TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 369, Number 7, July 2017, Pages 4817–4842
http://dx.doi.org/10.1090/tran/6819
Article electronically published on December 7, 2016

CAUCHY TRANSFORMS OF SELF-SIMILAR MEASURES: STARLIKENESS AND UNIVALENCE

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ABSTRACT. For the contractive iterated function system $S_k z = e^{2\pi ik/m} + \rho(z - e^{2\pi ik/m})$ with $0 < \rho < 1, k = 0, \cdots, m-1$, we let $K \subset \mathbb{C}$ be the attractor, and let μ be a self-similar measure defined by $\mu = \frac{1}{m}\sum_{k=0}^{m-1} \mu \circ S_k^{-1}$. We consider the Cauchy transform F of μ. It is known that the image of F at a small neighborhood of the boundary of K has very rich fractal structure, which is coined the Cantor boundary behavior. In this paper, we investigate the behavior of F away from K; it has nice geometry and analytic properties, such as univalence, starlikeness and convexity. We give a detailed investigation for those properties in the general situation as well as certain classical cases of self-similar measures.

CONTENTS

1. INTRODUCTION

For a finite Borel measure μ with support on K in $\mathbb{C}$, we define the *Cauchy transform* of μ to be

$$F(z) = \int_K \frac{d\mu(w)}{z - w} \ . \tag{1.1}$$

This transform plays a central role in classical complex analysis, harmonic analysis and geometric measure theory ([CMR], [Ma1], [To4]). It has phenomenal development in connection with the Caldéron-Zygmund type singular integral theory when μ is supported on a Lipschitz graph, which is also extended to doubling and

Received by the editors December 12, 2014 and, in revised form, July 16, 2015.
2010 *Mathematics Subject Classification.* Primary 28A80; Secondary 30C55, 30E20.
Key words and phrases. Self-similar measure, Cauchy transform, radius of starlikeness, convexity, univalence.
This research was supported in part by the NNSF of China (No. 11571099), an HKRGC grant, SRFDP of Higher Education (No. 20134306110003), and Scientific Research Fund of Hunan Provincial Education Department (No. 14K057). The first author is the corresponding author.

non-doubling measures [To1]–[To4], and also on self-similar sets ([Ch], [CU]). The Cauchy transform has also been used extensively in the study of removable sets (sets of analytic capacity zero) of bounded analytic functions; that a compact K with $\mathcal{H}^1(K) < \infty$ is removable for bounded analytic functions if and only if it is purely unrectifiable (Vitushkin's conjecture) was a celebrated conjecture in complex analysis in the last half of the 20th century ([G2], [Ma1]–[Ma2]). It was eventually settled in the 1990's (Mattila, Melnikov and Verdera [MMV], and David [Da]). The reader can refer to [CMT], [To4] for an excellent historic account of this. The research also led to characterizing certain Cantor type sets in $\mathbb{C}$ with non-σ-finite length being capacity zero ([G1], [Ma3], [MTV]), and further development of the removability for Lipschitz harmonic functions on Heisenberg groups and, more generally, on Carnot groups, with special examples of self-similar removable sets ([CMa], [CMT]).

In another direction, Strichartz *et al.* in [LSV] pioneered an innovative study of the Cauchy transform of the self-similar measures that have dimension ≥ 1, in particular, on the Sierpinski gasket. This initiation was taken up by Dong *et al.* in a series of papers ([DL1]–[DL4], [DLL], [LDP]). Using complex analytic methods, they investigated the continuous extension of F on K, the asymptotic behavior of the Laurent coefficients in terms of the fractal dimension, and the fractal behavior of F near the boundary of K. Of particular interest is the geometry of the image of F near K; from computer graphics of the Cauchy transform on the Sierpinski gasket [LSV], it is seen that the image of F at the outer boundary of K is amazingly intriguing. This observation was investigated in detail in [DLL] and [DL4] through a more general framework of the *Cantor boundary behavior*, which also includes the gap series and the complex Weierstrass functions.

In contrast to the chaotic behavior near the boundary, the Cauchy transform is analytic at ∞, and hence it is expected to behave well for z large. Indeed in [DL1], the sharp asymptotic formula of the Laurent coefficients of F was obtained. As for the image of F, the behavior can be described by the univalence, starlikeness and convexity. They are the main topics in the paper that we will investigate.

We consider the special class of self-similar measures

$$\mu = \frac{1}{m}\sum_{k=0}^{m-1} \mu \circ S_k^{-1} \quad \text{where} \quad S_k z = e^{2\pi i k/m} + \rho(z - e^{2\pi i k/m}) \tag{1.2}$$

with $0 < \rho < 1$ ([LSV], [DL1]). They include the Cantor measures when $m = 2$, $0 < \rho < 1/2$, the Hausdorff measure $\mathcal{H}^\alpha$ ($\alpha = \log 3/\log 2$) on the Sierpinski gasket when $m = 3, r = 1/2$, and many other typical examples [F]. Let $K = K_{\rho,m}$ denote the attractor of the above iterated function system (IFS) $\{S_j\}_{j=0}^{m-1}$; then $K \subset \{z : |z| \leq 1\}$, $e^{2\pi i/m}K = K$, and $\text{supp}\mu = K$.

Let $F := F_{m,\rho}$ be the Cauchy transform of μ defined by (1.2). Then F is analytic on $\mathbb{C} \setminus K$ with $F(\infty) = 0$, m-fold symmetric: $F(e^{i2\pi/m}z) = e^{-i2\pi/m}F(z)$, and

$$F(\frac{1}{z}) = z + \sum_{n=1}^{\infty} a_n z^{nm+1}\ , \quad |z| < 1 \tag{1.3}$$

(see Section 4). Our main goal is to study the domain of univalence, starlikeness and convexity of the image of $F_{m,\rho}$. A set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E. A starlike function f is a conformal mapping (analytic and univalent) of the unit disk $\mathbb{D}$ onto a domain $f(\mathbb{D})$ starlike with respect to the point $f(0)$ (see

[Du]). Consider $f(z) = F(\frac{1}{z}), z \in \mathbb{D}$. If for some $r_0 \in (0,1)$, f is starlike univalent in $\mathbb{D}_{r_0} = r_0\mathbb{D}$, we say F is starlike univalent in $|z| > \frac{1}{r_0}$. The following theorem gives a lower bound for the domain of the starlikeness (Section 4), which is a consequence of the more general Theorem 1.5 in the sequel.

Theorem 1.1. *For $m \geq 2$, $F_{m,\rho}$ is starlike univalent in $|z| > R$ for some $R \in [1, R_m^{-1})$ where $R_2 = (8\sqrt{2}-11)^{1/4}$, $R_3 = (3\sqrt{3}-5)^{1/6}$ and $R_m = (m-1)^{-1/m}$, $m \geq 4$.*

For general self-similar measure μ (or just a probability measure) with support lying in the unit disk $\overline{\mathbb{D}}$ without assuming any symmetry, it is easy to check that $f(z) = F_\mu(\frac{1}{z}) \in \mathcal{F}_1(\frac{1}{2})$ (see the definition in (1.5)). From Proposition 2.2, the radius of starlikeness of $f(z) \in \mathcal{F}_1(\frac{1}{2})$ is $\geq R_1 = \sqrt{2}/2$ [Mac], the lower bounded $\sqrt{2}/2$ is $< R_m, m \geq 2$; hence the symmetry improves the starlikeness (see the remark after Theorem 3.1 for more detail). We can further improve such geometry for specific $F_{m,\rho}, m \geq 2$, though the calculation can be complicated. For this we consider special cases for $m = 2, 3, 4$.

The class of self-similar measures $\mu = \mu_{m,\rho}$ with $m = 2$ is the most fundamental class of such measures. For $\rho = 1/2$, μ is the normalized Lebesgue measure on $[-1, 1]$; for $0 < \rho < 1/2$, it is the Cantor type measures, and for $0 < \rho < 1$, it is the Bernoulli convolution ([L], [LN], [PPS]). We recall that a convex function is one which maps the unit disk $\mathbb{D}$ conformally onto a convex domain. Let $f(z)$ be analytic in $\mathbb{D}$, with $f(0) = 0, f'(0) = 1$. It follows from [Du, p. 42] that f is convex univalent in $\mathbb{D}$ if and only if

$$\mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) \geq 0, \qquad z \in \mathbb{D}.$$

If the above real part is $\geq \alpha, z \in \mathbb{D}$ for some $\alpha \in [0, 1)$, then we say f is an α-order convex univalent function in $\mathbb{D}$. We say that a domain D is a *Steiner symmetric domain* [Hay, p. 112] with respect to the real axis if for any real ξ, the intersection of D with the line $x = \xi$ is an open interval symmetric about ξ (or an empty set). We prove (see Section 5)

Theorem 1.2. (i) *$F_{2,\frac{1}{2}}$ is $\frac{1}{2}$-order convex univalent on $|z| > 1$ and is analytic univalent in $\widehat{\mathbb{C}} \setminus \big([-1, 1]\big)$, with $F_{2,\frac{1}{2}}(\widehat{\mathbb{C}} \setminus \big([-1, 1]\big))$ a convex domain;*

(ii) *for $0 < \rho < 1$, $F_{2,\rho}$ is univalent on $|z| > 1$, and $F_{2,\rho}\big(|z| > 1\big)$ is a Steiner symmetric domain with respect to the x-axis;*

(iii) *for $0 < \rho < \frac{1}{2}$, $F_{2,\rho}$ is not convex univalent on $|z| > 1$.*

For $m = 4$ (Section 6), we consider $F_{4,\frac{1}{2}}$, which is the Cauchy transform of the normalized Lebesgue measure of the square $K = K_{4,\frac{1}{2}}$ with vertices at $\pm 1, \pm i$. It is an analog of $F_{2,\frac{1}{2}}$ in the last theorem, but has a far richer complex structure. It is known from [DL1] that $F_{4,\frac{1}{2}}$ is analytic in $\mathbb{C} \setminus K$ and continuous on $\mathbb{C}$ and is not analytic at each point $z \in K$. Here, we prove

Theorem 1.3. *$F_{4,\frac{1}{2}}$ has single-valued analytic extension F^* with domain $\mathbb{C} \setminus ([-1, 1] \cup [-i, i])$. Moreover,*

(i) *F^* is starlike univalent on $\mathbb{C} \setminus ([-1, 1] \cup [-i, i])$,*

(ii) *$F_{4,\frac{1}{2}}$ is $(\frac{4}{\pi} - 1)$-order convex univalent on $|z| > 1$, and*

(iii) *$F_{4,\frac{1}{2}}$ is analytic univalent in $\mathbb{C} \setminus K$, and $F(\partial K)$ is a Jordan curve.*

Similar to Theorem 1.2, we can also show that $F_{4,\rho}$ with $0 < \rho < \frac{1}{2}$ is not convex on $|z| > 1$. Note that when $\rho = \frac{1}{4}$, $K = K_\rho$ is Garnett's well-known "corner quarters" Cantor set [G1] that has capacity zero and positive length; its micro deformation with zero analytic capacity was investigated in detail by Mattila in [Ma3], and a question raised there was completely settled by Mateu, Tolsa and Verdera in [MTV]. Clearly, the zero analytic capacity of K_ρ for $\rho \in (0, \frac{1}{4}]$ implies that $F_{4,\rho}(z)$ is unbounded in $\mathbb{C} \setminus K_\rho$, and the precise rate of growth as z approaches $w \in K_\rho$ is given in [DL1]. For $\rho \in (\frac{1}{4}, \frac{1}{2}]$, $F_{4,\rho}$ has continuous extension on $\mathbb{C}$ and has Hölder continuous index $\alpha = \log 4/|\log \rho| - 1$; hence $F_{4,\rho}$ has bounded image (see [LSV], [DL1])).

The Sierpinski gasket K and the canonical self-similar measure on K correspond to $m = 3, \rho = 1/2$ in (1.1). It is the most studied case in the analysis of fractals and is also a pilot case for various considerations. For the Cauchy transform, we have the following estimate.

Proposition 1.4. *For $F_{3,\frac{1}{2}}$, the image of $\{z : |z| > 1.134\}$ is starlike.*

For the starlikeness of the $F_{m,\rho}$, the main analysis depends on the estimate $\inf_{|z|<1} \mathrm{Re}\big(\frac{F(\frac{1}{z})}{z}\big)$ [Du]. It is not difficult to show (Theorem 4.1) that

$$\mathrm{Re}\Big(\frac{F_{m,\rho}(1/z)}{z}\Big) \geq \frac{1}{2}(1 + \gamma_{m,\rho}) \tag{1.4}$$

for some $\gamma_{m,\rho} > 0$. To prove Theorem 1.1, we are led to consider the following class of functions in view of (1.3) and (1.4):

$$\mathcal{F}_m(\alpha) = \Big\{ f : f(z) = z + \sum_{n=1}^{\infty} a_n z^{nm+1} \text{ is analytic and } \mathrm{Re}\frac{f(z)}{z} > \alpha \text{ in } \mathbb{D} \Big\} \tag{1.5}$$

where $\mathbb{D} = \{z : |z| < 1\}$ and $\alpha \in [0, 1)$. Let $\mathcal{F}_m := \mathcal{F}_m(\frac{1}{2})$, and let $R^*(f)$ denote the *radius of starlikeness* of f. For $m = 1$, it is a well-known result of MacGregor [Mac] on the lower bound of $R^*(f)$: for every f in $\mathcal{F}_1$, $R^*(f) \geq 1/\sqrt{2} := R_1$, and the values R_1 are attainable. The result was also extended to the class of functions $\mathcal{F}_1(\alpha)$ with $0 < \alpha < 1$ ([Sh], [TA]).

For $\mathcal{F}_m$, $m \geq 2$, we have the following sharp estimate of the lower bound of the radius of starlikeness, which is one of the main theorems of the paper (see Section 3) and yields Theorem 1.1.

Theorem 1.5. *For $m \geq 2$ and $f \in \mathcal{F}_m (= \mathcal{F}_m(\frac{1}{2}))$, we have $R_m \leq R^*(f) \leq 1$ where the R_m is defined as in Theorem 1.3.*

Moreover, $R^*(f) = R_m$ if and only if $f(z) = e^{i\theta} k_m(e^{-i\theta} z)$ for some $\theta \in [0, 2\pi)$ where

(i) *for $m = 2, 3$,*

$$k_m(z) = \frac{z}{1 + z^m \varphi_m(z^m)} \quad \text{and} \quad \varphi_m(z) = \frac{z - c_m}{1 - c_m z}$$

with $c_2 = \frac{1}{7}(1 + 2\sqrt{2})R_2^2$ (≈ 0.3063), $c_3 = -\frac{1}{2}R_3^3$ (≈ -0.2214) (R_2, R_3 are in Theorem 1.1); and

(ii) *for $m \geq 4$, $k_m(z) = \dfrac{z}{1 + z^m}$.*

For the notation and approaches in proving Theorems 1.2, 1.3, we adopt the traditional setup as in [Du] (also [Mac]). We also make use of a classical result of

Spencer ([Sp], [Hay]) on the relation of mean p-valent functions and the order of growth at the boundary.

For the estimation of the case of the Sierpinski gasket in Theorem 1.4, we make a further estimate on the $\gamma_{3,\frac{1}{2}}$ in (1.4). It is used instead of the 1/2 in $\mathcal{F}_3$ and improves the value R_3^{-1} (≈ 1.3250) (in Theorem 1.1) to ≈ 1.134. The estimate is basic; certain inequalities involving elementary functions are easy to evaluate from Mathematica, but tedious to make a direct estimation. The result in Theorem 1.4 is not optimal; we make some comments in Section 7. In fact, from computer graphics, we conjecture that $F_{3,\frac{1}{2}}(\{|z|>1\})$ is starlike, i.e., the radius of starlikeness $R^*(F)=1$.

For the organization of the paper, we review some of the basic results for starlikeness and the related material. We prove Theorem 1.5 first, then the other theorems for the self-similar measures.

2. Preliminaries

Let $f(z) = z + a_2z^2 + \cdots$ be analytic in $\mathbb{D} = \{|z| < 1\}$ and $f(z) \neq 0$ for $0 < |z| < 1$. It is known ([Du, p. 42]) that such f is starlike in $|z| < r$ if and only if for $0 < \eta < r$, $\arg f(\eta e^{i\theta})$ is increasing as a function of θ, which is in turn equivalent to

$$\mathrm{Re}\frac{zf'(z)}{f(z)} = \frac{\partial}{\partial\theta}\arg f(\eta e^{i\theta}) > 0, \quad |z| = \eta < r. \tag{2.1}$$

In this case f is univalent, and $\frac{zf'(z)}{f(z)}$ is also analytic in $\mathbb{D}$. We will call

$$R^* = R^*(f) = \sup\{r : \ \mathrm{Re}\frac{zf'(z)}{f(z)} > 0, |z| < r\}$$

the *radius of starlikeness* of f. The following conclusions are obvious: (i) $R^* > 0$ since $f'(0) = 1$; (ii) f is starlike in $|z| < R^*$; (iii) if $R^* < 1$, then there exists z_0 with $|z_0| = R^*$ such that

$$\mathrm{Re}\frac{z_0f'(z_0)}{f(z_0)} = 0. \tag{2.2}$$

Also from the minimum principle for harmonic functions, we know that $R \in (0, R^*]$ if and only if

$$\min_{|z|\le R}\mathrm{Re}\frac{zf'(z)}{f(z)} = \min_{|z|=R}\mathrm{Re}\frac{zf'(z)}{f(z)} \ge 0; \tag{2.3}$$

hence f is starlike in $|z| < R$ for any $R \in (0, R^*]$.

Our first investigation is on the starlikeness of the class of functions

$$\mathcal{F}_m = \mathcal{F}_m(\frac{1}{2}) = \{f : \ f(z) = z+\sum_{n=1}^{\infty} a_n z^{nm+1} \text{ is analytic and } \mathrm{Re}\Big(\frac{f(z)}{z}\Big) > \frac{1}{2} \text{ in } \mathbb{D}\}$$

where $\mathbb{D} = \{|z| < 1\}$. By Herglotz's formula [Du, p. 22], it is easy to show that

$$\mathcal{F}_m = \Big\{f : f(z) = \int_{\partial\mathbb{D}} \frac{z}{1-wz^m}d\nu(w), \ \nu \in \Lambda\Big\}, \tag{2.4}$$

where Λ denotes the set of probability measures on the Borel subset of $\partial\mathbb{D} = \{|z| = 1\}$. In the following, we will give another representation for $\mathcal{F}_m$, which is a

direct modification of the case $\mathcal{F}_1$ proved by MacGregory [Mac]. This expression will be used throughout the paper.

Proposition 2.1. *For $f \in \mathcal{F}_m(= \mathcal{F}_m(\frac{1}{2}))$, there is an analytic $\varphi : \mathbb{D} \to \mathbb{D}$ such that*

$$f(z) = \frac{z}{1 + z^m \varphi(z^m)}.$$

Proof. Let $h(z^m) = f(z)/z$; then h is analytic and $\mathrm{Re}(h(z)) > \frac{1}{2}$ on $\mathbb{D}$. It is easy to show that h is subordinated to $g = (1+z)^{-1}$ (i.e., $f(\mathbb{D}) \subseteq g(\mathbb{D})$ and $f(0) = g(0)$). The Schwarz lemma implies that there exists an analytic $\psi : \mathbb{D} \to \mathbb{D}$ such that $|\psi(z)| \le |z|$ and $h(z) = g(\psi(z))$. Let $\varphi(z) = \psi(z)/z$. Then φ is analytic, $|\varphi(z)| \le 1$ in $\mathbb{D}$ and $f(z) = zg(z^m\varphi(z^m))$. □

Proposition 2.2. *If $f \in \mathcal{F}_1(= \mathcal{F}_1(\frac{1}{2}))$, then $\frac{1}{\sqrt{2}} \le R^*(f) \le 1$. Moreover $R^*(f) = \frac{1}{\sqrt{2}}$ if and only if $f(z) = e^{i\theta}k(e^{-i\theta}z)$ for some $\theta \in [0, 2\pi)$ where*

$$k(z) = \frac{z}{1 + z\varphi_1(z)} \qquad \text{and} \qquad \varphi_1(z) = \frac{z - 1/\sqrt{2}}{1 - 1/\sqrt{2}z}.$$

Proof. The first part is due to MacGregor [Mac]. The sufficiency of the second part is a direct calculation and is also in [Mac]. To prove the necessity, we suppose that $R^* = \frac{1}{\sqrt{2}}$; then by (2.2) there exists z_1 such that $|z_1| = \frac{1}{\sqrt{2}}$ and

$$\mathrm{Re}\big(z_1 f'(z_1)/f(z_1)\big) = 0. \tag{2.5}$$

The representation of f in Proposition 2.1 yields

$$\frac{zf'(z)}{f(z)} = \frac{1 - z^2\varphi'(z)}{1 + z\varphi(z)}. \tag{2.6}$$

We claim that $\varphi(z_1) = 0$. If otherwise, by the Schwarz-Pick Theorem [A, p. 3],

$$\frac{|\varphi'(z_1)|}{1 - |\varphi(z_1)|^2} \le \frac{1}{1 - |z_1|^2}.$$

This implies that

$$|z_1^2\varphi'(z_1)| \le \frac{|z_1|^2}{1 - |z_1|^2}(1 - |\varphi(z_1)|^2) = 1 - |\varphi(z_1)|^2 < 1.$$

By (2.6) we have $z_1 f'(z_1)/f(z_1) \ne 0$, and it follows from the proof of [Mac, p. 75] that

$$\Big|\arg \frac{z_1 f'(z_1)}{f(z_1)}\Big| \le \arcsin\sqrt{1 - \frac{|\varphi(z_1)|^2}{2}} < \frac{\pi}{2}.$$

This is impossible because (2.5) implies that $|\arg\big(z_1 f'(z_1)/f(z_1)\big)| = \pi/2$, and the claim $\varphi(z_1) = 0$ follows. We hence have $\mathrm{Re}(1 - z_1^2\varphi'(z_1)) = 0$ by (2.6), which implies that

$$|\varphi'(z_1)| = \frac{1 - |\varphi_1(z_1)|^2}{1 - |z_1|^2}.$$

From the Scharwz-Pick Theorem [A, p. 3], $\varphi(z)$ has to be a Möbius transformation that satisfies $\varphi(z_1) = 0$. Therefore there exists a $\theta \in [0, 2\pi)$ such that

$$\varphi(z) = e^{-i\theta}\frac{z - z_1}{1 - \overline{z_1}z}, \qquad |z| < 1.$$

Let $z_1 = \frac{1}{\sqrt{2}}e^{it}, t \in [0, 2\pi)$. By (2.5), (2.6), we get

$$0 = \mathrm{Re}\frac{z_1 f'(z_1)}{f(z_1)} = \mathrm{Re}(1 - z_1^2\varphi'(z_1)) = \mathrm{Re}(1 - e^{i(2t-\theta)}).$$

This implies that $e^{i2t} = e^{i\theta}$ and $\varphi(z) = e^{-it}\varphi_1(e^{-it}z)$ where φ_1 is defined in the statement of the theorem. The result follows by substituting this into $f(z) = z/(1 + z\varphi(z))$. □

3. Starlikeness and convexity of $\mathcal{F}_m$

In this section, we give the sharp lower bound R_m of the radii of starlikeness for the class of $f \in \mathcal{F}_m(= \mathcal{F}_m(\frac{1}{2}))$, which extends Proposition 2.2 for the case $m = 1$.

Theorem 3.1. *For $m \geq 2$ and $f \in \mathcal{F}_m(= \mathcal{F}_m(\frac{1}{2}))$, we have $R_m \leq R^*(f) \leq 1$ where*

$$R_m = \begin{cases} (8\sqrt{2} - 11)^{\frac{1}{4}}, & m = 2, \\ (3\sqrt{3} - 5)^{\frac{1}{6}}, & m = 3, \\ (m-1)^{-\frac{1}{m}}, & m \geq 4. \end{cases} \tag{3.1}$$

Remark. Numerically, $R_1 = \frac{1}{\sqrt{2}} \approx 0.7071$, and from the above, $R_2 \approx 0.74834$, $R_3 \approx 0.76225$, $R_4 \approx 0.75984$, and $R_m \nearrow 1$ as $m \to \infty$. Except for R_4, the lower bound of radius of starlikeness for $\mathcal{F}_m$ increases on m.

Proof. By Proposition 2.1, we can express

$$f(z) = \frac{z}{1 + z^m\varphi(z^m)}$$

for some φ analytic in $\mathbb{D}$ and $|\varphi(z)| \leq 1$ for $z \in \mathbb{D}$. In view of (2.3), we will estimate $\mathrm{Re}\frac{zf'(z)}{f(z)} \geq 0$, $|z| < R_m$. By a direct calculation,

$$\frac{zf'(z)}{f(z)} = \frac{1 - (m-1)z^m\varphi(z^m) - mz^{2m}\varphi'(z^m)}{1 + z^m\varphi(z^m)}, \qquad |z| < 1,$$

and the above inequality is reduced to

$$\mathrm{Re}\big\{(m-2)z\varphi(z) \ + \ mz^2\varphi'(z)\big(1 + \overline{z\varphi(z)}\big)\big\} \leq 1 - (m-1)|z\varphi(z)|^2 \tag{3.2}$$

for $|z| \leq R_m^m$. Using the Schwarz-Pick Theorem, $|\varphi'(z)| \leq (1 - |\varphi(z)|^2)/(1 - |z|^2)$ for $|z| < 1$, we see that (3.1) will be satisfied if we can prove that for $|z| \leq R_m^m$,

$$\begin{aligned} &(m-2)|z\varphi(z)| + m|z|^2\frac{1 - |\varphi(z)|^2}{1 - |z|^2}(1 + |z\varphi(z)|) \\ &\qquad \leq 1 - (m-1)|z\varphi(z)|^2. \end{aligned} \tag{3.3}$$

To this end, we let $a = |z|$, $x = |\varphi(z)|$, and define

$$\begin{aligned} g_m(a, x) = &-ma^3x^3 - (1 + (m-1)a^2)a^2x^2 \\ &+ (m - 2 + 2a^2)ax + (m+1)a^2 - 1. \end{aligned} \tag{3.4}$$

Then it is direct to show that (3.3) is equivalent to $g_m(a, x) \leq 0$ for $0 \leq a \leq R_m^m$, $0 \leq x \leq 1$. To complete the proof we will prove this statement for g_m in the following three lemmas for the cases $m = 2, 3$ and ≥ 4. □

It is easy to check that

$$\frac{\partial g_m}{\partial x} = -3ma^3(x-\alpha_m^+(a))(x-\alpha_m^-(a)), \tag{3.5}$$

where

$$\alpha_m^{\pm}(a) = \frac{1}{3ma}\Big(-1-(m-1)a^2 \pm \sqrt{(1+(m-1)a^2)^2+3m(m-2+2a^2)}\Big). \tag{3.6}$$

Also by a direct calculation

$$\alpha_m^+(a) \geq 1 \quad \Leftrightarrow \quad 0 \leq a \leq \frac{m-2}{2(m-1)}. \tag{3.7}$$

Lemma 3.2. *Let $0 \leq a \leq R_2^2$, $0 \leq x \leq 1$. Then $g_2(a,x) \leq 0$, and equality holds if and only if $a = R_2^2$, $x = \frac{1-a^2}{4a}$.*

Proof. By (3.4) we can write $g_2(a,x) = (1+ax)(-2a^2x^2 + a(1-a^2)x + 3a^2 - 1)$. We only need to prove that

$$h(a,x) := -2a^2x^2 + a(1-a^2)x + 3a^2 - 1 \leq 0.$$

For $0 \leq a \leq \sqrt{5}-2$, $0 \leq x \leq 1$,

$$h(a,x) \leq h(a,1) = -(1-a)^2(1+a) < 0.$$

For $\sqrt{5}-2 < a \leq R_2^2$, $0 \leq x \leq 1$,

$$h(a,x) \leq h(a, \frac{1-a^2}{4a}) = \frac{1}{8}(a^2+11+8\sqrt{2})(a^2+11-8\sqrt{2}),$$

and $h(a,x) = 0$ if and only if $a = R_2^2 = (8\sqrt{2}-11)^{\frac{1}{2}}, x = \frac{1-a^2}{4a}$. The lemma follows from these two observations. □

Lemma 3.3. *Let $0 \leq a \leq R_3^3$ and $0 \leq x \leq 1$. Then $g_3(a,x) \leq 0$, and equality holds if and only if $a = R_3^3$, $x = \frac{1-a^2}{3a}$.*

Proof. By (3.4) we have

$$g_3(a,x) = -3a^3x^3 - (1+2a^2)a^2x^2 + (1+2a^2)ax + 4a^2 - 1.$$

For $0 \leq a \leq \frac{1}{4}$, (3.7) implies that $\alpha_3^+(a) \geq 1$ and (3.5) implies that $g_3(a,\cdot)$ is increasing on $[0,1]$. Hence

$$g_3(a,x) \leq g_3(a,1) = -2a^4 - a^3 + 3a^2 + a - 1.$$

Since $dg_3(a,1)/da = -8(1+a)(a+\frac{\sqrt{57}-5}{16})(a-\frac{5+\sqrt{57}}{16}) > 0$ for $0 < a \leq \frac{1}{4}$, we have

$$g_3(a,x) \leq g_3(a,1) \leq g_3(\frac{1}{4},1) < 0.$$

For $\frac{1}{4} < a \leq R_3^3$, we have $0 < \alpha_3^+(a) < 1$ by (3.7). By simple calculus we can show that $g_3(a,x) \leq g_3(a,\alpha_3^+(a)) \leq 0$ for $0 \leq x \leq 1$ and that $g(a,x) = 0$ if and only if $a = (3\sqrt{3}-5)^{\frac{1}{2}}, x = \alpha_3^+(a)$.

By combining the two cases we have $g_3(a,x) \le 0$ in the required domain, and equality holds if and only if $a = R_3^3 = (3\sqrt{3}-5)^{\frac{1}{2}}$, $x = \alpha_3^+(a)$. Substituting $a = R_3^3 = (3\sqrt{3}-5)^{\frac{1}{2}}$ to (3.6), we have

$$\alpha_3^+(a) = \frac{1}{3a}\Big(3 - 2\sqrt{3} + \sqrt{6(2-\sqrt{3})}\Big) = \frac{1-a^2}{3a}$$

(note that $\sqrt{6(2-\sqrt{3})} = 3-\sqrt{3}$). This completes the proof of the lemma. □

Lemma 3.4. *For $m \ge 4$, $0 \le a \le R_m^m = \frac{1}{m-1}$ and $0 \le x \le 1$, we have $g_m(a,x) \le 0$, and equality holds if and only if $a = R_m^m$, $x = 1$.*

Proof. Since $0 \le a \le R_m^m = \frac{1}{m-1} \le \frac{m-2}{2(m-1)}$ for $m \ge 4$, we have by (3.7) that $\alpha_m^+(a) \ge 1$. Hence $g_m(a,\cdot)$ is increasing for $0 \le x \le 1$; i.e., $g_m(a,x) \le g_m(a,1)$ and equality holds if and only if $x = 1$. On the other hand,

$$\frac{d}{da} g_m(a,1) = -4(m-1)(a+1)(a-\lambda^+)(a-\lambda^-)$$

where

$$\lambda^\pm = \frac{m+2 \pm \sqrt{(m+2)^2 + 16(m-1)(m-2)}}{8(m-1)}.$$

Clearly $\lambda^- < 0$ and $\lambda^+ > R_m^m = \frac{1}{m-1}$. This implies that $g_m(\cdot,1)$ is increasing so that

$$g_m(a,1) \le g_m(R_m^m,1) = 0 \quad \text{for} \quad 0 \le a \le R_m^m.$$

This completes the proof. □

Theorem 3.1 follows from these three lemmas.

Similar to the second part of Proposition 2.2, in our next theorem, we characterize the extremal functions in $\mathcal{F}_m$ with respect to the radius of starlikeness.

Theorem 3.5. *For $m \ge 2$ and $f \in \mathcal{F}_m (= \mathcal{F}_m(\frac{1}{2}))$, $R^*(f) = R_m$ if and only if $f(z) = e^{i\theta} k(e^{-i\theta} z)$ for some $\theta \in [0, 2\pi)$ where*

(i) *for $m = 2, 3$,*

$$k(z) = \frac{z}{1 + z^m \varphi_m(z^m)} \quad \text{and} \quad \varphi_m(z) = \frac{z - c_m}{1 - c_m z}$$

with $c_2 = \frac{1}{7}(1+2\sqrt{2})R_2^2$ (≈ 0.3063), $c_3 = -\frac{1}{2}R_3^3$ (≈ -0.2214);

(ii) *for $m \ge 4$, $k(z) = \dfrac{z}{1+z^m}$.*

Proof. To prove the sufficiency, we can assume without loss of generality that $\theta = 0$ and hence $f(z) = k(z)$. We can write, for $m \ge 2$,

$$f(z) = \frac{z}{1 + z^m \varphi_m(z^m)} = \frac{1}{2} z \Big(1 + \frac{1 - z^m \varphi_m(z^m)}{1 + z^m \varphi_m(z^m)}\Big)$$

($\varphi_m \equiv 1$ for $m \ge 4$). Hence $\mathrm{Re}\big(f(z)/z\big) > 1/2$ for $|z| < 1$. Let R_m be defined as in Theorem 3.1. It is trivial to see that $f'(R_m) = 0$ for $m \ge 4$. For $m = 2, 3$, we observe that

$$f'(R_m) = \frac{1 + (m-3)c_m R_m^m + (1 - 2m + 2c_m^2) R_m^{2m} + (m-1) c_m R_m^{3m}}{(1 - 2c_m R_m^m + R_m^{2m})^2},$$

and it is direct to show that $f'(R_m) = 0$ also by using the specific values c_2 and c_3. We apply Theorem 3.1 and the minimal principle (see Section 2) to conclude that $R_m^*(f) = R_m$ for $m \geq 2$.

Next we prove the necessity. By Proposition 2.1, we can write

$$f(z) = \frac{z}{1 + z^m \varphi_m(z^m)}.$$

We know from (2.2) that there exists ξ_m with $|\xi_m| = R_m$ and $\mathrm{Re}\big(\xi_m f'(\xi_m)/f(\xi_m)\big) = 0$. Write $\xi_m = R_m e^{it_m}$ and $z_m = \xi_m^m$. Then by (3.2) and (3.3), we have for $m \geq 2$,

$$\mathrm{Re}\big(z_m \varphi_m(z_m)\big) = \big|z_m \varphi_m(z_m)\big| \tag{3.8}$$

and

$$\mathrm{Re}\big(z_m^2 \varphi_m'(z_m)\big) = |z_m^2 \varphi_m'(z_m)| = |z_m|^2 \frac{1 - |\varphi_m(z_m)|^2}{1 - |z_m|^2}. \tag{3.9}$$

Also by Lemma 3.2-3.4, we have

$$|\varphi_2(z_2)| = \frac{1 - R_2^4}{4R_2^2} := \rho_2, \quad |\varphi_3(z_3)| = \frac{1 - R_3^6}{3R_3^3} := \rho_3; \tag{3.10}$$
$$|\varphi_m(z_m)| = 1, \quad m \geq 4.$$

This together with (3.9) and the Schwarz-Pick Theorem implies that

$$\varphi_m(z) = e^{i\theta_m} \frac{z - d_m}{1 - \overline{d}_m z}, \quad m = 2, 3, \quad \text{and} \quad \varphi_m(z) \equiv e^{-imt_m}, \quad m \geq 4.$$

It follows easily that for $m \geq 4, f$ has the expression in (ii). We need to determine $\varphi_2(z)$ and $\varphi_3(z)$. Note that (3.9) yields $z_m^2 \varphi_m'(z_m) = |z_m^2 \varphi_m'(z_m)|$. If we substitute this with

$$\varphi_m'(z) = e^{i\theta_m} \frac{1 - |d_m|^2}{(1 - \overline{d}_m z_m)^2},$$

then we have

$$z_m^2 e^{i\theta_m} = R_m^{2m} \frac{1 - \overline{d}_m z_m}{1 - d_m \overline{z}_m}. \tag{3.11}$$

On the other hand, (3.8) and (3.10) imply that

$$z_m e^{i\theta_m}(z_m - d_m) = |z_m \varphi(z_m)|(1 - \overline{d}_m z_m) = R_m^m \rho_m (1 - \overline{d}_m z_m). \tag{3.12}$$

Now dividing (3.11) by the above equality, we have

$$z_m = \frac{R_m^m}{\rho_m} \frac{z_m - d_m}{1 - d_m \overline{z}_m},$$

and consequently,

$$d_m = \frac{R_m^m - \rho_m}{1 - \rho_m R_m^m} e^{imt_m}.$$

A direct calculation shows that the above coefficient of e^{imt_m} equals c_m in the theorem. If we substitute $d_m = c_m e^{mt_m}$ and $z_m = R_m^m e^{imt_m}$ into (3.11), we have $e^{i\theta_m} = e^{-2mt_m i}$. It follows easily that for $m = 2, 3$,

$$\varphi_m(z) = e^{-imt_m} \frac{e^{-imt_m} z - c_m}{1 - c_m e^{-imt_m} z}$$

and hence

$$f(z) = \frac{z}{1 + z^m \varphi_m(z^m)} = e^{it_m} k_m(e^{-it_m} z). \qquad \square$$

A function f is called *convex* in a domain D if it is analytic and one-to-one in D and if the image $f(D)$ is convex. Note that $f(z)$ is convex in $|z| < r$ if and only if $zf'(z)$ is starlike in $|z| < r$ [Du, p. 43], which is equivalent to

$$\mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) \geq 0, \quad z \in D. \tag{3.13}$$

Let $\hat{R}(f)$ denote the *radius of convexity* of f. The following corollary follows from Theorems 3.1 and 3.5. The case $m = 1$ is in [H, p. 291].

Corollary 3.6. *Let $f(z) = z + \sum_{n=1}^{\infty} a_n z^{nm+1}$ on $\mathbb{D}$ satisfying $\mathrm{Re} f'(z) > \frac{1}{2}$. Then $\hat{R}(f) \geq R_m$. Furthermore the equality holds if and only if $f(z) = e^{i\theta_m} h(e^{-i\theta_m} z)$ where $\theta_m \in [0, 2\pi)$,*

$$h(z) = \int_0^z k(t)\frac{dt}{t}, \qquad |z| < 1,$$

and k is given in Theorem 3.5.

Remark. As in (2.4), each $f \in \mathcal{F}_m (= \mathcal{F}_m(\frac{1}{2}))$ has a unique integral representation. It follows that f is an extreme point of $\mathcal{F}_m$ if and only if $f(z) = z/(1 - w_0 z^m)$, $|w_0| = 1$. Theorem 3.5 shows that for $m \geq 4$, the set $\{f \in \mathcal{F}_m : R^*(f) = R_m\}$ is the set of extreme points of $\mathcal{F}_m$, which contains exactly one function modulus the rotations.

It is also interesting to characterize the class $\{f \in \mathcal{F}_m : \ R^*(f) = 1\}$ (i.e., the class of $f \in \mathcal{F}_m$ starlike on $\mathbb{D}$) as in Theorem 3.5. As a partial result, it is known that this class contains the m-fold symmetry convex functions f in $\mathbb{D}$ with the normalization conditions $f(0) = 0$ and $f'(0) = 1$. They can be represented as

$$f(z) = \int_0^z \exp\Big(-\frac{2}{m}\int_{\partial\mathbb{D}} \log(1 - wt^m) d\nu(w)\Big) dt, \quad z \in \mathbb{D},$$

for some $\nu \in \Lambda$ [Du, pp. 72–73].

4. Cauchy transforms of self-similar measures

In this section, we consider the iterated function system (IFS)

$$S_k(z) = \varepsilon_k + \rho(z - \varepsilon_k), \quad k = 0, 1, \cdots, m-1, \tag{4.1}$$

where $\varepsilon_k = e^{2\pi i k/m}$, $m \geq 2$ and $0 < \rho < 1$. Let K be the attractor, and let μ be the self-similar measure defined by

$$\mu := \mu_{m,\rho} = \frac{1}{m}\sum_{k=0}^{m-1} \mu \circ S_k^{-1}. \tag{4.2}$$

Then $\mathrm{supp}\mu = K$, $e^{2\pi i/m} K = K$, and K is contained in the closed convex hull of the ε_k's, which is a subset of $\{|z| \leq 1\}$. Let

$$F(z) := F_{m,\rho}(z) = \int_K \frac{1}{z - \xi} d\mu(\xi) \tag{4.3}$$

be the Cauchy transform of μ. Note that the measure μ is invariant under $e^{2\pi i/m}$-rotation. In fact, consider the measure $\mu(\cdot) = \mu(\varepsilon_k \cdot)$. By making use of (4.1) and a direct calculation, it is easy to see that $\tilde{\mu}$ also satisfies (4.2). Hence $\tilde{\mu} = \mu$ by

the uniqueness of the self-similar measure for a given set of weights [Hut]. By this rotational invariance and a change of variable, we have

$$F(z) = \frac{1}{m}\int_K \sum_{k=0}^{m-1} \frac{1}{z - e^{2\pi ik/m}w} d\mu(w).$$

For fixed z, we define for $w \in \mathbb{C} \setminus \{z, e^{-2\pi i/m}z, \cdots, e^{-2(m-1)\pi i/m}z\}$,

$$h(w) = \frac{1}{m}\sum_{k=0}^{m-1} \frac{1}{z - e^{2\pi ik/m}w} - \frac{z^{m-1}}{z^m - w^m}.$$

Because $h(w)$ has a removable singularity for $w = e^{-2\pi ik/m}z$ and $h(\infty) = 0$, we have $h(w) \equiv 0$. This gives

$$F(z) = \int_K \frac{z^{m-1}}{z^m - w^m} d\mu(w). \tag{4.4}$$

If we let $f(z) = F(1/z)$, then $f(z) = \int_K \frac{z}{1-w^m z^m} d\mu(w)$, which is a similar expression as in (2.4). Also noting that $K \subset \{w : |w| \le 1\}$, we have for $z \in \mathbb{D}$, $f(z)$ has the expression

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^{mn+1}$$

as considered in the previous sections.

Theorem 4.1. *For $m \ge 2$, let $F = F_{m,\rho}$ be the Cauchy transform of the self-similar measure μ in* (4.2)*, and let $f(z) = F(z^{-1})$, $f(0) = 0$. Then*

(i) $\mathrm{Re}\frac{f(z)}{z} \ge \frac{1}{2}(1 + \gamma_{m,\rho})$ *where*

$$\gamma_{m,\rho} := \min_\theta \int_K \mathrm{Re}\frac{1 - e^{i\theta}w^m}{1 + e^{i\theta}w^m} d\mu(w) > 0.$$

(ii) *F is starlike in $|z| > R$ for some $R \in [1, R_m^{-1})$ where R_m is defined in Theorem* 3.1.

Proof. (i) Note that we have

$$\frac{f(z)}{z} = \frac{F(1/z)}{z} = \frac{1}{2} + \frac{1}{2}\int_K \frac{1 + z^m w^m}{1 - z^m w^m} d\mu(w).$$

The minimum principle for harmonic function gives that for $z \in \mathbb{D}$,

$$\mathrm{Re}\frac{f(z)}{z} \ge \frac{1}{2} + \frac{1}{2}\min_\theta \int_K \mathrm{Re}\frac{1 - e^{i\theta}w^m}{1 + e^{i\theta}w^m} d\mu(w) := \frac{1}{2}(1 + \gamma_{m,\rho}).$$

As μ is a continuous measure we have $\mu(|w| = 1) = 0$, hence

$$\gamma_{m,\rho} \ge \int_K \frac{1 - |w|^m}{1 + |w|^m} d\mu > 0.$$

(ii) By (i), Theorem 3.1 applies and $R_m \le R^*(f) \le 1$. The first $\le$ is actually $<$ because of Theorem 3.5 and that $f(z) \not\equiv k_m(z)$ or its rotations (since $\mathrm{Re}\{e^{-i\theta}k_m(e^{i\theta})\} = \frac{1}{2}$). $\square$

Remark. From (i), we can use the same proof as Proposition 2.1 to show that

$$f(z) = z\frac{1+\gamma_{m,\rho}z^m\varphi(z^m)}{1+z^m\varphi(z^m)}$$

for some $\varphi : \mathbb{D} \to \mathbb{D}$. This is used to improve the domain of starlikeness for the case of the Sierpinski gasket in Section 7.

We write

$$\rho_m = \sin\frac{\pi}{m}\Big/\Big(\sin\frac{\pi}{m} + \sin\frac{(2q+1)\pi}{m}\Big) \tag{4.5}$$

with $m = 4q + s \geq 2, s = 0, 1, 2, 3$. From [DL1, Proposition 5.1], we know that the iterated function system in (4.1) satisfies the open set condition (OSC) if $0 < \rho \leq \rho_m$. In this case, the Hausdorff dimension of K is $d = \log m/|\log\rho|$. Hence according to [LSV], for $1/m < \rho \leq \rho_m$, the Cauchy transform F in Theorem 4.1 is Hölder continuous of order $d-1$ on $\mathbb{C}$. Moreover the growth rate of F near $\varepsilon_0 = 1$ (and the same for the other ε_k's by symmetry) is also known [DL1, Corollary 5.3]:

Proposition 4.2. *Let F be the Cauchy transform of the self-similar measure defined in* (4.2)*. Suppose the contraction ratio ρ satisfies $\rho \in (0, \rho_m]$ (ρ_m is in* (4.5)*), and the dimension of K is $d = \log m/|\log\rho|$. Suppose for $2 - d > 0$ there exists $c > 0$ such that*

$$\frac{c^{-1}}{(1-t)^{2-d}} \leq |F'(t^{-1})| \leq \max_{|z|=t^{-1}}|F'(z)| \leq \frac{c}{(1-t)^{2-d}}, \qquad 1/2 \leq t < 1.$$

The modulus of continuity of F and the associate growth rate of F' (and F'') allow us to obtain more information on $\gamma_{m,\rho}$ and some other properties, and hence improve the domain of starlikeness and the convexity of the Cauchy transform F. We will carry this out for some special cases in the next three sections.

To conclude this section, we will provide a necessity condition for univalence. Let $f(z)$ be analytic in $\mathbb{D}$ and, for $\xi \in \partial\mathbb{D}$, we define the order $\alpha(\xi)$ at ξ to be the supremum of all δ such that

$$\liminf_{\substack{z \to \xi \\ \gamma(\xi)}} z(1-|z|)^\delta|f(z)| > 0,$$

where $\gamma(\xi)$ is a path lying inside $\mathbb{D}$ with one end point at ξ. We note that for the Cauchy transform F in Proposition 4.2, the function $f(z) = F'(1/z)$, $|z| < 1$, at each ε_k has order $\alpha(\varepsilon_k) = 2 - d$. It is the growth rate of $f(z)$ as $z \to \varepsilon_k$.

Proposition 4.3. *Suppose f is analytic and univalent in $\mathbb{D}$. Then the set E of distinct ξ on $\partial\mathbb{D}$ such that $\alpha(\xi) > 0$ satisfies $\sum_{\xi\in E}\alpha(\xi) \leq 2$.*

This is in fact a special case of a classical theorem of Spencer [Sp], [Hay, p. 42] that for a mean p-valent function in $\mathbb{D}$, the set E of distinct ξ on $\partial\mathbb{D}$, such that $\alpha(\xi) > 0$, is countable and satisfies $\sum_{\xi\in E}\alpha(\xi) \leq 2p$.

5. Self-similar measures with $m = 2$

In this section, we consider the Cauchy transform $F_{2,\rho}$. Note that $\mu_{2,1/2}$ is the normalized Lebesgue measure on $[-1, 1]$; for $0 < \rho < 1/2$, $\mu_{2,\rho}$ is the Cantor type measure; and for $0 < \rho < 1$, it is the Bernoulli convolution.

We say that a domain D is a *Steiner symmetric domain* [Hay, p. 112] with respect to the real axis if for any real ξ, the intersection of D with the line $x = \xi$ is an open interval symmetric about ξ (or an empty set).

Theorem 5.1. *Let $F(z) = F_{2,r}(z), 0 < r < 1$, be defined as in (4.3). Then $F(z)$ is univalent on $|z| > 1$, and $F(|z| > 1)$ is a Steiner symmetric domain with respect to the real axis.*

Proof. Let $f(z) = F(\frac{1}{z})$. Then by (4.5),

$$f(z) = F(\frac{1}{z}) = \int_K \frac{z}{1 - z^2x^2} d\mu(x)$$

where $K \subset [-1, 1]$. Obviously $f(z) = \overline{f(\overline{z})}$ and $f([0, 1)) \subset \mathbb{R}^+$. It suffices to prove that $F(|z| > 1)$ is a Steiner symmetric domain by checking for fixed $0 < r < 1$, $\mathrm{Im} f(re^{i\theta}) > 0$ and $\mathrm{Re} f(re^{i\theta})$ is decreasing on $0 < \theta < \pi$. Indeed, a direct check shows that

$$\mathrm{Im} f(re^{i\theta}) = \sin\theta \int_K \frac{r + r^3x^2}{|1 - r^2x^2e^{i2\theta}|^2} d\mu(x) > 0\ , \quad \theta \in (0, \pi),$$

and

$$\frac{\partial}{\partial\theta}\mathrm{Re} f(re^{i\theta}) = -\int_K \frac{|1 + r^2x^2e^{i2\theta}|}{|1 - r^2x^2e^{i2\theta}|^2}\ r\cos\varphi(y)\ d\mu(x) \tag{5.1}$$

where $\varphi(y) = -\frac{\pi}{2} + \theta + \arg(1 + ye^{i2\theta}) - 2\arg(1 - ye^{i2\theta})$ with $y = r^2x^2$. Since $x \in K \subseteqq [-1, 1]$, we have $0 \le y \le r^2 < 1$ and

$$\frac{\partial}{\partial y}\varphi(y) = \mathrm{Im}\frac{\partial}{\partial y}\log\frac{1 + ye^{i2\theta}}{(1 - ye^{i2\theta})^2} = \sin(2\theta)\Big(\frac{1}{|1 + ye^{i2\theta}|^2} + \frac{2}{|1 - ye^{i2\theta}|^2}\Big) > 0$$

for $0 < \theta < \pi/2$; hence $-\frac{\pi}{2} + \theta = \varphi(0) \le \varphi(y) < \varphi(1) = \pi/2$. This and (5.1) show that for $0 < r < 1$,

$$\frac{\partial}{\partial\theta}\mathrm{Re} f(re^{i\theta}) < 0\ , \quad 0 < \theta < \frac{\pi}{2}\ . \tag{5.2}$$

It is easy to see that for $\pi/2 \le \theta < \pi$,

$$f(re^{i\theta}) = -\overline{f(re^{i(\pi-\theta)})} \quad \text{and} \quad \mathrm{Re}(f(re^{i(\pi-\theta)})) > 0.$$

Hence by (5.2), $\mathrm{Re} f(re^{i\theta})$ is a strictly decreasing function on $\theta \in (0, \pi)$ for fixed $r \in (0, 1)$. We conclude that $f(|z| < r)$ is a Steiner symmetric domain. Furthermore, $f(z)$ is analytic on $\partial\mathbb{D}\backslash\{1, -1\}$ and the above proof holds for $r = 1$; hence $f(|z| < 1)$ is also a Steiner symmetric domain.

The above proof also shows that for fixed $r \in (0, 1)$, $\{f(re^{i\theta}) :\ \theta \in [0, 2\pi]\}$ is a simple closed curve; this implies that $f(z)$ is univalent in $|z| < r$. Consequently, $f(z)$ is an analytic univalent function in $|z| < 1$. □

Theorem 5.2. *Let $F(z) := F_{2,r}(z)$ be defined as in (4.3). Then:*

(i) *if $r = 1/2$, then F is convex univalent on $\mathbb{C} \setminus [-1, 1]$ and is a $\frac{1}{2}$-order convex univalent on $|z| > 1$, in the sense that for $f(z) = F(1/z)$, $\mathrm{Re}\big(1 + \frac{zf''(z)}{f'(z)}\big) > \frac{1}{2}$ for $|z| < 1$.*

(ii) *for $0 < r < \frac{1}{2}$, F is not a convex univalent on $|z| > 1$.*

Proof. (i) Note that for $r = 1/2$, F is the Cauchy transform of the normalized Lebesgue measure on $[-1, 1]$. It follows that

$$F(z) = \frac{1}{2}\log\frac{z + 1}{z - 1}\ , \qquad z \in \mathbb{C} \setminus [-1, 1],$$

where the logarithmic function is the principal branch, i.e., $\log(z+1)$ and $\log(z-1)$ are real for $z = x > 1$. It is easy to see that

$$F(|z| > 1) = \{w : \ |\mathrm{Im} w| < \frac{\pi}{4}\}, \quad F(\mathbb{C} \setminus [-1,1]) = \{w : \ |\mathrm{Im} w| < \frac{\pi}{2}\}.$$

Hence F is convex on $\mathbb{C} \setminus [-1,1]$. Let $f(z) = F(\frac{1}{z})$; then

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1}{1-z^2} \qquad \text{for } |z| < 1.$$

This implies that f is 1/2-order convex on $|z| < 1$.

(ii) Let $g(z) = zf'(z) = -\frac{1}{z}F'(\frac{1}{z})$. Then $f(|z| < 1)$ is a convex region if and only if $g(z)$ is a starlike univalent function on $|z| < 1$. Hence it suffices to prove $g(z)$ is not a univalent function on $|z| < 1$.

Note that the contraction ratio ρ satisfies $0 < \rho < \frac{1}{2} = \rho_2$ (see (4.5)). Hence by Proposition 4.2, at the boundary point $z = 1$, the order $\alpha(1)$ of g is $= 2 - d$ where $d = \log 2/|\log \rho| \in (0,1)$. Since F is 2-fold symmetric, $F(e^{2\pi i/2}z) = e^{2\pi i/2}F(z)$, we have

$$\sum_{\xi} \alpha(\xi) \geq \alpha(1) + \alpha(-1) = 2(2-d) > 2.$$

Proposition 4.3 implies that g is not a univalent function in $|z| < 1$. □

We can refine Theorem 5.2 (ii) to $m \geq 3$, but with more restriction on ρ.

Proposition 5.3. *For $m \geq 3$, $0 < \rho < m^{-m/(m-2)}$, $F_{m,\rho}$ is not univalent on $|z| > 1$.*

Proof. We need to prove that the given condition on ρ satisfies (4.4), i.e.,

$$m^{\frac{m}{m-2}} \geq 1 + \sin(\frac{2q+1}{m}\pi)\Big/\big(\sin\frac{\pi}{m}\big) := 1 + h(q,s) \tag{5.3}$$

with $m = 4q + s \geq 3$. Then Proposition 4.2 applies, and the order $\alpha(\varepsilon_k), k = 0, \cdots, m-1$, of $F_{m,\rho}$ is $1-d$, where $d = \log m/|\log \rho|$ is the dimension of K. Hence

$$\sum_{\ell=0}^{m-1} \alpha(e^{2\ell\pi i/m}) \geq m(1-d) = m(1 - \frac{\log m}{|\log \rho|}) > 2$$

for $0 < \rho < m^{-m/(m-2)}$ and $m \geq 3$. Hence f is not univalent in $|z| < 1$ by Proposition 4.3.

To this end, we observe that (5.3) is equivalent to

$$m^2 \geq \big(\frac{1}{m}(1 + h(q,s))\big)^{m-2}, \quad \text{where} \quad m = 4q + s \geq 3, \ s = 0,1,2,3.$$

This follows from $h(q,s) < m$, since $h(q,0) = \cot(\pi/m)$, $h(q,2) = \csc(\pi/m)$, and $h(q,s) = \frac{1}{2}\csc(\pi/2m)$ for $s = 1,3$. □

6. The square case, $m = 4$

In this section we consider the starlikeness and convexity of the Cauchy transform of the normalized Lebesgue measure $\mathcal{L}^2(w) = \frac{1}{2}dudv$ on the square K with vertices at $\pm 1, \pm i$. This corresponds to $F_{4,\frac{1}{2}}$, i.e.,

$$F(z) = F_{4,\frac{1}{2}}(z) = \int_K \frac{d\mathcal{L}^2(w)}{z-w}, \qquad z \in \mathbb{C}.$$

It is known that F is analytic on $\widehat{\mathbb{C}} \setminus K$ with $F(\infty) = 0$, but F is not analytic on K [DL1, Proposition 2.1]. A direct calculation yields [DL1, p. 96]

$$F(z) = \frac{1}{2}\sum_{n=0}^{3}(-1)^n(z - e^{2\pi i n/4})\log(1 - e^{2\pi i n/4}z^{-1}), \quad z \in \widehat{\mathbb{C}} \setminus K, \tag{6.1}$$

where the branches of the logarithmic functions $\log(1 - z^{-1})$, $\log(1 + z^{-1})$, $\log(1 + iz^{-1})$ and $\log(1 - iz^{-1})$ are such that for $z = x \in (1, \infty)$,

$$\arg(1 - z^{-1}) = \arg(1 + z^{-1}) = 0, \quad -\frac{\pi}{2} < \arg(1 - iz^{-1}) = -\arg(1 + iz^{-1}) < 0. \tag{6.2}$$

Also the Laurent series of F is

$$F(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(4n+1)} z^{-(4n+1)}, \qquad |z| > 1.$$

It follows that F in (6.1) can be extended to $\widehat{\mathbb{C}} \setminus \big([-1,1] \cup [-i,i]\big)$; we denote this extension by F^*. (We should note that $F^*(z) \not\equiv F(z)$ in any neighbors of $z \in K \setminus ([-1,1] \cup [-i,i])$ as F is not analytic for every point $z \in K$.)

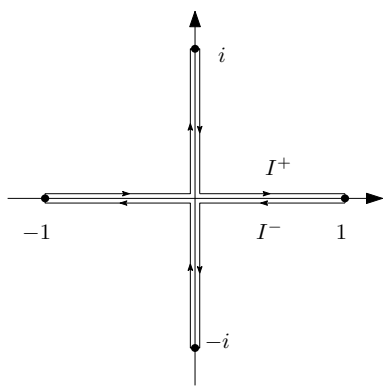

FIGURE 1. The curve Γ around $[-1,1] \cup [-i,i]$.

In the following, we will prove that F^* is starlike in $\widehat{\mathbb{C}} \setminus \big([-1,1] \cup [-i,i]\big)$ (see Figure 1). First we define the boundary curve of the domain defined by $[-1,1] \cup [-i,i]$. Let I^+ (I^-) denote the upper boundary (lower boundary, respectively) of the slit $(0,1)$; let $0_{I^\pm}$, $1_{I^\pm}$ denote the end points $0, 1$ in $I^\pm$ respectively. Similarly we define $I_i^\pm = iI^\pm$, $I_{-1}^\pm = i^2 I^\pm$, $I_{-i}^\pm = i^3 I^\pm$ and the end points $\{0_{I_i^\pm}, i_{I_i^\pm}\}, \{0_{I_{-1}^\pm}, (-1)_{I_{-1}^\pm}\}, \{0_{I_{-i}^\pm}, (-i)_{I_{-i}^\pm}\}$ for the six line segments $I_i^\pm, I_{-1}^\pm, I_{-i}^\pm$ respectively (see Figure 1). We define the boundary curve Γ of $\widehat{\mathbb{C}} \setminus \big([-1,1] \cup [-i,i]\big)$ to be

$$0_{I^+} \underset{I^+}{\longrightarrow} 1_{I^+}(= 1_{I^-}) \underset{I^-}{\longrightarrow} 0_{I^-}(\to 0_{I_{-i}^+}) \underset{I_{-i}^+}{\longrightarrow} (-i)_{I_{-i}^+} \cdots \underset{I_i^-}{\longrightarrow} 0_{I_i^-}(= 0_{I^+}). \tag{6.3}$$

Here, we identify $1_{I^+} = 1_{I^-}(= 1)$, $0_{I^+} = 0_{I_i^-}$; the same for $-i, -1, i$ and $0_{I_{-i}^+} = 0_{I^-}$, $0_{I_{-1}^+} = 0_{I_{-i}^-}$ and $0_{I_i^+} = 0_{I_{-1}^-}$ (defined in an obvious way). In this way, the curve Γ is a Jordan curve with eight line segments (at 0, it is considered as four different

points), which forms the boundary of $\widehat{\mathbb{C}} \setminus \big([-1,1] \cup [-i,i]\big)$. In the following, we will prove

Lemma 6.1. *F^* can be extended continuously to Γ, and $F^*(\Gamma)$ is a Jordan curve.*

Proof. It is obvious that F^* can be extended analytically passing the eight open line segments $I^\pm$, $I_i^\pm$, $I_{-1}^\pm$, $I_{-i}^\pm$. For example, for I^+, we restrict F^* on the domain $D^+ = \{z : |z - \frac{1}{2}| < \frac{1}{2},\ \mathrm{Im} z > 0\}$ and can choose an analytic branch of

$$\phi(z) := \frac{1}{2}\sum_{n=0}^{3}(-1)^n(z - e^{\frac{2\pi i n}{4}})\log(1 - e^{\frac{2\pi i n}{4}}z^{-1}) \ , \quad z \in D = \{z : |z - \frac{1}{2}| < \frac{1}{2}\},$$

such that $\phi(z) = F^*(z), z \in D^+$; i.e., F^* can be extended analytically passing the open line segment I^+. (Note that $F^*(z) \not\equiv \phi(z)$ in any neighbors of $z \in D^- = D \cap \{\mathrm{Im} z < 0\}$.) It follows from (6.2) that

$$\begin{aligned} &\arg(1 - x^{-1}) = \pm\pi, \ \ \arg(1 + x^{-1}) = 0, \\ &\arg(1 + ix^{-1}) = \arctan x^{-1} = -\arg(1 - ix^{-1}) \end{aligned} \tag{6.4}$$

for $x \in I^\pm$. Hence $F^*(x) := u(x) + iv_{I^\pm}(x),\ x \in I^\pm$, where

$$\begin{cases} u(x) = \frac{1}{2}\big((x-1)\log(1-x) + (1+x)\log(1+x) - x\log(1+x^2) + 2\arctan x^{-1}\big), \\ v_{I^\pm}(x) = \pm\frac{1}{2}(x-1)\pi. \end{cases} \tag{6.5}$$

In the following, we prove that F^* can be extended continuously to the end points of the above eight open line segments.

First, we consider the end points 0_{I^+}, $0_{I_i^-}$. Let $z_\varepsilon = \varepsilon e^{it}$ with $0 \le t \le \frac{\pi}{2}$ and $\varepsilon = \varepsilon(t) > 0$ small where $z_{\varepsilon(\pi/2)} \in I_i^-$ and $z_{\varepsilon(0)} \in I^+$. We claim that

$$F^*(z_\varepsilon) = \frac{1}{2}(\pi - \pi i) + O(\varepsilon \log\frac{1}{\varepsilon}), \quad \varepsilon \to 0^+ \ . \tag{6.6}$$

Thus we can define the value of $F^*(z)$ at $z = 0$ (denoted by $0^{(1)}$) in the first quadrant (containing the boundary I^+, I_i^-): $F^*(0^{(1)}) = \frac{1}{2}(\pi - \pi i)$. Obviously, $F^*(0_{I^+}) = F^*(0_{I_i^{-1}}) = \frac{1}{2}(\pi - \pi i)$.

Indeed by (6.2) and the continuity of the argument, we have $\arg(1 - z_\varepsilon^{-1}) \in [\arctan\ \varepsilon^{-1}, \pi]$ and $\arg(1 + z_\varepsilon^{-1}) \in [-\frac{\pi}{2}, 0]$, hence $\arg(1 - z_\varepsilon^{-1}) - \arg(1 + z_\varepsilon^{-1}) \in [\arctan\ \varepsilon^{-1}, \frac{3\pi}{2}]$. Noting that $\frac{1 - z_\varepsilon^{-1}}{1 + z_\varepsilon^{-1}} = -(1 - 2\varepsilon\cos t - i2\varepsilon\sin t) + O(\varepsilon^2)$, we have

$$\arg(1 - z_\varepsilon^{-1}) - \arg(1 + z_\varepsilon^{-1}) = \pi - \arctan(2\varepsilon\sin t) + O(\varepsilon^2). \tag{6.7}$$

Similarly, by $\arg(1 - iz_\varepsilon^{-1}) \in [-\pi, -\arctan(\varepsilon^{-1})]$ and $\arg(1 + z_\varepsilon^{-1}) \in [0, \arctan(\varepsilon^{-1})]$, we have

$$\arg(1 + iz_\varepsilon^{-1}) - \arg(1 - iz_\varepsilon^{-1}) = \pi - \arctan(2\varepsilon\cos t) + O(\varepsilon^2) \ . \tag{6.8}$$

Since F^* has the expression as in (6.1) on $\widehat{\mathbb{C}} \setminus ([-1,1] \cup [-i,i])$, we obtain

$$\begin{aligned} F^*(z_\varepsilon) = \frac{-1}{2}\Big\{ &\log\big|\frac{z_\varepsilon - 1}{z_\varepsilon + 1}\big| + i\log\big|\frac{z_\varepsilon + i}{z_\varepsilon - i}\big| + i\big(\arg(1 - z_\varepsilon^{-1}) - \arg(1 + z_\varepsilon^{-1})\big) \\ &- \big(\arg(1 + iz_\varepsilon^{-1}) - \arg(1 - iz_\varepsilon^{-1})\big)\Big\} + O(\varepsilon\log\frac{1}{\varepsilon}) \\ = \frac{1}{2}(\pi - \pi i) &+ O(\varepsilon\log\frac{1}{\varepsilon}), \quad \varepsilon \to 0^+ \qquad \text{(by (6.7) and (6.8))}. \end{aligned}$$

Next, we consider the end points $1_{I^+}, 1_{I^-}$. Let $\xi_\varepsilon = \varepsilon e^{it}$ with $-\pi \le t \le \pi$ and $\varepsilon = \varepsilon(t) > 0$ small where $1 + \xi_{\varepsilon(\pi)} \in I^+$ and $1 + \xi_{\varepsilon(-\pi)} \in I^-$. It follows from $\frac{1}{1+\xi_\varepsilon} = 1 - \varepsilon e^{it} + O(\varepsilon^2)$ and the same observation as the above that

$$\arg(1-(1+\xi_\varepsilon)^{-1}) = t + O(\varepsilon), \qquad \arg(1+(1+\xi_\varepsilon)^{-1}) = O(\varepsilon), \tag{6.9}$$

$$\arg(1-i(1+\xi_\varepsilon)^{-1}) = -\frac{\pi}{4} + O(\varepsilon), \qquad \arg(1+i(1+\xi_\varepsilon)^{-1}) = \frac{\pi}{4} + O(\varepsilon). \tag{6.10}$$

By substituting these into (6.1) and simplifying, we have

$$F^*(1+\xi_\varepsilon) = \frac{1}{2}(\log 2 + \frac{\pi}{2}) + O(\varepsilon \log \frac{1}{\varepsilon}). \tag{6.11}$$

We can hence define $F^*(1) := F^*(1_{I^+}) = F^*(1_{I^-}) = \frac{1}{2}(\log 2 + \frac{\pi}{2})$.

Finally by the 4-fold symmetry of F: $F(iz) = -iF(z)$, we can define F^* on the rest of the six end points of the slits. Hence F^* is well defined on Γ, and F^* can be extended to Γ continuously.

It is direct to check that $u(x)$, $v_{I^+}(x)$, $v_{I^-}(x)$ are monotonic. This together with the 4-fold symmetry of F^* imply that $F^*(\Gamma)$ is a Jordan curve (see Figure 2), which proves the lemma. □

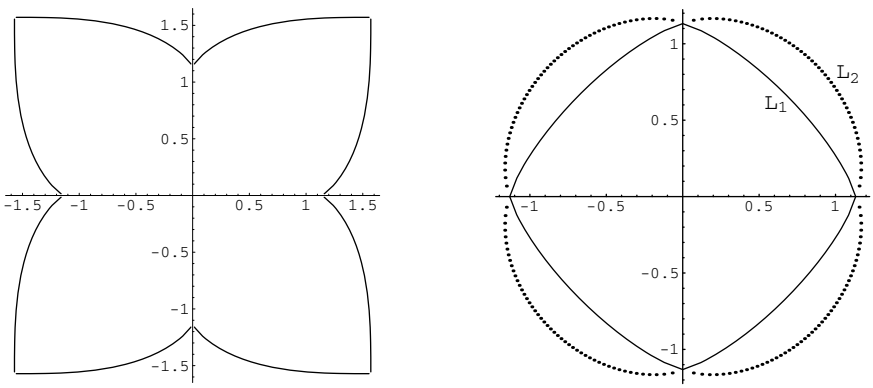

FIGURE 2. The left figure: $F^*(\mathbb{C} \setminus ([-1,1] \cup [-i,i]))$; the right figure: $L_1 = F(|z| = 1)$ and $L_2 = F(\partial K)$.

Theorem 6.2. *$F^*(z)$ is analytic, starlike univalent in $\mathbb{C} \setminus \big([-1,1] \cup [-i,i]\big)$.*

Proof. It suffices to prove $\arg F^*(z)$ is a monotone increasing function for z walks along the curve Γ in (6.3). Since F^* is 4-fold symmetric, $F^*(iz) = -iF^*(z)$, we only need to prove $\arg F^*(z)$ is a monotone increasing function for $z \in I^+ \cup I^-$. As $F^*(\overline{z}) = \overline{F(z)}$, we have for x in the slit $(0,1)$,

$$\arg F^*(x^+) = -\arg F^*(x^-) \tag{6.12}$$

where x^+, x^- mean $x \in I^+$ or $x \in I^-$ respectively. As $\arg F^*(x^+) = \arctan \frac{v_{I^+}(x)}{u(x)}$, by (6.5) and a direct calculation, we have

$$\Big(\frac{v_{I^+}(x)}{u(x)}\Big)' = \pi \, \frac{\log \frac{(1+x)^2}{1+x^2} + 2\arctan(x^{-1})}{u(x)^2} > 0, \quad x \in I^+.$$

Therefore $\arg F^*(x^+)$, $x^+ \in I^+$, is monotonic increasing from $-\frac{\pi}{4}$ to 0 as $x : 0 \nearrow 1$. Also by (6.12), $\arg F^*(x^-), x \in I^-$, is monotonic decreasing as $x : 0 \nearrow 1$. Counting the orientation of Γ in (6.3), we see that $\arg F^*(z)$ is increasing on $I^+ \cup I^-$ (see Figure 2). $\square$

As a direct consequence of Theorem 6.2, we have

Corollary 6.3. *$F(\partial K)$ is a Jordan curve; hence $F_{4,\frac{1}{2}}(z)$ is analytic univalent in $\mathbb{C} \setminus K$.*

Theorem 6.4. *$F(z) := F_{4,\frac{1}{2}}(z)$ is $(\frac{4}{\pi} - 1)$-order convex univalent on $|z| > 1$.*

Proof. Let $z = \frac{1}{\xi}$ and $f(\xi) = F(\frac{1}{\xi})$. We need to prove

$$\mathrm{Re}\big(1 + \frac{\xi f''(\xi)}{f'(\xi)}\big) = -1 - \mathrm{Re}\frac{zF''(z)}{F'(z)} \geq \frac{4}{\pi} - 1\,, \quad |z| > 1. \tag{6.13}$$

From the minimum principle for harmonic function, we intend to prove the theorem on the boundary $\partial\mathbb{D}$. However since F' and F'' have singularities at $\{\pm 1, \pm i\}$, our proof is adjusted to prove (6.13) on a new boundary Γ_ε that avoids the singularities, and F' and F'' are continuous on Γ_ε (see Figure 3), and then let $\varepsilon \to 0$.

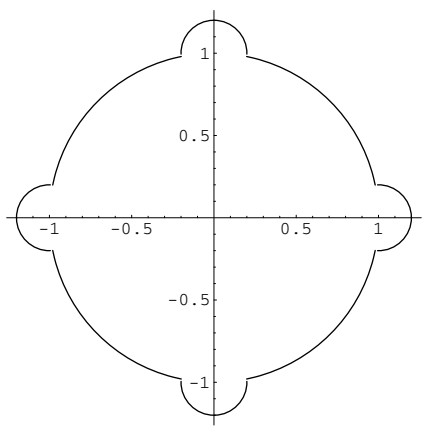

FIGURE 3. The curve Γ_ε that avoids ± 1 and $\pm i$.

First we will prove (6.13) on $\partial\mathbb{D} \setminus \{\pm 1, \pm i\}$. By the 4-fold symmetry and the conjugate symmetry, we only need to prove this for $0 < \theta < \pi/2$. By (6.1), it is easy to check that

$$F'(z) = \frac{1}{2}\sum_{n=0}^{3}(-1)^n \log(1 - e^{2\pi i n/4} z^{-1}) \ \text{ and } \ F''(z) = \frac{2z}{z^4 - 1}\,. \tag{6.14}$$

We rewrite $F'(e^{i\theta})$ as

$$F'(e^{i\theta}) = \frac{1}{2}\log\frac{|1 - e^{-i\theta}||1 + e^{-i\theta})|}{|1 - ie^{-i\theta}||1 + ie^{-i\theta}|} + \frac{i}{2}\arg\frac{(1 - e^{-i\theta})(1 + e^{-i\theta})}{(1 - ie^{-i\theta})(1 + ie^{-i\theta})}.$$

It follows from the choice (6.2) of the branch that

$$\varphi_1(\theta) = \arg(1 - e^{-i\theta}) \in (\frac{\pi}{4}, \frac{\pi}{2}), \qquad \varphi_2(\theta) = \arg(1 - ie^{-i\theta}) \in (-\frac{\pi}{2}, -\frac{\pi}{4}),$$
$$\varphi_3(\theta) = \arg(1 + e^{-i\theta}) \in (-\frac{\pi}{4}, 0), \qquad \varphi_4(\theta) = \arg(1 + ie^{-i\theta}) \in (0, \frac{\pi}{4}).$$

This shows $\varphi_1 + \varphi_3 - (\varphi_2 + \varphi_4) \in (0, \pi)$. Observe that $\dfrac{(1 - e^{-i\theta})(1 + e^{-i\theta})}{(1 - ie^{-i\theta})(1 + ie^{-i\theta})} = \dfrac{\sin\theta}{\sin(\frac{\pi}{2} - \theta)} i$. We have

$$\varphi_1 + \varphi_3 - (\varphi_2 + \varphi_4) = \arg \frac{(1 - e^{-i\theta})(1 + e^{-i\theta})}{(1 - ie^{-i\theta})(1 + ie^{-i\theta})} = \frac{\pi}{2},$$

hence $F'(e^{i\theta}) = \frac{1}{2}(\log\tan\theta + \frac{\pi}{2}i)$. By (6.14), $F''(e^{i\theta}) = \dfrac{-i}{\sin 2\theta} e^{-i\theta}$. Let $z = e^{i\theta}$; then

$$-1 - \mathrm{Re}\frac{zF''(z)}{F'(z)} = -1 + \frac{\pi}{\sin(2\theta)\big((\log\tan\theta)^2 + \pi^2/4\big)} := -1 + \frac{\pi}{h(\theta)}.$$

Obviously, $h(\pi/2 - \theta) = h(\theta)$. A direct check gives $h'(\theta) \geq 0$ for $0 < \theta < \pi/4$, hence $\pi/h(\theta) \geq \pi/h(\pi/4) = 4/\pi$. Thus (6.13) holds for $z \in \partial\mathbb{D} \setminus \{1, i, -1, -i\}$.

Next, we prove (6.13) holds on the small circular arc (see Figure 3)

$$\ell_\varepsilon = \{z_\varepsilon(t) = 1 + \varepsilon e^{it} : \ -t_\varepsilon \leq t \leq t_\varepsilon\}$$

where $t_\varepsilon = \frac{\pi}{2} + \arcsin\frac{\varepsilon}{2}$, and $\varepsilon > 0$ is sufficiently small and independent of t. It follows from (6.9)-(6.10) that

$$\begin{aligned}
&\log(1 - (1 + \varepsilon e^{it})^{-1}) = \log\varepsilon + it + O(\varepsilon),\\
&\log(1 + (1 + \varepsilon e^{it})^{-1}) = \log 2 + O(\varepsilon),\\
&\log(1 - i(1 + \varepsilon e^{it})^{-1}) = \frac{1}{2}\log 2 - \frac{\pi}{4}i + O(\varepsilon),\\
&\log(1 + i(1 + \varepsilon e^{it})^{-1}) = \frac{1}{2}\log 2 + \frac{\pi}{4}i + O(\varepsilon).
\end{aligned}$$

Hence by (6.14), we have

$$2F'(z_\varepsilon(t)) = \log\varepsilon + it + O(\varepsilon) = \log\varepsilon \cdot \big(1 + \frac{it}{\log\varepsilon} + O(\frac{\varepsilon}{\log\epsilon})\big).$$

Again by (6.14), we have

$$z_\varepsilon(t)F''(z_\varepsilon(t)) = \frac{(1 + \varepsilon e^{it})^2}{(1 + \varepsilon e^{it})^4 - 1} = \frac{1}{4\varepsilon}(e^{-it} + O(\varepsilon)).$$

Therefore

$$-\mathrm{Re}\frac{z_\varepsilon(t)F''(z_\varepsilon(t))}{F'(z_\varepsilon(t))} = \frac{-1}{2\varepsilon\log\varepsilon}\Big(\cos t - \frac{t\sin t}{\log\varepsilon} + O(\frac{1}{\log^2\varepsilon})\Big), \quad |t| \leq \frac{\pi}{2} + \arcsin\frac{\varepsilon}{2}.$$

Letting $\psi(t) = \cos t - \frac{t\sin t}{\log\varepsilon}$, it is easy to prove that $\psi'(t) \leq 0$ and

$$\psi(t) \geq \psi(t_\varepsilon) = -\frac{\varepsilon}{2} - \frac{\frac{\pi}{2} + \arcsin\frac{\varepsilon}{2}}{\log\varepsilon}\sqrt{1 - \frac{\varepsilon^2}{4}} = \frac{-\pi}{2\log\varepsilon} + O(\varepsilon)$$

for $|t| \le t_\varepsilon$ and small $\varepsilon > 0$, which gives

$$-\mathrm{Re}\frac{z_\varepsilon(t)F''(z_\varepsilon(t))}{F'(z_\varepsilon(t))} \ge \frac{1}{2\varepsilon \log^2 \varepsilon}\Big(\frac{\pi}{2} + O(\frac{1}{\log \varepsilon})\Big) \to +\infty$$

as $\varepsilon \to 0$. Hence (6.13) holds on ℓ_ε.

Combining the above proof for the two situations, we conclude that (6.13) holds on Γ_ε and hence on the unbounded domain determined by Γ_ε. By letting $\varepsilon \to 0$, the theorem follows. □

We end this section by giving a similar result as Theorem 5.2(ii) and Proposition 5.3. The proof is also similar and is omitted.

Proposition 6.5. *$F_{4,r}(z)$ with $0 < r < 1/2$ is not a convex univalent function on $|z| > 1$.*

7. The Sierpinski gasket, $m = 3$

In Theorem 4.1(i), we have seen that there is a term $\gamma_{m,\rho}$ in the inequality. It actually plays an important role in the geometry of the Cauchy transform. In this section we will give a more accurate estimation of such constant for the special case that $m = 3$ and $\rho = 1/2$. This allows us to give further improvement on the domain of starlikeness of $F = F_{3,\frac{1}{2}}$. Note that in this case, the attractor K is the Sierpinski gasket, and the self-similar measure μ is the normalized α-Hausdorff measure restricted on K with $\alpha = \log 3/\log 2$.

Proposition 7.1. *Let $\gamma_3 := \gamma_{3,\frac{1}{2}}$ be the constant defined as in Theorem 4.1(i). Then $\gamma_3 > 0.6834$, and hence $\mathrm{Re}\big(F_{3,\frac{1}{2}}(z^{-1})/z\big) > (1+\gamma_3)/2 > 0.8417$ for $z \in \mathbb{D}$.*

Proof. In view of Theorem 4.1(i),

$$\gamma_3 = \min_\theta \int_K \mathrm{Re}\frac{1+e^{i\theta}w^3}{1-e^{i\theta}w^3}d\mu(w).$$

Note that μ and K are invariant under $e^{2\pi i/3}$-rotation and that S_0K and μ are symmetric with respect to the real-axis,

$$\begin{aligned}\gamma_3 &= 3\min_\theta \int_{S_0K} \mathrm{Re}\frac{1+e^{i\theta}w^3}{1-e^{i\theta}w^3}d\mu(w)\\ &= 3\min_\theta \int_{S_0K^+} \mathrm{Re}\Big\{\frac{1+e^{i\theta}w^3}{1-e^{i\theta}w^3} + \frac{1+e^{-i\theta}w^3}{1-e^{-i\theta}w^3}\Big\}d\mu(w)\end{aligned}$$

where $S_0K^+ = \{w : w \in S_0K \text{ and } \mathrm{Im}w \ge 0\}$ (see Figure 4). By using $S_0K^+ = \bigcup_{n=1}^\infty S_0^n S_1 K \cup \{1\}$ and $|w| \le 1$ for $w \in S_0K$, it follows that for any fixed l,

$$\gamma_3 > 3\sum_{n=1}^{l} \min_\theta \int_{S_0^n S_1 K} \Big\{\frac{1-|w|^6}{|1-e^{i\theta}w^3|^2} + \frac{1-|w|^6}{|1-e^{-i\theta}w^3|^2}\Big\}d\mu(w). \tag{7.1}$$

Set $z = \varphi(w) = S_0^{n-1}w = 2^{1-n}(w-1)+1$. Then φ^{-1} exists and $|\varphi(w_1)-\varphi(w_2)| = 2^{1-n}|w_1-w_2|$. Since μ is the normalized α-Hausdorff measure restricted on K with $\alpha = \log 3/\log 2$, we have $\mu(\varphi(B)) = 2^{\alpha(1-n)}\mu(B) = 3^{1-n}\mu(B)$ for all Borel set B by [F, p. 27]. Write $E = S_0S_1(K)$; hence the right side of (7.1) is equal to

$$3\sum_{n=1}^{l} \frac{1}{3^{n-1}} \min_\theta \int_E \Big\{\frac{1-|(S_0^{n-1}w)|^6}{|1-e^{i\theta}(S_0^{n-1}w)^3|^2} + \frac{1-|(S_0^{n-1}w)|^6}{|1-e^{-i\theta}(S_0^{n-1}w)^3|^2}\Big\}d\mu(w). \tag{7.2}$$

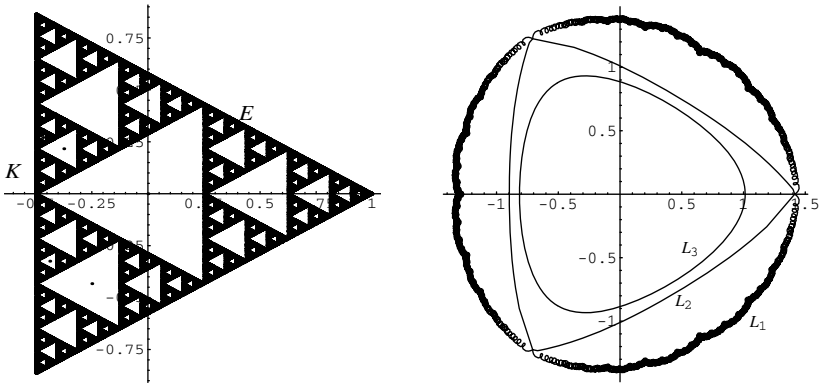

FIGURE 4. The left figure is the Sierpinski gasket K with vertices 1, $e^{2\pi i/3}$, $e^{4\pi i/3}$; $E = S_0S_1(K)$ denotes the 2nd-level cell with vertices $\frac{1}{4}, \frac{5}{8} + \frac{\sqrt{3}}{8}i, \frac{1}{4} + \frac{\sqrt{3}}{4}i$. In the right figure, $L_1 = F_{3,\frac{1}{2}}(T)$ where T is the outer perimeter of K, $L_2 = F_{3,\frac{1}{2}}(\{|z| = 1\})$; $L_3 = F_{3,\frac{1}{2}}(\{|z| = 1.134\})$ as in Proposition 7.2.

The first term in the above sum is

$$> 6\int_E (1-|w|^3)/(1+|w|^3)d\mu(w) \tag{7.3}$$

since $S_0^0 w = w$ and $|1 - e^{\pm i\theta}w^3| \le 1 + |w|^3$. For $n \ge 2$ and $w \in E$, $z = S_0^{n-1}w$ satisfies $0 \le \arg z < \frac{\pi}{6}$. Let $\xi = z^3 = (S_0^{n-1}w)^3$. By using the inequality of arithmetic mean and geometric mean, we have $\frac{1}{|1-e^{i\theta}\xi|^2} + \frac{1}{|1-e^{-i\theta}\xi|^2} \ge \frac{2}{d(\theta)}$ where $d(\theta) = |1 - 2\cos\theta\xi + \xi^2|$. It follows from

$$\frac{d'(\theta)}{d(\theta)} = 2\sin\theta\frac{(1+|\xi|^2)\mathrm{Re}\xi - 2\cos\theta|\xi|^2}{d^2(\theta)}$$

that $d(\theta) \le d(\pi) = |1+\xi|^2$ for $\theta \in [0, 2\pi]$ and $\xi = re^{it}$ with $t \in [0, \frac{\pi}{2}]$, $0 < r \le 1$. Hence

$$\frac{1}{|1-e^{i\theta}\xi|^2} + \frac{1}{|1-e^{-i\theta}\xi|^2} \ge \frac{2}{|1+\xi|^2} \quad \text{for} \quad \theta \in [0, 2\pi]. \tag{7.4}$$

By (7.1)-(7.4), we have

$$\gamma_3 > 6\int_E \frac{1-|w|^3}{1+|w|^3}d\mu(w) + 6\sum_{n=1}^{l-1}(\frac{1}{3})^n \int_E \frac{1-|S_0^n(w)|^6}{|1+(S_0^n(w))^3|^2}d\mu(w). \tag{7.5}$$

To make a numerical estimation of the expression, we observe that the smaller cell $E = S_0S_1(K)$ is the attractor of the maps

$$T_j z = z_j + 1/2(z - z_j), \quad j = 1, 2, 3,$$

where $z_1 = \frac{5}{8} + \frac{\sqrt{3}}{8}i$, $z_2 = \frac{1}{4} + \frac{\sqrt{3}}{4}i$, $z_3 = \frac{1}{4}$. Let $\mathcal{J}_k = \{J = (j_1, \cdots, j_k) : j_i = 1, 2, \text{or } 3\}$ be the index set of length k and let $T_J = T_{j_1}\cdots T_{j_k}$. Note that $S_0^n T_J E$ lies in the cell $\Delta_n(J)$ with vertices $S_0^n T_J(z_1), S_0^n T_J(z_2), S_0^n T_J(z_3)$. A simple geometric consideration shows (see Figure 4) that for $w \in T_J E$,

$$|S_0^n(w)| \le |S_0^n(T_J(z_1))|, \quad 0 < \mathrm{Re}(S_0^n w) \le \mathrm{Re}(S_0^n T_J(z_1)), \quad n = 0, 1, 2, \cdots,$$

and

$$0 \le \mathrm{Im}(S_0^n T_J(z_3)) \le \mathrm{Im}(S_0^n w) < \frac{\sqrt{3}}{3}\mathrm{Re}(S_0^n w), \quad n = 1, 2, \cdots .$$

Hence

$$\mathrm{Re}(S_0^n w)^3 \le \mathrm{Re}(S_0^n T_J(z_1))\Big[(\mathrm{Re}(S_0^n T_J(z_1)))^2 - 3(\mathrm{Im}(S_0^n T_J(z_3)))^2\Big] := p(J, n)$$

(we have used $\mathrm{Re}w^3 = \mathrm{Re}w[(\mathrm{Re}w)^2 - 3(\mathrm{Im}w)^2]$). Note that $E = \bigcup_{J\in\wp_k} T_J E$ and $\mu(T_J(E)) = 3^{-k-2}$. Hence for any fixed k, we can reduce the integral in (7.5) to a finite sum

$$\gamma_3 > \frac{2}{3^{k+1}} \sum_{J\in\wp_k} \frac{1 - |T_J(z_1)|^3}{1 + |T_J(z_1)|^3} + \sum_{n=1}^{l-1} \frac{2}{3^{k+n+1}} \sum_{J\in\wp_k} \frac{1 - |S_0^n(T_J(z_1))|^6}{1 + |S_0^n(T_J(z_1))|^6 + 2p(J, n)}.$$

We can now make use of Mathematica to calculate the sum. An accurate approximation is to take $l = 7, k = 8$; the value is > 0.6834. □

By using Proposition 7.1 we conclude that

Proposition 7.2. *The Cauchy transform $F_{3,\frac{1}{2}}$ on the Sierpinski gasket is starlike univalent in $|z| > 1.1340$.*

Sketch of proof. Let $f(z) = F_{3,\frac{1}{2}}(\frac{1}{z})$, and let $b = 0.8147$ as in Proposition 7.1. Then by Theorem 4.1, we see that $f \in \mathcal{F}_3(b)$ (see (1.5)). We first give an expression of $g \in \mathcal{F}_3(b)$ as in Proposition 2.1. Note that $p_b(z) = \frac{1+(2b-1)z}{1+z}$ maps conformally $\mathbb{D}$ onto the right-half plane $\mathrm{Re}w > b$ with $p_b(0) = 1$. For $g \in \mathcal{F}_3(b)$, let $h(z^3) = \frac{g(z)}{z}$. It is easy to show that $h(z)$ is subordinated to $p_b(z)$, i.e., $h(\mathbb{D}) \subseteq p_b(\mathbb{D})$ and $h(0) = p_b(0) = 1$. Let

$$\psi(z) = p_b^{-1} \circ h(z), \quad z \in \mathbb{D}. \tag{7.6}$$

Then, ψ is analytic in $\mathbb{D}$ and satisfies $\psi(\mathbb{D}) \subset \mathbb{D}$ and $\psi(0) = 0$. The Schwarz lemma shows that $|\psi(z)| \le |z|, z \in \mathbb{D}$. Thus $\varphi(z) := \frac{\psi(z)}{z}$ is analytic in $\mathbb{D}$ with $|\varphi(z)| \le 1, z \in \mathbb{D}$. It follows from (7.6) and $g(z) = zh(z^3)$ that

$$g(z) = zp_b(z^3\varphi(z^3)) = z\frac{1 + (2b-1)z^3\varphi(z^3)}{1 + z^3\varphi(z^3)}, \qquad |z| < 1.$$

Next, by substituting this into the starlikeness criterion $\mathrm{Re}\big(zg'(z)/g(z)\big) \ge 0$, $|z| = R$ in (2.1), and applying a similar proof as in Theorem 3.1, we can show that for $b \in (13/25, 1)$, $g(z)$ is starlike in $|z| < R$, where

$$R \ge \big((25b - 13)/q(b)\big)^{1/6} \tag{7.7}$$

with $q(b) = -31+79b-30b^2+6\sqrt{(2b-1)(24b^2-40b+17)}$. It applies for $b = 0.8417$ as we set up. The proof of (7.7) is, however, quite tedious and makes use of Mathematica for some estimate. The details are in [D]. □

Remark. (i) By using the expression of $f \in \mathcal{F}_1(\alpha)$ in [TA, Theorem 3.1] with $\alpha = 0.8417$ (by Proposition 7.1), it is quite direct to show that $F_{3,\frac{1}{2}}(\frac{1}{z})$ maps $\{|z| > 1.1974\}$ to a starlike set, which is weaker than Proposition 7.2.

(ii) Actually we can improve the result of [TA] to $f \in \mathcal{F}_m(\alpha)$ with $\alpha \in (0, 1), m \ge 2$, and obtain a complete result as an extension of Theorem 3.1. But the proof is lengthy and tedious, so we do not include it here.

We remark that our estimation in Proposition 7.2 is not optimal. In fact, in view of Figure 4 on the image of F, we make the following conjecture.

Conjecture. *$F(z)$ is starlike (or even convex) univalent on $|z| > 1$.*

The image of F on the exterior Δ_0 of the Sierpinski gasket K and near K has been discussed in [LSV], [DL3], [DL4]. Unlike $|z| > 1$, F is not univalent in Δ_0 (hence $F(\partial\Delta_0)$ has self-intersections), and the behaviors of F and F' become quite wild as z approaches the boundary $T = \partial\Delta_0$ of Δ_0. The image L_1 of $\partial\Delta_0$ under F reveals some very rich behavior of F. Detailed studies of these are in [DL3], [DL4], [DLL], and [LDP].

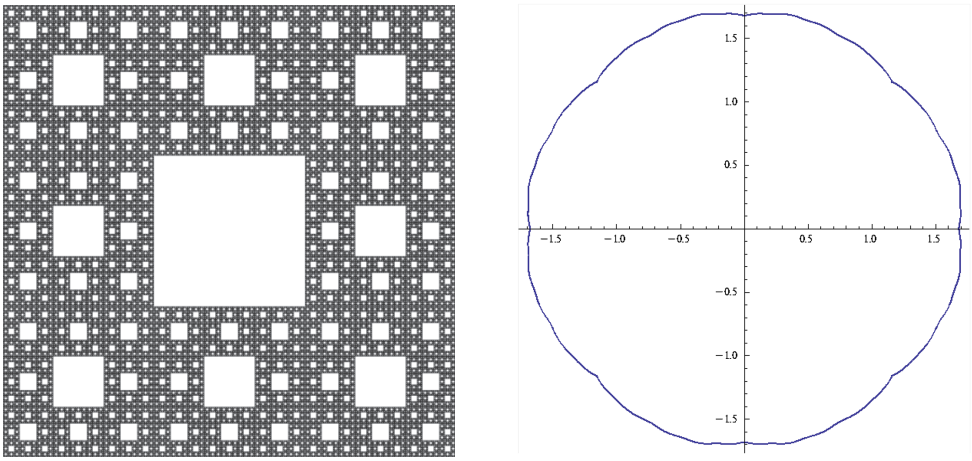

FIGURE 5. Sierpinski carpet K_c and the image $F_c(\partial K_c)$ of the outer boundary ∂K_c under F_c.

Another well-known self-similar set is the Sierpinski carpet K_c, which we have not included here (the IFS does not satisfy (4.1)). We are informed by J.C. Liu for the graph in Figure 5 of the Cauchy transform F_c of the normalized α-Hausdorff measure on K_c ($\alpha = \log 8/\log 3$): the image $F_c(\partial K_c)$ of the the outer boundary ∂K_c of K_c is a Jordan curve. This is quite unexpected, as for the Sierpinski gasket, the corresponding image is a fractal curve with infinitely many loops inside loops (see Figure 4 and more detail in [LSV], [DLL], [DL4]), which prompted us to study the *Cantor boundary behavior*. The difference in the two cases may by attributed to the way the subcells intersect.

Conjecture. *$F_c(\partial K_c)$ is a Jordan curve of Hausdorff dimension > 1.*

The same question of starlikeness and convexity can also be asked for the Sierpinski carpet. The techniques we developed to handle the square case in Section 5 may be useful in this new situation.

ACKNOWLEDGEMENT

The authors are grateful to the referee for going through the paper in detail, and for many valuable suggestions and references to improve the presentation.

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