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ON FOLIATIONS WITH NEF ANTI-CANONICAL BUNDLE

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Abstract. In this paper we prove that the anti-canonical bundle of a holomorphic foliation $\mathscr{F}$ on a complex projective manifold cannot be nef and big if either $\mathscr{F}$ is regular, or $\mathscr{F}$ has a compact leaf. Then we address codimension one regular foliations whose anti-canonical bundle is nef with maximal Kodaira dimension.

1. Introduction

In the last few decades, much progress has been made in the classification of complex projective varieties. The general viewpoint is that complex projective manifolds X should be classified according to the behavior of their canonical class K_X. Similar ideas can be applied in the context of *foliations* on complex projective manifolds. If $\mathscr{F} \subset T_X$ is a foliation on a complex projective manifold, we define its canonical class to be $K_{\mathscr{F}} = -c_1(\mathscr{F})$. In analogy with the case of projective manifolds, one expects the numerical properties of $K_{\mathscr{F}}$ to reflect geometric aspects of $\mathscr{F}$ (see for instance [Dru04], [LPT11], [AD13], [AD14], [AD16]).

In [PS14], Popa and Schnell proved that the canonical bundle of a codimension one regular foliation with trivial normal bundle cannot be big. In this paper we propose to investigate regular foliations on complex projective manifolds with $-K_{\mathscr{F}}$ nef. Codimension one regular foliations with trivial canonical bundle were classified by Touzet in [Tou08]. More recently, Pereira and Touzet have investigated regular foliations $\mathscr{F}$ of arbitrary rank on complex projective manifolds with $c_1(\mathscr{F}) = 0$ and $c_2(\mathscr{F}) = 0$ ([PT13]).

In [Miy93, Theorem 2], Miyaoka proved that the anti-canonical bundle of a smooth projective morphism $f : X \to C$ onto a smooth proper curve cannot be ample. In [Zha96, Proposition 1] (see also [Deb01, Theorem 3.12]), this result was generalized by dropping the smoothness assumption on f. In this note, we give a further generalization of this result to foliations (see also Theorem 7.1).

Theorem 1.1. *Let X be a complex projective manifold, and let $\mathscr{F} \subset T_X$ be a codimension q foliation with $0 < q < \dim X$. Suppose that either $\mathscr{F}$ is regular, or that $\mathscr{F}$ has a compact leaf. Then $-K_{\mathscr{F}}$ is not nef and big.*

In [AD14, Theorem 1.5], the authors proved that the anti-canonical bundle of a codimension one foliation on a complex projective manifold whose singular set has codimension $\geqslant 3$ cannot be nef and big.

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The following example shows that the statement of Theorem 1.1 becomes wrong if one drops the nefness assumption on $-K_{\mathscr{F}}$.

Example 1.2. Let $\mathscr{F}$ be the foliation on $\mathbb{P}^2$ induced by a linear projection $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. It is algebraically integrable with ample anti-canonical bundle, but does not possess any compact leaf. The foliation obtained by blowing up the unique singular point is regular with big but not nef anti-canonical bundle.

The next example shows that Theorem 1.1 is wrong if one drops the integrability assumption on $\mathscr{F}$.

Example 1.3. The null correlation bundle $\mathscr{N}$ on $\mathbb{P}^{2n+1}$ (see [OSS80, 4.2]) yields a contact distribution $\mathscr{D} = \mathscr{N} \otimes \mathscr{O}_{\mathbb{P}^{2n+1}}(1) \subset T_{\mathbb{P}^{2n+1}}$ on $\mathbb{P}^{2n+1}$ corresponding to a twisted 1-form $\theta \in H^0\big(\mathbb{P}^{2n+1}, \Omega^1_{\mathbb{P}^{2n+1}} \otimes \mathscr{O}_{\mathbb{P}^{2n+1}}(2)\big)$.

Recall that a contact structure on a complex manifold X is a corank 1 subbundle $\mathscr{D} \subset T_X$ such that the bilinear form on $\mathscr{D}$ with values in the quotient line bundle $\mathscr{L} = T_X/\mathscr{D}$ deduced from the Lie bracket on T_X is everywhere non-degenerate. This implies that the dimension of X is odd, say $\dim X = 2n+1$, that the anti-canonical bundle $\mathscr{O}_X(-K_X)$ is isomorphic to $\mathscr{L}^{\otimes n+1}$, and that $\det(\mathscr{D}) \cong \mathscr{L}^{\otimes n}$. Alternatively, the contact structure can be described by the twisted 1-form $\theta \in H^0(X, \Omega^1_X \otimes \mathscr{L})$ corresponding to the natural projection $T_X \twoheadrightarrow \mathscr{L}$.

An immediate consequence of Theorem 1.1 is the following upper bound on the Kodaira dimension of the anti-canonical bundle of a holomorphic codimension one foliation.

Corollary 1.4. *Let X be a complex projective manifold, and let $\mathscr{F} \subset T_X$ be a codimension one foliation. Suppose that either $\mathscr{F}$ is regular, or that $\mathscr{F}$ has a compact leaf. Suppose furthermore that $-K_{\mathscr{F}}$ is nef. Then $\kappa(X, -K_{\mathscr{F}}) \leqslant \dim X - 1$.*

Next, we investigate regular codimension one foliations where $-K_{\mathscr{F}}$ is nef and $\kappa(X, -K_{\mathscr{F}}) = \dim X - 1$. Note that $-K_{\mathscr{F}}$ is then nef and abundant.

Theorem 1.5. *Let X be a complex projective manifold, and let $\mathscr{F} \subset T_X$ be a codimension q foliation with $0 < q < \dim X$. Suppose that either $\mathscr{F}$ is regular, or that $\mathscr{F}$ has a compact leaf. Suppose furthermore that $-K_{\mathscr{F}}$ is nef and abundant. Then $\kappa(X, -K_{\mathscr{F}}) \leqslant \dim X - q$, and equality holds only if $\mathscr{F}$ is algebraically integrable.*

The following example shows to what extent Theorem 1.5 is optimal.

Example 1.6. Let T be a positive-dimensional projective manifold with $-K_T$ nef, and let A be a positive-dimensional abelian variety. Consider a linear foliation $\mathscr{G}$ on A, and let $\mathscr{F}$ be the pull-back of $\mathscr{G}$ on $T \times A$. Then $-K_{\mathscr{F}} \sim -K_{T\times A/A}$ is nef and $\nu(-K_{\mathscr{F}}) = \kappa(T \times A, -K_{\mathscr{F}}) = \kappa(T, -K_T) = \nu(-K_T)$. Moreover, if $\mathscr{G}$ is general enough and $\dim A \geqslant 2$, then $\mathscr{F}$ has no algebraic leaf.

Theorem 1.7. *Let X be a complex projective manifold with $h^1(X, \mathscr{O}_X) = 0$, and let $\mathscr{F} \subset T_X$ be a regular codimension one foliation. Suppose that $-K_{\mathscr{F}}$ is nef and $\kappa(X, -K_{\mathscr{F}}) = \dim X - 1$. Then $X \cong \mathbb{P}^1 \times F$, and $\mathscr{F}$ is induced by the natural morphism $X \cong \mathbb{P}^1 \times F \to \mathbb{P}^1$.*

The proof of the main results rely on the following observation (see also Proposition 6.1).

Proposition 1.8. *Let X be a complex projective manifold, and let $\mathscr{F} \subset T_X$ be a foliation. Suppose that $-K_{\mathscr{F}}$ is nef. Suppose furthermore that either $\mathscr{F}$ is regular, or that $\mathscr{F}$ has a compact leaf. There exist a foliation $\mathscr{H}$ on X induced by an almost proper map $\varphi\colon X \dashrightarrow Y$, and a foliation $\mathscr{G}$ on Y such that*

(1) *there is no positive-dimensional algebraic subvariety passing through a general point of Y that is tangent to $\mathscr{G}$,*
(2) *$\mathscr{F} = \varphi^{-1}\mathscr{G}$, and*
(3) *$K_{\mathscr{H}} \equiv K_{\mathscr{F}}$.*

The paper is organized as follows. When proving Theorems 1.1 and 1.5, we consider the algebraic and transcendental parts of $\mathscr{F}$, the latter being a foliation on the space of leaves of the former. Thus, we are led to consider foliations on singular varieties. In section 2, we review basic definitions and results about holomorphic foliations on normal varieties. In section 3, we extend a number of known results on slope-semistable sheaves from the classical case to the setting where polarisations are given by movable complete intersection curve classes. As application, we obtain a generalization of Metha-Ramanathan's theorem. In sections 4 and 5, we study the anti-canonical divisor of algebraically integrable foliations, and provide applications to the study of singularities of those foliations with $-K_{\mathscr{F}}$ nef. Section 6 is devoted to the proof of Proposition 1.8. Section 7 is devoted to the proofs of Theorem 1.1 and Corollary 1.4. In section 8 we prove Theorems 1.5 and 1.7.

We work over the field $\mathbb{C}$ of complex numbers.

2. Recollection: Foliations

In this section we recall the basic facts concerning foliations.

2.1. Foliations.

Definition 2.1. A *foliation* on a normal variety X is a coherent subsheaf $\mathscr{F} \subset T_X$ such that

- $\mathscr{F}$ is closed under the Lie bracket, and
- $\mathscr{F}$ is saturated in T_X. In other words, the quotient $T_X/\mathscr{F}$ is torsion-free.

The *rank* r of $\mathscr{F}$ is the generic rank of $\mathscr{F}$. The *codimension* of $\mathscr{F}$ is defined as $q := \dim X - r$.

Let $X^\circ \subset X_{\mathrm{ns}}$ be the maximal open set where $\mathscr{F}_{|X_{\mathrm{ns}}}$ is a subbundle of $T_{X_{\mathrm{ns}}}$. A *leaf* of $\mathscr{F}$ is a connected, locally closed holomorphic submanifold $L \subset X^\circ$ such that $T_L = \mathscr{F}_{|L}$. A leaf is called *algebraic* if it is open in its Zariski closure.

Definition 2.2. Let X be a complex manifold, and let $\mathscr{F}$ be a foliation on X. We say that $\mathscr{F}$ is *regular* if $\mathscr{F}$ is a subbundle of T_X.

Next, we define the *algebraic* and *transcendental* parts of a holomorphic foliation (see [AD14, Definition 2]).

Definition 2.3. Let $\mathscr{F}$ be a holomorphic foliation on a normal variety X. There exist a normal variety Y, unique up to birational equivalence, a dominant rational map with connected fibers $\varphi : X \dashrightarrow Y$, and a holomorphic foliation $\mathscr{G}$ on Y such

that the following holds (see [LPT11, Section 2.4]):

(1) $\mathscr{G}$ is purely transcendental, i.e., there is no positive-dimensional algebraic subvariety through a general point of Y that is tangent to $\mathscr{G}$; and
(2) $\mathscr{F}$ is the pull-back of $\mathscr{G}$ via φ (see section 2.9 for this notion).

The foliation on X induced by φ is called the *algebraic part* of $\mathscr{F}$.

2.4 (Foliations defined by q-forms)**.** Let $\mathscr{F}$ be a codimension q foliation on an n-dimensional complex manifold X. Suppose that $q \geqslant 1$. The *normal sheaf* of $\mathscr{F}$ is $\mathscr{N} := (T_X/\mathscr{F})^{**}$. The q-th wedge product of the inclusion $\mathscr{N}^* \hookrightarrow \Omega^1_X$ gives rise to a non-zero global section $\omega \in H^0\big(X, \Omega^q_X \otimes \det(\mathscr{N})\big)$ whose zero locus has codimension at least two in X. Such ω is *locally decomposable* and *integrable*. To say that ω is locally decomposable means that, in a neighborhood of a general point of X, ω decomposes as the wedge product of q local 1-forms $\omega = \omega_1 \wedge \cdots \wedge \omega_q$. To say that it is integrable means that for this local decomposition one has $d\omega_i \wedge \omega = 0$ for every $i \in \{1, \dots, q\}$. The integrability condition for ω is equivalent to the condition that $\mathscr{F}$ is closed under the Lie bracket.

Conversely, let $\mathscr{L}$ be a line bundle on X, $q \geqslant 1$, and $\omega \in H^0(X, \Omega^q_X \otimes \mathscr{L})$ a global section whose zero locus has codimension at least two in X. Suppose that ω is locally decomposable and integrable. Then one defines a foliation of rank $r = n - q$ on X as the kernel of the morphism $T_X \to \Omega^{q-1}_X \otimes \mathscr{L}$ given by the contraction with ω. These constructions are inverse of each other.

We will use the following notation.

Notation 2.5. Let $\varphi\colon X \to Y$ be a dominant morphism of normal varieties. Assume either that K_Y is $\mathbb{Q}$-Cartier or that φ is equidimensional. Write $K_{X/Y} := K_X - \varphi^* K_Y$. We refer to it as the *relative canonical divisor of X over Y*.

Remark 2.6. Let $\varphi\colon X \to Y$ be an equidimensional morphism of normal varieties, and let D be a Weil $\mathbb{Q}$-divisor on Y. The pull-back $\varphi^* D$ of D is defined as follows. We define $\varphi^* D$ to be the unique $\mathbb{Q}$-divisor on X whose restriction to $\varphi^{-1}(Y_{\mathrm{ns}})$ is $(\varphi_{|\varphi^{-1}(Y_{\mathrm{ns}})})^* D_{|Y_{\mathrm{ns}}}$. This construction agrees with the usual pull-back if D itself is $\mathbb{Q}$-Cartier.

Notation 2.7. Let $\varphi\colon X \to Y$ be a dominant morphism of normal varieties. Let $Y^\circ \subset Y$ be a dense open subset with $\operatorname{codim} Y \setminus Y^\circ \geqslant 2$ such that φ restricts to an equidimensional morphism $\varphi^\circ : X^\circ \to Y^\circ$, where $X^\circ := \varphi^{-1}(Y^\circ)$. Set

$$R(\varphi^\circ) = \sum_{D^\circ} \Big((\varphi^\circ)^* D^\circ - \big((\varphi^\circ)^* D^\circ\big)_{\mathrm{red}} \Big)$$

where D° runs through all prime divisors on Y°, and let $R(\varphi)$ denote the Zariski closure of $R(\varphi^\circ)$ in X. We refer to it as the *ramification divisor of φ*.

Definition 2.8. Let $\mathscr{F}$ be a foliation on a normal projective variety X. The *canonical class* $K_{\mathscr{F}}$ of $\mathscr{F}$ is any Weil divisor on X such that $\mathscr{O}_X(-K_{\mathscr{F}}) \cong \det(\mathscr{F})$.

We say that $\mathscr{F}$ is $\mathbb{Q}$-*Gorenstein* if $K_{\mathscr{F}}$ is a $\mathbb{Q}$-Cartier divisor.

2.9 (Foliations described as pull-backs)**.** Let X and Y be normal varieties, and let $\varphi\colon X \dashrightarrow Y$ be a dominant rational map that restricts to a morphism $\varphi^\circ\colon X^\circ \to Y^\circ$, where $X^\circ \subset X$ and $Y^\circ \subset Y$ are smooth open subsets.

Let $\mathscr{G}$ be a codimension q foliation on Y with $q \geqslant 1$. Suppose that the restriction $\mathscr{G}^\circ$ of $\mathscr{G}$ to Y° is defined by a twisted q-form $\omega_{Y^\circ} \in H^0\big(Y^\circ, \Omega^q_{Y^\circ} \otimes \det(\mathscr{N}_{\mathscr{G}^\circ})\big)$. Then

ω_{Y° induces a non-zero twisted q-form $\omega_{X^\circ} \in H^0\Big(X^\circ, \Omega^q_{X^\circ} \otimes (\varphi^\circ)^*\big(\det(\mathscr{N}_{\mathscr{G}})_{|Y^\circ}\big)\Big)$, which in turn defines a codimension q foliation $\mathscr{F}^\circ$ on X°. We say that the saturation $\mathscr{F}$ of $\mathscr{F}^\circ$ in T_X *is the pull-back of* $\mathscr{G}$ *via* φ, and write $\mathscr{F} = \varphi^{-1}\mathscr{G}$.

Suppose that X° is such that φ° is an equidimensional morphism. Let $(B_i)_{i\in I}$ be the (possibly empty) set of hypersurfaces in Y° contained in the set of critical values of φ° and invariant by $\mathscr{G}$. A straightforward computation shows that

$$\det\big(\mathscr{N}_{\mathscr{F}^\circ}\big) \;\cong\; (\varphi^\circ)^* \det\big(\mathscr{N}_{\mathscr{G}_{|Y^\circ}}\big) \otimes \mathscr{O}_{X^\circ}\Big(\sum_{i\in I}\big((\varphi^\circ)^*B_i\big)_{\mathrm{red}} - (\varphi^\circ)^*B_i\Big). \tag{2.1}$$

In particular, if $\mathscr{F}$ is induced by φ, then (2.1) gives

$$K_{\mathscr{F}^\circ} = K_{X^\circ/Y^\circ} - R(\varphi^\circ), \tag{2.2}$$

where $R(\varphi^\circ)$ denotes the ramification divisor of φ°.

Conversely, let $\mathscr{F}$ be a foliation on X, and suppose that the general fiber of φ is tangent to $\mathscr{F}$. This means that, for a general point x on a general fiber F of φ, the linear subspace $\mathscr{F}_x \subset T_xX$ determined by the inclusion $\mathscr{F} \subset T_X$ contains T_xF. Suppose moreover that φ° is smooth with connected fibers. Then, by [AD13, Lemma 6.7], there is a holomorphic foliation $\mathscr{G}$ on Y such that $\mathscr{F} = \varphi^{-1}\mathscr{G}$.

2.2. **Algebraically integrable foliations.**

Definition 2.10. Let X be normal variety. A foliation $\mathscr{F}$ on X is said to be *algebraically integrable* if the leaf of $\mathscr{F}$ through a general point of X is an algebraic variety.

Definition 2.11. Let $\mathscr{F}$ be an algebraically integrable $\mathbb{Q}$-Gorenstein foliation on a normal projective variety X. Let $i: F \to X$ be the normalization of the closure of a general leaf of $\mathscr{F}$. There is an effective $\mathbb{Q}$-divisor Δ on F such that $K_F + \Delta \sim i^*K_{\mathscr{F}}$ ([AD16, Definition 3.6]). The pair (F, Δ) is called a *general log leaf* of $\mathscr{F}$.

The case when (F, Δ) is *log canonical* is specially interesting (see [AD13] and [AD16]). We refer to [KM98, section 2.3] for the definition of klt and log canonical pairs. Here we only remark that if F is smooth and Δ is a reduced simple normal crossing divisor, then (F, Δ) is log canonical.

The same argument used in the proof of [AD16, Proposition 3.13] shows that the following holds. We leave the details to the reader. In particular, Proposition 2.12 below says that an algebraically integrable $\mathbb{Q}$-Gorenstein foliation with mild singularities and big anti-canonical divisor has a very special property: there is a common point contained in the closure of a general leaf.

Proposition 2.12. *Let X be a normal projective variety, let $\mathscr{F}$ be a $\mathbb{Q}$-Gorenstein algebraically integrable foliation on X, and let (F, Δ) be its general log leaf. Suppose that $-K_{\mathscr{F}} = A + E$ where A is a $\mathbb{Q}$-ample $\mathbb{Q}$-divisor and E is an effective $\mathbb{Q}$-divisor. Suppose furthermore that $(F, \Delta + E_{|F})$ is log canonical. Then there is a closed irreducible subset $T \subset X$ satisfying the following property. For a general log leaf (F, Δ), there exists a log canonical center S of $(F, \Delta + E_{|F})$ whose image in X is T.*

2.13 (The family of log leaves)**.** Let X be normal projective variety, and let $\mathscr{F}$ be an algebraically integrable foliation on X. We describe the *family of leaves* of $\mathscr{F}$ (see [AD14, Remark 3.12]). There is a unique normal projective variety Y contained in the normalization of the Chow variety of X whose general point parametrizes

the closure of a general leaf of $\mathscr{F}$ (viewed as a reduced and irreducible cycle in X). Let $Z \to Y \times X$ denote the normalization of the universal cycle. It comes with morphisms:

$$\begin{tikzcd} Z \arrow[r, "\nu"] \arrow[d, "\psi"'] & X, \\ Y & \end{tikzcd}$$

where $\nu\colon Z \to X$ is birational and, for a general point $y \in Y$, $\nu\big(\psi^{-1}(y)\big) \subset X$ is the closure of a leaf of $\mathscr{F}$. The variety Y is called the *family of leaves* of $\mathscr{F}$.

Suppose moreover that $\mathscr{F}$ is $\mathbb{Q}$-Gorenstein. Denote by $\mathscr{F}_Z$ the foliation induced by $\mathscr{F}$ (or ψ) on Z. There is a canonically defined effective Weil $\mathbb{Q}$-divisor B on Z such that

$$K_{\mathscr{F}_Z} + B = K_{Z/Y} - R(\psi) + B \sim_{\mathbb{Q}} \nu^* K_{\mathscr{F}}, \tag{2.3}$$

where $R(\psi)$ denotes the ramification divisor of ψ. Note that the equality $K_{\mathscr{F}_Z} = K_{Z/Y} - R(\psi)$ follows from (2.2). Suppose that $y \in Y$ is a general point, and set $Z_y := \psi^{-1}(y)$ and $\Delta_y := B_{|Z_y}$. Then (Z_y, Δ_y) coincides with the general log leaf (F, Δ) defined above.

Remark 2.14. In the setup of section 2.13, we claim that B is ν-exceptional. This is an immediate consequence of the equality $\nu_* K_{\mathscr{F}_Z} = K_{\mathscr{F}}$.

Remark 2.15. In the setup of section 2.13, suppose furthermore that $K_{\mathscr{F}}$ is a Cartier divisor. Then B is a Weil divisor, and (2.3) reads

$$K_{\mathscr{F}_Z} + B = K_{Z/Y} - R(\psi) + B \sim \nu^* K_{\mathscr{F}}.$$

We end this subsection with a useful criterion of algebraic integrability for foliations.

Theorem 2.16 ([BM16, Theorem 0.1], [Bos01, Theorem 3.5])**.** *Let X be a normal complex projective variety, and let $\mathscr{F}$ be a foliation on X. Let $C \subset X$ be a complete curve disjoint from the singular loci of X and $\mathscr{F}$. Suppose that the restriction $\mathscr{F}_{|C}$ is an ample vector bundle on C. Then the leaf of $\mathscr{F}$ through any point of C is an algebraic variety.*

3. Movable complete intersection curves and semistable sheaves

In order to prove our results, we construct subfoliations of the foliation $\mathscr{F}$ which inherit some of the positivity properties of $\mathscr{F}$. One way to construct such subfoliations is via Harder-Narasimhan filtrations (see Proposition 3.12 below).

The notion of slope-stability with respect to the choice of an ample line bundle is not flexible enough to allow for applications in birational geometry. The paper [GKP16] (see also [CP11]) extends a number of known results from the classical case to the setting where the ambient variety is normal and $\mathbb{Q}$-factorial, and polarisations are given by movable curve classes. Recall that a numerical curve class α on a normal projective variety is movable if $D \cdot \alpha \geqslant 0$ for all effective Cartier divisors D. In this paper, it is advantageous to generalize the notion of slope, replacing movable curve classes with movable complete intersection curve classes.

Let X be an n-dimensional normal projective variety. Consider the space $\mathrm{N}_1(X)_{\mathbb{R}}$ of numerical curve classes on X.

Notation 3.1. Let X be an n-dimensional normal projective variety, and let α be a numerical curve class on X. We say that α is a *complete intersection numerical curve class* if $\alpha = [D_1 \cdot \cdots \cdot D_{n-1}] \in \mathrm{N}_1(X)_{\mathbb{R}}$ for some Cartier divisors $D_1, \ldots, D_{n-1}$ on X.

Remark 3.2. Let $D_1, \ldots, D_{n-1}$ be Cartier divisors on X. Suppose that D_i is nef for all i. Then $\alpha := [D_1 \cdot \cdots \cdot D_{n-1}]$ is a movable complete intersection numerical curve class.

Definition 3.3. Let X be a normal projective variety. Let α be a complete intersection numerical curve class on X, and let $\mathscr{F}$ be a torsion-free sheaf of positive rank r.

We define the slope of $\mathscr{F}$ with respect to α to be $\mu_\alpha(\mathscr{F}) = \frac{\det \mathscr{F} \cdot \alpha}{r}$.

We say that $\mathscr{F}$ is *α-semistable* if for any subsheaf $\mathscr{E} \neq 0$ of $\mathscr{F}$ we have $\mu_\alpha(\mathscr{E}) \leq \mu_\alpha(\mathscr{F})$.

The same argument used in the proof of [GKP16, Corollary 2.26] shows that the following holds.

Proposition 3.4. *Given a torsion-free sheaf $\mathscr{F}$ on a normal projective variety X, and a movable complete intersection numerical curve class α, there exists a unique filtration of $\mathscr{F}$ by saturated subsheaves*

$$0 = \mathscr{F}_0 \subsetneq \mathscr{F}_1 \subsetneq \cdots \subsetneq \mathscr{F}_k = \mathscr{F},$$

with α-semistable quotients $\mathcal{Q}_i = \mathscr{F}_i/\mathscr{F}_{i-1}$, and such that $\mu_\alpha(\mathcal{Q}_1) > \mu_\alpha(\mathcal{Q}_2) > \cdots > \mu_\alpha(\mathcal{Q}_k)$.

This is called the *Harder-Narasimhan filtration* of $\mathscr{F}$. The sheaf $\mathscr{F}_1$ is called the maximal destabilizing subsheaf of $\mathscr{F}$ with respect to α, and it satisfies

$$\mu_\alpha(\mathscr{F}_1) = \mu_\alpha^{\max}(\mathscr{F}) := \sup\{\mu_\alpha(\mathscr{E}) \,|\, 0 \neq \mathscr{E} \subseteq \mathscr{F} \text{ a coherent subsheaf}\}.$$

We will need the following easy observations.

Lemma 3.5. *Let $\mathscr{F}$ be a torsion-free sheaf on a normal projective X, and let α be a complete intersection numerical curve class on X. Then $\mathscr{F}$ is α-semistable if and only if so is $\mathscr{F}^{**}$.*

Notation 3.6. Let $\varphi\colon X \to Y$ be any birational morphism of projective normal varieties, and let $\alpha = [D_1 \cdot \cdots \cdot D_{n-1}] \in \mathrm{N}_1(X)_{\mathbb{R}}$ be a complete intersection numerical curve class. Set $\varphi^*\alpha := [\varphi^* D_1 \cdot \cdots \cdot \varphi^* D_{n-1}] \in \mathrm{N}_1(X)_{\mathbb{R}}$. We refer to it as the *numerical pull-back of α*.

Remark 3.7. In the setup of Notation 3.6, we have $D \cdot \varphi^*\alpha = \varphi_* D \cdot \alpha$ for any Weil $\mathbb{Q}$-divisor D on X by the projection formula. It follows that this construction agrees with the usual numerical pull-back if X and Y are $\mathbb{Q}$-factorial.

Remark 3.8. The pull-back of any movable complete intersection numerical curve class is again a movable complete intersection numerical curve class.

Lemma 3.9. *Let $\varphi\colon X \to Y$ be any birational morphism of projective normal varieties, and let $\mathscr{F}$ be a torsion-free sheaf on X. Let α be a complete intersection numerical curve class on Y. Then $\mathscr{F}$ is $\varphi^*\alpha$-semistable if and only if the torsion-free sheaf $\varphi_*\mathscr{F}$ is α-semistable.*

The proof of the next lemma is similar to that of [HL97, Lemma 3.2.2], and so we leave the details to the reader.

Lemma 3.10. *Let $\varphi\colon X \to Y$ be a finite surjective morphism of projective normal varieties, and let $\mathscr{F}$ be a torsion-free sheaf on Y. Let α be a complete intersection numerical curve class on Y. Then $\mathscr{F}$ is α-semistable if and only if $\varphi^*\mathscr{F}/\mathrm{Tor}(\varphi^*\mathscr{F})$ is $\varphi^*\alpha$-semistable.*

We will make use of the following notation.

Notation 3.11. Let X be an n-dimensional normal projective variety, and let $|H|$ be a basepoint-free complete linear system on X. For each $1 \leqslant i \leqslant n-1$, let $D_i \in |H|$ be a general hypersurface, and set $C = D_1 \cap \cdots \cap D_{n-1}$. We say that C is a *general complete intersection curve on X of type H*.

We now provide a technical tool for the proof of the main results. The following result is a generalization of [AD13, Lemma 7.4] (see also [DP13, Lemma 12]).

Proposition 3.12. *Let $\psi\colon Z \to Y$ be a dominant and equidimensional morphism of normal projective varieties, let X be a normal projective variety, and let $\nu\colon Z \to X$ be a birational morphism. Let H be an ample divisor on X, and let $\mathscr{G}$ be a torsion-free sheaf on Y. Suppose that ν^*H is ample over Y. Then there exists a saturated subsheaf $\mathscr{E} \subseteq \mathscr{G}$ satisfying the following property. For a general complete intersection curve C on Z of type $m\nu^*H$ with m large enough, $\psi^*\mathscr{E}_{|C}$ is the maximally destabilizing subsheaf of $\psi^*\mathscr{G}_{|C}$.*

Proof. Set $n = \dim Z = \dim X$ and $l = \dim Y$. By replacing H with mH for some positive integer m, we may assume that H is very ample and that $m\nu^*H$ is very ample over Y.

Let m be a positive integer, and consider general hypersurfaces $H_i \in |mH|$ for $1 \leqslant i \leqslant n-1$. By replacing m with a larger integer, we may assume that the Harder-Narasimhan filtration of $\mathscr{F} := \big(\nu_*(\psi^*\mathscr{G})\big)^{**}$ with respect to H commutes with restriction to all complete intersections $H_1 \cap \cdots \cap H_i$, $1 \leqslant i \leqslant n-1$ (see [Fle84]).

Set $D_i := \nu^{-1}(H_i) \in |m\nu^*H|$, $Z_1 := D_1 \cap \cdots \cap D_{n-l}$, and $X_1 := H_1 \cap \cdots \cap H_{n-l}$. The restriction of ν to Z_1 yields a surjective birational morphism $\nu_1\colon Z_1 \to X_1$ of normal varieties. Denote by $\psi_1\colon Z_1 \to Y$ the restriction of ψ to Z_1. Note that ψ_1 is finite since ν^*H is very ample over Y.

Set $B = H_1 \cap \cdots \cap H_{n-1}$. Up to replacing $H_{n-l+1}, \ldots, H_{n-1}$ by linearly equivalent divisors on X, we may assume that $B \subset X \setminus \nu_1(\mathrm{Exc}(\nu_1))$. Set $C_1 = \nu_1^{-1}(B) = \nu^{-1}(B) = D_1 \cap \cdots \cap D_{n-1} \cong B$.

Set $\mathscr{T} := (\psi^*\mathscr{G})^{**}$, $\mathscr{T}_1 = (\psi_1^*\mathscr{G})^{**}$, and $\mathscr{F}_1 := \big((\nu_1)_*\mathscr{T}_1\big)^{**}$. Since $\mathscr{F}_{|X_1}$ is reflexive, we must have

$$\mathscr{F}_1 \cong \mathscr{F}_{|X_1}.$$

Let $\mathscr{K}$ be the maximally destabilizing subsheaf of $\mathscr{F}$ with respect to H. By Flenner's version of the Mehta-Ramanathan theorem and the choice of m, $\mathscr{K}_1 := \mathscr{K}_{|X_1}$ is the maximally destabilizing subsheaf of $\mathscr{F}_1 \cong \mathscr{F}_{|X_1}$ with respect to $H_{|X_1}$. Note that $\mathscr{K}_1$ is reflexive by [AD13, Remark 2.3]. Let $\mathscr{S}_1 \subset \mathscr{T}_1$ such that $\big((\nu_1)_*\mathscr{S}_1\big)^{**} = \mathscr{K}_1 \subset \mathscr{F}_1$. We may assume that $\mathscr{S}_1$ is saturated in $\mathscr{T}_1$. Since $B \subset X \setminus \nu_1(\mathrm{Exc}(\nu_1))$, we have

$$\mathscr{K}_{|B} = \mathscr{K}_{1|B} = \big((\nu_1)_*\mathscr{S}_1\big)^{**}{}_{|B} \cong (\nu_1)_*\mathscr{S}_{1|B} \cong \mathscr{S}_{1|C_1}$$

and

$$\mathscr{F}_{|B} = \mathscr{F}_{1|B} = \big((\nu_1)_*\mathscr{T}_1\big)^{**}_{|B} \cong (\nu_1)_*\mathscr{T}_{1|B} \cong \mathscr{T}_{1|C_1}.$$

By Flenner's version of the Mehta-Ramanathan theorem and the choice of m, we conclude as above that $\mathscr{S}_{1|C_1}$ is the maximally destabilizing subsheaf of $\mathscr{T}_{1|C_1}$.

Let K be a splitting field of the function field $\mathbb{C}(Z_1)$ over $\mathbb{C}(Y)$, and let $\nu_2\colon Z_2 \to Z_1$ be the normalization of Z_1 in K. Set $\psi_2 := \psi_1 \circ \nu_2\colon Z_2 \to Y$, and denote by G the Galois group of $\mathbb{C}(Z_2)$ over $\mathbb{C}(Y)$. We obtain a commutative diagram:

$$\begin{tikzcd}
Z_2 \arrow[r, "\nu_2,\ \text{finite}"] \arrow[d, "\psi_2,\ \text{finite}"'] & Z_1 \arrow[r, hook, "\nu_1,\ \text{birational}"] \arrow[d, "\psi_1,\ \text{finite}"'] \arrow[dr, hook] & X_1 \arrow[dr, hook] & \\
Y \arrow[r, equal] & Y \arrow[dr, equal] & Z \arrow[r, "\nu"'] \arrow[d, "\psi"] & X \\
& & Y. &
\end{tikzcd}$$

The numerical curve class $[C_1] \in \mathrm{N}_1(Z_1)_{\mathbb{R}}$ is a movable complete intersection curve class by Remark 3.2. By Lemmas 3.5 and 3.9, $\mathscr{S}_1$ is the maximally destabilizing sheaf of $\mathscr{T}_1$ with respect to the movable class $[C_1] = [\nu_1^* H_{|X_1}]^{l-1} \in \mathrm{N}_1(Z_1)_{\mathbb{R}}$. From Lemmas 3.5 and 3.10, we deduce that $\mathscr{S}_2 := (\nu_2^*\mathscr{S}_1)^{**}$ is the maximally destabilizing sheaf of $(\nu_2^*\mathscr{T}_1)^{**} \cong (\psi_2^*\mathscr{G}_1)^{**} =: \mathscr{T}_2$ with respect to the movable class $\nu_2^*[C_1] = [(\nu_1 \circ \nu_2)^* H_{|X_1}]^{l-1} \in \mathrm{N}_1(Z_2)_{\mathbb{R}}$.

Let $Y^\circ \subset Y$ be a dense open subset with $\operatorname{codim} Y \setminus Y^\circ \geqslant 2$ such that G acts on $Z_2^\circ := \psi_2^{-1}(Y^\circ)$ and such that $\mathscr{G}_{|Y^\circ}$ is locally free. Note that $Z_2^\circ/G \cong Y^\circ$. Because of its uniqueness, $\mathscr{S}_{2|Z_2^\circ}$ is invariant under the natural action of G on $\mathscr{T}_{2|Z_2^\circ} \cong \psi_2^*\mathscr{G}_{|Z_2^\circ}$. Therefore, there exists a saturated subsheaf $\mathscr{E} \subseteq \mathscr{G}$ such that $\psi_2^*\mathscr{E}_{|Z_2^\circ} \cong \mathscr{S}_{2|Z_2^\circ}$. By Lemma 3.10 again, we have that $(\psi_2^*\mathscr{E})^{**} \cong \mathscr{S}_1$. We conclude that $\psi^*\mathscr{E}_{|C} \cong \mathscr{S}_{1|C_1}$ is the maximally destabilizing subsheaf of $\psi^*\mathscr{G}_{|C} \cong \mathscr{T}_{1|C_1}$. This completes the proof of the proposition. □

4. The anti-canonical divisor of an algebraically integrable foliation

In this section, we apply Viehweg's weak positivity theorem to algebraically integrable foliations. We refer to [Vie95] for the definition of weak positivity. This notion was introduced by Viehweg, as a kind of birational version of being nef.

The following is a generalization of [Hör12, Lemma 2.14] and [CP13, Theorem 2.11] (see also Proposition 4.3 below). Recall that a $\mathbb{Q}$-divisor D on a normal projective variety X is said to be *pseudo-effective* if, for any big $\mathbb{Q}$-divisor B on X and any rational number $\varepsilon > 0$, there exists an effective $\mathbb{Q}$-divisor E on X such that $D + \varepsilon B \sim_{\mathbb{Q}} E$.

Proposition 4.1. *Let X be a normal projective variety, let $\mathscr{H}$ be a $\mathbb{Q}$-Gorenstein algebraically integrable (possibly singular) foliation on X, and let (F, Δ) be its general log leaf. Let L be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Suppose that $(F, \Delta + L_{|F})$ is log canonical, and that $\kappa\big(F, K_F + \Delta + L_{|F}\big) \geqslant 0$. Then $K_{\mathscr{H}} + L$ is pseudo-effective.*

Proof. Let $\psi\colon Z \to Y$ be the family of leaves, and let $\nu\colon Z \to X$ be the natural morphism (see section 2.13). By Lemma 4.2 below, there exists a finite surjective morphism $Y_1 \to Y$ with Y_1 normal and connected satisfying the following property.

If Z_1 denotes the normalization of the product $Y_1 \times_Y Z$, then the induced morphism $\psi_1 \colon Z_1 \to Y_1$ has reduced fibers over codimension one points in Y_1.

By section 2.13, there is a canonically defined effective $\mathbb{Q}$-Weil divisor B on Z such that

$$K_{Z/Y} - R(\psi) + B \sim_{\mathbb{Q}} \nu^* K_{\mathscr{H}} \tag{4.1}$$

where $R(\psi)$ denotes the ramification divisor of ψ. A straightforward computation shows that

$$\nu_1^*\big(K_{Z/Y} - R(\psi)\big) = K_{Z_1/Y_1}. \tag{4.2}$$

Set $\Gamma_1 = \nu_1^* B + (\nu \circ \nu_1)^* L$. Let F_1 be a general fiber of ψ_1. Note that $(F_1, \Gamma_{1|F_1})$ has log canonical singularities by assumption. It follows that (Z_1, Γ_1) has log canonical singularities over the generic point of Y_1 by inversion of adjunction (see [Kaw07]). Let $\mu_2 \colon Y_2 \to Y_1$ be a resolution of singularities, and let Z_2 be a resolution of singularities of the product $Y_2 \times_{Y_1} Z_1$. Up to replacing Z_2 with a birational model, we may assume that Z_2 is a log resolution of (Z_1, Γ_1). We have a commutative diagram:

$$\begin{tikzcd}
Z_2 \arrow[r, "\nu_2,\ \text{birational}"] \arrow[d, "\psi_2"] & Z_1 \arrow[r, "\nu_1,\ \text{finite}"] \arrow[d, "\psi_1"] & Z \arrow[r, "\nu,\ \text{birational}"] \arrow[d, "\psi"] & X \\
Y_2 \arrow[r, "\mu_2,\ \text{birational}"'] & Y_1 \arrow[r, "\mu_1,\ \text{finite}"'] & Y. &
\end{tikzcd}$$

We write

$$K_{Z_2} + \Gamma_2 = \nu_2^*(K_{Z_1} + \Gamma_1) + E \tag{4.3}$$

where Γ_2 and E are effective, with no common components, $(\nu_2)_*\Gamma_2 = \Gamma_1$, and E is ν_2-exceptional. Then (Z_2, Γ_2) has log canonical singularities over the generic point of Y_2. Let F_2 denote a general fiber of ψ_2. Then

$$\begin{aligned}
\kappa\big(F_2, K_{F_2} + \Gamma_{2|F_2}\big) &= \kappa\big(F_1, K_{F_1} + \Gamma_{1|F_1}\big) && \text{by (4.3)} \\
&= \kappa\big(F, K_F + B_{|F} + L_{|F}\big) && \text{by (4.2)} \\
&= \kappa\big(F, K_F + \Delta + L_{|F}\big) && \text{by section 2.13} \\
&\geqslant 0.
\end{aligned}$$

Thus, for any positive integer m which is sufficiently divisible, the natural morphism

$$\psi_2^*\Big((\psi_2)_*\mathscr{O}_{Z_2}\big(m(K_{Z_2/Y_2} + \Gamma_2)\big)\Big) \to \mathscr{O}_{Z_2}\big(m(K_{Z_2/Y_2} + \Gamma_2)\big)$$

is generically surjective. The sheaf $(\psi_2)_*\mathscr{O}_{Z_2}\big(m(K_{Z_2/Y_2} + \Gamma_2)\big)$ is weakly positive by [Cam04, Theorem 4.13]. Therefore, $K_{Z_2/Y_2} + \Gamma_2$ is pseudo-effective, and hence so is $(\nu \circ \nu_1 \circ \nu_2)_*(K_{Z_2/Y_2} + \Gamma_2)$. Since ψ_1 is equidimensional, there exist dense open subsets $Y_1^\circ \subset Y_1$ and $Z_1^\circ \subset Z_1$ with $\operatorname{codim} Y_1 \setminus Y_1^\circ \geqslant 2$ and $\operatorname{codim} Z_1 \setminus Z_1^\circ \geqslant 2$ such that $\psi_1(Z_1^\circ) \subset Y_1^\circ$, and such that ν_2 (respectively, μ_2) induces an isomorphism $\nu_2^{-1}(Z_1 \setminus Z_1^\circ) \cong Z_1 \setminus Z_1^\circ$ (respectively, $\mu_2^{-1}(Y_1 \setminus Y_1^\circ) \cong Y_1 \setminus Y_1^\circ$). Hence,

$$(\nu_2)_*(K_{Z_2/Y_2} + \Gamma_2) = K_{Z_1/Y_1} + \Gamma_1.$$

From (4.1) and (4.2), we conclude that

$$(\nu \circ \nu_1 \circ \nu_2)_*(K_{Z_2/Y_2} + \Gamma_2) = \deg(\nu_1)(K_{\mathscr{H}} + L)$$

is pseudo-effective, completing the proof of the proposition. $\square$

Lemma 4.2. *Let $\psi\colon Z \to Y$ be a dominant equidimensional morphism of normal varieties. There exists a finite surjective morphism $Y_1 \to Y$ with Y_1 normal and connected satisfying the following property. If Z_1 denotes the normalization of the product $Y_1 \times_Y Z$, then the induced morphism $\psi_1\colon Z_1 \to Y_1$ has reduced fibers over codimension one points in Y_1.*

Proof. This follows easily from [BLR95, Theorem 2.1'] (see also [AK00, Section 5]). □

As a first application of Proposition 4.1, we obtain a generalization of [Hör12, Lemma 2.14] and [CP13, Theorem 2.11].

Proposition 4.3. *Let $\varphi\colon X \to Y$ be a dominant morphism of normal projective varieties with general fiber F. Suppose that Y is smooth, and that φ has connected fibers. Let L be an effective $\mathbb{Q}$-divisor on X such that $K_X + L$ is $\mathbb{Q}$-Cartier in a neighborhood of F. Suppose furthermore that $(F, L_{|F})$ is log canonical, and that $\kappa\big(F, K_F + L_{|F}\big) \geqslant 0$. Then $K_{X/Y} - R(\varphi) + L$ is pseudo-effective, where $R(\varphi)$ denotes the ramification divisor of φ.*

Proof. Let $\nu\colon X_1 \to X$ be a log resolution of (X, L), and set $\varphi_1 := \varphi \circ \nu\colon X_1 \to Y$. Let $Y^\circ \subset Y$ be a dense open subset such that $K_{X^\circ} + L_{|X^\circ}$ is $\mathbb{Q}$-Cartier, where $X^\circ := \varphi^{-1}(Y^\circ)$. Set $X_1^\circ := \nu^{-1}(X^\circ)$. We write

$$K_{X_1^\circ} + L_1^\circ = (\nu_{|X_1^\circ})^*(K_{X^\circ} + L_{|X^\circ}) + E^\circ \tag{4.4}$$

where L_1° and E° are effective, with no common components, $(\nu_{|X_1^\circ})_* L_1^\circ = L_{|X^\circ}$, and E° is ν-exceptional. Denote by L_1 the Zariski closure of L_1° in X_1. Note that $\nu_* L_1 = L$. Let F_1 denote a general fiber of φ_1. Then $(F_1, L_{1|F_1})$ has log canonical singularities, and $\kappa\big(F_1, K_{F_1} + L_{1|F_1}\big) = \kappa\big(F, K_F + L_{|F}\big) \geqslant 0$ by (4.4). By Proposition 4.1 applied to the foliation $\mathscr{H}_1$ on X_1 induced by φ_1 and $L = L_1$, we conclude that $K_{\mathscr{H}_1} + L_1$ is pseudo-effective. There is an exact sequence

$$0 \to \mathscr{H}_1 \to T_{X_1} \to \varphi_1^* T_Y,$$

and thus $K_{\mathscr{H}_1} = K_{X_1/Y} - R(\varphi_1) - G$, where $R(\varphi_1)$ denotes the ramification divisor of φ_1 and G is an effective divisor. This in turn implies that $K_{X/Y} - R(\varphi) + L$ is pseudo-effective since $K_{X/Y} - R(\varphi) - \nu_*\big(K_{X_1/Y} - R(\varphi_1) - G\big) = \nu_* G$ is effective. □

Remark 4.4. It should be noted that Proposition 4.3 does not require $K_{X/Y} - R(\varphi) + L$ to be $\mathbb{Q}$-Cartier.

Corollary 4.5. *Let $\psi\colon Z \to Y$ be a dominant morphism of normal projective varieties with general fiber F. Suppose that ψ is equidimensional with connected fibers. Let L be an effective $\mathbb{Q}$-divisor on Z such that $K_Z + L$ is $\mathbb{Q}$-Cartier in a neighborhood of F. Suppose furthermore that $(F, L_{|F})$ is log canonical, and that $\kappa\big(F, K_F + L_{|F}\big) \geqslant 0$. Then $K_{Z/Y} - R(\psi) + L$ is pseudo-effective, where $R(\psi)$ denotes the ramification divisor of ψ.*

Proof. Let $\mu_1\colon Y_1 \to Y$ be a resolution of singularities, and let Z_1 be the normalization of the product $Y_1 \times_Y Z$, with natural morphisms $\psi_1\colon Z_1 \to Y_1$ and $\nu_1\colon Z_1 \to Z$. Apply Proposition 4.3 to ψ_1 and $\nu_1^* L$. We conclude that $K_{Z_1/Y_1} - R(\psi_1) + \nu_1^* L$ is pseudo-effective, where $R(\psi_1)$ denotes the ramification divisor of ψ_1. It follows that $(\nu_1)_*(K_{Z_1/Y_1} - R(\psi_1) + \nu_1^* L)$ is pseudo-effective as well. Since ψ is equidimensional, there exist dense open subsets $Y^\circ \subset Y$ and $Z^\circ \subset Z$ with $\operatorname{codim} Y \setminus Y^\circ \geqslant 2$

and $\operatorname{codim} Z \setminus Z^\circ \geqslant 2$ such that $\psi(Z^\circ) \subset Y^\circ$, and such that ν_1 (respectively, μ_1) induces an isomorphism $\nu_1^{-1}(Z \setminus Z^\circ) \cong Z \setminus Z^\circ$ (respectively, $\mu_1^{-1}(Y \setminus Y^\circ) \cong Y \setminus Y^\circ$). Hence, $(\nu_1)_*\big(K_{Z_1/Y_1} - R(\psi_1) + \nu_1^* L\big) = K_{Z/Y} - R(\psi) + L$, where $R(\psi)$ denotes the ramification divisor of ψ. This completes the proof of Corollary 4.5. □

Next, we obtain a generalization of [AD13, Proposition 5.8].

Proposition 4.6. *Let X be a normal projective variety, let $\mathscr{H}$ be a $\mathbb{Q}$-Gorenstein algebraically integrable foliation on X, and let (F, Δ) be its general log leaf. Suppose that $-K_{\mathscr{H}} = A + E$ where A is a $\mathbb{Q}$-ample $\mathbb{Q}$-divisor and E is an effective $\mathbb{Q}$-divisor. Then $(F, \Delta + E_{|F})$ is not klt.*

Proof. We argue by contradiction, and assume that $(F, \Delta + E_{|F})$ is klt. Let $\psi\colon Z \to Y$ be the family of leaves, and let $\nu\colon Z \to X$ be the natural morphism (see section 2.13). There is a canonically defined effective $\mathbb{Q}$-Weil divisor B on Z such that

$$K_{Z/Y} - R(\psi) + B \sim_{\mathbb{Q}} \nu^* K_{\mathscr{H}}$$

where $R(\psi)$ denotes the ramification divisor of ψ. We view F as a (general) fiber of ψ. Recall that $B_{|F} = \Delta$, and that $K_{Z/Y} - R(\psi)$ is a canonical divisor of the foliation $\mathscr{H}_Z$ on Z induced by $\mathscr{H}$ (or ψ). We write $\nu^* A = H + N$ where H is a $\mathbb{Q}$-ample $\mathbb{Q}$-divisor and N is an effective $\mathbb{Q}$-divisor. Let $0 < \varepsilon \ll 1$ be a rational number such that $(F, \Delta + E_{|F} + \varepsilon N_{|F})$ is klt, and set $H_\varepsilon = \nu^* A - \varepsilon N$. Note that H_ε is $\mathbb{Q}$-ample. Finally, let $D \neq 0$ be an effective $\mathbb{Q}$-Cartier divisor on Y, such that $H'_\varepsilon := H_\varepsilon - \psi^* D$ is $\mathbb{Q}$-ample. Then $K_F + \Delta + E_{|F} + H'_{\varepsilon|F} + \varepsilon N_{|F} \sim_{\mathbb{Q}} 0$. Now, apply Proposition 4.1 to conclude that $K_{\mathscr{H}_Z} + B + \nu^* E + H'_\varepsilon + \varepsilon N$ is pseudo-effective. But

$$K_{\mathscr{H}_Z} + B + \nu^* E + H'_\varepsilon + \varepsilon N \sim_{\mathbb{Q}} -\psi^* D,$$

yielding a contradiction. □

Remark 4.7. Note that Proposition 4.6 also follows from Proposition 2.12. We decided to include a proof to keep our exposition as self-contained as possible.

5. Singularities of foliations

In this section, we provide another application of Proposition 4.1: we study singularities of algebraically integrable foliations (see Proposition 5.5). First, we recall the definition of terminal and canonical singularities, inspired by the theory of singularities of pairs, developed in the context of the minimal model program.

5.1 ([McQ08, Definition I.1.2])**.** Let $\mathscr{F}$ be a foliation on a normal variety X. Given a birational morphism $\nu_1\colon X_1 \to X$ of normal varieties, there is a unique foliation $\mathscr{F}_1$ on X_1 that agrees with $\nu_1^*\mathscr{F}$ on the open subset of X_1 where ν_1 is an isomorphism. Let $B = \sum_{i \in I} a_i B_i$ be a (not necessarily effective) $\mathbb{Q}$-divisor on X. Suppose that B_i is not invariant by $\mathscr{F}$ for any $i \in I$.

Suppose moreover that $K_{\mathscr{F}} + B$ is $\mathbb{Q}$-Cartier and that ν_1 is projective. Then there are uniquely defined rational numbers $a_E(X, \mathscr{F}, B)$'s such that

$$K_{\mathscr{F}_1} + B_1 = \nu_1^*(K_{\mathscr{F}} + B) + \sum_E a_E(X, \mathscr{F}, B) E,$$

where E runs through all exceptional prime divisors for ν_1, and where B_1 denotes the proper transform of B in X_1. The $a_E(X, \mathscr{F}, B)$'s do not depend on the birational morphism ν_1, but only on the valuations associated to the E's.

We say that $(\mathscr{F}, B)$ is *terminal* (respectively, *canonical*) if $a_E(X, \mathscr{F}, B) > 0$ (respectively, $a_E(X, \mathscr{F}, B) \geqslant 0$) for all E exceptional over X.

We say that $\mathscr{F}$ is *terminal* (respectively, *canoncial*) if so is $(\mathscr{F}, 0)$.

Remark 5.2. A regular foliation on a smooth variety is canonical by [AD13, Lemma 3.10].

The following example shows that a regular foliation on a smooth variety may not be (log) terminal.

Example 5.3. Let F and T be smooth varieties with $\dim F \geqslant 1$, and $\dim T \geqslant 2$. Let $\mathscr{F}$ be the foliation on $X := F \times T$ induced by the projection $X \to T$. Fix a point $t \in T$, and let T_1 be the blow-up of T at t. Set $X_1 := F \times T_1$, with natural morphism $\nu_1 \colon X_1 \to X$. Note that $\nu_1 \colon X_1 \to X$ is the blow-up of X along $F \times \{t\} \subset F \times T$. Denote by E its exceptional set. Observe that E is invariant by $\mathscr{F}_1$. The foliation $\mathscr{F}_1$ induced by $\mathscr{F}$ on X_1 is given by the natural morphism $X_1 \to T_1$. We have $\mathscr{F}_1 = \nu_1^* \mathscr{F}$, and hence $a_E(X, \mathscr{F}, 0) = 0$.

The following result is probably well known to experts, though as far as we can see, the statement does not appear in the literature.

Lemma 5.4. *Let $\psi \colon Z \to Y$ be a dominant equidimensional morphism of normal varieties, and denote by $\mathscr{H}$ the foliation on Z induced by ψ. Suppose that Y is smooth. Let $B = \sum_{i \in I} a_i B_i$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Z. Suppose that B_i is not invariant by $\mathscr{H}$ for every $i \in I$. Suppose furthermore that $(\mathscr{H}, B)$ is terminal (respectively, canonical). Then $\big(Z, -R(\psi) + B\big)$ has terminal singularities (respectively, canonical singularities).*

Proof. By section 2.9, we have $K_{\mathscr{H}} = K_{Z/Y} - R(\psi)$, and thus $K_Z - R(\psi)$ is $\mathbb{Q}$-Cartier. Let $\nu_1 \colon Z_1 \to Z$ be a resolution of singularities, and set $\psi_1 := \psi \circ \nu_1$. Denote by $\mathscr{H}_1$ the foliation induced by $\mathscr{H}$ (or ψ_1) on Z_1. Write $K_{\mathscr{H}_1} + B_1 = \nu^*(K_{\mathscr{H}} + B) + E$ where E is a ν-exceptional $\mathbb{Q}$-divisor, and B_1 denotes the proper transform of B in Z_1. There is an exact sequence

$$0 \to \mathscr{H}_1 \to T_{Z_1} \to \psi_1^* T_Y,$$

and thus $K_{\mathscr{H}_1} = K_{Z_1/Y} - R(\psi)_1 - G$, where $R(\psi)_1$ denotes the proper transform of $R(\psi)$ in Z_1 and G is an effective ν-exceptional $\mathbb{Q}$-divisor. We obtain

$$K_{Z_1} - R(\psi)_1 + B_1 = \nu_1^*\big(K_Z - R(\psi) + B\big) + E + G.$$

The lemma follows easily. □

Proposition 5.5. *Let X be a normal projective variety, let $\mathscr{H}$ be a $\mathbb{Q}$-Gorenstein algebraically integrable foliation on X, and let (F, Δ) be its general log leaf. Suppose that $\Delta = 0$. Suppose moreover that $-K_{\mathscr{H}} \sim_{\mathbb{Q}} P + N$ where P is a nef $\mathbb{Q}$-divisor and $N = \sum_{i \in I} N_i$ is an effective $\mathbb{Q}$-divisor such that $(F, N_{|F})$ is canonical. Suppose furthermore that N_i is not invariant by $\mathscr{H}$ for any $i \in I$. Then $(\mathscr{H}, N)$ is canonical.*

Proof. Let $\psi \colon Z \to Y$ be the family of leaves, and let $\nu \colon Z \to X$ be the natural morphism (see section 2.13). Let $\nu_1 \colon Z_1 \to Z$ be a birational morphism with Z_1 smooth and projective. We obtain a commutative diagram

$$\begin{tikzcd} Z_1 \arrow[r, "\nu_1,\ \text{birational}"] \arrow[d, "\psi_1"] & Z \arrow[r, "\nu,\ \text{birational}"] \arrow[d, "\psi"] & X \\ Y \arrow[r, equal] & Y. & \end{tikzcd}$$

Denote by $\mathscr{H}_{Z_1}$ the foliation on Z_1 induced by $\mathscr{H}$. Let E_1 be the $\nu\circ\nu_1$-exceptional $\mathbb{Q}$-divisor such that

$$K_{\mathscr{H}_{Z_1}} + N_1 = (\nu\circ\nu_1)^*(K_{\mathscr{H}} + N) + E_1, \tag{5.1}$$

where N_1 denotes the proper transform of N in Z_1. To prove the lemma, it is enough to show that E_1 is effective. Denote by F_1 the proper transform of F in Z_1. Note that we have $K_{\mathscr{H}|F} \sim_{\mathbb{Q}} K_F$ since $\Delta = 0$. Since $(F, N_{|F})$ is canonical, the restriction $E_{1|F}$ of E_1 to F is effective, and $(F_1, N_{1|F_1})$ is canonical as well (see [Kol97, Lemma 3.10]). Let H be an ample (effective) divisor on Z_1, let $\varepsilon > 0$ be a rational number, and let $A_\varepsilon \sim_{\mathbb{Q}} (\nu\circ\nu_1)^*P + \varepsilon H$ such that $(F_1, N_{1|F_1} + A_{\epsilon|F_1})$ is canonical. We have

$$K_{F_1} + N_{1|F_1} = (\nu_{1|F_1})^*\big(K_F + N_{|F}\big) + E_{1|F_1} \sim_{\mathbb{Q}} -(\nu_{1|F_1})^*(P_{|F}) + E_{1|F_1}$$

by (5.1) using section 2.9 and the fact that $\Delta = 0$, and hence

$$\kappa\big(F_1, K_{F_1} + N_{1|F_1} + A_{\varepsilon|F_1}\big) = \kappa\big(F_1, E_{1|F_1} + \varepsilon H_{|F_1}\big) = \dim F_1 \geqslant 0.$$

By Proposition 4.1 applied to $\mathscr{H}_{Z_1}$ and $L = A_\varepsilon + N_1 \sim_{\mathbb{Q}} (\nu\circ\nu_1)^*P + \varepsilon H + N_1$, we conclude that

$$K_{\mathscr{H}_{Z_1}} + A_\varepsilon + N_1 \sim_{\mathbb{Q}} K_{\mathscr{H}_{Z_1}} + (\nu\circ\nu_1)^*P + \varepsilon H + N_1$$

is pseudo-effective for any $0 < \varepsilon \ll 1$, and hence $K_{\mathscr{H}_{Z_1}} + (\nu\circ\nu_1)^*P + N_1$ is pseudo-effective as well. Also, by (5.1), we have

$$K_{\mathscr{H}_{Z_1}} + (\nu\circ\nu_1)^*P + N_1 = E_1.$$

By a result of Lazarsfeld ([KL09, Corollary 13]), the $\nu\circ\nu_1$-exceptional $\mathbb{Q}$-divisor E_1 is pseudo-effective if and only if it is effective, completing the proof of the proposition. □

The proof of the next proposition is similar to that of Proposition 5.5. One only needs to replace the use of Proposition 4.1 with Proposition 4.3.

Proposition 5.6. *Let $\varphi\colon X \to Y$ be a dominant morphism of normal projective varieties with general fiber F. Suppose that Y is smooth, and that φ has connected fibers. Suppose moreover that $-K_{X/Y} \sim_{\mathbb{Q}} P + N$ where P is a nef $\mathbb{Q}$-divisor and N is an effective $\mathbb{Q}$-divisor such that $(F, N_{|F})$ is canonical. Then (X, N) has canonical singularities.*

6. Foliations with nef anti-canonical bundle

In this section, we provide another technical tool for the proof of the main results. The following result is the main observation of this paper. Note that Proposition 1.8 is an immediate consequence of Proposition 6.1 and Lemma 6.2 below.

Proposition 6.1. *Let X be a complex projective manifold, and let $\mathscr{F} \subset T_X$ be a foliation. Suppose that $-K_{\mathscr{F}} \equiv P + N$ where P is a nef $\mathbb{Q}$-divisor and N is an effective $\mathbb{Q}$-divisor such that (X, N) is log canonical. Suppose that the algebraic part $\mathscr{H}$ of $\mathscr{F}$ has a compact leaf. Let $\psi\colon Z \to Y$ be the family of leaves of $\mathscr{H}$, and let $\nu\colon Z \to X$ be the natural morphism. Set $\varphi := \psi\circ\nu^{-1}\colon X \dashrightarrow Y$, let $\mathscr{G}$ be the*

foliation on Y such that $\mathscr{F} = \varphi^{-1}\mathscr{G}$, and let $\mathscr{H}_Z$ be the foliation on Z induced by $\mathscr{H}$. Then

(1) $\varphi^* K_{\mathscr{G}} \equiv 0$,
(2) $K_{\mathscr{H}} \equiv K_{\mathscr{F}}$, *and*
(3) $K_{\mathscr{H}_Z} \sim \nu^* K_{\mathscr{H}}$.

Proof. We subdivide the proof into a number of relatively independent steps.

Step 1. By assumption, φ is an almost proper map: there exist dense Zariski open sets $X^\circ \subset X$ and $Y^\circ \subset Y$ such that the restriction φ° of φ to X° induces a proper morphism $\varphi^\circ \colon X^\circ \to Y^\circ$. Moreover, the following holds:

($\Diamond$) there is no positive-dimensional algebraic subvariety passing through a general point of Y that is tangent to $\mathscr{G}$.

Step 2. Let A be an ample divisor on X. Let F be a general fiber of φ. Notice that F is smooth. There exists an open set $U \supset F$ such that $\mathscr{H}_{|U}$ is a subbundle of $\mathscr{F}_{|U}$, $\varphi^*\mathscr{G}_{|U}$ is locally free, and $\left(\mathscr{F}/\mathscr{H}\right)_{|U} \cong \varphi^*\mathscr{G}_{|U}$. In particular, $K_{\mathscr{F}|F} \sim K_{\mathscr{H}|F} \sim K_F$. Let $\varepsilon > 0$ be a rational number. Then $K_F - K_{\mathscr{F}|F} + \varepsilon A_{|F} \equiv \varepsilon A_{|F}$ is $\mathbb{Q}$-ample, and hence $\kappa(F, K_F + P_{|F} + N_{|F} + \varepsilon A_{|F}) \geqslant 0$. Since $P + \varepsilon A$ is $\mathbb{Q}$-ample, there exists an effective $\mathbb{Q}$-divisor $A_\varepsilon \sim_{\mathbb{Q}} P + \varepsilon A$ such that the pair $(X, N + A_\varepsilon)$ is log canonical. From [Kol97, Theorem 4.8], we conclude that $(F, N_{|F} + A_{\varepsilon|F})$ is log canonical. By Proposition 4.1 applied to the foliation $\mathscr{H}$ on X and $L := A_\varepsilon + N$, we conclude that $K_{\mathscr{H}} + P + N + \varepsilon A \equiv K_{\mathscr{H}} - K_{\mathscr{F}} + \varepsilon A$ is pseudo-effective for any positive rational number $\varepsilon > 0$. It follows that $K_{\mathscr{H}} - K_{\mathscr{F}}$ is pseudo-effective.

Step 3. We will show that $(K_{\mathscr{H}} - K_{\mathscr{F}}) \cdot A^{\dim X - 1} \leqslant 0$. Note that $\psi \colon Z \to Y$ is equidimensional. Thus, by section 2.9, there is an effective divisor R on X such that

$$K_{\mathscr{H}} - K_{\mathscr{F}} = -(\varphi^* K_{\mathscr{G}} + R).$$

To prove that $(K_{\mathscr{H}} - K_{\mathscr{F}}) \cdot A^{\dim X - 1} \leqslant 0$, it is enough to show that $\varphi^* K_{\mathscr{G}} \cdot A^{\dim X - 1} \geqslant 0$. We argue by contradiction, and we assume that $\varphi^* K_{\mathscr{G}} \cdot A^{\dim X - 1} < 0$. By Proposition 3.12, there exists a saturated subsheaf $\mathscr{E} \subseteq \mathscr{G}$ satisfying the following property. Let C be a general complete intersection curve of type $m\nu^* A$ with m sufficiently large. Then $\psi^*\mathscr{E}_{|C}$ is the maximally destabilizing subsheaf of $\psi^*\mathscr{G}_{|C}$. By our current assumption, we have $\deg(\psi^*\mathscr{G}_{|C}) > 0$ so that we must have $\deg(\psi^*\mathscr{E}_{|C}) > 0$. This implies that $\psi^*\mathscr{E}_{|C}$ is ample. We conclude that $\mathscr{E}_{|B}$ is an ample vector bundle, where $B := \psi(C)$. By [KSCT07, Proposition 30], we have $\mathrm{Hom}_C(\psi^*\mathscr{E}_{|C} \otimes \psi^*\mathscr{E}_{|C}, \psi^*\mathscr{G}_{|C}/\psi^*\mathscr{E}_{|C}) = 0$ and thus $\mathrm{Hom}_Y(\mathscr{E} \otimes \mathscr{E}, \mathscr{G}/\mathscr{E}) = 0$. Since $\mathscr{G}$ is closed under the Lie bracket, it follows that $\mathscr{E}$ is a foliation. By Theorem 2.16, we conclude that the leaf of $\mathscr{E}$ through any point of B is algebraic. But this contradicts ($\Diamond$) above. This proves that $\varphi^* K_{\mathscr{G}} \cdot A^{\dim X - 1} \geqslant 0$, and $(K_{\mathscr{H}} - K_{\mathscr{F}}) \cdot A^{\dim X - 1} \leqslant 0$.

Step 4. Recall from Step 2 that $K_{\mathscr{H}} - K_{\mathscr{F}}$ is pseudo-effective. By Step 3, we have $(K_{\mathscr{H}} - K_{\mathscr{F}}) \cdot A^{\dim X - 1} \leqslant 0$, and hence, we must have

$$K_{\mathscr{H}} \equiv K_{\mathscr{F}}.$$

This proves (2). Note also that we must have $\varphi^* K_{\mathscr{G}} \equiv 0$, proving (1).

Step 5. Let $\mathscr{H}_Z$ be the folitation on Z induced by $\mathscr{H}$. We show that

$$K_{\mathscr{H}_Z} \sim \nu^* K_{\mathscr{H}}.$$

Note that $K_{\mathscr{H}_Z} = K_{Z/Y} - R(\psi)$ by section 2.9. Up to replacing P with $K_{\mathscr{H}} - N$ if necessary, we may assume that $K_{\mathscr{H}} = P + N$. By section 2.13, there is an effective Weil $\mathbb{Q}$-divisor B on Z such that

$$K_{\mathscr{H}_Z} + B \sim \nu^* K_{\mathscr{H}},$$

and hence

$$K_{\mathscr{H}_Z} + \nu^* P + \nu^* N \sim -B.$$

Note that $\operatorname{Supp}(B) \cap F = \emptyset$. The same argument used in Step 2 shows that $K_{\mathscr{H}_Z} + \nu^* P + \nu^* N$ is pseudo-effective. One only needs to replace the use of Proposition 4.1 with Corollary 4.5. Since $K_{\mathscr{H}_Z} + \nu^* P + \nu^* N \sim -B$, we conclude that $B = 0$. This completes the proof of the proposition. □

Lemma 6.2. *Let X be a complex projective manifold, and let $\mathscr{F} \subset T_X$ be a foliation. Suppose that either $\mathscr{F}$ is regular, or that $\mathscr{F}$ has a compact leaf. Then the algebraic part $\mathscr{H}$ of $\mathscr{F}$ has a compact leaf.*

Proof. Recall from Definition 2.3 that there exist a normal projective variety Y, a dominant rational map $\varphi\colon X \dashrightarrow Y$, and a foliation $\mathscr{G}$ on Y such that the following holds:

(a) there is no positive-dimensional algebraic subvariety passing through a general point of Y that is tangent to $\mathscr{G}$; and
(b) $\mathscr{F}$ is the pull-back of $\mathscr{G}$ via φ.

Note that $\mathscr{H}$ is the foliation on X induced by φ. After replacing Y with a birationally equivalent variety, we may assume that Y is the family of leaves of $\mathscr{H}$. Let Z be the normalization of the universal cycle over Y. It comes with morphisms $\psi\colon Z \to Y$, and $\nu\colon Z \to X$ (see section 2.13).

To prove the lemma, we have to show that $\psi\big(\operatorname{Exc}(\nu)\big) \subsetneq Y$. We argue by contradiction and assume that $\psi\big(\operatorname{Exc}(\nu)\big) = Y$.

Let F be the closure of a general fiber of φ. Denote by S the (possibly empty) singular locus of $\mathscr{F}$. Then $F \cap S = \emptyset$. Indeed, suppose that $S \neq \emptyset$, and consider a compact L leaf of $\mathscr{F}$. Pick $y \in Y$ such that $Z_y \cap L \neq \emptyset$. Then $Z_y \subset L$, and hence $Z_y \cap S = \emptyset$. Thus, if y' is sufficiently close to y, then $Z_{y'} \cap S = \emptyset$, proving our claim.

By Zariski's Main Theorem, there is an integral (complete) curve $C \subset Z$ with $\dim \nu(C) = 0$ and $\nu(C) \in F$. Set $B := \psi(C)$, and note that $\dim B = 1$. Then $\nu(\psi^{-1}(B)) \supset F$ is tangent to $\mathscr{F}$ since $\mathscr{F}$ is regular in a neighborhood of F, and $\dim \nu(\psi^{-1}(B)) = \dim F + 1$. This implies that B is tangent to $\mathscr{G}$, yielding a contradiction and proving the lemma. □

We believe that the following result will be useful when considering arbitrary foliations.

Proposition 6.3. *Let X be a complex projective manifold, and let $\mathscr{F} \subset T_X$ be a foliation. Let $\mathscr{H}$ be the algebraic part of $\mathscr{F}$. Suppose that $\mathscr{H}$ is induced by a rational map $\varphi\colon X \dashrightarrow Y$, and let $\mathscr{G}$ be the foliation on Y such that $\mathscr{F} = \varphi^{-1}\mathscr{G}$. Then $\mu_A^{\max}\big((\varphi^*\mathscr{G})^{**}\big) \leqslant 0$ for any ample divisor A on X.*

Proof. This follows easily from Steps 1 and 3 of the proof of Proposition 6.1 using the Mehta-Ramanathan theorem. □

7. Proofs

In this section we prove Theorem 1.1 and Corollary 1.4.

Theorem 7.1. *Let X be a complex projective manifold, and let $\mathscr{F} \subset T_X$ be a codimension q foliation with $0 < q < \dim X$. Suppose that the algebraic part $\mathscr{H}$ of $\mathscr{F}$ has a compact leaf. Suppose furthermore that $-K_{\mathscr{F}} \equiv A + E$ where A is a $\mathbb{Q}$-ample $\mathbb{Q}$-divisor and E is an effective $\mathbb{Q}$-divisor. Then (X, E) is not log canonical.*

Proof. Argue by contradiction, and assume that (X, E) is log canonical. There exist a dense open subset $X^\circ \subset X$, a normal variety Y°, and a proper morphism $\varphi^\circ \colon X^\circ \to Y^\circ$ such that $\mathscr{H}$ is the foliation on X induced by φ°. By Proposition 6.1, we have $K_{\mathscr{H}} \equiv K_{\mathscr{F}}$. Note that $\dim Y^\circ < \dim X^\circ$ since $K_{\mathscr{F}} \not\equiv 0$. Set $A' = -K_{\mathscr{H}} - E$. Then A' is $\mathbb{Q}$-ample, and $-K_{\mathscr{H}} = A' + E$.

Let y be a general point in Y, and denote by X_y the corresponding fiber of φ. The pair $(X_y, E_{|X_y})$ is log canonical by [Kol97, Theorem 4.8]. Also, note that X_y is smooth. Let $0 < \varepsilon \ll 1$ be a rational number such that $A' + \varepsilon E$ is $\mathbb{Q}$-ample, and such that $(X_y, (1-\varepsilon)E_{|X_y})$ is klt. Then $-K_{\mathscr{H}} = (A' + \varepsilon E) + (1-\varepsilon)E$. But this contradicts Proposition 4.6 applied to $\mathscr{H}$. □

Note that Theorem 1.1 and Corollary 1.4 are consequences of Theorem 7.2 and Corollary 7.3 respectively using Lemma 6.2.

Theorem 7.2. *Let X be a complex projective manifold, and let $\mathscr{F} \subset T_X$ be a codimension q foliation with $0 < q < \dim X$. Suppose that the algebraic part of $\mathscr{F}$ has a compact leaf. Then $-K_{\mathscr{F}}$ is not nef and big.*

Proof. Write $-K_{\mathscr{F}} = A + E$ where A is a $\mathbb{Q}$-ample $\mathbb{Q}$-divisor, and E is an effective $\mathbb{Q}$-divisor. Let $0 < \varepsilon < 1$ be a rational number such that $(X, \varepsilon E)$ is klt. Note that $A_\varepsilon := -K_{\mathscr{F}} - \varepsilon E$ is $\mathbb{Q}$-ample. This contradicts Theorem 7.1, proving Theorem 1.1. □

Corollary 7.3. *Let X be a complex projective manifold, and let $\mathscr{F} \subset T_X$ be a codimension one foliation. Suppose that the algebraic part of $\mathscr{F}$ has a compact leaf. Suppose furthermore that $-K_{\mathscr{F}}$ is nef. Then $\kappa(X, -K_{\mathscr{F}}) \leqslant \dim X - 1$.*

Proof. Denote by $\nu(-K_{\mathscr{F}})$ the numerical dimension of $-K_{\mathscr{F}}$ (see section 8.1 below for this notion). By Theorem 7.2, we must have $\nu(-K_{\mathscr{F}}) \leqslant \dim X - 1$. Thus $\kappa(X, -K_{\mathscr{F}}) \leqslant \nu(-K_{\mathscr{F}}) \leqslant \dim X - 1$ by [Kaw85, Proposition 2.2]. □

Question 7.4. Given a regular foliation $\mathscr{F}$ of rank r on a complex projective manifold with $-K_{\mathscr{F}}$ nef, do we have $\kappa(X, -K_{\mathscr{F}}) \leqslant r$?

8. Foliations with nef and abundant anti-canonical bundle

In this section, we answer Question 7.4 under an additional assumption. First, we recall the definition and basic properties of nef and abundant divisors.

8.1 (Abundant nef divisors)**.** Let $P \not\equiv 0$ be a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a normal projective variety X. Recall that the *numerical dimension* of P, denoted by $\nu(P)$, is the largest positive integer k such that $P^k \not\equiv 0$. We have $\kappa(X, P) \leqslant \nu(P)$ by [Kaw85, Proposition 2.2], and we say that P is *abundant* if equality holds.

By [Kaw85, Proposition 2.1], P is abundant if and only if there is a diagram of normal projective varieties

$$\begin{tikzcd} & Z \arrow[dl, "\nu"'] \arrow[dr, "f"] & \\ X & & T \end{tikzcd}$$

and a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divsor D on T such that $\nu^*P \sim_{\mathbb{Q}} f^*D$, where ν is a birational morphism, and f is surjective.

We will need the following easy observation.

Lemma 8.2. *Let P be a nef and abundant $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a normal projective variety X. With the above notation, write $D = A + E$ where A is a $\mathbb{Q}$-ample $\mathbb{Q}$-divisor, and E is an effective $\mathbb{Q}$-divisor. Let $F \subseteq X$ be a subvariety not contained in $\nu\Big(\mathrm{Exc}(\nu) \cup f^{-1}\big(\mathrm{Supp}(E)\big)\Big)$. Then $P_{|F}$ is nef and abundant, and $\nu(P_{|F}) = \dim f(\tilde{F})$, where $\tilde{F}$ denotes the proper transform of F in Z.*

Proof. Note that $f(\tilde{F}) \not\subset E$, and thus $D_{|f(\tilde{F})}$ is nef and big. The lemma follows easily. □

The following observation will prove to be crucial.

Lemma 8.3. *Let X and Y be normal projective varieties, and let $\varphi\colon X \dashrightarrow Y$ be an almost proper map with general fiber F. Let $\mathscr{H}$ be the induced foliation on X. Suppose that $\mathscr{H}$ is $\mathbb{Q}$-Gorenstein. Suppose furthermore that $-K_{\mathscr{H}} \equiv P$ where P is nef and abundant, and that $-K_{\mathscr{H}|F} = -K_F$ is nef and abundant. Then $\nu(-K_{\mathscr{H}}) = \nu(-K_F)$.*

Proof. Notice that $\nu(-K_{\mathscr{H}}) \geqslant \nu(-K_F)$ since $-K_{\mathscr{H}|F} = -K_F$ by assumption.

We have to show that $\nu(-K_{\mathscr{H}}) \leqslant \nu(-K_F)$.

Let $\psi\colon Z \to Y$ be the family of leaves of $\mathscr{H}$, and let $\nu\colon Z \to X$ be the natural morphism (see section 2.13). By Lemma 4.2, there exists a finite surjective morphism $\mu_1\colon Y_1 \to Y$ with Y_1 normal and connected satisfying the following property. If Z_1 denotes the normalization of the product $Y_1 \times_Y Z$, then the induced morphism $\psi_1\colon Z_1 \to Y_1$ has reduced fibers over codimension one points in Y_1. Denote by $\nu_1\colon Z_1 \to Z$ the natural morphism.

By Proposition 6.1 and section 2.9, we have

$$K_{Z/Y} - R(\psi) \sim \nu^* K_{\mathscr{H}}.$$

A straightforward computation shows that

$$\nu_1^*\big(K_{Z/Y} - R(\psi)\big) = K_{Z_1/Y_1}.$$

We conclude that $-K_{Z_1/Y_1}$ is $\mathbb{Q}$-Cartier, and that $-K_{Z_1/Y_1} \equiv P_1$, where $P_1 := (\nu_1 \circ \nu)^*P$.

By [Kaw85, Proposition 2.1] applied to P_1, there exist a surjective morphism $f_3\colon Z_3 \to T_3$ of normal projective varieties, a birational morphism $\nu_3\colon Z_3 \to Z_1$, and a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divsor D_3 on T_3 such that $\nu_3^*P_1 \sim_{\mathbb{Q}} f_3^*D_3$. We

obtain a commutative diagram

$$\begin{tikzcd} Z_3 \arrow[r, "\nu_3,\ \text{birational}"] \arrow[d, "f_3"] & Z_1 \arrow[r, "\nu_1,\ \text{finite}"] \arrow[d, "\psi_1"] & Z \arrow[r, "\nu,\ \text{birational}"] \arrow[d, "\psi"] & X \\ T_3 & Y_1 \arrow[r, "\mu_1,\ \text{finite}"'] & Y. & \end{tikzcd}$$

We view F as a fiber of ψ. Let F_1 be a fiber of ψ_1 such that $\nu_1(F_1) = F$. Note that the restriction of ν_1 to F_1 induces an isomorphism $F_1 \cong F$. Let $\tilde{F}_1$ be the proper transform of F_1 in Z_3. We also have that $P_{1|F_1} = P_{|F} \equiv -K_F$. By Lemma 8.2 above, we obtain that $\dim f_3(\tilde{F}_1) = \nu(P_{1|F_1}) = \nu(-K_F)$.

We argue by contradiction, and assume that $\dim T_3 = \nu(-K_{\mathscr{H}}) > \nu(-K_F)$. Let $Y_2 \subset Y_1$ be a general complete intersection curve, and set $Z_2 := \psi_1^{-1}(Y_2) \subset Z_1$. It comes with a morphism $\psi_2 \colon Z_2 \to Y_2$. Let $\tilde{Z}_2$ be the proper transform of Z_2 in Z_3. Then $\dim f_3(\tilde{Z}_2) = \dim f_3(\tilde{F}) + 1 = \nu(-K_F) + 1$, and thus $\nu\big(-K_{Z_1/Y_1|Z_2}\big) = \nu(P_{1|Z_2}) = \nu(-K_F) + 1$ by Lemma 8.2 again. By the adjunction formula, we have $K_{Z_1/Y_1|Z_2} \sim K_{Z_2/Y_2}$. This gives

$$\nu\big(-K_{Z_2/Y_2}\big) = \nu(-K_F) + 1.$$

By Proposition 5.6, Z_2 has canonical singularities, and by [Fuj11, Theorem 1.1], $-K_{Z_2/Y_2}$ is ψ_2-semiample. Therefore, there exist a Y_2-morphism $f_2 \colon Z_2 \to T_2$ with connected fibers onto a normal projective variety, and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor M on T_2 ample over Y_2 such that

$$-K_{Z_2/Y_2} \sim_{\mathbb{Q}} f_2^* M.$$

By Ambro's canonical bundle formula [Amb05, Theorem 4.1], there exists an effective $\mathbb{Q}$-divisor B_2 on T_2 such that (T_2, B_2) is klt and

$$K_{Z_2} \sim_{\mathbb{Q}} f_2^*(K_{T_2} + B_2).$$

Thus

$$K_{Z_2/Y_2} \sim_{\mathbb{Q}} f_2^*(K_{T_2/Y_2} + B_2),$$

and hence $-(K_{T_2/Y_2} + B_2)$ is nef with

$$\nu\big(-(K_{T_2/Y_2} + B_2)\big) = \nu(-K_{Z_2/Y_2}) = \nu(-K_F) + 1 = \dim T_2.$$

But this contradicts [AD13, Theorem 5.1] (see also Proposition 4.6), completing the proof of the lemma. □

Corollary 8.4. *Let X and Y be normal projective varieties, and let $\varphi \colon X \dashrightarrow Y$ be an almost proper map with general fiber F. Let $\mathscr{F}$ be the induced foliation on X. Suppose that $\mathscr{F}$ is $\mathbb{Q}$-Gorenstein. Suppose furthermore that $-K_{\mathscr{F}}$ is nef and abundant with $\nu(-K_{\mathscr{F}}) = \dim X - dimY$. Then $-K_F$ is nef and big, and $-K_{\mathscr{F}}$ is semiample.*

Proof. Note first that $-K_F$ is nef and abundant by Lemma 8.2. Then, apply Lemma 8.3 to conclude that $-K_F$ is nef and big. In particular, $-K_F$ is semiample.

By [Kaw85, Proposition 2.1] applied to $-K_{\mathscr{F}}$, there exist a surjective morphism $f \colon Z \to T$ of normal projective varieties, a birational morphism $\nu \colon Z \to X$, and a nef and big $\mathbb{Q}$-Cartier divisor D on T such that

$$-\nu^* K_{\mathscr{F}} \sim_{\mathbb{Q}} f^* D. \tag{8.1}$$

Let $\tilde{F}$ denote the proper transform of F in Z. The restriction $f_{|\tilde{F}}\colon \tilde{F} \to T$ of f to $\tilde{F}$ is generically finite by Lemma 8.2, and (8.1) reads

$$-\nu^*_{|\tilde{F}} K_F \sim_{\mathbb{Q}} f^*_{|\tilde{F}} D.$$

Since $-K_F$ is semiample, we conclude that D is semiample as well, and hence so is $-K_{\mathscr{F}}$. □

Corollary 8.5. *Let X be a complex projective manifold, and let $\varphi\colon X \to Y$ be a morphism onto a smooth complete curve with connected general fiber F. Denote by $\mathscr{F}$ the induced foliation on X. Suppose that $-K_{\mathscr{F}}$ is nef and $\kappa(X, -K_{\mathscr{F}}) = \dim X - 1$. Then $-K_F$ is nef and big, and $-K_{\mathscr{F}}$ is semiample.*

Proof. Note that $-K_{\mathscr{F}}$ is nef and abundant by Theorem 1.1. The claim then follows from Corollary 8.4. □

The proof of the next lemma is similar to that of Lemma 8.3.

Lemma 8.6. *Let $\varphi\colon X \to Y$ be a morphism of normal projective varieties with general fiber F. Suppose that there exists a $\mathbb{Q}$-Cartier divisor K on X such that $K_{|\varphi^{-1}(Y_{\mathrm{ns}})} \sim K_{\varphi^{-1}(Y_{\mathrm{ns}})/Y_{\mathrm{ns}}}$. Suppose furthermore that K is nef and abundant. Then $\nu(-K) = \nu(-K_F)$.*

The same argument used in the proof of Corollary 8.5 shows that the following holds. One only needs to replace the use of Theorem 1.1 with [AD13, Theorem 5.1], and the use of Lemma 8.3 with Lemma 8.6.

Corollary 8.7. *Let X be a complex projective manifold, and let $\varphi\colon X \to Y$ be a morphism onto a smooth complete curve with connected general fiber F. Suppose that $-K_{X/Y}$ is nef and $\kappa(X, -K_{X/Y}) = \dim X - 1$. Then $-K_F$ is nef and big, and $-K_{X/Y}$ is semiample.*

Note that Theorem 1.5 is an immediate consequence of Theorem 8.8 and Lemma 6.2.

Theorem 8.8. *Let X be a complex projective manifold, and let $\mathscr{F} \subset T_X$ be a codimension q foliation with $0 < q < \dim X$. Suppose that the algebraic part $\mathscr{H}$ of $\mathscr{F}$ has a compact leaf. Suppose furthermore that $-K_{\mathscr{F}}$ is nef and abundant. Then $\kappa(X, -K_{\mathscr{F}}) \leqslant \dim X - q$, and equality holds only if $\mathscr{F}$ is algebraically integrable.*

Proof. By assumption, $\mathscr{H}$ is induced by an almost proper map $\varphi\colon X \dashrightarrow Y$. By Proposition 6.1, we have $K_{\mathscr{H}} \equiv K_{\mathscr{F}}$. Let $\mathscr{G}$ be the foliation on Y such that $\mathscr{F} = \varphi^{-1}\mathscr{G}$. There exists an open set $U \supset F$ contained in X such that $\mathscr{H}_{|U}$ is a subbundle of $\mathscr{F}_{|U}$, $\mathscr{G}_{|U}$ is locally free, and $(\mathscr{F}/\mathscr{H})_{|U} \cong \varphi^*\mathscr{G}_{|U}$. In particular, we have $K_{\mathscr{F}|F} \sim K_{\mathscr{H}|F} \sim K_F$, and hence $-K_F$ is nef and abundant by Lemma 8.2. Then

$$\begin{aligned}
\kappa(X, -K_{\mathscr{F}}) &= \nu(-K_{\mathscr{F}}) && \text{since } -K_{\mathscr{F}} \text{ is nef and abundant}\\
&= \nu(-K_{\mathscr{H}}) && \text{since } K_{\mathscr{H}} \equiv K_{\mathscr{F}}\\
&\leqslant \nu(-K_F) && \text{by Lemma 8.3}\\
&\leqslant \dim F = \operatorname{rank} \mathscr{H} &&\\
&\leqslant \operatorname{rank} \mathscr{F} = \dim X - q && \text{since } \mathscr{H} \subset \mathscr{F}.
\end{aligned}$$

Suppose that $\kappa(X, -K_{\mathscr{F}}) = \dim X - q$. Then we must have $\operatorname{rank} \mathscr{H} = \operatorname{rank} \mathscr{F}$. Thus $\mathscr{H} = \mathscr{F}$, and hence $\mathscr{F}$ is algebraically integrable. This completes the proof of Theorem 8.8. □

Corollary 8.9. *Let X be a complex projective manifold, and let $\mathscr{F} \subset T_X$ be a codimension one foliation. Suppose that either $\mathscr{F}$ is regular, or that $\mathscr{F}$ has a compact leaf. Suppose furthermore that $-K_{\mathscr{F}}$ is nef and $\kappa(X, -K_{\mathscr{F}}) = \dim X - 1$. Then $\mathscr{F}$ is algebraically integrable.*

Proof. Note that $-K_{\mathscr{F}}$ is abundant by Corollary 1.4. The claim then follows from Theorem 1.5. □

The remainder of the present section is devoted to the proof of Theorem 1.7.

Lemma 8.10. *Let X be a complex projective manifold, and let $\mathscr{F} \subset T_X$ be a regular codimension one foliation. Suppose that $-K_{\mathscr{F}}$ is nef and $\kappa(X, -K_{\mathscr{F}}) = \dim X - 1$. Then $\mathscr{F}$ is induced by a smooth morphism $X \to Y$ onto a smooth complete curve Y of genus $g(Y) = h^1(X, \mathscr{O}_X)$.*

Proof. By Corollary 8.9, $\mathscr{F}$ is algebraically integrable. So let $\varphi\colon X \to Y$ be a first integral, with general fiber F. By Corollary 8.5, $-K_F$ is nef and big. Let $F_1 = mF_0$ be any fiber of φ, where m is a positive integer. By the adjunction formula, we have $K_{F_0} = (K_X + F_0)_{|F_0}$, and hence

$$\begin{aligned} K_{F_0}^{\dim X - 1} &= (K_X + F_0)^{\dim X - 1} \cdot F_0 = \big(K_{X/Y} - (m-1)F_0\big)^{\dim X - 1} \cdot F_0 \\ &= K_{\mathscr{F}}^{\dim X - 1} \cdot F_0 = \frac{1}{m} K_{\mathscr{F}}^{\dim X - 1} \cdot F_1 = \frac{1}{m} K_{\mathscr{F}}^{\dim X - 1} \cdot F = \frac{1}{m} K_F^{\dim X - 1} > 0. \end{aligned}$$

Moreover, $K_{F_0} = (K_X + F_0)_{|F_0} \sim \big(K_X - (m-1)F_0\big)_{|F_0} \sim K_{\mathscr{F}|F_0}$ is nef by assumption. In particular, $\pi_1(F_0) = \{1\}$ (see [Zha06]). By the holomorphic version of the Reeb stability theorem, we conclude that φ is a smooth morphism.

By the Kawamata-Viehweg vanishing theorem, we have $R^1\varphi_*\mathscr{O}_X = 0$. The Leray spectral sequence then yields $h^1(X, \mathscr{O}_X) = h^1(Y, \mathscr{O}_Y)$. This completes the proof of the lemma. □

Proof of Theorem 1.7. Let X be a complex projective manifold with $h^1(X, \mathscr{O}_X) = 0$, and let $\mathscr{F} \subset T_X$ be a codimension one regular foliation. Suppose that $-K_{\mathscr{F}}$ is nef and $\kappa(X, -K_{\mathscr{F}}) = \dim X - 1$.

By Lemma 8.10, $\mathscr{F}$ is induced by a smooth morphism $\varphi\colon X \to \mathbb{P}^1$. In particular, $K_{\mathscr{F}} = K_{X/\mathbb{P}^1}$. Now, observe that $-K_X = -K_{X/\mathbb{P}^1} - \varphi^* K_{\mathbb{P}^1}$ is nef, and that

$$(-K_X)^{\dim X} = (-K_{X/\mathbb{P}^1} - \varphi^* K_{\mathbb{P}^1})^{\dim X} = 2 \dim X (-K_F)^{\dim F} > 0.$$

In other words, X is a weak Fano manifold. Let F be a general fiber of φ. Let $R = \mathbb{R}_{\geqslant 0}[\ell] \subset \mathrm{NE}(X)$ be an extremal ray with $F \cdot \ell > 0$, and let $\psi_R\colon X \to Z$ be the corresponding contraction. Note that any fiber of ψ_R has dimension $\leqslant 1$. Thus, Z is smooth and either $\psi_R\colon X \to Z$ is the blow-up of a codimension two subvariety, or $\psi_R\colon X \to Z$ is a conic bundle by [Wiś91, Theorem 1.2].

Suppose first that $\psi_R\colon X \to Z$ is the blow-up of a codimension two subvariety. Then, up to replacing ℓ with a numerically equivalent curve on X, we may assume that $-K_X \cdot \ell = 1$. On the other hand,

$$-K_X \cdot \ell = (-K_{X/\mathbb{P}^1} - \varphi^* K_{\mathbb{P}^1}) \cdot \ell \geqslant 2F \cdot \ell \geqslant 2,$$

yielding a contradiction. This proves that $\psi_R\colon X \to Z$ is a conic bundle. Arguing as above, we conclude that $\psi_R\colon X \to Z$ is a smooth conic bundle, and that $F \cdot \ell = 1$. This implies that the morphism $\varphi \times \psi_R\colon X \to \mathbb{P}^1 \times Z$ is birational. On the other hand, it is finite, and hence $X \cong \mathbb{P}^1 \times F$, proving the theorem. □

Question 8.11. Given a smooth morphism $X \to Y$ onto a smooth complete curve Y with $g(Y) \geqslant 1$ such that $-K_{X/Y}$ is nef and $\kappa(X, -K_{X/Y}) = \dim X - 1$, do we have $X \cong Y \times F$?

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