

LONGTIME EXISTENCE OF THE KÄHLER-RICCI FLOW ON \mathbb{C}^n

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ABSTRACT. We produce longtime solutions to the Kähler-Ricci flow for complete Kähler metrics on \mathbb{C}^n without assuming the initial metric has bounded curvature, thus extending results in an earlier work of the authors. We prove the existence of a longtime bounded curvature solution emerging from any complete $U(n)$ -invariant Kähler metric with non-negative holomorphic bisectional curvature, and that the solution converges as $t \rightarrow \infty$ to the standard Euclidean metric after rescaling. We also prove longtime existence results for more general Kähler metrics on \mathbb{C}^n which are not necessarily $U(n)$ -invariant.

1. INTRODUCTION

The Kähler-Ricci flow is the following evolution equation for an initial Kähler metric g_0 on a complex manifold M :

$$(1.1) \quad \begin{cases} \frac{\partial g_{i\bar{j}}}{\partial t} = -R_{i\bar{j}}, \\ g(0) = g_0. \end{cases}$$

In this paper we establish longtime existence results for (1.1) in the case $M = \mathbb{C}^n$. For simplicity we will always assume g_0 to be smooth, although our results will only require g_0 to be Kähler and in some cases, also twice differentiable while satisfying some conditions. We are particularly interested in complete Kähler metrics with non-negative bisectional curvature.

One of our main results is (see Definition 1.1 below):

Theorem 1.1. *Let g_0 be a complete $U(n)$ -invariant Kähler metric on \mathbb{C}^n with non-negative bisectional curvature. Then*

- (i) *the Kähler-Ricci flow (1.1) has a smooth longtime $U(n)$ -invariant solution $g(t)$ which is equivalent to g_0 and has bounded non-negative bisectional curvature;*
- (ii) *$g(t)$ converges, after rescaling at the origin, to the standard Euclidean metric on \mathbb{C}^n ;*
- (iii) *$g(t)$ is unique in the class $\mathcal{S}(g_0)$.*

Received by the editors September 23, 2014 and, in revised form, August 5, 2015 and January 14, 2016.

2010 *Mathematics Subject Classification.* Primary 53C55, 58J35.

Key words and phrases. Kähler-Ricci flow, $U(n)$ -invariant Kähler metrics.

The research of the first author was partially supported by NSERC grant no. #327637-06.

The research of the third author was partially supported by Hong Kong RGC General Research Fund #CUHK 14305114.

See Corollary 3.1, Theorem 4.1, Theorem 2.1 for details. Here and throughout the rest of the paper, we use the following notation in reference to a solution $g(t)$ to (1.1) on $M \times [0, T)$.

Definition 1.1.

- We say the solution is *smooth* if $g(t) \in C^\infty(M \times (0, T) \cap C^0(M \times [0, T))$.
- We call the solution a *longtime solution* if $T = \infty$.
- We say the solution is *complete* if $g(t)$ is complete for each $t \in (0, T)$.
- We say that the solution is *equivalent to g_0* if for all $0 \leq T_1 \leq t \leq T_2 < T$ we have $c^{-1}g_0 \leq g(t) \leq cg_0$ for some $c > 0$.
- We say the solution has *non-negative bisectional curvature* if $g(t)$ has non-negative bisectional curvature for all $t \in [0, T)$, and the solution has *bounded curvature* if the curvature of $g(t)$ is uniformly bounded on $M \times [T_1, T_2]$ for all $0 < T_1 \leq T_2 < T$.
- We say that a longtime solution *converges to \tilde{g} after rescaling at $p \in M$* if for some $V \in T_pM$ the metrics $\frac{1}{|V|_t^2}g(t)$ converge to \tilde{g} smoothly and uniformly on compact subsets of M where $|V|_t^2 = g(t)(V, V)$.
- $\mathcal{S}(g_0)$ denotes the set of solutions of the Kähler-Ricci flow with initial data which is uniformly equivalent to g_0 .

We want to remark that using the result of Cabezas-Rivas and Wilking [6], Yang and Zheng [25] have produced a short time existence of the Kähler-Ricci flow for $U(n)$ invariant initial data g_0 with non-negative sectional curvature. See also [11]. However, it is unclear if the solution is uniformly equivalent to g_0 . Hence it is unclear if the solution is the same as the one constructed in the above theorem.

The short time existence of $g(t)$ in the above theorem was proved by Chau, Li and Tam in [3], while the fact that the solution has non-negative bisectional curvature was proved by Yang and Zheng in [25]. The existence of complete $U(n)$ -invariant Kähler metrics with unbounded non-negative bisectional curvature was shown in [23] and thus, in the $U(n)$ -invariant case, our results extend the longtime existence results of Shi [17]. These results are all motivated by Yau’s uniformization conjecture which states that: if g_0 is a complete non-compact Kähler manifold with positive holomorphic bisectional curvature, then $M = \mathbb{C}^n$ and we refer to [4], and the references therein, for a survey of Yau’s conjecture and the Kähler-Ricci flow.

In the case of non-negative bisectional curvature, the volume growth of g_0 is of particular importance in the proofs above. Let $V(r)$ be the volume of the geodesic r ball around the origin relative to g_0 having non-negative bisectional curvature. It was shown by Chen and Zhu [8] that

$$\frac{1}{c}r^n \leq V(r) \leq cr^{2n}$$

for some constant c . Following the notation in [23], we say (\mathbb{C}^n, g_0) is a conoid if $\limsup_{r \rightarrow \infty} V(r)/r^{2n}$ is positive, and that it is a cigar if $\limsup_{r \rightarrow \infty} V(r)/r^n < \infty$. The case when g_0 is a cigar is rather special, and our arguments rely heavily on the non-negativity of bisectional curvature. When g_0 is not a cigar, we may generalize the above theorem to more general Kähler metrics which may not have non-negative bisectional curvature, and may not even be $U(n)$ -invariant.

Theorem 1.2. *Let g_0 be a Kähler metric equivalent to a $U(n)$ -invariant Kähler metric which has non-negative bisectional curvature and is not a cigar. Suppose g_0 is either $U(n)$ -invariant, or else has bounded curvature. Then the Kähler-Ricci*

flow (1.1) has a unique smooth longtime solution $g(t)$ which is equivalent to g_0 , has bounded curvature and is $U(n)$ -invariant provided g_0 is $U(n)$ -invariant. The solution is unique in the class $\mathcal{S}(g_0)$.

We refer to Theorems 3.1 and 3.2 for details where this and more general results are proved.

We point out that when $n = 1$ above, characterization for the existence of a longtime solution to (1.1) has been obtained by Giesen and Topping [10]. In fact, it is shown that given any non-compact Riemann surface (M, g_0) with infinite volume, which may have unbounded curvature and may be incomplete, the Kähler-Ricci flow (1.1) admits a smooth longtime solution $g(t)$ which is complete for all $t > 0$. In particular, since a complete non-compact surface with non-negative Gaussian curvature always has infinite volume, (1.1) has longtime solution in this case. Shorttime and longtime existence results for the Ricci flow starting from complete non-compact Riemannian manifolds (M, g_0) with unbounded curvature have appeared in the works [6, 10, 12, 19, 20]. We also point out that examples of immortal solutions $g(t)$ (defined for all $t \in \mathbb{R}$) were constructed in [6]. We refer to [22] for a survey of related results.

The outline of the paper is as follows. In §2 we establish a uniqueness result for (1.1) on general Kähler manifolds. In §3 we prove our main longtime existence results on \mathbb{C}^n (Theorems 3.1 and 3.2) including our longtime existence result for $U(n)$ -invariant Kähler metrics with non-negative bisectional curvature (Corollary 3.1). In §4 we prove our convergence result for longtime $U(n)$ -invariant solutions with non-negative bisectional curvature (Proposition 4.1). In the appendix, we collect some basic facts of $U(n)$ -invariant Kähler metrics on \mathbb{C}^n for easy reference.

2. UNIQUENESS

Our first result is on the uniqueness of solutions to the Kähler-Ricci flow (1.1). It is well known that the Ricci flow on a complete non-compact Riemannian manifold is unique under the assumption that the curvature is bounded; see [7]. For the Kähler-Ricci flow, it is easier to obtain uniqueness. The following is a generalization of the result in [9]. Here we do not assume the curvature is bounded, and we do not assume that the Ricci form has a potential, which is assumed in [9].

Theorem 2.1. *Let (M^n, \hat{g}) be a complete non-compact Kähler manifold. Suppose there is an exhaustion function $\zeta > 0$ on (M^n, \hat{g}) with $\lim_{x \rightarrow \infty} \zeta(x) = \infty$ such that $|\partial\bar{\partial}\zeta|_{\hat{g}}$ and $|\widehat{\nabla}\zeta|_{\hat{g}}$ are bounded.*

Let $g_1(x, t)$ and $g_2(x, t)$ be two solutions of the Kähler-Ricci flow (1.1) on $M \times [0, T]$ with the same initial data $g_0(x) = g_1(x, 0) = g_2(x, 0)$. Suppose there is a positive function σ with $\lim_{x \rightarrow \infty} \log \sigma(x) / \log \zeta(x) = 0$ such that the following conditions hold for all $(x, t) \in M \times [0, T]$:

(i)

$$(2.1) \quad \widehat{g}(x) \leq \zeta(x)g_1(x, t); \quad \widehat{g}(x) \leq \zeta(x)g_2(x, t),$$

(ii)

$$\frac{\det((g_1)_{i\bar{j}}(x, t))}{\det((g_2)_{i\bar{j}}(x, t))} \leq \sigma(x).$$

Then $g_1 \equiv g_2$ on $M \times [0, T]$. In particular, if g_1 and g_2 are uniformly equivalent to \widehat{g} on $M \times [0, T]$, then $g_1 \equiv g_2$.

Proof. By adding a positive constant to ζ we may assume that $\eta := \log \zeta > 1$. Then

$$\eta_{i\bar{j}} = \frac{\zeta_{i\bar{j}}}{\zeta} - \frac{\zeta_i \zeta_{\bar{j}}}{\zeta^2}.$$

Since $|\partial\bar{\partial}\eta|_{\widehat{g}}$ and $|\widehat{\nabla}\eta|_{\widehat{g}}$ are uniformly bounded, there is $c_1 > 0$ such that

$$|\partial\bar{\partial}\eta|_{\widehat{g}} \leq \frac{c_1}{\zeta}$$

on M . Let $h(s, t)(x) = sg_1(x, t) + (1 - s)g_2(x, t)$, $0 \leq s \leq 1$. By (i), $\widehat{g}(x) \leq \zeta(x)h(s, t)(x, t)$ for all $(x, t) \in M \times [0, T]$ and for all s . Let (x, t) be fixed and diagonalize $\partial\bar{\partial}\eta$ with respect to \widehat{g} at x . Then $|\eta_{i\bar{i}}| \leq \frac{c_1}{\zeta}$. On the other hand,

$$(2.2) \quad \Delta_{h(s,t)}\eta = (h(s, t))^{i\bar{j}}\eta_{i\bar{j}} = (h(s, t))^{i\bar{i}}\eta_{i\bar{i}} \leq n\zeta \cdot \frac{c_1}{\zeta} = nc_1.$$

Let

$$w(x, t) = \int_0^t \left(\log \frac{\det((g_1)_{i\bar{j}}(x, s))}{\det((g_2)_{i\bar{j}}(x, s))} \right) ds.$$

Then

$$w_{i\bar{j}}(x, t) = \int_0^t ((R_1)_{i\bar{j}}(x, s) - (R_2)_{i\bar{j}}(x, s)) ds = -(g_1)_{i\bar{j}}(x, t) + (g_2)_{i\bar{j}}(x, t)$$

where $(R_k)_{i\bar{j}}$ is the Ricci tensor of g_k , $k = 1, 2$. Here we have used the Kähler-Ricci flow and the fact that $g_1 = g_2$ at $t = 0$. Hence in order to prove the proposition, it is sufficient to prove that $w \equiv 0$. Now

$$(2.3) \quad \begin{aligned} \frac{\partial}{\partial t} w(x, t) &= \int_0^1 \frac{\partial}{\partial s} \log \det(h_{i\bar{j}}(s, t)(x)) ds \\ &= \int_0^1 \Delta_{h(t,s)} w(x, t) ds. \end{aligned}$$

Let $W(x, t) = e^{At}\eta$ where $A = nc_1 + 1$. By (2.2),

$$\begin{aligned} \frac{\partial}{\partial t} W(x, t) - \int_0^1 \Delta_{h(t,s)} W(x, t) ds &\geq e^{At}(A\eta - nc_1) \\ &\geq e^{At}\eta \end{aligned}$$

where we have used the fact that $\eta > 1$. For any $\epsilon > 0$,

$$\frac{\partial}{\partial t} (\epsilon W - w)(x, t) - \int_0^1 \Delta_{h(t,s)} (\epsilon W - w)(x, t) ds \geq e^{At}\eta.$$

By (ii), $\lim_{x \rightarrow \infty} (\epsilon W - w)(x, t) = \infty$ uniformly in t . By the maximum principle, we conclude that $w \leq \epsilon W$. Letting $\epsilon \rightarrow 0$ gives $w \leq 0$. Similarly, one can prove that $-w \leq 0$ and hence $w \equiv 0$ on $M \times [0, T]$. This completes the proof of the proposition. \square

Remark 2.1.

- (i) Suppose \widehat{g} has bounded curvature; then ζ exists by [17]. See also [21].
- (ii) Suppose \widehat{g} has non-negative Ricci curvature and non-negative quadratic bisectional curvature; then ζ exists by [15].
- (iii) In particular, the solution constructed in Theorem 4.2 in [3] does not depend on the subsequence chosen.

3. LONGTIME EXISTENCE

In this section, we discuss the longtime existence of solutions to Kähler-Ricci flow (1.1) starting from $U(n)$ -invariant Kähler metrics on \mathbb{C}^n . We will make extensive use of the notation and results in Theorem A.1 (see Appendix A) on $U(n)$ -invariant Kähler metrics g and their associated auxiliary functions ξ, h, f , and we refer there for details. We also recall the notation in Definition 1.1.

Suppose g is a complete $U(n)$ -invariant Kähler metric on \mathbb{C}^n with non-negative bisectional curvature. Then the scalar curvature \mathcal{R} satisfies [23]:

$$\frac{1}{V_g(x, \rho)} \int_{B_g(x, \rho)} \mathcal{R} \leq \frac{c(x)}{1 + \rho}$$

for all $\rho > 0$ for some constant $c(x)$ which may depend on x . Note that if we can choose $c(x) = c$ which is independent of x and the curvature of g is bounded, then it is well known [17] (see also [14]) that the Kähler-Ricci flow has a longtime solution starting from g . In our main result below, Theorem 3.1, we do not assume that the curvature of g is bounded, and $c(x)$ may in general depend on x . In fact, we will prove longtime existence of the flow under more general conditions than non-negative bisectional curvature (see **(c1)**, **(c2)**, **(c3)** below). In order to obtain longtime existence, we make use of the following basic lemma from [3, Corollary 4.2]; see also the proof of [3, Theorem 4.2].

Lemma 3.1. *Let M^n be a complex non-compact manifold and let g_0, \hat{g} be complete Kähler metrics with bounded curvature on M . Suppose the holomorphic bisectional curvature of \hat{g} is bounded above by K and that $(1/\epsilon)\hat{g} \leq g_0 \leq C\hat{g}$ for some $C \geq 1$ and some $\epsilon > 0$. Let $T = 1/2nK\epsilon$ if $K > 0$, otherwise let $T = \infty$. Then the Kähler-Ricci flow (1.1) with g_0 as initial data has a smooth solution $g(t)$ on $M \times [0, T)$ which has bounded curvature and satisfies*

$$(3.1) \quad \left(\frac{1}{n} - 2\epsilon Kt\right) \hat{g} \leq g(t) \leq B(t)\hat{g}$$

for all $t \in [0, T)$ for some positive continuous function $B(t)$ on $[0, T)$.

For a given Kähler metric g and constant $\epsilon > 0$, it is very difficult in general to construct a metric \hat{g} satisfying the conditions in Lemma 3.1 such that K is independent of ϵ . When g is a $U(n)$ -invariant metric generated by ξ (see Appendix A), we show this is possible when **(c1)** or **(c2)** below holds (see Proposition 3.1), though it may not be possible when **(c3)** below holds (see Proposition 3.2).

(c1): There exist $0 \leq \alpha \leq \beta < 1$ and γ such that for all $0 < a < r$

$$\int_a^r \frac{\alpha - \xi}{s} ds, \quad \int_a^r \frac{\xi - \beta}{s} ds \leq \gamma.$$

(c2): $\lim_{r \rightarrow \infty} \xi(r) = 1$, $\int_1^\infty \frac{1 - \xi(s)}{s} ds = \infty$ and there exists $\delta > 0$ such that for all $0 < a < r$

$$\int_a^r \frac{1 - \xi}{s} ds \geq -\delta.$$

(c3): $\lim_{r \rightarrow \infty} \xi(r) = 1$, and $\int_1^\infty \frac{1 - \xi(s)}{s} ds < \infty$.

In [3, Theorem 5.4] it was proved that if ξ satisfies **(c1)** with $\beta \leq 1$, then the Kähler-Ricci flow has short time existence with initial metric g . It was also proved in [3] that if the bisectional curvature is non-positive, then we have longtime solutions.

In case the bisectional curvature of g is non-negative, then it satisfies either **(c1)** with $\alpha = 0$ and $\beta = \xi(\infty)$, **(c2)** or **(c3)** by Theorem A.1 (c), where **(c3)** is the case of a cigar and **(c1)** is the case of a conoid. So the conditions are weaker than the assumption that the bisectional curvature is non-negative.

The main result of this section is:

Theorem 3.1. *Let g_0 be a complete Kähler metric on \mathbb{C}^n satisfying one of the following:*

- (i) g_0 is $U(n)$ -invariant satisfying **(c1)** or **(c2)**.
- (ii) g_0 has bounded curvature and is equivalent to a $U(n)$ -invariant metric satisfying **(c1)** or **(c2)**.
- (iii) g_0 is $U(n)$ -invariant satisfying **(c3)**, and has non-negative bisectional curvature.

Then the Kähler-Ricci flow (1.1) with g_0 as initial data has a smooth longtime solution $g(t)$ which is equivalent to g_0 , has bounded curvature and is $U(n)$ -invariant provided g_0 is $U(n)$ -invariant. Moreover $g(t)$ is unique in the class $\mathcal{S}(g_0)$.

In particular, we get the following.

Corollary 3.1. *Let g_0 be a complete $U(n)$ -invariant Kähler metric on \mathbb{C}^n with non-negative bisectional curvature. Then the Kähler-Ricci flow (1.1) with g_0 as initial data has a smooth longtime $U(n)$ -invariant solution $g(t)$ which is equivalent to g_0 and has bounded non-negative bisectional curvature. Moreover $g(t)$ is unique in the class $\mathcal{S}(g_0)$.*

In order to prove parts (i) and (ii) of the theorem, we need the following:

Proposition 3.1. *Let g be a smooth $U(n)$ -invariant metric with bounded curvature generated by ξ satisfying **(c1)** or **(c2)**. Then given any $\epsilon > 0$ there exists \tilde{g} satisfying*

- (a) *The curvature of \tilde{g} is bounded by a constant independent of ϵ .*
- (b) *$(1/\epsilon)\tilde{g} \leq g \leq C\tilde{g}$ for some constant C .*

Remark 3.1. Suppose g has non-negative bisectional curvature, decaying on average like $\text{dist}(p, \cdot)^{-(1+a)}$, uniformly around all points on \mathbb{C}^n for some $a > 0$. Then the proposition also follows from estimates for the Kähler-Ricci flow in [17] and [14].

Proof of Proposition 3.1. Suppose ξ satisfies **(c1)**. For each $k \geq 1$, consider the linear automorphism of \mathbb{C}^n given by $\phi_k(z) = z/\sqrt{k}$ and consider the $U(n)$ -invariant Kähler metric $g_k := \phi_k^*g$ on \mathbb{C}^n . Consider the functions $h_k(r), \xi_k(r)$ and $h(r), \xi(r)$, etc., corresponding to g_k and g . Then for each $k \geq 1$ we have:

- (1) $h_k(r) = (1/k)h(r/k)$.
- (2) $\xi_k(r) = \xi(r/k)$.
- (3) The curvature of g_k is bounded by a constant independent of k because g_k is isometric to g_0 .

Now

$$(3.2) \quad \frac{h_k(r)}{h(r)} = \frac{1}{k} \frac{h(\frac{r}{k})}{h(r)} = \frac{1}{k} \exp\left(\int_{\frac{r}{k}}^r \frac{\xi(s)}{s} ds\right)$$

and thus by **(c1)** we have

$$(3.3) \quad e^{-\gamma k^{(\alpha-1)}} \leq \frac{h_k(r)}{h(r)} \leq e^{\gamma k^{(\beta-1)}}.$$

By Remark A.2, for any $k \geq 1$ we have

$$(3.4) \quad e^{-\gamma k^{(1-\beta)}} g_k \leq g \leq e^{\gamma k^{(1-\alpha)}} g_k.$$

Thus the proposition follows in this case by the fact that $\beta < 1$.

Suppose ξ satisfies **(c2)**. Let $\epsilon > 0$ be given. Let \widehat{g} be any $U(n)$ -invariant non-negative bisectional curvature metric with $\widehat{h}(0) = 1$ and generated by some $\widehat{\xi}$ with $\widehat{\xi}(r) = 1$ for $r \geq 1$. Let the curvature of \widehat{g} be bounded by \widehat{K} . For each $k \geq 1$ define the pullbacks $\widehat{g}_k := \phi_k^*(\widehat{g})$ as before. Let $\widehat{h}_k(r), \widehat{\xi}_k(r)$ and $\widehat{h}(r)$, etc., correspond to \widehat{g}_k and \widehat{g} . Then properties (1), (2) and (3) above still hold, but with h, ξ, h_k, ξ_k replaced with $\widehat{h}, \widehat{\xi}, \widehat{h}_k, \widehat{\xi}_k$.

Step 1. First note that by (3.2) (applied to $\widehat{h}(r)$) and the fact that $\widehat{\xi} \leq 1$, we see that $\widehat{h}_k(r)$ is non-increasing in k for all r . By **(c2)** there is $r_0 > 0$ such that if $r \geq r_0$, then for $k \geq 1$

$$\begin{aligned} h(r) &= \widehat{h}(r) \frac{h(r)}{\widehat{h}(r)} \\ &\geq \widehat{h}_k(r) \exp\left(\int_0^r \frac{\widehat{\xi}(s) - \xi(s)}{s} ds\right) \\ &\geq \frac{1}{\epsilon} \widehat{h}_k(r) \end{aligned}$$

where we have used the fact that $\widehat{\xi}(r) = 1$ for $r \geq 1$. On the other hand, by (3.2) one can see this is true for $r \leq r_0$ if k is large enough depending on r_0 . Hence by Remark A.2 we can find $k > 1$ depending on ϵ such that,

$$\frac{1}{\epsilon} g_k \leq g$$

on \mathbb{C}^n .

Step 2. Define

$$\widetilde{\xi}_k(r) := \widehat{\xi}_k(r) + o_k(r)$$

where $o_k : [0, \infty) \rightarrow \mathbb{R}$ is a non-positive smooth function with $|o_k| \leq \frac{1}{k}$, to be chosen. Let $\widetilde{h}_k(0) = 1/k$ and consider the corresponding metric \widetilde{g}_k .

Claim 1. There exists a constant $R_k > k$ and a smooth function $o_k(r)$ which is 0 on $[0, R_k]$ and satisfies:

$$|o'_k(r)| \leq \frac{4}{kr}$$

and

$$(3.5) \quad \left| \int_{R_k}^r \frac{\widetilde{\xi}_k(s) - \xi(s)}{s} ds \right| = \left| \int_{R_k}^r \frac{1 + o_k(s) - \xi(s)}{s} ds \right| \leq 1 + 2 \log 2$$

for $r \geq R_k$.

We may choose $R_k > k$ such that $1 - 1/k \leq \xi(r) \leq 1 + 1/k$ on $[R_k, \infty)$. The construction of $o_k(r)$ follows from the construction in the proof of Proposition 5.1 in [3]. We provide details here for the convenience of the reader. We first choose a

smooth non-increasing function $\rho(r) : [0, \infty) \rightarrow \mathbb{R}$ with $\rho = 0$ on $[0, 1]$, $\rho = 1/k$ on $[2, \infty)$ and $0 \leq \rho' \leq 2/k$. Let

$$I(r) := \int_{2R_k}^r \frac{1 + o_k(r) - \xi(s)}{s} ds$$

(note that $\tilde{\xi}_k(r) = 1 + o_k(r)$ for $r \geq R_k$). For any positive sequence $\{r_i\}$ such that $r_0 := R_k$ and $2r_i < r_{i+1}$, define $o_k(r) := 0$ if $r \in [0, r_0)$, $o_k(r) := \rho(r/r_0)$ if $r \in [r_0, 2r_0)$, and

$$(3.6) \quad o_k(r) : \begin{cases} = 1/k & \text{if } r \in [2r_0, r_1], \\ = 1/k - 2\rho(r/r_1) & \text{if } r \in [r_1, 2r_1], \\ = -1/k & \text{if } r \in [2r_1, r_2], \\ = -1/k + 2\rho(r/r_2) & \text{if } r \in [r_2, 2r_2], \\ = 1/k & \text{if } r \in [2r_2, r_3], \\ = 1/k - 2\rho(r/r_3) & \text{if } r \in [r_3, 2r_3]. \\ \dots \end{cases}$$

Now for $r \in [2r_0, \infty)$ we have $1 - 1/k \leq \xi(r) \leq 1 + 1/k$, and as long as $k \geq 2$ we have $|1 + o_k(r) - \xi(r)| \leq 1$ as well. The definition of $I(r)$ then gives the following for all $i \geq 0$:

$$I(r) = I(2r_i) + \int_{2r_i}^r \frac{(1 + (-1)^i/k) - \xi(s)}{s} ds$$

for $r \in [2r_i, r_{i+1})$,

$$I(r_{i+1}) - \log 2 \leq I(r) \leq I(r_{i+1}) + \log 2$$

for $r \in [r_{i+1}, 2r_{i+1})$.

By the fact $\xi(r) \rightarrow 1$, we may choose the r_i 's to be the smallest numbers with $I(r_1) = 1$, $I(r_2) = -1$, $I(r_3) = 1$, etc., and the estimates above give

$$(3.7) \quad -1 - \log 2 \leq I(r) \leq 1 + \log 2$$

for all $r \in [2R_k, \infty)$. The integral bound in the claim follows from (2.7) and the fact that $|1 + o_k(r) - \xi(r)| \leq 1$ for $r \in [R_k, 2R_k]$.

Finally, we also have $|o'_k(r)| = \frac{2}{r_i} \rho'(\frac{r}{r_i}) \leq \frac{4}{kr_i} \leq \frac{4}{kr}$ for all $r \in [r_i, 2r_i]$.

Claim 2. Let $o_k(r)$ be as in Claim 1. Then $(1/4e\epsilon)\tilde{g}_k \leq g \leq C_k\tilde{g}_k$ for some C_k and the curvature of \tilde{g}_k is bounded depending only on \tilde{g} .

To prove the first part of the claim, when $r \leq R_k$ we have $\tilde{g}_k(r) = \hat{g}_k(r)$, and so we only have to consider when $r \geq R_k$. In this case, we have $C_k \geq h(r)/\tilde{h}_k(r) = (h(R_k)/\tilde{h}_k(R_k))e^{\int_{R_k}^r \frac{1+o_k(s)-\xi(s)}{s} ds} \geq \frac{1}{2e\epsilon}$ for some C_k where we have used Step 1 and Claim 1.

To prove the second part of the claim, note that for $r \geq R_k$ we have $|\tilde{\xi}'_k(r)| = |o'_k(r)| \leq 4/kr$ and

$$\begin{aligned}
 \tilde{h}_k(r) &= \tilde{h}_k(1) \exp\left(-\int_1^r \frac{\tilde{\xi}_k(s)}{s} ds\right) \\
 &= \tilde{h}_k(1) \exp\left(-\int_1^{R_k} \frac{\tilde{\xi}_k(s)}{s} ds - \int_{R_k}^r \frac{\tilde{\xi}_k(s)}{s} ds\right) \\
 (3.8) \quad &\geq \tilde{h}_k(1) \frac{1}{R_k} \exp\left(-\int_{R_k}^r \left(\frac{\tilde{\xi}_k(s) - \xi(s)}{s} + \frac{\xi(s) - 1}{s} + \frac{1}{s}\right) ds\right) \\
 &\geq \tilde{h}_k(1) \frac{1}{R_k} \frac{R_k}{4e^{1+\delta}r} \\
 &= \hat{h}_k(1) \frac{1}{4e^{1+\delta}r} \\
 &\geq \frac{\hat{h}(1)}{k} \frac{1}{4e^{1+\delta}r}
 \end{aligned}$$

where in the third line we have used that $0 \leq \tilde{\xi}_k \leq 1$ by definition, and in the fourth line we have used Claim 1 and **(c2)** so that $\int_{R_k}^r \frac{1-\xi(s)}{s} ds \geq -\delta$.

Thus $|\tilde{\xi}'_k(r)/\tilde{h}_k(r)| \leq 16e^{1+\delta}/\hat{h}(1)$ for $r \geq R_k$. Since $\tilde{g}_k(r) = g_k(r)$ for $r \leq R_k$, by (3) we conclude that the curvature of \tilde{g}_k is bounded by a constant independent of k by [3, Lemma 5.1(i)]. □

Proposition 3.1 includes the case when g has non-negative bisectional curvature and is not a cigar. When g is a cigar, then the proposition is false in general. In particular, we have

Proposition 3.2. *Let g be a smooth $U(n)$ -invariant metric generated by ξ satisfying*

$$\int_1^r \frac{1-\xi}{s} ds \leq c$$

for some c for all $r \geq 1$. There exists a constant $c_1 > 0$ depending only on g such that if \tilde{g} is another $U(n)$ -invariant Kähler metric with bisectional curvature bounded above by 1 such that $g \geq \alpha\tilde{g}$ for some $\alpha > 0$, then $\alpha \leq c_1$.

Proof. Indeed, let g be a $U(n)$ -invariant complete Kähler metric generated by ξ normalized by $h(0) = 1$ such that

$$(3.9) \quad \int_1^r \frac{1-\xi}{s} ds \leq c$$

for some c for all $r \geq 1$. We may assume $0 \leq \xi \leq 1$ and $\xi = 1$ for $r \geq 1$, for if \hat{g} is some metric generated by $\hat{\xi}$ with $0 \leq \hat{\xi} \leq 1$ and $\hat{\xi} = 1$ for $r \geq 1$ and $\hat{h}(0) = 1$, then $\hat{g} \geq c'g$ for some $c' > 0$ by (3.9).

Let \tilde{g} be another $U(n)$ -invariant Kähler metric such that $g \geq \alpha\tilde{g}$ for some $\alpha > 0$ with bisectional curvature bounded above by 1. Assume \tilde{g} is generated by $\tilde{\xi}$. Then

$$(3.10) \quad \limsup_{r \rightarrow \infty} \tilde{\xi} \geq 1.$$

Otherwise, by (3.9), we would have

$$\frac{h(r)}{\tilde{h}(r)} = \frac{h(0)}{\tilde{h}(0)} \exp\left(\int_0^r \frac{\tilde{\xi} - \xi}{s} ds\right) \rightarrow 0$$

as $r \rightarrow \infty$.

Let $0 < \epsilon < 1$. By (3.10) and the fact that $\tilde{\xi}(0) = 0$, there is $r_1 > 0$ such that $\tilde{\xi}(r_1) = 1 - \epsilon$. By the fact that $\tilde{\xi}(0) = 0$, we can find $0 \leq r_0 < r_1$ such that $\tilde{\xi} > 0$ on $(r_0, r_1]$ and $\tilde{\xi}(r_0) = 0$. It is easy to see that $\tilde{h}(r) \leq \tilde{h}(r_0)$ for $r \in [r_0, r_1]$. Since the bisectional curvature of \tilde{g} is bounded above by 1,

$$\tilde{\xi}'(r) \leq \tilde{h}(r) \leq \tilde{h}(r_0)$$

for all $r \in [r_0, r_1]$. Hence $\tilde{\xi}(r) \leq \tilde{h}(r_0)(r - r_0)$ for all $r \in [r_0, r_1]$. In particular, $1 - \epsilon = \tilde{\xi}(r_1) \leq \tilde{h}(r_0)(r_1 - r_0)$ and so $r_2 = r_0 + (1 - \epsilon)/\tilde{h}(r_0) \leq r_1$. Then

$$\begin{aligned} h(r_2) &\geq \alpha \tilde{h}(r_2) \\ &= \alpha \tilde{h}(r_0) \exp\left(-\int_{r_0}^{r_2} \frac{\tilde{\xi}}{s} ds\right) \\ (3.11) \quad &\geq \alpha \tilde{h}(r_0) \exp\left(-\int_{r_0}^{r_1} \tilde{h}(r_0) ds\right) \\ &\geq \alpha \tilde{h}(r_0) \exp\left(\tilde{h}(r_0)(r_1 - r_0)\right) \\ &\geq \alpha \tilde{h}(r_0) \exp(-(1 - \epsilon)). \end{aligned}$$

On the other hand, by the definition of r_2 we have $\tilde{h}(r_0)r_2 \geq (1 - \epsilon)$ and thus

$$h(r_2) = h(1) \exp\left(-\int_1^{r_2} \frac{\xi}{s} ds\right) = \frac{h(1)}{r_2} \leq \frac{\tilde{h}(r_0)h(1)}{1 - \epsilon}.$$

By (3.11), we conclude that

$$\frac{\tilde{h}(r_0)h(1)}{1 - \epsilon} \geq \alpha \tilde{h}(r_0) \exp(-(1 - \epsilon)).$$

Since ϵ is arbitrary, $\alpha \leq h(1)$. □

Proof of Theorem 3.1(i) and (ii). First of all, uniqueness in each case (i), (ii) and (iii) follows from Theorem 2.1. Indeed, let $g_1(t), g_2(t)$ be two solutions as in the theorem. Then using the bounded curvature metric $\hat{g} = g_1(1)$ in Theorem 2.1, and noting the solutions $g_1(t), g_2(t)$ are both equivalent to g_0 and hence also \hat{g} , we conclude by Theorem 2.1 that $g_1(t) = g_2(t)$ for all t .

Now consider case (i). Suppose ξ satisfies **(c1)** or **(c2)**. By [3, Theorems 5.4, 4.2] there exists a smooth $U(n)$ -invariant solution $g(t)$ to Kähler-Ricci flow on $\mathbb{C}^n \times [0, T]$ for some $T > 0$ such that $g(0) = g_0$, the curvature of $g(t)$ is uniformly bounded by c/t and $c^{-1}g(t) \leq g_0 \leq cg(t)$ for some $c > 0$ for all $t \in (0, T]$. Fix $0 < t_0 < T$. By Proposition 3.1 and the fact that $g(t_0)$ is uniformly equivalent to g_0 , for any $\epsilon > 0$, there is a complete $U(n)$ -invariant metric \hat{g}_ϵ with curvature bounded by K with K being independent of ϵ such that

$$\frac{1}{\epsilon} \hat{g}_\epsilon \leq g(t_0) \leq c(\epsilon) \hat{g}_\epsilon$$

for some constant $c(\epsilon)$ which may depend on ϵ . By Lemma 3.1 and Theorem 2.1, the Kähler-Ricci flow $g(t)$ can be extended to $[0, T_\epsilon)$, where $T_\epsilon = \frac{1}{2nK\epsilon}$, such that $g(t)$

is uniformly equivalent to g_0 for all $t \in [0, T']$ for any $T' < T_\epsilon$ and has uniformly bounded curvature on (δ, T') for all $0 < \delta < T' < T_\epsilon$. Let $\epsilon \rightarrow 0$. One may conclude the theorem is true.

Now consider case (ii). Here we also have a short time solution $g(t)$ on $\mathbb{C}^n \times [0, T]$ as in case (i) by [17]. By the equivalence condition in (ii), it is easy to see that we can apply the exact same argument as above to conclude the theorem is true in this case as well. \square

Remark 3.2. It is easy to see that if ξ satisfies:

$$-c \leq \int_1^r \frac{\xi(s) - a}{s} ds \leq c$$

for some $c > 0$ and $0 < a < 1$ for all $r \geq 1$, then ξ satisfies **(c1)**. This generalizes [3, Theorem 5.5] in case $a < 1$.

If ξ satisfies **(c3)** in Theorem 3.1, then the previous argument does not work in light of Proposition 3.2. However, if g_0 also has non-negative bisectional curvature (and is thus a cigar) we may use other methods. Before we prove the theorem in this case we need a more general form of [14, Theorem 2.1], which will also be used later.

Lemma 3.2. *Let $(M^n, g(t))$ be a complete non-compact solution of the Kähler-Ricci flow (1.1) on $M \times [0, T]$ with bounded non-negative bisectional curvature. Let*

$$F(x, t) = \log \left(\frac{\det(g_{i\bar{j}}(x, t))}{\det(g_{i\bar{j}}(x, 0))} \right)$$

and for any $\rho > 0$, let $\mathbf{m}(\rho, x, t) = \inf_{y \in B_0(x, \rho)} F(y, t)$. Then there is $c > 0$ depending only on n such that for any $x_0 \in M$ and for all $\rho, t > 0$

$$(3.12) \quad -F(x_0, t) \leq c \left[\left(1 + \frac{t(1 - \mathbf{m}(\rho, x_0, t))}{\rho^2} \right) \int_0^{2\rho} sk(x_0, s) ds - \frac{t\mathbf{m}(\rho, x_0, t)(1 - \mathbf{m}(\rho, x_0, t))}{\rho^2} \right]$$

where $B_0(x, \rho)$ is the geodesic ball with respect to $g_0 = g(0)$, and $k(x, s)$ is the average of the scalar curvature \mathcal{R}_0 of g_0 over $B_0(x, s)$.

Proof. By [14, (2.6)], if G_ρ is the positive Green’s function on $B_0(x_0, \rho)$ with Dirichlet boundary value, then

$$(3.13) \quad \begin{aligned} & \int_{B_0(x_0, \rho)} G_\rho(x_0, y) \left(1 - e^{F(y, t)} \right) dv_0 \\ & \leq t \int_{B_0(x_0, \rho)} G_\rho(x_0, y) \mathcal{R}_0(y) dv_0 + \int_0^t \int_{B_0(x_0, \rho)} G_\rho(x_0, y) \Delta_0(-F(y, s)) dv_0 ds \\ & =: I + II. \end{aligned}$$

As in [14, p. 126],

$$(3.14) \quad II \leq -t\mathbf{m}(\rho, x_0, t).$$

As in [14, (2.8)] there is a constant c_1 depending only on n such that

$$\begin{aligned} & \rho^2 \int_{B_0(x_0, \frac{1}{5}\rho)} (-F(y, t)) dv_0 \\ & \leq c_1 t (1 - \mathbf{m}(\rho, x_0, t)) \left(\int_{B_0(x_0, \rho)} G_\rho(x_0, y) \mathcal{R}_0(y) dv_0 - \mathbf{m}(\rho, x_0, t) \right). \end{aligned}$$

Using the fact that $\Delta_0(-F) \geq -\mathcal{R}_0$ and [14, Lemma 2.1], we obtain

$$\begin{aligned} & -F(x_0, t) \\ (3.15) \quad & \leq \int_{B_0(x_0, \frac{1}{5}\rho)} G_\rho(x_0, y) \mathcal{R}_0(y) dv_0 \\ & + c_2 \rho^{-2} t (1 - \mathbf{m}(\rho, x_0, t)) \left(\int_{B_0(x_0, \rho)} G_\rho(x_0, y) \mathcal{R}_0(y) dv_0 - \mathbf{m}(\rho, x_0, t) \right) \end{aligned}$$

for some c_2 depending only on n . As in [14, p. 127], we get the result. □

We also need the following:

Lemma 3.3. *Let $g(t)$ be the complete $U(n)$ -invariant solution of the Kähler-Ricci flow (1.1) with non-negative bisectional curvature. Let*

$$F(r, t) := F(z, t) = \log \left(\frac{\det(g_{i\bar{j}}(z, t))}{\det(g_{i\bar{j}}(z, 0))} \right)$$

where $r = |z|^2$. Then for $r \geq 1$ and for all t

$$F(r, t) \geq -c - n \log r + F(1, t)$$

for some constant $c > 0$ depending only on $g(0)$. If in addition the generating function ξ of g_0 satisfies:

$$(3.16) \quad \lim_{r \rightarrow \infty} \int_1^r \frac{1 - \xi(s)}{s} ds = b < \infty,$$

then for $r \geq 1$ and for all t

$$F(r, t) \geq -c + nF(1, t)$$

for some constant $c > 0$ depending only on $g(0)$.

Proof. Consider the functions $\xi(r, t), h(r, t), f(r, t)$ corresponding to $g(r, t)$. Then $0 \leq \xi(r, t) \leq 1$ since $g(r, t)$ has non-negative bisectional curvature, and by Theorem A.1 we then get $0 \leq h(r, t), f(r, t) \leq 1$. Thus for $r \geq 1$ we have

$$\begin{aligned} f(r, t) &= \frac{1}{r} \int_0^r h(s, t) ds \geq \frac{1}{r} \int_0^1 h(s, t) dt = \frac{1}{r} f(1, t), \\ h(r, t) &= h(1, t) \exp \left(\int_1^r -\frac{\xi(s)}{s} ds \right) \geq \frac{1}{r} h(1, t), \end{aligned}$$

and using the formula $\det(g_{i\bar{j}}(r, t)) = h(r, t)f^{n-1}(r, t)$ we then get

$$\begin{aligned} \frac{\det(g_{i\bar{j}}(r, t))}{\det(g_{i\bar{j}}(r, 0))} &\geq \det(g_{i\bar{j}}(r, t)) \\ &\geq \frac{1}{r^n} h(1, t) f^{n-1}(1, t) \\ &= \frac{1}{r^n} \det(g_{i\bar{j}}(1, t)) \\ &= \frac{1}{r^n} \frac{\det(g_{i\bar{j}}(1, t))}{\det(g_{i\bar{j}}(1, 0))} \cdot \det(g_{i\bar{j}}(1, 0)). \end{aligned}$$

From this, it is easy to see the first result follows. Now suppose the generating function ξ of g_0 also satisfies (3.16). Then for $r \geq 1$,

$$\begin{aligned} \frac{h(r, t)}{h(r, 0)} &= \frac{h(1, t) \exp\left(-\int_1^r \frac{\xi(s, t)}{s} ds\right)}{h(1, 0) \exp\left(-\int_1^r \frac{\xi(s, 0)}{s} ds\right)} \\ &\geq \frac{h(1, t)}{h(1, 0)} \exp\left(\int_1^r \frac{\xi(s, 0) - 1}{s} ds\right) \\ &\geq c_1 \frac{h(1, t)}{h(1, 0)} \end{aligned}$$

for some constant $c_1 > 0$ independent of r, t , provided $r \geq 1$. Also for $r \geq 1$

$$\begin{aligned} \frac{f(r, t)}{f(r, 0)} &= \frac{\int_0^r h(s, t) ds}{\int_0^r h(s, 0) ds} \\ &= \frac{\int_0^1 h(s, t) ds + \int_1^r h(s, t) ds}{\int_0^1 h(s, 0) ds + \int_1^r h(s, 0) ds} \\ &\geq \frac{h(1, t) + c_1 \frac{h(1, t)}{h(1, 0)} \int_1^r h(s, 0) ds}{\int_0^1 h(s, 0) ds + \int_1^r h(s, 0) ds} \\ &\geq c_2 \frac{h(1, t)}{h(1, 0)} \end{aligned}$$

for some constant $c_2 > 0$ independent of r, t , provided $r \geq 1$.

Hence for $r \geq 1$, at the point $|z|^2 = r$

$$\begin{aligned} \frac{\det(g_{i\bar{j}}(z, t))}{\det(g_{i\bar{j}}(z, 0))} &= \frac{h(r, t) f^{n-1}(r, t)}{h(r, 0) f^{n-1}(r, 0)} \\ &\geq c_3 \left(\frac{h(1, t)}{h(1, 0)}\right)^n \\ &\geq c_3 \left(\frac{h(1, t) f^{n-1}(1, t)}{h(1, 0) f^{n-1}(1, 0)}\right)^n \\ &= c_3 \left(\frac{\det(g_{i\bar{j}}(1, t))}{\det(g_{i\bar{j}}(1, 0))}\right)^n \end{aligned}$$

for some $c_3 > 0$, where we have used the fact that $f(r, t) \leq f(r, 0)$. From this the second result follows. □

Proof of Theorem 3.1(iii). Let g_0 be generated by ξ normalized so that $h(0) = 1$ and so that **(c3)** is satisfied. We may assume that ξ is not identically zero, otherwise g_0 would be the standard Euclidean metric and the theorem is obviously true.

By [3, Theorem 5.4] and [25], the Kähler-Ricci flow has a $U(n)$ -invariant complete solution $g(t)$ on $M \times [0, T)$ which is equivalent to g_0 and has bounded non-negative bisectional curvature. Hence we may assume that g_0 has bounded curvature. Let T be the maximal such existence time.

Now, let $\mathbf{m}(t) = \inf_{z \in \mathbb{C}^n} F(z, t)$. By Lemma 3.3 and the fact $F \leq 0$,

$$\mathbf{m}(t) \geq -c_1 + n \inf_{\{z \in \mathbb{C}^n \mid |z|^2 \leq 1\}} F(z, t)$$

for some $c_1 > 0$ depending only on g_0 , and so since $F \leq 0$,

$$(3.17) \quad -\mathbf{m}(t) \leq c_1 - n \inf_{\{z \in \mathbb{C}^n, |z|^2 \leq 1\}} F(z, t).$$

On the other hand by the proof of [23, Theorem 7], there is a constant $c_2 > 0$ such that for all $\rho > 0$

$$\frac{1}{V_0(0, \rho)} \int_{B_0(0, \rho)} \mathcal{R}(0) \leq \frac{c_2}{1 + \rho}$$

where B_0 is the geodesic ball with respect to $g(0)$, where $\mathcal{R}(0)$ is the scalar curvature of g_0 . Hence by volume comparison, there is a constant $c_3 > 0$ such that

$$\frac{1}{V_0(z, \rho)} \int_{B_0(z, \rho)} \mathcal{R}(0) \leq \frac{c_3}{1 + \rho}$$

for all $r > 0$ and for all z with $|z|^2 \leq 1$. By Lemma 3.2, (3.17)

$$-\mathbf{m}(t) \leq c_1 + nc_4 \left[\left(1 + \frac{t(1 - \mathbf{m}(t))}{\rho^2} \right) c_5 \rho - \frac{t\mathbf{m}(t)(1 - \mathbf{m}(t))}{\rho^2} \right]$$

for some constants $c_4, c_5 > 0$ independent of t, ρ . Choose $\rho^2 = \frac{1}{2nc_1}t(1 - \mathbf{m}(t))$. Then one can conclude that for all $0 < t < T < \infty$

$$\frac{1}{2}(1 - \mathbf{m}(t)) \leq c_6 + c_7 T^{\frac{1}{2}}(1 - \mathbf{m}(t))^{\frac{1}{2}}$$

for some constants $c_6, c_7 > 0$ independent of T . So $(1 - \mathbf{m}(t))$ and hence $-\mathbf{m}(t)$ is uniformly bounded on $[0, T)$. Then one can proceed as in [17] to conclude that the theorem is true. □

The following theorem gives longtime existence under more general conditions, only in this case we do not know if the curvature of $g(t)$ is necessarily bounded for each time.

Theorem 3.2. *Let g_0 be a complete $U(n)$ -invariant Kähler metric on \mathbb{C}^n generated by ξ . Suppose*

$$(3.18) \quad \lim_{r \rightarrow \infty} \int_1^r \frac{1 - \xi(s)}{s} ds = \infty.$$

Then the Kähler-Ricci flow (1.1) with initial data g_0 has a smooth complete longtime $U(n)$ -invariant solution $g(t)$.

Proof. By (3.18) we may choose some increasing sequence $r_k \rightarrow \infty$ such that $\xi(r_k) < 1$. For each k , let ξ_k be smooth such that $\xi_k(r) = \xi(r)$ for $r \leq r_k - \delta_k$ for some $0 < \delta_k < 1$ and $\xi_k(r) = \xi(r_k)$ for $r \geq r_k$ such that

$$(3.19) \quad \exp \left(\int_{r_k - \delta_k}^{r_k} \left| \frac{\xi(s) - \xi_k(s)}{s} \right| ds \right) \leq 2.$$

Let g_k be the $U(n)$ -invariant Kähler metric associated to ξ_k and consider also the corresponding functions h_k, f_k . Then g_k has bounded curvature. By Theorem 3.1 or by [3, Theorem 5.5], for each k the Kähler-Ricci flow has a longtime solution $g_k(t)$ on $\mathbb{C}^n \times [0, \infty)$ with bounded curvature.

Now let $\epsilon > 0$ be given and consider the sequence $\widehat{g}_i = \phi_i^* \widehat{g}$ as in the first part of the proof of Proposition 3.1. Then as in Step 1 there, we may fix i_0 sufficiently large so that

$$(3.20) \quad \widehat{g}_{i_0}(r) \leq \epsilon g(r)$$

for all r . In particular, for every k we have

$$(3.21) \quad \widehat{g}_{i_0}(r) \leq 2\epsilon g_k(r)$$

for $r \leq r_k + \delta_k$ by (3.19) and the fact that $g_k(r) = g(r)$ for $r \leq r_k$. On the other hand, for some k_0 we have $\widehat{\xi}_{i_0}(r) = 1$ for $r \geq r_{k_0}$ and thus for all $k \geq k_0$,

$$\frac{\widehat{h}_{i_0}(r)}{h_k(r)} = \frac{\widehat{h}_{i_0}(r_k + \delta)}{h_k(r_k + \delta)} \exp \left(\int_{r_k + \delta_k}^r \frac{\xi_k(s) - 1}{s} ds \right) \leq 2\epsilon.$$

In particular, (3.21) holds for all $r \in [0, \infty)$ while the curvature of \widehat{g}_{i_0} is bounded by \widehat{K} . By [3, Lemma 3.1], we have

$$(3.22) \quad g_k(r, t) \geq \left(\frac{1}{n} - 4\widehat{K} \epsilon t \right) \widehat{g}_{i_0}(r)$$

on $\mathbb{C}^n \times [0, \frac{1}{4n^2 \widehat{K} \epsilon})$ for all $k \geq k_0$. Note that $g_k(x, 0) \rightarrow g_0(x)$ uniformly on compact sets. By the proof of [3, Theorem 4.2], $g_k(x, t)$ converges subsequentially uniformly on compact sets of $M \times [0, \frac{1}{4n \widehat{K} \epsilon})$, to a smooth solution of Kähler-Ricci flow (1.1). Since $\epsilon > 0$ was arbitrarily chosen, we conclude that the theorem is true by a diagonal process. \square

Remark 3.3. If $\xi(r) \leq 0$ near infinity in Theorem 3.2, then the result follows from [3, Th. 5.3].

In Theorem 3.2 it is unclear whether $g(t)$ has bounded curvature for $t > 0$, even if we assume g_0 has bounded curvature. This is in contrast to the solution constructed in Theorem 3.1.

4. CONVERGENCE

By Corollary 3.1, for any $U(n)$ -invariant complete Kähler metric g_0 with non-negative bisectional curvature, the Kähler-Ricci flow with initial data g_0 has a longtime $U(n)$ -invariant solution equivalent to g_0 with bounded non-negative bisectional curvature. In this section we discuss the convergence of such solutions. Again, we will make extensive use of the notation and results in Theorem A.1 in the appendix on $U(n)$ -invariant Kähler metrics g , their associated auxiliary functions ξ, h_ξ, f_ξ , and the associated components A, B, C of the curvature tensor. We also recall the notation in Definition 1.1.

Theorem 4.1. *Let $g(t)$ be a complete longtime $U(n)$ -invariant solution of the Kähler-Ricci flow (1.1) with bounded non-negative bisectional curvature, and assume $g(0)$ also has bounded curvature. Then $g(t)$ converges, after rescaling at the origin, to the standard Euclidean metric on \mathbb{C}^n .*

In order to prove the theorem, we need the following lemmas.

Lemma 4.1. *Let $g(t)$ be as in Theorem 4.1. Suppose the curvature of $g(t)$ is uniformly bounded by c_1 , in $D(R) \times [0, \infty)$, where $D(R) = \{|z|^2 < R\}$. Then there is a constant c_2 depending only on c_1 and R such that*

$$(4.1) \quad h(r, t) \leq h(0, t) \leq c_2 h(r, t); \quad f(r, t) \leq f(0, t) \leq c_2 f(r, t)$$

for all $0 < r < R$ and for all t .

Proof. By Remark A.2 and the fact that $g(t)$ has non-negative bisectional curvature, we have $h(0, t) \geq h(r, t)$ and $f(0, t) \geq f(r, t)$ for all $r > 0$. On the other hand, we have $A(r, t), B(r, t), C(r, t) \leq c_1$ in $D(R) \times [0, \infty)$ by hypothesis. Thus $C = -\frac{2f_r}{f^2}$ gives

$$\frac{1}{f(r, t)} - \frac{1}{f(0, t)} \leq \frac{c_1 R}{2}$$

in $D(R) \times [0, \infty)$, and by $f(0, t) \leq f(0, 0) = h(0, 0) = 1$ we get

$$f(0, t) \leq \left(\frac{c_1 R}{2} + 1 \right) f(r, t).$$

Also, $A = \frac{\xi_r(r, t)}{h(r, t)} \leq c_1$ gives

$$\xi_r(r, t) \leq c_1 h(r, t) \leq c_1 h(r, 0) \leq c_1 h(0, 0) = c_1$$

in $D(R) \times [0, \infty)$, and thus $\xi(r, t) \leq c_1 r$, giving

$$h(r, t) = h(0, t) \exp \left(\int_0^r \frac{-\xi(s, t)}{s} ds \right) \geq \exp(-c_1 R) h(0, t).$$

This completes the proof of the lemma. □

Lemma 4.2. *Let $g(t)$ be as in Theorem 4.1. Then for any $R > 0$ the curvature of $g(t)$ is uniformly bounded in $D(R) \times [0, \infty)$.*

Proof. Let

$$k(z, \rho) = \frac{1}{V_0(z, \rho)} \int_{B_0(z, \rho)} \mathcal{R}(0)$$

be the average of the scalar curvature $\mathcal{R}(0)$ of g_0 over the geodesic ball $B_0(z, \rho)$ with respect to g_0 . Let

$$k(\rho) = \sup_{|z| \leq 1} k(z, \rho).$$

By [23, Theorem 7], there is a constant c_1 such that

$$(4.2) \quad k(\rho) \leq \frac{c_1}{1 + \rho}.$$

Suppose $|z|^2 = r$; then the distance $\rho(z) = \rho(r)$ from z to the origin satisfies

$$(4.3) \quad \rho(z) = \rho(r) = \frac{1}{2} \int_0^r \frac{\sqrt{h}}{\sqrt{s}} ds \geq c_2 \log r$$

for some constant $c_2 > 0$ for all $r \geq 1$. Let

$$F(z, t) = \log \frac{\det(g_{i\bar{j}}(z, t))}{\det(g_{i\bar{j}}(z, 0))}$$

and let $\mathbf{m}(\rho, t) = \inf_{z \in \mathbb{C}^n, \rho(z) \leq \rho} F(z, t)$. Fix $r_0 > 1$ and let $\rho_0 = \rho(r_0)$. Denote $-\mathbf{m}(\rho_0, t)$ by $\eta(t)$. By Lemmas 3.2 and 3.3, there exist positive constants c_3, c_4 independent of t and ρ such that

$$\begin{aligned} \eta(t) &\leq c_4 \left[\left(1 + \frac{t(1 - \mathbf{m}(\rho + \rho_0, t))}{\rho^2} \right) K(\rho) - \frac{t\mathbf{m}(\rho + \rho_0, t)(1 - \mathbf{m}(\rho + \rho_0, t))}{\rho^2} \right] \\ &\leq c_4 \left[\left(1 + \frac{t(1 + c_1 + \log \tilde{r}(\rho) + \eta(t))}{\rho^2} \right) K(\rho) \right. \\ &\quad \left. + \frac{t(c_3 + \log \tilde{r}(\rho) + \eta(t))(1 + c_1 + \log \tilde{r}(\rho) + \eta(t))}{\rho^2} \right] \end{aligned}$$

where

$$K(\rho) = \int_0^{2\rho} sk(s) ds$$

and $\tilde{r}(\rho)$ is such that,

$$\rho + \rho_0 = \frac{1}{2} \int_0^{\tilde{r}(\rho)} \frac{\sqrt{h}}{\sqrt{s}} ds.$$

By (4.2) and (4.3), there is a constant c_5 independent of ρ and t , such that

$$1 + \eta(t) \leq c_5 \left(\rho + \frac{t(1 + \eta(t))}{\rho} + \frac{t(1 + \eta(t))^2}{\rho^2} + t \right).$$

Let $\rho^2 = 2c_5 t(1 + \eta(t))$; then

$$1 + \eta(t) \leq c_6 \left(t^{\frac{1}{2}}(1 + \eta(t))^{\frac{1}{2}} + t \right)$$

for some constant $c_6 > 0$ independent of t . From this we conclude that $\eta(t) \leq c_7(1 + t)$ for some constant c_7 independent of t . Hence

$$-F(z, t) \leq c_7(1 + t)$$

for all z with $|z|^2 \leq r_0$, which implies

$$\int_t^{2t} \mathcal{R}(z, s) ds \leq \int_0^{2t} \mathcal{R}(z, s) ds = -F(z, t) \leq c_7(1 + t).$$

On the other hand, by the Li-Yau-Hamilton inequality [2], $s\mathcal{R}(z, s) \geq t\mathcal{R}(z, t)$ for $s \geq t$. Hence we have

$$\mathcal{R}(z, t) \leq c_8$$

for all t and for all z with $|z|^2 \leq r_0$. This completes the proof of the lemma. □

Proof of Theorem 4.1. Let $a(t) = h(0, t)$. We claim that the curvature of $a(t)^{-1}g(z, t)$ converges to 0 uniformly on compact sets. Note that $a(t)^{-1}g(x, t)$ has non-negative bisectional curvature. Let $\mathcal{R}(z, t)$ be the scalar curvature of $g(t)$ at $z \in \mathbb{C}^n$. Suppose first that $\lim_{t \rightarrow \infty} \mathcal{R}(0, t) = 0$. Then by the Li-Yau-Hamilton inequality [2], we conclude that $\lim_{t \rightarrow \infty} \mathcal{R}(z, t) = 0$ uniform on compact sets. Since $a(t) \leq a(0) = h(0, 0) = 1$, the claim is true in this case.

Suppose on the other hand that there exist $k \rightarrow \infty$ and $c_1 > 0$ such that $\mathcal{R}(0, t_k) \geq c_1$ for all k . We may assume that $t_{k+1} \geq t_k + 1$. By the Li-Yau-Hamilton inequality again, there is $c_2 > 0$ such that $\mathcal{R}(0, t_k + s) \geq c_2$ for all k and for all $0 \leq s \leq 1$. Since the Ricci tensor of $g(t)$ at the origin is $\text{Ric} = \frac{\mathcal{R}}{n}g$, using the Kähler-Ricci flow equation, we have

$$h(0, t_{k+1}) \leq h(0, t_k + 1) \leq e^{-c_3} h(0, t_k)$$

for some $c_3 > 0$ for all k . Hence $h(0, t_k) \rightarrow 0$ as $k \rightarrow \infty$. Since $h(0, t)$ is non-increasing, we have $\lim_{t \rightarrow \infty} a(t) = \lim_{t \rightarrow \infty} h(0, t) = 0$. On the other hand, the curvature of $g(t)$ is uniformly bounded on compact sets by Lemma 4.2. Thus our claim is true in this case as well.

Consider any sequence $t_k \rightarrow \infty$. Let $a_k = h(0, t_k)$ and $\tilde{g}_k(x, t) = \frac{1}{a_k}g(x, a_k t + t_k)$. Then $\tilde{g}_k(t)$ is a $U(n)$ -invariant solution to the Kähler-Ricci flow on $\mathbb{C}^n \times [-\frac{t_k}{a_k}, \infty)$. Note that $-t_k/a_k \leq -t_k$ because $a_k \leq 1$. By Lemmas 4.1, 4.2, for any $R > 0$, $\tilde{g}_k(x, 0)$ is uniformly equivalent to the standard Euclidean metric g_e on $D(R)$ (with respect to k). By the claim above and the Li-Yau-Hamilton inequality [2], the curvature of the metrics $\tilde{g}_k(x, t)$ approach zero uniformly (with respect to k) on compact subsets of $\mathbb{C}^n \times (-\infty, 0]$.

In particular, we conclude that $\tilde{g}_k(t)$ is uniformly equivalent to g_e in $D(R)$ provided $-1 \leq t \leq 0$, and thus by [3, Theorem 2.2], we have for any $m \geq 0$, there is a c_4 depending on R such that

$$|\nabla_e^m \tilde{g}_k(0)| \leq c_4$$

on $D(\frac{R}{2})$, where ∇_e is the derivative with respect to the standard Euclidean metric. From this it is easy to conclude the subsequence convergence of $\tilde{g}_k(0)$ uniformly and smoothly on compact subsets of \mathbb{C}^n to a flat $U(n)$ -invariant Kähler metric g_∞ , generated by some ξ_∞ say. Since the curvature is zero, we have $\xi'_\infty \equiv 0$ and thus $\xi_\infty \equiv 0$. Moreover, at the origin $(g_\infty)_{i\bar{j}} = \delta_{ij}$. Hence $h_\infty(0) = 1$ which implies that $(g_\infty)_{i\bar{j}} = \delta_{ij}$ everywhere. From this the theorem follows as t_k was chosen arbitrarily. □

In some cases, we may remove the assumption that the metric is $U(n)$ -invariant.

Proposition 4.1. *Let g_0 be a $U(n)$ -invariant complete Kähler metric on \mathbb{C}^n with non-negative bisectional curvature with maximum volume growth (i.e. $\xi \rightarrow a < 1$). Let G_0 be another complete Kähler metric on \mathbb{C}^n with bounded and non-negative bisectional curvature. Suppose G_0 is uniformly equivalent to g_0 . Let $G(t)$ be the longtime solution of the Kähler-Ricci flow obtained in [17]. Then for any $t_k \rightarrow \infty$ and any fixed $v \in T^{1,0}(\mathbb{C}^n)$ at the origin with $v \neq 0$, there is a subsequence still denoted by t_k such that*

$$\frac{1}{G(t_k)(v, \bar{v})} G(t_k)$$

will converge uniformly and smoothly on compact sets to a complete flat metric on \mathbb{C}^n .

Proof. By the proof of Theorem 4.1, and for $r > 0$, there is a constant c_r such that

$$c_r^{-1}g_e \leq \frac{1}{g(t_k)(v, \bar{v})}g(t_k) \leq c_r g_e$$

on $|z|^2 < r$, where g_e is the standard metric on \mathbb{C}^n . By Lemma 4.3, we see that

$$(c_1 c_r)^{-1}g_e \leq \frac{1}{G(t_k)(v, \bar{v})}G(t_k) \leq c_1 c_r g_e$$

on $|z|^2 < r$, for some $c_1 > 0$ independent of k and r . Also, by the results in [17], the curvature of $G(t)$ is uniformly bounded on $\mathbb{C}^n \times [0, \infty)$. The argument for convergence is now similar to that in the proof of Theorem 4.1. \square

Lemma 4.3. *Let g_0 and G_0 be complete Kähler metrics on M with bounded non-negative bisectional curvature and maximum volume growth. Let $g(t), G(t)$ be the longtime solutions of the Kähler-Ricci flow (1.1) with initial data g_0, G_0 respectively as obtained in [17]. If g_0 and G_0 are equivalent, then there exist $c > 0$ such that for all $t > 0$*

$$cg(t) \leq G(t) \leq c^{-1}g(t).$$

Proof. Let $\mathcal{R}_g(x, t)$, and $\mathcal{R}_G(x, t)$ be the scalar curvatures of $g(t)$ and $G(t)$ at x respectively. By a result of Ni [13], $\mathcal{R}_g(\cdot, 0)$ and $\mathcal{R}_G(\cdot, 0)$ decay like r^{-2} on average uniformly at all points, and by [17], $g(t)$ and $G(t)$ exist for all time and there is $c_1 > 0$ such that for all x, t

$$(4.4) \quad t\mathcal{R}_g(x, t), t\mathcal{R}_G(x, t) \leq c_1.$$

By assumption, there is $c_2 > 0$ such that

$$c_2 g_0 \leq G(0) \leq c_2^{-1} g_0.$$

Since $g(t)$ and $G(t)$ are non-increasing, we have

$$G(0) \geq c_2 g(t), g(0) \geq c_2 G(t),$$

for all t . Fix $t_0 > 0$. The bisectional curvatures of $g(t_0), G(t_0)$ are bounded above by $c_3 t_0^{-1}$ for some c_3 independent of t_0 . By [3, Lemma 3.1], there exists a constant c_4 independent of t_0, t such that for all $t > 0$,

$$G(t) \geq c_2 \left(\frac{1}{n} - c_4 t_0^{-1} t \right) g(t_0),$$

and hence

$$G\left(\frac{1}{2nc_4}t_0\right) \geq \frac{c_2}{2n}g(t_0).$$

On the other hand, by the Kähler-Ricci flow equation and (4.4), we have

$$G(t_0) \geq c_5 G\left(\frac{1}{2nc_4}t_0\right)$$

for some $c_5 > 0$ independent of t_0 . Hence we have

$$G(t_0) \geq c_6 g(t_0)$$

for some $c_6 > 0$ independent of t_0 . Similarly, one can also prove that $g(t_0) \geq cG(t_0)$ for some $c > 0$ independent of t_0 . From this the lemma follows. \square

APPENDIX A

In this appendix, we collect some basic facts for $U(n)$ -invariant Kähler metrics on \mathbb{C}^n .

Theorem A.1.

- (a) (Wu-Zheng [23]) *Every smooth $U(n)$ -invariant Kähler metric g is generated by a function $\xi : [0, \infty) \rightarrow \mathbb{R}$ with $\xi(0) = 0$ such that if*

$$h_\xi(r) := Ce^{\int_0^r -\frac{\xi(s)}{s} ds}; \quad f_\xi(r) := \frac{1}{r} \int_0^r h_\xi(s) ds,$$

where $h_\xi(0) = C > 0$ and $f_\xi(0) = h_\xi(0)$, where $r = |z|^2$, then

$$g_{i\bar{j}} = f_\xi(r)\delta_{ij} + f'_\xi(r)\bar{z}_i z_j.$$

Here $g_{i\bar{j}}$ are the components of g in the standard coordinates $z = (z_1, \dots, z_n)$ on \mathbb{C}^n . Moreover g is complete if and only if

$$\int_0^\infty \frac{\sqrt{h_\xi(s)}}{\sqrt{s}} ds = \infty.$$

- (b) (Wu-Zheng [23]) *Let $h = h_\xi$, $f = f_\xi$. At the point $z = (z_1, 0, \dots, 0)$, relative to the orthonormal frame $e_1 = \frac{1}{\sqrt{h}}\partial_{z_1}$, $e_i = \frac{1}{\sqrt{f}}\partial_{z_i}$, $i \geq 2$, with respect to g , we have the curvature tensors*

$$A = R_{1\bar{1}1\bar{1}} = \frac{\xi'}{h}, \quad B = R_{1\bar{1}i\bar{i}} = \frac{1}{(rf(r))^2} \int_0^r \xi'(s) \left(\int_0^t G(s) ds \right) dt,$$

and

$$C = R_{i\bar{i}i\bar{i}} = 2R_{i\bar{i}j\bar{j}} = \frac{2}{(rf(r))^2} \int_0^r G(s)\xi(s) dt,$$

where $2 \leq i \neq j \leq n$. These are the only non-zero components of the curvature tensor at z except those obtained from A, B or C by the symmetric properties of R .

- (c) (Wu-Zheng [23], Yang [24]) *g has positive (non-negative) bisectional curvature if and only if $\xi' > 0$ ($\xi' \geq 0$). In particular, if g has non-negative bisectional curvature and is complete, then $\xi \leq 1$.*
- (d) ([3]) *A complete $U(n)$ -invariant Kähler metric generated by ξ on \mathbb{C}^n has bounded curvature if and only if $|\frac{\xi'}{h_\xi}|$ is uniformly bounded.*

Remark A.1. By the proof of [23, 24], it is easy to see that similar to (b) in the above theorem, g has non-positive bisectional curvature if and only if $\xi' \leq 0$.

Remark A.2. If g_1 and g_2 are two smooth $U(n)$ -invariant Kähler metrics on \mathbb{C}^n generated by ξ_1, ξ_2 respectively, and if the corresponding functions h_{ξ_1}, h_{ξ_2} satisfy $h_{\xi_1} \geq h_{\xi_2}$, then $g_1 \geq g_2$. Conversely, if $g_1 \geq g_2$, then $h_{\xi_1} \geq h_{\xi_2}$. This can be seen by comparing the metrics at the points $(a, 0, \dots, 0)$.

REFERENCES

- [1] Huai Dong Cao, *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. **81** (1985), no. 2, 359–372, DOI 10.1007/BF01389058. MR799272
- [2] Huai Dong Cao, *On Harnack's inequalities for the Kähler-Ricci flow*, Invent. Math. **109** (1992), no. 2, 247–263, DOI 10.1007/BF01232027. MR1172691
- [3] Albert Chau, Ka-Fai Li, and Luen-Fai Tam, *Deforming complete Hermitian metrics with unbounded curvature*, Asian J. Math. **20** (2016), no. 2, 267–292, DOI 10.4310/AJM.2016.v20.n2.a3. MR3480020
- [4] Albert Chau and Luen-Fai Tam, *A survey on the Kähler-Ricci flow and Yau's uniformization conjecture*, Surveys in differential geometry. Vol. XII. Geometric flows, Surv. Differ. Geom., vol. 12, Int. Press, Somerville, MA, 2008, pp. 21–46, DOI 10.4310/SDG.2007.v12.n1.a2. MR2488949
- [5] Albert Chau and Luen-Fai Tam, *On a modified parabolic complex Monge-Ampère equation with applications*, Math. Z. **269** (2011), no. 3-4, 777–800, DOI 10.1007/s00209-010-0760-z. MR2860264
- [6] Esther Cabezas-Rivas and Burkhard Wilking, *How to produce a Ricci flow via Cheeger-Gromoll exhaustion*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 12, 3153–3194, DOI 10.4171/JEMS/582. MR3429162
- [7] Bing-Long Chen and Xi-Ping Zhu, *Uniqueness of the Ricci flow on complete noncompact manifolds*, J. Differential Geom. **74** (2006), no. 1, 119–154. MR2260930
- [8] Bing-Long Chen and Xi-Ping Zhu, *Volume growth and curvature decay of positively curved Kähler manifolds*, Q. J. Pure Appl. Math. **1** (2005), no. 1, 68–108. MR2154333
- [9] Xu-Qian Fan, *A uniqueness result of Kähler Ricci flow with an application*, Proc. Amer. Math. Soc. **135** (2007), no. 1, 289–298 (electronic), DOI 10.1090/S0002-9939-06-08510-8. MR2280196
- [10] Gregor Giesen and Peter M. Topping, *Existence of Ricci flows of incomplete surfaces*, Comm. Partial Differential Equations **36** (2011), no. 10, 1860–1880, DOI 10.1080/03605302.2011.558555. MR2832165
- [11] S. Huang and L.-F. Tam, *Kähler-Ricci flow with unbounded curvature*, preprint, arXiv:1506.00322.
- [12] Herbert Koch and Tobias Lamm, *Geometric flows with rough initial data*, Asian J. Math. **16** (2012), no. 2, 209–235, DOI 10.4310/AJM.2012.v16.n2.a3. MR2916362
- [13] Lei Ni, *Ancient solutions to Kähler-Ricci flow*, Math. Res. Lett. **12** (2005), no. 5-6, 633–653, DOI 10.4310/MRL.2005.v12.n5.a3. MR2189227
- [14] Lei Ni and Luen-Fai Tam, *Kähler-Ricci flow and the Poincaré-Lelong equation*, Comm. Anal. Geom. **12** (2004), no. 1-2, 111–141. MR2074873
- [15] Lei Ni and Luen-Fai Tam, *Poincaré-Lelong equation via the Hodge-Laplace heat equation*, Compos. Math. **149** (2013), no. 11, 1856–1870, DOI 10.1112/S0010437X12000322. MR3133296
- [16] Wan-Xiong Shi, *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. **30** (1989), no. 1, 223–301. MR1001277
- [17] Wan-Xiong Shi, *Ricci flow and the uniformization on complete noncompact Kähler manifolds*, J. Differential Geom. **45** (1997), no. 1, 94–220. MR1443333
- [18] Miles Simon, *Deformation of C^0 Riemannian metrics in the direction of their Ricci curvature*, Comm. Anal. Geom. **10** (2002), no. 5, 1033–1074, DOI 10.4310/CAG.2002.v10.n5.a7. MR1957662
- [19] Oliver C. Schnürer, Felix Schulze, and Miles Simon, *Stability of Euclidean space under Ricci flow*, Comm. Anal. Geom. **16** (2008), no. 1, 127–158. MR2411470
- [20] Oliver C. Schnürer, Felix Schulze, and Miles Simon, *Stability of hyperbolic space under Ricci flow*, Comm. Anal. Geom. **19** (2011), no. 5, 1023–1047, DOI 10.4310/CAG.2011.v19.n5.a8. MR2886716
- [21] Luen-Fai Tam, *Exhaustion functions on complete manifolds*, Recent advances in geometric analysis, Adv. Lect. Math. (ALM), vol. 11, Int. Press, Somerville, MA, 2010, pp. 211–215. MR2648946
- [22] P. M. Topping, *Ricci flows with unbounded curvature*, Proceedings of the International Congress of Mathematicians, Seoul 2014, arXiv:1408.6866.

- [23] Hung-Hsi Wu and Fangyang Zheng, *Examples of positively curved complete Kähler manifolds*, Geometry and analysis. No. 1, Adv. Lect. Math. (ALM), vol. 17, Int. Press, Somerville, MA, 2011, pp. 517–542. MR2882437
- [24] Bo Yang, *On a problem of Yau regarding a higher dimensional generalization of the Cohn-Vossen inequality*, Math. Ann. **355** (2013), no. 2, 765–781, DOI 10.1007/s00208-012-0803-3. MR3010146
- [25] Bo Yang and Fangyang Zheng, *$U(n)$ -invariant Kähler-Ricci flow with non-negative curvature*, Comm. Anal. Geom. **21** (2013), no. 2, 251–294, DOI 10.4310/CAG.2013.v21.n2.a1. MR3043747

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