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GENERIC STATIONARY MEASURES AND ACTIONS

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Abstract. Let G be a countably infinite group, and let μ be a generating probability measure on G. We study the space of μ-stationary Borel probability measures on a topological G space, and in particular on Z^G, where Z is any perfect Polish space. We also study the space of μ-stationary, measurable G-actions on a standard, nonatomic probability space.

Equip the space of stationary measures with the weak* topology. When μ has finite entropy, we show that a generic measure is an essentially free extension of the Poisson boundary of (G, μ). When Z is compact, this implies that the simplex of μ-stationary measures on Z^G is a Poulsen simplex. We show that this is also the case for the simplex of stationary measures on $\{0, 1\}^G$.

We furthermore show that if the action of G on its Poisson boundary is essentially free, then a generic measure is isomorphic to the Poisson boundary.

Next, we consider the space of stationary actions, equipped with a standard topology known as the weak topology. Here we show that when G has property (T), the ergodic actions are meager. We also construct a group G without property (T) such that the ergodic actions are not dense, for some μ.

Finally, for a weaker topology on the set of actions, which we call the very weak topology, we show that a dynamical property (e.g., ergodicity) is topologically generic if and only if it is generic in the space of measures. There we also show a Glasner-King type 0-1 law stating that every dynamical property is either meager or residual.

Contents

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1. Introduction

This paper is motivated by two subjects: *genericity in dynamics* and *stationary actions*. We begin our introduction with background on genericity in dynamics.

1.1. Genericity.

There is a long history of topological genericity in dynamics beginning with Halmos [30], who showed that a generic automorphism is weakly mixing. More precisely, he studied the group $\mathrm{Aut}(X,\nu)$ of measure-preserving transformations of a standard Lebesgue probability space (X,ν) (in which two transformations that agree modulo measure zero are identified). This group is naturally equipped with a Polish topology. Halmos proved that the set of weakly mixing transformations is a dense G_δ subset of $\mathrm{Aut}(X,\nu)$. Since then, it has been proven that a generic measure-preserving transformation has zero entropy, rank one [17], is rigid, has interesting spectral properties [15, 49] and so on. There are also many interesting results about generic homeomorphisms [33].

Instead of studying the space of transformations, one may study the space of measures. Precisely, consider a homeomorphism $T : X \to X$ of a topological space X. Let $\mathcal{P}_T(X)$ denote the space of all T-invariant Borel probability measures on X with the weak* topology. If X is compact, then $\mathcal{P}_T(X)$ is a Choquet simplex: it is a convex, compact subset of a locally convex vector space, such that every measure in $\mathcal{P}_T(X)$ has a unique representation as the barycenter of a probability measure on the set of extreme points of $\mathcal{P}_T(X)$ (and the extreme points are exactly the ergodic measures for T). Given any abstract Choquet simplex Σ, there exists a compact metric space X and a homeomorphism $T : X \to X$ such that $\mathcal{P}_T(X)$ is affinely homeomorphic to Σ [16]. Therefore, it is natural to look for "special" homeomorphisms. Indeed, there is a canonical choice: consider the product space $[0,1]^{\mathbb{Z}}$ with the shift action $T : [0,1]^{\mathbb{Z}} \to [0,1]^{\mathbb{Z}}$ given by $(Tx)_i = x_{i+1}$. It follows from Rohlin's Lemma (for example) that the ergodic measures are dense (and therefore residual) in $\mathcal{P}_T([0,1]^{\mathbb{Z}})$. By [41] there is a unique Choquet simplex with this property (up to affine homeomorphisms) called the *Poulsen simplex*.

This result was greatly generalized in an influential paper of Glasner and Weiss [24]. They considered an arbitrary countable group G acting by homeomorphisms on a compact metrizable space X (they also considered locally compact groups, but we will not need that here). Let $\mathcal{P}_G(X)$ denote the space of G-invariant Borel probability measures on X with the weak* topology. As before this is a Choquet simplex and the ergodic measures are the extreme points. When G has property (T), they showed that the set of ergodic measures is closed in $\mathcal{P}_G(X)$, and therefore $\mathcal{P}_G(X)$ is a *Bauer simplex*. When G does not have property (T) then they show that $\mathcal{P}_G(\{0,1\}^G)$ is a Poulsen simplex, and G acts on $\{0,1\}^G$ by $(gx)_k = x_{g^{-1}k}$ for $x \in \{0,1\}^G, g, k \in G$. Their proof extends to any $\mathcal{P}_G(W^G)$, with W a nontrivial compact space.

The fact that ergodic transformations are residual in $\mathrm{Aut}(X,\nu)$ and ergodic measures are residual in $\mathcal{P}_T([0,1]^{\mathbb{Z}})$ is no accident: Glasner and King proved that for any dynamical property P, a generic element of $\mathrm{Aut}(X,\nu)$ has P if and only if a generic measure in $\mathcal{P}_T([0,1]^{\mathbb{Z}})$ has P [25]. In fact, their proof extends to measure-preserving actions of all countable groups.

To make a more precise statement, let G denote a countable group. The space $A(G,X,\nu) = \mathrm{Hom}(G, \mathrm{Aut}(X,\nu))$ of all homomorphisms from G to $\mathrm{Aut}(X,\nu)$ is equipped with the topology of pointwise convergence, under which it is a Polish

space. Glasner and King proved that if P is any dynamical property, then a generic action in $A(G, X, \nu)$ has P if and only if a generic measure in $\mathcal{P}_G([0,1]^G)$ has P. The precise statement is recounted in §8 of this paper.

As noted in [26], these results imply the following dichotomy: if G has (T), then the ergodic actions form a meager subset of $A(G, X, \nu)$, while if G does not have (T), then the ergodic actions form a residual subset of $A(G, X, \nu)$. This result was extended by Kerr and Pichot [39] to show that if G does not have (T), then the weakly mixing actions are a dense G_δ subset. Kerr and Pichot also generalize this result to C^*-dynamical systems.

Let us also mention here the *weak Rohlin property* as well as the 0-1 *law* of Glasner and King [25]. $\mathrm{Aut}(X, \nu)$ acts continuously on this space by conjugation, and a group has the weak Rohlin property if $A(G, X, \nu)$ has a dense $\mathrm{Aut}(X, \nu)$-orbit. By [26] all countable groups have the weak Rohlin property. Any group with the weak Rohlin property (and hence any countable group) obeys a 0-1 law: every Baire-measurable dynamical property of $A(G, X, \nu)$ is either residual or meager.

1.2. **Stationarity.** Let G be a countable group and μ a probability measure on G whose support generates G as a semigroup. An action $G \curvearrowright (X, \nu)$ on a probability space is *μ-stationary* (or just *stationary* if μ is understood) if

$$\sum_{g \in G} \mu(g) g_* \nu = \nu. \tag{1}$$

In this case, ν is said to be *μ-stationary*, and it follows that the action is nonsingular. Stationary actions are intimately related to random walks and harmonic functions on groups, as well as to the Poisson boundary [19–21], which is itself a stationary space.

In principle stationary actions exist in abundance: if G acts continuously on a compact metrizable space X, then there exists a μ-stationary probability measure on X. By contrast, if G is nonamenable, then an invariant measure need not exist. However, there are surprisingly few explicit constructions of stationary actions: aside from measure-preserving actions and Poisson boundaries, there are constructions from invariant random subgroups [10, 31] and methods for combining stationary actions via joinings [22]. There is a general structure theory of stationary actions [22] and a very deep structure theory in the case that G is a higher rank semisimple Lie group [45, 46]. There is also a growing literature on classifying stationary actions [6–8], and stationarity has found recent use in proving nonexistence of σ-compact topological models [13] for certain Borel actions.

The Poisson boundary of (G, μ) is the space of ergodic components of the shift action on $(G^{\mathbb{N}}, \mathbb{P}_\mu)$ where $\mathbb{P}_\mu$ is the law of the random walk on G with μ-increments [35]. We denote the Poisson boundary by $\Pi(G, \mu)$. This space was introduced by Furstenberg who showed that the space of bounded μ-harmonic functions on G is naturally isomorphic with $L^\infty(\Pi(G, \mu))$ [19, 20]. It plays a central role in the structure theory of stationary actions [22] and is important in rigidity theory [5, 32, 45]. It also plays a key role in this paper, and so we define it formally in §2.1.1.

1.3. **Main results.** Motivated by the above genericity results we ask the following questions: is a generic stationary action ergodic? Is a generic stationary measure ergodic? Does it depend on whether G has property (T)? Is there a Glasner-King type correspondence principle relating dynamical properties of generic stationary

actions and measures? Is there a Glasner-King 0-1 law for stationary actions? We answer some of these questions next.

1.3.1. *Spaces of measures.* Let G be a discrete, countable group. Our investigations begin with spaces of measures. So if G acts by homeomorphisms on a topological space Y, let $\mathcal{P}_\mu(Y)$ denote the space of all μ-stationary Borel probability measures on Y with the weak* topology. By (1) this is a closed subspace of $\mathcal{P}(Y)$, the space of all Borel probability measures on Y. Its extreme points are the ergodic measures and it is a Choquet simplex if Y is compact and Hausdorff [5]. Our first result is:

Theorem 1.1 (Generic stationary measures)**.** *Let Z be a perfect Polish space, and let μ have finite entropy. Then a generic measure in $\mathcal{P}_\mu(Z^G)$ is an ergodic, essentially free extension of the Poisson boundary, denoted $\Pi(G,\mu)$. Moreover, if the action $G \curvearrowright \Pi(G,\mu)$ is essentially free, then a generic measure $\nu \in \mathcal{P}_\mu(Z^G)$ is such that $G\curvearrowright(Z^G,\nu)$ is measurably conjugate to $G\curvearrowright\Pi(G,\mu)$.*

Here, $G\curvearrowright(B,\nu)$ is an extension of the Poisson boundary $\Pi(G,\mu)$ if there exists a G-equivariant factor $(B,\nu) \to \Pi(G,\mu)$. Note that an ergodic, essentially free extension of the Poisson boundary always exists; one can take, for example, the product of the Poisson boundary with a Bernoulli shift.

Corollary 1.2. *If Z is a compact perfect Polish space, then $\mathcal{P}_\mu(Z^G)$ is a Poulsen simplex.*

Observe that we do not put any conditions on the group G in the above results. In particular, G is allowed to have property (T). Perhaps the most unusual aspect of the result above occurs when G acts essentially freely on its Poisson boundary. For in this case, there is a generic measure-conjugacy class. This might be considered analogous to the Kechris-Rosendal result that there is a generic conjugacy class in the group of homeomorphisms of the Cantor set [3, 38]. See also [27] for other examples of transformation groups with generic conjugacy classes.

The main technical component of the proof of Theorem 1.1 is Theorem 4.1, which states that given an ergodic, essentially free extension of the Poisson boundary $G\curvearrowright(B,\nu)$ and a compact metric space W, the set of measures on $\mathcal{P}_\mu(W^G)$ that are G-factors of $G\curvearrowright(B,\nu)$ is dense.

To motivate our interest on groups that act freely on their Poisson boundaries, we provide the following straightforward claim.

Proposition 1.3. *Let G be a torsion-free, nonelementary word hyperbolic group. Then G acts essentially freely on its Poisson boundary $\Pi(G,\mu)$, for any generating measure μ.*

1.3.2. *Spaces of actions.* Next we turn our attention to spaces of actions. Here, it appears that there are two natural choices for the topology on the space of stationary actions. To be precise, let $\mathrm{Aut}^*(X,\nu)$ denote the group of nonsingular transformations of (X,ν) in which two such transformations are identified if they agree up to null sets. We embed this group into $\mathrm{Isom}(L^p(X,\nu))$ via

$$T \mapsto U_{T,p}, \quad U_{T,p}(f) = \left(\frac{dT_*\nu}{d\nu}(x)\right)^{1/p} f \circ T^{-1}.$$

We equip $\mathrm{Isom}(L^p(X,\nu))$ with either the weak or strong operator topology and $\mathrm{Aut}^*(X,\nu)$ with the subspace topology. From results in [12, 43], it follows that only

two different topologies on $\mathrm{Aut}^*(X,\nu)$ result from this construction: the topology derived from the weak operator topology on $\mathrm{Isom}(L^1(X,\nu))$ and the topology derived from any other choice of $1 \le p < \infty$ and (weak/strong). The latter topology has been studied previously [12, 34] and is called the *weak topology.* Therefore, we call the topology derived from the weak operator topology on $\mathrm{Isom}(L^1(X,\nu))$ the *very weak topology.* Both of these topologies are Polish topologies. However, only the weak topology is a group topology.

Next we let $A^*(G,X,\nu) = \mathrm{Hom}(G, \mathrm{Aut}^*(X,\nu))$ be the space of homomorphisms of G into $\mathrm{Aut}^*(X,\nu)$ with the topology of pointwise convergence and $A_\mu(G,X,\nu) \subset A^*(G,X,\nu)$ the subspace of μ-stationary actions with the subspace topology. This gives two distinct topologies on $A_\mu(G,X,\nu)$ (depending on the choice of topology on $\mathrm{Aut}^*(X,\nu)$), which we also call the weak and very weak topologies. Both topologies are Polish and both topologies restrict to the same topology on $A(G,X,\nu)$ (which is the usual one, as studied in [37, 39] for example). Note that as in the case of the measure-preserving actions, the group $\mathrm{Aut}(X,\nu)$ acts on $A_\mu(G,X,\nu)$ by conjugations. This action is continuous under both topologies on $A_\mu(G,X,\nu)$.

The weak topology on $A_\mu(G,X,\nu)$ is perhaps more natural, since it is derived from the group topology on $\mathrm{Isom}(L^1(X,\nu))$. However, under the very weak topology, $A_\mu(G,X,\nu)$ better resembles the space of measure-preserving actions $A(G,X,\nu)$. Indeed, as we will show below, under the very weak topology there is always a dense $\mathrm{Aut}(G,X,\nu)$-orbit in $A_\mu(G,X,\nu)$, and hence a 0-1 law. This is not true in the weak topology, unless the only stationary measures are invariant.

1.3.3. *The weak topology.* We will prove:

Theorem 1.4. *If G has property (T), then the set of ergodic actions in $A_\mu(G,X,\nu)$ is nowhere dense when $A_\mu(G,X,\nu)$ is endowed with the weak topology.*

Recall that the same result holds in the measure-preserving case [24]. Therefore, it makes sense to ask: if G does not have (T), then are the ergodic actions generic? In this generality, the answer is no: we provide an explicit counterexample.

Theorem 1.5. *There exist a countable group G that does not have property (T) and a generating probability measure μ on G such that the set of ergodic measures is not dense in $A_\mu(G,X,\nu)$ when $A_\mu(G,X,\nu)$ is endowed with the weak topology.*

At this point, it is natural to ask whether, because the ergodic measures in the example above are not dense, they must be meager. However there is no dense $\mathrm{Aut}(X,\nu)$-orbit and no Glasner-King type 0-1 law:

Proposition 1.6. *If (G,μ) has a nontrivial Poisson boundary, then, under the weak topology on $A_\mu(G,X,\nu)$, there does not exist in $A_\mu(G,X,\nu)$ a dense $\mathrm{Aut}(X,\nu)$-orbit, and there does exist an $\mathrm{Aut}(X,\nu)$-invariant Borel subset that is neither meager nor residual.*

On the other hand, if the Poisson boundary of (G,μ) is trivial, then all stationary actions are measure preserving, and there is a dense orbit and a 0-1 law. In this case, the group G is necessarily amenable. Incidentally, for amenable groups, it is an open question whether the ergodic actions are dense in $A_\mu(G,X,\nu)$ with respect to the weak topology.

1.3.4. *The very weak topology.* We prove that the Glasner-King correspondence principle generalizes to stationary actions with respect to the very weak topology:

Theorem 1.7 (Correspondence Principle)**.** *Let Z be a perfect Polish space, and let P be a dynamical property. A generic action in $A_\mu(G, X, \nu)$ has P iff a generic measure in $\mathcal{P}_\mu(Z^G)$ has P when $A_\mu(G, X, \nu)$ is endowed with the very weak topology.*

It follows from Theorems 1.1 and 1.7 that under this topology, a generic action is an essentially free extension of the Poisson boundary and is isomorphic to the Poisson boundary when the action on it is essentially free. Another interesting consequence of Theorem 1.7 is that if Z_1 and Z_2 are perfect Polish spaces, then a generic measure in $\mathcal{P}(Z_1^G)$ has a dynamical property iff a generic measure in $\mathcal{P}_\mu(Z_2^G)$ has this property. We use this in the proof of Theorem 1.1.

We prove that under the very weak topology, $A_\mu(G, X, \nu)$ does have a dense $\mathrm{Aut}(X, \nu)$-orbit.

Theorem 1.8. *For any discrete group G with a generating measure μ, there exists in $A_\mu(G, X, \nu)$ a dense $\mathrm{Aut}(X, \nu)$-orbit, with respect to the very weak topology.*

A consequence is a Glasner-King type 0-1 law.

Corollary 1.9 (0-1 law for stationary actions)**.** *Every $\mathrm{Aut}(X, \nu)$-invariant Baire measurable subset of $A_\mu(G, X, \nu)$ is either meager or residual in the very weak topology.*

2. Definitions and preliminaries

2.1. **Nonsingular and stationary measures.** Let $G \curvearrowright Y$ be a continuous action of a countable group on a compact Hausdorff space. A particularly interesting case is when $Y = W^G$ for some compact metric space W with cardinality > 1, the topology is the product topology, and G acts by left translations. We denote by $\mathcal{P}(Y)$ the space of Borel probability measures on Y, equipped with the weak* topology. This is a compact Polish space.

A subspace of $\mathcal{P}(Y)$ is $\mathcal{P}^*_G(Y)$, the space of G quasi-invariant measures on Y. Those are the measures $\nu \in \mathcal{P}(Y)$ such that $g_*\nu$ and ν are equivalent, that is, mutually absolutely continuous, for all $g \in G$.

A probability measure μ on G is said to be *generating* if its support generates G as a semigroup. Given such a measure μ, a subspace of $\mathcal{P}^*_G(Y)$ is the closed set of μ-stationary measures $\mathcal{P}_\mu(Y)$. Those are the measures that satisfy (1). This is also a Polish space.

Finally, $\mathcal{P}_G(Y) \subseteq \mathcal{P}_\mu(Y)$ is the space of G-invariant measures. This series of inclusions is summarized as follows:

$$\mathcal{P}_G(Y) \subseteq \mathcal{P}_\mu(Y) \subseteq \mathcal{P}^*_G(Y) \subseteq \mathcal{P}(Y).$$

2.1.1. *The Poisson boundary.* The Poisson boundary $\Pi(G, \mu)$ is an important measurable μ-stationary action on an abstract probability space. It was introduced by Furstenberg [20] in the context of Lie groups or, more generally, locally compact second countable groups; we will define it for countable groups.

So let G be a countable group and μ a probability measure on G whose support generates G as a semigroup. Let $\mathbb{P}_\mu$ be the push-forward of $\mu^{\mathbb{N}}$ under the map $(g_1, g_2, g_3, \ldots) \mapsto (g_1, g_1g_2, g_1g_2g_3, \ldots)$. The space $(G^{\mathbb{N}}, \mathbb{P}_\mu)$ is the space of random walks on G with μ-increments.

Consider the natural shift action on $G^{\mathbb{N}}$ given by $(g_1, g_2, g_3, \ldots) \mapsto (g_2, g_3, \ldots)$. The Poisson boundary $\Pi(G, \mu)$ is the Mackey point realization [42] of the shift-invariant sigma-algebra of $(G^{\mathbb{N}}, \mathbb{P}_\mu)$, and can be thought of as the set of possible asymptotic behaviors of the random walk. We refer the reader to Furman [18] for an in-depth discussion.

An important property of the Poisson boundary is that the G-action on it is amenable, in Zimmer's sense [51]. This fact is an important ingredient in the proof of Theorem 1.1, and we use it in the proof of Lemma 4.5. Another important property, which we discuss in the next section, is that the Furstenberg entropy of the Poisson boundary is maximal.

Let $G \curvearrowright (B, \beta)$ be the Poisson boundary of (G, μ). We call measure $\nu \in \mathcal{P}_\mu(Y)$ *Poisson* if $G \curvearrowright (Y, \nu)$ is measurably conjugate to $G \curvearrowright (B, \beta)$. Let $\mathcal{P}_\mu^{Poisson}(Y) \subset \mathcal{P}_\mu(Y)$ denote the subset of Poisson measures. A measure $\nu \in \mathcal{P}_\mu(Y)$ is an extension of the Poisson boundary if there exists a G-equivariant factor $\pi \colon Y \to B$ such that $\pi_* \nu = \beta$.

2.1.2. *Furstenberg entropy.* The *Furstenberg entropy* [19] of a μ-stationary measure $\nu \in \mathcal{P}_\mu(Y)$ is given by

$$h_\mu(Y, \nu) = \sum_{g \in G} \mu(g) \int_Y -\log \frac{d\nu}{dg_*\nu}(y) dg_*\nu(y).$$

We also refer to $h_\mu(\cdot)$ as *μ-entropy.*

Furstenberg entropy is an important measure-conjugacy invariant of stationary actions; for example, when the Shannon entropy of μ is finite, then the only proximal stationary space (i.e., a factor of the Poisson boundary) with maximal Furstenberg entropy is the Poisson boundary [35]. In general (i.e., even when the entropy of μ is infinite), every stationary space has Furstenberg entropy that is at most that of the Poisson boundary, and the latter is bounded by the Shannon entropy of μ. Because of this fact we say that a stationary action has *maximum μ-entropy* if its μ-entropy equals the μ-entropy of the Poisson boundary.

3. G_δ subsets of the space of measures

Let Y be a compact metric space on which G acts by homeomorphisms. Recall that a measure $\nu \in \mathcal{P}_\mu(Y)$ is

- *ergodic* if for every G-invariant measurable subset $E \subset Y$, $\nu(E) \in \{0, 1\}$,
- *maximal* if the μ-entropy of $G \curvearrowright (Y, \nu)$ is the same as the μ-entropy of G acting on the Poisson boundary,
- *proximal* if $G \curvearrowright (Y, \nu)$ is a measurable factor of the Poisson boundary action $G \curvearrowright \Pi(G, \mu)$,
- *Poisson* if $G \curvearrowright (Y, \nu)$ is measurably conjugate to the Poisson boundary action $G \curvearrowright \Pi(G, \mu)$,
- *essentially free* if for each $g \in G$, the set of G fixed points $\{y \in Y \,:\, gy = y\}$ has ν-measure zero.

The purpose of this section is to prove:

Theorem 3.1. *Let*

- $\mathcal{P}_\mu^e(Y) \subset \mathcal{P}_\mu(Y)$ *denote the subset of ergodic measures,*
- $\mathcal{P}_\mu^{max}(Y) \subset \mathcal{P}_\mu(Y)$ *denote the subset of maximum μ-entropy measures,*

- $\mathcal{P}_\mu^{proximal}(Y) \subset \mathcal{P}_\mu(Y)$ *denote the subset of proximal measures,*
- $\mathcal{P}_\mu^{free}(Y) \subset \mathcal{P}_\mu(Y)$ *denote the subset of essentially free measures,*
- $\mathcal{P}_\mu^{Poisson}(Y) \subset \mathcal{P}_\mu(Y)$ *denote the subset of Poisson measures.*

Then $\mathcal{P}_\mu^e(Y), \mathcal{P}_\mu^{max}(Y)$, $\mathcal{P}_\mu^{proximal}(Y)$ *and* $\mathcal{P}_\mu^{free}(Y)$ *are* G_δ *subsets of* $\mathcal{P}_\mu(Y)$.

If the Shannon entropy $H(\mu) < \infty$, *then* $\mathcal{P}_\mu^{Poisson}(Y)$ *is also a* G_δ*-subset of* $\mathcal{P}_\mu(Y)$.

To get started, we prove that $\mathcal{P}_\mu^e(Y)$, $\mathcal{P}_\mu^{max}(Y)$ and $\mathcal{P}_\mu^{free}(Y)$ are G_δ subsets after the next (standard) lemma.

Lemma 3.2. *Let* Δ *be a Choquet simplex and* $\Delta^e \subset \Delta$ *denote the subset of extreme points. Then* Δ^e *is a* G_δ *subset of* Δ.

Proof. Let d denote a continuous metric on Δ. For each integer $n > 0$ let Δ_n denote the set of all $x \in \Delta$ such that there exists $y, z \in \Delta$ with $d(y, z) \geq 1/n$ such that $x = (1/2)(y + z)$. Then Δ_n is closed in Δ and $\Delta^e = \bigcap_{n=1}^\infty \Delta \setminus \Delta_n$. □

The proof of the following corollary is straightforward and involves the application of known results from the measure-preserving case to the stationary case.

Corollary 3.3. $\mathcal{P}_\mu^e(Y)$, $\mathcal{P}_\mu^{max}(Y)$ *and* $\mathcal{P}_\mu^{free}(Y)$ *are* G_δ *subsets.*

Proof. By the ergodic decomposition theorem for stationary measures, $\mathcal{P}_\mu^e(Y)$ is the set of extreme points of $\mathcal{P}_\mu(Y)$ [5]. So the previous lemma implies $\mathcal{P}_\mu^e(Y)$ is a G_δ. By [48, Theorem 1], the map $\nu \mapsto h_\mu(Y, \nu)$ is lower semicontinuous on $\mathcal{P}_\mu(Y)$. This implies $\mathcal{P}_\mu^{max}(Y)$ is a G_δ. To see that $\mathcal{P}_\mu^{free}(Y)$ is a G-delta, note that for each $g \in G$, the set of g fixed points $F_g = \{y \in Y : gy = y\}$ is closed, by the continuity of the G-action. So the portmanteau theorem implies the set $M_{g,n} = \{\nu \in \mathcal{P}_\mu(Y) : \nu(F_g) < 1/n\}$ is open for all $n > 1$. Since $\mathcal{P}_\mu^{free}(Y) = \bigcap_{g \in G \setminus \{e\}} \bigcap_{n=1}^\infty M_g$, it is a G_δ. □

3.1. **$\mathbb{Z}$-invariant measures from stationary measures.** In order to prove that proximal measures form a G_δ subset of $\mathcal{P}_\mu(Y)$, we obtain an affine homeomorphism between $\mathcal{P}_\mu(Y)$ and a certain space of $\mathbb{Z}$-invariant measures. This idea is inspired by [22], and parts of what follows appear in [18] (see, e.g., Proposition 1.3 there, as well as section 2.3). We nevertheless provide complete proofs for the reader's convenience.

Given a measure μ on G, let $\check{\mu}$ be the measure on G given by $\check{\mu}(A) = \mu(\{g \in G : g^{-1} \in A\})$. To begin, we let G have the discrete topology, $G^\mathbb{Z}$ the product topology and $\mathcal{P}(G^\mathbb{Z} \times Y)$ the weak* topology. Let $\mathcal{P}(G^\mathbb{Z} \times Y | \check{\mu}^\mathbb{Z})$ denote the set of all measures $\lambda \in \mathcal{P}(G^\mathbb{Z} \times Y)$ whose projection to the first coordinate is $\check{\mu}^\mathbb{Z}$. We view $\mathcal{P}(G^\mathbb{Z} \times Y | \check{\mu}^\mathbb{Z})$ as a subspace of $\mathcal{P}(G^\mathbb{Z} \times Y)$ with the subspace topology. In Appendix A we show that this topology on $\mathcal{P}(G^\mathbb{Z} \times Y | \check{\mu}^\mathbb{Z})$ is independent of the choice of topology on $G^\mathbb{Z}$.

We will show that $\mathcal{P}_\mu(Y)$ is affinely homeomorphic with a subspace of $\mathcal{P}(G^\mathbb{Z} \times Y | \check{\mu}^\mathbb{Z})$. To define this subspace, let $r : \mathbb{Z} \times G^\mathbb{Z} \to G$ be the random walk cocycle:

$$r(n, \omega) = \begin{cases} (\omega_1 \cdots \omega_n)^{-1}, & n \geq 1, \\ 1_G, & n = 0, \\ \omega_{n+1} \cdots \omega_0, & n < 0. \end{cases}$$

Note that r satisfies the cocycle equation

$$r(n+m,\omega) = r(n,\sigma^m\omega)r(m,\omega)$$

where σ is the left shift-operator on $G^{\mathbb{Z}}$ defined by $\sigma(\omega)_i = \omega_{i+1}$ for $i \in \mathbb{Z}$.

Define the transformation $T : G^{\mathbb{Z}} \times Y \to G^{\mathbb{Z}} \times Y$ by $T(\omega, y) = (\sigma\omega, \omega_1^{-1}y)$. Observe that for any $n \in \mathbb{Z}$,

$$T^n(\omega, y) = (\sigma^n\omega, r(n,\omega)y).$$

This is a skew-product transformation. For $n \in \mathbb{Z}$ define

$$\begin{array}{rcccl} \phi_n & : & G^{\mathbb{Z}} \times Y & \longrightarrow & G^{\mathbb{N}} \times Y \\ & & (\omega, y) & \longmapsto & ((\omega_n, \omega_{n+1}, \ldots), y), \end{array}$$

and let $\mathcal{P}_{\mathbb{Z}}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$ be the set of all T-invariant Borel probability measures λ on $G^{\mathbb{Z}} \times Y$ such that

$$\phi_{1*}(\lambda) = \check{\mu}^{\mathbb{N}} \times \nu \tag{2}$$

for some $\nu \in \mathcal{P}(Y)$. Observe that because λ is T-invariant, (2) implies the projection of λ to the first coordinate is $\check{\mu}^{\mathbb{Z}}$. So $\mathcal{P}_{\mathbb{Z}}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}}) \subset \mathcal{P}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$. We give it the subspace topology.

The main result of this section is:

Theorem 3.4. *$\mathcal{P}_\mu(Y)$ is affinely homeomorphic with $\mathcal{P}_{\mathbb{Z}}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$. More precisely, define*

$$\alpha : \mathcal{P}_\mu(Y) \to \mathcal{P}_{\mathbb{Z}}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$$
$$\alpha(\nu) = \int \delta_\omega \times \nu_\omega \; d\check{\mu}^{\mathbb{Z}}(\omega)$$

where

$$\nu_\omega = \lim_{n\to\infty} r(-n,\omega)_*^{-1}\nu = \lim_{n\to\infty} \omega_0^{-1}\omega_{-1}^{-1}\cdots\omega_{-(n-1)}^{-1}\nu$$

for the full measure subset of $G^{\mathbb{Z}}$ for which this limit exists. Then α is an affine homeomorphism.

We need the following lemma.

Lemma 3.5.

(1) *For all $\lambda_1, \lambda_2 \in \mathcal{P}_{\mathbb{Z}}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$,*

$$\phi_{1*}(\lambda_1) = \phi_{1*}(\lambda_2) \quad \Rightarrow \quad \lambda_1 = \lambda_2.$$

(2) *Given $\lambda \in \mathcal{P}_{\mathbb{Z}}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$ there exists a measurable map $\omega \mapsto \lambda^\omega$ from $G^{\mathbb{Z}}$ into $\mathcal{P}(Y)$ such that*

$$\lambda = \int \delta_\omega \times \lambda^\omega \; d\check{\mu}^{\mathbb{Z}}(\omega). \tag{3}$$

Moreover, this map is unique up to null sets, and λ^ω depends only on $\{w_n : n \le 0\}$ for a.e. ω.

(3) *If $\lambda \in \mathcal{P}_{\mathbb{Z}}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$ and $\phi_{1*}(\lambda) = \check{\mu}^{\mathbb{N}} \times \nu$, then $\nu \in \mathcal{P}_\mu(Y)$.*

Proof. (1) Let $\mathcal{A}_n$ be the sub-sigma-algebra generated by ϕ_n, so that $\sigma(\bigcup_n \mathcal{A}_n)$ is the entire sigma-algebra. For $A \in \mathcal{A}_n$ and $i = 1, 2$, $\lambda_i(A) = \lambda_i(T^{-n+1}A)$, by T-invariance. But $T^{-n+1}A$ is $\mathcal{A}_1$-measurable, and, for sets in $\mathcal{A}_1$, λ_1 and λ_2 are identical, by (2).

(2) Existence and uniqueness follow from the disintegration theorem. By (2), λ^ω depends only on $\{w_n : n \leq 0\}$.

(3) By (3)

$$\phi_{1*}(T_*\lambda) = \phi_{1*}\left(\int \delta_{\sigma\omega} \times \omega_{1*}^{-1}\lambda^\omega \, d\check{\mu}^{\mathbb{Z}}(\omega)\right)$$
$$= \int \delta_{(\omega_2,\omega_3,\ldots)} \times \omega_{1*}^{-1}\lambda^\omega \, d\check{\mu}^{\mathbb{Z}}(\omega).$$

Since λ^ω depends only on $\{\omega_n : n \leq 0\}$, then

$$= \int \delta_{(\omega_2,\omega_3,\ldots)} \, d\check{\mu}^{\mathbb{Z}}(\omega) \times \int \omega_*'^{-1}\left(\int \lambda^\omega \, d\check{\mu}^{\mathbb{Z}}(\omega)\right) \, d\check{\mu}^{\mathbb{Z}}(\omega')$$
$$= \check{\mu}^{\mathbb{N}} \times \int g_*\nu \, d\mu(g),$$

where the last equality follows from the fact that $\nu = \int \lambda^\omega \, d\check{\mu}^{\mathbb{Z}}(\omega)$, a consequence of (2) and the definition of λ^ω. But $T_*\lambda = \lambda$, and so

$$\nu = \int g_*\nu \, d\mu(g).$$

□

Proof of Theorem 3.4. If $\omega \in G^{\mathbb{Z}}$ is chosen at random with law $\check{\mu}^{\mathbb{Z}}$, then $n \mapsto r(-n,\omega)_*^{-1}\nu$ is a martingale. By the Martingale Convergence Theorem, ν_ω exists for a.e. ω, and furthermore

$$\int \nu_\omega \, d\check{\mu}^{\mathbb{Z}}(\omega) = \nu. \tag{4}$$

Also,

$$r(m,\omega)_*\nu_\omega = r(m,\omega)_* \lim_{n\to\infty} r(-n,\omega)_*^{-1}\nu = \lim_{n\to\infty} r(-n,\sigma^m\omega)_*^{-1}\nu = \nu_{\sigma^m\omega}, \tag{5}$$

where the second equality follows from the weak* continuity of the G-action on $\mathcal{P}(Y)$ and the cocycle property of r.

Recall

$$\alpha(\nu) = \int \delta_\omega \times \nu_\omega \, d\check{\mu}^{\mathbb{Z}}(\omega).$$

So $\alpha(\nu)$ is indeed T-invariant by (5), and, since ν_ω depends only on $\{\omega_n : n \leq 0\}$, (2) is satisfied and so $\alpha(\nu) \in \mathcal{P}_{\mathbb{Z}}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$. Also define $\beta : \mathcal{P}_{\mathbb{Z}}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}}) \to \mathcal{P}_\mu(Y)$ by

$$\beta(\lambda) = \int \lambda^\omega \, d\check{\mu}^{\mathbb{Z}}(\omega),$$

where λ^ω is given by (3). The image of β is indeed in $\mathcal{P}_\mu(Y)$ by Lemma 3.5 (3).

Note that $\beta(\lambda)$ is simply the push-forward of λ under the projection on the second coordinate. Hence it follows from (4) that $\beta \circ \alpha$ is the identity, and α is one-to-one. By Lemma 3.5 (1) β is one-to-one, and so $\alpha \circ \beta$ is also the identity. Thus α and β are inverses.

It is clear that α and β are affine. So it suffices to show they are continuous. In fact, it suffices to show that α is continuous, since $\mathcal{P}_\mu(Y)$ is compact, and since every continuous bijection between compact spaces is a homeomorphism.

For each $n \in \mathbb{Z}$, let $\pi_n : G^{\mathbb{Z}} \to G$ be the n-th coordinate projection. Let $A \subset G^{\mathbb{Z}}$ be a Borel set contained in the sigma-algebra generated by $\{\pi_n : n \in [-m,m] \cap \mathbb{Z}\}$

where $m > 0$ is some integer. Because G acts continuously, if $\nu_n \to \nu_\infty$ in $\mathcal{P}_\mu(Y)$, then

$$\alpha(\nu_n)^A = \int_A (\nu_n)_\omega \, d\check{\mu}^{\mathbb{Z}}(\omega) = \int_A (\omega_0^{-1}\omega_{-1}^{-1}\cdots\omega_{-m}^{-1})_*\nu_n \, d\check{\mu}^{\mathbb{Z}}(\omega)$$

converges to $\alpha(\nu_\infty)^A$ as $n \to \infty$. Because the coordinate projections generate the sigma-algebra of $G^{\mathbb{Z}}$, this shows that $\alpha(\nu_n)^A$ converges to $\alpha(\nu_\infty)^A$ for all A in a dense subset of the measure algebra of $\check{\mu}^{\mathbb{Z}}$. So $\alpha(\nu_n)$ converges to $\alpha(\nu_\infty)$ by Corollary A.2. Because $\{\nu_n\}$ is arbitrary, α is continuous. □

3.2. **Proximal and Poisson measures.** In this section we finish the proof of Theorem 3.1. In order to prove that proximal measures form a G_δ subset, we need the next lemma.

Lemma 3.6. *Let $\mathcal{Q}$ be the set of all measures $\lambda \in \mathcal{P}_{\mathbb{Z}}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$ such that there exists some measurable map $f : G^{\mathbb{Z}} \to Y$ such that*

$$\lambda = \int \delta_\omega \times \delta_{f(\omega)} \, d\check{\mu}^{\mathbb{Z}}(\omega).$$

Then $\nu \in \mathcal{P}_\mu(Y)$ is proximal if and only if $\alpha(\nu) \in \mathcal{Q}$ (where the affine homeomorphism $\alpha : \mathcal{P}_\mu(Y) \to \mathcal{P}_{\mathbb{Z}}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$ is as in Theorem 3.4).

Proof. Recall from Theorem 3.4 that for any $\nu \in \mathcal{P}_\mu(Y)$,

$$\nu_\omega = \lim_{n\to\infty} r(-n,\omega)_*^{-1}\nu = \lim_{n\to\infty} \omega_0^{-1}\omega_{-1}^{-1}\cdots\omega_{-(n-1)}^{-1}\nu$$

exists for $\check{\mu}^{\mathbb{Z}}$-a.e. $\omega \in G^{\mathbb{Z}}$. It is well known [22] that ν is proximal if and only if ν_ω is a Dirac measure for a.e. ω. So the lemma follows from Theorem 3.4. □

Corollary 3.7. *$\mathcal{P}_\mu^{proximal}(Y)$ is a G_δ subset of $\mathcal{P}_\mu(Y)$.*

Proof. By the previous lemma it suffices to show that $\mathcal{Q}$ is a G_δ subset of $\mathcal{P}_{\mathbb{Z}}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$.

Let $\mathcal{P}^{ex}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$ be the set of extreme points of $\mathcal{P}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$. Observe that for every $\lambda \in \mathcal{P}^{ex}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$ there exists some measurable map $f : G^{\mathbb{Z}} \to Y$ such that

$$\lambda = \int \delta_\omega \times \delta_{f(\omega)} \, d\check{\mu}^{\mathbb{Z}}(\omega).$$

Therefore

$$\mathcal{Q} = \mathcal{P}^{ex}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}}) \cap \mathcal{P}_{\mathbb{Z}}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}}).$$

By Lemma 3.2, $\mathcal{P}^{ex}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$ is a G_δ subset of $\mathcal{P}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$. Of course, $\mathcal{P}_{\mathbb{Z}}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$ is closed in $\mathcal{P}(G^{\mathbb{Z}} \times Y|\check{\mu}^{\mathbb{Z}})$ (because it is compact since it is homeomorphic with $\mathcal{P}_\mu(Y)$). Since closed sets are G_δ subsets and intersections of G_δ's are also G_δ's, this proves that $\mathcal{Q}$ is a G_δ. □

Proof of Theorem 3.1. By Corollaries 3.3 and 3.7, it suffices to show that

$$\mathcal{P}_\mu^{Poisson}(Y) = \mathcal{P}_\mu^{max}(Y) \cap \mathcal{P}_\mu^{proximal}(Y)$$

whenever $H(\mu) < \infty$. This is proven in [35]. □

4. Dense measure conjugacy classes

Let $G \curvearrowright Y$ be an action by homeomorphisms on a compact metrizable space Y. Also let $\mathbf{z} = G \curvearrowright (Z, \zeta)$ be a μ-stationary action. Let $\mathrm{Fact}(\mathbf{z}, Y)$ be the set of all probability measures $\nu \in \mathcal{P}_\mu(Y)$ such that $\nu = \pi_* \zeta$ where $\pi : Z \to Y$ is a G-equivariant Borel map. This is the set of all *factor measures* of the action $G \curvearrowright (Z, \zeta)$. Denote by $\mathrm{Conj}(\mathbf{z}, Y)$ the set of all probability measures $\nu \in \mathcal{P}_\mu(Y)$ such that $G \curvearrowright (\nu, Y)$ is measurably conjugate to (Z, ζ). This is a subset of $\mathrm{Fact}(\mathbf{z}, Y)$.

The main technical result of this section is

Theorem 4.1. *Let W be a compact metric space. Let $\mathbf{b} = G \curvearrowright (B, \nu)$ be any stationary, essentially free extension of the Poisson boundary. Then* $\mathrm{Fact}(\mathbf{b}, W^G)$ *is dense in $\mathcal{P}_\mu(W^G)$.*

Note that such a (B, ν) always exists; for example, take the product of the Poisson boundary with a Bernoulli shift. Before proving this claim we draw a number of consequences and prove a "sharpness" claim. Let $\mathbb{X} = \{0, 1\}^{\mathbb{N}}$ be the Cantor space, equipped with the usual product topology.

Theorem 4.2. *Let $\mathbf{b} = G \curvearrowright (B, \nu)$ be a stationary, essentially free extension of the Poisson boundary. Then* $\mathrm{Conj}(\mathbf{b}, \mathbb{X}^G)$ *is dense in $\mathcal{P}_\mu(\mathbb{X}^G)$.*

Proof. By Theorem 4.1 it suffices to show that for every factor $\pi : (B, \nu) \to (\mathbb{X}^G, \pi_* \nu)$ there exists a sequence of measure-conjugacies $\Phi_i : (B, \nu) \to (\mathbb{X}^G, \Phi_{i*} \nu)$ such that

$$\lim_{i \to \infty} \Phi_{i*} \nu = \pi_* \nu$$

in the weak* topology on $\mathcal{P}_\mu(\mathbb{X}^G)$. For this purpose, choose a measure-conjugacy $\psi : (B, \nu) \to (\mathbb{X}^G, \psi_* \nu)$; the existence of such a measure-conjugacy follows from the fact that there exists a countable dense subset of the measure algebra of (B, ν). We are requiring that ψ is G-equivariant and a measure-space isomorphism. Define $\Phi_i \colon B \to \mathbb{X}^G$ by

$$\Phi_i(b)(g) = (x_1, x_2, \ldots)$$

where $x_j = \pi(b)(g)_j$ if $j \leq i$ and $x_j = \psi(b)(g)_{j-i}$ if $j > i$. If $\mathrm{Proj}_i \colon \mathbb{X}^G \to \mathbb{X}^G$ denotes the projection

$$\mathrm{Proj}_i(x)(g) = (x_{i+1}(g), x_{i+2}(g), \ldots),$$

then $\mathrm{Proj}_i \circ \Phi_i = \psi$. Hence Φ_i is an isomorphism. It is clear that $\lim_{i \to \infty} \Phi_{i*} \nu = \pi_* \nu$. $\square$

An immediate consequence is the following.

Corollary 4.3. *If the action of G on its Poisson boundary is essentially free, then $\mathcal{P}_\mu^{Poisson}(\mathbb{X}^G)$, the set of all measures $\lambda \in \mathcal{P}_\mu(\mathbb{X}^G)$ such that $G \curvearrowright (\mathbb{X}^G, \lambda)$ is measurably conjugate to the Poisson boundary, is dense in $\mathcal{P}_\mu(\mathbb{X}^G)$.*

See §5 for a discussion of conditions which guarantee freeness of the action of G on its Poisson boundary. Next, we observe that Theorem 4.1 is in a sense "best possible":

Proposition 4.4. *Let $\mathbf{b} = G \curvearrowright (B, \nu)$ be a stationary action. Suppose that this action either is not essentially free or does not have maximum μ-entropy. Then* $\mathrm{Fact}(\mathbf{b}, \mathbb{X}^G)$ *is not dense in $\mathcal{P}_\mu(\mathbb{X}^G)$.*

In particular, when the action on the Poisson boundary is not free, then the measure-conjugates of the Poisson boundary are not dense.

Proof. If $\mathbf{b}$ does not have maximum μ-entropy, then because μ-entropy is lower semicontinuous (see the proof of Corollary 3.3), $\overline{\mathrm{Fact}(\mathbf{b}, \mathbb{X}^G)} \cap \mathcal{P}_\mu^{max}(\mathbb{X}^G) = \emptyset$. However, $\mathcal{P}_\mu^{max}(\mathbb{X}^G)$ is nonempty, since it includes a Poisson measure (see, e.g., [50]).

Suppose $\mathbf{b}$ is not essentially free. Let $\mathcal{P}_\mu^{free}(\mathbb{X}^G)$ be the set of all $\eta \in \mathcal{P}_\mu(\mathbb{X}^G)$ such that $G \curvearrowright (\mathbb{X}^G, \eta)$ is essentially free. We claim that $\overline{\mathrm{Fact}(\mathbf{b}, \mathbb{X}^G)} \cap \mathcal{P}_\mu^{free}(\mathbb{X}^G) = \emptyset$.

Let $g \in G \setminus \{e\}$ be an element such that $\nu(\mathrm{Fix}(g : B)) > 0$ where $\mathrm{Fix}(g : B) = \{b \in B : \ gb = b\}$. Observe that if $\lambda \in \mathrm{Fact}(\mathbf{b}, \mathbb{X}^G)$, then $\nu(\mathrm{Fix}(g : \mathbb{X}^G)) \geq \nu(\mathrm{Fix}(g : B)) > 0$. So it suffices to show that the map $\lambda \mapsto \lambda(\mathrm{Fix}(g : \mathbb{X}^G))$ is an upper-semicontinuous function of $\lambda \in \mathcal{P}_\mu(\mathbb{X}^G)$. This follows from the portmanteau theorem because $\mathrm{Fix}(g : \mathbb{X}^G)$ is a closed subset of $\mathbb{X}^G$ and because the action $G \curvearrowright \mathbb{X}^G$ is continuous. □

The remainder of this section is devoted to the proof of Theorem 4.1. We begin with some preliminaries.

4.1. **Outline of the proof of Theorem 4.1.** The proof of Theorem 4.1 uses a technique analogous to painting names on Rohlin towers. First we use the amenability and freeness of the action $G \curvearrowright (B, \nu)$ to show that there exist partitions $\{\mathcal{Q}_{n,b}\}_{n \in \mathbb{N}, b \in B}$ of G satisfying:

- for each n, the assignment $b \mapsto \mathcal{Q}_{n,b}$ is measurable and G-equivariant,
- each partition element of $\mathcal{Q}_{n,b}$ is finite,
- if $\mathcal{Q}_{n,b}(g) \subset G$ denotes the partition element of $\mathcal{Q}_{n,b}$ containing $g \in G$, then for every finite subset $F \subset G$,
$$\lim_{n \to \infty} \nu(\{b \in B : \ F \subset \mathcal{Q}_{n,b}(1_G)\}) = 1,$$
where 1_G is the identity element of G.

This sequence carries information analogous to a Rohlin tower. For technical reasons, it is also useful to show the existence of subsets $R_{n,b} \subset G$ such that

- the assignment $b \mapsto R_{n,b}$ is measurable and G-equivariant,
- $R_{n,b}$ contains exactly one element from each partition element of $\mathcal{Q}_{n,b}$.

Elements of $R_{n,b}$ are *roots* of the partition elements of $\mathcal{Q}_{n,b}$.

The partitions $\mathcal{Q}_{n,b}$ and subsets $R_{n,b}$ induce a natural partition of a special subset of B. Namely, we define

- $B_n = \{b \in B : \ 1_G \in R_{n,b}\}$,
- $\mathcal{P}_n$ to be the partition of B_n defined by: $b, b' \in B_n$ are in the same partition element of $\mathcal{P}_n$ if and only if $\mathcal{Q}_{n,b}(1_G) = \mathcal{Q}_{n,b'}(1_G)$,
- $\psi_n : \mathcal{P}_n \to 2^G$ by $\psi_n(P) = \mathcal{Q}_{n,b}(1_G)$ for any $b \in P$.

Now let $\theta \in \mathcal{P}_\mu(W^G)$. It suffices to construct G-equivariant measurable maps $\pi_n : B \to W^G$ such that
$$\lim_{n \to \infty} \pi_{n*}\nu = \theta.$$
To achieve this, we first decompose θ and ν using the Poisson boundary. To be precise, let $\alpha : B \to \Pi(G, \mu)$ be a factor map to the Poisson boundary and define a map $B \to \mathcal{P}(W^G)$, $b \mapsto \theta_b$ by
$$\theta_b = \lim_{n \to \infty} g_n \theta \tag{6}$$

where $\{g_n\}$ is any sequence in G with $\mathrm{bnd}(\{g_n\}) = \alpha(b)$. Here bnd is the map that assigns to almost every sequence in $(G^{\mathbb{N}}, \mathcal{P}_\mu)$ the corresponding point in the Poisson boundary. This is well defined since $\lim_{n\to\infty} g_n\theta$ is measurable in the shift-invariant sigma-algebra of $(G^{\mathbb{N}}, \mathcal{P}_\mu)$, and therefore depends only on $\mathrm{bnd}(\{g_n\})$.

Because the map π_n must be G-equivariant, it suffices to define it on the subset B_n (since this subset intersects every G-orbit nontrivially). To define $\pi_n(b)$ (for $b \in B_n$), the rough idea is to take an element $x \in W^G$ which is "typical" with respect to θ_b and choose $\pi_n(b)(g) = x(g)$ for $g \in \psi_n(b)$. We use equivariance to define $\pi_n(b)$ on the rest of G. The element x should be a measurable function of the element b. This means we must choose a map $\beta_{n,P} : P \to W^G$ (for each $P \in \mathcal{P}_n$) such that

$$\beta_{n,P*}(\nu \restriction P) \sim \int_P \theta_b \, d\nu(b)$$

where $\sim$ means close in total variation norm. Actually, this is not good enough because we need a good approximation on translates of B_n. So what we really require is that

$$g_*^{-1}\beta_{n,P*}(g_*\nu \restriction P) \sim \int_{g^{-1}P} \theta_b \, d\nu(b) \tag{7}$$

for all $g \in \psi_n(P)$. Then we define $\pi_n(b)(g) = \beta_{n,P}(g)$ for $b \in P$ and $g \in \psi_n(b)$. It remains only to verify that π_n has the required properties.

4.2. **A random rooted partition from amenability.** Let $\mathrm{Part}(G)$ be the set of all (unordered) partitions of G. We may identify a partition $\mathcal{P} \in \mathrm{Part}(G)$ with the equivalence relation that it determines. Any equivalence relation on G is a subset of $G \times G$ and therefore may be identified with an element of $2^{G\times G}$ which is a compact metric space in the product topology. So we may view $\mathrm{Part}(G)$ as a subset of $2^{G\times G}$ and give it the subspace topology. In this topology, it is a compact metrizable space. Also, G acts continuously on $\mathrm{Part}(G)$ by $g\mathcal{P} = \{gP : \ P \in \mathcal{P}\}$.

Lemma 4.5. *Let $G \curvearrowright (B, \nu)$ be an essentially free, measure-preserving extension of the Poisson boundary. Then there exist partitions $\mathcal{Q}_{n,b} \in \mathrm{Part}(G)$ and subsets $R_{n,b} \subset G$ that have the properties detailed in* §4.1.

In fact, as the proof below shows, this lemma holds for any essentially free amenable G-action.

Proof. The action of G on its Poisson boundary is amenable in Zimmer's sense [51]. Because extensions of amenable actions are amenable, $G \curvearrowright (B, \nu)$ is amenable.

Let $E_G \subset B \times B$ denote the equivalence relation

$$E_G = \{(b, gb) : \ b \in B, g \in G\}.$$

Because the action of G on (B, ν) is amenable, E_G is hyperfinite. This means that there exists a sequence $\{\mathcal{R}_n\}_{n=1}^\infty$ of Borel equivalence relations $\mathcal{R}_n \subset B \times B$ such that

- for a.e. $b \in B$ and every n, the $\mathcal{R}_n$-class of b, denoted $[b]_{\mathcal{R}_n}$, is finite,
- $\mathcal{R}_n \subset \mathcal{R}_{n+1}$ for all n,
- $E_G = \bigcup_n \mathcal{R}_n$.

Define $\mathcal{Q}_{n,b}$ by: g_1, g_2 are in the same part of $\mathcal{Q}_{n,b}$ if and only if $g_1^{-1}b\mathcal{R}_n g_2^{-1}b$. Because $\mathcal{R}_n$ is Borel, the assignment $b \mapsto \mathcal{Q}_{n,b}$ is also Borel, hence measurable. For any $h \in G$, g_1, g_2 are in the same part of $\mathcal{Q}_{n,hb}$ if and only if $g_1^{-1}hb\mathcal{R}_n g_2^{-1}hb$ which

occurs if and only if $h^{-1}g_1, h^{-1}g_2$ are in the same part of $\mathcal{Q}_{n,b}$. So $\mathcal{Q}_{n,hb} = h\mathcal{Q}_{n,b}$ as required.

Because each $\mathcal{R}_n$-class is finite, each part of $\mathcal{Q}_{n,b}$ is finite. Let $F \subset G$ be finite. Because $E_G = \bigcup_n \mathcal{R}_n$ is an increasing union the probability that, for a randomly chosen $b \in B$, $\{fb\}_{f\in F}$ is contained in an $\mathcal{R}_n$-equivalence class tends to 1 as $n \to \infty$. Equivalently,

$$\lim_{n\to\infty} \nu(\{b \in B : \ F \subset \mathcal{Q}_{n,b}(1_G)\}) = 1,$$

and so the $\mathcal{Q}_{n,b}$ have the required properties.

As another consequence of the fact that each $\mathcal{R}_n$-class is finite, there exist measurable subsets $B_n \subset B$ such that for a.e. $b \in B$, $B_n \cap [b]_{\mathcal{R}_n}$ contains exactly one element. So define $R_{n,b} := \{g \in G : \ g^{-1}b \in B_n\}$. To check that $R_{n,b}$ contains exactly one element from each partition element of $\mathcal{Q}_{n,b}$, observe that if P is any part of $\mathcal{Q}_{n,b}$ and $g \in P$, then there exists a unique $b' \in B_n$ such that $g^{-1}b\mathcal{R}_n b'$. Because the action of G on B is essentially free, there is a unique $g_0 \in G$ such that $b' = g_0^{-1}b$. So g_0 is the unique element of $R_{n,b} \cap P$. □

4.3. **Painting names.** Following the outline in §4.1, we let $\theta \in \mathcal{P}_\mu(W^G)$ and define θ_b (for $b \in B$) by equation (6). In this subsection, we choose $\beta_{n,b}$ to satisfy (7). First we need to recall some basic facts about total variation distance.

Let λ_1, λ_2 be Borel measures on a space Z. Their *total variation distance* is defined by

$$\|\lambda_1 - \lambda_2\| = \sup_A |\lambda_1(A) - \lambda_2(A)|$$

where the supremum is over all Borel subsets $A \subset Z$. We will need two elementary facts:

Claim 4.6.

(1) *Suppose there exists a measure λ_3 such that λ_1, λ_2 are both absolutely continuous to λ_3. Then*

$$\|\lambda_1 - \lambda_2\| \le \int \left| \frac{d\lambda_1}{d\lambda_3}(z) - \frac{d\lambda_2}{d\lambda_3}(z) \right| \ d\lambda_3(z).$$

(2) *If $\Phi : Z \to Z'$ is any Borel map, then $\|\Phi_*\lambda_1 - \Phi_*\lambda_2\| \le \|\lambda_1 - \lambda_2\|$.*

Define $B_n, \mathcal{P}_n, \psi_n$ as in §4.1.

Lemma 4.7. *For every $P \in \mathcal{P}_n$ and $\delta_P > 0$ there exists a measurable map $\beta_{n,P} : P \to W^G$ such that*

$$\left\| g_*^{-1}\beta_{n,P*}(g_*\nu \restriction P) - \int_{g^{-1}P} \theta_b \ d\nu(b) \right\| < \delta_P$$

for every $g \in \psi_n(P)$.

Proof. Let $\mathcal{K}$ be a countable partition of P such that for every $g \in \psi_n(P)$

$$\sum_{K\in\mathcal{K}} \int_K \left| \frac{dg_*\nu}{d\nu}(b) - C(g,K) \right| \ d\nu(b) < \delta_P/2$$

for some constants $\{C(g,K) : \ g \in \psi_n(P), K \in \mathcal{K}\}$. Such a partition exists because step functions are dense in $L^1(B,\nu)$. By fact (1) of Claim 4.6,

$$\sum_{K\in\mathcal{K}} \|(g_*\nu \restriction K) - C(g,K)(\nu \restriction K)\| < \delta_P/2.$$

Choose a measurable map $\beta_{n,P} : P \to W^G$ so that

$$\beta_{n,P*}(\nu \restriction K) = \int_K \theta_b \, d\nu(b)$$

for every $K \in \mathcal{K}$. Here we are using the fact that (B, ν) has no atoms, which holds because the action $G \curvearrowright (B, \nu)$ is stationary and essentially free, and because G is countably infinite.[1]

Then for any $g \in \psi_n(P)$,

$$\begin{aligned}
& \left\| \beta_{n,P*}(g_*\nu \restriction P) - \int_P \theta_b \, dg_*\nu(b) \right\| \\
\leq \quad & \sum_{K \in \mathcal{K}} \left\| \beta_{n,P*}(g_*\nu \restriction K) - \int_K \theta_b \, dg_*\nu(b) \right\| \\
\leq \quad & \sum_{K \in \mathcal{K}} \|\beta_{n,P*}(g_*\nu \restriction K) - C(g,K)\beta_{n,P*}(\nu \restriction K)\| \\
& \quad + \left\| C(g,K)\beta_{n,P*}(\nu \restriction K) - C(g,K) \int_K \theta_b \, d\nu(b) \right\| \\
& \quad + \left\| C(g,K) \int_K \theta_b \, d\nu(b) - \int_K \theta_b \, dg_*\nu(b) \right\| \\
< \quad & \delta_P/2 + 0 + \delta_P/2 = \delta_P.
\end{aligned}$$

The lemma now follows from fact (2) of Claim 4.6 and

$$g_*^{-1} \left(\int_P \theta_b \, dg_*\nu(b) \right) = \int_{g^{-1}P} \theta_b \, d\nu(b).$$

□

4.4. **End of the proof.** Define $\pi_n(b)(g) = \beta_{n,P}(g)$ for $b \in P \in \mathcal{P}_n$ and $g \in \psi_n(b)$.

Lemma 4.8. *There exists a unique G-equivariant extension of π_n from B to W^G.*

Proof. Suppose $h \in G$, $b \in B_n$. We will define $\pi_n(b)(h)$ as follows. Let $Q \in \mathcal{Q}_{n,b}$ be the part containing h. There exists a unique $g \in R(n,b) \cap Q$. Because $R_{n,g^{-1}b} = g^{-1}R_{n,b} \ni 1_G$, it follows that $g^{-1}b \in B_n$. By definition, $\psi_n(g^{-1}b)$ is the part of $\mathcal{Q}_{n,g^{-1}b} = g^{-1}\mathcal{Q}_{n,b}$ containing 1_G. So $\psi_n(g^{-1}b) = g^{-1}Q$ contains $g^{-1}h$. So $\pi_n(g^{-1}b)(g^{-1}h)$ is well defined. We now define

$$\pi_n(b)(h) = \pi_n(g^{-1}b)(g^{-1}h)$$

and observe that we have now defined π_n on B_n so that $\pi_n(gb) = g\pi_n(b)$ whenever $g \in G$ and $b, gb \in B_n$.

Next we define π_n for arbitrary $b \in B$ as follows. Let Q be the part of $\mathcal{Q}_{n,b}$ with $1_G \in Q$. Let $g \in G$ be the unique element in $R_{n,b} \cap Q$. Because $R_{n,g^{-1}b} = g^{-1}R_{n,b} \ni 1_G$, it follows that $g^{-1}b \in B_n$. So $\pi_n(g^{-1}b)$ is well defined. We now define $\pi_n(b) = g\pi_n(g^{-1}n)$. This is the unique G-equivariant extension of π_n. □

Lemma 4.9. $\{g^{-1}P : P \in \mathcal{P}_n, g \in \psi_n(P)\}$ *is a partition of B (up to measure zero).*

[1] A finite, essentially free stationary measure cannot be atomic, since in any finite stationary measure the finite set of atoms of maximal measure must be invariant, and thus have nontrivial stabilizers.

The proof of this lemma follows directly from the definitions, and hence we omit it.

Lemma 4.10. *If $P \in \mathcal{P}_n$, $F = \psi_n(P) \subset G$ and $g \in F$, then*

$$\left\|\mathrm{Proj}_{W^{g^{-1}F}}(\pi_{n*}(\nu \restriction g^{-1}P)) - \mathrm{Proj}_{W^{g^{-1}F}}(\theta_{g^{-1}P})\right\| < \delta_P$$

where $\theta_P = \int_P \theta_b \, d\nu(b)$.

Proof. Let $b \in P$ be random with law $\nu \restriction g^{-1}P$ (normalized to have mass 1). Because π_n is G-equivariant,

$$\pi_n(b)(g^{-1}f) = g\pi_n(b)(f) = \pi_n(gb)(f) \quad \forall f \in F = \psi_n(P).$$

Because $gb \in P$,

$$\pi_n(gb)(f) = \beta_{P,n}(gb)(f) = g^{-1}\beta_{P,n}(gb)(g^{-1}f).$$

Thus $\pi_n(b)(h) = g^{-1}\beta_{P,n}(gb)(h)$ for all $h \in g^{-1}F$. The claim now follows from Lemma 4.7. □

Let $\epsilon > 0$ and $F \subset G$ be finite such that $1_G \in F$. It suffices to show that

$$\limsup_{n\to\infty} \| \mathrm{Proj}_{W^F}(\pi_{n*}\nu) - \mathrm{Proj}_{W^F}\, \theta\| \leq 3\epsilon.$$

By Lemma 4.5 there exists an N such that $n > N$ implies

$$\nu(\{b \in B : \ F \subset \mathcal{Q}_{n,b}(1_G)\}) > 1 - \epsilon.$$

For $P \in \mathcal{P}_n$, let $\psi'_n(P) = \{g \in \psi_n(P) : \ g^{-1}\psi_n(P) \supset F\}$. Let

$$\mathrm{GOOD}(n) = \bigcup\{g^{-1}P : \ P \in \mathcal{P}_n, g \in \psi'_n(P)\}.$$

Lemma 4.11. $\nu(\mathrm{GOOD}(n)) > 1 - \epsilon$.

Proof. By Lemma 4.5 there exists N such that $n > N$ implies

$$\nu(\{b \in B : \ F \subset \mathcal{Q}_{n,b}(1_G)\}) > 1 - \epsilon.$$

Suppose $b \in B$ is such that $F \subset \mathcal{Q}_{n,b}(1_G)$. By Lemma 4.9 there exist unique $P \in \mathcal{P}_n$ and $g \in \psi_n(P)$ such that $b \in g^{-1}P$. It suffices to show that $g \in \psi'_n(P)$, i.e., $F \subset g^{-1}\psi_n(P)$.

Because $gb \in P \subset B_n$, $1_G \in R_{n,gb} = gR_{n,b}$. So $g^{-1} \in R_{n,b}$. Let Q be the part of $\mathcal{Q}_{n,b}$ containing 1_G. So $F \subset Q$. Now $gQ \ni g \in \psi_n(P)$. So gQ is the part of $g\mathcal{Q}_{n,b} = \mathcal{Q}_{n,gb}$ containing g. Because $gb \in P$, $\psi_n(P)$ is a part of $\mathcal{Q}_{n,gb}$. By hypothesis $g \in \psi_n(P)$. So $gQ = \psi_n(P)$. Since $F \subset Q$, this implies $F \subset g^{-1}\psi_n(P)$ as claimed. □

We now have

$$\begin{aligned}
&\| \mathrm{Proj}_{W^F}(\pi_{n*}\nu) - \mathrm{Proj}_{W^F}\, \theta\| \\
&\quad\leq \sum_{P\in\mathcal{P}_n} \sum_{g\in\psi_n(P)} \| \mathrm{Proj}_{W^F}(\pi_{n*}(\nu \restriction g^{-1}P)) - \mathrm{Proj}_{W^F}(\theta_{g^{-1}P})\| \\
&\quad\leq 2\epsilon + \sum_{P\in\mathcal{P}_n} \sum_{g\in\psi'_n(P)} \| \mathrm{Proj}_{W^F}(\pi_{n*}(\nu \restriction g^{-1}P)) - \mathrm{Proj}_{W^F}(\theta_{g^{-1}P})\| \\
&\quad\leq 2\epsilon + \sum_{P\in\mathcal{P}_n} \sum_{g\in\psi'_n(P)} \delta_P.
\end{aligned}$$

The first inequality is implied by Lemma 4.9, the second by Lemma 4.11 and the last by Lemma 4.10. We may choose each δ_P so that $\sum_{P\in\mathcal{P}_n}\sum_{g\in\psi'_n(P)}\delta_P < \epsilon$. Since ϵ, θ, F are arbitrary, this implies Theorem 4.1.

5. Freeness of the Poisson boundary action

In this section we prove that every torsion-free nonelementary hyperbolic group acts essentially freely on its Poisson boundary. For standard references on hyperbolic groups see [4, 14, 23, 29]. Our main result is a consequence of the following more general result:

Theorem 5.1. *Let G be a countable group with a generating probability measure μ. Let $G\curvearrowright\Pi(G,\mu)$ denote the action of G on its Poisson boundary. Suppose G has only countably many amenable subgroups. Then there exists a normal amenable subgroup $N \lhd G$ such that the stabilizer of almost every $x \in \Pi(G,\mu)$ is equal to N.*

Proof. Let $(B,\nu) = \Pi(G,\mu)$ be the Poisson boundary. Let Sub_G be the space of subgroups of G, equipped with the Fell or Chabauty topology; in our case of discrete groups this is the same topology as the subspace topology inherited from the product topology on the space of subsets of G (see, e.g., [1]). Let $\mathrm{stab}\colon B \to \mathrm{Sub}_G$ be the stabilizer map given by $\mathrm{stab}(b) = \{g \in G : gb = b\}$. This map is G-equivariant, and so $\mathrm{stab}_*\,\nu$ is a μ-stationary distribution on Sub_G.

Since the action $G\curvearrowright(B,\nu)$ is amenable, $\mathrm{stab}_*\,\nu$ is supported on amenable groups ([28, Theorem 2] or, e.g., [2, Theorem A (v)]). However, G contains only countably many amenable subgroups so $\mathrm{stab}_*\,\nu$ has countable support. However, every countably supported stationary measure is invariant; this is because the set of points that have maximal probability is invariant.

Finally, every invariant factor of the Poisson boundary is trivial (see, e.g., [5, Corollary 2.20]), and so $\mathrm{stab}_*\,\nu$ is supported on a single subgroup, which has to be amenable and normal. □

Proof of Proposition 1.3. By the previous theorem, it suffices to prove:

(1) G has only countably many amenable subgroups,
(2) G does not contain a nontrivial normal amenable subgroup.

It is well known that hyperbolic groups satisfy a strong form of the Tits Alternative: every subgroup is either virtually cyclic or contains a nonabelian free subgroup [29]. In particular, every amenable subgroup is virtually cyclic and therefore must be finitely generated. Since there are only countably many finitely generated subgroups of G, this proves (1).

It is also well known that any normal amenable subgroup of a nonelementary word hyperbolic group must be finite. To see this, recall the Gromov compactification $\bar{G}$ of G. Let $\partial G = \bar{G} - G$ and for any subgroup $H \le G$, let $\partial H = \bar{H} - H$ where $\bar{H}$ is the closure of H in $\bar{G}$. It is well known that if H is virtually infinite cyclic, then ∂H consists of two points and H fixes ∂H (for example see [36, Theorem 12.2 (1)]). If H is normal, then it follows that $\partial G = \partial H$ [36, Theorem 12.2 (5)]. However, this implies that G is virtually cyclic, which contradicts our assumption that G is nonelementary.

Finally, since we are assuming that G is torsion-free, every finite subgroup is trivial. □

On the other hand, it is easy to construct examples of (G, μ) such that the action $G \curvearrowright \Pi(G, \mu)$ is not essentially free. For example, this occurs whenever G is nontrivial and abelian since in this case $\Pi(G, \mu)$ is trivial. For a less trivial example, suppose G_1, G_2 are two countable discrete groups. Let μ_i be a generating probability measure on G_i. Then $\mu_1 \times \mu_2$ is a generating measure on $G_1 \times G_2$. By [5, Corollary 3.2],

$$G_1 \times G_2 \curvearrowright \Pi(G_1 \times G_2, \mu_1 \times \mu_2) \cong G_1 \times G_2 \curvearrowright \Pi(G_1, \mu_1) \times \Pi(G_2, \mu_2).$$

Therefore if $G_1 \curvearrowright \Pi(G_1, \mu_1)$ is not essentially free, then $G_1 \times G_2 \curvearrowright \Pi(G_1 \times G_2, \mu_1 \times \mu_2)$ is also not essentially free. For example, if G_1 is finite or equal to $\mathbb{Z}^d$ for some $d \geq 1$, then this is always the case.

6. Nonsingular and stationary transformations and actions

6.1. **The group of nonsingular transformations.** Let (X, ν) denote a standard nonatomic Lebesgue probability space. A measurable transformation $T\colon X \to X$ is *nonsingular* if $T_*\nu$ is equivalent to ν; equivalently, ν is T-quasi-invariant. Let $\mathrm{Aut}^*(X, \nu)$ be the set of all nonsingular invertible transformations of (X, ν) in which we identify any two transformations that agree almost everywhere. More precisely: $\mathrm{Aut}^*(X, \nu)$ consists of equivalence classes of nonsingular transformations in which two such transformations are considered equivalent if they agree almost everywhere. To each $T \in \mathrm{Aut}^*(X, \nu)$ and $1 \leq p < \infty$ we assign the isometry $U_{T,p} \in \mathrm{Isom}(L^p(X, \nu))$ given by

$$[U_{T,p} f](x) = \left(\frac{d(T_*\nu)}{d\nu}(x) \right)^{1/p} f(T^{-1}x).$$

The map $T \mapsto U_{T,p}$ is an algebraic isomorphism of $\mathrm{Aut}^*(X, \nu)$ with the subgroup of $\mathrm{Isom}(L^p(X, \nu))$ that preserves the cone of positive functions (see, e.g., [40, Theorem 3.1]).

A topology τ on $L^p(X, \nu)$ can be lifted to a topology on $\mathrm{Isom}(L^p(X, \nu))$: a sequence $U_1, U_2, \ldots \in \mathrm{Isom}(L^p(X, \nu))$ converges to U if $U_n f$ converges in τ to Uf for every $f \in L^p(X, \nu)$.

In particular, the strong operator topology (SOT) and the weak operator topology (WOT) on $\mathrm{Isom}(L^p(X, \nu))$ are derived from the weak and norm (respectively) topologies on $L^p(X, \nu)$ (for $1 \leq p < \infty$). In the norm topology, $\lim_n f_n = f$ if $\lim_n \|f_n - f\|_p = 0$. In the weak topology, $\lim_n f_n = f$ if $\lim_n \langle f_n, g \rangle = \langle f, g \rangle$ for all g in $L^q(X, \nu)$, where $1/p + 1/q = 1$.

In [12] it is shown that the topologies induced on $\mathrm{Aut}^*(X, \nu)$ from the SOT on $\mathrm{Isom}(L^p(X, \nu))$ coincide, for all $1 \leq p < \infty$. In [43, Theorem 2.8] it is shown that for each p with $1 < p < \infty$, the topologies induced on $\mathrm{Aut}^*(X, \nu)$ from the SOT and the WOT on $\mathrm{Isom}(L^p(X, \nu))$ coincide, since then $L^p(X, \nu)$ is reflexive. The topology that these induce on $\mathrm{Aut}^*(X, \nu)$ is called the *weak topology*. In [12] it is shown that the weak topology on $\mathrm{Aut}^*(X, \nu)$ is a Polish group topology, which makes it a natural choice.

We are also concerned with the topology on $\mathrm{Aut}^*(X, \nu)$ induced from the WOT on $\mathrm{Isom}(L^1(X, \nu))$. We call this the *very weak topology*. This is not a group topology.

Lemma 6.1. *The topology on* $\mathrm{Aut}^*(X,\nu)$ *induced from the WOT on* $\mathrm{Isom}(L^1(X,\nu))$ *is not a group topology. More precisely, the multiplication map is not jointly continuous.*

Proof. Without loss of generality $X=[0,1]$ and ν is the Lebesgue measure. Below we consider $\mathrm{Aut}^*(X,\nu)$ with the WOT induced from its embedding into $\mathrm{Isom}(L^1(X,\nu))$.

For $n\in\mathbb{N}$, let $T_n\in\mathrm{Aut}^*(X,\nu)$ be the piecewise linear map such that

- T_n maps the interval $[k/n, k/n+1/n-1/n^2]$ linearly to the interval $[k/n, k/n+1/n^2]$ for each integer k with $0\le k<n$,
- T_n maps $[k/n+1/n-1/n^2, k/n+1/n]$ linearly to the interval $[k/2+1/n^2, k/n+1/n]$ for each k with $0\le k<n$.

Because T_n preserves the intervals $[k/n, k/n+1/n]$ for each k, T_n converges to the identity. Let S_n be the following measure-preserving map:

- S_n maps the interval $[k/n, k/n+1/n^2]$ to the interval $[k/n+1/2, k/n+1/n^2+1/2]$ where everything is considered mod 1. In other words, S_n behaves like a rotation when restricted to these intervals.
- S_n fixes all other points. That is, $S_n(x)=x$ if x is not contained in any interval of the form $[k/n, k/n+1/n^2]$.

The fixed point set of S_n has measure $1-1/n$. So S_n converges to the identity. However, the composition S_nT_n does not converge to the identity. Instead it converges to the rotation $x\mapsto x+1/2 \mod 1$. This is because T_n pushes most of the interval into the little subintervals $[k/n, k/n+1/n^2]$ which are then rotated by S_n. □

We observe that both the weak topology and the very weak topology coincide on $\mathrm{Aut}(X,\nu)$, the subgroup of measure-preserving transformations. We choose to study the weak topology because it is the natural Polish group topology on $\mathrm{Aut}^*(X,\nu)$. We also study the very weak topology since it is strongly related (by Theorem 1.7) to the weak* topology on $\mathcal{P}_\mu(Z^G)$; the weak topology is too fine for this purpose, and indeed Theorem 1.7 is not true for the weak topology.

6.1.1. *The weak topology.* A subbase of open sets for the weak topology on $\mathrm{Aut}^*(X,\nu)$ is

$$W_{A,\epsilon}(T)=\left\{S\in\mathrm{Aut}^*(X,\nu)\,:\,\|U_{T,1}\mathbb{1}_A-U_{S,1}\mathbb{1}_A\|_1<\epsilon\right\}$$

where $A\subseteq X$ is measurable, $\epsilon>0$ and $T\in\mathrm{Aut}^*(X,\nu)$.

If we let $\mathcal{S}$ be a countable base for the sigma-algebra of (X,ν), then $T_n\to T$, in this topology, if

$$\lim_n\int_X\left|\frac{dT_{n*}\nu}{d\nu}(x)\mathbb{1}_S(T_n^{-1}x)-\frac{dT_*\nu}{d\nu}(x)\mathbb{1}_S(T^{-1}x)\right|d\nu(x)=0,$$

for every $S\in\mathcal{S}$. Equivalently, $T_n\to T$ if $dT_{n*}\nu/d\nu$ converges to $dT_*\nu/d\nu$ in $L^1(X,\nu)$ and if furthermore

$$\lim_n\nu(S_1\cap T_nS_2)=\nu(S_1\cap TS_2), \tag{8}$$

for every $S_1,S_2\in\mathcal{S}$.

6.1.2. *The very weak topology.* A subbase of open sets for the very weak topology on $\mathrm{Aut}^*(X,\nu)$ is

$$W_{A,B,\epsilon}(T) = \left\{S \in \mathrm{Aut}^*(X,\nu) \,:\, |\langle U_{S,1}\mathbb{1}_A, \mathbb{1}_B\rangle - \langle U_{T,1}\mathbb{1}_A, \mathbb{1}_B\rangle| < \epsilon\right\}$$

where $A, B \subseteq X$ are measurable, $\epsilon > 0$, $T \in \mathrm{Aut}^*(X,\nu)$ and $\langle f, g\rangle = \int_X f(x) \cdot g(x)d\nu(x)$.

In the very weak topology it is enough that (8) holds for every $S_1, S_2 \in \mathcal{S}$ to guarantee that $T_n \to T$; the L^1-convergence of the Radon-Nikodym derivatives is not needed.

We note that:

Lemma 6.2. $\mathrm{Aut}^*(X,\nu)$ *is a Polish space with respect to the very weak topology.*

Proof. It suffices to observe that $\mathrm{Aut}^*(X,\nu)$ is closed in $\mathrm{Isom}(L^1(X,\nu))$ and that $\mathrm{Isom}(L^1(X,\nu))$ is a Polish space with respect to the weak operator topology. □

6.2. **Spaces of actions.** Given a countable group G, let

$$A^*(G,X,\nu) = \mathrm{Hom}(G, \mathrm{Aut}^*(X,\nu))$$

denote the set of all homomorphisms $a\colon G \to \mathrm{Aut}^*(X,\nu)$. We consider $A^*(G,X,\nu)$ as a subset of the product space $\prod_G \mathrm{Aut}^*(X,\nu)$. Given a topology on $\mathrm{Aut}^*(X,\nu)$, we endow $\prod_G \mathrm{Aut}^*(X,\nu)$ with the product topology and $A^*(G,X,\nu)$ with the subspace topology.

When $A^*(G,X,\nu)$ has the (very) weak topology, then the resulting topology on $A^*(G,X,\nu)$ will also be called the (very) weak topology.

6.2.1. *The space of stationary actions.* When $A^*(G,X,\nu)$ has the very weak topology, then $a_n \to a$ if

$$\lim_n \nu(S_1 \cap a_n(g)S_2) = \nu(S_1 \cap a(g)S_2), \tag{9}$$

for every $g \in G$ and measurable $S_1, S_2 \in \mathcal{S}$. In particular this implies that

$$\lim_n a_n(g)_*\nu = a(g)_*\nu \tag{10}$$

in the weak* topology on $\mathcal{P}(X)$ (regardless of what compatible topology X is endowed with).

Given a generating measure μ on G, let $A_\mu(G,X,\nu)$ be the set of all $a \in A^*(G,X,\nu)$ such that a is *μ-stationary*:

$$\nu = \sum_{g\in G} \mu(g)a(g)_*\nu.$$

The equation above, together with (10), implies that $A_\mu(G,X,\nu)$ is closed in $\prod_G \mathrm{Aut}^*(X,\nu)$ when the latter is equipped with the product weak topology. Because the product very weak topology is weaker than the product weak topology, $A_\mu(G,X,\nu)$ is also closed as a subspace of $A^*(G,X,\nu)$ with respect to the weak topology. Hence $A_\mu(G,X,\nu)$ is a Polish space when equipped with either the weak or the very weak topology.

A subset of $A_\mu(G,X,\nu)$ is $A(G,X,\nu)$, the set of measure-preserving actions, for which $a(g)_*\nu = \nu$ for all $g \in G$. Note that both the weak and the very weak topology on $A_\mu(G,X,\nu)$ coincide on $A(G,X,\nu)$, and in particular coincide with the topology studied by Kechris [37], Kerr and Pichot [39], and others.

7. The weak topology on $A_\mu(G, X, \nu)$

In this section we consider $A_\mu(G, X, \nu)$ with the weak topology. We will prove Theorems 1.4 and 1.5. To this end, the following proposition will be helpful.

Proposition 7.1 (Kaimanovich and Vershik [35]). *Fix a generating probability measure μ on G. Then for each $g \in G$ there exist constants $M_g, N_g > 0$ such that for every $a \in A_\mu(G, X, \nu)$ and ν-almost-every $x \in X$*

$$-N_g \leq \log \frac{da(g)_*\nu}{d\nu}(x) \leq M_g,$$

and furthermore

$$\sum_g \mu(g) M_g < H(\mu),$$

where $H(\mu)$ is the Shannon entropy of μ.

7.1. **Property (T) groups.** In this section we prove Theorem 1.4. We will need a few definitions and lemmas.

Let $G \curvearrowright (X, \nu)$ be a nonsingular ergodic action on a standard probability space, and, as above, let G act on $L^2(X, \nu)$ by

$$(gf)(x) = f(g^{-1}x)\sqrt{\frac{dg_*\nu}{d\nu}(x)}$$

for $g \in G, f \in L^2(X, \nu), x \in X$.

Lemma 7.2. *If ν is not equivalent to a G-invariant finite measure, then there does not exist a nonzero G-invariant vector in $L^2(X, \nu)$.*

Proof. To obtain a contradiction, suppose $f \in L^2(X, \nu)$ is a nonzero G-invariant vector. So

$$f(x) = f(g^{-1}x)\sqrt{\frac{dg_*\nu}{d\nu}(x)}$$

for a.e. $x \in X$ and every $g \in G$. Because f is nontrivial, the support $S = \{x \in X : f(x) \neq 0\}$ is a G-invariant subset which is not a null set. Because the action is ergodic, $\nu(S) = 1$.

Define the measure η on X by $d\eta(x) = f(x)^2 d\nu(x)$. Then

$$d\eta(g^{-1}x) = f(g^{-1}x)^2\ d\nu(g^{-1}x) = f(g^{-1}x)^2 \frac{dg_*\nu}{d\nu}(x)\ d\nu(x) = f(x)^2\ d\nu(x) = d\eta(x).$$

Thus η is a G-invariant measure equivalent to ν, and moreover has finite mass $\|f\|_2^2$, a contradiction. □

Corollary 7.3. *Let $G \curvearrowright (X, \nu)$ be an ergodic μ-stationary action on a standard probability space. If ν is not invariant, then there does not exist a nonzero G-invariant vector in $L^2(X, \nu)$.*

Proof. It is proven in [5, Proposition 2.6] that any noninvariant μ-stationary action on a standard probability space cannot be equivalent to an invariant action. Hence the corollary follows from the previous lemma. □

Denote the ergodic actions in $A_\mu(G, X, \nu)$ by $A^e_\mu(G, X, \nu)$. Let A' be the set of all actions $a \in A_\mu(G, X, \nu)$ such that there exists a subset $Y \subset X$ with the following properties:

(1) $0 < \nu(Y) < 1$,
(2) $a(g)(Y) = Y$ for all $g \in G$,
(3) a restricted to Y is ν measure preserving.

Lemma 7.4. *If G has property (T), then the closure of $A^e_\mu(G, X, \nu)$ is disjoint from A'.*

Proof. Choose $a \in A'$, and let Y have the properties listed above. In this case, the vector $\mathbb{1}_Y$ is an invariant vector in $L^2(X, \nu)$ for a. We will prove the lemma by showing that a cannot be a limit of ergodic actions.

Assume the contrary, so that $a_n \to_n a$ in the weak topology, with each $a_n \in A^e_\mu(G, X, \nu)$. Let $G \curvearrowright (X^{\mathbb{N}}, \eta)$ be an ergodic stationary joining (see, e.g., [22]) of $\{a_n \curvearrowright (X, \nu)\}_{n\in\mathbb{N}}$; the action on the n-th coordinate is by a_n. Since each a_n is ergodic, the projection of η on the n-th coordinate is ν. Define $f_n \colon X^{\mathbb{N}} \to \mathbb{R}$ by

$$f_n(x) = \mathbb{1}_Y(x_n)$$

where $x_n \in X$ is the n-th coordinate projection of x. Then,

$$\|gf_n - f_n\| = \|a_n(g)\mathbb{1}_Y - \mathbb{1}_Y\| = \|a_n(g)\mathbb{1}_Y - a(g)\mathbb{1}_Y\|,$$

where the first equality follows from the definition of f_n, and the second follows from the second and third properties of A'. Note also that $\|f_n\| = \nu(Y)^{1/2}$.

It follows from the definition of the weak topology that

$$\lim_n \|gf_n - f_n\| = 0,$$

and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of almost-invariant vectors in $L^2(X^{\mathbb{N}}, \eta)$. Since G has property (T), there must be a nonzero invariant vector in $L^2(X^{\mathbb{N}}, \eta)$. By Corollary 7.3, $G \curvearrowright (X^{\mathbb{N}}, \eta)$ must be measure preserving. This implies each action a_n is measure preserving and therefore the limit action a is also measure preserving. By [24, Theorem 1] (see also [37, Theorem 12.2]), because G has property (T), the set of ergodic measure-preserving actions is closed in the space of all measure-preserving actions. Since each a_n is ergodic this implies a must also be ergodic, a contradiction. □

Proof of Theorem 1.4. We will prove the theorem by showing that A' is dense in $A_\mu(G, X, \nu)$. It will then follow from Lemma 7.4 that $A^e_\mu(G, X, \nu)$ is nowhere dense.

Let a be any action in $A_\mu(G, X, \nu)$. Without loss of generality, we may assume that $X = [0, 1]$ and ν is the Lebesgue measure. Let $T_n \colon \mathbb{R} \to \mathbb{R}$ be the linear map $T_n(x) = (1 - 1/n) \cdot x$. Define $a_n \in A'$ by

$$a_n(g)(x) = \begin{cases} [T_n \circ a(g) \circ T_n^{-1}](x) & \text{for } x \in [0, 1 - 1/n], \\ x & \text{for } x \in (1 - 1/n, 1]. \end{cases}$$

Note that a_n is still stationary and indeed in A'; the set $(1 - 1/n, 1]$ is a nontrivial invariant set. We will show that $a_n \to a$ by showing that for every $g \in G$ and measurable $A \subseteq [0, 1]$ it holds that

$$\lim_n \|a_n(g)\mathbb{1}_A - a(g)\mathbb{1}_A\| = 0.$$

Fix $g \in G$, and denote

$$r(x) = \frac{da(g)_*\nu}{d\nu}(x), \quad r_n(x) = \frac{da_n(g)_*\nu}{d\nu}(x). \tag{11}$$

For any interval $A \subseteq [0,1]$ it holds that

$$\langle a_n(g)\mathbb{1}_A, a(g)\mathbb{1}_A\rangle = \int_{a_n(g)A\cap a(g)A} \sqrt{r_n(x)\cdot r(x)}d\nu(x).$$

Now, on $[0, 1-1/n]$, by the definition of a_n it holds that $r_n(x) = r(T_n^{-1}x)$. On $(1-1/n, 1]$, $r_n(x) = 1$, and by Proposition 7.1 there exists a constant $C = C(\mu, g)$ such that $\sqrt{r(x)} < C$. Hence

$$\begin{aligned}\langle a_n(g)\mathbb{1}_A, a(g)\mathbb{1}_A\rangle &= e_n + \int_{a_n(g)A\cap a(g)A} r(x)d\nu(x)\\ &= e_n + [a(g)_*\nu](a_n(g)A\cap a(g)A),\end{aligned}$$

where the error term e_n satisfies

$$\begin{aligned}|e_n| &\leq C/n + \int_0^{1-1/n}\left|\sqrt{r(T_n^{-1}x)r(x)} - r(x)\right| dx\\ &\leq C/n + C\int_0^{1/1-n}\left|\sqrt{r(T_n^{-1}x)} - \sqrt{r(x)}\right| dx.\end{aligned}$$

We will show that this error term tends to zero as $n \to \infty$. For this, let $\epsilon > 0$ and let f be a continuous function on $[0,1]$ such that $\|f - \sqrt{r}\|_1 < \epsilon$. Then

$$\begin{aligned}&\int_0^{1-1/n}\left|\sqrt{r(T_n^{-1}x)} - \sqrt{r(x)}\right| dx\\ \leq\ &\int_0^{1-1/n}\left|\sqrt{r(T_n^{-1}x)} - f(T_n^{-1}x)\right| + \left|f(T_n^{-1}x) - f(x)\right| + \left|f(x) - \sqrt{r(x)}\right| dx\\ \leq\ &\epsilon + \int_0^{1-1/n}\left|f(T_n^{-1}x) - f(x)\right| dx + \epsilon.\end{aligned}$$

The middle term tends to 0 as $n \to \infty$ by the Bounded Convergence Theorem. Since ϵ is arbitrary, this implies $\lim_{n\to\infty}|e_n| = 0$.

Thus

$$\lim_n \langle a_n(g)\mathbb{1}_A, a(g)\mathbb{1}_A\rangle = \lim_n [a(g)_*\nu](a_n(g)A\cap a(g)A).$$

Since $a_n(g)$ is measurable and ν-nonsingular, $\nu(a_n(g)A \triangle a(g)A)$ tends to zero. To be more precise, we observe that

$$\begin{aligned}\nu(a_n(g)A \triangle a(g)A) &\leq 1/n + \nu\big(a_n(g)(A\cap[0,1-1/n]) \triangle a(g)A\big)\\ &= 1/n + \nu\big(T_n a(g)T_n^{-1}(A\cap[0,1-1/n]) \triangle a(g)A\big)\\ &\leq 1/n + \nu\big(a(g)T_n^{-1}(A\cap[0,1-1/n]) \triangle T_n^{-1}a(g)A\big).\end{aligned}$$

So it suffices to show that if $B \subset [0, 1 - 1/n]$ is any measurable set, then $\lim_{n\to\infty} \nu(B \triangle T_n^{-1}B) = 0$. This follows by approximating the characteristic function 1_B by a continuous function. Thus

$$\begin{aligned}
\lim_n \langle a_n(g)\mathbb{1}_A, a(g)\mathbb{1}_A\rangle &= \lim_n [a(g)_*\nu](a_n(g)A \cap a(g)A) \\
&= \lim_n [a(g)_*\nu](a_n(g)A \cup a(g)A) \\
&\quad -[a(g)_*\nu](a_n(g)A \triangle a(g)A) \\
&= \lim_n [a(g)_*\nu](a_n(g)A \cup a(g)A) \\
&\geq \lim_n [a(g)_*\nu](a(g)A) = \nu(A).
\end{aligned}$$

Since $[a(g)_*\nu](a_n(g)A \cap a(g)A) \leq [a(g)_*\nu](a(g)A) = \nu(A)$, we must have

$$\lim_n \langle a_n(g)\mathbb{1}_A, a(g)\mathbb{1}_A\rangle = \nu(A),$$

and it follows immediately that

$$\lim_n \|a_n(g)\mathbb{1}_A - a(g)\mathbb{1}_A\| = 0.$$

□

7.2. **Entropy gaps and nonproperty (T) groups.** Recall that $A_\mu^e(G, X, \nu)$ is the set of ergodic actions in $A_\mu(G, X, \nu)$. Following Theorem 1.4, and in light of what is known about measure-preserving actions, it is natural to ask if $A_\mu^e(G, X, \nu)$ is residual when G does not have property (T). The purpose of this section is to give a negative answer to this question.

In particular, we will show that there exist groups G without property (T) and appropriately chosen μ for which $A_\mu^e(G, X, \nu)$ is not dense in $A_\mu(G, X, \nu)$ when the latter is equipped with the weak topology.

Recall that the *Furstenberg entropy* of an action $a \in A_\mu(G, X, \nu)$ is given by

$$h_\mu(a) = \sum_{g\in G} \mu(g) \int_X -\log\left(\frac{d\nu}{da(g)_*\nu}(x)\right)\, da(g)_*\nu(x).$$

By Nevo [44], if G has property (T), then any generating measure μ has an *entropy gap*. Namely, there exists a constant $\epsilon = \epsilon(\mu)$ such that the Furstenberg entropy of any non-measure-preserving μ-stationary ergodic action a is at least ϵ.

It is known that some nonproperty (T) groups do not have an entropy gap (e.g. free groups [10], some lamplighter groups, and $SL_2(\mathbb{Z})$ [31]). However, we will next describe groups with an entropy gap which fail to have property (T).

Let $\mu = \mu_1 \times \mu_2$ be a product of generating measures on a product group $G_1 \times G_2$ (i.e., the support of each μ_i generates G_i as a semigroup), and let $G \curvearrowright (X, \nu)$ be μ-stationary. Let $(X_i, \nu_i) = G_i \backslash\backslash X$ be the space of G_i-ergodic components of (X, ν) (i.e., the Mackey realization of the G_i-invariant sigma-algebra), and let $\pi_i \colon X \to X_i$ be the associated factor map such that $\pi_{i*}\nu = \nu_i$. Note that the G-action on (X_i, ν_i) factors through G_{3-i} for $i = 1, 2$ (that is, it has G_i in its kernel). Then by [5, Proposition 1.10], the map $\pi = \pi_1 \times \pi_2 \colon X \to X_1 \times X_2$ pushes ν to $\nu_1 \times \nu_2$, and is furthermore relatively measure preserving. It follows that

(1) $G \curvearrowright (X_1 \times X_2, \nu_1 \times \nu_2)$ is an ergodic, μ-stationary action, $G_1 \curvearrowright (X_2, \nu_2)$ is an ergodic, μ_1-stationary action, and likewise $G_2 \curvearrowright (X_1, \nu_1)$ is an ergodic, μ_2-stationary action.
(2) $h_\mu(X, \nu) = h_{\mu_1}(X_2, \nu_2) + h_{\mu_2}(X_1, \nu_1)$.

Given this, we are ready to state and prove the following claim.

Proposition 7.5. *There exist a group G that does not have property (T) and a generating probability measure μ on G (which may be taken to have finite entropy) such that μ has an entropy gap.*

Proof. Let $G = G_1 \times G_2 = \Gamma \times \mathbb{Z}$, where Γ has property (T). Let μ_1 be a generating measure on Γ, μ_2 a generating measure on $\mathbb{Z}$, and $\mu = \mu_1 \times \mu_2$. Note that μ can be taken to have finite entropy.

Let a be an ergodic, μ-stationary action on (X, ν), and let $(X_1 \times X_2, \nu_1 \times \nu_2)$ be given as above. Denote by a_i the induced action of G_i on (X_{3-i}, ν_{3-i}). Then $h_\mu(a) = h_{\mu_1}(a_1) + h_{\mu_2}(a_2)$.

Now, since $G_2 = \mathbb{Z}$ is abelian, $h_{\mu_2}(a_2) = 0$. Since G_1 has property (T), then μ_1 has an entropy gap, and so, since $G_1 \curvearrowright (X_2, \nu_2)$ is ergodic, $h_{\mu_1}(a_1)$ is either zero or larger than some ϵ that depends only on μ, and the proof is complete. □

Note that one can replace $(\mathbb{Z}, \mu_2)$ with any amenable group G_2 and a measure μ_2' such that (G_2, μ_2') has a trivial Poisson boundary.

Proposition 7.6. *If (G, μ) has an entropy gap, then $A_\mu^e(G, X, \nu)$ is not dense in $A_\mu(G, X, \nu)$.*

In order to prove the proposition, we first observe the following.

Lemma 7.7. *The Furstenberg entropy is a continuous map $h\colon A_\mu(G, X, \nu) \to \mathbb{R}$ where $A_\mu(G, X, \nu)$ is equipped with the weak topology.*

Proof. Define

$$h_\mu^g(a) = \int_X -\log\left(\frac{d\nu}{da(g)_*\nu}(x)\right)\, da(g)_*\nu(x),$$

so that

$$h_\mu(a) = \sum_g \mu(g) h_\mu^g(a).$$

It follows from Proposition 7.1 that the maps $h_\mu^g\colon A_\mu(G, X, \nu) \to \mathbb{R}^+$ are each bounded by a constant M_g, such that $\sum_g \mu(g) M_g < \infty$. We will henceforth prove the claim by showing that each of the maps h_μ^g is continuous and applying the Bounded Convergence Theorem.

Let $\{a_n\} \subset A_\mu(G, X, \nu)$ converge to $a \in A_\mu(G, X, \nu)$. Fix $g \in G$ and define $r(x)$ and $r_n(x)$ as in (11). Then

$$\lim_n \int_X |r_n(x) - r(x)|\, d\nu(x) = 0.$$

By Proposition 7.1, $-N_g \le \log r_n(x) \le M_g$, and likewise $-N_g \le \log r_n(x) \le M_g$. Hence

$$\lim_n \int_X |\log r_n(x) - \log r(x)|\, d\nu(x) = 0,$$

and since

$$h_\mu^g(a_n) = \int_X \log r_n(x)\, d\nu(x) \quad \text{and} \quad h_\mu^g(a) = \int_X \log r(x)\, d\nu(x)$$

we have shown that $\lim_n h_\mu^g(a_n) = h_\mu^g(a)$. □

Proof of Proposition 7.6. Let μ be a generating measure on G with finite Shannon entropy and an entropy gap. Choose $t \in [0, h_\mu(\Pi(G,\mu))]$, where $\Pi(G,\mu)$ is the action on the Poisson boundary. By weighting properly the disjoint union of a nonatomic measure-preserving action and the action on the Poisson boundary, we can construct a nonergodic μ-stationary action a with entropy t.

Assume that the ergodic actions are dense. Then there exists a sequence of ergodic actions $a_n \to a$, and by Lemma 7.7, $h_\mu(a_n) \to h_\mu(a)$. This means that the entropy values that can be realized by ergodic stationary actions are dense in $[0, h_\mu(\Pi(G,\mu))]$, which cannot be the case when μ has an entropy gap. □

The combination of Propositions 7.5 and 7.6 yields our desired result, Theorem 1.5. Incidentally, it is now easy to prove Proposition 1.6:

Proof of Proposition 1.6. Let t be any real number with $0 < t < h_\mu(\Pi(G,\mu))$ where, as above, $\Pi(G,\mu)$ denotes the Poisson boundary of (G,μ). Let $K \subset A_\mu(G,X,\nu)$ be the set of all actions a with $0 \le h_\mu(a) \le t$. Then K is $\mathrm{Aut}(X,\nu)$-invariant. Because entropy is continuous by Lemma 7.7, K is closed. However, since it does not contain any action measurably conjugate to the Poisson boundary, it is not dense. The closure of the complement of K is also not dense since it does not contain any measure-preserving actions (since these have entropy 0). Thus K is neither residual nor meager, and also a dense $\mathrm{Aut}(X,\nu)$-orbit cannot exist in $A_\mu(G,X,\nu)$. □

8. A correspondence principle

In this section, we endow $A_\mu(G,X,\nu)$ with the very weak topology and prove Theorem 1.7: a general correspondence between generic dynamical properties of $\mathcal{P}_\mu(X^G)$ and those of $A_\mu(G,X,\nu)$. This generalizes a result of Glasner and King [25] from the measure-preserving case to the stationary case and from the circle to an arbitrary perfect Polish space. We begin by studying the topology of the group $\mathrm{Aut}^*(X,\nu)$ of nonsingular transformations of a Lebesgue probability space (X,ν) (§6.1) from which we constructed a topology on $A_\mu(G,X,\nu)$ (§6.2).

8.1. **Dynamical generic-equivalence.** The group $\mathrm{Aut}(X,\nu)$ acts on $A^*(G,X,\nu)$ by conjugations:

$$(Ta)(g) = Ta(g)T^{-1}, \quad T \in \mathrm{Aut}(X,\nu), a \in A^*(G,X,\nu).$$

This action is by homeomorphisms. The orbit of a under this action is its *measure-conjugacy class*. More generally, if $G \curvearrowright (X',\nu')$ and $G \curvearrowright (X'',\nu'')$ are nonsingular actions and if there exists a G-equivariant measurable isomorphism $\phi : X' \to X''$ (ignoring sets of measure zero) such that $\phi_*\nu' = \nu''$, then we say these two actions are *measurably conjugate* and write $G \curvearrowright (X',\nu') \sim G \curvearrowright (X'',\nu'')$.

Suppose Ω is a topological space and for each $\omega \in \Omega$ there is assigned a nonsingular action $G \curvearrowright (X_\omega,\nu_\omega)$. Then $(\Omega, \{G \curvearrowright (X_\omega,\nu_\omega)\}_{\omega\in\Omega})$ is a *setting* [25]. For example, $A^*(G,X,\nu)$ is a setting. On the other hand, suppose $G \curvearrowright Y$ is a jointly continuous topological action of a countable group on a compact Hausdorff space. Recall that $\mathcal{P}^*_G(Y)$ is the set of Borel probability measures $\eta \in \mathcal{P}(Y)$ such that the action $G \curvearrowright (Y,\eta)$ is nonsingular. Then $\mathcal{P}^*_G(Y)$ is a setting: we associate to each $\eta \in \mathcal{P}^*_G(Y)$ the system $G \curvearrowright (Y,\eta)$.

Similarly, $A_\mu(G,X,\nu)$ and $\mathcal{P}_\mu(Y)$ are settings. Following [25] we are interested in comparing the dynamical properties of settings. To be precise, suppose

$(\Omega, \{G \curvearrowright (X_\omega, \nu_\omega)\}_{\omega \in \Omega})$ is a setting. A subset $P \subset \Omega$ is a *dynamical property* if for every $\omega_1, \omega_2 \in \Omega$

$$G \curvearrowright (X_{\omega_1}, \nu_{\omega_1}) \sim G \curvearrowright (X_{\omega_2}, \nu_{\omega_2}) \Leftrightarrow (\{\omega_1, \omega_2\} \subset P \text{ or } \{\omega_1, \omega_2\} \cap P = \emptyset).$$

Observe that if Ω' is another setting and $P \subset \Omega$ is dynamical, then there exists a corresponding dynamical set $P' \subset \Omega'$: it is the set of all $\omega' \in \Omega'$ such that there exists $\omega \in P$ with

$$G \curvearrowright (X_\omega, \nu_\omega) \sim G \curvearrowright ({X'}_{\omega'}, {\nu'}_{\omega'}).$$

A subset $P \subset \Omega$ is *Baire* if it can be written as the symmetric difference of O and M, where $O \subset \Omega$ is open and $M \subset \Omega$ is meager. The Baire sets form a sigma-algebra which contains the Borel sigma-algebra.

Two settings Ω_1, Ω_2 are *dynamically generically-equivalent* if for every dynamical property $P_1 \subset \Omega_1$ if $P_2 \subset \Omega_2$ is the corresponding property, then P_1 is Baire/residual/meager in Ω_1 iff P_2 is Baire/residual/meager in Ω_2.

The following is a formal rephrasing of Theorem 1.7. It is the main result of this section.

Theorem 8.1. *Let (G, μ) be a countable group with a generating probability measure, let Z be a perfect Polish space, and let $G \curvearrowright Z^G$ denote the shift action. Let (X, ν) be standard, nonatomic probability space. Then $A_\mu(G, X, \nu)$ and $\mathcal{P}_\mu(Z^G)$ are dynamically generically-equivalent.*

Remark 1. One can replace $\mathcal{P}_\mu(Z^G)$ with $\mathcal{P}_G(Z^G)$ and $A_\mu(G, X, \nu)$ with $A(G, X, \nu)$ in the proof of Theorem 1.7. So the proof shows that $A(G, X, \nu)$ and $\mathcal{P}_G(Z^G)$ are also dynamically generically-equivalent. This extends the Glasner-King Theorem [25], which shows that $A(G, X, \mu)$ and $\mathcal{P}_G(\mathbb{T}^G)$ are dynamically generically-equivalent where $\mathbb{T} = S^1$ represents the 1-dimensional torus. Our proof is based on [25]. We need a few additional arguments to generalize from $\mathbb{T}$ to Z and we fill a few gaps in the somewhat terse presentation in [25].

8.2. **Preliminaries and outline of the proof of Theorem 8.1.** Let

- $\mathbf{B} = \mathbb{N}^{\mathbb{N}}$ denote the Baire space,
- $\overline{\mathcal{P}}(\mathbf{B}) \subset \mathcal{P}(\mathbf{B})$ denote the subspace of fully-supported purely nonatomic Borel probability measures on $\mathbf{B}$,
- $\overline{\mathcal{P}}_\mu(\mathbf{B}^G) \subset \mathcal{P}_\mu(\mathbf{B}^G)$ denote the subspace of μ-stationary probability measures whose projections on each coordinate are in $\overline{\mathcal{P}}(\mathbf{B})$. Note that it is enough to require that the projection to the identity coordinate is in $\overline{\mathcal{P}}(\mathbf{B})$.

The first step in the proof of Theorem 1.7 is to prove:

Proposition 8.2. *$\overline{\mathcal{P}}_\mu(\mathbf{B}^G)$ and $\mathcal{P}_\mu(Z^G)$ are dynamically generically equivalent.*

This result is essentially due to the fact that any perfect Polish space contains a dense G_δ-subset homeomorphic to $\mathbf{B}$. It is proven in §8.3.

Because of Proposition 8.2, it suffices to prove that $\overline{\mathcal{P}}_\mu(\mathbf{B}^G)$ and $A_\mu(G, X, \nu)$ are dynamically generically-equivalent. Without loss of generality, we may assume that $(X, \nu) = (\mathbf{B}, \lambda)$, where λ is any fully-supported purely nonatomic Borel probability measure on $\mathbf{B}$.

Our proof proceeds as follows. We construct a Polish space $\mathbb{H}$ and consider the setting $\mathbb{H} \times A_\mu(G, \mathbf{B}, \lambda)$ (that depends on the second coordinate only). We show

(Proposition 8.7) that there exists a map $E\colon \mathbb{H}\times A_\mu(G,\mathbf{B},\lambda)\to \overline{\mathcal{P}}_\mu(\mathbf{B}^G)$ satisfying:

(1) for any $a\in A_\mu(G,\mathbf{B},\lambda)$ and $h\in\mathbb{H}$, the action $G\curvearrowright(\mathbf{B}^G,E(h,a))$ is measurably conjugate to a.
(2) The image of E is residual in $\overline{\mathcal{P}}_\mu(\mathbf{B}^G)$.
(3) E is a homeomorphism onto its image.

Given this, the proof of Theorem 8.1 is straightforward.

Proof of Theorem 8.1. Let $P_1\subseteq A_\mu(G,\mathbf{B},\lambda)$ be a dynamical property, and let $P_2\subseteq \overline{\mathcal{P}}_\mu(\mathbf{B}^G)$ and $P_3\subseteq \mathcal{P}_\mu(Z^G)$ be the corresponding properties. We would like to show that P_1 is $(*)$ iff P_3 is $(*)$, where $(*)$ stands for either Baire, residual or meager. By Proposition 8.2, P_2 is $(*)$ iff P_3 is $(*)$.

By the first property of E, $P_2=E(P_1\times\mathbb{H})\cup M$, for some M in the complement of the image of E. By the second property of E, M is meager. Hence, by the third property of E, P_2 is $(*)$ iff $\mathbb{H}\times P_1$ is $(*)$. Finally, by [47, page 57], $P_1\times\mathbb{H}$ is $(*)$ iff P_1 is $(*)$, and so the claim follows. □

8.3. **Reduction to Baire space.** The purpose of this subsection is to prove Proposition 8.2.

8.3.1. *The nonatomic, fully supported measures are a residual subset.* A well-known fact is that every perfect Polish space has a dense G_δ subset that is homeomorphic to the Baire space $\mathbf{B}=\mathbb{N}^{\mathbb{N}}$ (see, e.g., the proof of Proposition 2.1 in [11]). Denote by $\hat{Z}$ such a subset of Z. Recall that $\overline{\mathcal{P}}_\mu(\hat{Z}^G)$ denotes the set of μ-stationary measures on $\hat{Z}^G$ (a dense G_δ subset of Z^G that is homeomorphic to $\mathbf{B}^G$), which furthermore have a marginal on the identity coordinate that is fully supported and nonatomic.

To prove Proposition 8.2, we need the following lemma. Before stating it, we note that by the portmanteau theorem, if Y is a subset of X, then the space of all probability measures on X that are supported on Y is homeomorphic with $\mathcal{P}(Y)$. In particular, we think of $\overline{\mathcal{P}}_\mu(\hat{Z}^G)$ as a subset of $\mathcal{P}_\mu(Z^G)$.

Lemma 8.3. *$\overline{\mathcal{P}}_\mu(\hat{Z}^G)$ is dense in $\nu\in\mathcal{P}_\mu(Z^G)$.*

Proof. Given a measure $\nu\in\mathcal{P}_\mu(Z^G)$, we construct a sequence $\{\nu_n\}\subset\overline{\mathcal{P}}_\mu(\hat{Z}^G)$ such that $\lim_n\nu_n=\nu$.

Fix a nonatomic, fully supported $\lambda\in\mathcal{P}(\hat{Z})$. Fix a compatible metric on Z, and denote by $B(x,r)$ the ball of radius r around $x\in Z$. For $x\in Z$ and $n\in\mathbb{N}$, let $\lambda_{x,n}\in\mathcal{P}(Z)$ be equal to λ, conditioned on $B(x,1/n)$. That is, for any measurable $A\subseteq Z$, let

$$\lambda_{x,n}(A)=\frac{\lambda(A\cap B(x,1/n))}{\lambda(B(x,1/n))}.$$

This is well defined, since λ is fully supported, and so $\lambda(B(x,1/n))>0$. For $\xi\in Z^G$, let $\lambda_n^\xi=\prod_{g\in G}\lambda_{\xi(g),n}\in\mathcal{P}(\hat{Z}^G)$ be the product measure with marginal $\lambda_{\xi(g),n}$ on coordinate g. Note that $g_*\lambda_n^\xi=\lambda_n^{g\xi}$.

Let $\bar{\nu}_n\in\mathcal{P}(\hat{Z}^G)$ be given by

$$\bar{\nu}_n=\int_Z\lambda_n^\xi d\nu(\xi).$$

Then, since $g_*\lambda_n^\xi = \lambda_n^{g\xi}$,

$$\sum_{g\in G}\mu(g)g_*\bar{\nu}_n = \int_Z \lambda_n^\xi d\left(\sum_{h\in G}\mu(g)dg_*\nu\right)(\xi),$$

which, by the μ-stationarity of ν, is equal to $\bar{\nu}_n$. Hence $\bar{\nu}_n \in \mathcal{P}_\mu(\hat{Z}^G)$. Note that $\bar{\nu}_n$ is nonatomic, since each λ_n^ξ is nonatomic.

Let

$$\nu_n = \frac{1}{n}\lambda^G + \frac{n-1}{n}\bar{\nu}_n.$$

Since λ^G is invariant, ν_n is μ-stationary. Furthermore, $\ell_*\nu_n \in \overline{\mathcal{P}}(\hat{Z})$ where $\ell : Z^G \to Z$ is the projection map to the identity coordinate. Hence $\nu_n \in \overline{\mathcal{P}}_\mu(\hat{Z}^G)$.

It remains to be shown that $\lim_n \nu_n = \nu$. Clearly, this will follow if we show that $\lim_n \bar{\nu}_n = \nu$, since λ^G/n converges to the zero measure. Let $d(\cdot,\cdot)$ be the compatible metric on Z used to define λ_n^ξ, let $G = \{g_1, g_2, \ldots\}$ and let

$$\hat{d}(\xi,\xi') = \sum_i 2^{-i} d(\xi(g_i), \xi'(g_i))$$

be a compatible metric on Z^G. Then the support of λ_n^ξ is contained in a $\hat{d}$-ball of radius $1/n$ around ξ. Hence $\lim_n \lambda_n^\xi = \delta_\xi$. It follows by the bounded convergence theorem that

$$\lim_{n\to\infty}\bar{\nu}_n = \lim_{n\to\infty}\int_Z \lambda_n^\xi d\nu(\xi) = \int_Z \delta_\xi d\nu(\xi) = \nu.$$

□

Proof of Proposition 8.2. Let $\beta\colon \mathbf{B} \to \hat{Z}$ be a homeomorphism from the Baire space to the dense G_δ subset $\hat{Z} \subset Z$. We extend β to a map $\mathbf{B}^G \to \hat{Z}^G$ by acting independently in each coordinate. Thus β is a homeomorphism between $\mathbf{B}^G$ and $\hat{Z}^G$, which, furthermore, commutes with the G-action. Hence β_* is a homeomorphic measure conjugacy between $\mathcal{P}_\mu(\mathbf{B}^G)$ and $\mathcal{P}_\mu(\hat{Z}^G)$, and thus $\overline{\mathcal{P}}_\mu(\mathbf{B}^G)$ and $\overline{\mathcal{P}}_\mu(\hat{Z}^G)$ are dynamically generically equivalent. Clearly, the natural embedding $\overline{\mathcal{P}}_\mu(\hat{Z}^G) \hookrightarrow \mathcal{P}_\mu(Z^G)$ is a homeomorphic measure conjugacy. Accordingly, we prove the claim by showing that $\overline{\mathcal{P}}_\mu(\hat{Z}^G)$ is residual in $\mathcal{P}_\mu(Z^G)$.

By Lemma 8.3, $\overline{\mathcal{P}}_\mu(\hat{Z}^G)$ is dense in $\mathcal{P}_\mu(Z^G)$. It thus remains to be shown that $\overline{\mathcal{P}}_\mu(\hat{Z}^G)$ is a G_δ. We do this in two steps. First, we show that $\overline{\mathcal{P}}_\mu(\hat{Z}^G)$ is G_δ in $\mathcal{P}_\mu(\hat{Z}^G)$. Then, we show that $\mathcal{P}_\mu(\hat{Z}^G)$ is a G_δ in $\mathcal{P}_\mu(Z^G)$.

Identifying $\hat{Z}$ with $\mathbf{B}$, we show that $\overline{\mathcal{P}}_\mu(\mathbf{B}^G)$ is G_δ in $\mathcal{P}_\mu(\mathbf{B}^G)$. Let $\mathcal{S}$ be a countable base for the topology of $\mathbf{B}$. For $S \in \mathcal{S}$, the set

$$U_{S,n} = \{\nu \in \mathcal{P}_\mu(\mathbf{B}^G) \,:\, \ell_*\nu(S) > 1/n\}$$

is open by the portmanteau theorem, where, to remind the reader, $\ell_*\nu$ is the projection of ν on the identity coordinate. Let

$$F = \bigcap_{S\in\mathcal{S}}\bigcup_{n\in\mathbb{N}} U_{S,n}$$

be the measures with an identity marginal that is supported everywhere on $\mathbf{B}$. Note that F is a G_δ.

Having an atom of mass at least $1/n$ is a closed property. Hence the complementary set

$$W_n = \{\nu \in \mathcal{P}_\mu(\mathbf{B}^G) \,:\, \ell_*\nu(\{x\}) < 1/n \text{ for all } x \in \mathbf{B}\}$$

is open. Let $N = \bigcap_n W_n$ denote the measures with nonatomic marginals. Hence

$$\overline{\mathcal{P}}_\mu(\mathbf{B}^G) = F \cap N$$

is a G_δ in $\mathcal{P}_\mu(\mathbf{B}^G)$. It remains to be shown that $\mathcal{P}_\mu(\hat{Z}^G)$ is a G_δ in $\mathcal{P}_\mu(Z^G)$.

Let $Z^G \setminus \hat{Z}^G = \bigcup_{k\in\mathbb{N}} C_k$, where $C_k \subset Z^G$ is closed. These exist since $\hat{Z}^G$ is a G_δ in Z^G. Then

$$V_{k,n} = \{\nu \in \mathcal{P}_\mu(Z^G) \,:\, \nu(C_k) < 1/n\}$$

is open by the portmanteau theorem. Hence

$$\mathcal{P}_\mu(\hat{Z}^G) = \bigcap_{k\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} V_{k,n}$$

is a G_δ in $\mathcal{P}_\mu(Z^G)$. □

8.4. **The Polish group $\mathbb{H}$.** We now proceed to construct $\mathbb{H}$. We defer some of the proofs to Appendix B.

Denote by $\overline{\mathcal{P}}([0,1])$ the space of fully supported, nonatomic measures on the interval $[0,1]$, endowed with the weak* topology on $\mathcal{P}([0,1])$. Given $\nu \in \overline{\mathcal{P}}([0,1])$, we can define its "inverse cumulative distribution function" $h\colon [0,1] \to [0,1]$ by

$$h^{-1}(t) = \nu([0,t)).$$

It is easy to verify that h is an order-preserving homeomorphism of the interval $[0,1]$ that fixes 0 and 1. Let $\mathbb{H}$ denote the group of all such homeomorphisms, with the topology inherited from the space of continuous functions on $[0,1]$. Then $\mathbb{H}$ is Polish [25]. Since $h_*\nu \in \overline{\mathcal{P}}([0,1])$ for any $h \in \mathbb{H}$ and $\nu \in \overline{\mathcal{P}}([0,1])$, $\mathbb{H}$ acts on $\overline{\mathcal{P}}([0,1])$. This action is jointly continuous [25], free and transitive.

Denote by Irr the space of irrational numbers in the interval $[0,1]$, equipped with the subspace topology inherited from the interval. Because of continued fractions expansions this space is homeomorphic to the Baire space $\mathbf{B}$. Since Irr differs from $[0,1]$ by a countable number of points, $\overline{\mathcal{P}}([0,1])$ and $\overline{\mathcal{P}}(\mathrm{Irr})$ can be identified. Let $\alpha\colon \mathbf{B} \to \mathrm{Irr}$ be a homeomorphism. Then $\mathbb{H}$ acts on $\overline{\mathcal{P}}(\mathbf{B})$ by

$$h\nu := \alpha_*^{-1} h_* \alpha_* \nu.$$

Since $\mathbb{H}$ acts continuously, transitively and freely on $\overline{\mathcal{P}}(\mathrm{Irr})$, it also acts continuously, transitively and freely on $\overline{\mathcal{P}}(\mathbf{B})$.

Denote by $\overline{\mathcal{P}}(\mathbf{B}^G)$ (resp. $\overline{\mathcal{P}}([0,1]^G)$) the space of probability measures whose projections on each coordinate are in $\overline{\mathcal{P}}(\mathbf{B})$ (resp. $\overline{\mathcal{P}}([0,1])$).

Extending α to a map $\mathbf{B}^G \to \mathrm{Irr}^G$ by acting independently in each coordinate, it follows that $\alpha_*\colon \overline{\mathcal{P}}(\mathbf{B}^G) \to \overline{\mathcal{P}}([0,1]^G)$ is a homeomorphism. Hence, as above, we can define a continuous action of $\mathbb{H}$ on $\overline{\mathcal{P}}(\mathbf{B}^G)$ by

$$h\nu = \alpha_*^{-1} h_* \alpha_* \nu,$$

where here the action $\mathbb{H} \curvearrowright \mathrm{Irr}^G$ is also independent on each coordinate. As before, the action $\mathbb{H} \curvearrowright \overline{\mathcal{P}}(\mathbf{B}^G)$ is continuous. Since α acts on each coordinate separately, the $\mathbb{H}$-action on $\overline{\mathcal{P}}(\mathbf{B}^G)$ commutes with the G-action by shifts.

Let $\ell\colon \mathbf{B}^G \to \mathbf{B}$ be the projection on the identity coordinate, given by $\ell(\xi) = \xi(e)$. Denote by $\overline{\mathcal{P}}_\mu(\mathbf{B}^G) = \mathcal{P}_\mu(\mathbf{B}^G) \cap \overline{\mathcal{P}}(\mathbf{B}^G)$ the space of μ-stationary measures whose projection on each coordinate is in $\overline{\mathcal{P}}(\mathbf{B})$. Equivalently, one can just require that the projection on the identity coordinate be in $\overline{\mathcal{P}}(\mathbf{B})$, since the different marginals measures are mutually absolutely continuous.

Note that the $\mathbb{H}$ action on $\overline{\mathcal{P}}(\mathbf{B}^G)$ restricts to an action on $\overline{\mathcal{P}}_\mu(\mathbf{B}^G)$, since the $\mathbb{H}$- and G-actions on $\overline{\mathcal{P}}(\mathbf{B}^G)$ commute. This concludes our definition of $\mathbb{H}$ as a Polish group acting on $\overline{\mathcal{P}}_\mu(\mathbf{B}^G)$.

It thus remains to be shown that $A_\mu(G, \mathbf{B}, \lambda)$ and $\overline{\mathcal{P}}_\mu(\mathbf{B})$ are dynamically generically equivalent.

8.5. **The map E.** For $\zeta \in \mathcal{P}(\mathbf{B})$, denote by

$$\mathcal{P}_\mu^\zeta(\mathbf{B}^G) = \{\nu \in \mathcal{P}_\mu(\mathbf{B}^G) \,:\, \ell_*\nu = \zeta\}$$

the set of μ-stationary measures on $\mathbf{B}^G$ with marginal ζ on the identity coordinate. Recall that λ is any fully-supported purely nonatomic Borel probability measure on $\mathbf{B}$; we will be interested in $\mathcal{P}_\mu^\lambda(\mathbf{B}^G)$. Note that $\mathcal{P}_\mu^\lambda(\mathbf{B}^G) \subseteq \overline{\mathcal{P}}_\mu(\mathbf{B}^G)$, since λ is nonatomic and supported everywhere.

Define

$$\begin{array}{rcccl} D & : & \mathbb{H} \times \mathcal{P}_\mu^\lambda(\mathbf{B}^G) & \longrightarrow & \overline{\mathcal{P}}_\mu(\mathbf{B}^G) \\ & & (h, \nu) & \longmapsto & h_*\nu. \end{array}$$

In [25] (see specifically equation 14 and the preceding remark) it is shown that the map $\mathbb{H} \times \mathcal{P}_\mu^\zeta([0,1]^G) \to \overline{\mathcal{P}}_\mu([0,1]^G)$ given by $(h, \nu) \mapsto h_*\nu$ is a homeomorphism for ζ the Lebesgue measure on $[0,1]$. Since $\overline{\mathcal{P}}_\mu([0,1]^G)$ and $\overline{\mathcal{P}}_\mu(\mathbf{B}^G)$ are homeomorphic,[2] it follows that

Lemma 8.4. *D is a homeomorphism.*

Given $a \in A_\mu(G, \mathbf{B}, \lambda)$, let $\pi_a\colon \mathbf{B} \to \mathbf{B}^G$ be given by

$$[\pi_a(x)](g) = a(g^{-1})x. \tag{12}$$

For a fixed a, π_a is G-equivariant, since for all $k, g \in G$ and $x \in \mathbf{B}$,

$$[\pi_a(a(k)x)](g) = a(g^{-1})a(k)x = a(g^{-1}k)x = [\pi_a(x)](k^{-1}g) = [k\pi_a(x)](g).$$

It follows that $\pi_{a*}\lambda \in \mathcal{P}_\mu(\mathbf{B}^G)$. Furthermore, $\ell \circ \pi_a$ is the identity, and so in particular $\pi_{a*}\lambda \in \mathcal{P}_\mu^\lambda(\mathbf{B}^G)$. Define

$$\begin{array}{rcccl} F & : & A_\mu(G, \mathbf{B}, \lambda) & \longrightarrow & \mathcal{P}_\mu^\lambda(\mathbf{B}^G) \\ & & a & \longmapsto & \pi_{a*}\lambda. \end{array}$$

F is one to one, since disintegrating $\pi_{a*}\lambda$ with respect to the projection $f \mapsto f(e)$ from $\mathbf{B}^G$ to $\mathbf{B}$ yields point mass distributions as the fiber measures, from which the a-orbits of λ-a.e. $x \in \mathbf{B}$ can be reconstructed. Also, a and $F(a)$ are always measurable conjugate, with π_a being the G-equivariant measurable isomorphism. We furthermore prove in Appendix B the following two lemmas. Analogues of these lemmas appear in [25] (see pages 239 and 240) for the measure-preserving setting.

Lemma 8.5. *F is a homeomorphism onto its image.*

Lemma 8.6. *The image of F is residual in $\mathcal{P}_\mu^\lambda(\mathbf{B}^G)$.*

[2] As are $\mathcal{P}_\mu^\zeta([0,1]^G)$ and $\mathcal{P}_\mu^\lambda(\mathbf{B}^G)$; one can take $\lambda = \alpha_*^{-1}\zeta$.

Define

$$\begin{array}{rcccl} E & : & \mathbb{H} \times A_\mu(G, \mathbf{B}, \lambda) & \longrightarrow & \overline{\mathcal{P}}_\mu(\mathbf{B}^G) \\ & & (h, a) & \longmapsto & h_* \pi_{a*} \lambda. \end{array}$$

Alternatively, $E(h, a) = D(h, F(a))$.

The following proposition establishes the properties of E needed for Theorem 8.1.

Proposition 8.7. *E has the following properties:*

(1) *$a \in A_\mu(G, \mathbf{B}, \lambda)$ and $G \curvearrowright (\mathbf{B}^G, E(h, a))$ are measurably conjugate for all $h \in \mathbb{H}$.*
(2) *E is a homeomorphism onto its image.*
(3) *The image of E is residual in $\overline{\mathcal{P}}_\mu(\mathbf{B}^G)$.*

Proof. (1) This follows from the fact that both π_a and $\mathbb{H}$ commute with G; given an action $a \in A_\mu(G, \mathbf{B}, \lambda)$, the equivariant isomorphism (up to λ-null sets) between λ and $E(h, a)$ is given by $h \circ \pi_a \colon \mathbf{B} \to \mathbf{B}^G$.

(2) Since $E(h, a) = D(h, F(a))$, and since D and F are both homeomorphisms onto their images (Lemmas 8.4 and 8.5), it follows that E is a homeomorphism onto its image.

(3) This can be seen by considering the following sequence of embeddings, each of which, as we explain below, is a homeomorphic embedding with a residual image:

$$\mathbb{H} \times A_\mu(G, \mathbf{B}, \lambda) \quad \xrightarrow{\mathrm{id} \times F} \quad \mathbb{H} \times \mathcal{P}_\mu^\lambda(\mathbf{B}^G) \quad \xrightarrow{D} \quad \overline{\mathcal{P}}_\mu(\mathbf{B}^G).$$

By Lemmas 8.6 and 8.5, F embeds $A_\mu(G, \mathbf{B}, \lambda)$ homeomorphically into a residual subset of $\mathcal{P}_\mu^\lambda(\mathbf{B}^G)$. Hence $\mathrm{id} \times F$ embeds $\mathbb{H} \times A_\mu(G, \mathbf{B}, \lambda)$ into a residual subset of $\mathbb{H} \times \mathcal{P}_\mu^\lambda(\mathbf{B}^G)$, by [47, page 57].

Since D is a homeomorphism between $\mathbb{H} \times \mathcal{P}_\mu^\lambda(\mathbf{B}^G)$ and $\overline{\mathcal{P}}_\mu(\mathbf{B}^G)$ (Lemma 8.4), it follows that $E = D \circ (\mathrm{id} \times F)$ embeds $\mathbb{H} \times A_\mu(G, \mathbf{B}^G)$ into a residual subset of $\overline{\mathcal{P}}_\mu(\mathbf{B}^G)$. □

9. Applications of the correspondence principle

In this section we prove Theorems 1.1 and 1.8.

Proof of Theorem 1.1. Let $\mathbb{X} = \{0, 1\}^{\mathbb{N}}$ be the Cantor space, equipped with the usual product topology. Let $\mathcal{P}_\mu^{ext}(\mathbb{X}^G) \subset \mathcal{P}_\mu(\mathbb{X}^G)$ be the subset of all measures η such that $G \curvearrowright (\mathbb{X}^G, \eta)$ is an essentially free ergodic extension of the Poisson boundary. By Theorem 4.2 $\mathcal{P}_\mu^{ext}(\mathbb{X}^G)$ is dense in $\mathcal{P}_\mu(\mathbb{X}^G)$. It is well known that an action is an extension of the Poisson boundary if and only if it has maximal μ-entropy. So if $H(\mu) < \infty$ then by Theorem 3.1, $\mathcal{P}_\mu^{ext}(\mathbb{X}^G)$ is a G_δ subset of $\mathcal{P}_\mu(\mathbb{X}^G)$. Since a dense G_δ is residual, and since $\mathbb{X}$ is a perfect Polish space, it follows from Theorem 8.1 that the same holds for $A_\mu(G, X, \nu)$ (with the very weak topology). By another application of Theorem 8.1, the same also holds for $\mathcal{P}_\mu(Z^G)$, where Z is any perfect Polish space.

The second claim of this theorem follows in a similar way. □

Proof of Theorem 1.8. Recall from the proof of Theorem 1.1 above that $\mathcal{P}_\mu^{ext}(\mathbb{X}^G)$ is a residual subset of $\mathcal{P}_\mu(\mathbb{X}^G)$. So the Correspondence Principle (Theorem 1.7) implies that the subset $A_\mu^{ext}(G, X, \nu)$ of all actions $a \in A_\mu(G, X, \nu)$ that are ergodic essentially free extensions of the Poisson boundary is residual in $A_\mu(G, X, \nu)$.

Let $b \in A_\mu(G, X, \nu)$. We will show b is in the closure of $\mathrm{Aut}(X, \nu)a$. By §6.2.1, it suffices to show that for every $\epsilon > 0$, measurable partition $\mathcal{P} = \{P_1, \dots, P_n\}$ of X and finite $W \subset G$ there exists $a' \in \mathrm{Aut}(X, \nu)a$ such that

$$\sup_{1 \le i,j \le n} \sup_{g \in W} |\nu(b(g)P_i \cap P_j) - \nu(a'(g)P_i \cap P_j)| < \epsilon.$$

Let $\phi : X \to \{1, \dots, n\}$ be the map $\phi(x) = i$ if $x \in P_i$. Let $\Phi : X \to \{1, \dots, n\}^G$ be the map $\Phi(x)_g = \phi(b(g)^{-1}x)$. Observe that this is G-equivariant with respect to the b-action. Let

$$Y_i = \{y \in \{1, \dots, n\}^G : \ y_e = i\}.$$

By Theorem 4.1, there exists a μ-stationary probability measure κ on $\{1, \dots, n\}^G$ such that

- $G \curvearrowright (\{1, \dots, n\}^G, \kappa)$ is a G-factor of $G \curvearrowright^a (X, \nu)$;
- $\sup_{g \in W} \sum_{i,j=1}^n |\Phi_* \nu(Y_i \cap gY_j) - \kappa(Y_i \cap gY_j)| < \epsilon/n^2$.

Let $\Psi : X \to \{1, \dots, n\}^G$ be a G-factor of a so that $\kappa = \Psi_* \nu$. Let $Q'_i = \Psi^{-1}(Y_i)$. Observe that $\mathcal{Q}' = \{Q'_1, \dots, Q'_n\}$ is a measurable partition of X and

$$\sup_{g \in W} \sum_{i,j=1}^n |\nu(b(g)P_i \cap P_j) - \nu(a(g)Q'_i \cap Q'_j)| < \epsilon/n^2.$$

Because $\mathcal{P}$ is a partition, the equation above implies the existence of a measurable partition $\mathcal{Q} = \{Q_1, \dots, Q_n\}$ of X such that

- $\nu(P_i) = \nu(Q_i)$ for all i,
- $\nu(Q_i \triangle Q'_i) < \epsilon/n$ for all i.

Let $\psi \in \mathrm{Aut}(X, \nu)$ be any measure-preserving transformation such that $\psi(Q_i) = P_i$ for all i. Define $a' \in A_\mu(G, X, \nu)$ by $a'(g) = \psi a(g) \psi^{-1}$. It follows that

$$\sup_{1 \le i,j \le n} \sup_{g \in W} |\nu(b(g)P_i \cap P_j) - \nu(a'(g)P_i \cap P_j)| < \epsilon.$$

Thus $\mathrm{Aut}(X, \nu)a$ is dense as required. □

Proof of Corollary 1.9. In order to deduce the 0-1 law, recall the 0-1 lemma from [25]: Let $\mathcal{A}$ be a BaireCat space and let Φ be a group of homeomorphisms of $\mathcal{A}$ such that there exists a $T \in \mathcal{A}$ with a dense Φ-orbit. Then each Baire-measurable Φ-invariant subset of $\mathcal{A}$ is either residual or meager.

The term "BaireCat space" refers to a space that satisfies the Baire Category Theorem. In particular, Polish spaces are BaireCat. Since $\mathrm{Aut}(X, \nu)$ acts continuously on the Polish space $A_\mu(G, X, \nu)$, the corollary follows from Theorem 1.8. □

Appendix A. The weak* topology on a space of measures defined by a relative property

The purpose of this section is to define the weak* topology on a space of measures defined by a relative property. To be precise, let (V, ν) be a standard Borel probability space and W a Polish space. Let $\mathcal{P}(V \times W|\nu)$ denote the set of all Borel probability measures on $V \times W$ that project to ν. We will show that several natural topologies on this set are equal.

For this purpose, let us assume that V is also a Polish space. The weak* topology on $\mathcal{P}(V \times W)$ is defined by: a sequence $\{\lambda_n\}_{n=1}^\infty$ converges to λ_∞ if and only if for every compactly supported continuous function f on $V \times W$, $\int f \, d\lambda_n$ converges to $\int f \, d\lambda_\infty$. We regard $\mathcal{P}(V \times W|\nu)$ as a subspace of $\mathcal{P}(V \times W)$. We will show that the

subspace topology on $\mathcal{P}(V \times W|\nu)$ does not depend on the choice of Polish structure for V. This justifies the following definition: The *weak* topology* on $\mathcal{P}(V \times W|\nu)$ is the subspace topology inherited from the inclusion $\mathcal{P}(V \times W|\nu) \subset \mathcal{P}(V \times W)$ where V is endowed with an arbitrary Polish structure and $\mathcal{P}(V \times W)$ with the usual weak* topology.

Let $\mathrm{MALG}(\nu)$ denote the measure algebra of ν. To be precise, $\mathrm{MALG}(\nu)$ consists of all measurable subsets of V modulo null sets. For $A, B \in \mathrm{MALG}(\nu)$ we let $d(A, B) = \nu(A \triangle B)$. With this metric, $\mathrm{MALG}(\nu)$ is a complete separable metric space.

Let $\mathcal{M}(W)$ denote the space of all finite Borel measures on W with the weak* topology. Let $\mathrm{Map}(\mathrm{MALG}(\nu), \mathcal{M}(W))$ denote the space of all maps from $\mathrm{MALG}(\nu)$ to $\mathcal{M}(W)$ with the pointwise convergence topology. This space has a natural convex structure as it may be identified with the product space $\mathcal{M}(W)^{\mathrm{MALG}(\nu)}$.

Given a measure $\lambda \in \mathcal{P}(V \times W|\nu)$ let $v \mapsto \lambda^v$ be a measurable map from V to $\mathcal{P}(W)$ such that

$$\lambda = \int \delta_v \times \lambda^v \, d\nu(v).$$

It is a standard fact that such a map exists and is unique up to null sets.

Define

$$\Phi : \mathcal{P}(V \times W|\nu) \to \mathrm{Map}(\mathrm{MALG}(\nu), \mathcal{M}(W))$$

by

$$\Phi(\lambda)(A) = \lambda^A$$

where $\lambda^A = \int_A \lambda^v \, d\nu(v)$.

Proposition A.1. *The map Φ is an affine homeomorphism onto its image.*

Remark 2. The topology on $\mathrm{Map}(\mathrm{MALG}(\nu), \mathcal{M}(W))$ is independent of the topology on V. So this proposition shows that the topology on $\mathcal{P}(V \times W|\nu)$ is independent of the topology on V.

Proof. Any Borel measure λ on $V \times W$ is determined by its values on sets of the form $A \times B$ where $A \subset V, B \subset W$ are Borel. Note that

$$\lambda(A \times B) = \lambda^A(B) = \Phi(\lambda)(A)(B).$$

This proves that Φ is injective.

Suppose $\lambda_n \in \mathcal{P}(V \times W|\nu)$ and $\lim_n \lambda_n = \lambda_\infty$ in the weak* topology. To prove that Φ is continuous, it suffices to show that $\lambda_n^A \to \lambda_\infty^A$ for every $A \in \mathrm{MALG}(\nu)$. Actually, it suffices to show that this is true for every A in a dense subset of $\mathrm{MALG}(\nu)$ (because λ is completely determined by the values λ^A for A in a dense subset of $\mathrm{MALG}(\nu)$).

Recall that if Z is any topological space and $Y \subset Z$, then $\partial Y = \overline{Y} \cap \overline{Z \setminus Y}$. Given a measure ζ on Z we say Y is a *continuity set* of ν if $\nu(\partial Y) = 0$. It is not difficult to show that the collection of all continuity sets forms a dense subalgebra of $\mathrm{MALG}(\zeta)$ if Z is a Polish space (see, e.g., [9, Lemma 8.4]; the important requirement is the regularity of ζ).

The portmanteau theorem implies that $\lim_n \lambda_n(E) = \lambda_\infty(E)$ for any set $E \subset V \times W$ which is a continuity set for λ_∞. In particular, $\lim_n \lambda_n^A(B) = \lambda_\infty^A(B)$ if A is a continuity set for ν and B is a continuity set for the projection of λ^∞ to W. Because continuity sets of $\mathrm{Proj}_W(\lambda_\infty)$ are dense in $\mathrm{MALG}(\mathrm{Proj}_W(\lambda_\infty))$, this implies that $\lim_n \lambda_n^A = \lambda_\infty^A$ in the weak* topology on $\mathcal{P}(W)$ for every continuity

set A of ν. Because continuity sets of ν are dense in $\mathrm{MALG}(\nu)$, it follows that $\Phi(\lambda_n) \to \Phi(\lambda_\infty)$ as $n \to \infty$. Because λ is arbitrary, Φ is continuous.

It is clear that Φ is affine. If V is compact, then Φ^{-1} must be continuous on the image of Φ. This proves the proposition when V is compact.

Suppose V is noncompact and let $\{\lambda_n\} \subset \mathcal{P}(V \times W|\nu), \lambda_\infty \in \mathcal{P}(V \times W|\nu)$ be measures such that $\Phi(\lambda_n)$ converges to $\Phi(\lambda_\infty)$ as $n \to \infty$. If $K \subset V$ is a compact subset, then the considerations above imply that $\lambda_n \restriction K \times W \to \lambda_\infty \restriction K \times W$. Thus if $f \in C_c(V \times W)$ is a compactly supported continuous function, then

$$\lim_n \int f \, d\lambda_n = \int f \, d\lambda_\infty.$$

This implies λ_n converges to λ_∞ in the weak* topology. So the inverse of Φ is also continuous, which implies the proposition. □

Corollary A.2. *Let $(\lambda_n) \subset \mathcal{P}(V \times W|\nu)$ be a sequence of measures. Then the following are equivalent:*

1. *λ_n converges to a measure λ_∞ in the weak* topology on $\mathcal{P}(V \times W|\nu)$ with respect to any (every) Polish structure on V.*
2. *$\lambda_n^A \to \lambda_\infty^A$ for every $A \in \mathrm{MALG}(\nu)$.*
3. *$\lambda_n^A \to \lambda_\infty^A$ for every A in some dense subset of $\mathrm{MALG}(\nu)$.*

Moreover, if $\lambda_n^v \to \lambda_\infty^v$ for a.e. $v \in V$, then (1)–(3) above hold.

Proof. It follows immediately from the previous proposition that the first three items are equivalent. Suppose $\lambda_n^v \to \lambda_\infty^v$ for a.e. $v \in V$. Let V be endowed with a Polish topology. Let $f \in C_c(V \times W)$. Then

$$\begin{aligned} \lim_n \int f \, d\lambda_n &= \lim_n \iint f(v,w) \, d\lambda_n^v(w) d\nu(v) = \iint \lim_n f(v,w) \, d\lambda_n^v(w) d\nu(v) \\ &= \iint f(v,w) \, d\lambda_\infty^v(w) d\nu(v) = \int f \, d\lambda_\infty \end{aligned}$$

by the Bounded Convergence Theorem. Since f is arbitrary, this shows λ_n converges to λ_∞ in the weak* topology. □

Appendix B. Proofs of lemmas from §8.5

B.1. **A clopen base.** Let $\mathcal{S}_1, \mathcal{S}_2, \ldots$ be a sequence of finite partitions of $\mathbf{B}$ such that each $S \in \mathcal{S}_n$ is a clopen set, $\mathcal{S}_{n+1}$ is a refinement of $\mathcal{S}_n$, and $\mathcal{S} = \bigcup_n \mathcal{S}_n$ is a countable base of the topology of $\mathbf{B}$, and also of the associated Borel sigma-algebra.

Likewise, let $\mathcal{T}_1, \mathcal{T}_2, \ldots$ be a sequence of finite partitions of $\mathbf{B}^G$ with the same properties, and likewise denote $\mathcal{T} = \bigcup_n \mathcal{T}_n$. Furthermore, let each $T \in \mathcal{T}$ be of the form

$$T = \bigcap_{g \in K} g\ell^{-1}(S_g) = \bigcap_{g \in K} \{\xi \in \mathbf{B}^G \,:\, \xi(g) \in S_g\}$$

for some finite $K \subset G$ and a map $g \mapsto S_g \in \mathcal{S}_n$, for some $n \in \mathbb{N}$. This is, again, a countable base of the topology and of the Borel sigma-algebra.

B.2. F **is a homeomorphism onto its image.**

Proof of Lemma 8.5. It is shown in §8.5 that $F\colon A_\mu(G,\mathbf{B},\lambda) \to \mathcal{P}_\mu^\lambda(\mathbf{B}^G)$ is injective. Hence it remains to be shown that it is continuous and that its inverse is continuous.

Recall (see §B.1) that $\mathcal{S}$ is a base for the sigma-algebra of $\mathbf{B}$. For $S_1, S_2 \in \mathcal{S}$, define

$$A_{S_1,S_2,g} = \{\xi \in \mathbf{B}^G \,:\, \xi(e) \in S_1, \xi(g) \in S_2\}.$$

Let $\zeta = F(a)$. By the definition of F,

$$\zeta(A_{S_1,S_2,g}) = \lambda\left(S_1 \cap a(g)S_2\right). \tag{13}$$

Let $\{a_n\}$ be a sequence in $A_\mu(g,\mathbf{B},\lambda)$, and denote $\nu_n = F(a_n)$. Since each ν_n is a graph measure, it is determined by the two-dimensional marginals, or equivalently by the values of $\nu_n(A_{S_1,S_2,g})$. Hence $\lim_n \nu_n = \nu$ iff

$$\lim_n \nu_n(A_{S_1,S_2,g}) = \nu(A_{S_1,S_2,g}) \tag{14}$$

for all $S_1, S_2 \in \mathcal{S}$ and $g \in G$.

We first show that F^{-1} is continuous by showing that if $\lim_n \nu_n = \nu$, then $\lim_n a_n = a$, where $a = F^{-1}(\nu)$. Assume then that (14) holds. Combining it with (13) yields

$$\lim_n \lambda(S_1 \cap a_n(g)S_2) = \lambda(S_1 \cap a(g)S_2), \tag{15}$$

which implies $\lim_n a_n = a$. So F^{-1} is continuous.

Analogously, to see that F is continuous, assume that (15) holds. Combining this with (13) yields

$$\lim_n \nu_n(A_{S_1,S_2,g}) = \nu(A_{S_1,S_2,g}).$$

□

B.3. **The image of F is residual.** To prove Claim 8.6, we show that $\operatorname{Im} F$ is a dense G_δ in $\mathcal{P}_\mu^\lambda(\mathbf{B}^G)$; in Claim B.1 we show that it is dense, and in Claim B.2 we show that it is a G_δ.

Claim B.1. $\operatorname{Im} F$ *is dense in* $\mathcal{P}_\mu^\lambda(\mathbf{B}^G)$.

Proof. Recall (see §B.1) that $\mathcal{S}$ is a clopen base of the topology of $\mathbf{B}$, and $\mathcal{T}$ is a clopen base of the topology of $\mathbf{B}^G$.

Given $\nu \in \mathcal{P}_\mu^\lambda(\mathbf{B}^G)$, we construct for each $n \in \mathbb{N}$ an action $a_n \in A_\mu(G,\mathbf{B},\lambda)$ such that for all $T \in \mathcal{T}$ it holds that $[F(a_n)](T) = \nu(T)$ for n large enough. This will prove the claim, since it implies that $\lim_n F(a_n) = \nu$.

Fix n. Then $\{\ell^{-1}(S)\}_{S\in\mathcal{S}_n}$ is a finite partition of $\mathbf{B}^G$ where $\ell : \mathbf{B}^G \to \mathbf{B}$ is the projection map to the identity coordinate. Denote by $\nu|_{\ell^{-1}(S)}$ the measure ν restricted to $\ell^{-1}(S)$, and define $\lambda|_S$ analogously.

Let $\varphi_n\colon \mathbf{B}^G \to \mathbf{B}$ be a measurable map that, for each $S \in \mathcal{S}_n$, is a measure isomorphism between $(\ell^{-1}(S), \nu|_{\ell^{-1}(S)})$ and $(S, \lambda|_S)$. This is possible, since all of the spaces $(\ell^{-1}(S), \nu|_{\ell^{-1}(S)})$ and $(S, \lambda|_S)$ are standard nonatomic finite measure spaces, with the same total mass. It follows that φ_n is a measure isomorphism between $(\mathbf{B}^G, \nu)$ and $(\mathbf{B}, \lambda)$.

Now, let $a_n \in A_\mu(G, \mathbf{B}, \lambda)$ be given by, for every $g \in G$ and $x \in \mathbf{B}$,

$$[a_n(g)](x) = \varphi_n g \varphi_n^{-1} x, \tag{16}$$

where the G-action is here by shifts on $\mathbf{B}^G$. This is indeed a μ-stationary action, since it is conjugate to $(\mathbf{B}^G, \nu)$.

Finally, for $T \in \mathcal{T}$, we show that $[F(a_n)](T) = \nu(T)$ for n large enough. By the definition of $\mathcal{T}$, for $n \in \mathbb{N}$ large enough there exists a finite $K \subset G$ and a map $g \mapsto S_g$ from $K \to \mathcal{S}_n$ such that

$$T = \bigcap_{g \in K} g\ell^{-1}(S_g) = \bigcap_{g \in K} \{\xi \in \mathbf{B}^G : \xi(g) \in S_g\}. \tag{17}$$

By the definitions of F and π_{a_n},

$$[F(a_n)](T) = [\pi_{a_n *}\lambda](T) = \lambda(\{x \in \mathbf{B} : \pi_{a_n}(x) \in T\}).$$

By (17)

$$= \lambda(\{x \in \mathbf{B} : [\pi_{a_n}(x)](g) \in S_g \text{ for all } g \in K\}).$$

Hence by the definitions of π_{a_n} and a_n (in (16))

$$\begin{aligned} &= \lambda(\{x \in \mathbf{B} : a_n(g^{-1})(x) \in S_g \text{ for all } g \in K\}) \\ &= \lambda(\{x \in \mathbf{B} : \varphi_n g^{-1} \varphi_n^{-1} x \in S_g \text{ for all } g \in K\}). \end{aligned}$$

Now, $\varphi_n^{-1} S_g = \ell^{-1}(S_g)$, and so

$$= \lambda(\{x \in \mathbf{B} : \varphi_n^{-1} x \in g\ell^{-1}(S_g) \text{ for all } g \in K\}).$$

But $\varphi_{n*}^{-1}\lambda = \nu$ and so

$$= \nu(\{\xi \in \mathbf{B}^G : \xi \in g\ell^{-1}(S_g) \text{ for all } g \in K\}) = \nu(T).$$

Thus $\operatorname{Im} F$ is dense in $\mathcal{P}_\mu^\lambda(\mathbf{B}^G)$. □

Claim B.2. *$\operatorname{Im} F$ is a G_δ subset of $\mathcal{P}_\mu^\lambda(\mathbf{B}^G)$.*

Proof. A classical result states that if a subset of a metric space is completely metrizable, then it is a G_δ (see, e.g., [47, Theorem 12.3]). Since $\mathcal{P}_\mu^\lambda(\mathbf{B}^G)$ is Polish and since its subset $\operatorname{Im} F$ is the homeomorphic image (Lemma 8.5) of the Polish space $A_\mu(G, \mathbf{B}, \lambda)$, it follows that $\operatorname{Im} F$ is a G_δ subset of $\mathcal{P}_\mu^\lambda(\mathbf{B}^G)$. □

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