LOCALIZATION, METABELIAN GROUPS, AND THE ISOMORPHISM PROBLEM

GILBERT BAUMSLAG, ROMAN MIKHAILOV, AND KENT E. ORR

ABSTRACT. If G and H are finitely generated, residually nilpotent metabelian groups, H is termed para-G if there is a homomorphism of G into H which induces an isomorphism between the corresponding terms of their lower central quotient groups. We prove that this is an equivalence relation. It is a much coarser relation than isomorphism, our ultimate concern. It turns out that many of the groups in a given equivalence class share various properties, including finite presentability. There are examples, such as the lamplighter group, where an equivalence class consists of a single isomorphism class and others where this is not the case. We give several examples where we solve the Isomorphism Problem. We prove also that the sequence of torsion-free ranks of the lower central quotients of a finitely generated metabelian group is computable. In a future paper we plan on proving that there is an algorithm to compute the numerator and denominator of the rational Poincaré series of a finitely generated metabelian group and will carry out this computation in a number of examples, which may shed a tiny bit of light on the Isomorphism Problem. Our proofs use localization, class field theory and some constructive commutative algebra.

1. Introduction

1.1. **Preliminary remarks.** In a recent paper [BMO1], entitled "A new look at finitely generated metabelian groups", we outlined a number of ideas for exploring finitely generated metabelian groups. These ideas arise from several seemingly different sources: algebraic geometry, algebraic number theory, combinatorial group theory and constructive commutative algebra. Here we will provide some of the details briefly sketched in that paper and discuss and describe some additional theorems that our first paper has given rise to.

We have chosen in this paper to take a purely combinatorial view of our work. In the third paper of this series we will take a more homological, functorial approach using localization of groups, providing an alternative approach to the material which we hope will lead to further understanding.

1.2. **Finitely generated metabelian groups.** A group G is metabelian if its derived group, A, is abelian. In the first in a series of fundamental papers going back to 1954, Philip Hall [Ha1] observed that A can be viewed as a module over the

integral group ring $\mathbb{Z}[H]$ of the factor derived group $H = G/A$, where H acts on A by conjugation. In the event that G is finitely generated, H is a finitely generated abelian group and, as Hall noted, A is a finitely generated module over the finitely generated commutative ring $\mathbb{Z}[H]$. It follows from Hilbert's basis theorem that G satisfies the maximal condition for normal subgroups and hence that the number of isomorphism classes of finitely generated metabelian groups is countable. As a consequence every finitely generated metabelian group, viewed in the category of metabelian groups, has a finite description termed a *preferred presentation* in [BCR] (see also [LR]) defined as follows:

A *preferred presentation* of a finitely generated metabelian group G is a presentation which takes the form

$$G = \langle g_1, \ldots, g_t \mid R_1 \cup R_2 \rangle$$

where

(1) R_1 is a finite set of words of the form

$$w = \prod_{1 \leq i \leq j \leq t} [g_i, g_j]^{u_{ij}},$$

and we use the usual notation $[x,y]$ for $x^{-1}y^{-1}xy$, y^x for $x^{-1}yx$, and the u_{ij} are words of the form $g_1^{m_1} \ldots g_t^{m_t}$;

(2) R_2 is a finite set of words uw where u has the form $g_1^{m_1} \ldots g_t^{m_t}$ and

$$w = \prod_{1 \leq i \leq j \leq t} [g_i, g_j]^{v_{ij}},$$

with v_{ij} of the form $g_1^{n_1} \ldots g_t^{n_t}$.

Thus the words in R_1, together with the addition of all commutators $[x,y]$ where x and y take the form $[g_i, g_j]^{u_{ij}}$, are a regular presentation of A while those in R_2, read modulo A, yield a finite presentation of G/A. It follows that there is a recursive enumeration of all (preferred) presentations of finitely generated metabelian groups. These presentations are the finite descriptions needed in any discussion of algorithms involving finitely generated metabelian groups.

1.3. **Algorithmic problems about finitely generated metabelian groups.**
Both the word and conjugacy problems about finitely generated metabelian groups are solvable. The first is essentially due to Hall [Ha1] and the second to Noskov [N]. The Isomorphism Problem however remains almost untouched, except for the theorem of Groves and Miller [GM]: *The Isomorphism Problem for finitely generated free metabelian groups, i.e., the factor groups of finitely generated free groups by their second derived groups, is solvable.* In other words if we write down a preferred presentation of a finitely generated free metabelian group G and recursively enumerate all preferred presentations of finitely generated metabelian groups, then the subset consisting of those presentations of groups isomorphic to G is recursive. Another way of putting this is that there is an algorithm to decide whether or not any preferred presentation defines a group isomorphic to G.

There are many other solvable algorithmic problems about finitely generated metabelian groups:

(1) There is an algorithm to find a preferred presentation of a finitely generated subgroup of a finitely generated metabelian group.
(2) There is an algorithm to determine if a finitely generated metabelian group is torsion-free.
(3) There is an algorithm to find a module presentation of the derived group of a finitely generated metabelian group.
(4) There is an algorithm to decide if a finitely generated metabelian group is residually nilpotent.
(5) There is an algorithm to find the center of a finitely generated metabelian group.
(6) There is an algorithm to find the centralizer of a finitely generated subgroup in a finitely generated metabelian group.

We recommend Lennox and Robinson [LR] as an excellent general reference as well as Baumslag, Cannonito, and Robinson [BCR] and the papers of Seidenberg [Sei1] and [Sei2]. The specific algorithms above can be found, respectively, in [LR, page 185], [BCR, Cors. 4.4, 3.1, 9.2, Thms. 3.5, 6.1].

1.4. **The Isomorphism Problem.** Remeslennikov [R] has proved that every finitely generated metabelian group has a verbal subgroup of finite index which is residually nilpotent. So the Isomorphism Problem for finitely generated metabelian groups can be broken down into the Isomorphism Problem for finitely generated residually nilpotent metabelian groups, the study of the finite metabelian subgroups of their automorphism groups and the study of the finite extensions that come into play. This suggests that one approach to the Isomorphism Problem is to focus attention on the class $\mathcal{M}$ of finitely generated, residually nilpotent metabelian groups.

Definition 1.1. We term a group in the class $\mathcal{M}$ of finitely generated, residually nilpotent, metabelian groups an $\mathcal{M}$-*group.*

Since the preferred presentations of such $\mathcal{M}$-groups are not finite, in order to take advantage of the fact that they are residually nilpotent and the rich algorithmic properties of finitely presented nilpotent groups, we need to prove that the lower central sequences of $\mathcal{M}$-groups are computable. Hence the lower central sequences of $\mathcal{M}$-groups provide us with a computable set of invariants of $\mathcal{M}$-groups. This suggests that we focus on the following.

Definition 1.2. Two groups G and H have the *same lower central sequences* if $G/\gamma_n(G) \cong H/\gamma_n(H)$ for every n.

Here $\gamma_n(X)$ denotes the nth term of the lower central series of the group X.

Some of the properties of $\mathcal{M}$-groups can be encapsulated in a construction introduced by J. P. Levine [L1]. Levine called this an *algebraic closure of the group.* We restrict Levine's group closure to the class of $\mathcal{M}$-groups and use an alternate but equivalent definition in this case. We call this special case of Levine's group closure the *telescope of the group G,* as this name suggests the structure of the group closure when considering $\mathcal{M}$-groups. Nonetheless, Levine was aware of the telescoping structure of his group closure, at least for those $\mathcal{M}$-groups which are semi-direct products. The telescope and its applications will be discussed in detail in §2, where we will formulate some of our results, some of which were reported in our previous

paper [BMO1]. We also consider the better known and more studied pro-nilpotent completion of a group G, which we denote $\widehat{G}$. The telescope of G lies within $\widehat{G}$. In the event that the abelianization of G is finitely generated (in particular for $\mathcal{M}$-groups), $\widehat{G}$ has the same lower central sequences as G [Bo1, Theorem 13.3]. If a residually nilpotent group G is not nilpotent, $\widehat{G}$ is uncountable. Nonetheless in the case where G is an $\mathcal{M}$-group, we shall prove that $\widehat{G}$ does share an important and interesting property with G. We shall also discuss this further in §2.

2. Our main results

Recall that two groups G and H have the same lower central sequences if there is a sequence of isomorphisms

$$\phi_n : G/\gamma_n(G) \longrightarrow H/\gamma_n(H) \,(n = 1, 2, \ldots, n, \ldots)$$

between their lower central quotient groups. We have not been able to prove that there is a homomorphism $\phi : G \longrightarrow H$ which induces such a sequence of isomorphisms between their lower central sequences.

Definition 2.1. Let G and H be groups. We say that H is para-G if there is a homomorphism of G into H which induces isomorphisms between the corresponding quotients of their lower central series.

It is this relationship that we will explore in some detail here.

2.1. **Para-equivalence.** We begin with the following theorem:

Theorem 5.1. *Let G and H be residually nilpotent metabelian groups. If H is finitely generated and if H is para-G, then G is also finitely generated.*
  That is, there exists a para-G $\mathcal{M}$-group only if G is an $\mathcal{M}$-group.

Theorem 5.1 is an indication that there is a connection between groups with the same lower central sequences and their structure. A more important connection is contained in W. Magnus' fundamental paper [Mag] in 1935, where he proved the following.

Theorem 2.2 (Magnus)**.** *If ϕ is a homomorphism of a residually nilpotent group G into a group H which induces isomorphisms between the respective terms of their lower central sequences, then ϕ is a monomorphism.*

The relation of being para-G is much stronger than one might suspect, as one sees from the following theorem.

Theorem 5.8(4). *Let G and H be $\mathcal{M}$-groups. Then G is para-H if and only if H is para-G.*

The above theorem is part of our main result, our Telescope Theorem, Theorem 5.8, which we will discuss and prove in §5.
  It follows that this property, para-G, is an equivalence relation on the class $\mathcal{M}$.

Definition 2.3. If H is para-G and G is para-H, we say G and H are *para-equivalent.*

That is, our Telescope Theorem implies that the relation of being para-G is an equivalence relation when restricted to $\mathcal{M}$-groups.

We will construct in §9 examples of para-equivalence classes for $\mathcal{M}$-groups which consist of more than a single isomorphism class. Notice that Theorem 2.2 and 5.8(4) together imply that if the $\mathcal{M}$-groups G and H are para-equivalent, then G is isomorphic to a subgroup of H and H is isomorphic to a subgroup of G. So para-equivalence can be viewed as a coarse form of isomorphism and can be compared with isoclinism, an equivalence relation that Philip Hall introduced in an attempt to classify finite p-groups. Another consequence of the Telescope Theorem is the following.

Theorem 6.1. *Suppose that two $\mathcal{M}$-groups are para-equivalent. Then either both of them are finitely presented or neither of them is.*

The conclusion of Theorem 6.1 does not hold without restriction, since Bridson and Reid [BR] have recently constructed examples of finitely generated residually nilpotent groups with the same lower central sequences and with the following properties:

(1) There is a homomorphism of one of the groups into another which induces isomorphisms between their lower central sequences;
(2) one of the groups is finitely presented whereas the other is not;
(3) one of the groups has finitely generated second homology group whereas the other does not.

Their work follows our ongoing and earlier work [BMO1] and [BR], where a closer connection is established between certain groups with the same lower central sequences.

Another interesting consequence of Theorem 5.8(4) involves the subgroups of free metabelian groups.

Theorem 6.3. *Let F be a finitely generated free metabelian group and suppose that G is an $\mathcal{M}$-group with the same lower central sequences as F. Then G is isomorphic to a subgroup of F.*

Since there exists a wide variety of $\mathcal{M}$-groups with the same lower central sequences as a free metabelian group, it follows that the subgroup structure of finitely generated free metabelian groups is extremely complicated.

2.2. **Pro-nilpotent completions.** It is not hard to see that if a residually nilpotent group is torsion-free, then so too is its pro-nilpotent completion. As previously mentioned, the pro-nilpotent completion of an $\mathcal{M}$-group, G, has the same lower central sequence as G. It would be interesting to explore other properties of residually nilpotent groups which are inherited by their pro-nilpotent completions. We give one example of a theorem of this kind.

Theorem 8.1. *Let the $\mathcal{M}$-group G be polycyclic. Then $\widehat{G}$ is locally polycyclic; i.e., its finitely generated subgroups are polycyclic.*

Although we have not done so, in all likelihood our proof of Theorem 8.1 will carry over to polycyclic groups in general.

2.3. **The Isomorphism Problem and Poincaré series.** We recall that if G is a group and $r_n = r_n(G)$ is the torsion-free rank of $\gamma_n(G)/\gamma_{n+1}(G)$, then we term

$$P(G) = \sum_{n=1}^{\infty} r_n x^n$$

the *rational Poincaré series* of G. Such series have generally been studied in connection with graded modules over graded commutative rings. The Poincaré series of graded modules are rational functions; see, e.g., Atiyah and MacDonald [AM]. Baumslag first defined and investigated the Poincaré series of finitely generated residually torsion-free nilpotent metabelian groups and showed that this is a rational function in [Bau2] (see also Groves and Wilson [GW]). As a small contribution to the isomorphism problem for finitely generated metabelian groups, we add parts (2) and (3) of the theorem below to the first author's prior result (1), where (3) can be viewed as an addendum to the theorem of Groves and Miller [GM].

Theorem 7.3. *Let G be a finitely generated metabelian group. Then the following hold:*

(1) *The rational Poincaré series $P(G)$ of G is a rational function.*
(2) *There is an algorithm to compute the series $P(G)$.*
(3) *The quotient of the two polynomials which is the Poincaré series of a free metabelian group expressed as a rational function is computable.*

2.4. **More on the Isomorphism Problem.** As already noted, Theorem 5.8(4), provides a coarse classification called para-equivalence of certain finitely generated metabelian groups. It points the way to an approach to the Isomorphism Problem and gives rise to a number of questions. For example if the para-equivalence class of a given group is a singleton, can one solve the Isomorphism Problem for that group? In §9 we will construct some examples of groups which lie in a single equivalence class and solve the Isomorphism Problem for these groups. Furthermore, even if the equivalence class of a given group is not a singleton, we shall show that the Isomorphism Problem is sometimes solvable for such a group. We record here some samples of these kinds of results, which we will discuss more fully in §9. We also gather together in §9.2 some results and approaches to the Isomorphism Problem which make use of ideal theory and class field theory to distinguish some metabelian groups from one another. Although Gruenwald and Segal [Se] have solved the Isomorphism Problem for polycyclic groups, this approach is of independent interest and will be discussed in a further paper in this series.

2.5. **Examples.** First we give two families of $\mathcal{M}$-groups where each para-equivalence class contains a unique group up to isomorphism and for which the Isomorphism Problem is solvable.

Theorem 10.1. *Let $G_n = \langle a, t \mid t^{-1}at = a^n \rangle$, $n \neq 2$. Then the following hold:*

(1) *G_n is residually nilpotent;*
(2) *any $\mathcal{M}$-group with the same lower central sequence as G_n is isomorphic to G_n;*
(3) *the Isomorphism Problem is solvable for each G_n.*

In order to formulate our next two theorems, we use wreath products, which we will define in §9. These two theorems are combined in §9 into Theorem 9.1.

Theorem 9.1(1) and (2). *Let W be the wreath product of an abelian group A of prime order and an infinite cyclic group. Then*

(1) *W is residually nilpotent;*

(2) *any $\mathfrak{M}$-group with the same lower central sequences as W is isomorphic to W;*

(3) *the Isomorphism Problem is solvable for W.*

It is worth noting that this theorem includes the lamplighter group, the wreath product of a group of order two and an infinite cyclic group. Theorem 9.1(1) and (2) is capable of considerable generalization; however, because this will take us too far astray from our current work, we prefer to leave its formulation and proof to another time.

Our next example involves the wreath product $W = \langle a \rangle \wr \langle t \rangle$ of the infinite cyclic group A on a by the infinite cyclic group T on t. So W is generated by a and t, the conjugates of a by the different powers of t freely generate a free abelian group B, and $W = B \rtimes T$. We then have the following.

Theorem 9.1(3) and (4). *Let $W = A \wr T$. Then*

(1) *W is residually-torsion free nilpotent;*

(2) *the subgroup V of W generated by $a^{-1}(a^2)^t$, $a^2 a^{-t}$, and t has the same lower central sequences as W;*

(3) *$W \not\cong V$;*

(4) *the Isomorphism Problem is solvable for W.*

As a further example we will construct two polycyclic, residually nilpotent, metabelian $\mathfrak{M}$-groups with the same lower central sequences which are not isomorphic. We will not describe these groups here. They are constructed in §9 as infinite cyclic extensions of the additive groups of rings of integers of algebraic number fields with class numbers at least two. The argument used to prove that the groups involved are not isomorphic requires that certain ideals are not principal, an idea used in part (4) of the proof of Theorem 9.1 as well.

3. The arrangement of the rest of this paper

Before describing the main contents of this paper we will list some of the notation and standard definitions that will be used throughout.

3.1. Definitions and notation.
Let G be a group and let $x_1, x_2, \ldots$ be elements of G. We denote the commutator $x_1^{-1} x_2^{-1} x_1 x_2$ by $[x_1, x_2]$ and define, for $n > 0$,

$$[x_1, \ldots, x_{n+1}] = [[x_1, \ldots, x_n], x_{n+1}].$$

If H and K are subgroups of G, we define

$$[H, K] = gp([h, k] \mid h \in H, k \in K).$$

Its subgroup $[G, G]$ is termed the *derived group*. G is *metabelian* if its derived group is abelian. The *lower central series*

$$G = \gamma_1(G) \geq \gamma_2(G) \cdots$$

of G is defined inductively by setting

$$\gamma_{n+1}(G) = [\gamma_n(G), G],$$

and the sequence

$$G/\gamma_2(G), G/\gamma_3(G), \ldots, G/\gamma_n(G), \ldots$$

is called *the lower central sequence of G*. A group G is *residually nilpotent* if

$$\bigcap_{n=1}^{\infty} \gamma_n(G) = 1.$$

The rest of the paper is arranged as follows.

In §4 we will deal with localization of rings and modules, needed in the formulation and proof of our main theorem, the *Telescope Theorem*, Theorem 5.8.

We start §5 with the proof of Theorem 5.1. The rest of §5 will be devoted to a formulation of Theorem 5.8 together with a number of results about the telescope of a finitely generated residually nilpotent metabelian group which will be needed in this paper.

In §6 we will discuss a number of consequences of our Telescope Theorem, namely Theorems 6.1, 6.2 and 6.3.

In addition to our proof of Theorem 8.1, in §8 we will briefly discuss some possible implications of the algorithmic nature of the computation of the Poincaré series involved.

In §9 we give a number of examples of groups which are completely determined by their lower central sequences and solve the Isomorphism Problem for a few finitely generated metabelian groups. The use of the ideal theory needed to distinguish some of the groups constructed and the class field theory that comes into play will also be briefly discussed there. This aspect of our work will be dealt with in some detail in the third of this series of papers devoted to finitely generated metabelian groups.

4. Preparations for the Telescope Theorem

We term our main result of this paper the *Telescope Theorem*.

The *telescope of a metabelian group* is a type of group localization, or algebraic closure, whose purpose, from our view, is to turn para-equivalences into equivalences. The original construction arose from a radically different context in knot theory through work of J. P. Levine [L1, L2, L3]. We encourage the reader to investigate Levine's beautiful and powerful constructions independently of this work, and we acknowledge our debt.

When restricting to metabelian groups, Levine's construction has a particularly useful formulation, which we investigate here as our definition of the group telescope. In particular, we define the group telescope using the classical constructions of ring and module localization, that is, the result of adjoining inverses to elements in a commutative ring or module. This classical localization, in turn, extends the construction of a field from an integral domain making use of "fractions".

It was the first author who originally proposed employing module localization to examine the Isomorphism Problem for metabelian groups at an NSF funded conference at City College of New York, in March 2011, entitled *Finitely Presented Solvable Groups.* Algebraic geometry motivated this approach. Thus, ideas arising from algebraic geometry and knot theory unexpectedly merge in our exploration of metabelian groups.

4.1. **Localization of rings and modules.** The localizations referred to above are respectively:

- the construction of the ring of fractions $S^{-1}R$ of a unitary commutative ring R with respect to a multiplicatively closed subset S of R containing the identity element 1 of R
  and
- a related construction, the module of fractions $S^{-1}M$ of an R-module M with respect to such a multiplicatively closed set S.

We will sometimes refer to these constructions respectively as rings of fractions or modules of fractions or simply as localizations. Our discussion will follow closely that of Atiyah and MacDonald [AM]. We leave most details and proofs to the reader and refer simply to the cited reference.

We begin first with the construction of the ring $S^{-1}R$, which is defined to be the set of the equivalence classes of elements $(a, s) \in R \times S$ subject to the equivalence relation

$$(a, s) \sim (b, t) \text{ if there exists } u \in S \text{ such that } (at - bs)u = 0.$$

We denote the equivalence class of $(a, s) \in S^{-1}R$ by $\frac{a}{s}$ or, at times, by a/s. $S^{-1}R$ can be turned into a unitary commutative ring in the obvious way:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}, \qquad \frac{a}{s}\frac{b}{t} = \frac{ab}{st}.$$

The element $\frac{s}{1}$ is invertible in $S^{-1}R$ with inverse $\frac{1}{s}$. Notice that $S^{-1}R$ is again a commutative unitary ring.

There is an analogous construction to $S^{-1}R$ where R is replaced by an R-module M and the equivalence relation defined on $R \times S$ is replaced by an equivalence relation $\sim$ on $M \times S$ as follows:

$$(a, s) \sim (b, t) \text{ if there exists } u \in S \text{ such that } (at - bs)u = 0.$$

The set of equivalence classes of $M \times S$ is denoted by $S^{-1}M$ and we denote the equivalence class of (a, s) by $\frac{a}{s}$ or a/s. $S^{-1}M$ is then turned into an $S^{-1}R$ module in the obvious way by defining $a/s \cdot r/t = ar/st$. If we now fix $s \in S$ and consider the R-module $M_s = \{\frac{a}{s} \mid a \in M\}$, then the mapping $\mu_s : a \mapsto \frac{a}{s}$ is monic and maps M isomorphically onto $M_s \le S^{-1}M$ provided that $as \ne 0$ for every $a \in M, a \ne 0$. If this condition is satisfied we say that S does not contain any zero divisors of M. In particular if S does not contain any zero divisors of M, then the mapping $a \mapsto a/1$ is monic, and we can identify each element $a \in M$ with $a/1 \in S^{-1}M$. If $\alpha : M \longrightarrow N$ is a homomorphism of the R-module M into the R-module N, then it gives rise to an $S^{-1}R$-module homomorphism $S^{-1}\alpha : S^{-1}M \longrightarrow S^{-1}N$ defined by $S^{-1}\alpha : \frac{a}{s} \mapsto \frac{\alpha a}{s}$. Then it follows that $S^{-1}(\beta\alpha) = (S^{-1}\beta)(S^{-1}\alpha)$.

We will also denote $S^{-1}R$ by R_S and $S^{-1}M$ by M_S, and if ϕ is a homomorphism of the R-module M into the R-module N, we will denote $S^{-1}\phi$ by ϕ_S.

Lemma 4.1. *Suppose M, N and P are R-modules, R a commutative unitary ring. Then the following hold.*

(1) *If the sequence*

$$0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} P \to 0$$

is exact, so too is

$$0 \to M_S \xrightarrow{\alpha_S} N_S \xrightarrow{\beta_S} P_S \to 0.$$

(2) *If F is a free R-module with basis X, then F_S is a free R_S module with basis $\{x/1 \mid x \in X\}$.*

The proof of Lemma 4.1 is well known, straightforward, and omitted.

We now consider the special case where $R = \mathbb{Z}[H]$ is the integral group ring of the abelian group H and $S = 1 + I$, where I is the augmentation ideal of R, i.e., the ideal consisting of those elements of R with coefficient sum 0. The proof of the following lemma is also straightforward and will be omitted.

Lemma 4.2. *Let M and N be $R = \mathbb{Z}[H]$-modules. Then the following hold.*

(1) $(MI)_S = M_S I$.

(2) $M_S I^k = (MI^k)_S$.

(3) *If M is a submodule of the R-module N, then $\left(\frac{N}{M}\right)_S \cong \frac{N_S}{M_S}$.*

We come next to an important lemma which is needed to verify one of the properties of our Telescope Theorem.

Lemma 4.3. *Let H be an abelian group and let M and N be $R = \mathbb{Z}[H]$-modules, where H is an abelian group. Furthermore, let σ be a homomorphism of M to N. If σ induces a homomorphism from M onto N/NI and if N is finitely generated, then σ_S maps M_S onto N_S.*

Proof. Let $b_1, \ldots, b_k$ be a finite set of generators of N and and let F be the free $R = \mathbb{Z}[H]$-module with basis $x_1, \ldots, x_k$. Then the homomorphism ν from F to N, defined by sending x_i to b_i for each i, is onto N. Now σ induces a homomorphism from M onto N/NI. Consequently, for each i there exists an element $a_i \in M$ such that $\sigma(a_i) = b_i + \sum_{j=1}^{k} b_j r_{ij}$ where the $r_{ij} \in I$.

Define a homomorphism λ of F to F in the usual way by a $k \times k$ matrix $\Lambda = (\lambda(i,j))$ where $\lambda(i,j) = \delta_{ij} + r_{ij}$. Finally, define a homomorphism ρ from F to M by mapping x_i to a_i. It then follows that the following diagram is commutative:

$$
\begin{array}{ccc}
F & \xrightarrow{\lambda} & F \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \nu} \\
M & \xrightarrow{\sigma} & N
\end{array}
$$

Consequently on localizing each of the terms in the above diagram, we get another commutative diagram:

$$
\begin{array}{ccc}
F_S & \xrightarrow{\lambda_S} & F_S \\
\downarrow{\scriptstyle \rho_S} & & \downarrow{\scriptstyle \nu_S} \\
M_S & \xrightarrow{\sigma_S} & N_S
\end{array}
$$

The determinant of the matrix λ_S is invertible since it belongs to S. Consequently λ_S is an isomorphism. It follows that $\nu_S \lambda_S$ is onto N_S, and so the commutativity of the above diagram implies that σ_S is onto, as claimed. $\qquad \square$

5. The formulation and proof of the Telescope Theorem

We give an alternative to our original proof of the Telescope Theorem. We thank C.F. Miller III for the key Lemma 5.4 used in this newer proof.

Before proceeding to the formulation and proof of the Telescope Theorem, we prove the following theorem.

Theorem 5.1. *Let G and H be residually nilpotent metabelian groups. If H is finitely generated and if there is a homomorphism ϕ of G into H which induces an isomorphism between their lower central sequences, then G is also finitely generated.*

Proof. Let A be the derived group of G and B the derived group of H. Then ϕ maps A into B and induces a homomorphism of G onto H/B. So we can find a finite subset X of G whose image Y under ϕ generates H modulo B. Since A is invariant under conjugation by the elements of X, it follows that $\phi(A)$ is invariant under conjugation by the elements of Y. Since ϕ maps A into B it also follows that $\phi(A)$ is invariant under conjugation by the elements of H, i.e., that $\phi(A)$ is normal in H, and therefore an H/B-module. Since $\mathbb{Z}[H/B]$ is Noetherian, $\phi(A)$ is the normal closure of finitely many elements in H. By Theorem 2.2, ϕ is a monomorphism. Therefore A is the normal closure in G of finitely many elements. Since G/A is finitely generated, so too is G. $\qquad\square$

5.1. **Preliminaries leading to the proof of the Telescope Theorem.** Let G be an $\mathcal{M}$-group with derived group A, $S = 1+I$, where I is the augmentation ideal of the integral group ring R of $H = G/[G,G]$. A_S is a G-module and hence, as before, G acts on the localization A_S of A. We can then form the semi-direct product $P = G \ltimes A_S$ of A_S by G. So, denoting the elements of P by pairs $(g, a/s)$ (and noticing the peculiarities of this notation), g is here an element in a multiplicatively written group G while a/s is an element in a G-module now endowed with an additive notation. Let $K = \{(a^{-1}, a/1) \mid a \in A\}$. Then K is a normal subgroup of P. We now define what we term the *telescope* of G:

Definition 5.2. The *telescope of G* is the factor group $G_S = P/K$.

We need some further preparation before we can formulate in §5.3 our main theorem, the Telescope Theorem, Theorem 5.8.

5.2. **Some important subgroups of G_S.** We will need to identify some of the subgroups of G_S. First we have

Lemma 5.3. *Let G be an $\mathcal{M}$-group with derived group A and factor derived group $H = G/A$. Let R be the integral group ring of H and let $S = 1 + I$, where I is the augmentation ideal of R. Let G_S be the telescope of G. Then*

(1) *the mapping $\phi_1 : G \longrightarrow G_S$ defined by*

$$\phi_1 : g \mapsto (g, 0/1)\,(g \in G)$$

   is a monomorphism;
(2) *the mapping $\alpha : A_S \longrightarrow G_S$ defined by*

$$\alpha : a/s \mapsto (1, a/s)\,(a \in A, s \in S)$$

   is a monomorphism;
(3) *$G/A \cong G_S/A_S$.*

Proof. We begin by noting that if $a \in A$, $s \in S$, then $s = 1 - \alpha$ with $\alpha \in I$. So $as = a - a\alpha$. We note that if $a \neq 0$, then $as \neq 0$, for otherwise

$$a = a\alpha = a\alpha^2 = \cdots = a\alpha^n = \cdots.$$

Consequently $a \in \bigcap_{n=1}^{\infty} A(I^n)$. However $A(I^n) = \gamma_{n+2}(G)$, which implies that $a = 0$ since G is residually nilpotent. Thus no element of S is a zero-divisor of A. It follows, in particular, that $A_s \cong A$ for every $s \in S$ and in particular that $A_1 \cong A$.

So we can, as needed subsequently, identify a with $a/1$ and thence we identify A with A_1.

The proofs of (1) and (2) of Lemma 5.3 are straightforward. We need only note that if $(g, 0/1) \in K$, then $(g, 0/1) = (a^{-1}, a/1)$, which implies, interpreting the first a as an element in G, that $a = 1$ and hence that $g = 1$.

To prove (3) it is worthwhile to first clarify some of the identifications that one might take for granted working with G_S. We identify $(g, 0/1)$ with $g \in G$ and a/s with $(1, a/s)$. Then this identifies G with a subgroup of G_S and A_S with an abelian normal subgroup of G_S. Using these identifications, we see that $G_S = GA_S$, $G \cap A_S = A_1 = A = A/1$, and $G_S/A_S \cong G/A$.   □

Now recall that for each $s \in S$, $A_s = \{a/s \mid a \in A\}$. A_s in A_S is invariant under conjugation by elements of G. Viewing, once again, G and A_s as subgroups of G_S, we define G_s by the equation $G_s = GA_s$. Thus, the telescope, G_S, is an ascending union $G_S = \bigcup_{s \in S} G_s$ of the subgroups G_s.

In order to prepare for the proof of the Telescope Theorem we will need a number of lemmas.

5.3. A key lemma. Let G be a metabelian group.

Lemma 5.4 (Miller III). *Let $y_1, \ldots, y_n$ be elements of G and let θ be the mapping from G into G defined by*

$$x \mapsto x[x, y_1] \cdots [x, y_n] \, (x \in G).$$

Then θ is an endomorphism of G which induces the identity on $G/[G, G]$.

Proof. Suppose that x, y and z are elements of G. Then $[xz, y] = [x, y]^z [z, y]$. Hence, noting that commutators commute in G, we find that

$$
\begin{aligned}
\theta(xz) &= xz[xz, y_1][xz, y_2] \cdots [xz, y_n] \\
&= xz[x, y_1]^z [z, y_1] \cdots [x, y_n]^z [z, y_n] \\
&= xz([x, y_1][x, y_2] \cdots [x, y_n])^z [z, y_1][z, y_2] \cdots [z, y_n] \\
&= x[x, y_1][x, y_2] \cdots [x, y_n] z[z, y_1][z, y_2] \cdots [z, y_n] \\
&= \theta(x)\theta(z).
\end{aligned}
$$

□

Notice that the definition of θ does not depend on the ordering of the set $\{y_1, \ldots, y_n\}$.

We now have the following important consequences of Lemma 5.4.

Lemma 5.5. *Let G be an $\mathcal{M}$-group and let A be the derived group of G. Furthermore, let I be the augmentation ideal of the integral group ring $R = \mathbb{Z}[H]$ of $H = G/A$. Let $S = 1 + I$ and let $t \in S$. Then $t = 1 + \alpha$, where $\alpha \in I$, and so having chosen a set Y of representatives of the cosets of A in G, α can, disregarding order, be written uniquely as $\alpha = (-1 + y_1 A) + \cdots + (-1 + y_n A) \, (y_j \in Y)$. Then the mapping θ_t defined by*

$$x \mapsto x[x, y_1] \ldots [x, y_n] \, (x \in G_t)$$

is an endomorphism of G_t which is monic, maps A_t isomorphically onto A and G_t isomorphically onto G. Hence

$$G_t \cong G, \quad A\gamma_2(G_t) = A_t \cong A \cong \gamma_2(G).$$

Proof. The proof here is most easily given by using multiplicative notation for R-modules. Thus we use the notation $(a/t)^r$ in place of $(a/t)r$. Moreover, if $r = c_1g_1 + \cdots + c_ng_n$, then $(a/t)^r = (a^{g_1})^{c_1} \cdots (a^{g_n})^{c_n}$. Notice that

$$\theta_t(a/t) = a/t \cdot [a/t, y_1] \cdots [a/t, y_n] = (a/t)^{1+(-1+y_1)+\cdots+(-1+y_n)} = (a/t)^t = a.$$

Now every element $x \in G_t$ can be written in the form $x = g \cdot (a/t)$, where $g \in G$ and $a \in A$. Then

$$\theta_t(g \cdot a/t) = \theta_t(g)\theta_t(a/t) = \theta_t(g) \cdot (at/t) = \theta_t(g) \cdot a.$$

Since θ_t maps G into G it follows that θ_t maps G_t into G. Moreover modulo A_t, θ_t is the identity and is monic on A_t. So θ_t is an isomorphism as claimed. The rest of the lemma follows easily.   □

Notice that the definition of the mapping θ_t depends on the choice of the set of representatives of A in G.

Adopting the notation used above, we now are in a position to prove the following lemma.

Lemma 5.6. *Let*

$$S = \{s_1, s_2, \ldots, s_n, \ldots\}$$

and

$$T = \{t_1, t_2, \ldots, t_n, \ldots\} \text{ where for each } n,\ t_n = s_1 \ldots s_n.$$

If we denote $G = G_0$ and $G_i = G_{t_i}$, then the following hold:

(1) $G = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_S$ *is an ascending sequence of subgroups of G_S;*

(2) $G_i \cong G$ *for all i;*

(3) $G_S = \bigcup_{i=1}^{\infty} G_i$;

(4) $\gamma_2(G_S) = A_S$;

(5) $A/AI \cong A_S/A_SI$;

(6) $A/AI^n \cong A_S/A_SI^n$;

(7) $\gamma_{n+2}(G_S) = A_SI^n = (AI^n)_S$, $\gamma_{n+2}(G) = AI^n$, *for every n;*

(8) $\gamma_2(G)/\gamma_{n+2}(G) \cong \gamma_2(G_S)/\gamma_{n+2}(G_S)$;

(9) G_S *is residually nilpotent.*

Proof. We prove the above statements sequentially.

(1) This follows immediately from the very definitions of the G_i since

$$A_{i+1} = A_{t_{i+1}} = A_{t_i \cdot s_{i+1}} \geq A_{t_i} = A_i.$$

(2) This is a special case of Lemma 5.5.

(3) This follows from Lemma 5.3, as discussed in the paragraph subsequent to that lemma.

(4) Recall that the isomorphism $\theta_t \colon G_t \to G$ of Lemma 5.5 maps A_t to the derived subgroup A of G. It follows that the inverse of θ_t maps A to the derived subgroup of G_t and so $\gamma_2(G_t) = A_t$. Consequently

$$\gamma_2(G_S) = \gamma_2(\bigcup_{t_i} G_{t_i}) = \bigcup_{t_i} \gamma_2(G_{t_i}) = \bigcup(A_{t_i}) = A_S.$$

(5) We have the following sequence of isomorphisms:

$$A_S/A_SI \cong A_S \otimes_{Z[H]} \mathbb{Z} \cong (A \otimes_{\mathbb{Z}[H]} \mathbb{Z}[H]_S) \otimes_{\mathbb{Z}[H]} \mathbb{Z}$$
$$\cong A \otimes_{\mathbb{Z}[H]} (\mathbb{Z}[H]_S \otimes_{\mathbb{Z}[H]} \mathbb{Z}) \cong A \otimes_{\mathbb{Z}[H]} \mathbb{Z} \cong A/AI.$$

(6) The case $k = 1$ is taken care of by (5). Suppose inductively that $A_S/A_S I^k \cong A/AI^k$ for $k = n$. We have a commutative diagram where the second row is exact by the flatness of localization, Lemma 4.1:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{AI^n}{AI^{n+1}} & \longrightarrow & \dfrac{A}{AI^{n+1}} & \longrightarrow & \dfrac{A}{AI^n} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow{\scriptstyle\cong} & & \\
0 & \longrightarrow & \left(\dfrac{AI^n}{AI^{n+1}}\right)_S & \rightarrow & \left(\dfrac{A}{AI^{n+1}}\right)_S & \longrightarrow & \left(\dfrac{A}{AI^n}\right)_S & \longrightarrow & 0
\end{array}
$$

The right hand vertical homomorphism is an isomorphism by the inductive hypothesis and Lemma 4.2. The following shows that the left hand vertical homomorphism is also an isomorphism:

$$
\frac{AI^k}{AI^{k+1}} \cong AI^k \underset{\mathbb{Z}[H]}{\otimes} \mathbb{Z} \cong (AI^k)_S \underset{\mathbb{Z}[H]}{\otimes} \mathbb{Z} \cong (A_S)I^k \underset{\mathbb{Z}[H]}{\otimes} \mathbb{Z} \cong \frac{(A_S)I^k}{(A_S)I^{k+1}} \cong \left(\frac{AI^k}{AI^{k+1}}\right)_S .
$$

By the 5-Lemma, the result follows.

(7) By statement (4), already proven, $\gamma_2(G_S) = A_S$. By induction

$$
\gamma_{n+2}(G_S) = [G_S, \gamma_{n+1}(G_S)] = [G_S, A_S I^{n-1}] = [G, A_S I^{n-1}] = A_S I^n = (AI^n)_S.
$$

The middle equality follows since the derived subgroups of G and G_S act trivially on A_S and by statement (4), which implies that $G/A \cong G_S/A_S$.

(8) We have $A = \gamma_2(G)$, $AI^n = \gamma_{n+2}(G)$, $A_S = \gamma_2(A_S)$, $A_S I^n = \gamma_{n+2}(G_S)$, and so by (6)

$$
\gamma_2(G)/\gamma_{n+2}(G) = A/AI^n \cong A_S/A_S I^n = \gamma_2(G_S)/\gamma_{n+2}(G_S).
$$

(9) Since G is residually nilpotent the submodules $AI^n = \gamma_{n+2}(G)$ have trivial intersection, and so it follows from (7) that G_S is residually nilpotent. $\square$

Before we state and prove our Telescope Theorem, we establish some notation.

Notation 5.7. For an $\mathcal{M}$-group G, we denote the telescope of G by $\tau(G)$.

We are now in a position to formulate and prove our Telescope Theorem, much of which has already been proved in Lemma 5.6.

Theorem 5.8 (The Telescope Theorem). *Let G and H be $\mathcal{M}$-groups. Then the following hold:*

(1) *$\tau(G)$ is an ascending union of subgroups isomorphic to G.*
(2) *The telescope $\tau(G)$ is para-G.*
(3) *The group H is para-G if and only if $\tau(G) \cong \tau(H)$.*
(4) *If H is para-G, then G is para-H.*
(5) *If H is para-G, then H isomorphic to a subgroup of G and G is isomorphic to a subgroup of H.*

Proof. We prove these statements in sequence.

(1) We already know from Lemma 5.6(1) that $\tau(G)$ is an ascending union of the G_i, each of which is isomorphic to G.

(2) Let ϕ be the inclusion of G into $\tau(G) = G_S$ and let ϕ_n be the induced homomorphism of $G/\gamma_n(G)$ into $\tau(G)/\gamma_n(\tau(G))$. Observe that $\gamma_2(\tau(G)) = A_S$ by Lemma 5.6, and recall that $\gamma_2(G) = A$. Consider the following sequence of equalities and isomorphisms:

$$G/\gamma_2(G) = G/A \cong GA_S/A_S = GA_S/\gamma_2(G_S) = \tau(G)/\gamma_2(\tau(G)).$$

The composition of these maps is ϕ_2. It follows that if X is a finite set of generators of G, then $\{x\gamma_2(G_S) \mid x \in X\}$ generates $G_S/\gamma_2(G_S)$. Now in a nilpotent group, any set of elements which generates it modulo its derived group generates it modulo every term of its lower central series. Hence

$$\phi_{n+2} : G/\gamma_{n+2}(G) \longrightarrow G_S/\gamma_{n+2}(G_S)$$

is onto. Notice that since G is finitely generated so too are all of its quotients and subgroups. Moreover if H is any finitely generated nilpotent group and if $H/L \cong H$, then $L = 1$ by Magnus' Theorem 2.2. It follows that any homomorphism of a finitely generated nilpotent group onto an isomorphic nilpotent group is itself an isomorphism. Now ϕ_2 induces a homomorphism of $G/\gamma_2(G)$ onto the group $G_S/\gamma_2(G_S)$, which we have already seen is isomorphic to $G/\gamma_2(G)$. So ϕ_2 is an isomorphism. ϕ_n induces a homomorphism θ_n of $A/AI^n = \gamma_2(G)/\gamma_{n+2}(G)$ to $A_S/A_SI^n = \gamma_2(G_S)/\gamma_{n+2}(G_S)$, which by Lemma 5.6(6) is isomorphic to A/AI^n. Each of these groups is finitely generated since subgroups and quotient groups of finitely generated nilpotent groups are finitely generated, and hence θ_n is an isomorphism and so

$$\phi_n : G/\gamma_n(G) \longrightarrow \tau(G)/\gamma_n(\tau(G))$$

is an isomorphism for all n, which proves (2).

(3) Suppose H is para-G. Then there is a homomorphism ϕ from G into H which induces isomorphisms of the corresponding terms of the lower central sequences of G and H. In particular, $G/[G,G]$ and $H/[H,H]$ are isomorphic finitely generated abelian groups, which we identify and denote by Q.

So if we put $\gamma_2(G) = A$ and $\gamma_2(H) = B$, then it follows that ϕ induces a homomorphism of the Q-module A into the Q-module B. Moreover $\gamma_3(G) = AI$ and $\gamma_3(H) = BI$. So ϕ induces a homomorphism of A/AI onto B/BI. By Lemma 4.3, ϕ induces an epimorphism of A_S onto B_S. It follows that ϕ induces a homomorphism ϕ_S of G_S onto H_S. But G_S and H_S have the same lower central sequences and they are residually nilpotent by Lemma 5.6(8). Therefore, we have proved that $\tau(G) \cong \tau(H)$.

Choose any isomorphism $\phi\colon \tau(G) \to \tau(H)$. By Lemma 5.6(1) and (3), $\tau(H)$ is a union of subgroups H_k with $H_k \cong H$. Since G is finitely generated, ϕ sends G into one of these copies of H, say H_k. We will show that H_k is para-G. Since $H \cong H_k$, this will suffice to prove that H is para-G.

To show that $G \to H_k$ induces an isomorphism on lower central series quotients, first note that the following commutative diagram implies that $G/\gamma_n(G) \to H_k/\gamma_n(H_k)$ is one-to-one for all n:

$$
\begin{array}{ccc}
G/\gamma_n(G) & \xrightarrow{\ \phi\ } & H_k/\gamma_n(H_k) \\
\Big\downarrow{\scriptstyle\cong} & & \Big\downarrow \\
\tau(G)/\gamma_n(\tau(G)) & \xrightarrow{\ \cong\ } & \tau(H)/\gamma_n(\tau(H))
\end{array}
$$

Let B_k be the copy of B in H_k. The homomorphism $H_k \to \tau(H)$ is given by $H_k = HB_k \to HB_S = \tau(H)$. Thus,

$$H_k/\gamma_2(H_k) = HB_k/B_k \cong H/H \cap B_k$$
$$= H/H \cap B_S \cong HB_S/B_S = \tau(H)/\gamma_2(\tau(H)).$$

Hence, $G/\gamma_2(G) \to H_k/\gamma_2(H_k)$ is an isomorphism and, in particular, onto. This implies that $G/\gamma_n(G) \to H_k/\gamma_n(H_k)$ is onto for all n since, as previously noted, any set of elements in a nilpotent group which generate it modulo its derived group generates the group itself. It follows along the same lines in our previous discussions that $\phi\colon G/\gamma_n(G) \cong H_k/\gamma_n(H_k)$ for all n. Consequently, H_k, and hence H, is para-G.

(4) If H is para-G, then $\tau(G) \cong \tau(H)$. So $\tau(H) \cong \tau(G)$, which implies G is para-H.

(5) This follows immediately from statement (4) above and Magnus' Theorem 2.2.   □

6. Some consequences of the Telescope Theorem

Theorem 6.1. *Suppose that G and H are $\mathfrak{M}$-groups and that H is para-G. Then H is finitely presented if and only if G is finitely presented.*

We will need a theorem of Bieri and Strebel [BStr]. We recall the details. Let Q be a finitely generated abelian group and let $v \in Hom(Q, \mathbf{R})$. Define the submonoid $Q_v = \{q \in Q \mid v(q) \geq 0\}$. Now let A be a finitely generated $\mathbb{Z}[Q]$-module. Then for every $v \in Hom(Q, \mathbf{R})$, A can be viewed as a $\mathbb{Z}[Q_v]$-module. A is termed *tame* if for every $v \in Hom(Q, \mathbf{R})$ either A is finitely generated as a $\mathbb{Z}[Q_v]$-module or else it is finitely generated as a $\mathbb{Z}[Q_{-v}]$-module. The relevance of this is the following theorem of Bieri and Strebel.

The Bieri-Strebel Theorem. *Suppose Q is a finitely generated abelian group.*

(1) *If A is a finitely generated tame $\mathbb{Z}[Q]$-module, then every Q-submodule of A is also tame.*

(2) *Suppose that G is an extension of an abelian normal subgroup A by Q. Then G is finitely presented if and only if A is a tame $\mathbb{Z}[Q]$-module.*

We are now in a position to prove Theorem 6.1.

Proof of Theorem 6.1. Put $B = [H, H]$, $Q = H/[H, H]$. Since H is para-G, there exists a homomorphism ϕ from G into H which induces isomorphisms between the corresponding quotients of their lower central series. So ϕ induces an isomorphism between $G/[G, G]$ and Q. Since H is para-G, ϕ induces a monomorphism of $A = [G, G]$. It follows that $\phi(A)$ is a normal subgroup of H, i.e., a Q-submodule of the Q-module B.

Now suppose that H is finitely presented. Then B is tame and therefore so too is every submodule of B, in particular $\phi(A)$. It follows that the module A is also tame, which by the Bieri-Strebel Theorem implies that G is finitely presented.

Conversely since para-equivalence is an equivalence relation, G is para-H. So again as already noted, if G is finitely presented, so too is H.   □

We now record another simple consequence of the Telescope Theorem for polycyclic groups.

Theorem 6.2. *Suppose that H is a finitely generated residually nilpotent metabelian group. If G is polycyclic and H is para-G, then H is isomorphic to a subgroup of finite index in G and G is isomorphic to a subgroup of finite index in H.*

Proof. By the Telescope Theorem 5.8(5), H is isomorphic to a subgroup of G. It follows that the Hirsch number of H is less than or equal to that of G. But again by the Telescope Theorem 5.8(5), G is isomorphic to a subgroup of H, and so the Hirsch number of G is less than or equal to that of H. It follows that G and H have the same Hirsch numbers, and hence the image of H in G and that of G in H are of finite index. This completes the proof of Theorem 6.2. □

We record and prove one last consequence of the Telescope Theorem.

Theorem 6.3. *Let G be an $\mathcal{M}$-group. If G has the same lower central quotients as a finitely generated free metabelian group F, i.e., G is para-free-metabelian, then G is isomorphic to a subgroup of F.*

We will only sketch the proof of Theorem 6.3. Suppose that G is freely generated modulo $[G, G]$ by the set $g_1, \ldots, g_n$ and that F is freely generated by $x_1, \ldots, x_n$. Define a homomorphism ϕ from F into G by sending x_j to g_j for each j. Then ϕ induces isomorphisms between the corresponding quotients of the lower central series of F and G. Since free metabelian groups are residually nilpotent, it follows that G is para-F. Hence, by the Telescope Theorem 5.8(4), F is para-G. Consequently G is isomorphic to a subgroup of F.

□

We note that there exist finitely generated para-free-metabelian groups which are not free [Bau1]. We shall discuss the existence of more para-free metabelian groups in a separate paper.

7. Poincaré series for finitely generated, metabelian groups

Our objective here is to prove Theorem 7.3. We will need the following.

Lemma 7.1. *Let G be a finitely generated metabelian group given by a preferred presentation. There is an algorithm to compute a finite presentation for $G/\gamma_{n+1}(G)$ for every n.*

In view of the fact that preferred presentations are not finite but involve infinitely many relations that ensure that the derived group is abelian, some care is needed to obtain a finite presentation of $G/\gamma_{n+1}(G)$. To this end we first recall the definition of a preferred presentation.

Definition 7.2. A *preferred presentation* of a finitely generated metabelian group G, in the category of metabelian groups, is a presentation which takes the form

$$G = \langle g_1, \ldots, g_t \mid R_1 \cup R_2 \rangle$$

where

(1) R_1 is a finite set of words of the form

$$w = \prod_{1 \leq i \leq j \leq t} [g_i, g_j]^{u_{ij}},$$

and we use the usual notation $[x, y]$ for $x^{-1}y^{-1}xy$, y^x for $x^{-1}yx$, and the u_{ij} are words of the form $g_1^{m_1} \ldots g_t^{m_t}$;

(2) R_2 is a finite set of words uw where u has the form $g_1^{m_1} \ldots g_t^{m_t}$ and

$$w = \prod_{1 \leq i \leq j \leq t} [g_i, g_j]^{v_{ij}},$$

with v_{ij} of the form $g_1^{n_1} \ldots g_t^{n_t}$.

Thus the words in R_1, together with the addition of all commutators $[x, y]$ where x and y take the form $[g_i, g_j]^{u_{ij}}$, are a regular presentation of A while those in R_2 read modulo A yield a finite presentation of $H = G/A$.

In order to obtain a presentation for G *in the category of all groups* we need to add to the relations R_1 and R_2 the set R_3 of all relations of the form

$$[[w, x], [y, z]] = 1,$$

where w, x, y, z range over all words in the generators of G.

Proof of Lemma 7.1. Our objective is to prove that there is an algorithm to obtain a finite presentation of $G/\gamma_{c+1}(G)$. To this end we start by finding a finite presentation of a free nilpotent group of class c on the generators of G and add the relations R_1 and R_2 to the group H presented in this way. Now let $d_1, d_2, \ldots, d_n, \ldots$ be a recursive enumeration of the relators in R_3 and let D_n be the normal closure in H of $d_1, d_2, \ldots, d_n$. Then

$$D_1 \leq D_2 \cdots \leq D_n \ldots$$

is an increasing sequence of normal subgroups of the finitely generated nilpotent group H. It follows that for some m, adding the finite set of relations D_m will give a presentation for $G/\gamma_{c+1}(G)$. There is an algorithm to determine for each m whether $D_m = D_{m+1}$ [BCR, Lemma 2.2]. Let k be the smallest integer such that $D_k = D_{k+1}$. Thus, adding the finite set of relations in D_k gives the desired finite presentation for $G/\gamma_{c+1}(G)$ as desired and completes the proof of the lemma.   □

We continue our discussion concerning Theorem 7.3.

We have proved that if G is any finitely generated metabelian group given by a preferred presentation, then we can recursively enumerate finite presentations for the lower central series quotients. The basic commutators generate the subgroup $\gamma_c(G)/\gamma_{c+1}(G) \leq G/\gamma_{c+1}(G)$. Since quotients of the derived group are submodule computable (see, for instance, [BCM]), we can algorithmically generate a presentation for $\gamma_c(G)/\gamma_{c+1}(G)$ as an abelian group. Thus we can compute the rank of $\gamma_c(G)/\gamma_{c+1}(G)$. Consequently we can recursively enumerate the rational Poincaré series of a finitely generated metabelian group.

We are left for the proof of Theorem 7.3 to compute the denominator and numerator of $P(G)$ in the case where G is a free metabelian group of finite rank. This is straightforward and we leave it as an exercise for the reader.

8. Completions of polycyclic groups

We prove a sequence of theorems and lemmas culminating in the proof of the following.

Theorem 8.1. *Let the $\mathcal{M}$-group G be polycyclic. Then the pro-nilpotent completion of G, $\widehat{G}$, is locally polycyclic; i.e., its finitely generated subgroups are polycyclic.*

Our first theorem may have independent interest.

Theorem 8.2. *A finitely generated metabelian group is polycyclic if and only if its two-generator subgroups are polycyclic.*

Proof. Since subgroups of polycyclic groups are polycyclic, we only need prove one direction of this theorem.

Suppose that the two-generator subgroups of the finitely generated metabelian group G are polycyclic and that G is generated by $x_1, \ldots, x_\ell$. Then the commutator subgroup $A = [G, G]$ of G is the normal closure of finitely many elements, say $a_1, \ldots, a_m$. Since the subgroup of G generated by a_1 and x_1 is polycyclic, so is the subgroup A_1 of A which is generated by the conjugates of a_1 by the powers of x_1. Call these generators $a(1, 1), \ldots, a(1, n_1)$. The subgroup of A generated by the conjugates of the finitely many elements $a(1, 1), \ldots, a(1, n_1)$ by the powers of x_2 is finitely generated as well. Iterating this process we find that the subgroup B of A generated by the conjugates of the elements $a_1, \ldots, a_m$ by the finitely many elements $x_1, \ldots, x_\ell$ is finitely generated.

But $B = A$, the derived group of G. Consequently G is an extension of one finitely generated abelian group by another and is therefore polycyclic. This completes our proof. $\square$

Lemma 8.3. *Let G be a polycyclic metabelian group. Let A be an abelian normal subgroup of G with abelian factor group $Q = G/A$. View A as a module over the integral group ring R of Q. Then for each $t = sA \in Q, a \in A$, there exist polynomials*

$$\alpha = c_0 + c_1 t + \cdots + c_{m-1} t^{m-1} - t^m$$

and

$$\beta = -t^{-1} + d_0 + d_1 t + \cdots + d_{n-1} t^{n-1}$$

such that

$$a\alpha = a\beta = 0.$$

Moreover given two such polynomials α and β, if $a\alpha = a\beta = 0$, then it follows that the conjugates of a by the powers of s generate an abelian group which can be generated by $m + n$ elements.

Proof. We construct α and β in two steps.

(1) Consider the subgroup B_i of the subgroup $gp(a, s)$ of G generated by

$$a, a^s, \ldots, a^{s^i}.$$

Then since $gp(a, s)$ is polycyclic, it satisfies the maximal condition; i.e., every subgroup is finitely generated. So there exists an integer m such that $a^{s^m} \in B_{m-1}$. Hence there exists $c_0, \ldots, c_{m-1}$ such that

$$a^{s^m} = a^{c_0} + a^{c_1 s} + \cdots + a^{c_{m-1} s^{m-1}}.$$

Hence $a\alpha = 0$ as claimed.

(2) Consider the subgroup C_j generated by

$$a^{s^{-j}}, a^{s^{-j}+1}, \ldots, a^{s^{-1}}.$$

Since G satisfies the maximal condition there exists an integer n such that

$$a^{s^{-n}} \in C_{n-1}.$$

So there exist $d_0, d_1, \ldots, d_n$ such that

$$a^{s^{-n}} = a^{d_0 s^{-n+1}} + a^{d_1 s^{-n+2}} + \cdots + a^{d_{n-1}}.$$

Since the action of t on A is by conjugation by s, we can re-express what we have proved by writing $a\beta = 0$ as claimed. $\qquad\square$

Now let G be an $\mathcal{M}$-group. Our objective is to prove that the finitely generated subgroups of $\widehat{G}$ are polycyclic. In view of Theorem 8.2 it suffices to prove that the two-generator subgroups of $\widehat{G}$ are polycyclic. The following simple lemma will facilitate the proof.

Lemma 8.4. *Let H be a metabelian group generated by the elements s and a. Then H is polycyclic if the subgroup generated by conjugates of $[s,a]$ by the powers of s is finitely generated and the subgroup generated by the conjugates of $[s,a]$ by the powers of a is finitely generated.*

Proof. Notice that $[H, H]$ is the normal closure in H of $[s, a]$. Let $h_1, \ldots, h_m$ be a finite set of generators of the subgroup of H generated by the conjugates of $[s, a]$ by the powers of s. Notice that $([s,a]^s)^a = ([s,a]^a)^s$. So the subgroup K of H generated by the conjugates of the elements $h_1, \ldots, h_m$ by the powers of a is again finitely generated. But $K = [H, H]$. Thus H is an extension of one finitely generated abelian group by another finitely generated abelian group and therefore polycyclic. $\qquad\square$

To prove Theorem 8.1, we restrict our attention to the case of a two-generator metabelian group G since the general case follows along the same lines. If G is a metabelian group, then so is $\widehat{G}$. To prove that in $\widehat{G}$ the finitely generated subgroups are polycyclic, it suffices to show that the two-generator subgroups of $\widehat{G}$ are polycyclic by Theorem 8.2. By Lemma 8.4, it suffices to show that if $s, a \in \widehat{G}$ and $H = gp(s, a)$, then the subgroups of H generated by the conjugates of $b = [s, a]$ by s is finitely generated and, similarly, the subgroup of H generated by conjugates of b by the powers of a is finitely generated.

Toward this end, we have one last lemma before we prove Theorem 8.1.

Lemma 8.5. *Let $s \in \widehat{G}$ and let $b \in [\widehat{G}, \widehat{G}]$. Then the subgroup B of $\widehat{G}$ generated by the conjugates of b by the powers of s is a finitely generated abelian group.*

Proof. Recall that we assume G is generated by two elements, say, x_1, x_2. Then $\gamma_n(G)/\gamma_{n+1}(G)$ is generated by the right-normed commutators of the form

$$[x_1, x_2, y_1, \ldots, y_{n-2}]\gamma_{n+1}(G),$$

where the $y_j \in \{x_1, x_2\}$. Since G is metabelian, so is $\widehat{G}$. So in order to prove that the finitely generated subgroups of $\widehat{G}$ are polycyclic, it suffices to prove that the two-generator subgroups of $\widehat{G}$ are polycyclic.

Let

$$s(n) = s_1 \ldots s_n \gamma_{n+1}(G), \quad b(n) = b_1 \ldots b_n \gamma_{n+1}(G),$$

where here $s_j \in \gamma_j(G)$, $b_j \in \gamma_j(G)$ for each j. If $s_1 \in \gamma_2(G)$, then s and b commute and there is nothing to prove.

We consider, then, the case where $s_1 \notin \gamma_2(G)$. We need to consider the elements b_n. To this end, let us denote by Y_n the set of commutators of the form

$z(y_1, \ldots, y_{n-2}) = [x_1, x_2, y_1, \ldots, y_{n-2}]$ of weight $n > 1$. We adopt the convention that if $n = 2$, then $z = z(y_1, y_2, \ldots, y_{n-2}) = [x_1, x_2]$.

We proved in Lemma 8.3 that there exist two polynomials α, β in s_1, s_1^{-1}, where

$$\alpha = c_0 + c_1 s_1 + \cdots + c_{m-1} s_1^{m-1} - s_1^m$$

and

$$\beta = -s_1^{-1} + d_0 + d_1 s_1 + \cdots + d_{n-1} s_1^{n-1}$$

such that

$$[x_2, x_1]\alpha = [x_2, x_1]\beta = 0.$$

Now each $z(y_1, y_2, \ldots, y_{n-2})$ can be rewritten using exponential notation as

$$[x_1, x_2]^{(y_1 - 1)\ldots(y_{n-2}-1)}.$$

So it follows that the action of α on $z(y_1, y_2, \ldots, y_{n-2})$ can be re-expressed as follows:

$$([x_2, x_1]^{(y_1-1)\ldots(y_{n-2}-1)})^\alpha = [x_2, x_1]^{\alpha(y_1-1)\ldots(y_{n-2}-1)}.$$

So

$$([x_2, x_1]^{(y_1-1)\ldots(y_{n-2}-1)})^\alpha = 1.$$

It follows that $b\alpha = 0$ and similarly that $b\beta = 0$. Thus the conjugates of b by the powers of s generate a finitely generated group. This completes the proof. $\qquad \square$

We come now to the proof of Theorem 8.1.

Proof of Theorem 8.1. The same proof used above can be used to prove that the conjugates of b by the powers of a also generate a finitely generated group. So Lemma 8.4 applies as noted above. Thus we have proved that if G is polycyclic, the two-generator subgroups of $\widehat{G}$ are polycyclic, and hence the finitely generated subgroups of $\widehat{G}$ are also polycyclic by Theorem 8.2, as claimed. $\qquad \square$

9. Examples

We give here a number of examples of residually nilpotent metabelian groups with the same lower central sequences and with a variety of different properties.

9.1. Wreath products.
We recall that a group W is the *(restricted) wreath product* of its subgroups A and T, denoted by $A \wr T$, if W is generated by A and T and

  (1) the conjugates A^t of A by the distinct elements $t \in T$ generate a (restricted) direct product B, and
  (2) $A \cap B = 1$.

So $W = B \rtimes T$. From now on we will refer to these products as direct products and wreath products. By a direct product we mean the group of elements of the cartesian product $\prod_{t \in T} A^t$ where all but finitely many coordinates are the trivial element. We will prove that such wreath products have a number of interesting properties.

We will restrict our attention here to a number of special cases. It is likely that most of our results hold more generally, but we will not concern ourselves with greater generality here.

One can extend Theorem 9.1(1), below, to groups where A is any finite abelian group, as stated in §2. We prove the less general result here.

Theorem 9.1. *Let* $W = A \wr T$ *be the wreath product of its subgroups* A *and* T. *Then the following hold.*

(1) *If* A *is of prime order and* T *is infinite cyclic, then any finitely generated residually nilpotent metabelian group* H *with the same lower central sequences as* W *is isomorphic to* W; *i.e., the para-equivalence class of* W *consists of a single isomorphism class.*

(2) *If* A *is of prime order and* T *is infinite cyclic, then the Isomorphism Problem is solvable for* W.

(3) *If* A *and* T *are infinite cyclic, then the para-equivalence class of* W *contains at least two non-isomorphic groups.*

(4) *If* A *and* T *are infinite cyclic, then the Isomorphism Problem is solvable for* W.

These groups are residually nilpotent, as claimed in §2. We note, without proof, that the wreath product of a finite abelian group and a free abelian group is residually nilpotent. While harder to prove, the wreath product of two torsion-free abelian groups is residually nilpotent as well. We refer the reader to the papers by Gruenberg [GR], Lichtman [Li] and Hartley [Har], where proofs can be found or where the results described there can be used to prove them.

We are now in a position to prove the various parts of Theorem 9.1. In the remainder of this section we will use the term *direct product* and the product notation to denote the previously discussed restricted product.

Proof. We prove these statements sequentially.

(1) Let $W = A \wr T$ be the wreath product of a cyclic group, A, of prime order p generated by $a \in A$ and an infinite cyclic group, T, generated by t. We have already noted that W is residually nilpotent. Our objective now is to prove that any finitely generated, residually nilpotent, metabelian group H with the same lower central sequences as W is isomorphic to W. The normal closure B of a in W is the direct product of its subgroups $gp(a_i)$, $i = 1, 2, \ldots$, where $a_i = t^{-i} a t^i$ is order p. Since each of the a_i is of order p we can view B as a module over the group ring $\mathbb{Z}_p[T]$ of T over the field $\mathbb{Z}_p$ of p elements. Since the a_i generate their direct product, B is a free $\mathbb{Z}_p[T]$-module.

Since H has the same lower central sequences as W, $H/[H,H]$ is the direct product of an infinite cyclic group on $u[H,H]$ and a group of order p generated by $b[H,H]$. There is, for each c, an isomorphism ϕ_{c+1} mapping $H/\gamma_{c+1}(H)$ onto $W/\gamma_{c+1}(W)$. This gives rise to a monomorphism ϕ between the respective direct products

$$\prod_{c=1}^{\infty} H/\gamma_{c+1}(H) \longrightarrow \prod_{c=1}^{\infty} W/\gamma_{c+1}(W).$$

Since H is residually nilpotent, this monomorphism induces a monomorphism of H into $\prod_{c=1}^{\infty} W/\gamma_{c+1} W$. It follows that this induces a monomorphism of $[H,H]$ into $\prod_{c=1}^{\infty} W/\gamma_{c+1} W$, where the finite order elements of the latter group have order p. So $[H,H]$ is an abelian group of exponent p.

Now adjoin b to $[H,H]$. One easily checks that the only torsion in $\prod W/\gamma_{c+1}(W)$ has order p. Since b is of finite order modulo $[H,H]$ and since A is abelian of exponent p, it follows that b has finite order, and since the

above homomorphism is 1-1, b has finite order p. It follows from the residual nilpotence of H that b commutes with all of $[H, H]$. Otherwise there exists an element $h \in [H, H]$ such that $gp(b, h)$ is not abelian. However $gp(b, h)$ is then a finite subgroup of H and so is isomorphic to a finite subgroup of $\prod_{c=1}^{\infty} W/\gamma_{c+1}W$. Now the torsion subgroup of W can be expressed as an ascending union of abelian groups of exponent p so $gp(b, h)$ embeds into one of these subgroups in which the images of b and h are independent and, hence, commute. This implies that $K = gp(b, [H, H])$ is a normal abelian subgroup of H of exponent p.

So K is a module over $\mathbb{Z}_p[H/K] \cong \mathbb{Z}_p[T]$. Since this ring is a p.i.d., K is a sum of cyclic modules. In the event that the number of these cyclic summands is at least 2, then $H/[H, H]$ will be the direct product of an infinite cyclic group and at least two groups of order p, which is not the case.

So there is only one cyclic submodule generated by an element, say h. Since the submodule of K generated by h has infinite rank as a vector space over $\mathbb{Z}_p$, it follows that it is isomorphic to $\mathbb{Z}_p[T]$. So $H \cong gp(h) \wr gp(s)$, where sK generates H/K, and so is isomorphic to W. This completes the proof of (1).

(2) As above, we view B as a module over $\mathbb{Z}_p[T]$. B is a free $\mathbb{Z}_p[T]$-module on a single element a. We start by finding a preferred presentation of W [LR, page 185]. Our objective is to show that there is an algorithm which recursively enumerates all preferred presentations of groups isomorphic to W and recursively enumerates all preferred presentations of groups which are not isomorphic to W.

To this end let V be a finitely generated metabelian group defined by a given preferred presentation. At the outset we algorithmically check that V is residually nilpotent [BCR, Cor. 9.2]; otherwise the algorithm terminates. There is an algorithm to find a finite module presentation for $[V, V]$ [BCR, Thm. 3.1], and since the word problem is solvable for the finitely generated metabelian group V, one can algorithmically determine if $[V, V]$ is trivial. If $[V, V]$ is trivial, then $V \not\cong W$ and the algorithm halts. Otherwise, we can find a non-trivial element $v \in [V, V]$. There is then an algorithm to compute a preferred presentation of the centralizer C of $[V, V]$ [BCR, Thm. 6.1]. The group $C/[V, V]$ is a subgroup of $V/[V, V]$ and is therefore a finitely generated abelian group. We can algorithmically check if it is cyclic of order p. If not, then $V \not\cong W$. So we assume $C/[V, V]$ is finite of order p. Since C if generated by $[V, V]$ and just one more generator, and since C centralizes $[V, V]$ with a single additional generator, C must be abelian.

Since $[V, V] \subset C$, V/C is a finitely generated abelian group. So we can also check algorithmically whether it is infinite cyclic. If not, $V \not\cong W$. Otherwise, V/C is infinite cyclic. Since C is a finitely generated module over V/C and the word problem is solved for finitely generated metabelian groups, we can algorithmically check to see if each module generator has order p, and therefore whether C is abelian of exponent p. If not, $V \not\cong W$. If so, C is a module over the mod-p group ring $\mathbb{Z}_p[V/C] \cong \mathbb{Z}_p[T]$, a p.i.d. Thus, C is a sum of cyclic modules. If C has more than one summand, then as in part (1), $V/[V, V]$ has more than one cyclic summand of order

p, and $V \not\cong W$. Otherwise, C has only one summand as a $\mathbb{Z}_p[T]$-module. Since C has infinite rank as a vector space over $\mathbb{Z}_p$, C is a free rank one module over $\mathbb{Z}_p[T]$.

We have shown that V is generated by an element s which is of infinite order modulo C, and that C is a free rank one $\mathbb{Z}_p[V/C]$-module on an element b. Since the quotient homomorphism $V \to V/C$ splits with kernel C, it follows that $V \cong W$.

(3) Let $W = A \wr T$, where A is the infinite cyclic group on a and T is the infinite cyclic group on t. The normal closure B of a in W is freely generated by the conjugates a^{t^i} of a by the powers of t. So if we view B as a module over the integral group ring $\mathbb{Z}[T]$ of T, then B is a free module on a. We start then by finding a preferred presentation of W. Our objective is to show that there is an algorithm which recursively enumerates all preferred presentations of groups isomorphic to W and recursively enumerates all preferred presentations of groups which are not isomorphic to W.

To this end let V be a group defined by a given preferred presentation. As in the last argument, there is an algorithm which computes a module presentation of the derived group of V and a second algorithm that computes a presentation for the centralizer C of the derived group of V. As in the prior argument, another algorithm determines whether or not C is abelian. If C is not abelian, then V is not isomorphic to W and the algorithm comes to a halt. So we suppose that C is abelian. This means that C is an abelian normal subgroup of V containing the derived group of V. We can now determine whether or not V/C is infinite cyclic. If it is not, then again $V \not\cong W$ and the algorithm comes to an end. Suppose then that V/C is infinite cyclic. View C as a module over V/C. There is now an algorithm to decide whether or not C is projective. If it is not, then $V \not\cong W$. However if V is projective, then by the Quillen-Suslin theorem [QS], C is a free module over the infinite cyclic group. If the rank of C is different from 1, then $V \not\cong W$. On the other hand, if C is free of rank 1, then $V \cong W$ and the algorithm terminates.

(4) Let $W = A \wr T$, where A is infinite cyclic on a and T is infinite cyclic on t. Then W is residually nilpotent. Put $H = gp((a^2)^t a^{-1}, \ a^2 a^{t^{-1}}, \ t)$. Then $(a^2)^t a^{-1}, a^2 a^{t^{-1}}, t$ generate W modulo $\gamma_2(W)$, and hence they generate W modulo $\gamma_{c+1}(W)$ for every c, i.e., $H\gamma_{c+1}(W) = W$ for every c. Consequently

$$W/\gamma_{c+1}(W) = H\gamma_{c+1}(W)/\gamma_{c+1}(W) \cong H/H \cap \gamma_{c+1}(W).$$

Since $\gamma_{c+1}(W) \geq \gamma_{c+1}(H)$ in view of the fact that finitely generated nilpotent groups are Hopfian, $H \cap \gamma_{c+1}(W) = \gamma_{c+1}(H)$. So we have proved that H and W have the same lower central sequences. As a consequence W is para-H since now it follows that the inclusion of H into W induces isomorphisms between the factor groups $H/\gamma_{c+1}(H)$ and $W/\gamma_{c+1}(W)$. Thus, by the Telescope Theorem, H and W are para-equivalent.

However H and W are not isomorphic. To see that this is so, suppose the contrary. Any isomorphism from H to W will map the centralizer C of any non-trivial element of $[H, H]$ isomorphically onto the centralizer D of a non-trivial element of $[W, W]$. If we view C as a $\mathbb{Z}[T]$-module and D similarly also as a $\mathbb{Z}[T]$-module, then these modules must be isomorphic.

But D is a cyclic module, and it is not hard to prove that C is a two-generator module since the ideal of $\mathbb{Z}[T]$ generated by $2t - 1, 2 - t$ is not a principal ideal of $\mathbb{Z}[T]$; i.e., the class number of $\mathbb{Z}[T]$ is at least 2. □

It is this approach that we will take in our third paper, which takes advantage of class field theory to construct a number of interesting examples of residually nilpotent, metabelian, polycyclic groups whose para-equivalence classes are not singletons.

9.2. **Polycyclic metabelian groups with para-equivalence classes that are not singletons.** Recall from the Telescope Theorem, Theorem 5.8, that if $G \to H$ is a para-equivalence of residually nilpotent polycyclic metabelian groups, then G is isomorphic to a subgroup of finite index in H and H is isomorphic to a subgroup of finite index in G. This does not imply $G \cong H$ as the following theorem demonstrates.

We note that Gruenwald and Segal proved the Isomorphism Theorem in the special case of finitely generated nilpotent groups, and Segal completed the proof for polycyclic groups. See the complete proof of the solution of the Isomorphism Problem, which contains the joint work of Segal and Gruenwald followed by that of Segal, in the book [Se].

Theorem 9.2. *There exist finitely generated, residually nilpotent, para-equivalent polycyclic metabelian groups which are not isomorphic.*

Remark 9.3. Ideal class theory inspired this example. It's well known that the ideal class group of $\mathbb{Q}(\zeta_{23})$ has order 3 and is generated by the non-principal ideal $(2, 1 + P) \subset \mathbb{Z}[\zeta_{23}]$, where P is the Gaussian period described in the proof below. (See, for instance, [Mar, page 86].)

Proof. Let T be the infinite cyclic group on t. Consider the Dedekind domain

$$\mathbb{Z}[\zeta_{23}] \cong \mathbb{Z}[T]/(N(t)) \text{ where } N(t) = \sum_{k=0}^{22} t^k.$$

We view $\mathbb{Z}[\zeta_{23}]$ as a $\mathbb{Z}[T]$-module with t acting on $\mathbb{Z}[\zeta_{23}]$ by multiplication by ζ_{23}. Let G be the semi-direct product of $\mathbb{Z}[\zeta_{23}]$ by T using this action:

$$G = \mathbb{Z}[\zeta_{23}] \rtimes T.$$

Observe that the augmentation ϵ from $\mathbb{Z}[T]$ onto $\mathbb{Z}$ determines a commutative diagram where C_{23} denotes the cyclic group with 23 elements:

$$
\begin{array}{ccc}
\mathbb{Z}[T] & \xrightarrow{\epsilon} & \mathbb{Z} \\
\downarrow{\scriptstyle q} & & \downarrow \\
\mathbb{Z}[\zeta_{23}] & \longrightarrow & C_{23}
\end{array}
$$

Therefore if $p(t) \in \mathbb{Z}[T]$ and $p(1) = \pm 1$, then $q(p(t)) \neq 0$. Hence, since $\mathbb{Z}[\zeta_{23}]$ is an integral domain, the set $S \subset \mathbb{Z}[\zeta_{23}]$, that is, the image of $1 + \ker\{\epsilon \colon \mathbb{Z}[T] \to \mathbb{Z}\}$ in $\mathbb{Z}[\zeta_{23}]$, is a multiplicative set in $\mathbb{Z}[\zeta_{23}]$. One easily checks that G is polycyclic, metabelian and residually nilpotent as well.

With the above observations in mind, we now construct a residually nilpotent group H and a para-equivalence $G \to H$ such that G and H are not isomorphic.

Consider the *Gaussian period*

$$P = \sum_{k=1}^{11} \zeta_{23}^{k^2} = \zeta_{23} + \zeta_{23}^2 + \zeta_{23}^3 + \zeta_{23}^4 + \zeta_{23}^6 + \zeta_{23}^8 + \zeta_{23}^9 + \zeta_{23}^{12} + \zeta_{23}^{13} + \zeta_{23}^{16} + \zeta_{23}^{18}$$

$$= \frac{-1 + \sqrt{-23}}{2}$$

and the element

$$1 + P = \frac{1 + \sqrt{-23}}{2} \subset \mathbb{Z}[\zeta_{23}].$$

Consider the non-principal ideal I of $\mathbb{Z}[\zeta_{23}]$ generated by 2 and $1 + P$, and let H be the semi-direct product of I and T with t acting on I by multiplication by ζ_{23}:

$$H = I \rtimes T.$$

(We leave it to the reader to check that this ideal is non-principal.) Observe that the inclusion of I in $\mathbb{Z}[\zeta_{23}]$ induces an inclusion of H into G.

We now construct an element $s \in S$ such that the principle ideal of $\mathbb{Z}[\zeta_{23}]$ generated by s is properly contained in I.

Let

$$p(t) = 2(1 + t + t^2 + t^3 + t^4 + t^6 + t^8 + t^9 + t^{12} + t^{13} + t^{16} + t^{18}) - N(t).$$

Then

$$p(t) \in 1 + I,$$

since $p(1) = 1$. Now let $s = p(\zeta_{23}) \in S$. Also,

$$p(\zeta_{23}) = 2 + 2P = 1 + \sqrt{-23} \in \mathbb{Z}[\zeta_{23}].$$

Now consider the following diagram where α is the homomorphism such that $\alpha(1) = s$, that is, $\alpha(1) = 2(1 + P)$:

$$\mathbb{Z}[\zeta_{23}] \xrightarrow{\ \alpha\ } (2, 1 + P) \xrightarrow{\ \subset\ } \mathbb{Z}[\zeta_{23}] \xrightarrow{\ \alpha\ } (2, 1 + P).$$

Each composition of two homomorphisms is given by multiplication by $s \in S$. So all the above inclusions are isomorphisms after inverting the multiplicative set S. By Lemma 5.6(5), all homomorphisms in this diagram induce isomorphisms on I-adic quotients.

Therefore the homomorphisms in the corresponding diagram of groups induce isomorphisms on lower central series quotients:

$$G \xrightarrow{\ \alpha \rtimes id\ } H \subset G \xrightarrow{\ \alpha \times id\ } H$$

and

$$G \xrightarrow{\ \alpha \rtimes id\ } H$$

is a para-equivalence.

However

$$G = \mathbb{Z}[\zeta_{23}] \rtimes T \not\cong I \rtimes T = H,$$

since I is not a principal ideal in $\mathbb{Z}[\zeta_{23}]$, and therefore not isomorphic to $\mathbb{Z}[\zeta_{23}]$. $\square$

10. THE GROUPS $G_n = \langle a, t; a^t = a^n \rangle$

We shall prove that the G_n, $n \in \mathbb{Z}$, have a number of interesting properties, which we have collected together in the following theorem.

Theorem 10.1. *Consider the group G_n, $n \neq 2$, defined as above.*

  (1) *G_n is residually nilpotent.*
  (2) *If H is a residually nilpotent group with the same lower central sequences as G_n, then $H \cong G_n$.*
  (3) *The Isomorphism Problem is solvable for G_n.*

Proof. We prove these statements sequentially.

  (1) We will restrict our attention to the case where $n > 2$ since the remaining cases can be taken care of in much the same way.

   Put $G = G_n$. Notice first that a is of infinite order since by a theorem of Magnus, Karrass and Solitar, $a^t a^{-n}$ is not a proper power in the free group on a and t [MKS]. Let A be the normal closure in G of a and put $a_j = t^j a t^{-j}$ and $A_j = gp(a_j)$. Since $a^t = a^n$, it follows that $a_{j+1}^n = a_j$. So A is generated by $a = a_0, a_1, \ldots$ and is an ascending union of infinite cyclic groups. Consequently G is the semi-direct product of the torsion-free abelian group A and an infinite cyclic group and is therefore metabelian. Now $[a, t] = a^{n-1}$, and hence $\gamma_2(G)$ is the normal closure in G of a^{n-1} and $G/\gamma_2(G) = C_\infty \times C_{n-1}$, where C_∞ is an infinite cyclic group and C_{n-1} is a cyclic group of order $n - 1$.

   If we now put $b_j = [a, \underbrace{t, \ldots, t}_{j}]$ for $j = 2, \ldots$, and $B_j = gp(b_j)$, then $b_{j+1}^{n-1} = b_j$ and $\gamma_j(G) = gp(B_j, B_{j+1}, \ldots)$. It follows that $G/\gamma_{j+1}G = C_\infty \ltimes C_{(n-1)^j}$ and that $\gamma_\omega(G) = 1$ and therefore G is residually nilpotent.

  (2) Suppose then that H is a finitely generated, residually nilpotent group with the same lower central sequences as G. Then H embeds into the unrestricted direct product $\tilde{H}$ of the factor groups $H/\gamma_k(H)$. Consequently H is isomorphic to a subgroup of a direct product of metabelian groups and is therefore metabelian.

   By hypothesis, $H/\gamma_k(H) \cong G/\gamma_k(G)$. Let ϕ_k be a homomorphism mapping $H/\gamma_k(H)$ onto $G/\gamma_k(G)$ for each k. Choose $\tau \in H$ to be any element of H such that the image of $\tau\gamma_3(H)$ under ϕ_3 is $t\gamma_3(G)$ and choose $\alpha \in H$ such that the image of α under ϕ_3 is $a\gamma_3(G)$. Denote by τ_k the image of τ under ϕ_k and by α_k the image of α under ϕ_k. It follows that

$$\alpha_k^{\tau_k} \cong \alpha_k^n \text{ modulo } \gamma_k(H)$$

for every k, i.e.,

$$(\alpha)^\tau (\alpha)^{-n} \in \gamma_k(H)$$

for every $k > 2$. But the intersection of the $\gamma_k(H)$ is trivial. Therefore $\alpha^\tau = \alpha^n$.

   Since G is torsion-free, so too is $\widehat{G}$, and using the ϕ_k we can identify H with a subgroup of $\widehat{G}$. It follows that every abelian subgroup of H is torsion-free. Since H is finitely generated we can supplement $(\alpha)^{n-1} = c_1$ with finitely many elements $c_2, \ldots, c_\ell$ of $[H, H]$ which together with τ and α suffice to generate H. Now choose finitely many elements $d_1, \ldots, d_j$ in

$[H, H]$ which freely generate the subgroup of $[H, H]$ generated by $c_1, \ldots, c_\ell$. Since H is residually nilpotent and has the same lower central sequence as G, it follows that τ conjugates each of $d_1, \ldots, d_j$ to their n^{th} powers. It follows then from this discussion that H is generated by $\tau, \alpha, d_1, \ldots, d_k$ and is defined by the relations which specify that the elements $\alpha, d_1, \ldots, d_j$ commute, that τ conjugates each of the d_j into their n^{th}-powers and that $\alpha^\tau = \alpha^n$. This implies that $H/[H, H]$ is a direct product of an infinite cyclic group and $j + 1$ cyclic groups of order $n - 1$. This implies that $j = 0$ and so $H \cong G$.

(3) Let H now denote a finitely generated metabelian group given by a preferred presentation. We have to prove that there is an algorithm which determines whether or not $H \cong G$, which we describe in stages, as in the proof above.

There is an algorithm that decides whether or not H is residually nilpotent. If not, then $H \not\cong G$. Suppose then that H is residually nilpotent. There is an algorithm that decides whether H is torsion-free and also whether H is not abelian. If H is abelian or if H contains a non-trivial element of finite order, then $H \not\cong G$. So we can proceed under the assumptions that H is not abelian and that H is torsion-free. We now check to see if $H/[H, H]$ is the direct product of an infinite cyclic group and a cyclic group of order $n - 1$. If this is not the case, then $H \not\cong G$. So we can assume that $H/[H, H]$ is such a direct product. Now choose a non-trivial element $x \in [H, H]$. Then there is an algorithm to compute the centralizer C of x. We now check to see whether C is abelian and whether H/C is infinite cyclic. If not, then $H \not\cong G$. Suppose then that C is abelian and H/C is infinite cyclic generated by, say, xC. We now view C as a module over the integral group ring R of the infinite cyclic group generated by x. The R-module C is a finitely generated R-module. So we can find a finite set of elements $d_1, \ldots, d_q$ of C which generate C as an R-module.

If $C/[H, H]$ is not cyclic of order $n - 1$, then $H \not\cong G$. Let us assume that $C/[H, H]$ is cyclic of order $n - 1$. Choose an element $y \in C$ such that $y[H, H]$ generates $C/[H, H]$. Then y has order $n - 1$ modulo $[H, H]$. Now there is an algorithm to decide if the R submodule K of $[H, H]$ generated by $z = y^{n-1}$ is equal to $[H, H]$ [BCR, Cor. 5.3]. If it is not, then $H \not\cong G$. Suppose then that $K = [H, H]$. It follows that H is generated by x and y, that $y^x = y^n$ and that K is an ascending union of infinite cyclic groups, as needed.

$\square$

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CAISS and Department of Computer Science, City College of New York, Convent Avenue and 138th Street, New York, New York 10031

Chebyshev Laboratory, St. Petersburg State University, 14th Line, 29b, Saint Petersburg, 199178 Russia – and – St. Petersburg Department of the Steklov Mathematical Institute, Fontanka 27, Saint Petersburg, 191023 Russia
E-mail address: `rmikhailov@mail.ru`
URL: `http://chebyshev.spb.ru/roman-mikhailov`

Department of Mathematics, Indiana University, Bloomington, Indiana 47405
E-mail address: `korr@indiana.edu`