,j)-q)}d''(f^{n(k,j)}(y), f^{n(k,j)}(x(k,j;i,t))) \\
\le\ & \sqrt{2}\max\{e^{-(\beta_1''-\epsilon)q}, e^{-(\beta_2''-\epsilon)(n(k,j)-q)}\}(d''(f^{M_{k,i,j,t}}(z), y) \\
& +d''(f^{n(k,j)}(y), f^{n(k,j)}(x(k,j;i,t)))) \\
\le\ & \frac{2\sqrt{2}e^{\epsilon(k+1)}}{1-e^{-\epsilon}}(d(f^{M_{k,i,j,t}}(z), y) + d(f^{n(k,j)}(y), f^{n(k,j)}(x(k,j;i,t)))) \\
\le\ & \frac{2\sqrt{2}e^{\epsilon(k+1)}}{1-e^{-\epsilon}}2\operatorname{diam}(\xi_k) \\
<\ & b_k
\end{aligned}$$

for $0 \le q \le n(k,j)-1$. Now by (5) we can deduce that

$$|\varphi_p(f^{M_{k,i,j,t}+q}(z) - \varphi_p(f^q(x(k,j;i,t)))| < \frac{1}{k}\|\varphi_p\|, \quad 1 \le p \le k,$$

which implies that

$$D(\mathcal{E}_{n(k,j)}(f^{M_{k,i,j,t}}(z)), \mathcal{E}_{n(k,j)}(x(k,j;i,t))) < \frac{1}{k} + \frac{1}{2^{k-1}} < \frac{2}{k}, \tag{12}$$

[2]This hyperbolic property, which is applicable to the estimation of distance along adjacent segments, is crucial together with the weak shadowing lemma 2.2 (which is actually stated in a topological way) to conclude Theorem 1.1.

for sufficiently large k. By the triangle inequality, we have

$$\begin{aligned}
&D(\mathcal{E}_{Y_k}(f^{M_{k,i}}(z)),\, \nu_k) \le D(\mathcal{E}_{Y_k}(f^{M_{k,i}}(z)),\, \mu_k) + \frac{1}{k} \\
&\quad \le D(\mathcal{E}_{Y_k}(f^{M_{k,i}}(z)),\, \frac{1}{Y_k - X_k}\sum_{j=1}^{p_k} N_k C_{k,j} n(k,j)\mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z))) \\
&\quad + D(\frac{1}{Y_k - X_k}\sum_{j=1}^{p_k} N_k C_{k,j} n(k,j)\mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z)),\, \mu_k) + \frac{1}{k}.
\end{aligned}$$

Note that for any $\varphi \in C^0(M)$, it holds that

$$\begin{aligned}
&\| \int \varphi d\mathcal{E}_{Y_k}(f^{M_{k,i}}(z)) - \int \varphi d\frac{1}{Y_k - X_k}\sum_{j=1}^{p_k} N_k C_{k,j} n(k,j)\mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z))\| \\
&= \|\frac{1}{Y_k}\sum_{q=1}^{Y_k-1}\varphi(f^{M_{k,i}+q}(z)) - \frac{1}{Y_k - X_k}\sum_{j=1}^{p_k} N_k C_{k,j}\sum_{q=1}^{n(k,j)-1}\varphi(f^{M_{k,i,j}+q}(z))\| \\
&\le \|\frac{1}{Y_k}\sum_{j=1}^{p_k} N_k C_{k,j}\sum_{q=1}^{n(k,j)-1}\varphi(f^{M_{k,i,j}+q}(z)) \\
&\qquad\qquad - \frac{1}{Y_k - X_k}\sum_{j=1}^{p_k} N_k C_{k,j}\sum_{q=1}^{n(k,j)-1}\varphi(f^{M_{k,i,j}+q}(z))\| \\
&+ \|\frac{1}{Y_k}(\sum_{j=1}^{p_k-1}\sum_{q=1}^{s(m_{k,j},\, m_{k,j+1})-1}\varphi(f^{M_{k,i,j}-s(m_{k,j},\, m_{k,j+1})+q}(z)) \\
&\qquad\qquad + \sum_{q=1}^{s(m_{k,p_k},\, m_{k,1})-1}\varphi(f^{M_{k,i,j}-s(m_{k,p_k},\, m_{k,1})+q}(z)))\| \\
&\le [|(\frac{1}{Y_k} - \frac{1}{Y_k - X_k})(Y_k - X_k)| + \frac{X_k}{Y_k}]\|\varphi\|.
\end{aligned}$$

Then by the definition of D, the above inequality implies that

$$\begin{aligned}
&D(\mathcal{E}_{Y_k}(f^{M_{k,i}}(z)),\, \frac{1}{Y_k - X_k}\sum_{j=1}^{p_k} N_k C_{k,j} n(k,j)\mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z))) \\
\le\ & |(\frac{1}{Y_k} - \frac{1}{Y_k - X_k})(Y_k - X_k)| + \frac{X_k}{Y_k}.
\end{aligned}$$

Thus, by the affine property of D, together with the property $a_{k,j} = n(k,j)C_{k,j}$ and $N_k = Y_k - X_k$, we have

$$\begin{aligned}
&D(\mathcal{E}_{Y_k}(f^{M_{k,i}}(z)),\, \nu_k)\\
&\quad\le D(\frac{1}{Y_k - X_k}\sum_{j=1}^{p_k} N_k C_{k,j} n(k,j)\mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z)), \sum_{j=1}^{p_k} a_{k,j} m_{k,j})\\
&\quad+ |(\frac{1}{Y_k} - \frac{1}{Y_k - X_k})(Y_k - X_k)| + \frac{X_k}{Y_k} + \frac{1}{k}\\
&\quad\le \frac{N_k}{Y_k - X_k}\sum_{j=1}^{p_k} a_{k,j} D(\mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z), m_{k,j})) + \frac{2X_k}{Y_k} + \frac{1}{k}\\
&\quad= \sum_{j=1}^{p_k} a_{k,j} D(\mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z), m_{k,j})) + \frac{2X_k}{Y_k} + \frac{1}{k}.
\end{aligned}$$

Noting that

$$\begin{aligned}
&\sum_{j=1}^{p_k} a_{k,j} D(\mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z), m_{k,j}))\\
&\qquad\le \sum_{j=1}^{p_k} a_{k,j} D(\mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z)), \mathcal{E}_{n(k,j)}(x(k,j;i,t)))\\
&\qquad+ \sum_{j=1}^{p_k} a_{k,j} D(\mathcal{E}_{n(k,j)}(x(k,j;i,t)),\, m_{k,j})
\end{aligned}$$

and by the definition of $\Lambda^t(m_{k,j})$ to which all $x(k,j;i,t)$ belong and by (12), we can further deduce that

$$\begin{aligned}
D(\mathcal{E}_{Y_k}(f^{M_{k,i}}(z)),\, \nu_k) &\le \sum_{j=1}^{p_k} a_{k,j} D(\mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z)), \mathcal{E}_{n(k,j)}(x(k,j;i,t)))\\
&\quad+ \frac{1}{k} + \frac{2X_k}{Y_k} + \frac{1}{k}\\
&\le \frac{2}{k} + \frac{1}{k} + \frac{2X_k}{Y_k} + \frac{1}{k}\\
&\le \frac{6}{k} \quad \text{(by (9))}.
\end{aligned}$$

For sufficiently large $M_{k,i} \le n \le M_{k,i+1}$, by affine property, we have that

$$\begin{aligned} D(\mathcal{E}_n(z),\, \nu_k) \quad \le \quad & \frac{M_{k-2}}{n} D(\mathcal{E}_{M_{k-2}}(z), \nu_k) + \frac{Y_{k-1}}{n} \sum_{r=1}^{T_{k-1}} D(\mathcal{E}_{Y_{k-1}}(f^{M_{k-1,r-1}}(z)), \nu_k) \\ & + \frac{s(m_{k-1,1}, m_{k,1})}{n} D(\mathcal{E}_{s(m_{k-1,1}, m_{k,1})}(f^{M_{k-1,T_{k-1}}}(z)), \nu_k) \\ & + \frac{Y_k}{n} \sum_{r=1}^{i-1} D(\mathcal{E}_{Y_k}(f^{M_{k,r-1}}(z)), \nu_k) \\ & + \frac{n - M_{k,i}}{n} D(\mathcal{E}_{n-M_{k,i}}(f^{M_{k,i}}(z)), \nu_k). \end{aligned}$$

Noting that

$$D(\mathcal{E}_{Y_{k-1}}(f^{M_{k-1,i-1}}(z)), \nu_k) \le D(\mathcal{E}_{Y_{k-1}}(f^{M_{k-1,i-1}}(z)), \nu_{k-1}) + D(\nu_{k-1}, \nu_k)$$

and using the fact that $D(\nu_k, \nu_{k-1}) \le 2\zeta_k + 2\zeta_{k-1}$ and inequalities (10) and (11), one can deduce that

$$D(\mathcal{E}_n(z),\, \nu_k) \le \frac{1}{k} + (\frac{6}{k-1} + 2\zeta_k + 2\zeta_{k-1}) + \frac{1}{k} + \frac{6}{k} + \frac{1}{k}. \tag{13}$$

Letting $n \to +\infty$, we get $V(z) \subset K$. On the other hand, note that

$$\bigcap_{N \ge 1} \bigcup_{n \ge N} U_n = K,$$

so $\mathcal{E}_n(z)$ can enter any neighborhood of each $\nu \in K$ infinitely many times, which implies $K \subset V(x)$. Consequently, $V(z) = K$. That is, $\mathcal{J} \in G_K$. For any $z' \in \overline{\mathcal{J}}$, we take $z_\tau \in \mathcal{J}$ with $\lim_\tau z_\tau = z'$. Observing that for $M_{k,i} \le n \le M_{k,i+1}$, it always holds (13) for each z_τ, by continuity we obtain the inequality (13) for z'. This completes the proof of Lemma 4.1. □

To finish the proof of Theorem 3.2, we need to compute the entropy of $\overline{\mathcal{J}} \subset G_K$. Notice that the choices of the position labeled by $x(k, j; i, t)$ in (12) have at least

$$e^{n(k,j)(1-\frac{1}{k})(h^{Kat}_{m_{k,j}}(f,\varepsilon|\delta)-\frac{4}{k})}$$

by (8). Moreover, fixing the position indexed k, j, t, for distinct $x(k, j; i, t)$, $x'(k, j; i, t)$ in $V_{n(k,j)} \cap A_{n(k,j)}(m_{k,j})$, the corresponding shadowing points z, z' satisfy

$$\begin{aligned} & d(f^{M_{k,i,j,t}+q}(z), f^{M_{k,i,j,t}+q}(z')) \\ \ge \quad & d(f^q(x(k, j; i, t)), f^q(x'(k, j; i, t))) - d(f^{M_{k,i,j,t}+q}(z), f^q(x(k, j; i, t))) \\ & - d(f^{M_{k,i,j,t}+q}(z'), f^q(x'(k, j; i, t))) \\ \ge \quad & d(f^q(x(k, j; i, t)), f^q(x'(k, j; i, t))) - \frac{\varepsilon}{2}. \end{aligned}$$

Since $x(k,j;i,t), x'(k,j;i,t)$ are $(n(k,j),\varepsilon)$-separated, so $f^{M_{k,i,j,t}}(z)$, $f^{M_{k,i,j,t}}(z')$ are $(n(k,j),\frac{\varepsilon}{2})$-separated. For the choice of quasi-orbits in M_{ki} denote by

$$H_{k,i}=\{(x(k,j;i,1),\cdots,x(k,j;i,N_kC_{k,j}),\cdots,x(k,p_k;i,1),\cdots,x(1,p_k;i,N_kC_{k,p_k}))$$
$$\mid\ x(k,j;i,t)\in V_{n(k,j)}\cap A_{n(k,j)}\}.$$

Then

$$\sharp H_{k,i}\geq e^{\sum_{j=1}^{p_k}N_kC_{k,j}n(k,j)(1-\frac{1}{k})(h^{Kat}_{m_{k,j}}(f,\varepsilon|\delta)-\frac{4}{k})}.$$

Write $r_0=\inf\{h_\mu(f)\mid\ \mu\in K\}$. If $r_0=0$, we have nothing to prove. So suppose $r_0>0$. Hence,

$$\begin{aligned}
\frac{1}{Y_k}\log\sharp H_{k,i} &\geq \frac{Y_k-X_k}{Y_k}\sum_{j=1}^{p_k}a_{k,j}(1-\frac{1}{k})(h^{Kat}_{m_{k,j}}(f,\varepsilon\mid\delta)-\frac{4}{k}) \qquad (14)\\
&\geq (1-\frac{1}{k})\sum_{j=1}^{p_k}a_{k,j}(1-\frac{1}{k})(h^{Kat}_{m_{k,j}}(f,\varepsilon\mid\delta)-\frac{4}{k})\\
&= (1-\frac{1}{k})^2h^{Kat}_{\mu_k}(f,\varepsilon\mid\delta)-\frac{4}{k}(1-\frac{1}{k})^2\\
&\geq (1-\frac{1}{k})^2(h^{Kat}_{\nu_k}(f,\varepsilon\mid\delta)-\frac{1}{k})-\frac{4}{k}(1-\frac{1}{k})^2\\
&\geq (1-\frac{1}{k})^2(r_0-\gamma-\frac{5}{k}). \qquad (15)
\end{aligned}$$

Since $\overline{\mathcal{J}}$ is compact we can take only finite covers of $\overline{\mathcal{J}}$ in $\mathcal{C}(\overline{\mathcal{J}},\varepsilon/2)$ while calculating topological entropy $h_{\text{top}}(\overline{\mathcal{J}},\frac{\varepsilon}{2})$. Let $0<\frac{r_1}{1-\gamma}+\gamma<r_0-\gamma\leq\inf_{\mu\in K}h^{Kat}_\mu(f,\varepsilon\mid\delta)$ and $0<r<r_1$. Here γ and r_1 are chosen small enough, and then the choice of ε depends on γ.

For each $\mathcal{A}\in\mathcal{C}(\overline{\mathcal{J}},\varepsilon/2)$ we define a new cover $\mathcal{A}'$ in which for $M_{k,i}\leq m\leq M_{k,i+1}$, $B_m(z,\varepsilon/2)$ is replaced by $B_{M_{k,i}}(z,\varepsilon/2)$, where we suppose $M_{k,0}=M_{k-1,p_{k-1}}$, $M_{k,p_k+1}=M_{k+1,1}$. Therefore,

$$\mathcal{Y}(\overline{\mathcal{J}};r,n,\varepsilon/2)=\inf_{\mathcal{A}\in\mathcal{C}(\overline{\mathcal{J}},\varepsilon/2)}\sum_{B_m(z,\varepsilon/2)\in\mathcal{A}}e^{-rm}\geq\inf_{\mathcal{A}\in\mathcal{C}(\overline{\mathcal{J}},\varepsilon/2)}\sum_{B_{M_{k,i}}(z,\varepsilon/2)\in\mathcal{A}'}e^{-rM_{k,i+1}}.$$

Denote

$$b=b(\mathcal{A}')=\max\{M_{k,i}\mid\ B_m(z,\frac{\varepsilon}{2})\in\mathcal{A}'\quad\text{and}\quad M_{k,i}\leq m<M_{k,i+1}\}.$$

Noticing that $\mathcal{A}'$ is a cover of $\mathcal{J}$, each point of $\mathcal{J}$ belongs to some $B_{M_{k,i}}(x,\frac{\varepsilon}{2})$ with $M_{k,i}\leq b$. Moreover, if $z,z'\in\mathcal{J}$ with some position $x(k,j;i,t)\neq x'(k,j;i,t)$, then z,z' cannot stay in the same $B_{M_{k,i}}(x,\frac{\varepsilon}{2})$. Define

$$W_{k,i}=\{B_{M_{k,i}}(z,\frac{\varepsilon}{2})\in\mathcal{A}'\}.$$

It follows that

$$\sum_{M_{k,i}\leq b}\sharp W_{k,i}\ \Pi_{M_{k,i}<M_{k',i'}\leq b}\ \sharp H_{k',i'}\ \geq\ \Pi_{M_{k',i'}\leq b}\ \sharp H_{k',i'}.$$

So,

$$\sum_{M_{k,i}\le b} \sharp W_{k,i}\ (\Pi_{M_{k',i'}\le M_{k,i}}\ \sharp H_{k',i'})^{-1}\ \ge\ 1.$$

From (14) it is easily seen that

$$\limsup_{k\to\infty} \frac{\Pi_{M_{k',i'}\le M_{k,i}}\ \sharp H_{k',i'}}{\exp(r_1 M_{k,i})} \ge 1.$$

Since $r < r_1$ and $\lim_{k\to\infty} \frac{M_{k,i+1}}{M_{k,i}} = 1$, we can take k large enough so that

$$\frac{M_{k,i+1}}{M_{k,i}} \le \frac{r_1}{r}.$$

Thus there is some constant $c_0 > 0$ for large k,

$$\begin{aligned}
\sum_{B_{M_{k,i}}(z,\varepsilon/2)\in\mathcal{A}'} e^{-rM_{k,i+1}} &= \sum_{M_{k,i}\le b} \sharp W_{k,i}\ e^{-rM_{k,i+1}} \\
&\ge \sum_{M_{k,i}\le b} \sharp W_{k,i}\ \exp(-r_1 M_{k,i}) \\
&\ge c_0 \sum_{M_{k,i}\le b} \sharp W_{k,i}\ (\Pi_{M_{k',i'}\le M_{k,i}}\ \sharp H_{k',i'})^{-1} \\
&\ge c_0,
\end{aligned}$$

which, together with the arbitrariness of r, leads to the required inequality

$$h_{\mathrm{top}}(\overline{\mathcal{J}}, \frac{\varepsilon}{2}) \ge (1-\gamma)(r_0 - 2\gamma).$$

Finally, the arbitrariness of γ yields

$$h_{\mathrm{top}}(f, G_K) \ge r_0.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.3. Consider the quasi-orbits approximate to K_1 under positive iterations and approximate to K_2 under negative iterations. Using Lemma 2.2 and the computation method in the proof of Theorem 1.1 we can conclude that

$$h_{\mathrm{top}}(f^{-1}, G_{K_1} \cap G_{K_2}^{-1}) \ge \inf\{h_\mu(f) \mid \mu \in K_2\}.$$

□

5. The structure of NUH set $\widetilde{\Lambda}$

Here, we will show for several classes of diffeomorphisms beyond Anosov that $\mathcal{M}_{inv}(\widetilde{\Lambda}, f)$ contains many more members.

Let f_0 be a transitive Anosov diffeomorphism on some tours $M = \mathbb{T}^n$. For $x \in M$, $\varepsilon_0 > 0$, we have the stable manifold $W^s_{\varepsilon_0}(x)$ and unstable manifold $W^u_{\varepsilon_0}(x)$ defined by

$$W^s_{\varepsilon_0}(x) = \{y \in M \mid d(f_0^n(x), f_0^n(y)) \le \varepsilon_0, \quad \text{for all} \quad n \ge 0\},$$

$$W^u_{\varepsilon_0}(x) = \{y \in M \mid d(f_0^{-n}(x), f_0^{-n}(y)) \le \varepsilon_0, \quad \text{for all} \quad n \ge 0\}.$$

Fixing small $\varepsilon_0 > 0$ there exists a $\delta_0 > 0$ so that $W^s_{\varepsilon_0}(x) \cap W^u_{\varepsilon_0}(y)$ contains a single point $[x, y]$ whenever $d(x, y) < \delta_0$. Furthermore, the function

$$[\cdot,\cdot] : \{(x, y) \in M \times M \mid d(x, y) < \delta_0\} \to M$$

is continuous. A rectangle R is understood as a subset of M with small diameter and $[x, y] \in R$ whenever $x, y \in R$. For $x \in R$ let

$$W^s(x, R) = W^s_{\varepsilon_0}(x) \cap R \quad \text{and} \quad W^u(x, R) = W^u_{\varepsilon_0}(x) \cap R.$$

For Anosov diffeomorphism f_0 one can obtain the following structure known as a Markov partition $\mathcal{R} = \{R_1, R_2, \cdots, R_l\}$ of M with properties:

(1) $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ for $i \neq j$;
(2) $f_0 W^u(x, R_i) \supset W^u(f_0 x, R_j)$ and
$f_0 W^s(x, R_i) \subset W^s(f_0 x, R_j)$ when $x \in \operatorname{int} R_i$, $fx \in \operatorname{int} R_j$.

Using the Markov partition $\mathcal{R}$ one can define the transition matrix $B = B(\mathcal{R})$ by

$$B_{i,j} = \begin{cases} 1 & \text{if} \quad \operatorname{int} R_i \cap f_0^{-1}(\operatorname{int} R_j) \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

The subshift (Σ_B, σ) associated with B is given by

$$\Sigma_B = \{\underline{q} \in \Sigma_l \mid B_{q_i, q_{i+1}} = 1 \quad \forall i \in \mathbb{Z}\}.$$

For each $\underline{q} \in \Sigma_B$ by the hyperbolic property the set $\bigcap_{i\in\mathbb{Z}} f_0^{-i} R_{q_i}$ contains a single point, denoted by $\pi_0(\underline{q})$. We denote

$$\Sigma_B(i) = \{\underline{q} \in \Sigma_B \mid q_0 = i\}.$$

The following properties hold for the map π_0 (see Sinai [40] and Bowen [5, 6]).

Proposition 5.1.

(1) *The map $\pi_0 : \Sigma_B \to M$ is a continuous surjection satisfying $\pi_0 \circ \sigma = f_0 \circ \pi_0$;*

(2) $\pi_0(\Sigma_B(i)) = R_i, \quad 1 \leq i \leq l;$

(3) $h_{\text{top}}(\sigma, \Sigma_B) = h_{\text{top}}(f_0, M)$.

Since B is a $(0, 1)$-matrix, using the Perron Frobenius Theorem the maximal eigenvalue λ of B is positive and simple. λ has the row eigenvector $u = (u_1, \cdots, u_l)$, $u_i > 0$, and the column eigenvector $v = (v_1, \cdots, v_l)^T$, $v_i > 0$. We assume $\sum_{i=1}^l u_i v_i = 1$ and denote $(p_1, \cdots, p_l) = (u_1 v_1, \cdots, u_l v_l)$. Define a new matrix

$$\mathcal{P} = (p_{i,j})_{l\times l}, \qquad \text{where} \quad p_{i,j} = \frac{B_{i,j}\, v_j}{\lambda\, v_i}.$$

Then $\mathcal{P}$ can define a Markov chain with probability μ_0 satisfying

$$\mu_0([a_0 a_1 \cdots a_i]) = p_{a_0} p_{a_0, a_1} \cdots p_{a_{i-1}, a_i}.$$

Then μ_0 is σ-invariant and Gurevich [16, 17] proved that μ_0 is the unique maximal measure of (Σ_B, σ), that is,

$$h_{\text{top}}(\sigma, \Sigma_B) = h_{\mu_0}(\sigma, \Sigma_B) = \log \lambda.$$

In addition, Bowen [5] proved that $\pi_{0*}(\mu_0)$ is the unique maximal measure of f_0 and $\pi_{0*}(\mu)(\partial\mathcal{R}) = 0$, where $\partial\mathcal{R}$ consists of all boundaries of R_i, $1 \leq i \leq l$. Let $\mu_1 = \pi_{0*}(\mu_0)$ and $p_i = \mu_1(R_i)$ for $1 \leq i \leq l$.

The following shadowing lemma is Corollary 4.3 in [10].

Lemma 5.2. *Let $f_0 : \mathbb{T}^n \to \mathbb{T}^n$ be a linear Anosov map. Then there exists $C > 0$ such that for any small r and any $f : \mathbb{T}^n \to \mathbb{T}^n$ with $\mathrm{dist}_{C^0}(f, g) < r$ there exists $\pi : \mathbb{T}^n \to \mathbb{T}^n$ continuous and onto, $\mathrm{dist}_{C^0}(\pi, \mathrm{id}) < Cr$, and*

$$f_0 \circ \pi = \pi \circ f.$$

Suppose there is a fixed point $O \in \mathrm{int}\, R_1$. Take small r satisfying the ball $B(O, Cr) \subset R_1$ and $d(B(O, Cr), \partial R_1) > Cr$, where C is taken as Lemma 5.2.

Denote by $TM = E_0^s \oplus E_0^u$ the hyperbolic splitting for f_0 and let

$$\lambda_s := \|Df\mid_{E_0^s}\|, \quad \lambda_u := \|Df^{-1}\mid_{E_0^u}\|.$$

Fix positive numbers $\delta, \gamma \ll \lambda := \max\{\lambda_s, \lambda_u\}$. Let

$$\begin{aligned} p_0 &= \frac{1}{2}\min\{1 - p_i \mid 1 \le i \le l\}, \\ \beta &= -(1 - p_1 - p_0 - \gamma)\log\lambda - \delta. \end{aligned}$$

Let $\mathcal{V}$ consists of f satisfying the following conditions:

(1) There exist a Df-invariant continuous splitting $TM = E \oplus F$.

(2) f is C^1 close to f_0 in the complement of $B(O, r)$, so that for $x \in \mathbb{T}^n \setminus B(O, r)$:

$$\|Df^{-1}|F(x)\| < e^{\delta}\lambda \text{ and } \|Df|E(x)\| < e^{\delta}\lambda.$$

(3) For $x \in B(O, r)$:

$$\|Df^{-1}|F(x)\| < e^{\delta} \text{ and } \|Df|E(x)\| < e^{\delta}.$$

The structure of $\mathcal{V}$ applies to (i) the nonuniformly hyperbolic diffeomorphisms constructed by Katok [19]; (ii) the robustly transitive partially hyperbolic diffeomorphisms constructed by Mañé [28]; (iii) the robustly transitive nonpartially hyperbolic diffeomorphisms constructed by Bonatti and Viana [4]. For such a function f, Buzzi, Fisher, Sambarino, and Vásquez [10], [9] have shown that there exists a unique maximal measure ν_0 of f with $\pi_*\nu_0 = \mu_1$. We are going to prove that the measure ν_0 conforms to a rigidity structure of NUH set $\widetilde{\Lambda}$ by the following theorem.

Theorem 5.3. *There exists a neighborhood U of ν_0 in $\mathcal{M}_{inv}(\mathbb{T}^n, f)$ such that for any ergodic $\nu \in U$ it holds that $\nu \in \mathcal{M}_{inv}(\widetilde{\Lambda}(\beta, \beta, \epsilon), f)$ for any $0 \le \epsilon \ll \beta$.*

Let's begin with the analysis of the transitive property of Anosov system f_0. For $0 < \gamma < 1$, $N \in \mathbb{N}$ define

$$\begin{aligned} \Gamma_N(i, \gamma) = \{x \in M \mid \sharp\{n \le j \le n + k - 1 \mid f_0^j(x) \in R_i\} \le N + k(p_i + \gamma) + |n|\gamma, \\ \sharp\{n \le j \le n + k - 1 \mid f_0^{-j}(x) \in R_i\} \le N + k(p_i + \gamma) + |n|\gamma \\ \forall\, k \ge 1,\ \forall\, n \in \mathbb{Z}\}. \end{aligned}$$

Then $f_0^{\pm}(\Gamma_N(i, \gamma)) \subset \Gamma_{N+1}(i, \gamma)$. Let $\Gamma(i, \gamma) = \bigcup_{N \ge 1} \Gamma_N(i, \gamma)$.

Lemma 5.4. *For any $m \in \mathcal{M}_{erg}(M, f_0)$, if $m(R_i) < p_i + \gamma/2$, then $m(\Gamma(i, \gamma)) = 1$.*

Proof. Since $m(R_i) < p_i + \gamma/2$, for almost m, then for every x one can find $N(x) > 0$ such that

$$n(m(R_i) - \frac{\gamma}{2}) \le \sharp\{0 \le j \le n - 1 \mid f_0^j(x) \in R_i\} \le n(m(R_i) + \frac{\gamma}{2}), \quad \forall\, n \ge N(x);$$

$$n(m(R_i) - \frac{\gamma}{2}) \le \sharp\{0 \le j \le n - 1 \mid f_0^{-j}(x) \in R_i\} \le n(m(R_i) + \frac{\gamma}{2}), \quad \forall\, n \ge N(x).$$

Take $N_0(x)$ to be the smallest number such that for every $n \geq 1$,

$$\begin{aligned} -N_0(x) + n(m(R_i) - \frac{\gamma}{2}) &\leq \sharp\{0 \leq j \leq n-1 \mid f_0^j(x) \in R_i\} \\ &\leq N_0(x) + n(m(R_i) + \frac{\gamma}{2}); \\ -N_0(x) + n(m(R_i) - \frac{\gamma}{2}) &\leq \sharp\{0 \leq j \leq n-1 \mid f_0^{-j}(x) \in R_i\} \\ &\leq N_0(x) + n(m(R_i) + \frac{\gamma}{2}). \end{aligned}$$

Then for any $k \geq 1$,

$$\begin{aligned} & \sharp\{n \leq j \leq n+k-1 \mid f_0^j(x) \in R_i\} \\ = \quad & \sharp\{0 \leq j \leq n+k-1 \mid f_0^j(x) \in R_i\} - \sharp\{0 \leq j \leq n-1 \mid f_0^j(x) \in R_i\} \\ \leq \quad & N_0(x) + (n+k)(m(R_i) + \frac{\gamma}{2}) - (-N_0(x) + n(m(R_i) - \frac{\gamma}{2})) \\ = \quad & 2N_0(x) + k(m(R_i) + \frac{\gamma}{2}) + n\gamma. \end{aligned}$$

In this manner we can also show that

$$\sharp\{n \leq j \leq n+k-1 \mid f_0^{-j}(x) \in R_i\} \leq 2N_0(x) + k(m(R_i) + \frac{\gamma}{2}) + n\gamma.$$

Thus, $x \in \Gamma_{2N_0(x)}(i, \gamma)$. □

Since $\mu_1(R_i) = p_i$ for $1 \leq i \leq l$, by Lemma 5.4 $\mu_1(\Gamma(i,\gamma)) = 1$. We further define

$$\widetilde{\Gamma}_N(i,\gamma) = \mathrm{supp}(\mu_1 \mid_{\Gamma_N(i,\gamma)}) \quad \text{and} \quad \widetilde{\Gamma}(i,\gamma) = \bigcup_{N \geq 1} \widetilde{\Gamma}_N(i,\gamma).$$

It holds that $\widetilde{\Gamma}(i,\gamma)$ is f-invariant and $\mu_1(\widetilde{\Gamma}(i,\gamma)) = 1$.

Proposition 5.5. *There is a neighborhood U of μ_1 in $\mathcal{M}_{inv}(M, f_0)$ such that for any ergodic measure $m \in U$ we have $m \in \mathcal{M}_{inv}(\widetilde{\Gamma}(i,\gamma), f_0)$.*

Proof. Observing that $\mu_1(\partial R_i) = 0$, for $\gamma > 0$ there exists a neighborhood U of μ_1 in $\mathcal{M}_{inv}(M, F)$ such that for any $m \in U$ one has

$$m(R_i) < p_i + \frac{\gamma}{2}.$$

We then claim: There is an ergodic measure $m_0 \in \mathcal{M}_{inv}(\Sigma_B, \sigma)$ satisfying $\pi_{0*}m_0 = m$.

The proof of the claim goes as follows. Denote the basin of m by

$$Q_m(M, f_0) = \Big\{x \in M \mid \lim_{n \to \pm\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_0^i x) = \int_M \varphi dm, \quad \forall \varphi \in C^0(M)\Big\}.$$

Take and fix a point $x \in Q_m(M, f_0)$ and choose $q \in \Sigma_B$ with $\pi_0(q) = x$. Define a sequence of measures ν_n on Σ_B by

$$\int \psi d\nu_n := \frac{1}{n} \sum_{i=0}^{n-1} \psi(\sigma^i(q)), \quad \forall \psi \in C^0(\Sigma_B).$$

By taking a subsequence when necessary we can assume that $\nu_n \to \nu_0$. It is standard to verify that ν_0 is a σ-invariant measure and ν_0 covers m, i.e., $\pi_{0*}(\nu_0) = m$. Set

$$Q(\sigma) := \bigcup_{\nu \in \mathcal{M}_{erg}(\Sigma_B, \sigma)} Q_\nu(\Sigma_B, \sigma).$$

Then $Q(\sigma)$ is a σ-invariant total measure subset in Σ_B. We have

$$\begin{aligned} & m(Q_m(M, F) \cap \pi_0 Q(\sigma)) \\ \geq & \nu_0(\pi_0^{-1} Q_m(M, f_0) \cap Q(\sigma)) \\ = & 1. \end{aligned}$$

Then the set

$$\mathcal{A}_0 := \Big\{ \nu \in \mathcal{M}_{erg}(\Sigma_B, \sigma) \mid \exists\, q \in Q(\sigma), \pi_0(q) \in Q_m(M, f_0), s.t. \\ \lim_{n \to \pm\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(\sigma^i(q)) = \int_{\Sigma_B} \psi \, d\nu, \quad \forall \psi \in C^0(\Sigma_B) \Big\}$$

is nonempty. It is clear that ν covers m, $\pi_{0*}(\nu) = m$, for all $\nu \in \mathcal{A}_0$. Hence the claim.

We now continue the proof of Proposition 5.5. Since $\pi_{0*}(m_0) = m$, $m_0(\pi_0^{-1}(R_i)) = m(R_i) < p_i + \gamma/2$, together with $\Sigma_B(i) \subset \pi_0^{-1}(R_i)$, we get

$$m_0(\Sigma_B(i)) < p_i + \frac{\gamma}{2}.$$

For $0 < \gamma < 1$, $N \in \mathbb{N}$ define

$$\begin{aligned} \Upsilon_N(i, \gamma) = \{ \underline{q} \in \Sigma_B \mid & \sharp\{n \leq j \leq n + k - 1 \mid q_j = i\} \leq N + k(p_i + \gamma) + |n|\gamma, \\ & \sharp\{n \leq j \leq n + k - 1 \mid q_{-j} = i\} \leq N + k(p_i + \gamma) + |n|\gamma \\ & \forall\, k \geq 1,\ \forall\, n \in \mathbb{Z} \}. \end{aligned}$$

Let $\Upsilon(i, \gamma) = \bigcup_{N \geq 1} \Upsilon_N(i, \gamma)$. Then $\mu_0(\Upsilon(i, \gamma)) = 1$. Further define

$$\widetilde{\Upsilon}_N(i, \gamma) = \mathrm{supp}(\mu_0 \mid_{\Upsilon_N(i,\gamma)}) \quad \text{and} \quad \widetilde{\Upsilon}(i, \gamma) = \bigcup_{N \geq 1} \widetilde{\Upsilon}_N(i, \gamma).$$

It also holds that $\widetilde{\Upsilon}(i, \gamma)$ is σ-invariant and $\mu_0(\widetilde{\Upsilon}(i, \gamma)) = 1$.

Lemma 5.6. *Given $m_0 \in \mathcal{M}_{erg}(\Sigma_B, \sigma)$, if $m_0(\Sigma_B(i)) < p_i + \gamma/2$, then $m_0 \in \mathcal{M}_{inv}(\widetilde{\Upsilon}(i, \gamma), \sigma)$.*

Proof. Since $m_0(\Sigma_B(i)) < p_i + \gamma/2$ we obtain $m_0(\bigcup_{N \in \mathbb{N}} \Upsilon_N(i, \gamma)) = 1$. We can take N_0 so large that $m_0(\Upsilon_{N_0}(i, \gamma)) > 0$ and $\mu_0(\widetilde{\Upsilon}_{N_0}(i, \gamma)) > 0$. Define

$$\tilde{\Upsilon}^j_{N_0}(i, \gamma) = \{ \underline{q} \in \widetilde{\Upsilon}_{N_0}(i, \gamma) \mid q_0 = j \}.$$

Then there exists $j \in [1, l]$ such that $\mu_0(\tilde{\Upsilon}^j_{N_0}(i, \gamma)) > 0$.

Noting that (Σ_B, σ) is mixing, there exists $L_0 > 0$ such that for each pair j_1, j_2 one can choose a sequence $L(j_1, j_2) = (c_1 \cdots c_L)$ satisfying $c_1 = j_1$, $c_L = j_2$, $B_{c_i, c_{i+1}} = 1$, $1 \leq i \leq L - 1$, and $2 \leq \sharp L(j_1, j_2) \leq L_0$.

Arbitrarily taking $\underline{q} \in \Upsilon_{N_0}(i,\gamma)$, $\underline{z} \in \tilde{\Upsilon}^j_{N_0}(i,\gamma)$, $n \in \mathbb{N}$, define a new point $\underline{y} = \underline{y}(\underline{q},\underline{z},n) \in \Sigma_B$ as follows:

$$\underline{y} = (\cdots z_{-3}z_{-2}z_{-1}L(z_0,q_{-n})q_{-n+1}\cdots q_{-1};\overset{0}{q_0}\, q_1\cdots q_{n-1}L(q_n,z_0)z_1z_2z_3\cdots).$$

Denote $N_1 = 2L_0 + 2N_0 + 1$. For any $\theta > 0$, we can take large n satisfying $n > N_1$ and $d(\underline{y}(\underline{q},\underline{z},n),\underline{q}) < \theta$. Define a new subset of Σ_B:

$$Y(\underline{q},n) = \{\underline{y}(\underline{q},\underline{z},n) \in \Sigma_B \mid \underline{z} \in \tilde{\Upsilon}^j_{N_0}(i,\gamma)\}.$$

Consider the positive and negative components of $\tilde{\Upsilon}^j_{N_0}(i,\gamma)$ as given by

$$\begin{aligned}\Upsilon^+(i,j) &= \{\underline{w} \in \Sigma_B \mid w_k = z_k,\ k \geq 0, \quad \text{for some } \ \underline{z} \in \tilde{\Upsilon}^j_{N_0}(i,\gamma)\},\\ \Upsilon^-(i,j) &= \{\underline{w} \in \Sigma_B \mid w_k = z_k,\ k \leq 0, \quad \text{for some } \ \underline{z} \in \tilde{\Upsilon}^j_{N_0}(i,\gamma)\}.\end{aligned}$$

Clearly $\Upsilon^+(i,j) \supset \tilde{\Upsilon}^j_{N_0}(i,\gamma)$, $\Upsilon^-(i,j) \supset \tilde{\Upsilon}^j_{N_0}(i,\gamma)$, so

$$\mu_0(\Upsilon^+(i,j)) > 0, \quad \mu_0(\Upsilon^-(i,j)) > 0.$$

Write

$$L(z_0,q_{-n})q_{-n+1}\cdots q_{-1};\overset{0}{q_0}\, q_1\cdots q_{n-1}L(q_n,z_0) = (j,g_\tau,\cdots,g_{-1};\overset{0}{g_0}\, g_1,\cdots,g_\varsigma,j).$$

Then, by the Markov property of μ_0, we obtain

$$\mu_0(Y(\underline{q},n)) = \mu_0(\Upsilon^-(i,j))p_{j,g_\tau}p_{g_\tau,g_{\tau+1}}\cdots p_{g_{\varsigma-1},g_\varsigma}p_{g_\varsigma,j}\, p_j^{-1}\mu_0(\Upsilon^+(i,j)) > 0.$$

Moreover, for any $\underline{y} \in Y(\underline{q},n)$ and $k \geq 1$, $s \in \mathbb{Z}$, we have the following cases.

Case 1. $-n - \sharp L \leq s \leq n + \sharp L$, $s + k - 1 \leq n + \sharp L$, so it follows that

$$\begin{aligned}\sharp\{s \leq t \leq s+k-1 \mid y_t = i\} &\leq 2L_0 + \sharp\{s \leq t \leq s+k-1 \mid q_t = i\}\\ &\leq 2L_0 + N_0 + k(p_i+\gamma) + |s|\gamma.\end{aligned}$$

Case 2. $-n - \sharp L \leq s \leq n + \sharp L$, $s + k - 1 > n + \sharp L$, so it follows that

$$\begin{aligned}\sharp\{s \leq t \leq s+k-1 \mid y_t = i\} &\leq L_0 + N_0 + (n + \sharp L - s)(p_i+\gamma) + s|\gamma| + N_0\\ &\quad +(s+k-1-n-\sharp L)(p_i+\gamma)\\ &\leq L_0 + 2N_0 + k(p_i+\gamma) + |s|\gamma.\end{aligned}$$

Case 3. $s > n + \sharp L$, so it follows that

$$\sharp\{s \leq t \leq s+k-1 \mid y_t = i\} \leq N_0 + k(p_i+\gamma) + |s|\gamma.$$

Case 4. $s < -n - \sharp L$, so it follows that

$$\sharp\{s \leq t \leq s+k-1 \mid y_t = i\} \leq 2L_0 + 2N_0 + k(p_i+\gamma) + |s|\gamma.$$

The situation of $\sharp\{s \leq t \leq s+k-1 \mid y_{-t} = i\}$ is similar. Let $N_1 = 2L_0 + 2N_0 + 1$; then $Y(\underline{q},n) \subset \Upsilon_{N_1}(i,\gamma)$. The arbitrariness of θ entails that

$$\underline{q} \in \operatorname{supp}(\mu_0 \mid \Upsilon_{N_1}(i,\gamma)).$$

That is, $\Upsilon_{N_0}(i,\gamma) \subset \widetilde{\Upsilon}_{N_1}(i,\gamma)$. Since $m_0(\Upsilon_{N_0}(i,\gamma)) > 0$, $m_0(\widetilde{\Upsilon}_{N_1}(i,\gamma)) > 0$, which, by the ergodicity of m_0, implies $m_0(\widetilde{\Upsilon}(i,\gamma)) = 1$. $\square$

Noting that $\pi_0(\widetilde{\Upsilon}_N(i,\gamma)) \subset \widetilde{\Gamma}_N(i,\gamma)$, by Lemma 5.6, we obtain

$$m(\widetilde{\Gamma}(i,\gamma)) = m_0(\pi_0^{-1}(\widetilde{\Gamma}(i,\gamma))) \geq m_0(\widetilde{\Upsilon}(i,\gamma)) = 1,$$

which concludes Proposition 5.5. □

Proof of Theorem 5.3. By Proposition 5.5 we can take a neighborhood U_1 of μ_1 in $\mathcal{M}_{inv}(\mathbb{T}^n, f_0)$ such that every ergodic $\mu \in U_1$ also belongs to $\widetilde{\Gamma}(i,\gamma)$, where $\widetilde{\Gamma}(i,\gamma)$ is given by Proposition 5.5. Since π is continuous, there is a neighborhood U of ν_0 in $\mathcal{M}_{inv}(\mathbb{T}^n, f)$ such that $\pi_* U \subset U_1$. For $N \in \mathbb{N}, \gamma > 0$, define

$$\begin{aligned} T_N(i,\gamma) = \{x \in M \mid\ & \sharp\{n \leq j \leq n+k-1 \mid f^j(x) \in B(O,r)\} \leq N + k(p_i+\gamma) + |n|\gamma, \\ & \sharp\{n \leq j \leq n+k-1 \mid f^{-j}(x) \in B(O,r)\} \leq N + k(p_i+\gamma) + |n|\gamma \\ & \forall\, k \geq 1,\ \forall\, n \in \mathbb{Z}\}. \end{aligned}$$

For large N we have $\nu_0(T_N(i,\gamma)) > 0$ and let

$$\widetilde{T}_N(i,\gamma) = \mathrm{supp}(\nu_0 \mid_{T_N(i,\gamma)}).$$

Define

$$\begin{aligned} T_N(i,\gamma) = \{x \in M \mid\ & \sharp\{n \leq j \leq n+k-1 \mid f^j(x) \in B(O,r)\} \leq N + k(p_i+\gamma) + |n|\gamma, \\ & \sharp\{n \leq j \leq n+k-1 \mid f^{-j}(x) \in B(O,r)\} \leq N + k(p_i+\gamma) + |n|\gamma \\ & \forall\, k \geq 1,\ \forall\, n \in \mathbb{Z}\}. \end{aligned}$$

For large N we have $\nu_0(T_N(i,\gamma)) > 0$ and let

$$\widetilde{T}_N(i,\gamma) = \mathrm{supp}(\nu_0 \mid_{T_N(i,\gamma)}).$$

Let $N_2 = [\frac{\beta N}{\gamma p_0}] + 1$, $\epsilon = \max\{\delta(p_1+\gamma+p_0), \gamma\}$. Then

$$T_N(1,\gamma) \subset \Lambda_{N_2}(\beta,\beta,\epsilon) \quad \text{and} \quad T_N(1,\gamma) \subset \widetilde{\Lambda}_{N_2}(\beta,\beta,\epsilon).$$

For any $x \in \Gamma_N(1,\gamma)$, $z \in \pi^{-1}(x)$, we have

$$d(f^i(z), f_0^i(x)) = d(f^i(z), \pi(f^i(x))) < Cr,$$

which implies that if $f_0^i(x) \notin R_1$, then $f^i(z) \notin B(O,r)$ because $d(B(O,r), \partial R_1) > Cr$. Thus

$$\pi^{-1}(\Gamma_N(1,\gamma)) \subset T_N(1,\gamma) \subset \Lambda_{N_2}(\beta,\beta,\epsilon),$$

which yields that

$$\pi^{-1}(\widetilde{\Gamma}_N(1,\gamma)) \subset \widetilde{\Lambda}_{N_2}(\beta,\beta,\epsilon).$$

For any ergodic $\nu \in U$, $\pi_*\nu \in U_1$. So $\pi_*\nu(\widetilde{\Gamma}_N(1,\gamma)) > 0$. We obtain

$$\nu(\widetilde{\Lambda}_{N_2}(\beta,\beta,\epsilon)) \geq \nu(\pi^{-1}(\widetilde{\Gamma}_N(1,\gamma))) = \pi_*\nu(\widetilde{\Gamma}_N(1,\gamma)) > 0.$$

Once more, the ergodicity of ν_0 implies $\nu(\widetilde{\Lambda}(\beta,\beta,\epsilon)) = 1$.

For any $z \in T_N(i,\gamma)$, $n \in \mathbb{Z}$, $k \geq 1$ we have the following cases.

Case 1. $k(p_1+\gamma+p_0) \leq N + k(p_1+\gamma) + |n|\gamma$; then

$$k \leq \frac{N+|n|\gamma}{p_0}.$$

So,

$$\begin{aligned}\|Df^{-k}\mid_{F(f^n x)}\| &\leq e^{-k\beta}\exp(\frac{\beta}{p_0}(N+|n|\gamma)),\\ \|Df^{k}\mid_{E(f^n x)}\| &\leq e^{-k\beta}\exp(\frac{\beta}{p_0}(N+|n|\gamma)).\end{aligned}$$

Case 2. $k(p_1+\gamma+p_0) > N + k(p_1+\gamma) + |n|\gamma$; then

$$\begin{aligned}\|Df^{-k}\mid_{F(f^n x)}\| &\leq (\lambda e^{\delta})^{(1-p_1-p_0-\gamma)k}e^{\delta k(p_1+\gamma+p_0)} \leq e^{-\beta k}e^{\delta k(p_1+\gamma+p_0)},\\ \|Df^{k}\mid_{E(f^n x)}\| &\leq (\lambda e^{\delta})^{(1-p_1-p_0-\gamma)k}e^{\delta k(p_1+\gamma+p_0)} \leq e^{-\beta k}e^{\delta k(p_1+\gamma+p_0)}.\end{aligned}$$

Let $N_2 = [\frac{\beta N}{\gamma p_0}] + 1$, $\epsilon = \max\{\delta(p_1+\gamma+p_0),\, \gamma\}$. Then

$$T_N(1,\gamma) \subset \Lambda_{N_2}(\beta,\beta,\epsilon) \quad \text{and} \quad \widetilde{T}_N(1,\gamma) \subset \widetilde{\Lambda}_{N_2}(\beta,\beta,\epsilon).$$

For any $x \in \Gamma_N(1,\gamma)$, $z \in \pi^{-1}(x)$, it results that

$$d(f^i(z), f_0^i(x)) = d(f^i(z),\, \pi(f^i(x))) < Cr,$$

which implies that if $f_0^i(x) \notin R_1$, then $f^i(z) \notin B(O,r)$ because $d(B(O,r), \partial R_1) > Cr$. Thus

$$\pi^{-1}(\Gamma_N(1,\gamma)) \subset T_N(1,\gamma) \subset \Lambda_{N_2}(\beta,\beta,\epsilon),$$

which yields

$$\pi^{-1}(\widetilde{\Gamma}_N(1,\gamma)) \subset \widetilde{\Lambda}_{N_2}(\beta,\beta,\epsilon).$$

For any ergodic $\nu \in U$, $\pi_*\nu \in U_1$. So $\pi_*\nu(\widetilde{\Gamma}_N(1,\gamma)) > 0$. We obtain

$$\nu(\widetilde{\Lambda}_{N_2}(\beta,\beta,\epsilon)) \geq \nu(\pi^{-1}(\widetilde{\Gamma}_N(1,\gamma))) = \pi_*\nu(\widetilde{\Gamma}_N(1,\gamma)) > 0.$$

The ergodicity of ν concludes that $\nu(\widetilde{\Lambda}(\beta,\beta,\epsilon)) = 1$. □

Here, all ergodic measures near the maximal measure share the same nonuniform hyperbolic structure $\widetilde{\Lambda}$, which in some sense exhibits the rigidity part of the resulting Anosov ones. We further have:

Question 5.7. In Theorem 5.3, does $\mathcal{M}_{inv}(\widetilde{\Lambda}, f)$ admit the same hyperbolic rate?

6. Entropy of the set with historical behavior

In this section we prove that the topological entropy of the set $\widehat{M}$ with historical behavior is bounded below by the metric entropy of hyperbolic measure ω, leading to a chaotic $\widehat{M}$.

Proposition 6.1. *If a given continuous function $\varphi : M \to \mathbb{R}$ satisfies*

$$\inf_{\mu\in\mathcal{M}_{inv}(\widetilde{\Lambda},f)} \int \varphi\, d\mu < \sup_{\mu\in\mathcal{M}_{inv}(\widetilde{\Lambda},f)} \int \varphi\, d\mu,$$

then we have

$$h_\omega(f) \leq \sup\{h_\mu(f) \mid \mu \in \mathcal{M}_{inv}(\widetilde{\Lambda}, f)\} \leq h_{\text{top}}(f, \widehat{M}). \tag{16}$$

Proof. Denote $\sup\{h_\mu(f) \mid \mu \in \mathcal{M}_{inv}(\widetilde{\Lambda}, f)\}$ by a number a. If $a = 0$, then it is trivial. Now suppose $a > 0$. For $\varepsilon > 0$ take $\mu \in \mathcal{M}_{inv}(\widetilde{\Lambda}, f)$ such that $h_\mu(f) > a - \varepsilon$. By assumption, we can take $\mu_1 \neq \mu_2 \in \mathcal{M}_{inv}(\widetilde{\Lambda}, f)$ such that

$$\int \varphi \, d\mu_1 < \int \varphi \, d\mu_2.$$

Then we can choose $\nu_i = \theta_0 \mu + (1 - \theta_0)\mu_i$ for some $\theta_0 \in (0, 1)$ such that

$$h_{\nu_i}(f) = \theta_0 h_\mu(f) + (1 - \theta_0) h_{\mu_i}(f) \geq \theta_0 h_\mu(f) > a - 2\varepsilon, \; i = 1, 2.$$

Note that ν_1, ν_2 differ and still belong to $\mathcal{M}_{inv}(\widetilde{\Lambda}, f)$. Then the set

$$K := \{m | m = \theta \nu_1 + (1 - \theta)\nu_2, \theta \in [0, 1]\} \subseteq \mathcal{M}_{inv}(\widetilde{\Lambda}, f)$$

has the hyperbolic rate for the sequence

$$\eta_l := 1 - \min\{\nu_1(\widetilde{\Lambda}_l), \nu_2(\widetilde{\Lambda}_l)\}.$$

By Theorem 1.1,

$$h_{\text{top}}(f, G_K) = \inf\{h_m(f) \mid m \in K\}.$$

Observe that $\nu_1, \nu_2 \in K$ and

$$\int \varphi d\nu_1 = \theta_0 \int \varphi d\mu + (1 - \theta_0) \int \varphi d\mu_1 < \theta_0 \int \varphi \, d\mu + (1 - \theta_0) \int \varphi d\mu_2 = \int \varphi \, d\nu_2;$$

then $G_K \subseteq \widehat{M}$. Otherwise, for some $x \in G_K$,

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \quad \text{exists.}$$

Take $n_{1,l}, n_{2,l} \to +\infty$ such that

$$\lim_{n \to +\infty} \frac{1}{n_{i,l}} \sum_{j=0}^{n_{i,l}-1} \delta_{f^j(x)} = \nu_i, \quad i = 1, 2.$$

So $\int \varphi d\nu_1 = \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi d\nu_2$, which is a contradiction. $G_K \subseteq \widehat{M}$ implies

$$\begin{aligned} h_{\text{top}}(f, \widehat{M}) &\geq h_{\text{top}}(f, G_K) = \inf\{h_m(f) \mid m \in K\} \\ &= \inf\{\theta h_{\nu_1}(f) + (1 - \theta) h_{\nu_2}(f) | \theta \in [0, 1]\} \\ &> a - 2\varepsilon. \end{aligned}$$

Then (16) follows from the arbitrariness of ε. □

Remark 6.2. If $\mathcal{M}_{inv}(\widetilde{\Lambda}, f)$ contains at least two measures, then the required φ always exists.

Proof of Theorem 1.6. From Theorem 5.3, the NUH sets $\widetilde{\Lambda}$ relate to the maximal entropy measure, so applying Proposition 6.1 gives

$$h_{\text{top}}(f, \widehat{M}) = h_{\text{top}}(f).$$

The transitive nonpartially hyperbolic diffeomorphisms given by Bonatti and Viana [4] conclude Theorem 1.6. □

Linking with a Palis conjecture [32] that every $f \in \mathrm{Diff}^r(M)$ can be C^r-approximated by a Morse-Smale diffeomorphism or one exhibiting a transverse homoclinic intersection, the property $h_{\mathrm{top}}(f, \widehat{M}) > 0$ holds for open dense C^r systems excluded by Morse-Smale diffeomorphisms. This conjecture has been answered positively for C^1 topology. See [3, 11, 36]. This fact leads us to consider:

Question 6.3. Is the property that

$$h_{\mathrm{top}}(f, \widehat{M}) = h_{\mathrm{top}}(f)$$

generic with respect to the C^r topology?

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