TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 369, Number 6, June 2017, Pages 4311–4346
http://dx.doi.org/10.1090/tran/6852
Article electronically published on February 10, 2017

THE FUNDAMENTAL THEOREM OF CUBICAL SMALL CANCELLATION THEORY

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Abstract. We give a new proof of the main theorem in the theory of C(6) small cancellation complexes. We prove the fundamental theorem of cubical small cancellation theory for C(9) cubical small cancellation complexes.

Introduction

Small cancellation theory studies groups with the property that the relators in their group presentation have small overlaps with each other. The theory, initiated by Tartakovskii [7], was developed by Greendlinger and others in the 1960s; however some ideas appeared much earlier, in the work of Dehn, among others. The geometric approach in the study of small cancellation groups, i.e. the use of disc diagrams, was introduced by Lyndon and can be found in Chapter V of [4]. In geometric language, a combinatorial 2-complex satisfies the metric small cancellation condition $C'(\frac{1}{p})$ if each *piece*, i.e. a path arising in two ways as a subpath of 2-cell attaching maps, has length less than $\frac{1}{p}$ of the length of the boundary path of a 2-cell containing the piece. The non-metric small cancellation condition $C(p)$ requires that the boundary path of each 2-cell cannot be expressed as a concatenation of fewer than p pieces. Note that the condition $C'(\frac{1}{p})$ implies the condition $C(p+1)$. The fundamental theorem in the theory takes the following form.

Theorem 1. *Let X be a* C(6)*-complex and $D \to X$ a minimal disc diagram. One of the following holds (see Figure 1):*

- *D is a single cell,*
- *D is a ladder, or*
- *D has at least three spurs or shells of degree ≤ 3.*

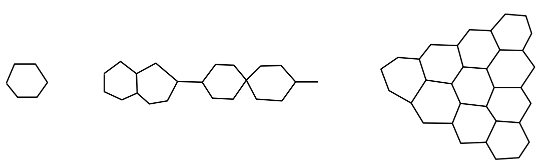

Figure 1. Disc diagrams in C(6)-complexes.

Received by the editors March 30, 2015 and, in revised form, October 10, 2015.
2010 *Mathematics Subject Classification.* Primary 20F65, 20F67.

For details, see Theorem 2.3; all definitions can be found in Section 1. Theorem 1 incorporates a variation of Greendlinger's Lemma as well as a "ladder result" that is a variant of a classical result on annular diagrams. In its initial form, Greendliger's Lemma was used to prove that Dehn's algorithm solves the word problem in $C'(\frac{1}{6})$ small cancellation groups (see [1]). The above formulation was given by McCammond and Wise in [5]. Unlike the proof presented in this paper, their proof uses combinatorial curvature and the combinatorial Gauss-Bonnet formula.

Cubical small cancellation theory is a generalization of classical theory, and it was introduced and developed by Wise in [8]. This builds upon the theory of non-positively curved cube complexes. Instead of a standard group presentation, we use cubical presentation, i.e. express a group as $\pi_1(X)/\langle\langle\{\phi_{i*}(\pi_1(Y_i))\}\rangle\rangle$, where X is a non-positively curved cube complex and each $\phi_i : Y_i \to X$ is a local isometry of cube complexes. Immersed complexes Y_i play the role of relators in classical theory. We now introduce two types of pieces, contained in the intersection either of two "relators" or of a "relator" and the carrier of a hyperplane. Our main result is the following:

Theorem 2. *Let $\langle X, \{Y_i\}\rangle$ satisfy* C(9) *and let $(D, \partial D) \to (X^*, X)$ be a minimal disc diagram. Then one of the following holds:*

- *D is a single vertex or single cone-cell,*
- *D is a ladder, or*
- *D has at least three shells of degree ≤ 4 and/or corners and/or spurs.*

The notation is explained in Section 4.1. The theorem in the case of C(12) is due to Wise and can be found in [8]. Our result partially answers the question on the limits of the theory posed by Wise in section 3.r in [8]. Compared to the proof in [8] our explanation is shorter, self-contained and works for the more general condition C(9) instead of C(12). Wise's approach generalizes the classical case in ways we have not engaged with, but the most important result there is covered here.

The paper is divided into five sections. Section 1 presents some preliminaries; we set up notation and terminology that is used throughout the paper. It also provides an exposition of classical small cancellation theory. In Section 2 we give a new proof of Theorem 1. In Section 3 we will look more closely at non-positively curved cube complexes and prove the following theorem:

Theorem 3. *Let X be a non-positively curved cube complex and $D \to X$ be a minimal disc diagram. Then D is a path graph or it has at least three corners and/or spurs.*

This proof is intended to motivate our approach in the proof of Theorem 2. Section 4 provides the exposition of cubical small cancellation theory. Finally, Section 5 is devoted to our main result, Theorem 2. We first introduce the notion of D-walls, which are the crucial tool in our approach, and then after a few lemmas we proceed with the proof.

1. Basic definitions

In this section we give definitions of classical small cancellation theory, following mainly [4] and [9].

1.1. **Cell complexes.** A map $\phi : X \to Y$ between CW-complexes X, Y is called *combinatorial* if its restriction to any open cell of X is a homeomorphism onto an open cell of Y. A CW-complex X is called *combinatorial* if the attaching map of each open cell in X is combinatorial for some subdivision of the sphere. We will refer to a closed cell as a *cell*. A cell of dimension 0 is called a *vertex*, and a cell of dimension 1 is called an *edge*. Combinatorial map $\phi : X \to Y$ between combinatorial complexes X, Y is a *combinatorial immersion* if it is locally injective.

An *n-cube* is a copy of $[-1, 1]^n$. A *face* of a cube is a subspace obtained by restricting some coordinates to ± 1; faces are cubes of lower dimension. A *cube complex* is a combinatorial complex whose cells are cubes (with subdivision of the boundary consisting of all faces of the cube). A cube of dimension 2 is called a *square*.

A *valence* of a vertex $v \in X$ is the number of edges in X incident to v with loops counted twice. A *path graph* is a 1-complex P which is homeomorphic to an interval (possibly degenerated, i.e. a single point). The value $\#\{vertices\} - 1$ is called the *length* of P and is denoted by $l(P)$. A combinatorial immersion $P \to X$ where P is a path graph is called a *combinatorial path*. The images of vertices of valence 1 in P are called *endpoints* of P. A path graph of length n is denoted by I_n.

1.2. **Disc diagrams.** A *disc diagram* D is a compact, contractible 2-complex with a fixed embedding in the plane. A *disc diagram D in X* is a combinatorial map $D \to X$ where D is a disc diagram. The *boundary path* of D is the attaching map of the 2-cell containing the point at ∞ (regarding $S^2 = \mathbb{R}^2 \cup \infty$). See Figure 2.

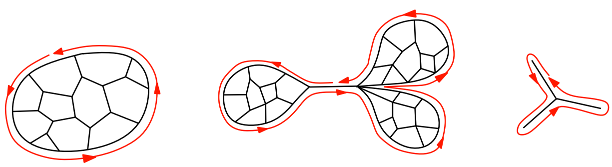

FIGURE 2. Boundary paths.

The number of 2-cells in D is called the *area* of D and is denoted by $\mathrm{area}(D)$. A *minimal disc diagram* is a disc diagram $D \to X$ with the boundary path $P \to X$ such that $\mathrm{area}(D)$ is minimal among all disc diagrams with boundary path P. If a disc diagram is homeomorphic to a disc, it is called *non-singular*. A 2-cell C is a *boundary cell* if $C \cap \partial D \neq \emptyset$ and C is an *internal cell* otherwise. An edge e is a *boundary edge* if $e \subset \partial D$, e is *semi-internal* if $e \cap \partial D \neq \emptyset$ but $e \not\subset \partial D$ and e is *internal* if $e \cap \partial D = \emptyset$. A vertex v is a *boundary vertex* if $v \in \partial D$, and v is an *internal vertex* otherwise. A combinatorial path $P \to D$ of length ≥ 1 with endpoints of valence ≥ 3 in D and all other vertices of valence 2 in D is called an *arc*. Note that every arc is embedded except for endpoints possibly. We call an arc P in D a *boundary arc* if $P \subset \partial D$, and an *internal arc* otherwise. The *internal subdiagram* of D, denoted by Int_D, is the subcomplex consisting of all internal 2-cells and all arcs that intersect ∂D trivially. See Figure 3. A disc diagram which is a cube complex is called a *squared disc diagram*. It has cells of three types: vertices, edges and squares.

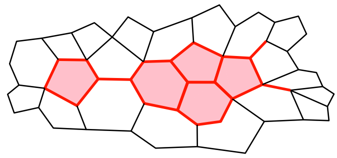

FIGURE 3. The internal subdiagram.

1.3. C(p)**-small cancellation condition.** Let X be a combinatorial 2-complex. A non-trivial combinatorial path $P \to X$ is a *piece* if there are 2-cells C_1, C_2 such that $P \to X$ factors as $P \to \partial C_1 \to X$ and $P \to \partial C_2 \to X$ but there does not exist a homeomorphism $\partial C_1 \to \partial C_2$ such that the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \partial C_1 \arrow[d] \arrow[dl] \\ \partial C_2 \arrow[r] & X \end{tikzcd}$$

commutes. A *maximal piece* is a piece that is not a proper subpath of any piece. Note that in a minimal disc diagram D notions of maximal pieces and of internal arcs coincide. Every internal and semi-internal edge in such a minimal disc diagram is contained in a unique arc, hence in a unique maximal piece.

Let p be a natural number. A 2-complex X is C(p)*-complex* (or it satisfies C(p)*-condition*) if the boundary path of each 2-cell cannot be expressed as a concatenation of fewer than p pieces in X.

1.4. **Spurs and shells.** Let D be a disc diagram. A k*-shell* of D is a 2-cell $C \to D$ whose boundary path $\partial C \to D$ is the concatenation $P_0P_1 \cdots P_k$ for some $k \geq 0$ where P_0 is a boundary arc in D and $P_1, \dots, P_k$ are non-trivial internal arcs in D. The concatenation $P_1 \cdots P_k$ is called the *inner path* of C. The value k is called the *degree* of C. A *spur* in D is a vertex of valence one in D. See Figure 4.

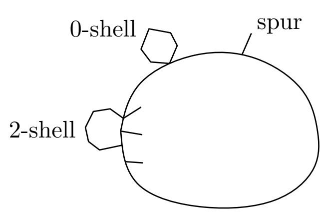

FIGURE 4. Spur, 2-shell and 0-shell.

A cell C in D is called a *disconnecting* cell if $D - C$ is not connected.

1.5. **Ladders.** A *ladder* is a disc diagram L consisting of a sequence of 2-cells and/or vertices $C_1, C_2, \dots, C_n$ ($n \geq 2$) and edges joining them in the following way:

- if $n = 2$ one of the following holds:
  - $L = C_1 \cup_P C_2$, where C_1, C_2 are 2-cells and $P \to C_i$ is an arc for $i = 1, 2$,

  - L consists of C_1, C_2 and an edge e such that $e \cap C_1, e \cap C_2$ are two endpoints of e;
- if $n > 2$ for every $1 < i < n$ there are exactly two connected components L', L'' of $L - C_i$, and subdiagrams $L' \cup C_i, L'' \cup C_i \subset L$ are both ladders.

Cells C_1 and C_n are called *end-cells*. See Figure 5.

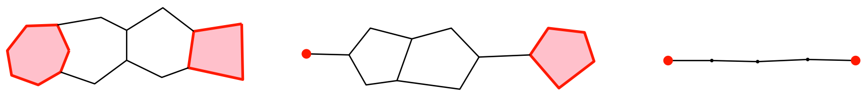

FIGURE 5. Examples of ladders. End-cells are marked.

1.6. ***n*-Greendlinger condition.** We say that a disc diagram D satisfies the *n-Greendlinger condition* if one of the following holds:

- D is a single cell,
- D is a ladder, or
- D has at least three spurs and/or shells of degree $\leq n$, called *exposed cells*.

2. Fundamental theorem of classical small cancellation

The aim of this section is to give a proof of Theorem 1. First we state and prove two lemmas, which will be useful in this proof, as well as later in the case of non-positively curved cube complexes and diagrams in cubical small cancellation complexes. Then we proceed with the proof of Theorem 1.

Lemma 2.1. *Let D be a disc diagram without disconnecting cells. Then either Int_D is a non-trivial disc diagram or D consists of at most two cells. If D is minimal, then so is Int_D.*

Proof. The embedding of Int_D in the plane is induced by the embedding of D. First, suppose that D has a trivial internal subdiagram. We will show that D has a disconnecting cell or consists of ≤ 2 cells. Suppose D is not a single vertex. Let $b : S^1 \to D$ denote the boundary path of D. If b is not an embedding, then either D is a single 1-cell or there exists vertex v such that $|b^{-1}(v)| > 1$. But then v is disconnecting, which contradicts the assumption that D has no disconnecting cells. Suppose b is an embedding. If all the vertices in ∂D have valence two, then D is a single 2-cell. Suppose that there is a vertex $v \in \partial D$ of valence ≥ 3 and denote by P an internal arc in D starting at v. Since D has a trivial internal subdiagram, the other endpoint of P also lies in ∂D. There are two 2-cells C_1, C_2 containing P. Observe that $D - P$ is not connected. If there are any 2-cells in D other than C_1, C_2, then one of C_1, C_2 is disconnecting, a contradiction. Thus if D has more than two cells, then Int_D is non-trivial.

Now let us prove that in this case Int_D is compact and contractible. Let $H : D \times I \to D$ be a homotopy between $H_0 = id_D$ and a constant map $H_1 = p_x$ mapping D to $x \in D$ which exists since D is contractible. There is a well-defined retraction $r : D \to \mathrm{Int}_D$ mapping each internal arc P such that $P \cap \partial D \neq \emptyset$ to its endpoint contained in Int_D (such an endpoint exists since P is not disconnecting) and projecting each boundary 2-cell C onto $C \cap \mathrm{Int}_D$. See Figure 6. We have $r \circ \iota = id_{\mathrm{Int}_D}$ where $\iota : \mathrm{Int}_D \to D$ is the inclusion. Then $r \circ H \circ \iota : \mathrm{Int}_D \times I \to \mathrm{Int}_D$ is a homotopy between $r \circ id_D \circ \iota = id_{\mathrm{Int}_D}$ and a constant map $r \circ p_x \circ \iota = p_{r(x)}$.

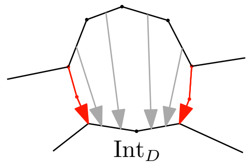

FIGURE 6. Retraction $D \to \mathrm{Int}_D$.

Thus Int_D is contractible. Finally, Int_D is compact since it is the image of the compact space D under the continuous map r.

The minimality of Int_D assuming the minimality of D is immediate. $\square$

We write $D = D_1 \cup_C D_2$ if $D = D_1 \cup D_2$ and $C = D_1 \cap D_2$.

Lemma 2.2. *Let $D = D_1 \cup_C D_2$ be a disc diagram where D_1, D_2 are disc diagrams and C is a single cell. If D_1 and D_2 satisfy the n-Greendlinger condition, then so does D.*

Proof. Suppose C is a 1-cell. If for $i = 1, 2$ the disc diagram D_i is a single 2-cell and $C \subset D_i$, then D is a ladder. Otherwise either C contains a 0-cell which is disconnecting in D or C is contained in a 2-cell which is disconnecting in D. Thus it suffices to consider cases where C is a 0-cell or a 2-cell. If one of D_1, D_2 consists only of C, then there is nothing to show, so we assume that $C \subsetneq D_i$ for $i = 1, 2$. Observe that C is a boundary cell in D_i for $i = 1, 2$, because otherwise D would not embed in the plane. If one of D_1 and D_2 has at least three exposed cells, say D_1, then at least two exposed cells of D_1 remain exposed in D, since we glue along a single cell. Also, at least one of the exposed cells of D_2 remains an exposed cell of D. See Figure 7. If D_1 and D_2 are both ladders or single cells, then one of the

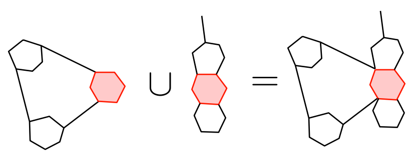

FIGURE 7. We glue disc diagram with three exposed cells and ladder along the marked cell.

following holds:

- the diagram D is a ladder if for both $i = 1, 2$ either D_i is a single 2-cell or C is contained in an end-cell of D_i;
- the diagram D has three exposed cells, otherwise. To see this suppose that D_1 is a ladder and C is not contained in any end-cell of D_1. Then the end-cells of D_1 remain exposed in D, and there is at least one end-cell of D_2 which remains exposed in D. $\square$

Theorem 2.3. *Let X be a* $\mathrm{C}(6)$*-complex and $D \to X$ a minimal disc diagram. Then D satisfies the* 3*-Greendlinger condition.*

Proof. We prove the theorem inductively on the number of 2-cells. It suffices to check the 3-Greendlinger condition for a disc diagram D with no disconnecting cells.

Indeed, if D has a disconnecting cell, i.e. $D = D_1 \cup_C D_2$, then by the induction assumption they both satisfy the 3-Greendlinger condition and by Lemma 2.2 so does D. If D is a single cell, there is nothing to prove. If D consists of two cells, then it is a ladder. From now on we assume that D has no disconnecting cells and that it has ≥ 3 cells. By Lemma 2.1 the internal subdiagram Int_D is a minimal disc diagram with fewer internal cells than D, and by the induction assumption Int_D satisfies the 3-Greendlinger condition.

We now consider different cases depending on what Int_D is and we show that in every case there are at least three exposed cells in D. The case of a ladder is the last one.

- (Int_D is a single vertex v) The valence of v in D is at least 3 (if it was 2, then v would belong to the arc with endpoints in ∂D, so v would not belong to Int_D), so there are at least three 2-cells attached to Int_D; they are 2-shells in D.
- (Int_D is a single 2-cell) There is a 2-cell attached to each arc in Int_D; it is a 3-shell of D. By C(6) there are at least six arcs; hence there are at least six 3-shells in D.
- (Int_D has at least three exposed cells) Let C be an exposed cell of D. If C is a spur, then there is at least one 2-shell in D attached to the endpoint of C. Suppose C is a shell. Let $P_1, \dots P_k$ be arcs in D such that the concatenation $P_1 \cdots P_k$ is a boundary arc of C in Int_D. Since C is an exposed cell in Int_D and D is a C(6) diagram, we have $k \geq 3$. The 2-cell attached to P_2 is a 3-shell in D. See Figure 8. Hence, for any exposed cell C of Int_D there is an exposed cell of D attached to C and to no other cell of Int_D.

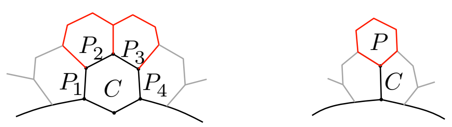

FIGURE 8. The cell C is exposed in the internal subdiagram. The cells P_2, P_3 on the left and P on the right are exposed in D.

- (Int_D is a ladder) First suppose that Int_D is a path graph of non-zero length. Let $\mathrm{Int}_D = P_1 \cdots P_k$ where P_i are arcs in D. There is at least one 2-shell in D attached to each endpoint of Int_D. At most one of the 2-shells attached to P_1 is also attached to P_2, so there is one which is a 3-shell. See Figure 9.

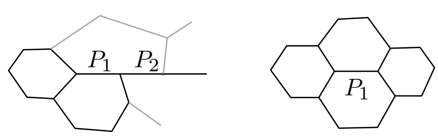

FIGURE 9. The internal subdiagram is a path graph of length ≥ 2 and of length 1.

If Int_D is not a path graph, then there is a 2-cell in Int_D which we assume now. Suppose one of the end-cells of Int_D, say C_1, is a 2-cell in Int_D. Let $P_1, \dots, P_k$ be arcs in D such that $P_1 \cdots P_k$ is a boundary arc of C_1 in Int_D.

There is a 3-shell in D attached to P_i for each $i = 2, \ldots, k-1$. By C(6) we have $k \geq 5$, so there are at least three such 3-shells. If both end-cells are vertices, then there is at least one exposed cell attached to each of them. Denote by i the minimal index such that C_i from the definition of ladder is a 2-cell. There is a boundary arc $P = P_1 \cdots P_k$ of C_i in Int_D where $P_1 \ldots, P_k$ are arcs in D. As before there is a 3-shell in D attached to P_i for each $i = 2, \ldots, k-1$. By C(6) we have $k \geq 3$, so there is at least one such 3-shell, and in total there are at least three 3-shells in D. $\square$

3. Non-positively curved cube complexes

In this section we give a brief exposition of cube complexes, following [2] or [8], and prove Theorem 3. The approach applied here will be later adapted in the proof of the main theorem.

3.1. **Non-positively curved cube complexes.** A *link* of a vertex v in a cube complex X is a complex whose vertices correspond to oriented edges incident to v, and there is an n-simplex spanned on a collection of vertices whenever there is an $(n+1)$-cube $C \to X$ such that corresponding edges in X are images of faces of C containing a vertex $\bar{v}$ which is mapped to v. It can be thought of as an intersection of a small radius sphere around the vertex v in X. A *flag complex* is a simplicial complex such that each set of vertices pairwise connected by edges spans a simplex. A cube complex is called *non-positively curved* if all its vertex links are flag. A CAT(0) *cube complex* is a simply connected, non-positively curved cube complex.

Let X be a non-positively curved cube complex and $D \to X$ a disc diagram in X consisting of three squares incident to a vertex v that are pairwise intersecting along one edge. Since X is non-positively curved, there exists the disc diagram $D' \to X$ with $\partial D = \partial D'$ such that $D \cup_{\partial D = \partial D'} D'$ is the 2-skeleton of a 3-dimensional cube $Q \to X$. The replacement of D by D' is called a *hexagon move*. See Figure 10. For every square $C \to D$ there exists the unique square $\hat{C} \to D'$ such that $C \cap \hat{C} = \emptyset$ in Q. Such $\hat{C}$ is called *opposite* to C.

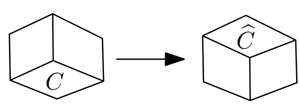

Figure 10. Hexagon move.

3.2. **Corners.** Let $D \to X$ be a disc diagram in a cube complex X. A boundary vertex v of valence 2 contained in some square C in D is a *corner*. The square C is called a *corner-square*. See Figure 11. Note that a corner-square may contain more than one corner.

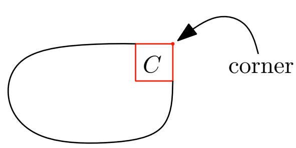

Figure 11. Corner-square C.

3.3. **Hyperplanes.** Let X be a cube complex. A *midcube* is a subspace of a cube $[-1,1]^n$ obtained by restricting one coordinate to 0. A midcube of an edge is called a *midpoint*. Let H be a new cube complex whose cubes are midcubes of X, and attaching maps are restrictions of attaching maps in X to midcubes. A connected component Γ of H is called an *immersed hyperplane*.

There is a natural immersion $\Gamma \to X$, and we will often think of hyperplanes as subspaces of X. See Figure 12. An immersed hyperplane Γ is said to be *dual* to an edge e if a midpoint of e is a vertex of Γ. The immersed hyperplane dual to e is denoted by $\Gamma(e)$. We say that edges e, e' are *parallel* if $\Gamma(e) = \Gamma(e')$.

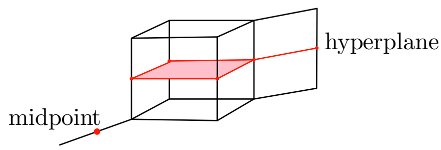

FIGURE 12. Hyperplane.

The *carrier* $N(\Gamma)$ of an immersed hyperplane Γ in X (or of a subcomplex Γ of an immersed hyperplane) is a cube complex defined as follows: for each cube C in Γ we take the copy of the cube in X whose midcube is C, and two such cubes are attached to each other along faces if corresponding midcubes are attached to each other in Γ along midcubes of these faces. By the construction, we have a map $\iota : N(\Gamma) \to X$. Whenever ι is an embedding we write $N(\Gamma)$ instead of $\iota\big(N(\Gamma)\big)$.

The immersed hyperplanes in a squared disc diagram are immersed path graphs. Suppose Γ is an immersed hyperplane in a squared disc diagram D such that $\iota : N(\Gamma) \to D$ is an embedding. Denote by K one of two connected components of $D - \Gamma$. We define the Γ-*component* corresponding to K as $K \cup N(\Gamma)$; i.e. this is the minimal subdiagram of D which contains K. See Figure 13.

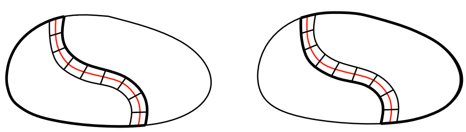

FIGURE 13. The Γ-components.

Let D be a squared disc diagram with no disconnecting cells and non-trivial internal subdiagram. A hyperplane Γ is *collaring* if it is not dual to any internal edge in D. We say that D is *collared* if $\Gamma(e)$ is collaring for every semi-internal edge e. We say that D is *collared by* $\{\Gamma_1, \dots, \Gamma_n\}$ if for every semi-internal e there exists i such that $\Gamma(e) = \Gamma_i$ and all Γ_i are collaring.

Lemma 3.1. *Let D be a squared disc diagram with no disconnecting cells that has at least one corner. The following are equivalent* (*see Figure* 14)*:*

(1) *D is collared,*
(2) *every hyperplane dual to an edge containing a corner of D is collaring,*
(3) *all boundary vertices of D have valence ≤ 3.*

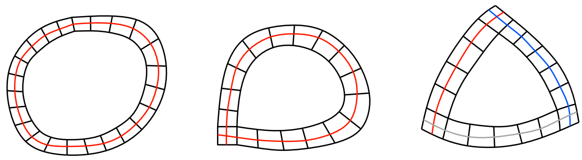

FIGURE 14. First two disc diagrams are collared by a single hyperplane, and the last disc diagram is bounded by three hyperplanes.

Proof. (1) $\Rightarrow$ (2) This implication is trivial.

(2) $\Rightarrow$ (3) If a boundary vertex v has valence > 3, then no hyperplane dual to a semi-internal edge containing v is collaring. Let P be a minimal subpath of ∂D with corners of D as endpoints and denote by $v_1, \ldots, v_n$ the consecutive vertices of P. Let $1 < i < n$ be the minimal number such that valence of v_i in D is > 3. Let e be the edge in D that contains v_1 but is not contained in P. The hyperplane $\Gamma(e)$ is not collaring. See Figure 15.

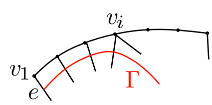

FIGURE 15. The hyperplane $\Gamma(e)$ dual to an edge e containing corner v_1 is not collaring.

(3) $\Rightarrow$ (1) If D is not collared, then there exists an immersed hyperplane dual to some internal edge and some semi-internal edge in D. There exists a square C in D such that one of its edges e is semi-internal and the opposite one $\bar{e}$ is internal. Denote by v the boundary vertex contained in e. The valence of v is ≥ 4, because otherwise $\bar{e}$ would contain a boundary vertex. $\square$

3.4. **Disc diagrams in non-positively curved cube complexes.**

Theorem 3.2. *Let X be a non-positively curved cube complex and $D \to X$ a minimal disc diagram. Then D is a path graph or it has at least three corners and/or spurs.*

Proof.

Step 1. It suffices to verify the 2-Greendlinger condition.

Indeed,

- if D is a single 0-cell or a ladder consisting only of 1-cells, then D is a path graph;
- if D is a single square or a ladder with at least one 2-cell or D has at least three shells of degree ≤ 2 and/or spurs, then D has at least three corners and/or spurs. To see that note that shells of degree ≤ 2 in D are corner-squares.

We show that the 2-Greendlinger condition is satisfied by induction on the number of cells.

Step 2. All cells in D are embedded and the intersection of two cells consists of exactly one cell.

Since X is non-positively curved, no square in D has two consecutive edges glued, because otherwise one of the vertex links in X would contain a loop. Similarly if there are two squares with ≥ 2 consecutive common edges in D, then since X is non-positively curved they are mapped to the same square in X; thus D is not minimal. See Figure 16. Suppose that S is a non-embedded square in D. Let

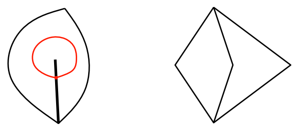

FIGURE 16. These cannot be minimal disc diagrams in a non-positively curved cube complex.

$P \to \partial S$ be a minimal subpath whose endpoints are mapped to the same point in D such that S is not contained in the subdiagram D' of D bounded by P. See the left diagram in Figure 17. By the minimality of P the diagram D' has no

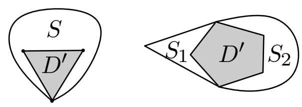

FIGURE 17. Subdiagram D'.

spurs. Since two squares cannot have two consecutive edges in common, the only possible corner of D' is the endpoint of P. Thus D' contradicts the induction assumption. Now suppose that there are two cells S_1, S_2 in D whose intersection is not connected. Let $P_1 \to \partial S_1, P_2 \to \partial S_2$ be minimal subpaths with common endpoints in D such that none of S_1, S_2 is contained in the subdiagram D' of D bounded by the concatenation of P_1 and P_2. See the right diagram in Figure 17. As before there are no spurs in D' and there are at most two corners, so D' contradicts the induction assumption. Similarly all edges are embedded and every two edges have at most one common vertex.

Step 3. It suffices to verify the 2-Greendlinger condition for $D \to X$ that has no disconnecting cells.

It follows from Lemma 2.2. If D is a single cell, there is nothing to prove. If D consists of two cells, then it contains a disconnecting cell. Thus we can restrict our attention to diagrams with ≥ 3 squares. In such case we need to show that D has ≥ 3 corner-squares.

Step 4. From now on, we assume that D has no disconnecting cells. The carriers of immersed hyperplanes in D embed.

Suppose to the contrary that Γ is an immersed hyperplane such that $N(\Gamma)$ does not embed. Let Γ' be a minimal subpath of Γ such that $N(\Gamma')$ does not embed. See Figure 18. Denote by $D_{\Gamma'}$ the minimal disc diagram that contains $N(\Gamma')$ and

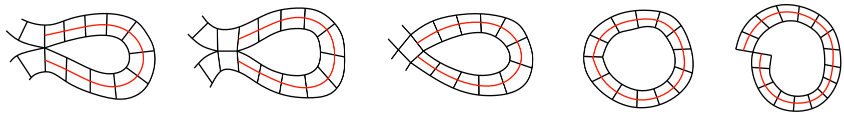

FIGURE 18. The hyperplane Γ where $N(\Gamma)$ does not embed.

the *internal* connected component of $D - N(\Gamma')$, i.e. the unique component which has trivial intersection with ∂D. One of the following holds:

- The diagram $D_{\Gamma'}$ is a proper subdiagram of D, so $D_{\Gamma'}$ has fewer cells than D. But D' has at most two corner-squares (images of end-cells of $N(\Gamma')$ possibly) and no spurs; hence we get a contradiction with the induction assumption.
- We have $D_{\Gamma'} = D$. Set S to be any square in $N(\Gamma')$ that is not a corner-square and let Γ_S be the immersed hyperplane dual to the unique boundary edge of S. See Figure 19. If $N(\Gamma_S)$ is not embedded, then we proceed as

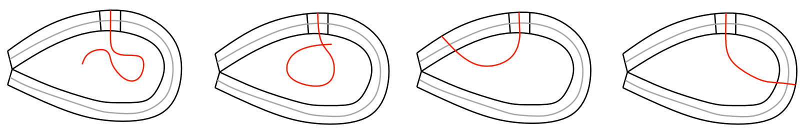

FIGURE 19. The diagram $D = D_{\Gamma'}$ and hyperplane Γ_S.

in the previous case and get a proper subdiagram D_{Γ_S} of D with at most two corner-squares and no spurs, a contradiction. Thus $N(\Gamma_S)$ embeds. One of Γ_S-components has 2 exposed squares and fewer cells than D, a contradiction with the induction assumption.

Thus carriers of hyperplanes embed in D.

Step 5. The diagram D satisfies the 2-Greendlinger condition.

Suppose, contrary to the 2-Greendlinger condition, that D has at most two corner-squares. We now show that in this case D is collared. If D had no corner-squares, then any $\Gamma(e)$-component, for any boundary edge e, would have ≤ 2 corner-squares, no spurs and obviously fewer cells than D, which would contradict the induction assumption. Thus D has some corner-squares. By Lemma 3.1 it suffices to verify that all hyperplanes dual to edges containing corners are collaring. Let e' be such an edge. If $\Gamma(e')$ was not collaring, then one of $\Gamma(e')$-components would have at most two corner-squares, no spurs and fewer cells than D, which is impossible by the induction assumption. See Figure 20. Thus D is collared.

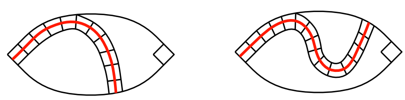

FIGURE 20. The hyperplane $\Gamma(e')$.

By Lemma 2.1 the internal subdiagram Int_D is a minimal disc diagram, so by the induction assumption it satisfies the 2-Greendlinger condition, and therefore it is a path graph or it has at least three corners and/or spurs. Let us consider these two cases separately.

- (Int_D has at least three corners and/or spurs) There is a corner-square of D attached to each spur of Int_D, since valence in D of a spur of Int_D is ≥ 3. Suppose that there is a corner v in Int_D, and let S be a square in Int_D containing v. If the valence of v in D is ≥ 4, then there is a corner-square of D containing v. See Figure 21. If the valence is 3, then by a hexagon move applied to squares containing v we obtain a disc diagram D' with the same number of cells as D and $\partial D = \partial D'$. Since D is collared, by Lemma 3.1 all boundary vertices in D have valence ≤ 3, so square $\widehat{S}$ opposite to S is a corner-square of D' and the diagram $\overline{D' - \widehat{S}}$ has ≤ 2 corners and no spurs, a contradiction. See Figure 21.

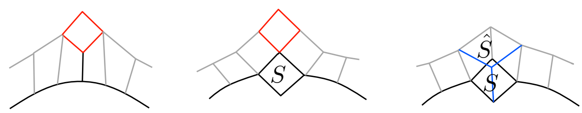

FIGURE 21. Internal diagram contains a spur, a corner whose valence in D is ≥ 4, or a corner whose valence in D is 3.

- (Int_D is a path graph) There are corner-squares S_1, S_2 each incident to one endpoint of Int_D. Since D is collared by Lemma 3.1, the diagram D has no boundary vertices of valence > 3. Since S_1, S_2 are the only corner-squares in D, we have $\overline{D - (S_1 \cup S_2)} = I_2 \times I_n$ where $n \geq 1$ is the length of Int_D. See the left diagram in Figure 22. Applying a hexagon move to squares containing an endpoint of Int_D we obtain a diagram with a proper subdiagram D' with two corners and no spurs, which is a contradiction and completes the proof. See the right diagram in Figure 22. □

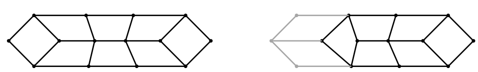

FIGURE 22. On the left, the diagram D where Int_D is a path graph of length 3. On the right, the diagram D' obtained from D by a hexagon move.

Corollary 3.3. *If X is a non-positively curved cube complex and $D \to X$ is a minimal non-singular disc diagram, then D has at least three corners.*

3.5. **Convexity.** Let X be a CAT(0) cube complex. A subcomplex $Y \subset X$ is *convex* if for any vertices $v, v' \in Y$ every combinatorial path $P \to X$ of minimal length with endpoints v and v' is contained in Y. A combinatorial immersion $\phi : Y \to X$ of cube complexes is called a *local isometry* provided that for any pair of edges e, e' incident to a vertex v in X, if the vertices in the link of $\phi(v)$ corresponding to $\phi(e), \phi(e')$ are adjacent, then the vertices corresponding to e, e' in the link of v are also adjacent. Note that if X is a non-positively curved cube complex and Γ is an immersed hyperplane in X, then $\iota : N(\Gamma) \to X$ is a local isometry.

Lemma 3.4. *Let X, Y be non-positively curved cube complexes and let $\phi : Y \to X$ be a local isometry. Then the lift $\widetilde{\phi} : \widetilde{Y} \to \widetilde{X}$ is an embedding and the image is a convex subcomplex of $\widetilde{X}$, where $\widetilde{X}, \widetilde{Y}$ are universal covers of X, Y.*

Proof. Let $v, v' \in \widetilde{Y}$ be vertices. It suffices to verify that any minimal length combinatorial path in $\widetilde{X}$ joining $\widetilde{\phi}(v), \widetilde{\phi}(v')$ is the image under $\widetilde{\phi}$ of a minimal length combinatorial path in $\widetilde{Y}$ joining v, v'. Let $D \to \widetilde{X}$ be a minimal disc diagram with boundary path $\widetilde{\phi}(\beta)\bar{\gamma}$, where β is a minimal length combinatorial path in $\widetilde{Y}$ joining v, v' and γ is a minimal length combinatorial path joining $\widetilde{\phi}(v), \widetilde{\phi}(v')$ ($\bar{\gamma}$ denotes the path γ with reversed direction). By induction on area(D) over pairs β, γ we show that there exists a minimal length combinatorial path α in $\widetilde{Y}$ such that $\widetilde{\phi}(\alpha) = \gamma$. If area$(D) = 0$, then by Theorem 3.2:

- either D is a path graph with endpoints $\widetilde{\phi}(v), \widetilde{\phi}(v')$, and what follows $\widetilde{\phi}(\beta) = \gamma$,
- or there is a spur w in D, distinct from v, v'. If $w \in \gamma$, then the length of γ is not minimal. Otherwise, if $w \in \widetilde{\phi}(\beta)$, then since $\widetilde{\phi}$ is a combinatorial immersion, the length of β is not minimal.

Now suppose that area$(D) = n > 0$. By Theorem 3.2 there are at least three corners and/or spurs in D, so at least one corner or spur distinct from v, v'; let us denote it by w. If w is a spur, we conclude as before that the length of one of γ, β is not minimal. Thus w is a corner; denote by S the square in $\widetilde{X}$ containing w. If $w \in \widetilde{\phi}(\beta)$, then since $\widetilde{\phi}$ is a local isometry, there is a square S' in $\widetilde{Y}$ such that $\widetilde{\phi}(S') = S$. Let $e_1e_2e_3e_4 = \partial S'$ such that $\phi(e_1), \phi(e_2)$ both contain w and their concatenation e_1e_2 is a subpath of β. Let β' be the path obtained from β by replacing e_1e_2 by $\bar{e_4}\bar{e_3}$ ($\bar{e}$ denotes the edge e with reversed direction). See Figure 23. The path β' is joining v, v' in $\widetilde{Y}$, and the minimal area of the disc diagram with

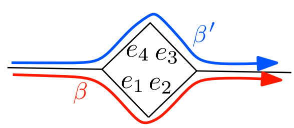

FIGURE 23. Replacing e_1e_2 by $\bar{e_4}\bar{e_3}$.

the boundary path $\widetilde{\phi}(\beta')\bar{\gamma}$ is equal to $n - 1$. By induction assumption there is a minimal length combinatorial path α in $\widetilde{Y}$ such that $\widetilde{\phi}(\alpha) = \gamma$.

Now suppose $w \in \gamma$ and let $e_1e_2e_3e_4 = \partial S$ such that e_1, e_2 both contain w and their concatenation e_1e_2 is a subpath of γ. Let γ' be the path obtained from γ by replacing subpath e_1e_2 by $\bar{e_4}\bar{e_3}$. The minimal area of a disc diagram with the boundary path $\widetilde{\phi}(\beta)\bar{\gamma}'$ is $n-1$. Thus by the induction assumption there is a minimal length combinatorial path α' in $\widetilde{Y}$ such that $\widetilde{\phi}(\alpha') = \gamma'$. Since $\widetilde{\phi}$ is a local isometry, there exists a minimal length combinatorial path α in $\widetilde{Y}$ such that $\widetilde{\phi}(\alpha) = \gamma$. $\square$

Corollary 3.5. *The carrier $N(\Gamma)$ of a hyperplane Γ in a* CAT(0) *cube complex X is a convex subcomplex.*

4. Cubical small cancellation theory

The aim of this section is to describe the cubical small cancellation theory, due to Wise, which is a generalization of the classical small cancellation theory. In the beginning we define cubical presentations introduced by Wise, following [6], [10] or [8]. Secondly, we introduce the notion of pseudorectangles and use it to define ladders. This definition is consistent with the one given in [8], but is more general than the one in [10] and [6], which is equivalent to our definition with the restriction that the joining pseudorectangles are actual rectangles. Then, we define cone-pieces and hyperplane-pieces. There are several equivalent definitions of pieces in cubical presentation complexes; here we follow [6]. This allows us to formulate cubical small cancellation conditions. Notice that we use different terminology than in [6]; i.e. our hyperplanes-pieces are referred to as *wall-pieces* there. Finally, we introduce D-pieces for a disc diagram D in a cube complex X, which are aimed to correspond to pieces in X.

4.1. **Cubical presentation.** A *cubical presentation* $\langle X, \{Y_i\}\rangle$ consists of a non-positively curved cube complex X and a family of local isometries of cube complexes $\phi_i : Y_i \to X$. The group G assigned to a cubical presentation is the quotient

$$G = \pi_1(X)/\langle\langle\{\phi_{i*}\big(\pi_1(Y_i)\big)\}\rangle\rangle.$$

Let

$$X^* = X \cup \bigcup_i \mathrm{Cone}(Y_i)/\{(y_i, 0) \sim \phi_i(y_i) \text{ for all } y_i \in Y_i\},$$

where $\mathrm{Cone}(Y) = Y \times [0,1]/Y \times \{1\}$. Then we have $G = \pi_1(X^*)$. We regard X^* as a cell complex with cells divided into two families: cubes and pyramids (i.e. cones on single cubes). We will refer to X^* as a *presentation complex*. There is a natural combinatorial inclusion $X \to X^*$. The vertices of X^* which are not contained in X are called *cone-points*. By the van Kampen lemma (see [3]), for every closed combinatorial path $P \to X$ such that the composition $P \to X \to X^*$ is null-homotopic, there exist a disc diagram $(D, \partial D) \to (X^*, X)$ with boundary path P. The 2-cells of D are either squares of X or triangles (i.e. cones on edges) in the cone $\mathrm{Cone}(Y_i)$ for some i. The points which are mapped to cone-points in X^* are also called *cone-points* of D. Triangles in D are grouped together into cyclic families meeting around a cone-point v; such families form polygons which we call *cone-cells*. From now on we regard D as a cell complex with 2-cells divided into two families: squares and cone-cells. We define the *complexity* of a disc diagram D as the following:

$$\mathrm{Comp}(D) = (\#\text{cone-cells}, \#\text{squares}).$$

The disc diagram $(D, \partial D) \to (X^*, X)$ is called *minimal* if $\mathrm{Comp}(D)$ is minimal in the lexicographical order among disc diagrams with the same boundary path as D. Whenever the boundary path of a cone-cell C in D has a spur, we can replace C by a cone-cell with this spur removed without changing the complexity of D. Thus we can assume that the boundary path of each cone-cell is immersed.

Example. Let X be a wedge of circles labelled by $x_1, \ldots, x_n$. Suppose Y_i are immersed closed combinatorial paths; i.e. Y_i corresponds to a cyclically reduced word r_i in alphabet $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$. Then X^* of the cubical presentation $\langle X, \{Y_1, \ldots, Y_m\}\rangle$ is the standard presentation complex associated to the group presentation $\langle x_1, \ldots, x_n | r_1, \ldots, r_m\rangle$.

4.2. **Pseudorectangles and ladders.**

Definition 4.1. A *rectangle* is a squared disc diagram isometric to $I_n \times I_m$ for some natural numbers n, m. A *pseudorectangle* is a square disc diagram R with $\partial R = e_1 \cdots e_n f_1 \cdots f_k e'_n \cdots e'_1 g_l \cdots g_1$ (where $n \geq 1$, $k, l \geq 0$) such that

- for every $i = 1, \dots, n$ we have $\Gamma(e_i) = \Gamma(e'_i)$,
- for $i \neq j$ we have $\Gamma(e_i) \cap \Gamma(e_j) = \emptyset$, and
- the concatenation $e_n f_1 \cdots f_k e'_n$ is a path in $N\big(\Gamma(e_n)\big)$, and $e'_1 g_l \cdots g_1 e_1$ is a path in $N\big(\Gamma(e_1)\big)$.

See the left diagram in Figure 24. Paths $e_1 \cdots e_n$ and $e'_n \cdots e'_1$ are called the *(opposite) sides* of a pseudorectangle R.

Let D be a squared disc diagram with $e_1 e_2, e'_1 e'_2 \subset \partial D$ such that for $i = 1, 2$ we have $\Gamma(e_i) = \Gamma(e'_i)$ and $\Gamma(e_1) \cap \Gamma(e_2) = \emptyset$. The subdiagram K *lying between* $\Gamma(e_1)$ *and* $\Gamma(e_2)$ is the maximal subdiagram in the unique connected component of $\overline{D - \Gamma(e_1) \cup \Gamma(e_2)}$ that intersects both $N\big(\Gamma(e_1)\big), N\big(\Gamma(e_2)\big)$. See the right diagram in Figure 24.

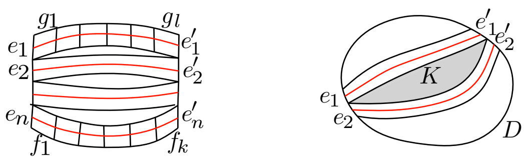

FIGURE 24. On the left, a pseudorectangle. On the right, the subdiagram K consists of all squares lying between $\Gamma(e_1)$ and $\Gamma(e_2)$ in D .

Lemma 4.2. *Let $D \to X$ be a minimal disc diagram in a non-positively curved cube complex X. Suppose D is a pseudorectangle, as in Definition* 4.1*. Then $k = l$ and all squares lying between hyperplanes $\Gamma(e_i)$ $(i = 1, \dots, n)$ can be pushed upward; i.e. there exists a disc diagram D' obtained from D by a sequence of hexagon moves such that one of the $\Gamma(e_1)$-components of D' is a rectangle with sides $e_1 \cdots e_n$ and $e'_n \dots e'_1$. See Figure* 25*.*

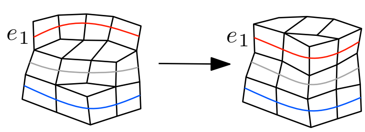

FIGURE 25. Pushing squares upward.

Proof. It is immediate that it suffices to prove this lemma for $n = 2$. Denote by m the number of squares in the subdiagram lying between $\Gamma(e_1)$ and $\Gamma(e_2)$. We will construct a sequence of diagrams $D = D_m, \dots, D_0$ with the following properties:

- D_i is obtained from D_{i+1} by a single hexagon move,
- the subdiagram K_i of D_i lying between $\Gamma(e_1)$ and $\Gamma(e_2)$ has exactly i squares.

By definition of m the diagram D_m satisfies the second property. Suppose we have already defined $D_m, \ldots, D_{i+1}$ (where $i = 0, \ldots, m-1$); let us define D_i. By Theorem 3.2 the diagram K_{i+1} has at least three corners and/or spurs. Denote by v_{i+1} one that is distinct from $e_1 \cap e_2$ and $e_1' \cap e_2'$. If v_{i+1} was a spur, there would be two squares with two consecutive common edges, which is impossible by the minimality of D. See Figure 26. Thus v_{i+1} is a corner. Denote by C the square

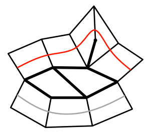

FIGURE 26. A spur in K_i violates the minimality of D.

in K_{i+1} containing v_{i+1}. There are two more squares in D_{i+1} containing v_{i+1}; they are contained in the carrier of exactly one of the hyperplanes $\Gamma(e_1), \Gamma(e_2)$. We perform a hexagon move at v_{i+1}. We set D_i to be the resulting diagram and we denote by C_i the square opposite to C. See Figure 27. In D_0 the diagram

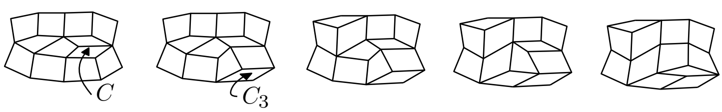

FIGURE 27. The sequence of diagrams $D = D_4, \ldots, D_0$.

$N\big(\Gamma(e_1)\big) \cup N\big(\Gamma(e_2)\big)$ is a rectangle; i.e. K_0 is a path graph, because otherwise there would be three spurs in K_0, which would contradict the minimality of D_0. Note that C_j remains in all D_t for $t \le j$, in particular in D_0. Let $n_1, \ldots, n_h$ be a subsequence of $1, \ldots, m$ consisting of exactly those numbers for which C_{n_j} was obtained by pushing a square downward; i.e. they appeared in steps where the hexagon move was applied to squares such that two of them are contained in $N(\Gamma(e_2))$. In other words, C_{n_j} intersects $N(\Gamma(e_2))$ in D_{n_j} $\Big($and what follows, C_{n_j} does not intersect $N\big(\Gamma(e_1)\big)$ in D_{n_j} and what follows also in D_t for $t < n_j\Big)$. Now squares pushed downward will be pushed "back". We define a sequence of disc diagrams $D_0 = D_0', \ldots, D_h'$ in the following way: the diagram D_{j+1}' is obtained from D_j' by pushing square C_{n_j} upward; i.e. we first apply a hexagon move to C_{n_j} and two uniquely determined squares in $N\big(\Gamma(e_2)\big)$ which meet C_{n_j} and then we apply a second hexagon move to the square $\widehat{C}_{n_j}$ opposite to C_{n_j} and two uniquely determined squares in $N\big(\Gamma(e_1)\big)$ which meet $\widehat{C}_{n_j}$. See Figure 28. Note that the first hexagon move is the "inverse" of the hexagon moves performed in the definition of

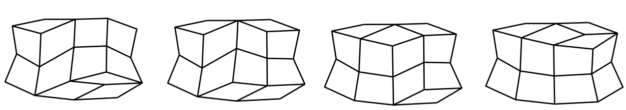

FIGURE 28. Pushing squares "back". The sequence of diagrams $D_0', \ldots, D_h'$.

D_{n_j} and can be performed because all other hexagon moves performed in the first part (i.e. those used in the definition of D_i for $i = n_j$ $j = 1, \dots, h$) leave unchanged $N(\Gamma(e_2))$ and all squares already pushed downward. In each D'_j the subdiagram $N\big(\Gamma(e_1)\big) \cup N\big(\Gamma(e_2)\big)$ is a rectangle, and so the second hexagon move is also well defined. We set $D' = D'_h$ and we are done. □

Definition 4.3. A *ladder* is a minimal disc diagram $(L, \partial L) \to (X^*, X)$ in a presentation complex X^* consisting of a sequence of cone-cells and/or vertices $C_1, \dots, C_n$ $(n \geq 2)$ and square complexes joining them in the following sense:

- if $n = 2$ one of the following holds (see Figure 29):
  (1) C_1 and C_2 are cone-cells glued along a vertex v, i.e. $L = C_1 \cup_v C_2$, or
  (2) C_1 and C_2 are joined by a single edge e where $e \cap C_1, e \cap C_2$ are two vertices of e, or
  (3) the diagram consists of a pseudorectangle R and cone-cells C_1, C_2 each attached to one side of R;

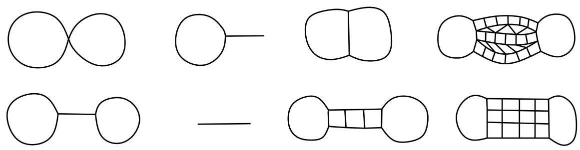

FIGURE 29. Ladders for $n = 2$.

- if $n \geq 3$, then for each $1 < i < n$ there are exactly two connected components L' and L'' of $L - C_i$, and subdiagrams $L' \cup C_i, L'' \cup C_i \subset L$ are both ladders (see Figure 30).

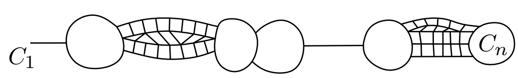

FIGURE 30. Example of a ladder.

The cells C_1, C_n are called *end-cells* of L.

4.3. **Pieces.** Given a map $\phi : Y \to X$ an *elevation* of Y to the universal cover $\widetilde{X}$ of X is a map $\widetilde{Y} \to \widetilde{X}$ which covers $Y \to X$ such that $\widetilde{Y}$ is the covering space corresponding to $\ker \phi_*$, where $\phi_* : \pi_1(Y) \to \pi_1(X)$ is induced by ϕ. Note that such $\widetilde{Y}$ is the universal cover of Y whenever ϕ_* is injective. An *abstract cone-piece* in Y_i of Y_j is the intersection $P = \widetilde{Y}_i \cap \widetilde{Y}'_j$ for some elevations $\widetilde{Y}_i, \widetilde{Y}'_j$ of Y_i, Y_j to the universal cover $\widetilde{X}$ of X. In the case where $i = j$ we require that for the projections $P \to Y_i, P \to Y_j$ there is no automorphism $Y_i \to Y_j$ such that the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y_i \arrow[d] \arrow[dl] \\ Y_j \arrow[r] & X \end{tikzcd}$$

commutes. An *abstract hyperplane-piece* in Y_i is the intersection $\widetilde{Y}_i \cap N(\widetilde{A})$, where $\widetilde{A}$ is a hyperplane in $\widetilde{X}$ such that $\widetilde{A} \cap \widetilde{Y}_i = \emptyset$. An *abstract piece* is an abstract cone-piece or an abstract hyperplane-piece. A path $\alpha \to Y_i$ is a *piece* (respectively, a *cone-piece* or a *hyperplane-piece*) in Y_i if it lifts to $\widetilde{Y}_i$ into an abstract piece (respectively, an abstract cone-piece, or an abstract hyperplane-piece) in Y_i. A closed path is *essential* if it is not homotopic to a constant map. The cubical presentation $\langle X, \{Y_i\}\rangle$ satisfies the C(p)-*small cancellation condition* if no essential closed path in Y_i can be expressed as a concatenation of fewer than p pieces.

Let $(D, \partial D) \to (X^*, X)$ be a minimal disc diagram. A D-*cone-piece* in a cone-cell C is a subpath P of ∂C which lies in $C \cap C'$ for some cone-cell $C' \neq C$ in D. See Figure 31. A D-*hyperplane-piece* in C is a subpath P of ∂C such that there exists a

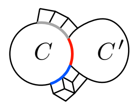

FIGURE 31. D-cone-piece and D-hyperplane-pieces.

diagram D' obtained from D by a sequence of hexagon moves and a rectangle $I_n \times I_1$ in D' with $P = I_n \times \{1\}$. A D-*piece* is a D-cone-piece or a D-hyperplane-piece. See Figure 31. Note that any subpath of a D-piece is a D-piece.

For every D-cone-piece in C there exists a unique maximal D-cone-piece containing it, but in general this is not true for the D-hyperplane-pieces. See Figure 32.

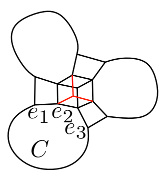

FIGURE 32. The maximal D-hyperplane-pieces in C are e_1e_2 and e_2e_3.

Lemma 4.4. *Let $\psi : (D, \partial D) \to (X^*, X)$ be a minimal disc diagram in the presentation complex X^*. Then every D-piece in a cone-cell C corresponding to Y_i is mapped under ψ to a piece in Y_i.*

Proof. First suppose that P is a D-cone-piece of C' in C, where C corresponds to Y_i and C' to Y_j. If $i \neq j$, then there is nothing to check. Suppose that $i = j$ and P is not mapped to a piece; i.e. there is an automorphism $\phi : Y_i \to Y_j$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "cp"] \arrow[d, "c'p'"'] & Y_i \arrow[d] \arrow[dl, "\phi"] \\ Y_j \arrow[r] & X \end{tikzcd}$$

commutes, where $p : P \to \partial C$, $c : \partial C \to Y_i$, $p' : P \to \partial C'$, $c' : \partial C' \to Y_j$. By the universal property of amalgamated sum, there exists $\mu : C_1 \cup_P C_2 \to \mathrm{Cone}(Y_j)$ such that the diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & C \arrow[d] \arrow[ddr, bend left, "\mathrm{Cone}(\phi)\mathrm{Cone}(p)"] & \\
C' \arrow[r] \arrow[drr, bend right, "\mathrm{Cone}(p')"'] & C \cup_P C' \arrow[dr, "\mu"] & \\
& & \mathrm{Cone}(Y_j)
\end{tikzcd}$$

commutes. We can replace C and C' by a single cone-cell contained in $\mathrm{Cone}(Y_j)$ and get a diagram $D' \to X$, with $\partial D' = \partial D$ and $\mathrm{Comp}(D') <_{lex} \mathrm{Comp}(D)$, which contradicts the minimality of D.

Now suppose P is a D-hyperplane-piece in C, and let $R = I_n \times I_1$ be the rectangle in diagram D' obtained from D by a sequence of hexagon moves such that $P = I_n \times \{1\}$. Let e be any edge in $\psi(R)$ which has a vertex in Y_i, but is not contained in Y_i. Let $\widetilde{Y}_i$ be some elevation of Y_i and $\tilde{e}$ a lift of e with a vertex $\tilde{v}$ in $\widetilde{Y}_i$. Then $\tilde{e}$ is not contained in $\widetilde{Y}_i$, since e was not contained in Y_i. It suffices to check that $\Gamma(\tilde{e})$ does not intersect $\widetilde{Y}_i$. Suppose the contrary and denote by $\tilde{e}'$ an edge in $\widetilde{Y}_i$ dual to $\Gamma(\tilde{e})$. See Figure 33. Observe that, by Lemma 3.4, $\widetilde{Y}_i$ is a convex subcomplex of $\widetilde{X}$ and by Corollary 3.5, also $N\big(\Gamma(\tilde{e})\big)$ is convex. The intersection $N\big(\Gamma(\tilde{e})\big) \cap \widetilde{Y}_i$ contains $\tilde{v}$ and $\tilde{e}'$, and since it is convex as an intersection of convex complexes it also contains $\tilde{e}$, which is a contradiction. $\square$

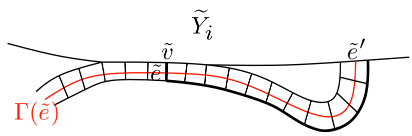

FIGURE 33. The thickened path is minimal, so it is contained in $\widetilde{Y}_i$, since $\widetilde{Y}_i$ is convex.

Having defined D-pieces in a disc diagram D, we can adapt the notion of shells. A non-disconnecting boundary cell C is a *shell of degree k* (or a *k-shell* for short) if k is the minimal number such that the inner path P of C can be expressed as a concatenation of k D-pieces. The degree of a cone-cell C in D is denoted by $\deg_D(C)$. In the statement of the main result we will also use the notion of corners as defined in Section 3.2 and spurs as in Section 1.4.

Lemma 4.5. *If C is a k-shell, then for every $i = 1, \dots, k$ there exists an edge e in the inner path P of C, such that $e \subset P_i$ for any decomposition $P = P_1 \cdots P_k$ into D-pieces.*

Proof. Suppose the contrary and let i be minimal such that the intersection $\bigcap P_i$ over all decompositions $P = P_1 \cdots P_k$ into D-pieces contains no edges. Since all P_i are connected subpaths of P it follows that there exist two decompositions $P_1 \cdots P_k$

and $Q_1 \cdots Q_k$ of P into D-pieces such that $P_i \cap Q_i$ contains no edges. Without loss of generality we can assume that Q_i occurs in P before P_i. Then

$$P_j \cap Q_l \neq \emptyset \text{ for some } j < i \text{ and } l > i.$$

Observe that

$$P_1 \cdots P_j \cdot \overline{Q_l - P_j} \cdot Q_{l+1} \cdots Q_k$$

is a decomposition of P into at most $k-1$ D-pieces, which contradicts the assumption that C is a k-shell. $\square$

Every embedded boundary cone-cell C that does not disconnect is a k-shell for some k, since every edge of ∂C internal in D is contained in some D-piece. The degree of a boundary cone-cell in D is stable under applying hexagon moves in D. This follows immediately from the definition of D-piece.

Example. By Lemma 4.2, if R is a pseudorectangle with a side P contained in ∂C, then P is a D-hyperplane-piece in C. In particular if L is a ladder, then every end-cell C is a vertex or 1-*shell* in L. All other cone-cells are disconnecting, so there are no shells of degree > 1 in L.

5. The main theorem

Our goal is to prove the following theorem:

Theorem 5.1. *Let $\langle X, \{Y_i\}\rangle$ be a cubical presentation satisfying the* C(9) *small cancellation condition and let $(D, \partial D) \to (X^*, X)$ be a minimal disc diagram. Then one of the following holds:*

- *D is a single vertex or a single cone-cell,*
- *D is a ladder,*
- *D has at least three shells of degree ≤ 4 and/or corner-squares and/or spurs. However, if D contains a 4-shell, there are at least four shells of degree ≤ 4 and/or corner-squares and/or spurs.*

We will refer to shells of degree ≤ 4, corner-squares and spurs as *exposed cells.* If $D = C$ is a single cone-cell, then C is also called an exposed cell of D. This theorem for the condition $C'(\frac{1}{12})$ with suitable notion of exposed cone-cells is Theorem 9.3 in [10] or Theorem 3.38 in [8]. In these papers generalized corners are allowed in the place of corners, which gives a slightly weaker statement than here. See Lemma 5.12 for the definition of generalized corners and note that in [10] they are called *cornsquares.*

Definition 5.2. A *generalized ladder* (see Figure 34) is a disc diagram D such that

- either D is a rectangle $I_n \times I_1$ with $n \geq 1$,
- or $D = R_1 \cup_{C_1} L \cup_{C_2} R_2$, where
  - L is a ladder or a single cone-cell or vertex (called the *ladder part* of D) and C_1, C_2 are vertices or edges that in the case where L is a ladder are contained in two different end-cells of L.
  - for $i = 1, 2$ the diagram R_i is one of the following:
    - a vertex equal to C_i, or
    - a rectangle $I_{n_i} \times I_1$ with $n_i \geq 1$ (called an *attached rectangle* of D) with $C_i = v_i \times I_1$ where v_i is an endpoint of I_{n_i}, or
    - a square (called an *attached square*) and C_i is a vertex of R_i.

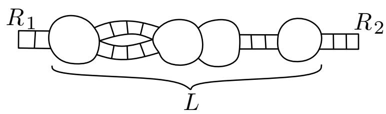

FIGURE 34. Generalized ladder.

A single cell meets the definition of a generalized ladder, while it is not a genuine ladder. Note that all generalized ladders have ≤ 2 exposed cells. In fact, as we will see later, every minimal disc diagram $(D, \partial D) \to (X, X^*)$ with ≤ 2 exposed cells is a generalized ladder. We will inductively prove for D as in Theorem 5.1 that the following condition (which will be referred to as *condition* $(\star)$) is satisfied:

- D is a generalized ladder, or
- D has at least three exposed cells (shells of degree ≤ 4 and/or corner-squares and/or spurs). However, if D contains a 4-shell, there are at least four exposed cells.

Lemma 5.3. *Let $(D, \partial D) \to (X^*, X)$ be a disc diagram such that $D = D_1 \cup_C D_2$ where C is a single cell. If both D_1, D_2 satisfy $(\star)$, then so does D.*

Proof. This proof is much like the proof of Lemma 2.2. We can assume that $C \subsetneq D_i$ for $i = 1, 2$, because otherwise there is nothing to prove. If any of the components, say D_1, has ≥ 3 exposed cells, then ≥ 2 of them are disjoint from C, so they remain exposed in D. Together with one exposed cell in D_2 (possibly D_2 itself in the case where D_2 is a single cell) there are ≥ 3 exposed cells in D. Similarly, if there is a 4-shell in D, then it is a 4-shell of one of the components D_1, D_2, say D_1. Thus D_1 has ≥ 4 exposed cells, so ≥ 3 remain exposed in D, and we have ≥ 4 exposed cells in total in D.

Now suppose that D_1, D_2 are both generalized ladders. Then D is a generalized ladder if one of the following holds:

- C is a vertex and for $i = 1, 2$ either the vertex C is contained in an exposed cone-cell/spur of D_i (in that case, in particular, D_i has at most one attached rectangle or square) or D_i is a single square. See the first diagram in Figure 35.
- C is a cone-cell and for $i = 1, 2$ it is exposed in D_i.
- C is an edge and for $i = 1, 2$ one of the following holds:
  - C is contained in an exposed cone-cell of D_i,
  - C is a side of an attached rectangle R_i in D_i opposite to the side contained in the ladder part L_i of D_i,
  - D_i is a rectangle and C is one of its sides.

  See the second diagram in Figure 35.
- C is a square and for $i = 1, 2$ the square C is a corner-square contained in a rectangle R_i that is either an attached rectangle of D_i or $R_i = D_i$. Moreover $R_1 \cup_C R_2$ is a rectangle whose two opposite sides are contained in cone-cells and/or corner-squares of D. See the third figure in Figure 35.

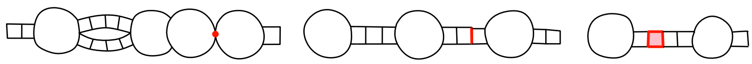

FIGURE 35. Gluing generalized ladders along marked cells.

In other cases, there are at least three exposed cells in D. Indeed one of the following holds:

- C is not contained in an exposed cell in one of D_1, D_2, say D_1. In such case there are two exposed cells in D_1, which remain exposed in D, so there are at least three exposed cells in total.
- C is contained in an attached square R of one of D_1, D_2. Then R is a corner-square of D.
- For $i = 1, 2$ the cell C is contained in a rectangle R_i that is either an attached rectangle of D_i or $R_i = D_i$, but $R_1 \cup_C R_2$ is not a rectangle with two opposite sides entirely contained in cone-cells and/or corner-squares of D_i. See Figure 36. □

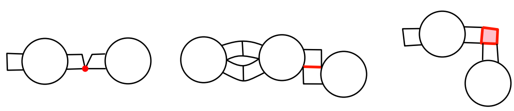

FIGURE 36. Gluing generalized ladders along marked cells.

Before the proof of Theorem 5.1 we define D-walls which will play a similar role in cubical small cancellation complexes as hyperplanes play in cube complexes. This is the crucial tool we use in proving Theorem 5.1. Next we discuss the notion of Γ-components, similar to one introduced in Section 3.3, and finally proceed with the proof.

5.1. **D-walls.** Throughout this section $(D, \partial D) \to (X^*, X)$ is a minimal disc diagram with no disconnecting cells such that all cone-cells embed. The complex X^* is the presentation complex corresponding to a presentation $\langle X, \{Y_i\}\rangle$ satisfying the C(9) condition.

Definition 5.4. Let $e_0, \ldots, e_n$ be a sequence of edges of D and $C_1, \ldots, C_n$ a sequence of 2-cells in D (with $e_i \neq e_{i+1}$ and $C_i \neq C_{i+1}$) such that for all $i = 1, \ldots, n$ we have

(1) either C_i is a square with e_{i-1}, e_i a pair of opposite edges, i.e. $e_{i-1} \cap e_i = \emptyset$,
(2) or C_i is a cone-cell with edges $e_{i-1}, e_i \subset \partial C_i$ such that there does not exist a subpath of ∂C_i containing both e_{i-1} and e_i that can be expressed as a concatenation of < 5 D-pieces.

Such a pair of sequences $\Gamma = \{(e_i), (C_i)\}$ is called a *D-wall*. Note that if C_n is a boundary cone-cell of D and e_{n-1} is an internal or semi-internal edge contained in C_n, then condition (2) is satisfied for any $e_n \subset \partial D \cap \partial C_i$.

The D-wall Γ might be identified with a path graph locally embedded (not combinatorially) in D in the following way:

- the vertices of Γ correspond to edges $e_0, \dots, e_n$ and they are mapped to the midpoints of corresponding edges;
- the edges of Γ correspond to cells $C_1, \dots, C_n$ and each edge of Γ is mapped to a midcube in the square, or respectively to the union of two intervals joining cone-points with midpoints of the appropriate edges in the cone-cell. See Figure 37.

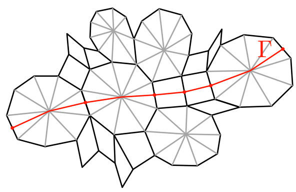

FIGURE 37. Example of a D-wall Γ.

The D-wall $\{(e_{k-1}, \dots, e_l), (C_k, \dots, C_l)\}$ for some $1 \leq k < l \leq n$ is called a *sub-D-wall* of Γ. The D-wall $\Gamma = \{(e_0, \dots, e_n), (C_1, \dots, C_n)\}$ with $n \geq 1$ is called

- *maximal* if $e_n \subset \partial D$ or $e_n = e_k$ for some $k < n$,
- *bimaximal* if both $\{(e_0, \dots, e_n), (C_1, \dots, C_n)\}$ and $\{(e_n, \dots, e_0), (C_n, \dots, C_1)\}$ are maximal.

Let e be an edge in D and $K \subset D$ a subcomplex. We say that

- Γ is *dual* to e if there exists k such that $e_k = e$ (then we also say that e is dual to Γ),
- Γ *starts* at e (respectively, in K) if $e_0 = e$ (respectively, if $e_0 \subset K$),
- Γ *terminates* at e (respectively, in K) if $e_n = e$ (respectively, if $e_n \subset K$).

Let $\Gamma' = \{(e'_0, \dots, e'_m), (C'_1, \dots, C'_m)\}$. We say that Γ and Γ' *intersect* if $C_i = C'_j$ for some i, j.

Lemma 5.5. *Let e be an edge in D. There exists a bimaximal D-wall dual to e.*

Proof. Since D is compact, there are finitely many edges in D, so it suffices to prove that for every cone-cell C and edge $e \subset \partial C$ there exists $e' \subset \partial C$ such that e, e' satisfy condition (2) from Definition 5.4. Indeed, we construct a bimaximal D-wall step by step until it terminates in ∂D or itself. If $e \subset \partial D$ we set e' to any other edge in ∂C. If $e \not\subset \partial D$, but $C \cap \partial D$ contains some edges, we set e' to any boundary edge in C. Assume that C is a cone-cell with at most one boundary vertex in ∂C. Suppose that for every edge e' in ∂C there is a path in ∂C containing both e, e' that is a concatenation of ≤ 4 D-pieces. Then there exists a pair of such paths that covers whole ∂C; thus ∂C can be expressed as a concatenation of 8 D-pieces, which is a contradiction to the C(9) condition. Thus there exists a required e'. $\square$

Note that in general a maximal D-wall dual to an edge e is not unique. The notions of hyperplanes and maximal D-walls starting at boundary edges coincide if $D \to X^*$ factors as $D \to X \to X^*$, i.e. if D consists of squares only.

A D-wall Γ is *collaring* if Γ is not dual to any internal edge in D. We say that D is *collared* if for every semi-internal edge e every D-wall Γ dual to e is collaring.

We say that D is *collared by a collection of D-walls* $\Gamma_1, \dots, \Gamma_n$ if D is collared and every semi-internal edge e is dual to Γ_i for some i. See Figure 38.

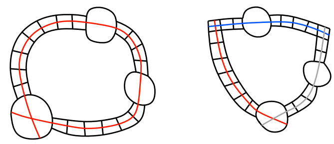

FIGURE 38. Collared disc diagrams.

Remark 5.6. If D is collared, then for every boundary cone-cell C with the inner path $e_1 \cdots e_n$, paths $e_2 \cdots e_n$ and $e_1 \cdots e_{n-1}$ both can be expressed as concatenations of < 5 D-pieces. In particular, $\deg_D C \leq 5$.

Proof. Indeed, otherwise the D-wall $\{(e_1, e_{n-1}), (C)\}$ or $\{(e_2, e_n), (C)\}$ would contradict the assumption that D is collared. □

Lemma 5.7. *The diagram D with at least one exposed cell is collared if and only if all D-walls starting in semi-internal edges of exposed cells are collaring.*

Proof. The implication from left to right is trivial. For the other implication, assume that all D-walls starting in semi-internal edges of exposed cells are collaring. Observe that for every semi-internal edge e contained in non-exposed cone-cell C there is an edge e' which is not in ∂D such that $\{(e, e'), (C)\}$ is a D-wall. Thus for every semi-internal edge e of exposed cell there exists a D-wall starting at e which also terminates at a semi-internal edge of an exposed cell. It follows that there exists a unique collection $\mathcal{G}$ of D-walls with both endpoints in semi-internal edges of exposed cells, such that every semi-internal edge in D is dual to an element of $\mathcal{G}$.

If D is not collared, there are a semi-internal edge e, an internal edge e' and a 2-cell C such that $\Gamma = \{(e, e'), (C)\}$ is a D-wall. But e is also dual to some $\Gamma' \in \mathcal{G}$, so taking the suitable sub-D-wall of Γ' and composing it with Γ, we obtain a non-collaring D-wall starting in a semi-internal edge of an exposed cell of D. This is a contradiction. □

Remark 5.8. Let D be a collared disc diagram with $n \geq 1$ exposed cells. We have $|\mathcal{G}| = n$ where $\mathcal{G}$ is the collection of D-walls from the proof of Lemma 5.7. The diagram D is collared by $\mathcal{G}$. For any collection $\mathcal{H}$ such that D is collared by $\mathcal{H}$ we have $|\mathcal{H}| \geq n$.

We define the *D-carrier* $N(\Gamma)$ of a D-wall $\Gamma = \{(e_i), (C_i)\}$ as follows:

$$N(\Gamma) := \coprod_{j=1}^{n} C_i \,/_{\sim},$$

where C_i, C_{i+1} are glued along the maximal D-piece of C_i in C_{i+1} containing e_{i+1}. It is immediate that $N(\Gamma)$ is a ladder. There is a natural combinatorial immersion $\iota : N(\Gamma) \to D$ whose image is the minimal subcomplex of D that contains Γ regarded as an immersed path graph. Whenever ι is an embedding we write $N(\Gamma)$ for $\iota\big(N(\Gamma)\big)$.

Let Γ be a bimaximal D-wall with embedded D-carrier. Observe that $D - \Gamma$ has exactly two connected components; denote one of them by K. The Γ-*component* corresponding to K is defined as $K \cup N(\Gamma)$. It is the minimal subdiagram of D containing $N(\Gamma)$ and K.

Lemma 5.9. *Let C be a cone-cell which is an end-cell of a ladder L contained in D. Denote by P the inner path of C in L. For any maximal D-piece Q in C, which has a common edge with P, we have $P \subset Q$. See the left diagram in Figure* 39.

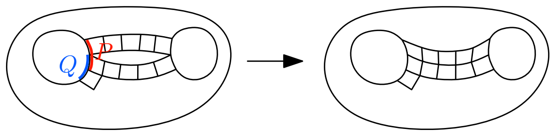

FIGURE 39. If Q is a maximal D-piece, then it contains whole P.

Proof. We may assume that there is a rectangle in D with side Q. Let R be a pseudorectangle from the definition of ladder with side P. By Lemma 4.2 we may push all squares lying between hyperplanes dual to edges in P upward, so we get a diagram containing a rectangle with side $P \cup Q$. See Figure 39. □

Lemma 5.10. *Let Γ be a bimaximal D-wall with embedded carrier and denote by D', D'' Γ-components of D. If D' is a ladder (see Figure* 40*), then $\deg_D(C) \leq \deg_{D''}(C)$ for every boundary cone-cell C in D''. In particular, all exposed cells of D'' are also exposed in D.*

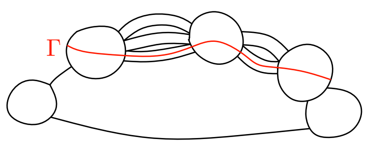

FIGURE 40. D-wall Γ.

Proof. Let $C_1, \cdots, C_n$ denote the cone-cells in the ladder D'. By Lemma 5.9 we have $\deg_D(C) = \deg_{D''}(C)$. For any other cone-cell $C \neq C_i$ in D'' we have $\deg_D(C) \leq \deg_{D''}(C)$, because every hexagon move in D'' can also be performed in the bigger diagram D which contains D'' and what follows every D''-piece of C is a D-piece. It remains to verify that there are no exposed cells in D'', which are internal in D. All boundary cells of D'' which are internal in D lie in $N(\Gamma)$, so they are not exposed in D'', by the definition of D-wall. □

Remark 5.11. By the definition of a ladder, if both Γ-components in Lemma 5.10 are ladders, then so is D.

5.2. **Preliminaries.** Let us now prove the following lemmas useful in the proof of Theorem 5.1.

Lemma 5.12 (Lemma 3.6 in [10])**.** *Let $D \to X$ be a minimal disc diagram in a non-positively curved cube complex X and let $e, e' \subset \partial D$ be a pair of adjacent edges such that $\Gamma(e)$ and $\Gamma(e')$ intersect in a square S in D. Suppose that S is the only square of the intersection of $\Gamma(e), \Gamma(e')$ in D and that $\Gamma(e), \Gamma(e')$ are collaring. See*

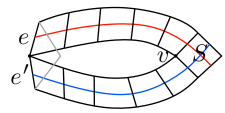

FIGURE 41. Generalized corner.

Figure 41*. Such square S together with edges e, e' will be referred to as a* generalized corner with edges e, e'.

There exists a diagram $D_0 \to X$ obtained from D by a sequence of hexagon moves such that there is a square S' in D_0 with $e'e$ a subpath of $\partial S'$.

Proof. Denote by v the unique vertex in S which is internal in D. First suppose that $\overline{D - S} = I_2 \times I_n$ for some $n \geq 0$; i.e. the internal subdiagram Int_D is a path graph. Denote by $v_1, \ldots, v_n$ all consecutive vertices of Int_D with $v_n = v$. Set $D_n = D$ and define D_{k-1} as a diagram obtained from D_k by a hexagon move applied to squares containing vertex v_k; see Figure 42. The diagram D_0 contains a square S such that ee' is a subpath of ∂C.

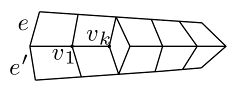

FIGURE 42. The diagram D_k.

In the general case by Lemma 4.2 applied to $\overline{D - S}$, we know that there exists a disc diagram $D' \to X$ obtained from D by a sequence of hexagon moves such that the subdiagram lying between $\Gamma(e)$ and $\Gamma(e')$ in $\overline{D - S}$ is a path graph, so there is a subdiagram containing e, e' and S which has a form as in the first step. This completes the proof. $\square$

Let us state two corollaries of Lemma 5.12, which are useful in the proof of Theorem 5.1. We assume that $(D, \partial D) \to (X^*, X)$ is a disc diagram in the presentation complex X^* corresponding to a presentation $\langle X, \{Y_i\}\rangle$ which satisfies the C(9) condition. The first corollary is an immediate consequence of the assumption that maps $Y_i \to X$ are local isometries:

Corollary 5.13. *Suppose D contains a cone-cell C and a generalized corner with edges $e, e' \subset \partial C$. See Figure 43. Then D is not minimal.*

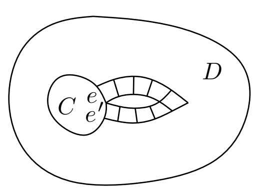

FIGURE 43. This diagram is not minimal.

Corollary 5.14. *Suppose that $(D, \partial D) \to (X^*, X)$ is a minimal disc diagram, $C \subset D$ is a cone-cell, $P = e_1 \cdots e_n$ is a subpath of ∂C and e is an edge which* *has a common vertex with e_1, but is not contained in ∂C. Suppose that there is a subdiagram D_P of D consisting of squares only and containing P and e such that hyperplanes $\Gamma(e)$ and $\Gamma(e_n)$ intersect in D_P. See the left diagram in Figure 44. Then there exists a diagram D'_P obtained from D_P by a sequence of hexagon moves such that $P \subset N\big(\Gamma(e)\big)$ in D'_P; see the right diagram in Figure 44. In particular, P is a D-piece.*

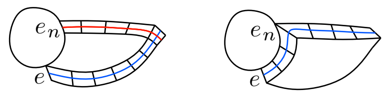

FIGURE 44. The diagrams D_P and D'_P.

Proof. The proof is by induction on the length of P. If $n = 1$, i.e. $P = e_1$, then the assertion follows immediately from Lemma 5.12 applied to the generalized corner with edges e, e_1. Suppose $n > 1$. By Corollary 5.13 we know that $\Gamma(e_i)$ and $\Gamma(e_{i+1})$ do not intersect for any $i = 1, \dots, n-1$. It follows that $\Gamma(e_1)$ intersects $\Gamma(e)$. Hence by Lemma 5.12 there is a disc diagram D' obtained from D_P by a sequence of hexagon moves such that there is a square S in D' with $ee_1 \subset \partial S$. Denote by e' the edge opposite to e in S and set $P' = e_2 \cdots e_n$. Note that $\Gamma(e_n)$ and $\Gamma(e) = \Gamma(e')$ intersect in D', since $\Gamma(e_n)$ and $\Gamma(e)$ intersect in D_P. By the induction assumption applied to P', e' and the appropriate diagram there is a diagram D'_P obtained from D' by a sequence of hexagon moves leaving S unchanged such that P' is a path in $N\big(\Gamma(e')\big)$; by construction so is P. $\square$

5.3. **Proof of the main theorem.** In this section we prove Theorem 5.1. The proof is divided into nine steps. The first three steps allow us to reduce the problem to diagrams with a non-trivial internal subdiagram and all cells embedded and not disconnecting. In the fourth step we show that the D-carriers of D-walls embed. In the next two steps we restrict our attention to collared diagrams of two types: diagrams with exactly two exposed cells (in that case we intend to verify that they are ladders) and diagrams with three exposed cells with a 4-shell among them (in that case we intend to obtain a contradiction). In Step 7 we prove that the internal subdiagrams are squared. Finally, in the last two steps we show that there are no non-exposed cone-cells in our diagrams, and in what follows they are ladders in the first case and in the second case we obtain a contradiction.

It is immediate that condition $(\star)$ (formulated after Definition 5.2) implies the hypothesis of the theorem. We will prove by induction on the number of cells that all minimal disc diagrams satisfy $(\star)$. We assume that all disc diagrams having fewer cells than D satisfy $(\star)$ and deduce that so does D.

Step 1. All cone-cells in D are embedded. The intersection of two cone-cells is connected.

Proof. First suppose that C is a cone-cell that does not embed. Let P be the minimal subpath of ∂C such that its endpoints are mapped to the same point p in D. The path P is a boundary path of the disc diagram D', which is the closure of a connected component of $D - C$. See the left diagram in Figure 45. There are no spurs in D' by the minimality of P. Observe that for any shell C' in D', the connected intersection $C' \cap \partial D'$ is a path that can be expressed as a concatenation

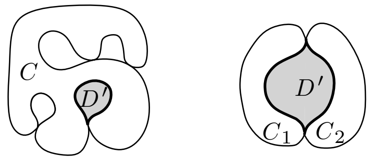

FIGURE 45. Cell C is not embedded. Cells C_1, C_2 intersect in two points.

of ≤ 2 D-pieces. Thus there are no exposed cone-cells in D' and D' is not a single cone-cell. No vertex of P, except for p possibly, is a corner of D'. Hence, D' does not satisfy $(\star)$, which contradicts the induction assumption.

Let us now prove the second statement. Suppose that the intersection of cone-cells C_1, C_2 is not connected. Let $P_1 \to \partial C_1, P_2 \to \partial C_2$ be minimal paths such that P_1, P_2 have common endpoints in D. Their concatenation is a boundary path of the disc diagram D', which is the closure of a connected component of $D - C_1 \cup C_2$. See the right diagram in Figure 45. Similarly as before, we conclude that D' does not contain exposed cone-cells and has no more than two corners; hence D' contradicts the induction assumption. □

Step 2. We may assume that D has no disconnecting cells.

Proof. This follows from Lemma 5.3. □

Step 3. We may assume that $\mathrm{Int}_D \neq \emptyset$.

Proof. Since D has no disconnecting cells, by Lemma 2.1 either D consists of at most two cells or $\mathrm{Int}_D \neq \emptyset$. If D is a single cell, there is nothing to prove. If D consists of two cells, then there is a path of length ≥ 2 contained in their intersection, since there are no disconnecting cells. If D consists of two cone-cells, then D is a ladder. Otherwise, if D contains a square, D is not minimal. □

To prove $(\star)$ we will verify that

- either D is a ladder consisting of two cone-cells joined by a pseudorectangle,
- or there are at least three exposed 2-cells in D.

Step 4. For any D-wall Γ the D-carrier $N(\Gamma)$ embeds in D.

Proof. Suppose to the contrary that $\iota : N(\Gamma) \to D$ is not an embedding. Let $\Gamma' = \{(e_0, \ldots, e_n), (C_1, \ldots, C_n)\}$ be a minimal sub-D-wall of Γ such that $N(\Gamma')$ does not embed. Since the intersection of any two cells is connected, we have $n \geq 3$. Denote by K the component of $D - \iota\big(N(\Gamma')\big)$ which intersects ∂D trivially. Let $D_{\Gamma'}$ be the minimal subdiagram containing $\iota\big(N(\Gamma')\big)$ and K. See Figure 46.

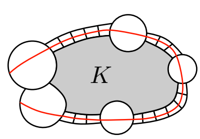

FIGURE 46. The diagram $D_{\Gamma'}$.

By the minimality of Γ' the diagram $D_{\Gamma'}$ is a disc diagram. One of the following holds:

- The diagram $D_{\Gamma'}$ is a proper subdiagram of D. The images of end-cells of $N(\Gamma')$ are the only possible exposed cells in $D_{\Gamma'}$. Since they are shells of degree ≥ 2, the diagram $D_{\Gamma'}$ cannot be a ladder, so since $D_{\Gamma'}$ has fewer cells than D we obtain a contradiction with the induction assumption.
- We have $D_{\Gamma'} = D$. Choose any boundary cell C distinct from C_1, C_n and consider a bimaximal D-wall Γ'' dual to an edge of C that is contained in any piece intersecting ∂D. If $N(\Gamma'')$ is not embedded, then proceeding as in the first case, we obtain a proper subdiagram which contradicts the induction assumption. If $N(\Gamma'')$ is embedded, then one of the Γ''-components D' is a diagram collared by Γ'' and sub-D-wall of Γ, so D' has ≤ 2 exposed cells. The cell C is either a corner-square or a shell of degree ≥ 2 in D', so D' is not a ladder, a contradiction. □

Step 5. The diagram D has at least two exposed cells.

Proof. Suppose D has ≤ 1 exposed cells. Let C be any non-exposed boundary cell in D and let Γ be a bimaximal D-wall dual to an edge of C that is not contained in any piece intersecting ∂D. There is at least one Γ-component D' that has ≤ 2 exposed cells. Since C is a corner-square or a shell of degree ≥ 2 in D', we conclude that D' is not a ladder. The diagram D' has fewer cells than D, so we obtain a contradiction with the induction assumption. □

From now on we assume that

(A) the diagram D has exactly 2 exposed cells C_1, C_2, or
(B) the diagram D has three exposed cells C_1, C_2, C_3 and C_3 is a 4-shell.

To complete the proof of the theorem we will verify that D is a ladder in Case (A) and we will obtain a contradiction in Case (B).

Step 6. The diagram D is collared and both C_1, C_2 have inner paths of length 2 or D is a ladder.

Proof. By Lemma 5.7, to verify that D is collared and the internal paths of C_1, C_2 have length 2 it suffices to check that all D-walls starting in C_1, C_2 $\big($or in a semi-internal edge of C_3 in Case (B)$\big)$ are collaring. Suppose that Γ is a non-collaring D-wall starting in one of C_1, C_2 $\big($or in a semi-internal edge of C_3 in Case (B)$\big)$. Let us consider Cases (A) and (B) separately:

(A) One of the Γ-components has ≤ 2 exposed cells, so by the induction assumption it is a ladder. By Lemma 5.10 the other Γ-component has ≤ 2 exposed cells, so by the induction assumption it is also a ladder (this happens only if Γ starts and terminates in C_1 and C_2). By Remark 5.11 the diagram D is a ladder.
(B) If there is a Γ-component with ≤ 2 exposed cells, then by the induction assumption it is a ladder, and by Lemma 5.10 the other Γ-component D'' has either 3 exposed cells with a shell C_3 of degree 4 among them or ≤ 2 exposed cells and at least one non-exposed shell. The second case occurs if $\deg_{D''}(C_3) > \deg_D(C_3) = 4$; i.e. some D-piece in the inner path of C_3 in D'' is not a single D''-piece. In both cases D'' contradicts the induction assumption. If none of the Γ-components has ≤ 2 exposed cells, then both have three exposed cells and one of them contains a 4-shell, a contradiction.

It follows that D is collared and C_1, C_2 have inner paths of length 2. $\square$

By Remark 5.8:

In Case (A) there exist D-walls Γ, Γ' such that D is collared by Γ, Γ'.

In Case (B) there exist D-walls $\Gamma, \Gamma'_1, \Gamma'_2$ such that D is collared by $\Gamma, \Gamma'_1, \Gamma'_2$ and Γ'_i starts in C_i and terminates in C_3. See Figure 47.

Set $\Gamma = \{(e_0^\Gamma, \ldots, e_m^\Gamma), (C_1^\Gamma, \ldots, C_m^\Gamma)\}$ such that $e_o^\Gamma \subset C_1$ and $e_m^\Gamma \subset C_2$. Note that C_1, C_2 can be corner-squares and/or shells.

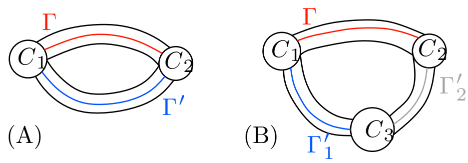

FIGURE 47. In Case (A) the diagram is collared by two D-walls; in Case (B) the diagram is collared by three D-walls.

Step 7. Internal subdiagram Int_D of D is a squared diagram.

Proof. Suppose to the contrary that there is an internal cone-cell in D and denote it by C. First suppose that there exists a D-wall starting in C and terminating in $N(\Gamma)$. We will discuss the other case in the very end of this step. Let $e_1 \cdots e_n$ be the boundary path of C where e_1 is chosen so that there is a D-wall Γ_{e_1} starting at e_1 which terminates in $C_{i_1}^\Gamma$ for minimal i_1 (i.e. closest to C_1 in $N(\Gamma)$). Let e_k be the edge in ∂C such that there exists a D-wall Γ_{e_k} starting at e_k which terminates in $C_{i_k}^\Gamma$ for maximal i_k. If $e_1 \cdots e_k$ cannot be expressed as a concatenation of < 5

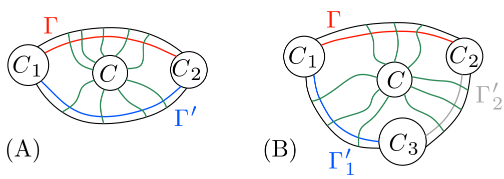

FIGURE 48. We suppose there is an internal cone-cell in D.

D-pieces, then the diagram D' collared by $\Gamma_{e_1}, \Gamma_{e_k}$ and the appropriate sub-D-wall of Γ has ≤ 2 exposed cells and $\deg_{D'}(C) \geq 5$, so D' is not a ladder, which is a contradiction with the induction assumption. Hence $e_1 \cdots e_k$ can be expressed as a concatenation of ≤ 4 D-pieces. Observe that for every $l > k$ such that there is a D-wall Γ_{e_l} starting at e_l which intersects Γ_{e_k} (respectively, Γ_{e_1}), the diagram collared by the appropriate sub-D-walls of Γ_{e_k} and Γ_{e_l} (respectively, Γ_{e_l} and Γ_{e_1}) has ≤ 2 exposed cells. By the induction assumption it is a ladder; in particular, by Lemma 5.9 any maximal D-piece containing e_k (respectively e_1) contains also e_l. It follows that the minimal path containing $e_1 \cdots e_k$ and every edge e dual to some D-wall intersecting one of $\Gamma_{e_1}, \Gamma_{e_k}$, can be expressed as a concatenation of ≤ 4 pieces. Denote by P the maximal subpath of $e_{k+1} \cdots e_n$ such that all D-walls starting in P intersect none of $\Gamma_{e_1}, \Gamma_{e_k}$. By C(9) and our last observation P cannot be expressed as a concatenation of ≤ 5 D-pieces. Denote by Γ_1^P, Γ_2^P D-walls starting at two different end-cells of P. They do not intersect, because otherwise there would be a diagram with only one exposed cell collared by them. Since they do not intersect any of $\Gamma_{e_1}, \Gamma_{e_k}$, they both terminate in

(A) $N(\Gamma')$,
(B) $N(\Gamma_1') \cup C_3 \cup N(\Gamma_2')$

in the following way:

(A) The diagram D' collared by Γ_1^P, Γ_2^P and the appropriate sub-D-wall of Γ' has ≤ 2 exposed cells and $\deg_{D'}(C) \geq 5$. Thus D' is not a ladder, and this is a contradiction with the induction assumption.
(B) The diagram D' collared by Γ_1^P, Γ_2^P and the appropriate sub-D-walls of Γ_1', Γ_2' either has ≤ 2 exposed cells and contains a shell of degree ≥ 5 or has ≤ 3 exposed cells with a 4-shell among them. In both cases this is a contradiction with the induction assumption.

Now suppose that no D-wall starting in C terminates in $N(\Gamma)$. In Case (A) proceed exactly as before replacing Γ by Γ'. In Case (B) for $i = 1, 2$ denote by Γ_i^P the D-wall starting in C and terminating in the closest cell to C_i in $N(\Gamma_1') \cup C_3 \cup N(\Gamma_2')$. Similarly as before, the subdiagram collared by Γ_1^P, Γ_2^P and the appropriate sub-D-walls of Γ_1', Γ_2' contradicts the induction assumption. Thus we have shown that D has no internal cone-cells. □

Step 8. In Case (A) the diagram D is a ladder.

Proof. First, let us show that D has no cone-cells at all except for C_1, C_2 possibly. Suppose to the contrary that C is a non-exposed boundary cone-cell. Without loss of generality we can assume that $C \subset N(\Gamma)$. By Remark 5.6 we know that C is a 5-shell. By Lemma 4.5 there exists an edge e_2 which is contained in P_2 for any decomposition of the inner path $P = P_1 \cdots P_5$ of C into D-pieces. Denote by Γ_{e_2} a maximal D-wall starting at e_2. See Figure 49. One of the following holds:

- If Γ_{e_2} terminates in $N(\Gamma)$, then the Γ_{e_2}-component D' collared by Γ_{e_2} and the appropriate sub-D-wall of Γ has ≤ 2 exposed cells. The inner path of C in D' cannot be expressed as a concatenation of 2 (respectively 4) D-pieces, so it cannot be expressed as a concatenation of ≥ 2 (respectively ≥ 4) D'-pieces. Thus D' is not a ladder. This is a contradiction.
- If Γ_{e_2} terminates in $N(\Gamma')$, then the Γ_{e_2}-component D'' such that $\deg_{D''}(C) \geq 4$ has either three exposed cells with a 4-shell among them (if $\deg D''(C) = 4$) or two exposed cells and a non-exposed cone-cell (if $\deg_{D''}(C) > 4$). By the induction assumption we obtain a contradiction.

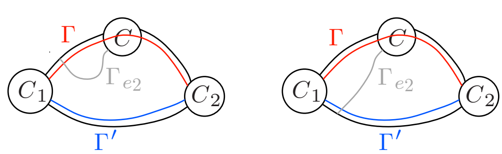

FIGURE 49. Γ_{e_2} terminates in $N(\Gamma)$ or $N(\Gamma')$.

Hence there are no cone-cells in D except for C_1, C_2 possibly. Let us now consider different cases depending on what C_1, C_2 are:

- If C_1, C_2 are both squares, then D is a squared diagram with only two corners, which is impossible by Theorem 3.2.
- If one of C_1, C_2 is a square and the other one is a cone-cell, then D is not minimal by Corollary 5.13.
- If C_1, C_2 are both cone-cells, then by definition D is a ladder. □

Step 9. Case (B) is not possible.

Proof. First let us show that there are no cone-cells in $N(\Gamma'_1) \cup N(\Gamma'_2)$. Suppose to the contrary that C is a non-exposed cone-cell, say in $N(\Gamma'_1)$; i.e. there exists i such that $C_i^{\Gamma'_1} = C$ where $\Gamma'_1 = \{(e_0^{\Gamma'_1}, \dots, e_m^{\Gamma'_1}), (C_1^{\Gamma'_1}, \dots, C_m^{\Gamma'_1})\}$ such that $e_0^{\Gamma'_1} \subset C_1$. Let P be the inner path of C such that its first edge is $e_i^{\Gamma'_1}$ and its last edge is $e_{i-1}^{\Gamma'_1}$. By Lemma 4.5 there is an edge e contained in P_2 for any decomposition of the inner path $P = P_1 \cdots P_5$ of C into D-pieces. Denote by Γ_e a maximal D-wall starting at e. The degree of C in a Γ_e-component is ≥ 2 (respectively ≥ 4), since the inner path of C in Γ_e-components cannot be expressed as a concatenation of 2 (respectively 4) D-pieces. If Γ_e terminates in $N(\Gamma'_1)$, then one of the Γ_e-components has ≤ 2 exposed cells and contains a shell of degree ≥ 2, so it is not a ladder, a contradiction. If Γ_e terminates in $N(\Gamma'_2)$, then the Γ_e-component containing C_3 has ≤ 3 exposed cell and a 4-shell among them, a contradiction. If Γ_e terminates in $N(\Gamma)$, then the Γ_e-component that does not contain C_3 has either 3 exposed cells with a 4-shell C among them or ≤ 2 exposed cells and a non-exposed cone-cell C, a contradiction. See Figure 50. We have just proved that there are no cone-cells in $N(\Gamma'_1) \cup N(\Gamma'_2)$.

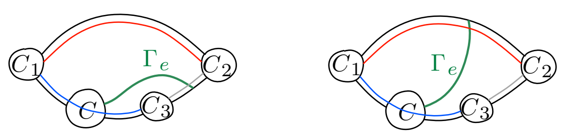

FIGURE 50. One of the Γ_e-components has either 3 exposed cells with a 4-shell among them or ≤ 2 exposed cells.

Now we will show that the only cone-cells in D are C_3 and C_1, C_2 possibly. It remains to verify that there are no cone-cells in $N(\Gamma)$. Suppose to the contrary that C is a non-exposed cone-cell in $N(\Gamma)$. By Remark 5.6 we know that C is a 5-shell.

Denote by e_3 an edge contained in P_3 for every decomposition of the inner path $P = P_1 \cdots P_5$ of C into pieces. Let e_2 (e_4 respectively) be the first (the last respectively) edge in P that is not contained in P_1 (P_5 respectively) for any decomposition of P into pieces. Note that e_2 and e_3 (e_3 and e_4) are not contained in a single piece. Let P' be the minimal subpath of P containing e_2 and e_4. Observe that every D-wall starting in P' terminates in $N(\Gamma'_1) \cup N(\Gamma'_2)$. Otherwise, there would be a subdiagram with two exposed cells and a shell of degree ≥ 2 among them, which is impossible by the induction assumption. Let us consider three cases:

- We have $e_3 \cap \big(N(\Gamma'_1) \cup C_3 \cup N(\Gamma'_2)\big) = \emptyset$; see Figure 51. Denote by S a square in D which contains e_3. Let $\Gamma_S = \{(e_0^{\Gamma_S}, \dots, e_m^{\Gamma_S}), (C_1^{\Gamma_S}, \dots, C_m^{\Gamma_S})\}$

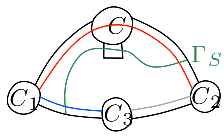

FIGURE 51. The edge e_3 is not contained in $N(\Gamma_1') \cup C_3 \cup N(\Gamma_2')$.

be a bimaximal D-wall intersecting S but not e_3; i.e. $S = C_i^{\Gamma_S}$ for some i and $e_{i-1}^{\Gamma_S}, e_i^{\Gamma_S} \neq e_3$. See Figure 51. Observe that no endpoint of Γ_S lies in $N(\Gamma)$, because otherwise by Corollary 5.14 the minimal subpath of ∂C containing e_3 and one of e_2, e_4 would be a D-piece, but this is not the case. Thus both endpoints of Γ_S lie in $N(\Gamma_1') \cup N(\Gamma_2')$. The diagram collared by Γ_S and the appropriate sub-D-walls of Γ_1' and Γ_2' has either three exposed cells with a 4-shell C_3 among them or only two exposed cells and at least one of them is a corner-square. In both cases we obtain a contradiction.

- We have $e_3 \subset N(\Gamma_1') \cup N(\Gamma_2')$; see Figure 52. Without loss of generality, we may assume that $e_3 \subset N(\Gamma_1')$. Let e be one of two edges intersecting e_3 and dual to Γ_1' that is closer to C_3 in $N(\Gamma_1')$. Let Γ_{e_2} be a D-wall dual to e_2. Since Γ_{e_2} terminates in $N(\Gamma_1')$, by Corollary 5.14 applied to the minimal subpath of ∂C containing e_2 and e_3 and to the edge e we conclude that e_2 and e_3 are contained in a single D-piece, a contradiction. See Figure 52.

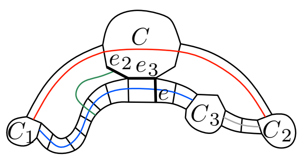

FIGURE 52. The edge e_3 is contained in $N(\Gamma_1')$.

- We have $e_3 \subset C_3$; see Figure 53. Let $Q = Q_1 \cdots Q_4$ be some decomposition into D-pieces of the inner path Q of C_3, where Γ_1' is dual to an edge in Q_1. The edge e_3 is contained in one of Q_2, Q_3. Without loss of generality we can assume that it is in Q_2. Note that Q_3 is a D-hyperplane-piece. Denote by Q'', Q' paths such that $Q = Q'' e_3 Q'$, and note that $Q_3 Q_4 \subset Q'$. Let e be the edge that occurs right after e_3 in P. Since the D-wall starting at e terminates in $N(\Gamma_2')$, by Corollary 5.14 applied to the path Q' and edge e, we conclude that $Q_3 Q_4$ is a single D-piece, which is a contradiction. See Figure 53.

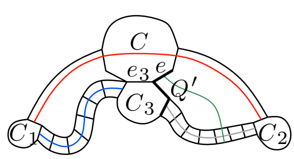

FIGURE 53. The edge e_3 is contained in C_3.

Thus, there are no cone-cells in D except for C_3 and C_1, C_2 possibly.

Finally, we show that such D cannot exist. Let Q be the inner path of C with Γ_1' dual to the first edge e_1 of Q. Denote by e_2 (e_3 respectively) the first (the last respectively) edge in Q that is not contained in Q_1 (respectively Q_4) for any decomposition $Q = Q_1 \cdots Q_4$ into pieces. At most one of e_2, e_3 is contained in $N(\Gamma)$. Without loss of generality, assume that $e_2 \not\subset N(\Gamma)$. Denote by S the square in D containing e_2 and by e the one of two edges of S dual to Γ_S that intersect Q farther from e_1. See Figure 54. By Corollary 5.14 applied to the minimal subpath of

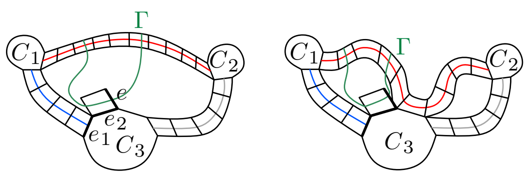

FIGURE 54. Corollary 5.14 is applied to the thickened path.

Q containing e_1 and e_2 and to the edge e we know that Γ_S cannot have an endpoint in $N(\Gamma_1')$, because otherwise e_1 and e_2 would lie in the single piece of C, which is not the case. Similarly, we conclude that Γ_S do not have endpoints in $N(\Gamma_2')$. It follows that both endpoints of Γ_S lie in $N(\Gamma)$. Hence there is a squared diagram collared by Γ_S and a sub-D-wall of Γ, which has only two corners and consists of squares only, which is a contradiction and completes the proof. $\square$

Acknowledgements

The author is deeply grateful to his advisor, Piotr Przytycki, for being simply the best. The author would also like to thank Damian Osajda and Daniel Wise for helpful discussions and Maciej Zdanowicz for his support.

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