TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 369, Number 6, June 2017, Pages 4385–4413
http://dx.doi.org/10.1090/tran/6925
Article electronically published on February 13, 2017

GAP PHENOMENA AND CURVATURE ESTIMATES FOR CONFORMALLY COMPACT EINSTEIN MANIFOLDS

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ABSTRACT. In this paper we obtain first a gap theorem for a class of conformally compact Einstein manifolds with a renormalized volume that is close to its maximum value. We also use a blow-up method to derive curvature estimates for conformally compact Einstein manifolds with large renormalized volume. The major part of this paper is the study of how a property of the conformal infinity influences the geometry of the interior of a conformally compact Einstein manifold. Specifically we are interested in conformally compact Einstein manifolds with conformal infinity whose Yamabe invariant is close to that of the round sphere. Based on the approach initiated by Dutta and Javaheri we present a complete proof of the relative volume inequality

$$\left(\frac{Y(\partial X,[\hat{g}])}{Y(\mathbb{S}^{n-1},[g_{\mathbb{S}}])}\right)^{\frac{n-1}{2}} \leq \frac{Vol(\partial B_{g^+}(p,t))}{Vol(\partial B_{g_{\mathbb{H}}}(0,t))} \leq \frac{Vol(B_{g^+}(p,t))}{Vol(B_{g_{\mathbb{H}}}(0,t))} \leq 1,$$

for conformally compact Einstein manifolds. This leads not only to the complete proof of the rigidity theorem for conformally compact Einstein manifolds in arbitrary dimension without spin assumption but also a new curvature pinching estimate for conformally compact Einstein manifolds with conformal infinities having large Yamabe invariant. We also derive curvature estimates for such manifolds.

1. INTRODUCTION

The study of conformally compact Einstein manifolds is fundamental in establishing the mathematical theory of the so-called AdS/CFT correspondence proposed in the theory of quantum gravity in theoretical physics. It is well known that there is a rigidity phenomenon for conformally compact Einstein manifolds. Namely, a conformally compact Einstein manifold whose conformal infinity is the round sphere is isometric to the hyperbolic space [3, 30]. On the other hand, in [7, 19], it was shown that for each conformal class $[\hat{g}]$ that is sufficiently close to the round sphere there exists a conformally compact Einstein metric on the ball whose conformal infinity is $[\hat{g}]$. Those conformally compact Einstein metrics are automatically close to the hyperbolic space in some appropriate way. In an attempt to understand if those conformally compact Einstein metrics are globally unique, we describe some

Received by the editors March 24, 2015 and, in revised form, June 26, 2015, November 4, 2015, and January 3, 2016.

2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 58J05.

Key words and phrases. Conformally compact Einstein manifolds, gap phenomena, rigidity, curvature estimates, renormalized volumes, Yamabe constants.

The research of the first author was supported by China Postdoctoral Science Foundation grant 2014M550540.

The research of the second author was supported by NSF grant DMS-1303543.

The research of the third author was supported by NSF of China grant 10990013.

gap phenomena and derive curvature estimates either when the renormalized volume is close to that of the hyperbolic space or when the Yamabe invariant of the conformal infinity is close to that of the conformal round sphere.

The first part of this paper is on a gap theorem for renormalized volumes of conformally compact Einstein 4-manifolds that grows out of the gap theorem in [12] for closed Bach flat 4-manifolds. As a consequence of recent work of Cheeger and Naber [13], we remove the dependence of the L^2-norm of the Weyl curvature from the gap theorem in [12]. Our proof of the improved gap theorem (cf. Theorem 3.5) is slightly different from that in [12]. It relies on the control of the Yamabe invariant instead. For a given complete non-compact Riemannian manifold, the Yamabe invariant is defined as follows:

$$Y(M,\ g) = \inf \left\{ \frac{\int_M (\frac{4(n-1)}{n-2}|\nabla u|^2 + R[g]u^2) dv[g]}{(\int_M |u|^{\frac{2n}{n-2}} dv[g])^{\frac{n-2}{n}}} : u \in C_c^\infty(M)\backslash\{0\} \right\}, \tag{1.1}$$

where $C_c^\infty(M)$ is the space of smooth functions with compact support on M. It is easily seen that $Y(M,\ g)$ is a conformal invariant when $(M,\ g)$ is compact with no boundary, in which case we denote it by $Y(M,\ [g])$ instead. Note that the Yamabe constant is as convenient to use as the Euclidean volume growth bound when rescaling metrics because it is easily seen that a complete non-compact Ricci flat 4-manifold is isometric to the Euclidean 4-space if its Yamabe constant is large enough (cf. Lemma 3.4).

It is known from [18] that, on a conformally compact Einstein 4-manifold (X^4, g^+) with conformal infinity $(\partial X, [\hat{g}])$,

$$\mathrm{Vol}(\{x > \epsilon\}) = \frac{1}{3}\mathrm{Vol}(\partial X, \hat{g})\epsilon^{-3} - \frac{1}{8}\int_{\partial X} (Rdv)[\hat{g}]\epsilon^{-1} + V(X^4,\ g^+) + o(1) \tag{1.2}$$

where $R[\hat{g}]$ is the scalar curvature of the metric $\hat{g}$ and x is the geodesic defining function associated with a representative $\hat{g}$ of the conformal infinity. It turns out that $V(X^4,\ g^+)$ in (1.2) is independent of the representative $\hat{g}$ and is called the renormalized volume (cf. [18, 21]). The expansion (1.2) uses the expansion of the Einstein metric g^+ given in [17]:

$$x^2 g^+ = dx^2 + g_x = dx^2 + \hat{g} + g^{(2)}x^2 + g^{(3)}x^3 + o(x^3) \tag{1.3}$$

where $g^{(2)}$ is a curvature tensor of $\hat{g}$ while $g^{(3)}$ is non-local. Our gap theorem for conformally compact Einstein 4-manifolds with very large renormalized volume is the following:

Theorem 1.1. *There exists a small positive number ϵ such that a conformally compact Einstein 4-manifold $(X^4,\ g^+)$ with conformal infinity of positive Yamabe type has to be isometric to the hyperbolic space if its renormalized volume satisfies*

$$V(X^4,\ g^+) \geq (1-\epsilon)\frac{4\pi^2}{3} = (1-\epsilon)V(\mathbb{H}^4,\ g_{\mathbb{H}}) \tag{1.4}$$

and the non-local term $g^{(3)}$ in (1.3) for g^+ vanishes.

Theorem 1.1 indicates that the hyperbolic space is the only "critical point" of the renormalized volume among all conformally compact Einstein manifolds that satisfy (1.4), since the Euler-Lagrange equation for the renormalized volume is $g^{(3)} = 0$ by the calculation made in [2]. On the other hand, Theorem 1.1 clearly does not hold if one drops the assumption $g^{(3)} = 0$. In fact the conformally compact Einstein metrics constructed in [7, 19] provide plenty of examples of conformally compact Einstein metrics that satisfy (1.4) for arbitrarily small ϵ and with conformal infinity of positive Yamabe type. Nevertheless, we are able to use a blow-up method to obtain the following curvature estimate for conformally compact Einstein metrics when the renormalized volume is large enough.

Theorem 1.2. *For any $\epsilon \in (0, \frac{1}{2})$, there is a positive constant C such that*

$$\|Rm\|[g^+] \leq C$$

on any conformally compact Einstein 4-manifold $(X,\ g^+)$, provided that the conformal infinity is of positive Yamabe type and that (1.4) holds.

The second part of this paper is on conformally compact Einstein manifolds whose conformal infinity has large positive Yamabe invariant. The problems in the second part are much more challenging and our results in the second part are more interesting. This is because the renormalized volume is the geometric property of the bulk space $(X^4,\ g^+)$, while the Yamabe constant $Y(\partial X, [\hat{g}])$ is an invariant of the conformal infinity only.

It is very interesting to see in [31] and [16] how the Yamabe constant of the conformal infinity is used to control the relative volume growth of geodesic balls in conformally compact Einstein manifolds. We recognize the important contribution of [16], but we would like to give a detailed proof of the following fact:

Theorem 1.3. *Assume that $(X^n,\ g^+)$ is an AH manifold of C^3 regularity whose conformal infinity is of positive Yamabe type. Let $p \in X^n$ be a fixed point. Assume further that*

$$Ric[g^+] \geq -(n-1)g^+ \ \text{ and } \ R[g^+] + n(n-1) = o(e^{-2t}) \tag{1.5}$$

for the distance function t from p. Then

$$\left(\frac{Y(\partial X, [\hat{g}])}{Y(\mathbb{S}^{n-1}, [g_{\mathbb{S}}])}\right)^{\frac{n-1}{2}} \leq \frac{Vol(\partial B_{g^+}(p,t))}{Vol(\partial B_{g_{\mathbb{H}}}(0,t))} \leq \frac{Vol(B_{g^+}(p,t))}{Vol(B_{g_{\mathbb{H}}}(0,t))} \leq 1, \tag{1.6}$$

where $B_{g^+}(p,t)$ and $B_{g_{\mathbb{H}}}(0,t)$ are geodesic balls.

The C^3 regularity is used to construct the geodesic defining function x and a C^2-conformal compactification $\bar{g} = x^2 g^+$, for each given representative $\hat{g}$ of the conformal class of the conformal infinity. Note that in the proof of existence of the geodesic defining function, there is a loss of regularity of one order; see [18] and [14]. To start the proof we first need to clear a technical issue.

Lemma 1.4. *Assume that $(X^n,\ g^+)$ is AH of C^2 regularity and that x is a defining function. Assume further that*

$$Ric[g^+] \geq -(n-1)g^+ \ \text{ and } \ R[g^+] + n(n-1) = o(x). \tag{1.7}$$

Then there is a constant C_0 such that

$$|K[g^+] + 1| \leq C_0 x^2 \tag{1.8}$$

for any sectional curvature K.

It takes substantial arguments to prove Theorem 1.3 based on the approach initiated in [31] and later modified by [16]. The first issue is that the identity (4.8) (cf. (3.17) in [16]) may not be available as claimed in [16], since the distance function t is only Lipschitz in general. We devote Section 5 to solving this issue by a careful study of the cut loci based on [23,28]. The second issue is that the estimate (4.1) (cf. (2.3) in [16]) is not known to hold without assuming the convexity of the geodesic spheres (cf. [22,31]). We devote Section 6 to deriving the total scalar curvature estimate (4.10) without (4.1). Our argument in Section 6 uses more delicate analysis of the Riccati equations on AH manifolds and volume estimates along the normal flow where the mean curvature of the geodesic sphere is small.

Clearly, Theorem 1.3 leads to the rigidity of conformally compact Einstein manifolds when the conformal infinity is exactly the round sphere in any dimension (cf. [3,8,15,16,25,27,30,31,33]).

Corollary 1.5. *Suppose that $(X^n,\ g^+)$ is AH of C^3 regularity and that (1.5) holds. Then (X^n, g^+) is isometric to the hyperbolic space $(\mathbb{H}^n,\ g_{\mathbb{H}})$, provided that the conformal infinity $(\partial X,\ [\hat{g}])$ is the round conformal sphere.*

Notice that when the conformal infinity is the round sphere, a conformally compact Einstein metric g^+ is always smooth according to [14], provided that it is at least of C^2 regularity. Therefore Corollary 1.5 is the most general rigidity theorem in the literature for conformally compact Einstein manifolds whose conformal infinity is the round sphere in any dimension. Theorem 1.3 also leads to the following interesting curvature pinching estimates.

Theorem 1.6. *For any $\epsilon > 0$, there exists $\delta > 0$; for any conformally compact Einstein manifold (X^n, g^+) $(n \geq 4)$, one gets*

$$|K[g^+] + 1| \leq \epsilon, \tag{1.9}$$

for all sectional curvature K of g^+, provided that

$$Y(\partial X, [\hat{g}]) \geq (1-\delta) Y(S^{n-1}, [g_{\mathbb{S}}]).$$

In particular, any conformally compact Einstein manifold with conformal infinity of Yamabe constant sufficiently close to that of the round sphere is necessarily negatively curved.

This result is even more interesting than the rigidity because it gives a curvature pinching estimate which only relies on the Yamabe constant of the conformal infinity. Particularly, from Theorem 1.6 we now know that any conformally compact Einstein manifold whose conformal infinity is prescribed to be sufficiently close to the round conformal sphere is negatively curved, which was only known to be true for those conformally compact Einstein manifolds constructed in [7,19].

As a consequence of the proof of Theorem 1.6 we also get the following curvature estimate for conformally compact Einstein manifolds whose conformal infinity has large Yamabe constant.

Corollary 1.7. *For any $L > 0$, $\tau > \frac{1}{2}$ and any $n \geq 4$, there is a number $C = C(L, \tau)$ such that*

$$|W|[g^+] \leq C$$

on any conformally compact Einstein manifold $(X^n,\ g^+)$, provided that

$$Y(\partial X, [\hat{g}]) \geq \tau^{\frac{2}{n-1}} Y(\mathbb{S}^n, [g_{\mathbb{S}}]),$$

and in addition

$$\int_{X^n} (|W|^{\frac{n}{2}} dv)[g^+] \leq L$$

when $n \geq 5$.

The organization of this paper is as follows: in Section 2, we introduce some basics about AH manifolds and conformally compact Einstein manifolds. In particular, we prove Lemma 1.4. In Section 3, we use the renormalized volume to control the Yamabe invariant of the conformally compact Einstein 4-manifolds and prove Theorem 1.1 about the gap phenomenon and Theorem 1.2 on curvature estimates. In Section 4, based on the approach initiated in [16], we first sketch a proof of the main lemma (cf. Lemma 4.4) that leads to the proof of Theorem 1.3. Then we will use Theorem 1.3 to obtain Theorem 1.6 and Corollary 1.7. Finally, in Section 5 and Section 6 we address all the issues left in the proof of Lemma 4.4 in Section 4 and complete the proof of Theorem 1.3.

2. Preliminaries

Let us recall some basic facts about AH manifolds and conformally compact Einstein manifolds. First we use the following definition for conformally compact Einstein manifolds.

A smooth defining function x for the boundary ∂X of a smooth manifold $\overline{X^n}$ is a smooth non-negative function from $\overline{X^n}$ to $\mathbb{R}$ such that

- $x > 0$ in the interior X^n;
- $x = 0$ on the boundary ∂X^{n-1};
- $dx \neq 0$ on the boundary ∂X^{n-1}.

Definition 2.1. Assume that X^n is the interior of a smooth compact manifold $\overline{X^n}$ with boundary ∂X^{n-1}. A Riemannian metric g^+ on X^n is said to be conformally compact of $C^{k,\alpha}$ regularity if, given a smooth defining function x for the boundary ∂X^{n-1} in $\overline{X^n}$, $\bar{g} = x^2 g^+$ can be extended to a $C^{k,\alpha}$ Riemannian metric on $\overline{X^n}$. If, in addition, $|dx|^2_{x^2 g^+} \equiv 1$ on ∂X^{n-1}, then we say that $(X^n,\ g^+)$ is asymptotically hyperbolic (AH in short) of $C^{k,\alpha}$ regularity. Further if g^+ is at least of C^2 regularity and Einstein, that is,

$$\mathrm{Ric}[g^+] = -(n-1)g^+, \tag{2.1}$$

then we say that $(X^n,\ g^+)$ is a conformally compact Einstein manifold.

The compactification $\bar{g}$ induces a metric $\hat{g}$ on the boundary ∂X^{n-1} that changes conformally when changing the defining function x. Hence a conformally compact metric g^+ always induces a conformal structure $[\hat{g}]$ on the boundary ∂X^{n-1}. The conformal manifold $(\partial X^{n-1},\ [\hat{g}])$ is called the conformal infinity of the conformally compact manifold $(X^n,\ g^+)$. For an AH manifold (M, g), for any metric $\hat{g}$ in its conformal infinity $(\partial M, [\hat{g}])$, there exists a unique geodesic defining function x near infinity so that $|dx|_{x^2 g} = 1$ in a neighborhood of ∂X^{n-1} and $\hat{g} = x^2 g\big|_{\partial M}$; see [18].

Before recalling basic properties of conformally compact Einstein manifolds, we give a proof of Lemma 1.4 based on the proof of Lemma 3.1 in [14].

Proof of Lemma 1.4. The first step completely follows the proof of Lemma 3.1 in [14] and concludes that there is a coordinate at infinity (up to a C^3 collar diffeomorphism in the language of [14]) such that, for a defining function x,

$$\bar{g} = x^2 g^+ = dx^2 + \hat{g} + g^{(1)}x + O(x^2) \in C^2(\overline{X^n}) \tag{2.2}$$

for some symmetric 2-tensor $g^{(1)}$ and a representative $\hat{g} = x^2 g^+|_{T\partial X}$ on ∂X.

We then calculate the transformation of the Riemann curvature, the Ricci curvature and the scalar curvature based on $g^+ = x^{-2}\bar{g}$:

$$R_{ijkl}[g^+] = -(g^+_{ik}g^+_{jl} - g^+_{jk}g^+_{il}) + x^{-1}(g^+_{ik}\nabla^{\bar{g}}_j\nabla^{\bar{g}}_l x + g^+_{jl}\nabla^{\bar{g}}_i\nabla^{\bar{g}}_k x - g^+_{il}\nabla^{\bar{g}}_j\nabla^{\bar{g}}_k x - g^+_{jk}\nabla^{\bar{g}}_i\nabla^{\bar{g}}_l x) + O(x^2),$$

$$R_{ik}[g^+] = -(n-1)g^+ + x^{-1}((n-2)\nabla^{\bar{g}}_i\nabla^{\bar{g}}_k x + \Delta[\bar{g}]x\bar{g}_{ik}) + O(x^2),$$

and

$$R[g^+] = -n(n-1) + 2x(n-1)\Delta[\bar{g}]x + O(x^2).$$

At the same time we can calculate

$$\nabla^{\bar{g}}_i\nabla^{\bar{g}}_k x = \frac{1}{2}\partial_x\bar{g}_{ik} + O(x).$$

Therefore condition (1.7) gets translated to

$$(n-2)\partial_x\bar{g}_{ik} + \bar{g}^{jl}\partial_x\bar{g}_{jl}\bar{g}^{ik} \geq 0 \text{ and } \bar{g}^{jl}\partial_x g_{jl} = 0 \text{ at } x = 0,$$

which implies $g^{(1)} = 0$ and thus estimate (1.8) follows. □

The fundamental properties of conformally compact Einstein 4-manifolds that are useful to us are summarized in the following:

Lemma 2.2 ([14, 17, 18, 26])**.** *Let $(X^4,\ g^+)$ be a conformally compact Einstein manifold and x be the geodesic defining function associated with a representative $\hat{g}$ of the conformal infinity $(\partial X^3, [\hat{g}])$. In a neighborhood of the infinity*

$$g = x^{-2}\bar{g} = x^{-2}(dx^2 + g_x) \tag{2.3}$$

with the expansion

$$g_x = \hat{g} + g^{(2)}x^2 + g^{(3)}x^3 + \sum_{k=4}^{m} g^{(k)}x^k + o(x^m) \tag{2.4}$$

for any $m \geq 4$, where $g^{(k)}$ is a symmetric $(0,2)$ tensor on ∂X for all k and $g^{(3)}$ is the so-called non-local term. Moreover, the expansion (2.4) *only has even power terms when $g^{(3)}$ vanishes.*

As a consequence of the expansion (2.4) one gets the following volume expansion:

Lemma 2.3 ([18, 21])**.** *Let $(X^4,\ g^+)$ be a conformally compact Einstein manifold and x be the geodesic defining function associated with a representative $\hat{g}$ of the conformal infinity $(\partial X^3, [\hat{g}])$. One has*

$$Vol(\{x > \epsilon\}) = \frac{1}{3}Vol(\partial X, \hat{g})\epsilon^{-3} - \frac{1}{8}\int_{\partial X}(Rdv)[\hat{g}]\epsilon^{-1} + V(X^4,\ g^+) + o(1).$$

More importantly $V(X^4,\ g^+)$ is independent of the choice of representative $\hat{g}$ and is called the renormalized volume.

To appreciate the global invariant $V(X^4,\ g^+)$ of a conformally compact Einstein 4-manifold $(X^4,\ g^+)$ we recall the Gauss-Bonnet formula observed in [2, 10].

Lemma 2.4. *Let $(X^4,\ g^+)$ be a conformally compact Einstein 4-manifold. Then*

$$8\pi^2\chi(X) = \frac{1}{4}\int_X (|W|^2dv)[g^+] + 6V(X^4,\ g^+). \tag{2.5}$$

Comparing the Gauss-Bonnet formula (2.5) with the Gauss-Bonnet formula for compact 4-manifold $(X, \bar{g})$ with totally geodesic boundary:

$$8\pi^2\chi(X) = \frac{1}{4}\int_X (|W|^2 dv)[\bar{g}] + \int_X (\sigma_2(A)dv)[\bar{g}], \tag{2.6}$$

we arrive at

$$\int_X (\sigma_2(A)dv)[\bar{g}] = 6V(X^4,\ g^4), \tag{2.7}$$

for any compactification $\bar{g} = x^2 g^+$, where $\sigma_2(A[\bar{g}])$ is the second symmetric function of the eigenvalues of the Schouten curvature tensor $A[\bar{g}] = \frac{1}{n-2}(\mathrm{Ric} - \frac{R}{2(n-1)}\bar{g})$ of the metric $\bar{g}$.

To discuss the conformal gap theorem in [12] we recall the definition of Bach curvature tensor and the ϵ-regularity theorem for Bach flat Yamabe metrics. Here Yamabe metric means a metric with constant scalar curvature whose Yamabe quotient realizes the Yamabe constant in the conformal class. On 4-manifolds the Bach curvature tensor is a symmetric 2-tensor defined as follows:

$$B_{ij} = W_{kijl,}{}^{lk} + \frac{1}{2}R^{kl}W_{kijl}. \tag{2.8}$$

The following ϵ-regularity theorem has been established in [32]:

Lemma 2.5 ([32]). *Suppose that (M^4, g) is a Bach flat 4-manifold with Yamabe constant $Y > 0$ and g is a Yamabe metric. Then there exist positive numbers τ_k and C_k depending on Y such that, for each geodesic ball $B_{2r}(p)$ centered at $p \in M$, if*

$$\int_{B_{2r}(p)} |Rm|^2 dv \le \tau_k,$$

then

$$\sup_{B_r(p)} |\nabla^k Rm| \le \frac{C_k}{r^{2+k}} \left(\int_{B_{2r}(p)} |Rm|^2 dv \right)^{\frac{1}{2}}. \tag{2.9}$$

3. Conformally compact Einstein manifolds with large renormalized volumes

In this section we use Cheeger and Naber's result (see [13]) in dimension 4 to drop the L^2-bound in the conformal gap theorem in [12] and prove a conformal gap theorem for the renormalized volume on conformally compact Einstein manifolds. First we state Cheeger and Naber's result.

Theorem 3.1 (Theorem 1.5 [13]). *For any $v > 0$, there exists $C = C(v)$ such that if M^4 satisfies $|Ric_{M^4}| \le 3$ and $Vol(B_1(p)) > v > 0$, then*

$$\fint_{B_1(p)} |Rm|^2 dV \le C, \tag{3.1}$$

where p is a point in M and $B_1(p)$ is a geodesic ball on (M^4, g).

We then recall the following theorem on the analysis of ends of manifolds from [1, 6].

Theorem 3.2 ([1, 6]). *Let (M^n, g) $(n \geq 4)$ be a complete non-compact Ricci flat Riemannian manifold satisfying that*

$$\exists C > 0, \ s.t. \ Vol(B_r) \geq Cr^n \text{ for all } r > 0, \tag{3.2}$$

$$\int_M (|Rm|^{\frac{n}{2}} dV)[g] < \infty. \tag{3.3}$$

Then given $q \in M$, there is an $R_0 > 0$ such that

- *$M \setminus B_{R_0}(q)$ is diffeomorphic to the cone $(R_0, \infty) \times S^{n-1}/\Gamma$;*
- *$g = g_F + O(r^{-2})$, where g_F is the flat metric on the cone.*

Moreover, if M is simply connected at infinity, then $(M^n, \ g)$ is isometric to the Euclidean space.

We remark that if a complete Riemannian manifold (M^n, g) with non-negative Ricci curvature admits two ends, then by the splitting theorem, $M = \mathbb{R} \times N^{n-1}$ where N is of non-negative Ricci curvature. Therefore, by (3.2), there is only one end for the manifold M in Theorem 3.2.

It turns out, as a straightforward consequence of Theorem 3.1 of Cheeger and Naber, that one can drop the assumption (3.3) in Theorem 3.2 in dimension 4.

Lemma 3.3. *Let (M^4, g) be a complete non-compact Ricci flat Riemannian 4-manifold satisfying the Euclidean volume growth assumption* (3.2). *Then, given $q \in M$, there is an $R_0 > 0$ such that*

- *$M \setminus B_{R_0}(q)$ is diffeomorphic to the cone $(R_0, \infty) \times S^3/\Gamma$;*
- *$g = g_F + O(r^{-2})$, where g_F is the flat metric on the cone.*

Moreover, if M is simply connected at infinity, then $(M^4, \ g)$ is isometric to the Euclidean 4-space.

Proof. In light of Theorem 3.2 ([1, 6]) it suffices to show that there is a constant C such that

$$\int_{B_R} (|Rm|^2 dv)[g] \leq C$$

for any $R > 0$. For any fixed $R > 0$, we consider the metric $g_R = R^{-2} g$. It is easily verified that Theorem 3.1 is applicable to g_R. Hence the proof is complete. $\square$

Theorem 3.5 is an improved version of the conformal gap theorem in [12] based on the above Lemma 3.3. But we will present a complete proof of Theorem 3.5 that is slightly different from that in [12]. Let g_Y be the Yamabe metric in the conformal class $[g]$, i.e. $R_{g_Y} = 12$. Our approach relies on the control of the Yamabe constant via

$$\int_M (\sigma_2(A) dv)[g_Y] = \int_M \left(\left(\frac{1}{24} R^2 - \frac{1}{2} |\overset{\circ}{\mathrm{Ric}}|^2 \right) dv \right) [g_Y] \leq \frac{1}{24} (Y(M, [g]))^2 \tag{3.4}$$

and the following observation that describes the influence of the end structure from the Yamabe constant similar to that of the lower bound of the Euclidean volume growth.

Lemma 3.4. *Let $(M^n,\ g)$ $(n \geq 4)$ be a complete non-compact Ricci flat Riemannian manifold. Assume that condition* (3.3) *holds when $n \geq 5$. Then $(M^n,\ g)$ is isometric to the Euclidean space $\mathbb{R}^n$, provided that*

$$Y(M,\ g) \geq \tau Y(\mathbb{R}^n, g_E) \tag{3.5}$$

for some $\tau > 2^{-\frac{2}{n}}$.

Proof. First of all, according to Lemma 3.2 in [20], one indeed has a lower bound for the Euclidean volume growth from the lower bound on the Yamabe constant. Then from our Lemma 3.3 and the arguments in [1, 6] it is known that the tangent cone $(M^n_\infty,\ g_\infty)$ at infinity for the Ricci flat manifold $(M^n,\ g)$ is a flat cone $\mathbb{R}^n/\Gamma$. This comes from the following blow-down argument: the rescaled manifolds $(M^n,\ \lambda_i^2 g, p)$, as $\lambda_i \to 0$ with $p \in M^n$ a fixed point, converge to the tangent cone $(M_\infty,\ g_\infty, p_\infty)$ at infinity in the Cheeger-Gromov topology away from the singular point p_∞ (cf. [1, 6]). It is rather easily seen from (1.1) that the Yamabe quotient of the blow-down sequence converges to that of the tangent cone at infinity for any smooth function with compact support away from the vertex of the cone. For a smooth function with compact support in general, we simply use cut-off functions to modify it as follows. For any small $s > 0$, we consider cut-off functions:

$$\begin{cases} \phi = 0 \text{ when } r \leq s, \\ \phi = 1 \text{ when } r \geq 2s, \\ \phi \in C^\infty. \end{cases}$$

We may require that $|\nabla \phi| \leq C s^{-1}$ for some constant C. Then, for a function u with compact support, which is smooth away from the vertex and Lipschitz across the vertex, we calculate

$$\begin{aligned} \int_{\mathbb{R}^n/\Gamma} |\nabla u|^2 &= \int_{\mathbb{R}^n/\Gamma} |\nabla(\phi u)|^2 + \int_{\mathbb{R}^n/\Gamma} (2\nabla(\phi u)\cdot\nabla((1-\phi)u) + |\nabla((1-\phi)u)|^2) \\ &= \int_{\mathbb{R}^n/\Gamma} |\nabla(\phi u)|^2 + O(s^{n-2}) \end{aligned}$$

and

$$\int_{\mathbb{R}^n/\Gamma} u^{\frac{2n}{n-2}} = \int_{\mathbb{R}^n/\Gamma} (\phi u)^{\frac{2n}{n-2}} + O(s^n)$$

as $s \to 0$. Therefore, due to the scaling invariance of the Yamabe constant, we also know that

$$Y(M^n_\infty,\ g_\infty) \geq \liminf Y(M, \lambda_i g) \geq \tau Y(\mathbb{R}^n, g_E)$$

for some $\tau > 2^{-\frac{2}{n}}$ from our assumption (3.5). Suppose that

$$D_\Gamma : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$$

is the desingularization of the cone metric. Then it is easily seen that

$$\frac{\int_{\mathbb{R}^n} \frac{4(n-1}{n-2}(|\nabla u \circ D_\Gamma|^2 dv)[g_E]}{(\int_{\mathbb{R}^n} (u\circ D_\Gamma)^{\frac{2n}{n-2}} dv[g_E])^{\frac{n-2}{n}}} = |\Gamma|^{\frac{2}{n}} \frac{\int_{M^n_\infty} \frac{4(n-1}{n-2}(|\nabla u|^2 dv)[g_\infty]}{(\int_{M^n_\infty} u^{\frac{2n}{n-2}} dv[g_\infty])^{\frac{n-2}{n}}}. \tag{3.6}$$

Hence, when using appropriately modifications of the standard function

$$u \circ D_\Gamma = (\frac{2}{1+|x|^2})^{\frac{n-2}{2}}$$

in (3.6), one easily gets

$$Y(M^n_\infty,\ g_\infty) \le |\Gamma|^{-\frac{2}{n}} Y(\mathbb{R}^n,\ g_E),$$

which, compared to (3.5), forces the group Γ to be trivial and the Ricci flat manifold $(M^n,\ g)$ to be isometric to the Euclidean space $(\mathbb{R}^n,\ g_E)$ according to Lemma 3.3 in dimension 4 and Theorem 3.2 in general dimension. So the proof is complete. □

Now we are ready to state and prove the gap theorem for closed Bach flat 4-manifolds.

Theorem 3.5. *There exists a small positive number ϵ such that a closed Bach flat 4-manifold $(M,\ g)$ of positive Yamabe type has to be conformally equivalent to the round 4-sphere if*

$$\int_M \sigma_2(A[g]) dV_g \ge (1-\epsilon) 16\pi^2,$$

where $\sigma_2(A[g])$ is the second symmetric function of the eigenvalues of the Schouten curvature tensor $A[g]$ of the metric g.

In the course of the proof we shall need the following interesting result:

Lemma 3.6 ([12]). *There is $\epsilon > 0$ such that a Bach flat metric g on 4-sphere $\mathbb{S}^4$ is conformal to the standard round metric $g_{\mathbb{S}}$, provided that*

$$\int_{\mathbb{S}^4} (|W|^2 dv)[g] \le \epsilon.$$

Proof of Theorem 3.5. We argue by contradiction. Assume that there exists a sequence of Bach flat 4-manifolds $(M^4_j,\ (g_Y)_j)$ that are not conformally equivalent to the round 4-sphere, where $(g_Y)_j$ is the Yamabe metric with $R[(g_Y)_j] = 12$ and

$$\int_M (\sigma_2(A) dv)[(g_Y)_j] = (1-\epsilon_j)^2 16\pi^2 \to 16\pi^2$$

as $j \to \infty$. Hence, in light of (3.4), we have

$$(1-\epsilon_j) Y(\mathbb{S}^n,\ [g_{\mathbb{S}}]) \le Y(M^4_j, [(g_Y)_j]) \le Y(\mathbb{S}^n,\ [g_{\mathbb{S}}]) \tag{3.7}$$

and

$$\int_{M_j} (|\overset{\circ}{\mathrm{Ric}}|^2 dv)[(g_Y)_j] \le \frac{1}{12}\epsilon_j \to 0. \tag{3.8}$$

Here we use the fact that, on the round sphere $(\mathbb{S}^4, g_{\mathbb{S}})$,

$$\int_{S^4} (\sigma_2(A) dv)[g_{\mathbb{S}}] = \frac{1}{24} Y(\mathbb{S}^4,\ [g_{\mathbb{S}}])^2.$$

By the Gauss-Bonnet-Chern formula

$$8\pi^2 \chi(M_j) = \frac{1}{4}\int_{M_j} (|W|^2 dv)[g_j] + \int_{M_j} (\sigma_2(A) dv)[g_j]$$

and by our condition on $\int_{M_j} (\sigma_2(A) dv)[g_j]$, if M_j is diffeomorphic to $\mathbb{S}^4$, then

$$\int_{M_j} (|W|^2 dv)[g_j] \le 64\pi^2 \epsilon_j (2-\epsilon_j).$$

From the argument in [12], to get a contradiction it suffices to show that M^4_j is diffeomorphic to $\mathbb{S}^4$ for some subsequence of j.

We now use a more or less standard rescaling argument based on our Lemma 3.4. We first derive a contradiction if there was curvature blow-up. Again, we use Lemma 3.2 in [20] to get the uniform lower bound on the Euclidean volume growth for such sequence of manifolds. Then, standing at the point of curvature blow-up, that is, $p_j \in M_j$ such that

$$\lambda_j = |Rm|(p_j)[(g_Y)_j] = \max_{M_j} |Rm|[(g_Y)_j] \to \infty,$$

we consider the sequence of pointed Riemannian manifold (M_j, g_j, p_j) with the rescaled metric $g_j = \lambda_j^2 (g_Y)_j$. Therefore, according to the curvature estimates established, for example, in [32], one derives a subsequence that converges to complete non-compact manifold $(M_\infty, g_\infty, p_\infty)$ for the Cheeger-Gromov topology. As a consequence of (3.7) and (3.8) one knows that

- $Y(M_\infty, g_\infty) = Y(\mathbb{S}^n, [g_{\mathbb{S}}]) = Y(\mathbb{R}^n, g_E)$ and
- $\overset{\circ}{\mathrm{Ric}}[g_\infty] = 0$ and $R[g_\infty] = 0$.

Here we use an argument similar to that in the proof in Lemma 3.4 to derive the equality $Y(M_\infty, g_\infty) = Y(\mathbb{R}^n, g_E)$. Therefore (M_∞, g_∞) is isometric to the Euclidean 4-space according to Lemma 3.4. At the same time $|Rm|(p_\infty)[g_\infty] = 1$, which is a contradiction. On the other hand, if there is no curvature blow-up for the sequence $(M_j,\ (g_Y)_j)$, then the same compactness argument implies that there would exist a subsequence that converges to the round sphere for the Cheeger-Gromov topology and M_j is diffeomorphic to the round sphere for j large, which is impossible due to Lemma 3.6. Thus the proof of Theorem 3.5 is complete. $\square$

Next we use the facts collected in Section 2 to prove Theorem 1.1:

Proof of Theorem 1.1. Since $g^{(3)} = 0$, it follows from the expansion (2.4) that the doubling

$$(X_D,\ \tilde{g}) = (\bar{X} \bigcup \bar{X}, \bar{g})$$

is a smooth Bach flat 4-manifold (for more details about the doubling see [11], which uses [9]). In fact, it is also shown in [11] that if the conformal infinity $(\partial X^3, [\hat{g}])$ is of positive Yamabe type, then the doubling $(X_D, \tilde{g})$ is of positive Yamabe type. In the meantime we recall from (2.7) that

$$\int_{X_D} (\sigma_2(A)dv)[\tilde{g}] = 2\int_X (\sigma_2(A)dv)[\bar{g}] = 12V(X^4,\ g^+)$$

to conclude that

$$\int_{X_D} (\sigma_2(A)dv)[\tilde{g}] \geq (1-\epsilon)16\pi^2$$

by the assumption (1.4). Now one applies Theorem 3.5 to the doubling $(X_D,\ \tilde{g})$ and derives that $(X_D, \tilde{g})$ is conformally equivalent to the round 4-sphere when ϵ is sufficiently small. Particularly one obtains that $(X^4,\ g^+)$ is a simply connected Riemannian manifold of constant curvature -1 (cf. [11]) when ϵ is sufficiently small, which completes the proof. $\square$

To derive Theorem 1.2 we carry the above rescaling scheme with Lemma 3.4 on conformally compact Einstein manifolds. In fact we are able to derive a curvature bound for conformally compact Einstein manifolds in arbitrary dimension. Theorem 1.2 is then a corollary of this result.

Theorem 3.7. *Given constants $B > 0$ and $\tau > 2^{-\frac{2}{n}}$, there exists a constant $C = C(n, \tau, B) > 0$ such that*

$$|Rm|[g^+] \le C, \tag{3.9}$$

for any conformally compact Einstein manifold $(X^n,\ g^+)$ $(n \ge 4)$ with

- $Y(X^n,\ g^+) \ge \tau Y(\mathbb{H}^n,\ g_{\mathbb{H}})$ *and*
- $\int_{X^n} (|W|^{\frac{n}{2}} dv)[g^+] \le B$ *when* $n \ge 5$.

Proof. Assume, by contradiction, that there is a sequence of conformally compact Einstein manifolds $(X_j^n,\ g_j^+)$ satisfying the assumptions in the theorem with curvature blowing up. Since conformally compact Einstein manifolds are always asymptotically hyperbolic, we may extract a sequence of points $p_j \in X_j^n$ such that

$$\lambda_j = |Rm|(p_j)[g_j^+] = \max_{X_j^n} |Rm|[g_j^+] \to \infty$$

and consider the pointed rescaled manifolds $(X_j^n,\ \lambda_j g_j^+, p_j)$. It is then easily seen that there is a subsequence (X_j^n, g_j^+, p_j) converging in the Cheeger-Gromov topology to a complete non-compact Ricci flat manifold $(X_\infty^n, g_\infty, p_\infty)$ with

- $Y(X_\infty^n,\ g_\infty) \ge \tau Y(\mathbb{H}^n, g_{\mathbb{H}}) = \tau Y(\mathbb{R}^n,\ g_E)$;
- $|Rm|(p_\infty)[g_\infty] = 1$;
- $\int_{X_\infty^n} (|W|^{\frac{n}{2}} dv)[g_\infty] \le B$ when $n \ge 5$,

which is a contradiction in light of Lemma 3.4. Note that the lower bound $\tau > 2^{-\frac{2}{n}}$ is used to guarantee that the cone in the blow-down argument is trivial as in Lemma 3.4. □

Corollary 3.8. *For any $\epsilon \in \left(0, \frac{1}{2}\right)$, there exists a number $C > 0$ such that*

$$|Rm|[g^+] \le C,$$

for any conformally compact Einstein 4-manifold $(X^4,\ g^+)$ with conformal infinity of positive Yamabe type such that

$$V(X^4,\ g^+) \ge (1-\epsilon)\frac{4\pi^2}{3} = (1-\epsilon)V(\mathbb{H}^4,\ g_{\mathbb{H}}).$$

Proof. Using the Gauss-Bonnet formula (2.5) and our assumption on the renormalized volume, we get the upper bound of L^2 norm of the Weyl tensor. To apply Theorem 3.7 to get the conclusion, it suffices to verify that

$$Y(X^4, g^+) \ge (1-\epsilon)^{\frac{1}{2}} Y(\mathbb{H}^4, g_{\mathbb{H}}). \tag{3.10}$$

From (3.7) in the proof of Lemma 3.4 we know that

$$Y(X_D,\ \tilde{g}) \ge (1-\epsilon)^{\frac{1}{2}} Y(\mathbb{H}^4,\ g_{\mathbb{H}}),$$

which implies (3.10) by the conformal invariance of the Yamabe constant. □

From the proof of Corollary 3.8, the local Yamabe constant $Y(X^4,\ g^+)$ for a conformally compact Einstein 4-manifold approaches that of the hyperbolic space as the renormalized volume $V(X^4,\ g^+)$ approaches that of the hyperbolic space.

Before we end this section we state an easy observation based on the construction of Aubin about the impact of the local Yamabe constant to the local geometry in higher dimensions.

Proposition 3.9. *For any $\epsilon > 0$, there is $\delta > 0$ such that the sectional curvature $K[g^+]$ of any conformally compact Einstein manifold $(X^n,\ g^+)$, $n \geq 6$, satisfies*

$$|K + 1| \leq \epsilon,$$

provided that

$$Y(X^n,\ g^+) \geq (1 - \delta)Y(\mathbb{H}^n,\ g_{\mathbb{H}}).$$

Proof. We argue again by contradiction and assume that, for some $\epsilon_0 > 0$, there are a sequence of conformally compact Einstein manifolds $(X_j^n,\ g_j^+)$ and a sequence $\delta_j \to 0$ such that

$$Y(X_j^n,\ g_j^+) \geq (1 - \delta_j)Y(\mathbb{H}^n,\ g_{\mathbb{H}}),$$

but

$$|K + 1| > \epsilon_0,$$

for a sectional curvature K at some point on X_j^n. We are going to derive a contradiction in two steps. First, if there is a subsequence $(X_j^n,\ g_j^+)$ whose sectional curvature is not bounded as $j \to \infty$ (for convenience, we continue to use the same index j), then we may rescale the metrics and set

$$\tilde{g}_j = \lambda_j^2 g_j^+$$

where $\lambda_j = |W|(p_j)[g_j^+] = \max_{X_j^n} |W|[g_j^+]$. Then, due to the curvature estimates for Einstein manifolds, it is easily seen that, at least for a subsequence, $(X_j^n,\ \tilde{g}_j, p_j)$ converges to a complete non-compact Ricci flat manifold $(X_\infty^n,\ g_\infty, p_\infty)$ in the Cheeger-Gromov topology. But $|W|(p_\infty)[g_\infty] = 1$ contradicts the fact that $Y(X_\infty^n,\ g_\infty) = Y(\mathbb{R}^n, g_E)$ in light of the estimate of the Yamabe constant in paragraph 6.10 in [4] (cf. also [24]).

Secondly, if there is no curvature blow-up, then one may extract a subsequence such that

$$|W|(p_j)[g_j^+] \to w > 0$$

and $(X_j^n,\ g_j^+, p_j)$ converges to a complete non-compact Einstein manifold $(X_\infty^n,\ g_\infty^+, p_\infty)$ in Cheeger-Gromov topology. But, again, $|W|(p_\infty)[g_\infty^+] = w > 0$ contradicts the fact that $Y(X_\infty^n,\ g_\infty^+) = Y(\mathbb{H}^n, g_{\mathbb{H}})$ in light of the estimate of the Yamabe constant in paragraph 6.10 in [4]. □

A natural question is whether Proposition 3.9 still holds in dimension 5 and dimension 4. Answering this question would be much more significant.

4. Conformally compact Einstein manifolds whose conformal infinities have large Yamabe constants

In this section we introduce the idea of [16] to establish the relative volume growth bounds (1.6). We present here a self-contained argument for a complete and vigorous proof of (1.6) to the best of our knowledge.

We point out what is not clear and not correct in [16] and finish filling those gaps in the subsequent sections. Then we will carry out the rescaling argument to derive the curvature estimates in Theorem 1.6 in two steps similar to that in the proof of Proposition 3.9.

We recall that a Riemannian manifold $(X^n,\ g^+)$ is said to be AH (short for asymptotically hyperbolic) if it is conformally compact and its curvature goes to

-1 at infinity. Obviously a conformally compact Einstein manifold is always AH. Let us consider the distance function from a given point $p_0 \in X^n$:

$$t = \mathrm{dist}(\cdot, p_0)$$

and the geodesic sphere $\Gamma_t = \{p \in X^n : \mathrm{dist}(p, p_0) = t\}$. The important initial step in the proof of estimate (1.6) is estimate (2.3) in Lemma 2.1 of [16]:

$$\nabla_g^2 t(v, v) = 1 + O(e^{-\beta t}), \tag{4.1}$$

where v is any unit vector perpendicular to ∇t and β is a positive number less than 2. It is not clear to us how Section 6.2 in [29] is applied in the proof (4.1) in [16]. In fact this estimate does not seem to be correct without convexity of the geodesic spheres (cf. [22, 31]).

It is observed in [16] that one may employ the Bishop-Gromov relative volume comparison theorem and get

$$\frac{\mathrm{Vol}(\Gamma_t, g^+)}{\mathrm{Vol}(\Gamma_t, g_{\mathbb{H}})} \leq \frac{\mathrm{Vol}(B(t, p_0), g^+)}{\mathrm{Vol}(B(t, 0), g_{\mathbb{H}})} \leq 1$$

for all $t > 0$ on a conformally compact Einstein manifold. Hence the real issue for proving (1.6) is the lower bound and the key is to establish the relative volume lower bound by the limit

$$\lim_{t\to\infty} \frac{\mathrm{Vol}(\Gamma_t, g^+)}{\mathrm{Vol}(\Gamma_t, g_{\mathbb{H}})}.$$

The idea of [16] is very clever. It consists of noticing that one may use the Yamabe quotient to bound the relative volume $\frac{\mathrm{Vol}(\Gamma_t, g^+)}{\mathrm{Vol}(\Gamma_t, g_{\mathbb{H}})}$ from below. Suppose that $(X^n,\ g^+)$ is an AH manifold and that x is the geodesic defining function associated with a representative $\hat{g}$ of the conformal infinity $(\partial X^{n-1},\ [\hat{g}])$. Set

$$r = -\log \frac{x}{2} \tag{4.2}$$

and let $\Sigma_r = \{p \in X^n : r(p) = r\}$ be the level set of the geodesic defining function. We would like to mention that the inequality (2.2) in Lemma 2.1 of [16] is a well known fact about AH manifolds of C^3 regularity, which is

$$Ddr(v, v) = 1 + O(e^{-2r}), \tag{4.3}$$

where v is any unit vector perpendicular to ∇r, provided that (1.8) holds. Let $\bar{g} = x^2 g^+$ be the conformal compactification from the geodesic defining function x and let

$$\bar{g}_t = \bar{g}|_{\Gamma_t} \text{ and } \bar{g}_r = \bar{g}|_{\Sigma_r}.$$

Also let $\tilde{g} = 4e^{-2t} g^+ = \psi^{\frac{4}{n-3}} \bar{g}$ be the conformal compactification from the distance function and let

$$\tilde{g}_t = \tilde{g}|_{\Gamma_t} \text{ and } \tilde{g}_r = \tilde{g}|_{\Sigma_r},$$

where $\psi = e^{\frac{n-3}{2}(r-t)}$. First of all it is easily seen that $u = r - t$ is bounded on X^n by the triangle inequality for distance functions. It is then observed in Lemma 3.1 of [16] that $|\nabla u|[\bar{g}]$ is bounded. For the convenience of readers we present a complete proof and an easy fix for a gap in [16].

Lemma 4.1. *Assume that* $(X^n,\ g^+)$ *is an AH manifold of* C^3 *regularity and that the curvature decay estimate* (1.8) *holds. Then there is a constant* C *such that*

$$|du|[\bar{g}] \leq C, \tag{4.4}$$

when r *is large enough.*

Proof. It suffices to show that

$$g^+(\nabla t, \nabla r) = 1 + O(e^{-2t}). \tag{4.5}$$

First, as noticed in the proof of Lemma 3.1 in [16], if $\phi = g^+(\nabla t, \nabla r)$, then

$$\frac{d}{dt}\phi = \nabla^2 r(\nabla t, \nabla t) = (1 - \phi^2)\nabla^2 r(v, v), \tag{4.6}$$

where $\nabla t = \sqrt{1 - \phi^2} v + \phi \nabla r$. We want to point out that it is not enough to derive (4.5) just from (4.6) and (4.3). One needs the next lemma, which turns out to be an easy fact but not a consequence of (4.5) as presented in the proof of Lemma 4.1 of [16]. □

Lemma 4.2. *Assume that* $(X^n,\ g^+)$ *is AH of* C^3 *regularity and that estimate* (1.8) *holds. Let* x *be the geodesic defining function associated to a representative* $\hat{g}$ *of the conformal infinity* $(\partial X,\ [\hat{g}])$*. Let* $p_0 \in X^n$ *be a fixed point and let* t *be the distance from* p_0 *with respect to the metric* g^+*. And let* r *be given as in equation* (4.2)*. Then*

$$g^+(\nabla r, \nabla t) > 0$$

at any point where t *is smooth and if* r *is sufficiently large.*

Proof. It is known (cf. (2.2) in Lemma 2.1 of [16]) that there exists $r_0 > 0$ such that, for $r > r_0$,

$$\nabla^2 r(v, v) > \frac{1}{2}$$

for all unit vectors v that are perpendicular to ∇r. Then we claim that

$$\phi = g^+(\nabla t, \nabla r) > 0$$

for any point p where t is smooth and $r > r_0$. To see this, consider the minimal geodesic γ that connects p to p_0 and realizes the distance. Then, in light of (4.6), it is not hard to see that $\phi > 0$ from the time t_0 when the geodesic γ goes out of Σ_{r_0} (notice that $0 \leq \phi \leq 1$). □

By Lemma 4.2, for large r, a geodesic starting from p_0 can touch Σ_r at most once till the time it hits the cut locus of p_0. Hence by (4.4) the metric $\tilde{g} = \psi^{\frac{4}{n-3}} \bar{g}$ extends to $\overline{X^n}$ with Lipschitz regularity up to the boundary. In fact, as a consequence of estimate (4.3) and Lemma 4.1 as observed in Corollary 2.2 of [16], one has the following:

Corollary 4.3. *Assume that* $(X^n,\ g^+)$ *is AH of* C^3 *regularity and that estimate* (1.8) *holds. Then the second fundamental form of the level set* Σ_r *in* $(X^n,\ \tilde{g})$ *converges to zero as* $r \to \infty$*.*

Corollary 4.3 is used in the calculation of the scalar curvature $R[\tilde{g}_r]$ for $(\Sigma_r, \tilde{g}_r)$ (cf. (5.5) in [16]). Lemma 4.4 that follows is one of the key steps of the approach in [16]. We present a complete proof of Lemma 4.4 in the subsequent sections of this paper. But we first outline the proof and point out what is incomplete and incorrect in the arguments in [16].

Lemma 4.4. *Assume that $(X^n,\ g^+)$ is an AH manifold of C^3 regularity and that the Ricci curvature bound* (1.5) *holds. Assume further that the conformal infinity $(\partial X^{n-1}, [\hat g])$ has positive Yamabe type. Then*

$$\left(\frac{Y(\partial X, [\hat g])}{(n-2)(n-1)}\right)^{\frac{n-1}{2}} \leq Vol(\partial X, \tilde g_0) = \lim_{t\to\infty} Vol(\Gamma_t, \tilde g_t),$$

where $\tilde g_0$ is the continuous extension of $\tilde g$ to the boundary.

Proof. We are going to describe the steps of the proof of this lemma in the following. We will specify what is known from [16] and more importantly what we will do in the subsequent sections to achieve the goal in each step.

The first step is to derive the lower bound for the limit

$$\liminf_{r\to\infty} \frac{\left(\int_{\Sigma_r\setminus C_p} Rdv\right)[\tilde g_r]}{\left(\int_{\Sigma_r} dv[\tilde g_r]\right)^{\frac{n-3}{n-1}}},$$

which is a modification of (3.8) in [16], where C_p is the cut locus of p. Using the Lipschitz extension of ψ based on Lemma 4.1, we have

$$\frac{\int_{\partial X}\left(\left(\frac{4(n-2)}{n-3}|\nabla\psi|^2 + R\psi^2\right)dv\right)[\hat g]}{\left(\int_{\partial X}\psi^{\frac{2(n-1)}{n-3}}dv[\hat g]\right)^{\frac{n-3}{n-1}}} \leq \liminf_{r\to\infty}\frac{\int_{\Sigma_r}\left(\left(\frac{4(n-2)}{n-3}|\nabla\psi|^2 + R\psi^2\right)dv\right)[\bar g_r]}{\left(\int_{\Sigma_r}\psi^{\frac{2(n-1)}{n-3}}dv[\bar g_r]\right)^{\frac{n-3}{n-1}}}. \tag{4.7}$$

On the other hand, we remark that, though

$$\int_{\Sigma_r}\left(\left(\frac{4(n-2)}{n-3}|\nabla\psi|^2 + R\psi^2\right)dv\right)[\bar g_r] = \int_{\Sigma_r}(Rdv)[\tilde g_r] \tag{4.8}$$

easily holds when ψ is smooth, equation (4.8) (cf. (3.7) in [16]) may not be correct when ψ is only known to be Lipschitz. Our approach is to use the deep understanding of the structure of cut locus based on [23, 28] to overcome the challenge. We will deal with this issue and complete this first step in Section 5 (cf. Theorem 5.4).

The second step is to obtain the pointwise scalar curvature estimate

$$R[\tilde g_r] \leq (n-1)(n-2) + o(1) \tag{4.9}$$

(cf. (5.1) in [16]). The proof of Lemma 5.1 in [16] uses Corollary 4.3 and the Laplacian comparison theorem. More importantly it uses estimate (4.1) in calculating

$$\nabla^2 t(\nabla r, \nabla r) = 1 - (g^+(\nabla r, \nabla r))^2 + O(e^{-3t}),$$

which indeed would imply (4.9) and

$$\begin{aligned}\liminf_{r\to\infty}\frac{\int_{\Sigma_r}(Rdv)[\tilde g_r]}{\left(\int_{\Sigma_r}dv[\tilde g_r]\right)^{\frac{n-3}{n-1}}} &\leq (n-1)(n-2)\lim_{r\to\infty}\mathrm{Vol}(\Sigma_r,\tilde g)^{\frac{2}{n-1}}\\ &= (n-1)(n-2)\mathrm{Vol}(\partial X,\tilde g_0)^{\frac{2}{n-1}}.\end{aligned} \tag{4.10}$$

We will present a proof of (4.10) without assuming estimate (4.1) at each smooth point of t on Σ_r. Our key idea is to show that the part of Σ_r where estimate (4.1) does not holds has arbitrarily small volume. We will present a complete proof of the second step in Section 6 (cf. Theorem 6.7).

The last step is to show that

$$\lim_{t\to\infty} \mathrm{Vol}(\Gamma_t, \tilde{g}_t) = \mathrm{Vol}(\partial X, \tilde{g}_0). \tag{4.11}$$

To do so, we use an argument similar to the discussions in Section 6 of [16] that is based on Lemma 4.1. We consider the geodesic sphere Γ_t as a Lipschitz graph over ∂X in $(X^n,\ \tilde{g})$, where

$$x(t) \to 0 \text{ as } t \to \infty$$

in the $W^{1,p}$-topology for any $p \in [1, \infty)$. It is then easily seen that estimate (4.11) holds, as shown in [16]. □

Using Lemma 4.4, for conformally compact Einstein manifolds, we now have the lower bound for the relative volume growth.

Proof of Theorem 1.3. As calculated in [16], we have the following equality:

$$\frac{\mathrm{Vol}(\Gamma_t, g^+)}{\mathrm{Vol}(\Gamma_t, g_{\mathbb{H}})} = \frac{\mathrm{Vol}(\Gamma_t, \tilde{g}_t)(\frac{e^t}{2})^{n-1}}{\omega_{n-1}\sinh^{n-1} t} = \frac{\mathrm{Vol}(\Gamma_t, \tilde{g}_t)}{\omega_{n-1}} + o(1) \tag{4.12}$$

as $t \to \infty$. □

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. First we want to show that there exist constants $\delta_0 > 0$ and $C > 0$ such that

$$|W|[g^+] \le C$$

for any conformally compact Einstein manifolds that satisfy the assumptions in Theorem 1.6 for $0 < \delta \le \delta_0$. Assume by contradiction that there exists a sequence of conformally compact Einstein manifolds $(X^n_j,\ g^+_j)$ such that

$$|W|[g^+_j] \to \infty \text{ and } Y(\partial X_j, [\hat{g}_j]) \to Y(\mathbb{S}, [g_{\mathbb{S}}])$$

as $j \to \infty$. By Theorem 1.3, we know that

$$\left(\frac{Y(\partial X_j, [\hat{g}_j])}{Y(\mathbb{S}, [g_{\mathbb{S}}])}\right)^{\frac{n-1}{2}} \le \frac{\mathrm{Vol}(\Gamma_t, g^+_j)}{\mathrm{Vol}(\Gamma_t, g_{\mathbb{H}})} \le \frac{\mathrm{Vol}(B(p_j, t), g^+_j)}{\mathrm{Vol}(B(0, t), g_{\mathbb{H}})} \le 1 \tag{4.13}$$

for $t > 0$. Since $|W|[g^+_j](p) \to 0$ as $p \to \infty$ on each conformally compact Einstein manifold $(X^n_j,\ g^+_j)$, there exists a point $p_j \in X^n_j$ so that

$$\tau_j = |W|[g^+_j](p_j) = \max_{p\in X^n_j} |W|[g^+_j](p) \to \infty$$

as $j \to \infty$. We then consider the rescaled metric $g_j = \tau_j g^+_j$ on the pointed manifold (X^n_j, p_j). From (4.13), one may conclude that the sequence of pointed Einstein manifolds $(X^n_j, \tau_j g^+_j, p_j)$ converges to a Ricci flat manifold $(X^n_\infty, g_\infty, p_\infty)$ for the Cheeger-Gromov topology. In particular, one gets, again from (4.13), that

$$\mathrm{Vol}(\Gamma_t, g_\infty) = \mathrm{Vol}(\Gamma_t, g_E)$$

for all $t > 0$, which implies that (X^n_∞, g_∞) is isometric to the Euclidean space $(\mathbb{R}^n,\ g_E)$ and hence contradicts $|W|(p_\infty)[g_\infty] = 1$.

To finish the proof of Theorem 1.6 we assume again by contradiction that there are $\epsilon_0 > 0$ and a sequence of conformally compact Einstein manifolds $(X^n_j,\ g^+_j)$ such that

$$|W|(p_j)[g^+_j] \ge \epsilon_0 \text{ and } Y(\partial X,\ [\hat{g}_j]) \to Y(\mathbb{S}^n, [g_{\mathbb{S}}])$$

as $j \to \infty$. According to the above uniform bound for the curvature for such a sequence, we may extract a subsequence of pointed Einstein manifolds $(X_j^n,\ g_j^+, p_j)$ with $|W|(p_j)[g_j^+] \geq \epsilon_0 > 0$, which converges to an Einstein manifold $(X_\infty^n, g_\infty, p_\infty)$ for the Cheeger-Gromov topology. Then the same argument as the one at the end of Section 7 in [16] produces a contradiction to $|W|(p_\infty)[g_\infty] \geq \epsilon_0 > 0$. So the proof is complete. □

It is obvious that the argument in the first step in the above proof of Theorem 1.6 implies Corollary 1.7, in the same spirit as in the proof of Lemma 3.4. As an easy consequence of Theorem 1.6 and Theorem B in [10], we have

Corollary 4.5. *There exists $\epsilon > 0$ such that, for a conformally compact Einstein 4-manifold $(X,\ g^+)$ that satisfies*

$$V(X, g^+) > \frac{1}{2} V(\mathbb{H}^4, g^+) = \frac{2\pi^2}{3}$$

and

$$Y(\partial X, [\hat{g}]) > (1-\epsilon) Y(S^3, [g_{\mathbb{S}}]),$$

the closure $\bar{X}$ of X is diffeomorphic to the closed unit ball in Euclidean space.

5. Normal cut loci

In this section we focus on the issue in the first step of the proof of Lemma 4.4. First ψ is smooth away from the cut locus of the point p in a conformally compact Einstein manifold $(X^n,\ g^+)$. Hence it is necessary to understand the fine structure of the cut locus and the behavior of the distance function near the cut locus in order to study equation (4.8).

One might think that, for a fixed p in a complete and non-compact manifold $(M^n,\ g)$, the cut locus C_p may be contained in a compact set. In fact, to the contrary, C_p cannot be separated as a disjoint union of a compact subset and another closed subset (possibly empty). This is because $M^n \setminus C_p$ is always diffeomorphic to a star-shaped domain in the Euclidean space $\mathbb{R}^n$.

On a complete Riemannian manifold $(M^n,\ g)$ with a fixed point $p \in M^n$, the set C_p consists of the set Q_p of conjugate points and the set A_p of non-conjugate cut locus. Among the points in A_p we call those from which there are exactly two minimal geodesics connecting to p and realizing the distance to p in $(M^n,\ g)$ the normal cut locus, according to [23,28]. We denote the normal cut locus N_p and the rest of non-conjugate cut locus by L_p. In those notations we have

$$C_p = Q_p \cup L_p \cup N_p.$$

We recall from [23,28] the following facts about the cut locus on Riemannian manifolds in general.

Lemma 5.1 ([23,28]). *Suppose that $(M^n,\ g)$ is a complete Riemannian manifold and that $p \in M^n$. Then:*

- *The closed set $Q_p \cup L_p$ is of Hausdorff dimension no more than $n-2$.*
- *The normal cut locus N_p consists of possibly countably many disjoint smooth hypersurfaces in M^n.*
- *Moreover, at each point $q \in N_p$ in the normal cut locus, there is a small open neighborhood U of q such that $U \cap C_p = U \cap N_p$ is a piece of smooth hypersurface in M^n.*

In our case, on a conformally compact Einstein manifold $(X^n,\ g^+)$ with a given point p, we are concerned with the set

$$\gamma_r = \Sigma_r \cap C_p = \underbrace{(\Sigma_r \cap (Q_p \cup L_p))}_{:=\gamma^{QL}} \cup \underbrace{(\Sigma_r \cap N_p)}_{:=\gamma_r^N},$$

where ψ is not smooth as a Lipschitz function on Σ_r. Before we move to look closely at equation (4.8) we mention some more facts about the distance function t and the geodesic spheres in our context.

Lemma 5.2. *Assume that $(X^n,\ g^+)$ is AH of C^3 regularity and that estimate* (1.8) *holds. Then when t is sufficiently large,*

- *the geodesic sphere Γ_t is a Lipschitz graph over ∂X;*
- *the outward angle of the corner at the normal cut locus on Γ_t is always less than π.*

Proof. The first statement can be proven using the same argument as in Section 4 of [16] (cf. Lemma 4.1 and Lemma 4.2 in the previous section).

From [23, 28] we know that the singularities for the geodesic sphere at normal cut locus points are corners that are, at least locally, the intersection of two smooth hypersurfaces. To see the outward angle of such a corner at each normal cut locus point is always less than π, let us argue by contradiction. Let γ be a (distance realizing) minimal geodesic from the fixed p to a normal cut locus point $q \in \Gamma_t$ where the inward angle is less than π. We may push toward the geodesic sphere Γ_t from inside a small geodesic ball centered along γ. Clearly the geodesic ball will definitely touch the geodesic sphere Γ_t at some point $\bar{q} \in \Gamma_t$ before it reaches the corner $q \in \Gamma_t$. At this moment the center of the geodesic ball is a point p_1 on γ. Then

$$\mathrm{dist}_g(p,\bar{q}) \le \mathrm{dist}_g(p,p_1) + \mathrm{dist}_g(p_1,\bar{q}) < \mathrm{dist}_g(p,p_1) + \mathrm{dist}_g(p_1,q) = \mathrm{dist}_g(p,q) = t.$$

This gives a contradiction to the fact that $\bar{q} \in \Gamma_t$. □

By compactness of γ_r^{QL}, there are finitely many points $p_i \in \gamma_r^{QL}$ so that $B_r^\epsilon = \bigcup_i B_\epsilon(p_i)$ covers γ_r^{QL}. Then B_r^ϵ together with γ_r^N covers the set γ_r. Then we perform integration by parts to calculate the Yamabe functional for ψ as follows:

$$\begin{aligned}(5.1)\qquad &\int_{\Sigma_r\setminus(B_r^\epsilon\cup\gamma_r^N)} \left(\left(\frac{4(n-2)}{n-3}|\nabla\psi|^2 + R\psi^2\right)dv\right)[\bar{g}_r] \\ &\quad = \int_{\Sigma_r\setminus(B_r^\epsilon\cup\gamma_r^N)} (Rdv)[\tilde{g}_r] + \oint_{\partial(\Sigma_r\setminus(B_r^\epsilon\cup\gamma_r^N))} \frac{4(n-2)}{n-3}\psi\left(\frac{\partial}{\partial n}\psi d\sigma\right)[\bar{g}_r].\end{aligned}$$

In light of Lemma 5.1, for almost every r as $r\to\infty$ and almost every ϵ as $\epsilon \to 0$, one may assume γ_r^{QL} is of Hausdorff dimension no more than $n-3$ and $\gamma_r^N \setminus B_r^\epsilon$ is a union of finitely many disjoint hypersurfaces in Σ_r. Hence

$$\begin{aligned}&\oint_{\partial(\Sigma_r\setminus(B_r^\epsilon\cup\gamma_r^N))} \psi\left(\frac{\partial}{\partial n}\psi d\sigma\right)[\bar{g}_r] \\ &\quad = \oint_{\partial B_r^\epsilon} \psi\left(\frac{\partial}{\partial n}\psi d\sigma\right)[\bar{g}_r] + \oint_{\gamma_r^N\setminus B_r^\epsilon} \psi\left(\left(\frac{\partial}{\partial n^+}\psi + \frac{\partial}{\partial n^-}\psi\right)d\sigma\right)[\bar{g}_r]\end{aligned}$$

where n^+ and n^- are the two outward normal directions to γ_r^N from the inside of $\Sigma_r \setminus (B_r^\epsilon \cup \gamma_r^N)$. It is not hard to see that

$$\oint_{\partial B_r^\epsilon} \psi \left(\frac{\partial}{\partial n} \psi d\sigma \right) [\bar{g}_r] \to 0 \tag{5.2}$$

as $\epsilon \to 0$. Indeed, γ_r^{QL} is compact and of Hausdorff dimension no more than $n-3$ and ψ is uniformly Lipschitz on Σ_r (cf. (4.4) in Lemma 4.1). Recall that $\psi = e^{\frac{n-3}{2}(r-t)}$ at least for almost every r. Therefore

$$\frac{\partial}{\partial n^\pm}\psi = -\frac{n-3}{2}\psi\frac{\partial t}{\partial n^\pm} = -\frac{n-3}{2}\psi(\nabla t)^\pm \cdot n^\pm$$

and

$$\frac{\partial}{\partial n^+}\psi + \frac{\partial}{\partial n^-}\psi = -\frac{n-3}{2}\psi(\frac{\partial t}{\partial n^+} + \frac{\partial t}{\partial n^-}) = -\frac{n-3}{2}\psi((\nabla t)^+ - (\nabla t)^-) \cdot n^+, \tag{5.3}$$

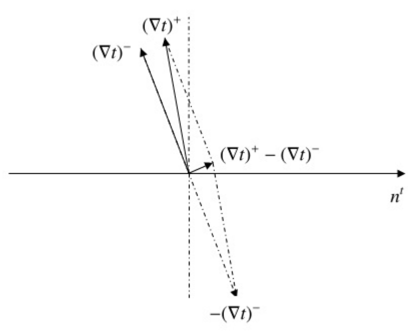
where $(\nabla t)^\pm$ is the gradient of the distance function t with respect to the metric $\bar{g}$ from either side of the corner γ_r^N.

Lemma 5.3. *For almost every r, when γ_r^N is the union of disjoint hypersurfaces in Σ_r,*

$$\frac{\partial t}{\partial n^+} + \frac{\partial t}{\partial n^-} \geq 0 \tag{5.4}$$

at each point on γ_r^N.

Proof. Given a point $q \in \gamma_r^N$, let us consider the plane P spanned by $(\nabla t)^+$ and $(\nabla t)^-$. We denote by n^t the normalized orthogonal projection of n^+ on the plane P. One notices that, from estimate (4.5), the angle between $(\nabla t)^+$ and $(\nabla t)^-$ is arbitrarily small as well as the angle between $(\nabla t)^\pm$ and n^t is arbitrarily close to $\frac{\pi}{2}$, as $r \to \infty$. In light of equation (5.3), to verify inequality (5.4) it is equivalent to verify that the angle between $(\nabla t)^-$ and n^t is not smaller than the one between $(\nabla t)^+$ and n^t, since $\|(\nabla t)^+\| = \|(\nabla t)^-\|$.



This turns out to be true because the outward angle of the corner at any normal cut locus on the geodesic sphere is always less than π according to Lemma 5.2. $\square$

From the proof of Lemma 5.3, we can observe that for $r > 0$ large, N_p intersects Σ_r transversely, so that for $q \in N_p$, there exists $v \in T_q N_p$ such that the angle between v and $\nabla_g r(q)$ is bounded by Ce^{-r} with a uniform constant $C > 0$. Therefore, $\mathcal{H}^{n-1}(\Sigma_r \cap C_p) = 0$. Similarly, $\mathcal{H}^{n-1}(\Gamma_t \cap C_p) = 0$ for $t > 0$ large. To summarize what we have so far in this section we state the following proposition.

Theorem 5.4. *Assume that $(X^n,\ g^+)$ is an AH manifold of C^3 regularity and that the decay condition (1.8) holds. Let p be a fixed point on X^n. Let t be the distance function from p on $(X^n,\ g^+)$ and let $r = -\log\frac{x}{2}$, where x is the geodesic defining function associated with a representative $\hat{g}$ of the conformal infinity $(\partial X,\ [\hat{g}])$. Then for almost every large $r > 0$ so that $\mathcal{H}^{n-2}(\gamma_r^{QL}) = 0$, it holds that*

$$\int_{\Sigma_r}\left(\left(\frac{4(n-2)}{n-3}|\nabla\psi|^2 + R\psi^2\right)dv\right)[\bar{g}_r] \le \int_{\Sigma_r\setminus(B_r^\epsilon\cup\gamma_r^N)}(Rdv)[\tilde{g}_r] + o_\epsilon(1), \tag{5.5}$$

where $\psi = e^{\frac{n-3}{2}(r-t)}$ and $o_\epsilon(1)\to 0$ as $\epsilon\to 0$.

6. Estimates for the total scalar curvature

In this section we focus on the issue in the second step of the proof of Lemma 4.4. Let us first be very clear on how estimate (4.1) is used in the argument in [16] and what one can hope to get for an upper bound for the scalar curvature $R[\tilde{g}_r]$. Recall from [16] that to estimate the scalar curvature $R[\tilde{g}_r]$, one starts with (5.5) in [16],

$$R[\tilde{g}_r] = R[\tilde{g}] - 2\mathrm{Ric}[\tilde{g}](N,N) + o(1),$$

and

$$\mathrm{Ric}[\tilde{g}](N,N) = \frac{e^{2t}}{4}(\mathrm{Ric}[g^+](\nabla r,\nabla r) + \Delta t + (n-2)(1-(g^+(\nabla r,\nabla t))^2)(\nabla^2 t(v,v)-1))$$

where $N = \frac{1}{2}e^t\nabla r$ and

$$\nabla r = g^+(\nabla r,\nabla t)\nabla t + \sqrt{1-(g^+(\nabla r,\nabla t))^2}\,v$$

for some unit vector $v\perp\nabla t$. Hence, following the calculation in [16] and assuming

$$\mathrm{Ric}[g^+] \ge -(n-1)g^+ \ \text{and}\ \ R[g^+] = -n(n-1) + o(e^{-2t}), \tag{6.1}$$

one arrives at

$$R[\tilde{g}_r] \le (n-1)(n-2) + \frac{n-2}{2}e^{2t}(1-(g^+(\nabla r,\nabla t))^2)(1-\nabla^2 t(v,v)) + o(1), \tag{6.2}$$

which implies (4.9) whenever (4.1) is available. In fact it is clear that any lower bound of the principal curvature would yield an upper bound for the scalar curvature $R[\tilde{g}_t]$ from (6.2) and (4.5). To our best knowledge, one does not have any a prior lower bound on the principal curvature, though one can manage indeed to get the desired upper bound for the principal curvature by the nature of the Riccati equations in general (cf. [22, 31]). What we observe is that the smaller the principal curvature is, the smaller the mean curvature is and, therefore, the smaller the surface volume element is, at any smooth point of the distance function. Thus we will still be able to obtain the desired estimate for the total scalar curvature

$$\int_{\Sigma_r\setminus C_{p_0}}(Rdv)[\tilde{g}_r] \le (n-1)(n-2)\int_{\Sigma_r}dv[\tilde{g}_r] + o(1) \tag{6.3}$$

where $o(1)\to 0$ as $r\to\infty$. We organize this section into three subsections.

6.1. **Curvature estimates based on the Riccati equations.** In this section we derive the curvature estimates based on the Riccati equations on AH manifolds. Let us start with the Riccati equation for the shape operator of the geodesic spheres along a geodesic $\gamma(t)$ (cf., for example, (2.1) in [16] or [22, 29, 31])

$$\nabla_{\nabla t}S + S^2 = -(R_{\nabla t})[g^+] \tag{6.4}$$

where $S = \nabla^2 t$ is the shape operator and $(R_{\nabla t})[g^+](v) = (R(v, \nabla t)\nabla t)[g^+]$ is the normal curvature. In particular, one has

$$1 - C_0 e^{-2t} \leq \mu'(t) + \mu^2(t) \leq 1 + C_0 e^{-2t} \tag{6.5}$$

for the principal curvature μ on an AH manifold satisfying estimate (1.8) (cf. (2.6) in [16]). We will use μ_m and μ_M to stand for the smallest and the biggest principal curvature respectively. The step following (2.6) in the proof of (2.3) in [16] does not seem to be correct to us. In the rest of this subsection we will present a careful study of the Riccati equations and derive the upper bounds and lower bounds of the principal curvature that hold on AH manifolds.

We now give a sharp upper bound for the principal curvature of geodesic spheres in AH manifolds; see also [5].

Lemma 6.1. *Assume that $(X^n,\ g^+)$ is AH and $p \in X^n$ is a fixed point. Also suppose that the curvature condition* (1.8) *holds on $(X^n,\ g^+)$. Then, for $t_0 > 0$, there exists a constant C such that*

$$\mu_M(q) \leq 1 + C(t+1)e^{-2t} \tag{6.6}$$

for all $q \in \Gamma_t \setminus C_p$ and all $t \geq t_0$.

Proof. From equation (6.5) one has

$$\mu_M' + \mu_M^2 \leq 1 + C_0 e^{-2t}.$$

Then one may consider $z = \mu_M - 1$ along the minimizing geodesic $\gamma(s)$ from p to $q \in \Gamma_t$ and the equation (6.5) for z,

$$z' + 2z \leq C_0 e^{-2s} - z^2 \leq C_0 e^{-2s},$$

which easily implies (6.6). We remark that the constant C may vary in different places. ☐

Roughly speaking, the reason why one has the sharp upper bound (6.6) is because 1 is a sink for the equation

$$\mu' + \mu^2 = 1. \tag{6.7}$$

On the other hand, since -1 is a source for the equation (6.7), one could not expect a lower bound in general. In fact the best one can say about the lower bound for the principal curvature is as follows:

Proposition 6.2. *Assume that $(X^n,\ g^+)$ is AH and $p \in X^n$ is a fixed point. Assume also that the curvature condition* (1.8) *holds on $(X^n,\ g^+)$. Then*

$$\mu_m(q) \geq -\sqrt{1 + C_0 e^{-2t}}$$

for $q \in \Gamma_t$ and q is on a minimizing geodesic ray that runs from p to infinity without intersecting the cut locus C_p of p.

Proof. Let γ be a minimizing geodesic that runs from p to infinity without intersecting the set C_p. We argue by contradiction. Assume that for the point $q \in \Gamma_t$ on γ,

$$\mu_m(\gamma(t)) < -\sqrt{1 + C_0 e^{-2t}}.$$

By (6.5), along the geodesic γ we have that

$$\mu_m'(s) + \mu_m^2(s) \le 1 + C_0 e^{-2s}, \text{ for } s \ge t. \tag{6.8}$$

But $\mu_m(t) < -\sqrt{1 + C_0 e^{-2t}}$; then

$$\mu_m(t)' < 0$$

near $s = t$. Combining with (6.8) we have that for $s > t$,

$$\mu_m^2(s) > 1 + C_0 e^{-2t}. \tag{6.9}$$

Let $t_1 > t$ be a point close to t. By (6.8) for $s \ge t_1$,

$$\mu_m(s)' + \mu_m(s)^2 < 1 + C_0 e^{-2t}. \tag{6.10}$$

By (6.9) we solve (6.10) for $s \ge t_1$ and obtain that μ_m goes to infinity in a finite time, which is a contradiction. ☐

6.2. **Volume estimates where the mean curvature is not big enough.** In this section we are concerned with the set

$$U_r^\delta = \{q \in \Sigma_r \setminus C_p : H(q) < (n-1)(1-\delta)\}$$

where $H = \Delta t$. We would like to show that

$$\int_{U_r^\delta} dv[\tilde{g}_r] \to 0$$

as $r \to \infty$.

We first notice from (4.5) in the proof of Lemma 4.1 that

$$\|dv[g^+](\nabla t) - dv[g^+](\nabla r)\|_{g^+} \le Ce^{-t} \tag{6.11}$$

and

$$\left| \frac{2^{n-1} e^{-(n-1)t} dv[g^+](\nabla r)}{2^{n-1} e^{-(n-1)t} dv[g^+](\nabla t)|_{(\nabla r)^\perp}} - 1 \right| = \left| \frac{dv[g^+](\nabla r)}{dv[g^+](\nabla t)|_{(\nabla r)^\perp}} - 1 \right| \le Ce^{-2t}, \tag{6.12}$$

where

$$dv[\tilde{g}_r] = 2^{n-1} e^{-(n-1)t} dv[g^+](\nabla r) \tag{6.13}$$

and $(\nabla r)^\perp$ stands for the hyperplane perpendicular to ∇r. This gives us a clue as to how the mean curvature $H = \Delta t$ is related to the volume element $dv[\tilde{g}_r]$ on Σ_r. We now focus on the behavior of the $(n-1)$-form $dv[g^+](\nabla t)$ along geodesics γ from p.

In a small neighborhood of a geodesic γ starting from the fixed point p before touching the cut locus of p, we use polar normal coordinates (t, θ) at p, where $d\theta$ is the area element of the unit $(n-1)$-sphere. Then we may write the area element of Γ_t at the intersection point with γ as

$$dv[g^+](\nabla t) = \mathcal{J} d\theta.$$

Now we recall from the first variation formula in Riemannian geometry that

$$\frac{d\mathcal{J}}{dt} = H\mathcal{J}. \tag{6.14}$$

To state the key observation for the volume estimate, for each small δ, we let t_δ be a fixed large number such that

$$(n-2)C(t+1)e^{-2t} \leq \frac{\delta}{4} \text{ and } 1 - C_0 e^{-2t} \geq (1-\frac{1}{2}\delta)^2$$

for all $t \geq t_\delta$, where C is the constant in (6.6) and C_0 is the constant in (1.8).

Lemma 6.3. *Assume that* (X^n, g^+) $(n \geq 4)$ *is AH of* C^3 *regularity and* $p \in X^n$ *is a fixed point. Also suppose that the curvature condition* (1.8) *holds. Let* $q \in U_r^\delta$ *and* $\gamma(s)$ *be the minimizing geodesic from* p *to* $q = \gamma(t)$ *for* $t > t_\delta$*. Then*

$$H(\gamma(s)) \leq (n-1)\left(1 - \frac{\delta}{4(n-1)}\right) \tag{6.15}$$

for all $s \in (t_\delta, t)$*.*

Proof. Assume by contradiction that there exists $s \in (t_\delta, t)$ such that

$$H(\gamma(s)) > (n-1)\left(1 - \frac{\delta}{4(n-1)}\right).$$

Using the sharp upper bound (6.6) we conclude that

$$\mu_m(\gamma(s)) > 1 - \frac{\delta}{2}. \tag{6.16}$$

It is then easily seen that (6.16) implies that

$$\mu_m(q) = \mu_m(\gamma(t)) \geq 1 - \frac{3}{4}\delta > 1 - \delta$$

because, from (6.5), one has

$$\mu_m' + \mu_m^2 \geq 1 - C_0 e^{-2t} \geq \left(1 - \frac{1}{2}\delta\right)^2$$

in (t_δ, t), which means μ_m' is positive whenever $\mu_m \in (0, 1 - \frac{1}{2}\delta)$. □

Consequently, from (6.13), (6.12), and (6.14), and considering U_r^δ as a graph on a subset of Γ_{t_δ} induced by the exponential map at p, we have the following proposition.

Proposition 6.4. *Assume that* (X^n, g^+) *is an AH manifold of* C^3 *regularity and that* $p \in X^n$ *is a fixed point. Assume also that the curvature condition* (1.8) *holds. Then*

$$\mathcal{J}(q) \leq \mathcal{J}(\gamma(t_\delta))e^{(n-1)(1-\frac{\delta}{4(n-1)})(t-t_\delta)} \tag{6.17}$$

for $q \in \Gamma_t \setminus C_p$ *with* $H < (n-1)(1-\delta)$ *and where* γ *is the minimizing geodesic from* p *to* q*. Thus, for each fixed small* $\delta > 0$*,*

$$\int_{U_r^\delta} dv[\tilde{g}_r] \to 0 \tag{6.18}$$

as $r \to \infty$*.*

6.3. **Volume estimates where the mean curvature is very negative.** If there were a lower bound for the mean curvature, then (6.2) and (6.6) would yield an upper bound the scalar curvature $R[\tilde{g}_r]$ and one would have by now completed the proof of (6.3). Therefore we need to estimate the lower bound of $Hdv[g^+](\nabla t)$ at least when the mean curvature H is very negative. First of all one easily derives from the first variational formula (6.14) that, along a geodesic $\gamma(s)$,

$$\frac{d}{ds}(Hdv[g^+](\nabla t)) = (H' + H^2)dv[g^+](\nabla t). \tag{6.19}$$

The key observation for this subsection is the following more detailed statement of Proposition 6.2 about the Riccati equation:

Lemma 6.5. *Suppose that $(X^n,\ g^+)$ is AH and $p \in X^n$ is a fixed point. Also suppose that the curvature condition (1.8) holds. Then, for any $t_0 > 0$, there is a constant $C > 0$, for $q \in \Gamma_t \setminus C_p$, and γ being the minimizing geodesic from p to q, one always has*

$$\mu_m(\gamma(t-1)) \geq -C, \tag{6.20}$$

provided that $t > t_0 + 1$.

Proof. To start, one considers the Riccati equation (6.5) along the geodesic γ when $\mu(s_0) < -\sqrt{1 + C_0 e^{-2s_0}}$ with $s_0 = t - 1 > t_0$. Recall that

$$\mu_m'(s) + \mu_m^2(s) \leq 1 + C_0 e^{-2s}, \text{ for } s \geq s_0,$$

which implies that

$$\mu_m'(s) + \mu_m^2(s) \leq 1 + C_0 e^{-2s_0}, \text{ for } s \geq s_0.$$

By solving the differential inequality we have that

$$\mu(s) \leq -a \frac{1 + \chi e^{2a(s-s_0)}}{1 - \chi e^{2a(s-s_0)}},$$

where $a = \sqrt{1 + C_0 e^{-2s_0}}$ and χ is a constant depending on $s_0 = t - 1$ but independent of s such that $\chi = -\frac{a+\mu(s_0)}{a-\mu(s_0)} \in (0, 1)$. Therefore it is clear that μ reaches $-\infty$ at

$$s = s_0 - \frac{1}{2a}\log\chi = t - 1 - \frac{1}{2a}\log\chi,$$

and by the definition of χ,

$$\lim_{\mu(t-1)\to -a} \chi = \lim_{\mu(s_0)\to -a} \chi = 0 \text{ and } \lim_{\mu(t-1)\to -\infty} \chi = 1.$$

Moreover, if

$$\mu(t-1) \leq -\frac{a(e^a + 1)}{(e^a - 1)},$$

we have that

$$-\frac{1}{2a} log\chi < \frac{1}{2},$$

and then $s_1 \leq t - 1 + \frac{1}{2}$ so that μ reaches $-\infty$ before $q = \gamma(t)$, which is a contradiction. Therefore, there exists a constant C independent of $t > t_0 + 1$ so that (6.20) holds. □

We are now ready to deal with the volume estimate at places the mean curvature is very negative.

Proposition 6.6. *Assume that $(X^n,\ g^+)$ is AH and $p \in X^n$ is a fixed point. Assume further that the curvature condition (1.8) holds. Then there is a constant $C > 0$ such that, for $q \in \Gamma_t \setminus C_p$ with $H(q) \le -2(n-1)$ and γ being the minimizing geodesic from p to q,*

$$H\mathcal{J}(\gamma(t)) \ge -Ce^{(n-1)(1-\frac{\delta}{4(n-1)})t}, \tag{6.21}$$

provided that $t > t_\delta + 1$.

Proof. Using the Riccati equation and (6.19), we obtain

$$\frac{d}{ds}(H\mathcal{J}) = (-\mathrm{Ric}[g^+](\nabla t, \nabla t) - |\nabla^2 t|^2 + H^2)\mathcal{J}$$

for $s \in (t-1, t) \subset (t_\delta, t)$. Hence we are looking for a lower bound for $(\Delta t)^2 - |\nabla^2 t|^2$. Let

$$\{\mu_i : i = 1, \cdots, n-1\}$$

be the eigenvalues of $\nabla^2 t$ with $\mu_i \le \mu_{i+1}$ so that μ_1 is μ_m. We calculate that

$$(\Delta t)^2 - |\nabla^2 t|^2 = \sum_{i \ne j} \mu_i \mu_j \ge \sum_{\mu_i \cdot \mu_j < 0} \mu_i \mu_j \ge 2(n-1)^2 \mu_m - C_1 \ge 2(n-1)^2 H - C_2$$

for some positive constants $C_1 = C_1(n)$, $C_2 = C_2(n)$, where the last two inequalities use the sharp upper bound (6.6). For instance, by (6.6), C_1 can be chosen uniformly when $\mu_m \ge 0$, while we simply let $C_1 = 0$ when $\mu_m < 0$. Therefore by (1.8), we obtain

$$\frac{d}{ds}(H\mathcal{J}) \ge 2(n-1)^2 H\mathcal{J} - C\mathcal{J} \tag{6.22}$$

for some constant $C > 0$, which can be rewritten as follows:

$$\frac{d}{ds}(e^{-2(n-1)^2 s} H\mathcal{J}) \ge -Ce^{-2(n-1)^2 s}\mathcal{J}$$

for $s \in (t-1, t)$. Thus

$$H\mathcal{J}(\gamma(t)) \ge e^{2(n-1)^2} H\mathcal{J}(\gamma(t-1)) - Ce^{2(n-1)^2 t}\int_{t-1}^{t} e^{-2(n-1)^2 s}\mathcal{J}(\gamma(s))ds. \tag{6.23}$$

Now, in light of Lemma 6.5, (6.21) is proven if (6.17) in Proposition 6.4 is applicable to all points $\gamma(s) : s \in [t-1, t]$. To see that (6.17) holds at each point $\gamma(s) : s \in [t-1, t]$, one uses that $\mu_m(\gamma(t)) < -2$ and the Riccati equation (6.5) for $t - 1 > t_\delta$. Indeed one gets that $\mu_m(\gamma(s)) < -1 + C_0 e^{-2t_\delta}$ and hence $H(\gamma(s)) \le (n-1)(1-\delta)$ for all $s \in [t-1, t]$, at least when δ is small enough. □

Combining Proposition 6.4 and Proposition 6.6 we thus have obtained the following result.

Theorem 6.7. *Assume that $(X^n,\ g^+)$ is AH of C^3 regularity and $p \in X^n$ is a fixed point. Assume further that (6.1) holds. Then for almost all large $r > 0$ so that $\mathcal{H}^{n-2}(\gamma_r^{QL}) = 0$, it holds that*

$$\int_{\Sigma_r \setminus (B_\epsilon^r \cup N_p)} (Rdv)[\tilde{g}_r] \le (n-1)(n-2)\int_{\Sigma_r} dv[\tilde{g}_r] + o_r(1), \tag{6.24}$$

where $o_r(1)$ is independent of $\epsilon > 0$ and $o_r(1) \to 0$ as $r \to \infty$.

Proof. At any point $q \in \Sigma_r \setminus C_p$, by (6.12),

$$\left| \frac{Hdv[g^+](\nabla r)}{Hdv[g^+](\nabla t)|_{(\nabla r)^\perp}} - 1 \right| \leq Ce^{-2t}.$$

Since $|t - r|$ is uniformly controlled, for $q \in \Sigma_r \setminus C_p$ so that $H \leq -2(n-1)$ we get the decay of $H\, dv[g^+](\nabla r)$ by Proposition 6.6.

For any $\delta > 0$ and for $r > 0$ large as in the statement of the theorem, considering Σ_r as a graph of Γ_{t_δ} induced by the exponential map at p, we have the splitting of the integral:

$$\begin{aligned}
&\int_{\Sigma_r \setminus (B^r_\epsilon \cup N_p)} (RdV)[\tilde{g}_r] \\
&= \int_{\{q \in \Sigma_r \setminus (B^r_\epsilon \cup N_p):\, H(q) > (n-1)(1-\delta)\}} (RdV)[\tilde{g}_r] \\
&+ \int_{\{q \in \Sigma_r \setminus (B^r_\epsilon \cup N_p):\, -2(n-1) < H \leq (n-1)(1-\delta)\}} (RdV)[\tilde{g}_r] \\
&+ \int_{\{q \in \Sigma_r \setminus (B^r_\epsilon \cup N_p):\, H \leq -2(n-1)\}} (RdV)[\tilde{g}_r].
\end{aligned}$$

Note that $Ddt(v,v) \geq \mu_m \geq H - C$. By Proposition 6.4 and Proposition 6.6 combined with (6.13) and (6.14) for an arbitrary choice of $\delta > 0$, we complete the proof of (6.24). $\square$

Acknowledgments

The authors are very grateful to the Beijing International Center for Mathematical Research (BICMR) for its hospitality. Most of the research in this paper was conducted during the summer when the authors visited BICMR. The authors are also indebted to Professor Romain Gicquaud for careful readings and helpful suggestions for the improvement of the writing of this paper. The first author would like to thank Professor Matthew Gursky for his interest of the problems and his encouragement. The first author would also like to thank Professor Leon Simon for enlightening discussions on geometric measure theory.

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