TORUS INVARIANT TRANSVERSE KÄHLER FOLIATIONS

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Abstract. In this paper, we show the convexity of the image of a moment map on a transverse symplectic manifold equipped with a torus action under a certain condition. We also study properties of moment maps in the case of transverse Kähler manifolds. As an application, we give a positive answer to the conjecture posed by Cupit-Foutou and Zaffran.

1. Introduction

In [11], a family of complex manifolds (say LV manifolds here) which includes classical Hopf manifolds and Calabi-Eckmann manifolds are constructed by López de Medrano and Verjovsky. Most of LV manifolds are non-Kähler; it is shown that an LV manifold admits a Kähler form if and only if it is a compact complex torus of complex dimension 1. In [10], as a contrast, it is shown by Loeb and Nicolau that each LV manifold carries a transverse Kähler vector field. In [12], Meersseman generalizes the construction of LV manifolds and gives a new family of complex manifolds which are known as LVM manifolds. As well as LV manifolds, an LVM manifold admits a Kähler form if and only if it is a compact complex torus. He also constructs a foliation $\mathcal{F}$ on each LVM manifold and shows that $\mathcal{F}$ is transverse Kähler. In [13], the condition (K) is introduced for LVM manifolds by Meersseman and Varjovsky. They also show that if an LVM manifold $M$ satisfies the condition (K), then the leaf space $M/\mathcal{F}$ is a projective toric orbifold. In [4], Bosio generalizes LVM manifolds and now they are known as LVMB manifolds. As well as LVM manifolds, if an LVMB manifold is Kähler, then it is a compact complex torus. In [3], Battisti gives an explanation of the difference between LVM manifolds and LVMB manifolds in terms of toric geometry.

In [6], Cupit-Foutou and Zaffran construct a foliation $\mathcal{F}$ on each LVMB manifold as a natural generalization of the case of LVM manifolds. They also generalize the condition (K) to LVMB manifolds. They show that the condition (K) is invariant under biholomorphic equivalence and if the foliation $\mathcal{F}$ on an LVMB manifold $M$ satisfying the condition (K) is transverse Kähler, then $M$ is an LVM manifold. Using this, they produce examples of LVMB manifolds that are not biholomorphic.
to LVM manifolds. In particular, the family of LVMB manifolds properly contains the family of LVM manifolds.

From this point of view, they give the following conjecture which is the motivation of this paper:

**Conjecture 1.1.** An LVMB manifold is an LVM manifold if and only if the foliation $\mathcal{F}$ is transverse Kähler.

Our approach to Conjecture 1.1 uses techniques of Hamiltonian torus actions on symplectic manifolds. Especially, (an analogue of) the convexity theorem plays an important role. The convexity theorem shown by Atiyah, Guillemin and Sternberg in [1] and [7] states that if a compact torus $G$ acts on a compact connected symplectic manifold $M$ in Hamiltonian fashion, then the image of a corresponding moment map is a convex polytope.

A closed 2-form on a smooth manifold is called a presymplectic form (see [9]). A smooth manifold $M$ equipped with a presymplectic form $\omega$ is called a presymplectic manifold. Let $\mathcal{F}$ be a smooth foliation on $M$. A transverse symplectic form $\omega$ with respect to $\mathcal{F}$ is a presymplectic form on $M$ whose kernel coincides with $TF$. As well as the symplectic case, we can consider a moment map for an action of a Lie group $G$ on a presymplectic manifold.

In this paper, first we show the following:

**Theorem 1.2** (See also Theorem 2.7). Let $M$ be a compact connected manifold equipped with an action of a compact torus $G$. Let $g'$ be a subspace of the Lie algebra $g$ of $G$ such that the action of $g'$ is locally free. Let $\mathcal{F}_{g'}$ be the foliation on $M$ whose leaves are $g'$-orbits. Let $\omega$ be a $G$-invariant transverse symplectic form on $M$ with respect to $\mathcal{F}_{g'}$. If there exists a moment map $\Phi : M \to g^\ast$ with respect to $\omega$, then the image of $M$ by $\Phi$ is the convex hull of the image of common critical points of $h_v$ for $v \in g$, where $h_v : M \to \mathbb{R}$ is given by $h_v(x) = \langle \Phi(x), v \rangle$.

**Example 1.3.** The setting of Theorem 1.2 naturally appears in symplectic manifolds. Let $N$ be a compact connected symplectic manifold equipped with an effective Hamiltonian action of a compact torus $G$. Let $g'$ be any subspace of $g$ and let $i : g' \to g$ denote the inclusion. Let $c \in (g')^\ast$ be a regular value of $i^\ast \circ \Phi : N \to (g')^\ast$ and let $M := (i^\ast \circ \Phi)^{-1}(c)$. Then, $g'$ acts on $M$ locally freely and the restriction $\omega|_M$ of the symplectic form on $N$ is a transverse symplectic form on $M$ with respect to $\mathcal{F}_{g'}$. The image $\Phi(M)$ coincides with $(i^\ast)^{-1}(c) \cap \Phi(N)$, and it is a convex polytope.

For a connected complex manifold $M$ equipped with an effective action of a compact torus $G$ which preserves the complex structure $J$ on $M$, the subspace

$$g_J := \{ v \in g \mid X_v = -J X_{v'}, \exists v' \in g \}$$

of $g$ acts on $M$ locally freely (see Proposition 3.3). The foliation $\mathcal{F}_{g_J}$ whose leaves are $g_J$-orbits coincides with the foliation which has been considered in [6] in the case of LVMB manifolds and the foliation which has been considered in [15] in the case of moment-angle manifolds equipped with complex-analytic structures. $\mathcal{F}_{g_J}$ gives a lower bound of $G$-invariant foliations that admit $G$-invariant transverse Kähler forms such that there exist moment maps with respect to the forms (see Proposition 4.2).

An effective action of a compact torus $G$ on a connected manifold $M$ is said to be maximal if there exists a point $x \in M$ such that $\dim G + \dim G_x = \dim M$ (see...
If $M$ is a compact connected complex manifold equipped with a maximal action of a compact torus $G$ preserving the complex structure $J$ on $M$, one can associate a complete fan $q(\Delta)$ in $\mathfrak{g}/\mathfrak{g}_J$ with $M$. On the other hand, if there exists a $G$-invariant transverse Kähler form on $M$ with respect to $\mathcal{F}_{\mathfrak{g}_J}$ and if there exists a moment map $\Phi : M \to \mathfrak{g}^*$ with respect to the form, then one can find a lift $\tilde{\Phi} : M \to (\mathfrak{g}/\mathfrak{g}_J)^*$ of $\Phi$. The following is the main theorem in this paper:

**Theorem 1.4** (See also Theorem 5.8). Let $M$ be a compact connected complex manifold equipped with a maximal action of a compact torus $G$ which preserves the complex structure $J$ on $M$. If $\mathcal{F}_{\mathfrak{g}_J}$ is transverse Kähler, then a moment map $\Phi$ with respect to a $G$-invariant transverse Kähler form exists and $\Phi(M)$ is a convex polytope normal to $q(\Delta)$. Conversely, if $q(\Delta)$ is polytopal, then $\mathcal{F}_{\mathfrak{g}_J}$ is transverse Kähler.

As an application of Theorem 1.4, we show that Conjecture 1.1 is true.

**Remark 1.5.** It is shown that if the fan $q(\Delta)$ is weakly normal, then there exists a closed 2-form on $M$ whose restriction to an open dense subset is transverse Kähler; see [16].

This paper is organized as follows. In Section 2 we investigate Hamiltonian functions for almost periodic vector fields on transverse symplectic manifolds and show the convexity of the image of a moment map with almost the same argument as Atiyah. In Section 3 we construct a $G$-invariant foliation $\mathcal{F}_{\mathfrak{g}_J}$ on a complex manifold equipped with an action of a compact torus $G$. In Section 4 we show that the foliation $\mathcal{F}_{\mathfrak{g}_J}$ is a lower bound of $G$-invariant foliations that admit moment maps. In Section 5 we consider the case of maximal torus action and give a proof of the conjecture posed by Cupit-Foutou and Zaffran.

**Conventions and notation.** For a smooth manifold $M$ and a smooth foliation $\mathcal{F}$ on $M$, we denote by $T\mathcal{F}$ the subbundle of the tangent bundle $TM$ consisting of vectors tangent to leaves of $\mathcal{F}$. Let $G$ be a compact torus acting on $M$ smoothly. We say that $\mathcal{F}$ is $G$-invariant if $T\mathcal{F}$ is a $G$-equivariant subbundle of $TM$. We denote by $\mathfrak{g}$ the Lie algebra of $G$. Through the exponential map, $\mathfrak{g}$ also acts on $M$. For a subspace $\mathfrak{g}'$ that acts on $M$ locally freely, we denote by $\mathcal{F}_{\mathfrak{g}'}$ the foliation on $M$ whose leaves are $\mathfrak{g}'$-orbits. For a point $x \in M$, we denote by $G_x$ the isotropy subgroup of $G$ at $x$. We also denote by $\mathfrak{g}_x$ the Lie algebra of $G_x$. Remark that $\mathfrak{g}_x$ is not the isotropy subgroup of $\mathfrak{g}$ at $x$. For $v \in \mathfrak{g}$, we denote by $X_v$ the fundamental vector field generated by $v$ on $M$. For a vector field $X$ on $M$, we denote by $X_x$ the value of $X$ at $x$. For the fundamental vector field, we denote by $(X_v)_x$ the value of $X_v$ at $x$. For a differential form $\omega$, we denote by $\omega_x$ the value of $\omega$ at $x$. We denote by $\mathcal{L}_X \omega$ the Lie derivative for $\omega$ along $X$ and by $i_X \omega$ the interior product of $X$ and $\omega$. We identify $\mathbb{R}$ with the Lie algebra of $S^1$ by the differential of the map $t \mapsto e^{2\pi \sqrt{-1}t}$. For $\alpha : G \to S^1$, we denote by $d\alpha$ the differential at the unit of $G$ and $d\alpha$ is regarded as an element in $\mathfrak{g}^*$. We denote by $H^*_F(M)$ the first basic cohomology group with coefficients in $\mathbb{R}$.

2. The convexity theorem

Let $\omega$ be a transverse symplectic form on a smooth manifold $M$ with respect to a smooth foliation $\mathcal{F}$ on $M$. Let a compact torus $G$ act on $M$ effectively. We assume
that the action of $G$ preserves $\omega$ (and hence, $\mathcal{F}$ is $G$-invariant). In this case, by the Cartan formula we have that
\[
0 = \mathcal{L}_{X_v} \omega = dt_{X_v} \omega + \iota_{X_v} d\omega = dt_{X_v} \omega
\]
for $v \in \mathfrak{g}$. We say that a smooth map $\Phi : M \to \mathfrak{g}^*$ is a moment map if the function $h_v : M \to \mathbb{R}$ given by $\langle \Phi(x), v \rangle = h_v(x)$ satisfies that $dh_v = -\iota_{X_v} \omega$. A moment map $\Phi$ with respect to $\omega$ exists if and only if $\iota_{X_v} \omega$ is exact for any $v$. In particular, the obstruction for the existence of a moment map sits in $H^1_X(M)$. The purpose of this section is to show the convexity of the image of a moment map under certain conditions by almost the same argument as Atiyah (see [1]).

Let $\{\psi_\alpha : U_\alpha \to V_\alpha\}_\alpha$ be foliation charts of $(M, \mathcal{F})$. The local leaf space $U_\alpha / \mathcal{F}$ is diffeomorphic to an open subset of $\mathbb{R}^{\dim M - \dim \mathcal{F}}$. The quotient map $\pi_\alpha : U_\alpha \to U_\alpha / \mathcal{F}$ is a fiber bundle whose fibers are diffeomorphic to open balls of dimension $\dim \mathcal{F}$. The transition functions $\psi_{\alpha \beta} : \psi_\beta(U_\alpha \cap U_\beta) \to \psi_\alpha(U_\alpha \cap U_\beta)$ can be written as
\[
\psi_{\alpha \beta}(x, z) = (\psi^T_{\alpha \beta}(x), \psi^\beta_\alpha(x, z)) \in \psi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^{\dim M - \dim \mathcal{F}} \times \mathbb{R}^{\dim \mathcal{F}}
\]
for $(x, z) \in \psi_\beta(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^{\dim M - \dim \mathcal{F}} \times \mathbb{R}^{\dim \mathcal{F}}$.

Let $x \in M$ and let $G_x$ denote the isotropy subgroup at $x$ of $G$. Let $\psi : U \to V$ be a local foliation chart on an open neighborhood at $x$. The local leaf space $U / \mathcal{F}$ is diffeomorphic to an open subset of $\mathbb{R}^{\dim M - \dim \mathcal{F}}$. Let $\pi : U \to U / \mathcal{F}$ be the quotient map. Since $G_x$ is compact, the intersection $U' := \bigcap_{g \in G_x} g(U)$ is a $G_x$-invariant open neighborhood at $x$. Since $\mathcal{F}$ is $G$-invariant, $\pi(U')$ is a $G_x$-manifold of dimension $\dim M - \dim \mathcal{F}$. Let $\omega$ be a $G$-invariant transverse symplectic form on $M$ with respect to $\mathcal{F}$ and suppose that $\dim \mathcal{F} = \ell$ and $\dim M = 2n + \ell$. $\omega$ descends to a symplectic form $\tilde{\omega}$ on $\pi(U')$. By the equivariant Darboux theorem, there exist $G_x$-invariant open subsets $U_x$ of $\pi(U')$ and $V_x$ of $T_x M / T_x \mathcal{F}$, a $G_x$-equivariant diffeomorphism $\varphi : U_x \to V_x$ and local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $T_x M / T_x \mathcal{F}$ such that
\[
(\varphi^{-1})^* \tilde{\omega} = \sum_{i=1}^n dx_i \wedge dy_i
\]
and
\[
X_v = 2\pi \sum_{i=1}^n \langle d\alpha_i, v \rangle \left( x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right)
\]
for $v \in \mathfrak{g}_x$, where $\alpha_1, \ldots, \alpha_n \in \text{Hom}(G_x, S^1)$ are weights at $0 \in T_x M / T_x \mathcal{F}$. Remark that $\psi_{|\pi^{-1}(U_x)} : \pi^{-1}(U_x) \to \psi(\pi^{-1}(U_x))$ is a local foliation chart near $x$. We state this fact as a lemma for later use.

**Lemma 2.1.** Let $M$ be a smooth manifold of dimension $2n + \ell$ equipped with an action of a compact torus $G$. Let $\mathcal{F}$ be a $G$-invariant smooth foliation on $M$ of dimension $\ell$ and let $\omega$ be a $G$-invariant transverse symplectic form on $M$ with respect to $\mathcal{F}$. Then, for any $x \in M$, there exist
- a local foliation chart $\psi_x : \tilde{U}_x \to \tilde{V}_x$ on an open neighborhood $\tilde{U}_x$ at $x$ such that $\tilde{U}_x / \mathcal{F}$ carries the action of $G_x$,
- a $G_x$-invariant open neighborhood $V_x$ at $0 \in T_x M / T_x \mathcal{F}$,
- a $G_x$-equivariant diffeomorphism $\varphi_x : \tilde{U}_x / \mathcal{F} \to V_x$, and
- local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $T_x M / T_x \mathcal{F}$.
such that \((\varphi_x^{-1})^*\omega = \sum_{i=1}^n dx_i \wedge dy_i\) and

\[
X_v = 2\pi \sum_{i=1}^n (d\alpha_i, v) \left( x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right)
\]

for \(v \in g_x\), where \(\alpha_1, \ldots, \alpha_n \in \text{Hom}(G_x, S^1)\) are weights at \(0 \in T_x M/T_x F\).

Let \(M, G, F, \omega\) be as Lemma 2.1. Let \(v \in g\) and suppose that there exists a smooth function \(h_v : M \to \mathbb{R}\) such that \(-\iota_{X_v} \omega = dh_v\). Since \(\omega\) is transverse symplectic with respect to \(F\), a point \(x \in M\) is a critical point of \(h_v\) if and only if \((X_v)_x \in T_x F\). Because of this, unfortunately, we cannot deduce the property that \(h_v\) is nondegenerate\(^2\) for a general foliation, unlike the symplectic case. Let \(g'\) be a subspace of \(g\) such that \(g'\) acts on \(M\) locally freely. Since \(G\) is abelian, \(F_{g'}\) is a \(G\)-invariant foliation.

**Lemma 2.2.** Let \(M\) be a smooth manifold equipped with an action of a compact torus \(G\). Let \(g'\) be a subspace of \(g\) such that \(g'\) acts on \(M\) locally freely. Let \(\omega\) be a \(G\)-invariant transverse symplectic form on \(M\) with respect to \(F_{g'}\). Let \(v \in g\) and suppose that there exists a smooth function \(h_v : M \to \mathbb{R}\) such that \(dh_v = -\iota_{X_v} \omega\). Then, \(h_v\) is a nondegenerate function and the index of each critical submanifold is even.

**Proof.** Let \(x \in M\) be a critical point of \(h_v\). Then, \((X_v)_x \in T_x F_{g'}\) implies that there exist \(v_x \in g_x\) and \(v' \in g'\) such that \(v = v_x + v'\). Since \(\iota_{X_v} \omega = 0\), we have that \(\iota_{X_v} \omega = \iota_{X_{v'}} \omega\). Since \(\iota_{X_v} \omega\) is basic for \(F_{g'}\), so is \(h_v\). Let \(\psi_x : \widetilde{U}_x \to V_x, V_x, \varphi_x, (x_1, \ldots, x_n, y_1, \ldots, y_n)\) be as in Lemma 2.1. Since \(h_v\) is basic for \(F_{g'}\), \(h_v\) descends to a smooth function \(h_v : \widetilde{U}_x/F_{g'} \to \mathbb{R}\) such that \(\pi^* h_v = h_v\), where \(\pi : \widetilde{U}_x \to \widetilde{U}_x/F_{g'}\) denotes the quotient map. By definition of \(F_{g'}\), \(\pi\) sends the fundamental vector field \(X_v\) generated by \(v\) on \(\widetilde{U}_x\) to the fundamental vector field \(X_{v_x}\) generated by \(v_x\) on \(\widetilde{U}_x/F_{g'}\). Also, since \(\varphi_x\) is \(G_x\)-equivariant, \(\varphi_x\) sends \(X_{v_x}\) on \(\widetilde{U}_x/F_{g'}\) to \(X_{v_x}\) on \(T_x M/T_x F_{g'}\).

Therefore

\[
dh_v = -\iota_{X_v} \omega = -\iota_{X_{v_x}} \omega = \pi^* (-\iota_{X_{v_x}} \omega) = \pi^* \circ \varphi_x^*(-\iota_{X_{v_x}} (\varphi_x^{-1})^* \omega).
\]

On the other hand,

\[
dh_v = d(\pi^* h_v) = \pi^* (dh_v) = \pi^* \circ (\varphi_x^{-1})^*(d((\varphi_x^{-1})^* h_v)).
\]

Since \(\pi^*\) is injective and \(\varphi_x\) is a diffeomorphism, we have that \(d((\varphi_x^{-1})^* h_v) = -\iota_{X_{v_x}} (\varphi_x^{-1})^* \omega\). Let \(\alpha_1, \ldots, \alpha_n \in \text{Hom}(G_x, S^1)\) be the weights at the origin in \(T_x M/T_x F_{g'}\). Then, \(X_{v_x}\) on \(T_x M/T_x F_{g'}\) can be represented as

\[
X_{v_x} = 2\pi \sum_{i=1}^n (d\alpha_i, v_x) \left( x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right)
\]

with the coordinates \((x_1, \ldots, x_n, y_1, \ldots, y_n)\). Therefore

\[
-\iota_{X_{v_x}} (\varphi_x^{-1})^* \omega = 2\pi \sum_{i=1}^n (d\alpha_i, v_x) (x_i dx_i + y_i dy_i)
\]

\(^2\)A smooth function \(f : M \to \mathbb{R}\) is said to be non-degenerate (in the sense of Bott) if its critical set is a submanifold and the second derivative is a nondegenerate quadratic form in the transverse direction.
and hence
\begin{equation}
(\varphi_x^{-1})^* h_v = (\varphi_x^{-1})^* h_v(0) + \pi \sum_{i=1}^n \langle d\alpha_i, v_x \rangle (x_i^2 + y_i^2).
\end{equation}

Therefore \((\varphi_x^{-1})^* h_v\) is nondegenerate at 0 and the index at 0 is twice as many as the number of \(\alpha_i\) such that \(\langle d\alpha_i, v_x \rangle < 0\). Since \(\varphi_x \circ \pi : \tilde{U}_x \to V_x\) is a fiber bundle, \(h_v\) is nondegenerate at \(x\) and the index at \(x\) is twice as many as the number of \(\alpha_i\) such that \(\langle d\alpha_i, v_x \rangle < 0\), proving the lemma.

**Remark 2.3.** In the proof of Lemma 2.2 it follows from (2.1) that \(h_v\) attains a local minimum at \(x\) if and only if \(\langle d\alpha_i, v_x \rangle \geq 0\) for all \(\alpha_i\).

**Remark 2.4.** We are not sure whether Lemma 2.2 holds even if we replace \(F_g\) in any \(G\)-invariant foliation \(F\) or not.

The following is the key to the convexity theorem.

**Lemma 2.5 ([1] Lemma 2.1).** Let \(\phi : N \to \mathbb{R}\) be a non-degenerate function (in the sense of Bott) on the compact connected manifold \(N\), and assume that neither \(\phi\) nor \(-\phi\) has a critical manifold of index 1. Then \(\phi^{-1}(c)\) is connected (or empty) for every \(c \in \mathbb{R}\).

By Lemmas 2.2 and 2.5 if \(M\) is compact, then the level set \(h^{-1}_v(c)\) is connected unless empty. Moreover, if \(c\) is a regular value, then \(h^{-1}_v(c)\) is a connected submanifold of \(M\). Since \(G\) is abelian, \(dh_v = -\iota_{X_v}\omega\) is \(G\)-invariant. Therefore \(h_v\) is also \(G\)-invariant. Therefore \(h^{-1}_v(c)\) is \(G\)-invariant for \(c \in \mathbb{R}\).

**Lemma 2.6.** Let \(M\) be a compact connected manifold equipped with an action of a compact torus \(G\). Let \(g'\) be a subspace of \(g\) such that \(g'\) acts on \(M\) locally freely. Let \(\omega\) be a \(G\)-invariant transverse symplectic form on \(M\) with respect to \(F_{g'}\). Let \(v_1, \ldots, v_k \in g\) and suppose that there exists a smooth function \(h_{v_i}\) such that \(dh_{v_i} = -\iota_{X_{v_i}}\omega\) for \(i = 1, \ldots, k\). Let \(c \in \mathbb{R}^k\) be a regular value of \(h = (h_{v_1}, \ldots, h_{v_k}) : M \to \mathbb{R}^k\). Then, \(g'' = g' + \mathbb{R} v_1 + \cdots + \mathbb{R} v_k\) acts on \(h^{-1}(c)\) locally freely and \(\omega|_{h^{-1}(c)}\) is a transverse symplectic form with respect to the foliation \(F_{g''}\).

**Proof.** Let \(x \in h^{-1}(c)\). Since \(c\) is a regular value, \((-\iota_{X_v} \omega)_x\) are linearly independent. This together with the fact that the action of \(g''\) is locally free yields that \((X_{v'})_x = 0\) if and only if \(v'' = 0\) for \(v'' \in g''\). Therefore the action of \(g''\) is locally free.

\(T_x(h^{-1}(c))\) is given by \(\ker(dh)_x = (T_x F_{g''})^\perp\), where \((T_x F_{g''})^\perp\) denotes the annihilator of \(T_x F_{g''}\) with respect to \(\omega\). Therefore \(\omega_x\) descends to a symplectic form on \(T_x(h^{-1}(c))/T_x F_{g''}\). It turns out that \(Y_x \in \ker(\omega|_{h^{-1}(c)})_x\) if and only if \(Y_x \in T_x F_{g''}\). Therefore \(\omega|_{h^{-1}(c)}\) is a transverse symplectic form with respect to \(F_{g''}\), proving the lemma.

Now we are in a position to prove the convexity theorem.

**Theorem 2.7.** Let \(M\) be a compact connected manifold equipped with an action of a compact torus \(G\). Let \(g'\) be a subspace of \(g\) such that \(g'\) acts on \(M\) locally freely. Let \(\omega\) be a \(G\)-invariant transverse symplectic form on \(M\) with respect to \(F_{g'}\). Let \(v_1, \ldots, v_k \in g\) and suppose that there exists a smooth function \(h_{v_i}\) such
that $dh_{vi} = -i_{X_{vi}} \omega$ for $i = 1, \ldots, k$. Put $h = (h_1, \ldots, h_k) : M \to \mathbb{R}^k$. Then the following hold:

(A) For $c \in \mathbb{R}^k$, the fiber $h^{-1}(c)$ is connected unless empty.
(B) $h(M)$ is convex.
(C) If $Z_1, \ldots, Z_N$ are the connected components of the set of common critical points of $h_{vi}$, then $h(Z_j)$ is a point $c_j$ and $h(M)$ is a convex hull of $c_1, \ldots, c_N$.

Proof. The proof consists of the following steps.

Step 1. Assume that (A) holds. Let $\pi : \mathbb{R}^{k+1} \to \mathbb{R}^k$ be any linear projection given by $\pi(e_i) = \sum_{j=1}^{k+1} a_{ij} e_j$ for $i = 1, \ldots, k+1$. The composition $h' := \pi \circ h : M \to \mathbb{R}^k$ satisfies the assumption of the theorem. Namely, the $j$-th component of $h'$ is a smooth function $\sum_{i=1}^{k+1} a_{ij} h_{vi}$, but

$$d \left( \sum_{i=1}^{k+1} a_{ij} h_{vi} \right) = -i_{X_{\sum_{i=1}^{k+1} a_{ij} v_i}} \omega.$$ 

Therefore each fiber of $h'$ is connected unless empty. Let $x, y \in h(M)$ and assume that $\pi$ is surjective and $\pi(x) = \pi(y) = c$. Since the fiber $\pi^{-1}(c)$ is a line in $\mathbb{R}^{k+1}$, it suffices to see that $h(M) \cap \pi^{-1}(c)$ is connected. Since $h' = \pi \circ h$, we have that $h(M) \cap \pi^{-1}(c) = h(h'^{-1}(c))$. Since $h$ is continuous and $h'^{-1}(c)$ is connected, $h(M) \cap \pi^{-1}(c)$ is connected, proving that (A) implies (B$_{k+1}$).

Step 2. It follows from Lemmas 2.2 and 2.5 that (A) holds. Assume that (A) holds. Let $v_1, \ldots, v_{k+1} \in \mathfrak{g}$ and assume that there exists a smooth function $h_{vi} : M \to \mathbb{R}$ such that $dh_{vi} = -i_{X_{vi}} \omega$ for $i = 1, \ldots, k+1$. Let $h = (h_{v_1}, \ldots, h_{v_{k+1}})$ and let $c = (c_1, \ldots, c_{k+1})$ be a point in $\mathbb{R}^{k+1}$. We want to show that $h^{-1}(c) = h_{v_1}^{-1}(c_1) \cap \cdots \cap h_{v_{k+1}}^{-1}(c_{k+1})$ is connected unless empty. If $h$ has no regular value, then one of $dh_{vi}$ is a linear combination of the others. By the assumption that (A) holds, we are done. Assume that $h$ has a regular value. Then, the set of regular values is dense in $h(M)$. By continuity, we only need to show that $h^{-1}(c)$ is connected for any regular value $c$. Then, $N := h_{v_1}^{-1}(c_1) \cap \cdots \cap h_{v_{k+1}}^{-1}(c_{k+1})$ is a connected submanifold by (A). Moreover, it follows from Lemma 2.4 that $\omega|_N$ is a transverse symplectic form on $N$ with respect to $\mathcal{F}_g^\ast$ on $N$, where $g'' = g + \mathbb{R} v_1 + \cdots + \mathbb{R} v_k$. The function $h_{v_{k+1}}|_N$ satisfies that $dh_{v_{k+1}}|_N = -i_{X_{v_{k+1}}} \omega|_N$. Therefore by Lemmas 2.2 and 2.5, $h_{v_{k+1}}|_N$ is connected. Therefore $h^{-1}(c) = h_{v_1}^{-1}(c_1) \cap \cdots \cap h_{v_{k+1}}^{-1}(c_{k+1})$ is connected, proving that (A) holds for all $k$.

Step 3. The former assertion that states that $h(Z_j)$ is a point $c_j$ is obvious. Let $H$ be the closure of $\exp(g'')$ in $G$. Let $x$ be a common critical point of $h_{v_1}, \ldots, h_{v_k}$. Since $(dh_{v_i})_x = (-i_{X_{vi}} \omega)_x = 0$, we have that $(X_{vi})_x \in T_x \mathcal{F}_g$ for $i = 1, \ldots, k$. Therefore there exist $v_i, v'_i \in h_x$ such that $v_i = v_i + v'_i$ for $i = 1, \ldots, k$. Let $H^0_x$ denote the identity component of $H_x$. Then, $\{\exp(t_1 v_{i_1}) \cdots \exp(t_k v_{k,x}) : t_i \in \mathbb{R}\}$ is dense in $H^0_x$ and $\exp(h_x + g')$ is dense in $H$. Conversely, for a subtorus $H'$ of $H$, if
Since \( h_v := \sum_{i=1}^{k} a_i h_v^i \). We claim that each critical point of \( h_v \) is a common critical point of \( h_{v_1}, \ldots, h_{v_k} \). Let \( x \) be a critical point of \( h_v \). Since \((dh_v)_x = (−t_x, ω)_x = 0\), there exists \( v_x \in \mathfrak{h}_x \) and \( v' \in \mathfrak{g}' \) such that \( v = v_x + v'\). Since \{exp(tv) \mid t \in \mathbb{R}\} is dense in \( H\), we have that \{exp(tv) \mid t \in \mathbb{R}\} is also dense in \( H_x^0\). By definition of \( v \) and \( v_x \), the closure of \( \exp(\mathfrak{h}'_x + \mathfrak{g}'_x) \) is \( H\). Therefore the critical point \( x \) of \( h_v \) is a common critical point of \( h_{v_1}, \ldots, h_{v_k} \). In particular, \( h_v \) takes the minimum value in a common critical point of \( h_{v_1}, \ldots, h_{v_k} \). It turns out that the linear form \( \alpha := \sum_{i=1}^{k} a_i \mathbf{e}_i^* \) restricted to \( h(M) \) takes the minimum value at one of \( c_j\)'s. Therefore

\[
\tag{2.2}
\quad h(M) \subset \bigcap_{(a_1, \ldots, a_k) \in A} \{y = (y_1, \ldots, y_k) \in \mathbb{R}^k \mid \langle \alpha, y \rangle \geq \min(\langle \alpha, c_j \rangle \mid j = 1, \ldots, N)\},
\]

where

\[
A := \{(a_1, \ldots, a_k) \mid \{\exp(tv) \mid t \in \mathbb{R}\} \text{ is dense in } H\}.
\]

Since \( A \) is dense in \( \mathbb{R}^k \), the right hand side of \((2.2)\) is the convex hull of \( c_j\)'s. It follows from \((B_h)\) and \( c_j \in h(M) \) for all \( j \) that \( h(M) \) is the convex full of \( c_j\)'s, proving the theorem. \( \square \)

\section{Holomorphic foliations from torus actions}

Let \( M \) be a complex manifold and let \( G \) be a compact torus acting on \( M \) as holomorphic transformations. In this section, we define a subspace \( \mathfrak{g}_J \) of \( \mathfrak{g} \) which acts on \( M \) locally freely by using the complex structure \( J \) on \( M \) and the action of \( G \). We begin with the following lemmas.

\textbf{Lemma 3.1.} Let \( M \) be a complex manifold equipped with an action of a compact torus \( G \) which acts by holomorphic transformations. For \( x \in M \), there exist \( G_x\)-invariant open neighborhoods \( U \) at \( x \in M \) and \( V \) at \( 0 \in T_x M \) such that \( U \) and \( V \) are \( G_x\)-equivariantly biholomorphic.

\textbf{Proof.} Let \( U_0 \) be an open neighborhood at \( x \) and let \( \varphi : U_0 \to V_0 \) be a local holomorphic coordinate centered at \( x \), where \( V_0 \) is an open subset of \( \mathbb{C}^n \). Since \( G_x \) is compact, the intersection \( \bigcap_{g \in G_x} g(U_0) \) is a \( G_x\)-invariant open neighborhood at \( x \). By restricting the domain of definition, we may assume that \( U_0 \) is \( G_x\)-invariant. Through the differential \((d\varphi)_x : T_x M \to T_0 \mathbb{C}^n = \mathbb{C}^n\), we identify \( \mathbb{C}^n \) with \( T_x M \). Then we have a biholomorphism \((d\varphi)_x^{-1} \circ \varphi : U_0 \to (d\varphi)_x^{-1}(V_0) \subset T_x M \). By averaging on \( G_x \), we have a \( G_x\)-equivariant holomorphic map

\[
\varphi' := \int_{g \in G_x} (dg)_x \circ ((d\varphi)_x^{-1} \circ \varphi) \circ g^{-1}dg : U_0 \to T_x M.
\]

\( \varphi' \) is no longer injective, but \((d\varphi)_x = \text{id}_{T_x M\} \). Therefore, it follows from the implicit function theorem that there exists an open subset \( U \) of \( U_0 \) such that \( \varphi'|_U : U \to \varphi'(U) \) is biholomorphic. As before, we may assume that \( U \) is \( G_x\)-invariant and then \( V := \varphi'(U) \) is also \( G_x\)-invariant. Therefore there exists a \( G_x\)-equivariant biholomorphism \( \varphi' : U \to V \subset T_x M \), proving the lemma. \( \square \)
Lemma 3.2. Let $M$ be a connected complex manifold with the complex structure $J$. Let $X$ be a nonzero almost periodic vector field on $M$ whose flows preserve $J$. Assume that $JX$ is complete. If $X$ vanishes at a point $x \in M$, then $JX$ is not almost periodic.

Proof. Assume that $JX$ is almost periodic. Since $X$ is an infinitesimal automorphism of $J$, $[X, JX] = 0$. Therefore we may assume that a compact torus $G$ acts on $M$ effectively and as holomorphic transformations and there exist $v, v' \in \mathfrak{g}$ such that $X = X_v, JX = X_{v'}$ and the subgroup $\{ \exp(sv) \exp(tv') \mid s, t \in \mathbb{R} \}$ is dense in $G$. Since $X_x = (JX)_x = 0$, $x$ is a $G$-fixed point. Let $\alpha_1, \ldots, \alpha_n \in \text{Hom}(G, S^1)$ be the weights of the $G$-representation $T_x M$. By Lemma 3.1 there exists an equivariant biholomorphic map $U \to V \subseteq T_x M$, where $U$ is an open neighborhood at $x$. Combining with the decomposition of $T_x M$ into 1-dimensional representations of weights $\alpha_1, \ldots, \alpha_n$, we have a local coordinate $(z_1, \ldots, z_n) : U \to \mathbb{C}^n$ such that $z_i(g \cdot p) = \alpha_i(g) z_i(p)$ for $p \in U$. Let $x_i$ and $y_i$ denote the real and imaginary parts of $z_i$, respectively. Then, through the local coordinate $(z_1, \ldots, z_n)$ we can represent $X$ and $JX$ as

$$X = 2\pi \sum_{i=1}^{n} \langle d\alpha_i, v \rangle \left( -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i} \right)$$

and

$$JX = 2\pi \sum_{i=1}^{n} \langle d\alpha_i, v' \rangle \left( -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i} \right).$$

On the other hand, $J$ is represented as

$$J = \sum_{i=1}^{n} \left( \frac{\partial}{\partial y_i} \otimes dx_i - \frac{\partial}{\partial x_i} \otimes dy_i \right).$$

Therefore

$$0 = X + J^2 X = 2\pi \sum_{i=1}^{n} \left( \langle d\alpha_i, v \rangle \left( -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i} \right) + \langle d\alpha_i, v' \rangle \left( -x_i \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_i} \right) \right)$$

$$= 2\pi \sum_{i=1}^{n} \left( \langle d\alpha_i, -y_i v - x_i v' \rangle \frac{\partial}{\partial x_i} + \langle d\alpha_i, x_i v - y_i v' \rangle \frac{\partial}{\partial y_i} \right).$$

Therefore, by substituting $\epsilon, 0 < |\epsilon| \ll 1$ for $x_i$ and $y_i$, we have that

$$0 = \langle d\alpha_i, -\epsilon v - \epsilon v' \rangle = -\epsilon \langle d\alpha_i, v + v' \rangle$$

and

$$0 = \langle d\alpha_i, \epsilon v - \epsilon v' \rangle = \epsilon \langle d\alpha_i, v - v' \rangle$$

for all $i = 1, \ldots, n$. Thus we have $\langle d\alpha_i, v \rangle = 0$ for all $i = 1, \ldots, n$. Since the action of $G$ on $M$ is effective and $M$ is connected, $d\alpha_i \in \mathfrak{g}^*$ for $i = 1, \ldots, n$ spans $\mathfrak{g}^*$. This together with the fact that $\langle d\alpha_i, v \rangle = 0$ for all $i$ shows that $v = 0$. This contradicts the assumption that $X = X_v$ is nonzero and hence $JX$ is not almost periodic, as required. 

\[\square\]
Proposition 3.3. Let $M$ be a connected complex manifold with the complex structure $J$. Let $G$ be a compact torus acting on $M$ effectively and as holomorphic transformations. Define

$$g_J := \{ v \in g \mid \text{there exists } v' \in g \text{ such that } X_v = -JX_{v'} \}.$$ 

Then:

1. $g_J$ is a Lie subalgebra of $g$.
2. $g_J$ has the complex structure $J_0$ which satisfies $X_{J_0(v)} = JX_v$.
3. $g_J$ acts on $M$ holomorphically and locally freely.

Proof. Part (1) follows from the fact that $G$ is commutative. For part (2), let $v \in g_J$. Assume that $v', v'' \in g_J$ satisfy that $X_v = -JX_{v'} = -JX_{v''}$. Then, $X_{v'} = X_{v''}$. It follows from the effectiveness of the $G$-action that $v' = v''$. Therefore for $v \in g_J$, there exists unique $J_0(v) \in g_J$ such that $X_{J_0(v)} = JX_v$. The map $J_0 : g_J \to g_J$ is linear and $J_0^2 = -1$, proving part (2). Part (3) follows from part (2) and Lemma 3.2. The proposition is proved.

Let $M$ be a connected complex manifold with the complex structure $J$ and let a compact torus $G$ act on $M$ effectively and as holomorphic transformations. By Proposition 3.3, $g_J$ acts on $M$ holomorphically and locally freely. Therefore we have a holomorphic foliation $F_{g_J}$ whose leaves are $g_J$-orbits.

4. Torus invariant transverse Kähler foliations

A transverse Kähler form is a special kind of transverse symplectic form. Let $M$ be a complex manifold with the complex structure $J$. Let $F$ be a holomorphic foliation on $M$. A real 2-form $\omega$ on $M$ is called transverse Kähler with respect to $F$ if the following conditions are satisfied:

1. $\omega$ is transverse symplectic with respect to $F$.
2. $\omega$ is of type $(1,1)$. Namely, for $Y_x, Z_x \in T_xM$, $\omega_x(JY_x, JZ_x) = \omega_x(Y_x, Z_x)$.
3. $\omega$ is positive. Namely, $\omega_x(Y_x, JY_x) \geq 0$ for all $Y_x \in T_xM$.

The conditions (1) and (3) imply that $\omega_x(Y_x, JY_x) = 0$ if and only if $Y_x \in T_xF$. For a holomorphic foliation $F$ on $M$, if a transverse Kähler form $\omega$ exists, we say that $F$ is transverse Kähler.

Proposition 4.1. Let $M$ be a complex manifold with the complex structure $J$. Let $G$ be a compact torus acting on $M$ as holomorphic transformations. Let $F$ be a $G$-invariant foliation and let $\omega$ be a transverse Kähler form with respect to $F$. Then,

$$\int_{g \in G} g^* \omega dg$$

is a transverse Kähler form with respect to $F$ and invariant under the $G$-action on $M$.

Proof. For short, denote

$$\omega' = \int_{g \in G} g^* \omega dg.$$

Since $\omega$ is closed, so is $\omega'$. Since $G$ acts on $M$ preserving the complex structure $J$, $\omega'$ is a positive $(1,1)$-form. It remains to show that $\ker \omega'_x = T_xF$ for all $x \in M$. 

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By definition, for $Y_x \in T_x M$,
\[
\omega'_x(Y_x, JY_x) = \int_{g \in G} (g^* \omega)_x(Y_x, JY_x) dg \\
= \int_{g \in G} \omega_{g \cdot x}((dg)_x(Y_x), (dg)(JY_x)) dg \\
= \int_{g \in G} \omega_{g \cdot x}((dg)_x(Y_x), J((dg)_x(Y_x))) dg
\]
because $J$ is $G$-invariant. Since $\omega_{g \cdot x}((dg)_x(Y_x), J((dg)_x(Y_x))) \geq 0$ and the equality holds if and only if $(dg)_x(Y_x) \in T_{g \cdot x} F$, it follows from the $G$-invariance of $F$ that $\omega'_x(Y_x, JY_x) = 0$ if and only if $Y_x \in T_x F$, proving the proposition.

Thanks to Proposition 4.1 if a $G$-invariant foliation $F$ is transverse Kähler, we may always assume that the transverse Kähler form with respect to $F$ is $G$-invariant without loss of generality.

For foliations $F_1$ and $F_2$ on a smooth manifold $M$, we denote by $F_1 \subseteq F_2$ if $TF_1 \subseteq TF_2$. Our next purpose is to give a lower bound of $G$-invariant transverse Kähler foliations that admit moment maps.

**Proposition 4.2.** Let $M$ be a connected complex manifold with the complex structure $J$. Let a compact torus $G$ act on $M$ effectively and as holomorphic transformations. Let $F$ be a $G$-invariant holomorphic foliation and let $\omega$ be a $G$-invariant transverse Kähler form with respect to $F$. If there exists a moment map with respect to $\omega$, then $F_{\mathfrak{g}J} \subseteq F$.

**Proof.** Let $\Phi : M \to \mathfrak{g}^*$ be a moment map. We denote by $h_v$ the smooth function given by $h_v(x) = \langle \Phi(x), v \rangle$ for $v \in \mathfrak{g}$ and $x \in M$. Since $h_v$ is $G$-invariant for $v \in \mathfrak{g}$, we have that
\[
0 = L_{X_{v_1}} h_{v_2} = \iota_{X_{v_1}} dh_{v_2} = -\iota_{X_{v_1}} \iota_{X_{v_2}} \omega = 2\omega(X_{v_1}, X_{v_2})
\]
for any $v_1, v_2 \in \mathfrak{g}$. Assume that $v \in \mathfrak{g}_J$. Then,
\[
0 \leq \omega(X_v, JX_v) = \omega(X_v, X_{J_0(v)}) = 0
\]
by (4.1). Therefore $(X_v)_x \in T_x F$ for all $x$ and hence $F_{\mathfrak{g}J} \subseteq F$, as required.

**Theorem 4.3.** Let $M, J, G, F, \omega$ be as in Proposition 4.2. Let $q : \mathfrak{g} \to (\mathfrak{g} / \mathfrak{g}_J)^*$ be the quotient map. Assume that there exists a moment map $\Phi : M \to \mathfrak{g}^*$ with respect to $\omega$. Then, there exist $c \in \mathfrak{g}^*$ and a smooth map $\overline{\Phi} : M \to (\mathfrak{g} / \mathfrak{g}_J)^*$ such that $\Phi + c = q^* \circ \overline{\Phi}$.

**Proof.** For $v \in \mathfrak{g}$, $h_v$ denotes the smooth function given by $h_v(x) = \langle \Phi(x), v \rangle$. It follows from Proposition 4.2 that $dh_v = -\iota_{X_v} \omega = 0$ for $v \in \mathfrak{g}_J$. It turns out that $h_v$ is constant on $M$. Let $i : \mathfrak{g}_J \to \mathfrak{g}$ denote the inclusion. Then, there exists $\overline{c} \in \mathfrak{g}_J^*$ such that $i^* \circ \Phi(x) = \overline{c}$ for any $x \in M$. The sequences
\[
0 \longrightarrow \mathfrak{g}_J \stackrel{i}{\longrightarrow} \mathfrak{g} \stackrel{q}{\longrightarrow} (\mathfrak{g} / \mathfrak{g}_J)^* \longrightarrow 0
\]
and
\[
0 \longleftarrow \mathfrak{g}_J^* \stackrel{i^*}{\longleftarrow} \mathfrak{g}^* \stackrel{q^*}{\longleftarrow} ((\mathfrak{g} / \mathfrak{g}_J)^*)^* \longleftarrow 0
\]
are exact. Since $i^*$ is surjective, there exists $c \in \mathfrak{g}^*$ such that $i^*(c) = \overline{c}$. In particular, $i^*(\Phi(x) - c) = 0$ for all $x \in M$. Therefore, there uniquely exists $\overline{\Phi}(x) \in (\mathfrak{g} / \mathfrak{g}_J)^*$
such that \( q^*(\tilde{\Phi}(x)) = \Phi(x) - c \) for all \( x \in M \). The smoothness is obvious. The theorem is proved. \( \square \)

We call \( \tilde{\Phi} : M \to (\mathfrak{g}/\mathfrak{g}_J)^* \) an induced moment map.

**Remark 4.4.**

1. Assume that \( \mathfrak{g}_J \) is a Lie algebra of a subtorus (that is, closed) in \( G \). Let \( G_J \) be the subtorus of \( G \) corresponding to \( \mathfrak{g}_J \). Then the leaf space \( M/F_{\mathfrak{g}_J} \), is an orbifold \( M/G_J \) equipped with an action of the compact torus \( G/G_J \). A transverse Kähler form on \( M \) with respect to \( F_{\mathfrak{g}_J} \) descends to a Kähler form on \( M/G_J \). Conversely, the pullback of a Kähler form on \( M/G_J \) is a transverse Kähler form on \( M \). An induced moment map \( \Phi : M \to (\mathfrak{g}/\mathfrak{g}_J)^* \) is a composition of the quotient map \( M \to M/G_J \) and a moment map \( \overline{\Phi} : M/G_J \to (\mathfrak{g}/\mathfrak{g}_J)^* \).

2. In the case when \( M \) is an LVM manifold satisfying the condition (K) with a specific transverse Kähler form which is the so-called canonical Euler form on \( M \), it has been shown that a moment map \( \Phi : M \to \mathfrak{g}^* \) descends to a moment map \( \overline{\Phi} : M/G_J \to (\mathfrak{g}/\mathfrak{g}_J)^* \). Moreover the images of moment maps are described; see [13, Theorem E, (ii)]. The context of Theorem 4.3 is more general than the LVM case, but it says nothing about the image of a moment map. In the next section, we will see the image of an LVMB manifold \( M \) by an induced moment map, without the assumption that \( M \) satisfies the condition (K).

As a corollary of Theorems 2.7 and 4.3 we have the following.

**Corollary 4.5.** Let \( M \) be a compact connected complex manifold. Let a compact torus \( G \) act on \( M \) effectively and preserving the complex structure \( J \) on \( M \). Assume that \( F_{\mathfrak{g}_J} \) is transverse Kähler and there exists a moment map \( \Phi : M \to \mathfrak{g}^* \) with respect to a \( G \)-invariant transverse Kähler form. Then, the image of \( M \) by an induced moment map \( \Phi : M \to (\mathfrak{g}/\mathfrak{g}_J)^* \) is a convex polytope in \( (\mathfrak{g}/\mathfrak{g}_J)^* \).

### 5. The extreme case

In this section, we consider the extreme case. First we recall the notion of maximal torus action introduced in [8]. Let \( M \) be a connected smooth manifold equipped with an effective action of a compact torus \( G \). Then, for any point \( x \), we have that \( \dim G_x + \dim G \leq \dim M \). The \( G \)-action on \( M \) is maximal if there exists a point \( x \in M \) such that

\[
\dim G + \dim G_x = \dim M.
\]

Any compact connected complex manifold \( M \) equipped with a maximal action of a compact torus \( G \) which preserves the complex structure can be described with a fan \( \Delta \) in \( \mathfrak{g} \) and a complex subspace \( \mathfrak{h} \) of \( \mathfrak{g}^C \).

**Theorem 5.1** (See [8]). Let \( M \) be a compact connected complex manifold \( M \) equipped with a maximal action of a compact torus \( G \) which preserves the complex structure \( J \). Then, there exists a nonsingular fan \( \Delta \) in \( \mathfrak{g} \) and a complex subspace \( \mathfrak{h} \) such that \( M \) is \( G \)-equivariantly biholomorphic to \( X(\Delta)/H \), where \( X(\Delta) \) denotes the toric variety associated with \( \Delta \) and \( H := \exp(\mathfrak{h}) \subseteq G^C \subset X(\Delta) \).

We shall recall how to deduce \( \Delta \) and \( \mathfrak{h} \) from \( M \) briefly. Each connected component of the set of fixed points of a circle subgroup of \( G \) is a closed complex submanifold of \( M \). If such a submanifold has complex codimension one, then we call
it a characteristic submanifold of $M$. The number of characteristic submanifolds is at most finite. Let $N_1, \ldots, N_k$ be characteristic submanifolds of $M$. Each characteristic submanifold $N_i$ is fixed by a circle subgroup $G_i$ of $G$ by definition. To each characteristic submanifold $N_i$, we assign a group isomorphism $\lambda_i : S^1 \to G_i \subseteq G$ such that

$$(\lambda_i(g))_* (\xi) = g\xi \quad \text{for all } g \in S^1 \text{ and } \xi \in TM|_{N_i}/TN_i.$$ 

We can think of $\lambda \in \text{Hom}(S^1, G)$ as a vector in $\mathfrak{g}$ by $d\lambda(1) \in \mathfrak{g}$. We have a collection $\Delta$ of cones

$$\Delta := \left\{ \text{pos}(\lambda_i \mid i \in I) \mid \bigcap_{i \in I} N_i \neq \emptyset \right\},$$

where $\text{pos}(\lambda_i \mid i \in I)$ is the cone spanned by $\lambda_i$ for $i \in I$. It has been shown that $\Delta$ is a nonsingular fan in $\mathfrak{g}$ with respect to the lattice $\text{Hom}(S^1, G)$. Since the action of $G$ preserves the complex structure $J$ on $M$, it extends to a holomorphic action of $G^\mathbb{C}$ on $M$. Then the complex subspace $\mathfrak{h}$ of $\mathfrak{g}^\mathbb{C} = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} \otimes 1 + \mathfrak{g} \otimes \sqrt{-1}$ is defined to be the Lie algebra of global stabilizers of the $G^\mathbb{C}$-action on $M$. Namely,

$$\mathfrak{h} = \{ \upsilon \otimes 1 + v \otimes \sqrt{-1} \in \mathfrak{g}^\mathbb{C} \mid X_u + JX_v = 0 \}.$$ 

The pair of $\Delta$ and $\mathfrak{h}$ satisfies the following.

1. The restriction $p|_{\mathfrak{h}}$ of the projection $p : \mathfrak{g}^\mathbb{C} \to \mathfrak{g} \otimes 1 \cong \mathfrak{g}$ is injective.
2. The quotient map $q : \mathfrak{g} \to \mathfrak{g}/p(\mathfrak{h})$ sends $\Delta$ to a complete fan $q(\Delta)$ in $\mathfrak{g}/p(\mathfrak{h})$.

Conversely, if $\Delta$ and $\mathfrak{h}$ satisfy the conditions (1) and (2), then the quotient $X(\Delta)/H$ is a compact connected complex manifold and the action of $G$ on $X(\Delta)$ descends to a maximal action on $X(\Delta)/H$.

**Proposition 5.2.** Let $M$ be a compact connected complex manifold $M$ equipped with an action of a compact torus $G$ which preserves the complex structure $J$. Let $\mathfrak{h}$ be the Lie algebra of global stabilizers of the $G^\mathbb{C}$-action on $M$. Then, $p(\mathfrak{h}) = \mathfrak{g}_J$.

**Proof.** This follows from the definitions of $\mathfrak{h}$ and $\mathfrak{g}_J$ immediately. $\square$

**Lemma 5.3.** Let $\Delta, \mathfrak{h}, q$ be as above and let $J$ denote the complex structure on $X(\Delta)/H$. Assume that $F^\mathbb{R}$ is a transverse Kähler foliation on $X(\Delta)/H$ and let $\omega$ be a $G$-invariant transverse Kähler form with respect to $F^\mathbb{R}$. In addition, assume that there exists a moment map $\Phi : X(\Delta)/H \to \mathfrak{g}^\mathbb{R}$ with respect to $\omega$. Then, the image of $X(\Delta)/H$ by an induced moment map $\tilde{\Phi}$ is a convex polytope and $\tilde{\Phi}(X(\Delta)/H)$ is a normal polytope of $q(\Delta)$.

Before the proof of Lemma 5.3, we shall recall notions of normal fan and normal polytope. Let $P$ be an $n$-dimensional polytope in a vector space $V^*$ of dimension $n$. For a vector $v \in V$, we put

$$F_v := \{ \alpha \in P \mid \langle \alpha, v \rangle \leq \langle \alpha', v \rangle \text{ for all } \alpha' \in P \}.$$ 

Then $F_v$ is a face of $P$, and it is the locus where $\langle \dot{\bullet}, v \rangle|_P$ attains its minimum. For a face $F$ of $P$, the (inner) normal cone $\sigma_F$ of $F$ is given by

$$\sigma_F := \{ v \in V \mid F_v \subseteq F \}.$$ 

Its relative interior is given by $\{ v \in V \mid F_v = F \}$. The (inner) normal fan of $P$ is the collection $\Delta_P := \{ \sigma_F \}_F$ of cones $\sigma_F$ for the faces $F$ of $P$. Conversely, for given fan $\Delta$ in $V$, a polytope $P$ in $V^*$ whose (inner) normal fan coincides with $\Delta$
is called an \textit{(inner) normal polytope} of $\Delta$. If such a polytope $P$ exists for $\Delta$, then $\Delta$ is said to be \textit{polytopal}.

Now assume that $\Delta$ is a nonsingular fan in $\mathfrak{g}$. We also prepare some notation of submanifolds. For a cone $\sigma \in \Delta$, we denote by $X_\sigma$ the closed toric subvariety of $X(\Delta)$ corresponding to $\sigma$. If $\lambda_1, \ldots, \lambda_k \in \text{Hom}(S^1, G)$ are the primitive generators of 1-cones of $\Delta$, then $\sigma$ can be written as $\sigma = \text{pos}(\lambda_i \mid i \in I)$ for some $I \subseteq \{1, \ldots, k\}$ and $(\lambda_i)_{i \in I}$ is a part of the $\mathbb{Z}$-basis of $\text{Hom}(S^1, G)$. More precisely, each point of $X_\sigma$ is fixed by a subtorus $G_\sigma$ of $G$, and $(\lambda_i)_{i \in I}$ is a $\mathbb{Z}$-basis of $\text{Hom}(S^1, G_\sigma)$. The image $Y_\sigma$ of $X_\sigma$ by the quotient map $X(\Delta) \to X(\Delta)/H$ is a closed submanifold of $X(\Delta)/H$ because $X_\sigma$ and $Y_\sigma$ are both connected components of the set of fixed points by the $G_\sigma$-actions. If we denote by $(\alpha_i^1)_{i \in I}$ the dual basis of $(\lambda_i)_{i \in I}$, the set of nonzero weights of $T_{\tau}Y_\sigma$ coincides with $(\alpha_i^1)_{i \in I}$ for all $x \in Y_\sigma$.

\textbf{Proof of Lemma 5.3} We may assume that $q^* \circ \Phi = \Phi$ without loss of generality. For $v \in \mathfrak{g}$, define $h_v : \tilde{X}(\Delta)/H \to \mathfrak{g}^*$ by $h_v(x) = \langle \Phi(x), v \rangle$. Then, $h_v(x) = \langle \Phi(x), q(v) \rangle$. We shall see that each connected component of the set of critical points of $h_v$ is one of $Y_\sigma$ for some $\sigma \in \Delta$. Let $x \in \tilde{X}(\Delta)/H$. Since $dh_v = -\iota_{X_\sigma} \omega$, $x$ is a critical point of $h_v$ if and only if $(X_v)_x \in T_x F_{\sigma_j}$, in particular, $v \in \mathfrak{g}_x + \mathfrak{g}_J$. Therefore, the set of critical points is $\bigcup_{\sigma; v \in \mathfrak{g}_\sigma + \mathfrak{g}_J} Y_\sigma$.

Assume that $h_v$ takes the minimum value $a_v$ on $Y_\sigma$. Let $v_\sigma \in \mathfrak{g}_\sigma$ such that $q(v) = q(v_\sigma)$. Then, $\langle \alpha_i^1, v_\sigma \rangle > 0$ for all $i \in I$, where $(\lambda_i)_{i \in I}$ is the set of primitive generators of $\sigma$ (see Remark 2.3). Therefore $v_\sigma$ sits in the relative interior of $\sigma$. In particular, $q(v)$ sits in the relative interior of $q(\sigma)$. The converse is also true: if $q(v)$ sits in the relative interior of $q(\sigma)$, then $h_v$ takes the minimum value $a_v$ on $Y_\sigma$.

By Corollary 4.5, $\Phi(X(\Delta)/H)$ is a convex polytope $P$ in $(\mathfrak{g}/\mathfrak{g}_J)^*$. We claim that, for each $\sigma$, the image of $Y_\sigma$ by $\Phi$ is a face of $P$. Let $v \in \mathfrak{g}$ such that $q(v)$ sits in the relative interior of $\sigma$. Then, $h_v$ attains the minimum value $a_v$ on $Y_\sigma$. But $h_v(x) = \langle \Phi(x), q(v) \rangle$ implies that $h_v$ attains its minimum value $a_v$ at $x$ if and only if $q(v)|_P$ attains its minimum value $a_v$ at $\Phi(x)$. Since $h_v^{-1}(a_v) = Y_\sigma$, we have that $\langle q(v)|_P \rangle^{-1}(a_v) = \Phi(Y_\sigma)$. Since $(q(v)|_P)(\alpha) \geq a_v$ for all $\alpha \in P$, $\Phi(Y_\sigma)$ is a face of $P$ that is given by $P \cap H_{q(v), a_v}$, where $H_{q(v), a_v}$ is the hyperplane in $(\mathfrak{g}/\mathfrak{g}_J)^*$ defined as $H_{q(v), a_v} := \{ \alpha \in (\mathfrak{g}/\mathfrak{g}_J)^* \mid \langle \alpha, q(v) \rangle = a_v \}$.

Conversely, if a face $F$ of $P$ is given by $P \cap H_{q(v), a_v}$, then $F$ is the image of $Y_\sigma$ by $\Phi$, where $\sigma$ is the cone such that $q(v)$ sits in the relative interior of $q(\sigma)$. It turns out that for each face $F$ of $P$ there exists a cone $\sigma$ such that the inner normal cone of $F$ coincides with $q(\sigma)$. Hence $P$ is a normal polytope of $q(\Delta)$, as required.

Now we consider the obstruction for the existence of a moment map in the case of an LVMB manifold with at least one indispensable index. Let $\Sigma$ be an abstract simplicial complex on $\{0, 1, \ldots, m\}$ (a singleton $\{i\}$ does not need to be a member of $\Sigma$). If an index $i$ satisfies that $\{i\} \notin \Sigma$, we say that $i$ is \textit{indispensable}, according to the literature of LVMB manifolds (see [4], [12] and [13]). Let $G = (S^1)^m$. Then $\mathfrak{g} = \mathbb{R}^m$ and $\mathbb{Z}^m$ is identified with $\text{Hom}(S^1, G)$. $G$ acts on $\mathbb{C}P^m$ via $(g_1, \ldots, g_m) \cdot [z_0, z_1, \ldots, z_m] := [z_0 g_1 z_1, \ldots, g_m z_m]$ for $(g_1, \ldots, g_m) \in G$ and $[z_0, z_1, \ldots, z_m] \in \mathbb{C}P^m$. Put $e_0 := -e_1 - \cdots - e_m$ and define $\Delta := \{ \text{pos}(e_i \mid i \in I) \mid I \in \Sigma \}$. 

\[ \Delta \]
Remark 5.4. By comparing with [11] Section 3, we can see that if 0 is an indispensable index, then the LVMB manifold \( X(\Delta) / H \) is equivariantly biholomorphic to a moment-angle manifold with a complex structure constructed by Panov and Ustinovsky.

In the case when there is an indispensable index, we call \( X(\Delta) / H \) an LVMB manifold with an indispensable index.

It has been shown in [2] that the odd degree basic cohomology groups of \( X(\Delta) / H \) with respect to \( F_{\mathfrak{g}, J} \) vanish for shellable \( \Sigma \). Therefore for shellable \( \Sigma \), there exists a moment map for any transverse Kähler form on \( X(\Delta) / H \) with respect to \( F_{\mathfrak{g}, J} \). We can avoid the assumption on \( \Sigma \) for the vanishing of first basic cohomology groups with a straightforward computation.

Lemma 5.5. Let \( X(\Delta) / H \) be an LVMB manifold with at least one indispensable index. Let \( J \) be the complex structure on \( X(\Delta) / H \). Then, \( H^{1}_{F_{\mathfrak{g}, J}}(X(\Delta) / H) = 0 \).

Proof. By renumbering, we may assume that \( \{i\} \) is a member of \( \Sigma \) for \( i = 1, \ldots, r \) but not for \( i = 0, r + 1, r + 2, \ldots, m \) (\( r \) could be \( m \)). Then, \( X(\Delta) = X(\Delta') \times (\mathbb{C} \setminus \{0\})^{m-r} \), where \( \Delta' \) is an abstract simplicial complex on \( \{0, 1, \ldots, r\} \) such that if \( I \in \Sigma \), then \( I \in \Sigma' \). \( X(\Delta') \) is a complement of coordinate subspaces of real codimension \( \geq 4 \) in \( \mathbb{C}^{r} \). Thus, \( X(\Delta') \) is simply connected. Let \( c_{i} : S^{1} \to X(\Delta) \) be the curve defined by

\[
c_{i}(t) = (1, \ldots, 1, t, 1, \ldots, 1) \in X(\Delta) \subseteq \mathbb{C}^{m}
\]

for \( i = 1, \ldots, m \). \( c_{i} \) is null-homologous for \( i = 1, \ldots, r \) and the homology classes \( [c_{i}] \) determined by \( c_{i} \) for \( i = r + 1, \ldots, m \) form a basis of \( H_{1}(X(\Delta)) \).

Let \( \beta \) be a 1-form on \( X(\Delta) / H \). \( \beta \) is closed and basic for \( F_{\mathfrak{g}, J} \) if and only if \( d\beta = 0 \) and \( \iota_{X_{v}}\beta = 0 \) for any \( v \in \mathfrak{g}_{J} \). Let \( \pi : X(\Delta) \to X(\Delta) / H \) be the quotient map. \( \pi^{*}\beta \) is a 1-form basic for \( F_{\mathfrak{h}} \). That is, \( \iota_{X_{v}}\pi^{*}\beta = 0 \) and \( \iota_{X_{v}}d\pi^{*}\beta = 0 \) for
\( u \in \mathfrak{h} \). Therefore we need to show that a closed 1-form \( \gamma \) on \( X(\Delta_S) \) satisfying

- \( \iota_X u \gamma = 0 \) for \( u \in \mathfrak{h} \),
- \( \iota_X v \gamma = 0 \) for \( v \in \mathfrak{g}_J \)

is exact. Let \( \gamma \) be such a 1-form on \( X(\Delta_S) \). By averaging \( \gamma \) with the action of \( G \), we may assume that \( \gamma \) is \( G \)-invariant without loss of generality. Since \( \gamma \) is real, we can represent

\[
\gamma = \sum_{i=1}^m f_i dz_i + \bar{f}_i d\bar{z}_i
\]

with smooth functions \( f_i : X(\Delta_S) \to \mathbb{C} \). Let \( v = (v_1, \ldots, v_m) \in \mathfrak{g} = \mathbb{R}^m \). Then \( X_v \) can be represented as

\[
X_v = 2\pi \sum_{i=1}^m \sqrt{-1} v_i \left( z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right).
\]

Since \( \gamma \) is \( G \)-invariant, we have that there exists \( a_i \in \mathbb{R} \) such that

\[
2\pi \sqrt{-1} (z_i f_i - \bar{z}_i \bar{f}_i) = a_i \quad \text{for } i = 1, \ldots, m.
\]

Since \( H_1(X(\Delta_S)) \) is generated by \( c_i \) for \( i = r+1, \ldots, m \) and the Kronecker pairing is given by \( \langle [c_i], [\gamma] \rangle = a_i \), it suffices to show that \( a_i = 0 \) for \( i = r+1, \ldots, m \).

If \( u = (u_{1,1} + \sqrt{-1} u_{1,\bar{1}}, \ldots, u_{m,1} + \sqrt{-1} u_{m,\bar{1}}) \in \mathbb{C}^m \), \( X_u \) can be represented as

\[
X_u = 2\pi \sum_{i=1}^m \left( \sqrt{-1} u_{i,1} \left( z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right) - u_{i,\bar{1}} \left( z_i \frac{\partial}{\partial z_i} + \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right) \right).
\]

Since \( p(\mathfrak{h}) = \mathfrak{g}_J \) by Proposition 5.2, it follows from (5.1) and (5.2) that the conditions \( \iota_X u \gamma = 0 \) for \( v \in \mathfrak{g}_J \) and \( \iota_X v \gamma = 0 \) for \( u \in \mathfrak{h} \) are equivalent to

\[
\sum_{i=1}^m v_i z_i f_i = 0 \quad \text{for all } v = (v_1, \ldots, v_m) \in \mathfrak{g}_J.
\]

Assume that \( \{1, \ldots, n\} \in \Sigma \) is a maximal simplex. Then, \( q(e_1), \ldots, q(e_n) \) form a basis of \( \mathfrak{g}/\mathfrak{g}_J \). Let \( \alpha_1, \ldots, \alpha_n \) be the dual basis of \( q(e_1), \ldots, q(e_n) \). Then, we have a basis

\[
e_j - \sum_{i=1}^n \langle \alpha_i, q(e_j) \rangle e_i \quad \text{for } j = n+1, \ldots, m
\]

of \( \mathfrak{g}_J = \ker q \). Therefore we have that (5.3) is equivalent to

\[
z_j f_j - \sum_{i=1}^n \langle \alpha_i, q(e_j) \rangle z_i f_i = 0 \quad \text{for } j = n+1, \ldots, m.
\]

The Kronecker pairing \( \langle [c_i], [\gamma] \rangle = 0 \) for \( i = 1, \ldots, r \) because \( c_i \) for \( i = 1, \ldots, r \) is null-homologous. Therefore \( a_i = 2\pi \sqrt{-1} (z_i f_i - \bar{z}_i \bar{f}_i) = 0 \) for \( i = 1, \ldots, n \). This together with (5.4) yields that \( a_j = 0 \) for all \( j = n+1, \ldots, m \). Therefore \( \gamma \) is exact, proving the lemma.

**Corollary 5.6.** Let \( X(\Delta_S)/H \) be an LVMB manifold with at least one indispens-able index. Assume that \( F_{\mathfrak{g}_J} \) on \( X(\Delta_S)/H \) is transverse Kähler. Then, the complete fan \( q(\Delta) \) in \( \mathfrak{g}/\mathfrak{g}_J \) is polytopal. Namely, \( X(\Delta_S)/H \) is an LVM manifold.
Proof. Let \( \omega \) be a transverse Kähler form on \( X(\Delta_S)/H \) with respect to \( F_{g_J} \). Since \( F_{g_J} \) is \( G \)-invariant, we may assume that \( \omega \) is \( G \)-invariant by Proposition 4.1. The closed 1-form \( -\iota_X \omega \) is exact for all \( v \in \mathfrak{g} \) by Lemma 5.3. Therefore there exists a moment map on \( X(\Delta_S)/H \) with respect to \( \omega \). Let \( \Phi : X(\Delta_S)/H \to (\mathfrak{g}/\mathfrak{g}_J)^* \) be an induced moment map. By Lemma 5.3 the image of \( X(\Delta_S)/H \) by \( \Phi \) is a normal polytope of \( q(\Delta) \). Therefore \( q(\Delta) \) is polytopal, as required.

Conversely, we can construct a transverse Kähler form on \( X(\Delta_S)/H \) with respect to \( F_{g_J} \) from a normal polytope \( P \) of \( q(\Delta) \). Essentially, this fact has been shown in [10] and [12]. But, the “language” in this paper is slightly different from them. For the reader’s convenience, we give a brief explanation of the construction of a transverse Kähler form without a proof. Let \( P \) be a normal polytope of \( q(\Delta_S) \) represented as

\[
P = \{ \alpha \in (\mathfrak{g}/\mathfrak{g}_J)^* \mid \langle \alpha, q(e_i) \rangle \geq a_i \}.
\]

The map \( q^* : (\mathfrak{g}/\mathfrak{g}_J)^* \to \mathfrak{g}^* \) is an injective map. We consider the embedding \( P \to \mathfrak{g}^* \) given by

\[
\alpha \mapsto \sum_{i=1}^m (\langle \alpha, q(e_i) \rangle - a_i) e_i^* = q^*(\alpha) - \sum_{i=1}^m a_i e_i^*,
\]

where \( e_i^* \) denotes the \( i \)-th dual basis vector of the standard basis \( e_1, \ldots, e_m \) of \( \mathfrak{g} = \mathbb{R}^m \). Let \( i : \mathfrak{g}_J \to \mathfrak{g} \) be the inclusion and consider the dual map \( i^* : \mathfrak{g}^* \to \mathfrak{g}_J^* \). The image of embedded \( P \) is the point \( i^*(\sum_{i=1}^m -a_i e_i^*) =: \beta \). \( X(\Delta_S) \) is an open subset of \( \mathbb{C}^m \). So \( X(\Delta_S) \) has the standard Kähler form

\[
\omega_{\text{st}} = \frac{\sqrt{-1}}{2} \sum_{i=1}^m dz_i \wedge d\bar{z}_i.
\]

\( G \) acts on \( X(\Delta_S) \) preserving \( \omega_{\text{st}} \). The map \( \Phi : X(\Delta_S) \to \mathfrak{g}^* \) given by

\[
\Phi(z_1, \ldots, z_m) = \pi \sum_{i=1}^m |z_i|^2 e_i^*
\]

is a moment map with respect to \( \omega_{\text{st}} \). For the composition \( i^* \circ \Phi : X(\Delta_S) \to \mathfrak{g}_J^* \), the value \( \beta \in \mathfrak{g}_J^* \) is a regular value and \((i^* \circ \Phi)^{-1}(\beta) =: \mathcal{Z}_P \) is a smooth manifold equipped with an action of \( G \) and the \( G \)-invariant transverse symplectic form \( \omega := \omega_{\text{st}}|_{\mathcal{Z}_P} \) with respect to \( F_{g_J} \). Each orbit of \( H \) intersects with \( \mathcal{Z}_P \) at exactly one point in \( \mathcal{Z}_P \), and hence the inclusion \( \mathcal{Z}_P \to X(\Delta_S) \) induces an equivariant diffeomorphism \( \varphi : \mathcal{Z}_P \to X(\Delta_S)/H. \) The form \((\varphi^{-1})^* \omega \) on \( X(\Delta_S)/H \) is what we wanted. The image of an induced moment map is nothing but \( P \) up to translations.

The construction above and Corollary 5.6 yield the following.

Theorem 5.7. The holomorphic foliation \( F_{g_J} \) on an LVMB manifold \( X(\Delta_S)/H \) with at least one indispensable index is transverse Kähler if and only if \( X(\Delta_S)/H \) is an LVM manifold with at least one indispensable index.

We give remarks on the foliation \( F_{g_J} \) and equivariant holomorphic principal bundles. Let \( M_1 \) and \( M_2 \) be complex manifolds with the complex structures \( J_1 \) and \( J_2 \), respectively. Assume that compact tori \( G_1 \) and \( G_2 \) act on \( M_1 \) and \( M_2 \) respectively. If we have an equivariant principal holomorphic bundle \( \pi : M_1 \to M_2 \), it is easy to see that \( T_x F_{g_{J_1}} = (d\pi)_x^{-1}(T_{\pi(x)} F_{g_{J_2}}) \) for all \( x \in M_1 \). Therefore we can obtain every basic form for \( F_{g_{J_1}} \) from a basic form for \( F_{g_{J_2}} \) by the pull-back operator \( \pi^* \). Moreover, every basic form for \( F_{g_{J_1}} \) is also basic for the action of
ker $\alpha$ (that is, invariant under the action of ker $\alpha$ and the interior product with fundamental vector fields generated by the action of ker $\alpha$ vanishes). Therefore there exists the inverse operator $(\pi^*)^{-1}$ of $\pi^*$ defined for basic forms for $F_{\mathfrak{g}_1, J_1}$. In particular, $F_{\mathfrak{g}_1, J_1}$ on $M_1$ is transverse Kähler if and only if so is $F_{\mathfrak{g}_2, J_2}$ on $M_2$. Also, there exists a moment map $\Phi_1 : M_1 \to \mathfrak{g}_1^*$ with respect to a $G_1$ invariant transverse Kähler form $\omega_1$ if and only if there exists a moment map $\Phi_2 : M_2 \to \mathfrak{g}_2^*$ with respect to $(\pi^*)^{-1} \omega_1$.

It has been shown in \cite{8} that a compact connected complex manifold $M$ equipped with a maximal action of a compact torus $G$ is obtained as a quotient of an LVMB manifold with at least one indispensable index. Therefore, we can characterize the manifold with a maximal torus action which admits a transverse Kähler form with respect to $F_{\mathfrak{g}, J}$.

**Theorem 5.8.** Let $M$, $G$, $J$, $\Delta, \mathfrak{h}$ be as in Theorem 5.1. Then, the following are equivalent:

1. $F_{\mathfrak{g}, J}$ on $M$ is transverse Kähler.
2. $q(\Delta)$ is polytopal.

In this case, for any $G$-invariant transverse Kähler form $\omega$, there exists a moment map $\Phi : M \to \mathfrak{g}^*$ with respect to $\omega$ and the image of $M$ by an induced moment map $\Phi : M \to (\mathfrak{g}/\mathfrak{g}_J)^*$ is an inner normal polytope of $q(\Delta)$.

As a corollary, we show that the conjecture posed in \cite{6} holds.

**Corollary 5.9.** For an LVMB manifold $M$, the holomorphic foliation $F_{\mathfrak{g}, J}$ is transverse Kähler if and only if $M$ is an LVM manifold.

**References**


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