ISOCATEGORICAL GROUPS
AND THEIR WEIL REPRESENTATIONS

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Abstract. Two groups are called isocategorical over a field $k$ if their respective categories of $k$-linear representations are monoidally equivalent. We classify isocategorical groups over arbitrary fields, extending the earlier classification of Etingof-Gelaki and Davydov for algebraically closed fields. In order to construct concrete examples of isocategorical groups a new variant of the Weil representation associated to isocategorical groups is defined. We construct examples of non-isomorphic isocategorical groups over any field of characteristic different from two and rational Weil representations associated to symplectic spaces over finite fields of characteristic two.

1. Introduction

1. Two groups are called isocategorical over a field $k$ if their respective categories of $k$-linear representations are monoidally equivalent. The goals of this paper are to classify isocategorical groups over arbitrary fields, extending a previous classification of Etingof-Gelaki [EG01] and Davydov [Dav01] for algebraically closed fields and to introduce a new variant of the Weil representation associated to simple Galois algebras and isocategorical groups. These Weil representations include as particular cases the unitary Weil representations associated to symplectic and quadratic spaces over finite fields of characteristic two.

2. Let $G$ be an algebraic group over a field $k$. The category $\text{Rep}_k(G)$ of finite dimensional $k$-linear representations is a symmetric tensor category over $k$. Symmetric tensor categories of this kind can be characterized using Tannaka duality theory as those admitting a symmetric fiber functor [SR72]. Moreover, if $k$ is algebraically closed of characteristic 0, the symmetric tensor category $\text{Rep}_k(G)$ determines the group $G$ up to isomorphism; see [DM82 Theorem 3.2]. Despite this, there exist examples of non-isomorphic isocategorical groups; for example, the affine symplectic group $\text{ASp}(V) := V \rtimes \text{Sp}(V)$ and the pseudo-symplectic group $\text{APs}(V)$ ($V$ a symplectic space over the field of two elements) are non-isomorphic isocategorical groups over $\mathbb{C}$ [Dav01, EG01] (however, they are not isocategorical over $\mathbb{R}$; see Proposition 5.4).

3. The Weil representation is a unitary projective representation of the symplectic group over a local field [Wei64]. When the local field is non-archimedean with residual field of characteristic two the Weil representation is a real projective representation of $\text{Ps}(V)$, the pseudo-symplectic group; see [GH12]. Isocategorical groups over arbitrary fields are closely related to Weil representations; for example,
the affine orthogonal group and the pseudo-symplectic group of a quadratic space over a finite field of characteristic two are non-isomorphic isocategorical over \( \mathbb{Q} \); see Proposition 5.5.

As was pointed out in [GH12] and [EG01] the unitary Weil representation of \( \text{APs}(V) \) is an extension of the Weil representation of \( \text{Ps}(V) \). However, the Weil representation of \( \text{APs}(V) \) is not real. Is natural to ask if there exists a real Weil representation associated to a symplectic space over a finite field of characteristic two. We give a positive answer to this question and provide some examples of isocategorical groups over \( \mathbb{Q} \) of a nature slightly different from the affine orthogonal and pseudo-symplectic group.

4. We would like to finish the introduction by pointing out an interesting relation between isocategorical groups and stringy orbifold theory. The Drinfel’d double \( D(k[G]) \) of a group algebra plays an important roll in stringy orbifold theory, for example (see [KP] and [MS]):

- The category of \( G \)-Frobenius algebras arising in global orbifold cohomology or \( K \)-theory is the category of Frobenius algebras in the modular category of finite dimensional \( D(k[G]) \)-modules.

- The Grothendieck ring of the modular category of finite dimensional \( D(k[G]) \)-modules can be realized geometrically as \( K_{\text{orb}}^k([*/G]) \), the stringy \( K \)-theory of the orbifold \([*/G]\).

The category of representation of the Drinfel’d double has a conceptual interpretation as the Drinfel’d center of the category \( \text{Rep}_k(G) \) of \( k \)-linear representations of \( G \). Thus, the Drinfel’d doubles of isocategorical groups are braided equivalents. Then as an application of our main results we explicitly construct some family of example of pairs of non-isomorphic groups with the same category of \( G \)-Frobenius algebras and the same stringy \( K \)-theories for every field \( k \) of characteristic different than two.

5. The paper is organized as follows: In Section 2 we discuss the classification of Galois algebras of finite groups and their relation with isocategorical groups. In Section 3 we give a classification of isocategorical groups over arbitrary fields and we study in detail isocategorical groups over formally real fields. The main results of this section are Theorem 3.9 and Corollary 3.15. In Section 4 we introduce the Weil representation associated to a simple Galois algebra over a finite abelian group. We develop in detail the case of Weil representations associated to arbitrary symplectic and quadratic modules. In Section 5 we present some concrete examples of non-isomorphic isocategorical groups over \( \mathbb{Q} \) and an application to the Weil representation associated to a quadratic space over a finite field of characteristic two.

2. Preliminaries

Our main reference for the general theory of Hopf algebra is [Mon93]. We make free use of Sweedler’s notation for comultiplications and comodule structures, omitting the summation symbols: \( \Delta(c) = c_1 \otimes c_2 \), \( \rho(v) = v_0 \otimes v_1 \) for right comodules and \( \lambda(v) = v_{-1} \otimes v_0 \) for left comodules.

2.1. Galois objects of Hopf algebras and Galois algebras of finite groups.

In this section we review some definitions and results on Hopf Galois extensions
that we will need later. We refer the reader to [Sch04] and [Mon09] for a detailed exposition on the subject.

**Definition 2.1.** Let $k$ be a commutative base ring and $H$ a Hopf algebra over $k$. A right $H$-Galois object is a right $H$-comodule algebra $(A, ρ)$ such that $A$ is a faithfully flat $k$-module, $A^{	ext{co}H} = k$ and the canonical map

$$\text{can} : A \otimes A \to A \otimes H, \ x \otimes y \mapsto xy(0) \otimes y(1)$$

is bijective. Left $H$-Galois objects are defined similarly.

A morphism of $H$-Galois objects is an $H$-colinear algebra map. It is known that a morphism of $H$-Galois objects is an isomorphism.

**Definition 2.2.** An $(L, H)$-bi-Galois object is an $(L, H)$-bicomodule algebra $A$ which is simultaneously a left $L$-Galois object and a right $H$-Galois object.

For any right $H$-Galois object $A$ there is an associated Hopf algebra $L = L(A, H)$, called the left Galois Hopf algebra, such that $A$ is in a natural way an $(L, H)$-bi-Galois object.

**Remark 2.3.** An $H$-Galois object $A$ is called cleft if there is an $H$-colinear convolution invertible map $H \to A$. In the case that $A$ is cleft, the Hopf algebra $L(A, H)$ is obtained by a cocycle deformation [Sch96 Theorem 3.9]; in particular for finite dimensional Hopf algebras over a field every Galois object is cleft [KC76 Proposition 2], so $\dim_k(L(A, H)) = \dim_k(H)$.

If $G$ is a finite group, we will denote by $\mathcal{O}_k(G)$ the Hopf algebra of regular function of the constant group scheme over $k$; that is, $\mathcal{O}_k(G)$ is the free $k$-module with a basis $\{δ_g\}_{g ∈ G}$, multiplication $m(δ_g, δ_h) = δ_{gh}$, and comultiplication $\Delta(δ_g) = \sum_{x,y ∈ G} δ_x \otimes δ_y$, for all $g, h ∈ G$.

The category of $\mathcal{O}_k(G)$-comodule algebras is the same as the category of $k$-algebras endowed with an action of $G$ by $k$-algebra automorphisms. If $A$ is a $G$-algebra, then $A^\mathcal{O}_k(G) = A^G$, the subalgebra of $G$-invariants.

For simplicity, an $\mathcal{O}_k(G)$-Galois object will be called just a $G$-Galois algebra over $k$ or just a $G$-Galois algebra if the base ring is clear. Analogously a $(G_1, G_2)$-bi-Galois algebra is just a $(\mathcal{O}_k(G_2) - \mathcal{O}_k(G_1))$-bi-Galois object.

### 2.2. Isocategorical groups and bi-Galois algebras

We will use freely the basic language of monoidal categories theory; for more reference see [BK01] and [ML98].

If $H$ is a Hopf algebra we denote by $\mathcal{M}^H$ the $k$-linear monoidal category of all right $H$-comodules.

Given a finite group $G$, we will denote by $\text{Rep}_k(G)$ the monoidal category of all (left) $k[G]$-modules. Note that $\text{Rep}_k(G) = \mathcal{M}^{\mathcal{O}_k(G)}$ is the tensor category of $k$-linear representations of $G$.

**Definition 2.4 ([EG01]).** Let $G_1$ and $G_2$ be two finite groups. We say that $G_1$ and $G_2$ are isocategorical over a commutative ring $k$ if the monoidal categories $\text{Rep}_k(G_1)$ and $\text{Rep}_k(G_2)$ are equivalent as monoidal $k$-linear categories.

If $A$ is a $G$-algebra over $k$, we will denote the group of $G$-equivariant algebra automorphisms by

$$\text{Aut}_G(A) = \{ f ∈ \text{Aut}_{\text{Alg}}(A) | f(g \cdot a) = g \cdot f(a), \text{ for all } g ∈ G \}.$$
Lemma 2.5. If \( k \) is a field and \( A \) is a \( G \)-Galois algebra, then \( A \) is a \((G - \text{Aut}_G(A))\)-bi-Galois algebra if and only if \( |\text{Aut}_G(A)| = |G| \).

Proof. If \( A \) is a \( G \)-Galois algebra, there is a unique Hopf algebra (up to isomorphisms) such that \( A \) is an \( L-\mathcal{O}_k(G) \)-bi-Galois object, \( \text{[Sch96] Theorem 3.5} \). Remark 2.3 implies that \( |G| = \dim_k(L) \).

By \( \text{[Sch04 Corollary 3.1.4]} \), \( \text{Alg}(L,k) \simeq \text{Aut}_G(A) \), so \( L \cong \mathcal{O}_k(\text{Aut}_G(A)) \) if and only if \( |G| = |\text{Aut}_G(A)| \). \( \square \)

Proposition 2.6. Let \( G_1 \) and \( G_2 \) be two finite groups and let \( k \) be a commutative ring. The following are equivalents:

(a) \( G_1 \) and \( G_2 \) are isocategoric.

(b) There is a \((G_1, G_2)\)-bi-Galois algebra \( A \) over \( k \).

(c) If \( k \) is a field, (a) and (b) are equivalent to: there is a \( G_1 \)-Galois algebra \( A \) such that \( \text{Aut}_{G_1}(A) \cong G_2 \) and \( |G_1| = |G_2| \).

Proof. Equivalence between (a) and (b) is \( \text{[Sch96 Corolario 5.7]} \), and the equivalence between (b) and (c) follows by Lemma 2.3. \( \square \)

2.3. Group cohomology. In order to fix notation, we will recall the usual description of group cohomology associated to the normalized Bar resolution of \( \mathbb{Z} \); see \( \text{[EML53]} \) for more details. Let \( N \) be a group and let \( A \) be a \( \mathbb{Z}[N] \)-module written in multiplicative notation. Define \( C^n(N, A) = A \), and for \( n \geq 1 \)

\[
C^n(N, A) = \{ f: N \times \cdots \times N \to A \mid f(x_1, \ldots, x_n) = 1, \text{ if } x_i = 1_N \text{ for some } i \}.
\]

Consider the following cochain complex:

\[
0 \longrightarrow C^0(N, A) \xrightarrow{\delta_0} C^1(N, A) \xrightarrow{\delta_1} C^2(N, A) \cdots C^n(N, A) \xrightarrow{\delta_n} C^{n+1}(N, A) \cdots
\]

where

\[
\delta_n(f)(x_1, x_2, \ldots, x_{n+1}) = x_1 \cdot f(x_2, \ldots, x_{n+1})
\]

\[
\times \prod_{i=1}^{n} f(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_{n+1})^{(-1)^i}
\]

\[
\times f(x_1, \ldots, x_n)^{(-1)^{n+1}}.
\]

Then, \( Z^n(N, A) := \ker(\delta_n) \), \( B^n(N, A) := \text{Im}(\delta_{n-1}) \) and

\[
H^n(N, A) := Z^n(N, A)/B^n(N, A) \quad (n \geq 1)
\]

is the group cohomology of \( N \) with coefficients in \( A \).

2.3.1. Group cohomology associated to a group exact sequence. Let

\[
1 \to N \to S \to Q \to 1
\]

be a group exact sequence. Let \( A \) be a \( Q \)-module, and consider \( A \) as an \( N \)-module with trivial action and \((C^n(N, A), \delta_n)\) as a cochain complex of \( S \)-modules, where \( C^n(N, A) \) has \( S \)-module structure given by

\[
(g f)(x_1, \ldots, x_n) = g f( g^{-1} x_1 g, \ldots, g^{-1} x_n g),
\]
Let $x, y$ for all $x \in N, g \in S, f \in C^n(N, A)$. Since the maps $\delta_n : C^n(N, A) \to C^{n+1}(N, A)$ are $S$-equivariant we have the double cochain complex $C^p_S(N, A) := C^q(S, C^{p+1}(N, A))$, $p, q \geq 0$. Then we define the complex

$$C^p_S(N, A) := \text{Tot}^{n-1}(C^{p,*}_S(N, A)), \quad n > 1,$$

and the cohomology groups

$$H^n_S(N, A) := H^n(C^*_S(N, A)), \quad n \geq 0.$$

For future reference it will be useful to describe the equations that define a 2-cocycle and the coboundary of 1-cochains:

- The 1-cochains are $C^1_S(N, A) = C^1(N, A)$, and the 2-cochains are $C^2_S(N, A) = C^2(N, A) \oplus C^1(S, C^1(N, A))$, so a 2-cochain is a pair of normalized functions $\sigma : N \times N \to A, \gamma : S \times N \to A$.
- A 2-cocycle is a 2-cochain $(\sigma, \gamma)$ such that

$$\begin{align*}
\sigma(x, y)\sigma(xy, z) &= \sigma(y, z)\sigma(x, yz), \\
g\sigma(x, y)\gamma(g, xy) &= \sigma(gx, gy)\gamma(g, x), \\
g\gamma(h, x) &= g\gamma(h, x),
\end{align*}$$

for all $x, y \in N, g, h \in S$.
- The coboundary of a 1-cochain $\gamma : N \to A$ is given by

$$\begin{align*}
\partial(\gamma)(x, y) &= \dfrac{\gamma(y)\gamma(x)}{\gamma(xy)}, \\
\partial(\gamma)(x, g) &= \dfrac{g\gamma(x)}{\gamma(gx)},
\end{align*}$$

for all $g \in S, x, y \in N$.

**Proposition 2.7.** Let $Z^2_S(N, A)_n$ be the subgroup of all 2-cocycles $(\sigma, \gamma)$ such that

$$
\gamma(x, y) = \sigma(x, y)\sigma(xy, x^{-1})\sigma(x, x^{-1})^{-1}
$$

for all $x, y \in N$. Then $B^2_S(N, A) \subset Z^2_S(N, A)_n$ and

$$H^2_S(N, A) \cong Z^2_S(N, A)_n/B^2_S(N, A).$$

**Proof.** A straightforward calculation shows that $B_S(N, A) \subset Z^2_S(N, A)_n$.

Note that if $Q = 1$, then $H^n_S(N, A) = H^n(N, A)$ and a particular quasi-isomorphism $Z^2(N, A) \to Z^2_S(N, A)$ is given by $\sigma \mapsto (\sigma, \gamma)$, where

$$\gamma(x, y) = \sigma(x, y)\sigma(xy, x^{-1})\sigma(x, x^{-1})^{-1},$$

for all $x, y \in N$. Let $(\sigma, \gamma) \in Z^2_S(N, A)$ be an arbitrary 2-cocycle. Since $(\sigma, \gamma|_{N \times N}) \in Z^2_N(N, A)$, there is $\gamma : N \to A$ such that $\partial(\gamma)(\sigma, \gamma) = (\sigma, \gamma\sigma) \in Z^2_N(N, A)$. Then $\partial(\gamma)(\sigma, \gamma) \in Z^2_S(N, A)$ is a 2-cocycle in $Z^2_S(N, A)_n$ cohomologous to $(\sigma, \gamma)$. 

2.3.2. The second cohomological group for abelian groups. Let $V$ be a finite abelian group and let $k$ be a field such that $k^*$ is a divisible group. A bicharacter $\omega : V \times V \to k^*$ is called a skew-symmetric form if $\omega(x, x) = 1$ for all $x \in V$. Let us denote by $\bigwedge^2 \hat{V}$ the abelian group of all skew-symmetric forms over $k^*$. 

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Proposition 2.8. The group morphism
\[ \text{Alt} : Z^2(V, k^*) \to \wedge^2 \hat{V} \]
\[ \alpha \mapsto [(x, y) \mapsto \frac{\alpha(x, y)}{\alpha(y, x)}] \]
induces an isomorphism \( H^2(V, k^*) \cong \wedge^2 \hat{V} \).

Proof. See [Tam00, Proposition 2.6]. \( \square \)

2.4. Galois algebras over arbitrary fields. In this section we review the main result of [GM12]. In this section \( k \) will denote a field.

Let \( S \) be a subgroup of \( G \) and \( B \) an \( S \)-algebra. The induced algebra is defined as the algebra of functions
\[ \text{Ind}_S^G(B) = \{ r : G \to B | r(sg) = s \cdot r(g) \ \forall s \in S, g \in G \}, \]
with \( G \)-action \((g \cdot r)(x) = r(xg)\), for \( r \in \text{Ind}_S^G(B) \), \( g, x \in G \).

Remark 2.9. Induction is a covariant functor, where each homomorphism of \( S \)-algebras \( f : A \to B \) is sent to the homomorphism of \( G \)-algebras \( \text{Ind}_S^G(f) : \text{Ind}_S^G(A) \to \text{Ind}_S^G(B) \), \( \text{Ind}_S^G(f)(r) = f \circ r \).

Let \( S \) be a finite group, let \( k \) be a field and let \( \sigma \in Z^2(S, k^*) \) be a 2-cocycle. For each \( s \in S \), we will use the notation \( u_s \in k_\sigma[S] \) to indicate the corresponding element in the twisted group algebra \( k_\sigma[S] \). Thus \( \{u_s\}_{s \in S} \) is a \( k \)-basis of \( k_\sigma[S] \), and in this basis \( u_s u_t = \sigma(s, t) u_{st} \) for all \( s, t \in S \).

Definition 2.10. An element \( s \in S \) is called \( \sigma \)-regular if \( \sigma(s, t) = \sigma(t, s) \) for all \( t \in C_S(s) \). The 2-cocycle \( \sigma \) is called non-degenerate if and only if \( 1 \in S \) is the only \( \sigma \)-regular element.

Remark 2.11. If \( S \) is an abelian group a 2-cocycle \( \sigma \in Z^2(S, k^*) \) is non-degenerate if and only if the skew-symmetric form \( \text{Alt}(\sigma) \) is a non-degenerate bicharacter.

Definition 2.12. Let \( G \) be a finite group and let \( k \) be a field. A Galois datum associated to \( G \) is a collection \((S, K, N, \sigma, \gamma)\) such that
i) \( S \) is a subgroup of \( G \) and \( N \) is a normal subgroup of \( S \),
ii) \( K \supseteq k \) is a Galois extension with Galois group \( S/N \),
iii) \( \text{char}(k) \nmid |N| \),
iv) \( (\sigma, \gamma) \in Z_2^S(N, K^*)_n \) is a 2-cocycle such that \( \sigma \in Z^2(N, K^*) \) is non-degenerate.

Let \((S, K, N, \sigma, \gamma)\) be a Galois datum associated to \( G \). We will denote by \( A(K_\sigma[N], \gamma) \) the twisted group algebra \( K_\sigma[N] \) with \( S \)-action defined by
\[ g \cdot (\alpha u_x) = (g \cdot \alpha)(g \cdot u_x) = \tilde{\gamma}(\alpha) \gamma(g, x) u_{\alpha x}, \]
for \( g \in S, x \in N, \) and \( \alpha \in K \).

We will denote by \( \text{Ind}_S^G(A(K_\sigma[N], \gamma)) \) the induced \( G \)-algebra from the \( S \)-algebra \( A(K_\sigma[N], \gamma) \).

Remark 2.13. If \((S, K, N, \sigma, \gamma)\) is a Galois datum where \( K = k \), then \( S = N \) and \( \gamma \) is determined by \( \sigma \). Then a Galois datum with \( K = k \) is really just a pair \((S, \sigma)\) where \( S \) is a subgroup of \( G \) and \( \sigma \in Z^2(S, k^*) \) is a non-degenerate 2-cocycle.
Now we can reformulate the main results of [GM12].

**Theorem 2.14.** Let $G$ be a finite group and let $k$ be a field.

1. Let $(S, K, N, \sigma, \gamma)$ be a Galois datum associated to $G$. Then the $G$-algebra $\text{Ind}^G_S(A(K_\sigma[N], \gamma))$ is a $G$-Galois algebra over $k$.

2. Let $A$ be a $G$-Galois algebra over $k$. Then $A \simeq \text{Ind}^G_S(A(K_\sigma[N], \gamma))$ for a Galois datum $(S, K, N, \sigma, \gamma)$. \hfill \Box

### 3. Isocategorical groups over arbitrary fields

Considering Proposition 2.7 and Lemma 2.5 in order to construct all groups isocategorical to a fixed group $G$ it is enough to describe the Galois data of $G$ such that $|\text{Aut}_G(A)| = |G|$, where $A$ is the associated $G$-Galois algebra.

#### 3.1. Equivariant automorphisms of Galois objects

Let $G$ be a finite group, $S$ a subgroup of $G$, and $(B, \cdot)$ an $S$-algebra. For each $g \in G$, we consider the $g^{-1}Sg$-algebra $(B^{(g)}, \cdot_g)$, where $B^{(g)} = B$ are algebras and $g^{-1}Sg$-action is given by

$$h \cdot_g b = (ghg^{-1}) \cdot b,$$

for all $h \in g^{-1}Sg$ and $b \in B^{(g)}$.

Note that for all $g \in G$, the map

$$\psi_g : \text{Ind}^G_S(B) \rightarrow \text{Ind}^G_{g^{-1}Sg}(B^{(g)})$$

$$f \mapsto \psi_g(f) = [h \mapsto f(gh)]$$

is a $G$-algebra isomorphism.

For a pair $(b, g) \in B \times G$, we define an element $\chi^b_g \in \text{Ind}^G_S(B)$ as the function

$$\chi^b_g(x) = \begin{cases} 0, & \text{if } x \notin Sg, \\ s \cdot b, & \text{if } x = sg, \text{ where } s \in S. \end{cases}$$

Note that $(b, g), (s', g') \in B \times G$ define the same element in $\text{Ind}^G_S(B)$ if and only if there is $s \in S$ such $g^1 = sg$ and $b' = s \cdot b$.

The elements $\chi^1_g$ are central idempotents that depend only on the coset $Sg$, which we will denote by $\chi_{Sg}$. The action of $G$ on $\text{Ind}^G_S(B)$ defines a transitive action of $G$ on $\{\chi_x\}_{x \in S/G}$ by $h \cdot \chi_{Sg} = \chi_{Sgh^{-1}}$. If $B$ is a simple algebra the central primitive idempotents of $\text{Ind}^G_S(B)$ are exactly $\{\chi_x\}_{x \in S/G}$. Therefore, in this case each $G$-algebra automorphism of $\text{Ind}^G_S(B)$ determines a unique automorphism of the $G$-set $\{\chi_x\}_{x \in S/G}$ and a group homomorphism

$$\pi : \text{Aut}_G(\text{Ind}^G_S(B)) \rightarrow N_G(S)/S$$

$$F \mapsto Sg$$

where $Sg$ is the unique coset such that $F(\chi_S) = \chi_{Sg}$.

The following proposition can be seen as a generalization of [Dav01] Theorem 5.5.

**Proposition 3.1.** Let $G$ be a finite group, $S$ a subgroup of $G$, and $(B, \cdot)$ an $S$-Galois algebra. Then, the sequence

$$1 \rightarrow \text{Aut}_S(B) \rightarrow \text{Aut}_G(\text{Ind}^G_S(B)) \rightarrow N_G(S)/S$$
is exact. The map $\text{Aut}_S(B) \to \text{Aut}_G(\text{Ind}_S^G(B))$ is the induction, and $\text{Aut}_G(\text{Ind}_S^G(B)) \to N_G(S)/S$ is the group morphism \([\mathbf{3}]\). Moreover, the sequence

$$1 \to \text{Aut}_S(B) \to \text{Aut}_G(\text{Ind}_S^G(B)) \to N_G(S)/S \to 1$$

is exact if and only if $B \cong B^{(g)}$ for all $g \in N_G(S)$.

Proof. First, we will see that induction is injective. Let $T \in \text{Aut}_S(B)$ such that $\text{Ind}_S^G(T) = \text{id}_{\text{Ind}_S^G(B)}$. Then, for all $r \in \text{Ind}_S^G(B)$, $T \circ r = r$. In particular, for $\chi_g$, we have $T \circ \chi_g = \chi_g$, so $T = \text{id}_B$.

Now, let $F \in \text{Aut}(\text{Ind}_S^G(B))$ such that $\pi(F) = S$, that is, $F(\chi_{sg}) = \chi_{sg}$ for all $g \in G$. Using the injective map

$$B \to \text{Ind}_S^G(B)$$

we can and will identify $B$ with its image in $\text{Ind}_S^G(B)$. Note that $f \in B \subset \text{Ind}_S^G(B)$ if and only if $\chi_{sg}f = 0$ for all $g \notin S$. Thus, $F(\chi_g) \in B$, because $\chi_{sg}F(\chi_g) = F(\chi_{sg})F(\chi_g) = 0$ if $g \notin S$. Therefore, $F$ defines an automorphism $F|_B: B \to B$ by $F(\chi_g) = \chi_{eg}$. We will see that $F = \text{Ind}_S^G(F|_B)$. Let

$$\text{Ind}_S^G(F|_B)(\chi_g) = F|_B \circ \chi_g$$

$$= F|_B(g^{-1} \cdot \chi_g)$$

$$= g^{-1}F(\chi_g)$$

$$= F(g^{-1} \cdot \chi_g)$$

$$= F(\chi_{eg}).$$

Since every $f \in \text{Ind}_S^G(B)$ is a sum of elements of the form $\chi_g$, it follows that $\text{Ind}_S^G(F|_B) = F$.

Now we want to show that $\pi$ is surjective if and only if $B \cong B^{(g)}$ for all $g \in G$. Suppose that $\pi$ is surjective. Then, for any $g \in N_G(S)$ there exists $F_g \in \text{Aut}_G(\text{Ind}_S^G(B))$ such that $\pi(\chi_S) = \chi_{sg}$. Using \([4]\), we have $G$-algebra isomorphisms

$$\text{Ind}_S^G(B) \xrightarrow{F_g} \text{Ind}_S^G(B) \xrightarrow{\psi_g} \text{Ind}_S^G(B^{(g)}),$$

and the restriction $(\psi_g \circ F_g)|_B: B \to B^{(g)}$ defines an $S$-algebra isomorphism. Conversely, if $\gamma_g: B^{(g)} \to B$ is an $S$-algebra isomorphism, then

$$\text{Ind}_S^G(B) \xrightarrow{\psi_g} \text{Ind}_S^G(B^{(g)}) \xrightarrow{\text{Ind}_S^G(\gamma_g)} \text{Ind}_S^G(B)$$

is an algebra isomorphism such that $\pi(\text{Ind}_S^G(\gamma_g) \circ \psi_g) = Sg$. \[\square\]

**Corollary 3.2.** Let $G$ be a finite group and $S \subset G$ be a subgroup. Let $B$ be a simple $S$-Galois algebra and $A = \text{Ind}_S^G(B)$. Then $|\text{Aut}_G(A)| = |G|$ if and only if

1. $|\text{Aut}_S(B)| = |S|$,
2. $S$ is a normal subgroup of $G$,
3. for all $g \in G$, $B^{(g)} \cong B$ as $S$-algebras.
Lemma 3.6. Let $G$ be a finite group. A Galois datum $(S, K, N, \sigma, \gamma)$ will be called a torsor datum if

1. $S$ is abelian,
2. $S = N \oplus \text{Gal}(K|k)$,
3. $k$ has a primitive root of unity of order equal to the exponent of $N$,
4. $\sigma$ takes values in $k^*$,
5. $\gamma : S \times N \to k^*$ is a pairing where $\gamma|_{\text{Gal}(K|k) \times N} = 1$ and $\gamma|_{N \times N} = \text{Alt}(\sigma)$,
6. $N$ and $S$ are normal subgroups of $G$,
7. $[(\sigma, \gamma)] = [(\sigma^g, \gamma^g)]$ for all $g \in G/S$.

Remark 3.4. In the information of Galois datum we have the exact sequence of groups

$$(7) \quad 1 \to N \to S \to \text{Gal}(K|k) \to 1.$$ 

Thus, the meaning of condition $S = N \oplus \text{Gal}(K|k)$ in a torsor datum is just the choice of a particular splitting of the sequence (7).

Lemma 3.5. Let $B$ be a simple $S$-Galois algebra with Galois data $(S, K, N, \sigma, \gamma)$. Then the group $\text{Aut}_S(B)$ is isomorphic to

$$\{ (\eta, \omega) \in C^1(N, K^*) \times Z(\text{Gal}(K|k)) \mid \partial(\eta) = \left( \frac{\omega(\sigma)}{\sigma}, \frac{\omega(\gamma)}{\gamma} \right) \}$$

with product $(\eta, \omega)(\eta', \omega') = (\eta^\omega \eta', \omega \omega')$.

Proof. See [GM12] Proposition 5.6. \hfill \Box

Lemma 3.6. Let $B$ be a simple $S$-Galois algebra with Galois data $(S, K, N, \sigma, \gamma)$. Then $|\text{Aut}_S(B)| = |S|$ if and only if the following conditions are satisfied:

1. $N$ is abelian and $K$ contains a primitive root of unity of order equal to the exponent of $N$.
2. $\text{Hom}_{\text{Gal}(K|k)}(N, K^*) = \text{Hom}(N, K^*)$, where

$$\text{Hom}_{\text{Gal}(K|k)}(N, K^*) = \{ f : N \to K^* : f(nn') = f(n)f(n') \}$$

$$(f(qnq^{-1})) = qf(n), \quad \forall n, n' \in N, q \in \text{Gal}(K|k) \cong G/N.$$ 

3. $\text{Gal}(K|k)$ is abelian and for all $\omega \in \text{Gal}(K|k)$,

$$[(\sigma, \eta)] = [(\omega(\sigma), \omega(\eta))],$$

as elements in $H^2_{\text{Gal}(K|k)}(N, K^*)$. 

Proof. By Proposition 3.1 we have

$$|\text{Aut}_G(A)| = |\text{Aut}_S(B)||\text{Im}(\pi)|$$

$$\leq |\text{Aut}_S(B)||N_G(S)/S|$$

$$\leq |S|\left|\frac{N_G(S)}{|S|}\right|$$

$$= |N_G(S)| \leq |G|.$$

Thus, if $|\text{Aut}_G(A)| = |G|$, $S$ is a normal subgroup, $|\text{Aut}_S(B)| = |S|$ and $\text{Im}(\pi) = G/S$. Therefore, by Proposition 3.1, $B^g \cong B$ for all $g \in G$.

Conversely, the third condition implies that $|\text{Aut}_G(A)| = |\text{Aut}_S(B)||N_G(S)/S|$. Since $B$ is normal and $|\text{Aut}_S(B)| = |S|$, we have $|G| = |\text{Aut}_G(A)|$. \hfill \Box
Proof. The group homomorphism

\[ \pi_2 : \text{Aut}_S(B) \rightarrow \mathcal{Z}(\text{Gal}(k|K)) \]

\[ (\eta, \omega) \mapsto \omega \]

induces the exact sequence

\[ 1 \rightarrow \text{Hom}_{\text{Gal}(k|K)}(N, K^*) \rightarrow \text{Aut}_S(B) \xrightarrow{\pi_2} \mathcal{Z}(\text{Gal}(k|K)). \]

Therefore,

\[ |\text{Aut}_S(B)| \leq |\text{Hom}_{\text{Gal}(k|K)}(N, K^*)| |\text{Im}(\pi_2)| \]
\[ \leq |\text{Hom}(N, K^*)| |\mathcal{Z}(\text{Gal}(k|K))| \]
\[ \leq |N| |\text{Gal}(k|K)| \]
\[ = |N| |S/N| = |S|. \]

Thus, \( |\text{Aut}_S(B)| = |S| \) if and only if \( \text{Im}(\pi_2) = \mathcal{Z}(\text{Gal}(k|K)) \), \( \mathcal{Z}(\text{Gal}(k|K)) = \text{Gal}(k|K) \), \( \text{Hom}_{\text{Gal}(k|K)}(N, K^*) = \text{Hom}(N, K^*) \) and \( |N| = |\text{Hom}(N, K^*)| \), and these conditions are precisely (1), (2) and (3). \( \square \)

Lemma 3.7. Let \( B \) be a simple \( S \)-Galois algebra with Galois data \( (S, K, N, \sigma, \gamma) \). If \( |\text{Aut}_S(B)| = |S| \), then

- \( k \) has a primitive root of unity of order equal to the exponent of \( N \),
- there is a canonical decomposition \( S = N \oplus \text{Gal}(K|k) \).

Proof. Let \( \omega \in \text{Gal}(K|k) \). Then, there is \( \eta_\omega : N \rightarrow K^* \) such that \( \omega(\sigma) = \delta(\eta_\omega)\sigma \). Therefore, \( \text{Alt}(\sigma) = \text{Alt}(\omega(\sigma)) = \omega(\text{Alt}(\sigma)) \) for all \( \omega \in \text{Gal}(K|k) \). Since \( N \) has exponent \( n \) and \( \text{Alt}(\sigma) \) is a non-degenerate bicharacter there are \( x, y \in N \) such that \( q = \text{Alt}(\sigma)(x, y) \) is a primitive root of unity of order the exponent of \( N \). Therefore, \( \omega(q) = q \) for all \( \omega \in \text{Gal}(K|k) \), so \( q \in k \).

Since \( k \) has primitive root of unity, \( \text{Gal}(K|k) \) acts trivially on \( \text{Hom}(N, K^*) = \text{Hom}(N, k^*) \). Therefore, by Lemma 3.6, \( S = \text{Aut}_{\text{Aut}_S(B)}(B) = N \oplus \text{Gal}(K|k) \). \( \square \)

Lemma 3.8. Let \( B \) be a simple \( S \)-Galois algebra with Galois data \( (S, K, N, \sigma, \gamma) \) where \( S = N \oplus \text{Gal}(K|k) \). The Galois data is equivalent to the data \( (S, K, N, \sigma', \gamma') \), where \( \gamma'|_{\text{Gal}(K|k) \times N} = 1 \), \( \sigma' \in Z^2(N, k^*) \) and \( \gamma'|_{N \times N} = \text{Alt}(\sigma') = \text{Alt}(\sigma) \).

Proof. Since \( S \) is abelian the equation \( \Box \) implies that for every \( x \in N \), \( \gamma(-, x) : \text{Gal}(K|k) \rightarrow K^* \) is a 1-cocycle. By Hilbert’s Theorem 90 \( H^1(\text{Gal}(K|k), K^*) = 0 \); therefore there exists \( \tau : N \rightarrow K^* \) such that \( \gamma(g, x) = g(\tau(x)) / \tau(x) \) for all \( x \in N, g \in \text{Gal}(K|k) \). Therefore, if \( \sigma' = \sigma \delta(\tau) \) and \( \gamma' = \gamma \delta(\tau) \), it follows that \( (S, K, N, \sigma', \gamma') \) is a new Galois datum where \( \gamma'|_{\text{Gal}(K|k) \times N} = 1 \), thus \( \sigma' \in Z^2(N, k^*) \) and \( \gamma'|_{N \times N} = \text{Alt}(\sigma') = \text{Alt}(\sigma) \). By equation \( \Box \), \( g \sigma'(x, y) = \sigma'(x, y) \) for all \( g \in \text{Gal}(K|k) \); then \( \sigma'(x, y) \in k^* \) for all \( x, y \in N \). \( \square \)

Theorem 3.9. Let \( G \) be a finite group and let \( k \) be a field.

i) If \( (S, K, N, \sigma, \gamma) \) is a torsor datum associated to \( G \), then

\[ |\text{Aut}_G(\text{Ind}_S^G(A(K_\sigma[N], \gamma)))| = |G|. \]

ii) Let \( A \) be a \( G \)-Galois algebra such that \( |\text{Aut}_G(A)| = |G| \). There exists a torsor datum \( (S, K, N, \sigma, \gamma) \) such that \( A \cong \text{Ind}_S^G(A(K_\sigma[N], \gamma)) \).
Proof. Let $A$ be a $G$-Galois algebra such that $|\text{Aut}_G(A)| = |A|$. By Lemma 3.7 and Lemma 3.8, $A$ has a Galois datum $(S, K, N, \sigma, \gamma)$ which satisfies the first five conditions of Definition 3.3.

By Corollary 3.2, if $|\text{Aut}_G(A)| = |G|$, then $S$ is a normal subgroup of $G$. We want to find conditions that imply $A(K_\sigma[N], \gamma) \cong A(K_\sigma[N], \gamma)^{(\sigma)}$ for all $\sigma \in G$. The Galois datum of the simple $S$-algebra $A(K_\sigma[N], \gamma)^{(\sigma)}$ is

$$(g^{-1}Sg, g^{-1}Ng, \sigma^{(\sigma)}, \gamma^{(\sigma)}),$$

where

$$
\sigma^{(\sigma)}(x, y) := \sigma(gxg^{-1}, gyg^{-1}), \quad \gamma^{(\sigma)}(h, x) := \gamma(ghg^{-1}, gxg^{-1})
$$

for all $h \in g^{-1}Sg, x, y \in g^{-1}Ng$. By [CM12, Proposition 5.6] $A(K_\sigma[N], \gamma) \cong A(K_\sigma[N], \gamma)^{(\sigma)}$ for all $g \in G$ if and only if $N$ is normal in $G$ and there is $\omega \in \text{Gal}(K|k)$ such that $(\omega(\sigma), \omega(\eta))$ is cohomologous to $(\sigma^g, \eta^g)$. By condition (3) of Lemma 3.6, $(\omega(\sigma), \omega(\eta))$ is cohomologous to $(\sigma, \eta)$, so $A(K_\sigma[N], \gamma) \cong A(K_\sigma[N], \gamma)^{(\sigma)}$ if and only if for all $g \in G$,

$$
[\langle \sigma, \eta \rangle] = [\langle \sigma^g, \eta^g \rangle],
$$

as elements in $H^2_{\text{Gal}(K|k)}(N, K^*)$, where $\sigma^g(x, y) = \sigma(gxg^{-1}, gyg^{-1})$ and $\eta^g(s, x) = \eta(gs^{-1}g^{-1}, gs^{-1}g^{-1})$, for all $x, y \in N, g \in G$. Therefore, the new Galois datum constructed using Lemma 3.8 is a torsor datum.

Conversely, if $(S, K, N, \sigma, \gamma)$ is a torsor datum, then the first five conditions of Definition 3.3 and Lemma 3.6 imply that $|\text{Aut}_S(A(K_\sigma, \gamma))| = |S|$. Conditions (6) and (7) of Definition 3.3 and Corollary 3.2 imply that $|\text{Aut}_G(\text{Ind}_S^G(A(K_\sigma[N], \gamma)))| = |G|$. □

The next corollary is a generalization of [EG01, Theorem 1.3] and [Dav01, Corollary 6.2].

Corollary 3.10. Let $G_1$ and $G_2$ be finite groups and let $k$ be a field. Then, $G_1$ and $G_2$ are isocategorical over $k$ if and only if there is a torsor datum of $G_1$ with $G_1$-Galois algebra associated to $A$ such that $\text{Aut}_{G_1}(A) \cong G_2$. □

3.2. Isocategorical groups over formally real fields. A field $k$ is called formally real if $-1$ is not a sum of squares. A formally real field with no formally real proper algebraic extensions is called a real closed field. Examples of real fields are $\mathbb{Q}$ and $\mathbb{R}$. The field of real numbers is a real closed field.

Every formally real field $k$ has characteristic zero, and the field extension $k \subset k(i)$ (where $i^2 = -1$) is a Galois extension with $\text{Gal}(k(i)|k) \cong \mathbb{Z}/2\mathbb{Z}$.

Definition 3.11. Let $G$ be a finite group and $k$ a formally real field.

1. A real torsor datum for $G$ is a pair $(N, \sigma)$ where:
   - $N$ is a normal abelian elementary 2-subgroup of $G$,
   - $\sigma \in Z^2(N, \mu_2)$ is a non-degenerate 2-cocycle, where $\mu_2 = \langle -1 \rangle$,
   - $[[\sigma]] = [\sigma^g]$ for all $g \in G/N$, as elements in $H^2(N, \mu_2)$.

2. A semi-real torsor datum for $G$ is a torsor datum $(S, k(i), N, \sigma, \gamma)$ where:
   - $N$ is an abelian 2-group,
   - $\sigma \in Z^2(N, \mu_2)$, where $\mu_2 = \langle -1 \rangle$,
   - $[[\sigma, \gamma]] = [\langle \sigma, \gamma \rangle]$ for all $g \in G/S$, as elements in $H^2_S(N, \mu_4)$, where $\mu_4 = \langle i \rangle$.

Remark 3.12. A real torsor datum $(N, \sigma)$ defines a torsor datum $(N, k, N, \sigma, \gamma_\sigma)$, where $\gamma_\sigma(x, y) = \sigma(x, y)\sigma(xy, x^{-1})\sigma(x, x^{-1})^{-1}$, for all $x, y \in V$. 

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Lemma 3.13. Let N be an elementary abelian 2-group, k a real closed field and S = N ⊕ Gal(k(i)|k). Every 2-cocycle in Z^2(S,N,k(i)^*) is cohomologous to a 2-cocycle in σ ∈ Z^2(S,N,µ_2), and α, β ∈ Z^2(S,N,k(i)^*) are cohomologous as elements in Z^2(S,N,µ_4) if and only if α, β are cohomologous as elements in Z^2(S,N,µ_4).

Proof. Since k is a real closed field, k(i) is an algebraically closed field of characteristic zero. It is well known that for finite abelian groups every 2-cocycle with values in k(i)^* is cohomologous to a bicharacter [Tam00, Proposition 2.6], so we can suppose that σ is a bicharacter, and since N is an elementary 2-group σ takes values in µ_2. Then, the equation (2) implies that γ(g,−):N → k(i)^* is a character for all g ∈ S. Again, since N is an elementary abelian 2-group, γ only takes values in µ_2. Now, equation (3) implies that γ:S × N → µ_2 in fact is a pairing and therefore every 2-cocycle in Z^2(S,N,µ_2) is cohomologous to a 2-cocycle in Z^2(S,N,µ_2) ⊂ Z^2(S,N,µ_4). Let (σ,γ) ∈ Z^2(S,N,µ_2) ⊂ Z^2(S,N,µ_4) and η : N → µ_∞, such that ∂(η) = (σ,γ). Let g ∈ Gal(K(i)|k) ⊂ S be the conjugation on k(i). Then γ(g,η(x)/η(x)) = η(x) for all x ∈ N. Since γ(g,x) ∈ µ_2, then η(x) ∈ µ_4 for all x ∈ N. Then, if α, β ∈ Z^2(S,N,µ_2) ⊂ Z^2(S,N,µ_4) and they are cohomologous as elements in Z^2(S,N,k(i)^*), they are cohomologous as elements in Z^2(S,N,µ_4).

Theorem 3.14. Let G be a finite group and k a formally real closed field. Let A be a Galois algebra over k such that |Aut_G(A)| = |G|. Then, A ≃ Ind^G_S(A(k(i)|σ[N], γ)) or A ≃ Ind^G_S(A(k(i)|σ[N], γ)) for a semi-real or real torsor datum, respectively.

Proof. Since k is a formally real field the unique primitive root of unity is −1. Let A be a G-Galois algebra over k such that |Aut_G(A)| = |G|. By Theorem 3.9 there exists a torsor datum (S,K,N,σ,γ), where N is an abelian group of exponent 2. Therefore N is an elementary abelian group.

Since H^2(N,k^*) ≃ H^2(N,µ_2) for any formally real closed field, if K = k the torsor datum defines a real torsor. Now, if K is a proper extension of k, then K = k(i), and by Lemma 3.13 the torsor datum is equivalent to a semi-real torsor for G.

Corollary 3.15. Let G_1 and G_2 be finite groups and let k be a formally real field. Then:

1. G_1 and G_2 are isocategorical over k if and only if there is a real or semi-real torsor datum such that Aut_{G_1}(A) ≅ G_2, where A is the associated Galois algebra to the datum.

2. G_1 and G_2 are isocategorical over k if and only if they are isocategorical over Q.

Proof. Let k' be a formally real closed extension of k. If G_1 and G_2 are isocategorical over k, then by Proposition 2.6 there exists a G_1-Galois algebra A such that Aut_{G_1}(A) ≃ G_2. The G_1-algebra A' = A ⊗ k k' is also Galois and Aut_{G_1}(A') ≃ G_2. By Theorem 3.14 there exists a real or semi-real torsor datum such that A' ≃ Ind^{G_1}_S(A(k(i)|σ[N], γ)). By the definition of real or semi-real datum, we have that A' has a k-form D := Ind^{G_1}_S(A(k(i)|σ[N], γ)) and by Lemma 3.13 Aut_{G_1}(D) ≃ Aut_{G_1}(A') ≃ G_2. Therefore G_1 and G_2 are isocategorical over k. For the second part note also that A' has a rational form L := Ind^{G_1}_S(A(Q(i)|σ[N], γ)) and Aut_{G_1}(L) ≃ Aut_{G_1}(A') ≃ G_2.
4. Galois algebras and Weil representations

4.1. Categorical setting. Let $G$ be a small groupoid. We will denote by $\text{Aut}(G)$ the monoidal groupoid where objects are autoequivalences of $G$, morphisms are natural isomorphisms, and the monoidal structure is given by the composition of functors.

Given a group $X$, we will denote by $X$ the discrete monoidal category where objects are elements of $X$ and the monoidal structure is given by the product of $X$.

A normalized (right) action of a group $X$ on a category $G$ is a monoidal functor $\rho^*: X^{\text{op}} \to \text{Aut}(G)$, where $X^{\text{op}}$ denotes the opposite group of $X$. More concretely, an action of $X$ on $G$ consists of the following data:

- functors $(-)^{(x)}: G \to G, A \mapsto A^x$ for all $x \in X$,
- natural isomorphisms $\gamma^A_{x,y}: A^x(y) \to A^x(y(z))$ for all $x, y \in X, A \in G$,

such that $(-)^1 = \text{id}_G$, $\gamma^A_{1,x} = \gamma^A_{1,x} = \text{id}_{A^x}$ for all $x \in X$ and the diagrams

$$
\begin{array}{ccc}
A^{xyz} & \xrightarrow{\gamma^A_{xy,z}} & A^{xyz} \\
\downarrow_{\gamma^A_{x,yz}} & & \downarrow_{\gamma^A_{x,yz}} \\
A^{x(yz)} & \xrightarrow{\gamma^A_{x,yz}} & A^{x(yz)} \\
\end{array}
$$

commute for all $x, y, z \in G, A \in G$.

There are two different groupoids associated to a normalized action of a group $X$ on a groupoid $G$: the groupoid of equivariant objects $G^X$ and the quotient groupoid $G//X$.

4.1.1. The groupoid of $X$-equivariant objects. The groupoid $G^X$ of $X$-equivariant objects is defined as follows: an object in $G^X$ is a pair $(A, u)$, where $A \in G$ is an object and $u_x : A^x \to A$ is a family of morphisms such that $u_1 = \text{id}_A$ and the diagrams

$$
\begin{array}{ccc}
A^{xy} & \xrightarrow{u_{xy}} & A \\
\downarrow_{\gamma^A_{x,y}} & & \downarrow_{u_y} \\
A^{x(y)} & \xrightarrow{(u_x)^{(y)}} & A^{y} \\
\end{array}
$$

commute for all $x, y \in X$. A morphism from $(A, u)$ to $(A', u')$ is a morphism $f \in \text{Hom}_G(A, A')$ such that the diagrams

$$
\begin{array}{ccc}
A^x & \xrightarrow{u} & A \\
\downarrow_{f^x} & & \downarrow_{f} \\
A'^x & \xrightarrow{u'} & A \\
\end{array}
$$

commute for all $x \in X$. 

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4.1.2. The quotient groupoid of an $X$-action. The groupoid $G\!/\!X$ has objects $\text{Obj}(G)$ and morphisms

$$\text{Hom}_{G\!/\!X}(A,B) = \{(g, \alpha)|g \in X \text{ and } \alpha \in \text{Hom}_{G}(A^{(g)}, B)\}.$$ 

The composition of $(g, \alpha_g) \in \text{Hom}_{G\!/\!X}(A,B)$ and $(h, \alpha_h) \in \text{Hom}_{G\!/\!X}(B,C)$ is defined by $(g, \alpha_g) \circ (h, \alpha_h) := (gh, \alpha_g \circ \alpha_h) \in \text{Hom}_{G\!/\!X}(A,C)$ where $\alpha_g \circ \alpha_h : A^{(gh)} \rightarrow C$ is defined by the commutativity of the diagram

$$
\begin{array}{ccc}
A^{(gh)} & \xrightarrow{\alpha_g \circ \alpha_h} & C \\
\gamma_{g,h} & & \alpha_h \\
\downarrow & & \downarrow \\
A^{(g)} & \xrightarrow{\alpha_g(h)} & B^{(h)}
\end{array}
$$

The identity of an object $A$ is given by $(1, \text{id}_A)$, and the associativity of the composition follows from the naturality of $\gamma$ and the commutativity of the diagrams $\mathbb{N}$:

$$
\alpha_g \circ (\alpha_h \circ \alpha_k) = \alpha_k \alpha_h^{(k)} \gamma_{h,k} A \alpha_g^{(h)} \gamma_{g,hk}
= \alpha_k \alpha_h^{(k)} \left(\gamma_{h,k} A g \right) \gamma_{g,hk}
= \alpha_k \alpha_h^{(k)} \left(\alpha_g^{(h)} \gamma_{h,k} A \right) \gamma_{g,hk}
= \alpha_k \alpha_h^{(k)} \left(\alpha_g^{(h)} \gamma_{g,hk} A \right)
= \alpha_k \alpha_g^{(h)} \gamma_{gh,k} A
= (\alpha_g \circ \alpha_h) \circ \alpha_k.
$$

The next proposition gives a complete description of the structure of the groupoids $G^X$ and $G\!/\!X$.

**Proposition 4.1.** Let $X$ be a group acting on a groupoid $G$. Then

(a) The set of isomorphism classes of $G\!/\!X$ is the set of orbits of the $X$-set of isomorphism classes of objects of $G$.

(b) For every $A \in \text{Obj}(G)$, the sequence

$$1 \rightarrow \text{Aut}_G(A) \rightarrow \text{Aut}_{G\!/\!X}(A) \rightarrow \text{St}([A]) \rightarrow 1$$

is exact, where $\text{Aut}_G(A) \rightarrow \text{Aut}_{G\!/\!X}(A), f \mapsto (1, f)$, $\text{Aut}_{G\!/\!X}(A) \rightarrow \text{St}([A]), (g, \alpha_g) \mapsto g$ and $\text{St}([A]) = \{g \in X | A^{(g)} \cong A\}$.

(c) Suppose that $\text{St}([A]) = X$. There is a bijective correspondence between splittings of the exact sequence (10) and families of morphisms $\{u_g : A^{(g)} \rightarrow A\}_{g \in X}$ such that $(A, u_g) \in G^X$.

(d) If $(A, u_g) \in G^X$, then $X$ acts on the right on $\text{Aut}_G(A)$ by

$$f \cdot g := u_g(f)^{(g)} u^{-1}_g,$$

for all $g \in X, f \in \text{Aut}_G(A)$. Moreover, $\text{Aut}_{G\!/\!X}((A, u_g)) = \text{Aut}_G(A)^X$, and the isomorphism classes of objects in $G^X$ with underlying object $A$ are in correspondence with elements of the pointed set $H^1(X, \text{Aut}_G(A))$. 


Proof. The parts (a), (b) and (c) are intermediate from the definitions.

Let \((A, u_g) \in \mathcal{G}^X\). The group morphism \(\hat{u} : X \rightarrow \text{Aut}_G/(A), g \mapsto (g, u_g)\) is a splitting of the exact sequence \((\text{II})\). Therefore, the group \(X\) acts (on the right) on \(\text{Aut}_G(X)\) by

\[
f \cdot g := u_{g^{-1}} \circ f \circ u_g
\]

for all \(g \in G, f \in \text{Aut}_G(X)\). Using the diagrams \((\text{III})\), we have \(\text{id}_A = u_g u_{g^{-1}}^{-1} \gamma_{g^{-1},g}^A\).

Hence

\[
f \cdot g := u_{g^{-1}} \circ f \circ u_g
= u_g f(g) u_{g^{-1}} \gamma_{g^{-1},g}^A
= u_g f(g) u_{g^{-1}}^{-1},
\]

for all \(g \in G, f \in \text{Aut}_G(A)\). Now it is clear that \(\text{Aut}_G \times ((A, u_g)) = \text{Aut}_G(A)^X\).

The pointed set \(H^1(X, \text{Aut}_G(A))\) is in bijective correspondence with the set of conjugacy classes of splittings of the exact sequence \((\text{II})\). Therefore, by (c) and the first part of (d), \(H^1(X, \text{Aut}_G(A))\) is in bijective correspondence with the set of isomorphism classes of objects in \(\mathcal{G}^X\). \(\square\)

4.2. Actions on the groupoid of \(N\)-Galois algebras.

4.2.1. Group extensions. Let \(X\) and \(N\) be groups. An \(X\)-crossed system over \(N\) is a pair of maps

- \(\cdot : X \times N \rightarrow N, (n, x) \mapsto x n,\)
- \(\theta : X \times X \rightarrow N,\)

such that the set \(N \times X\) with the product given by

\[
(n, x)(m, y) := (n(xm) \theta(x, y), xy), \quad \text{for all } (n, x), (m, y) \in N \times X,
\]

is a group with unit \((1, 1)\). It is easy to see that a pair \((\cdot, \theta)\) is an \(X\)-crossed system if and only if

- \(x(nm) = (x n)(y m),\)
- \(\theta(x, y) xy n = x(y n) \theta(x, y),\)
- \(\theta(x, y) \theta(xy, z) = x \theta(y, z) \theta(x, y z),\)
- \(\theta(x, 1) = \theta(1, x) = 1,\)
- \(1 n = n,\)

for all \(x, y, z \in X, m, n \in N\).

We will denote by \(N \#_\theta X\) the group and will call it a crossed product of \(X\) by \(N\). Note that we have an exact sequence

\[
1 \rightarrow N \rightarrow N \#_\theta X \rightarrow X \rightarrow 1,
\]

where \(N \rightarrow N \#_\theta X, n \mapsto (n, 1)\) and \(N \#_\theta X \rightarrow X, (n, x) \mapsto x\).

Let \(N\) be a finite group and let \(k\) be a field. It is well known that every morphism between Galois algebras over \(k\) is an isomorphism. Therefore, the category \(\text{Gal}(N, k)\) of \(N\)-Galois algebras over \(k\) is a groupoid.

Let \(X\) be a finite group and let \((N, \cdot, \theta)\) be an \(X\)-crossed system. The crossed system defines an action of \(X\) on \(\text{Gal}(N, k)\) as follows. Let \(A\) be an \(N\)-Galois algebra.
over $k$. Let $A^{(x)} = A$ be algebras with new $N$-action given by $n \cdot a := x \cdot n \cdot a$, for all $x \in X, n \in N, a \in A$ and let

$$\gamma_{x, y} : A^{(xy)} \to A^{(x)(y)}$$

$$a \mapsto \theta(x, y)a$$

for all $a \in A$. It is straightforward to see that the above formulas define a normalized right $X$-action on $\text{Gal}(N, k)$.

**Proposition 4.2.** Let $X$ be a finite group and let $(N, \cdot, \theta)$ be an $X$-crossed system. Let $A \in \text{Gal}(N, k)$. Then, there is an isomorphism of exact sequences

$$1 \to \text{Aut}_N(A) \to \text{Aut}_{N \# X}^\theta(\text{Ind}_N^{N \# X}(A)) \to X$$

Is naturally

$$1 \to \text{Aut}_{\text{Gal}(N, k)}(A) \to \text{Aut}_{\text{Gal}(N, k) \# X}(A) \to X$$

where the first sequence is (23) and the second one is (10).

**Proof.** Let $G := N \# \theta X$ be the crossed product group. Let $A$ be an $N$-Galois algebra. For each $x \in X$ the map

$$\psi_x : \text{Ind}_N^G(A) \to \text{Ind}_N^G(A^{(x)})$$

$$f \mapsto \psi_x(f) = [(n, z) \mapsto f(x \cdot n \cdot \theta(x, z), xz)]$$

defines a $G$-algebra isomorphism. The map

$$F : \text{Aut}_{\text{Gal}(N, k) \# X}(A) \to \text{Aut}_{N \# \theta X}(\text{Ind}_N^{N \# \theta X}(A))$$

$$(\alpha, x) \mapsto \text{Ind}(\alpha_x) \circ \psi_x$$

is a morphism of groups. In fact,

$$[F(\alpha_y, y) \circ F(\alpha, x)](f)(n, z) = \text{Ind}(\alpha_y) \psi_y \text{Ind}(\alpha_x) \psi_x(f)(n, z)$$

$$= \text{Ind}(\alpha_y) \text{Ind}(\alpha_x)(f)(y \cdot n \cdot \theta(y, z), yz)$$

$$= \text{Ind}(\alpha_y) \text{Ind}(\alpha_x)(f)(x \cdot n \cdot \theta(x, y, z) \cdot \theta(x, yz), xyz)$$

$$= \text{Ind}(\alpha_y) \text{Ind}(\alpha_x)(f)(x \cdot n \cdot \theta(x, y, z), x, yz)$$

$$= \text{Ind}(\alpha_y) \text{Ind}(\alpha_x)(f)(x \cdot n \cdot \theta(x, y, z), x, yz)$$

$$= \text{Ind}(\alpha_y) \text{Ind}(\alpha_x)(f)(x \cdot n \cdot \theta(x, y, z), x, yz)$$

$$= \text{Ind}(\alpha_y) \text{Ind}(\alpha_x)(f)(x \cdot n \cdot \theta(x, y, z), x, yz)$$

$$= F(\alpha_x \oplus \alpha_y, xy)(f)(n, z)$$

for all $(n, z) \in N \# \theta X$.

It follows from the proof of Proposition 3.1 that $F$ is injective. We claim that $F$ is surjective. Let $W \in \text{Aut}_{N \# X}(\text{Ind}_N^{N \# X}(A))$. There is a unique $x \in X$ such that $W(\chi_x^y) = \chi_1^x$, so $W \psi_x^{-1}(\chi_1^y) = \chi_1^y$ for all $y \in X$. If $\alpha_x = (W \psi_x^{-1})|_{A}$, then $W \psi_x^{-1} = \text{Ind}(\alpha_x)$, thus $W = F(\alpha_x, x)$. 

$$\square$$
4.3. The finite Weil representation associated to a simple Galois algebra. Let $S$ be a finite abelian group and let $A$ be an $S$-Galois algebra. Then, $\text{Aut}_S(A)$ is an abelian group. Let

$$\text{St}(A) = \{g \in \text{Aut}(S)|A^g \cong A\text{ as }S\text{-algebras}\}.$$ 

**Proposition-Definition 4.3.** Let $S$ be a finite abelian group and let $A$ be an $S$-Galois algebra. If we choose an isomorphism of $S$-algebras $\alpha_g:A^g \rightarrow A$ for each $g \in \text{St}(A)$, then

$$g \cdot f := \alpha_g^{-1}f\alpha_g,$$

$$\theta(x,y) := \alpha_x^{-1}\alpha_y^{-1}\alpha_{xy} \in \text{Aut}_S(A)$$

define a crossed system of $\text{St}(A)$ over $\text{Aut}_S(A)$. The crossed product $\text{Aut}_S(A)\#_\theta \text{St}(A)$ acts on $A$ by algebra automorphisms as

$$(\psi,x) \cdot a = \psi(\alpha_x^{-1}(a)).$$

This action will be called the Weil action. If the cohomology class

$$\theta \in Z^2(\text{St}(A), \text{Aut}_S(S))$$

is zero, $\text{St}(A)$ acts on $A$, and this action will also be called the Weil action.

**Proof.** Straightforward. \qed

If $A$ is a simple algebra, once a simple $A$-module $M$ is fixed, there is a canonical isomorphism $A \cong M_n(D)$, where $D = \text{End}_A(M)$. Using the Skolem-Noether theorem, the Weil action defines and is defined by a unique (up to isomorphism) projective representation $\rho : \text{Aut}_S(A)\#_\theta \text{St}(A) \rightarrow \text{PGL}(M) := \text{GL}_k(M)/k^*$ by the equation

$$\alpha_g(f)(v) = \rho_g f \rho_g^{-1}(v)$$

for all $f \in M_n(D), v \in M$.

**Example 4.4.** Let $V$ be an abelian group of odd order and $\omega \in \text{Hom}(\bigwedge^2 V, k^*)$ a non-degenerate skew-symmetric bicharacter. Let $A = k_\omega[V]$ be the twisted group algebra. Then $A$ is a $V$-Galois algebra,

$$(12) \quad \text{Sp}(V,\omega) = \{g \in \text{Aut}(V)|\omega(x,y) = \omega(g(x),g(y))\} = \text{St}(A),$$

and $\text{Aut}_V(A) = \hat{V}$. The group $\text{Sp}(V,\omega)$ acts on $A$ by $\alpha_g : A \rightarrow A, u_x \mapsto u_{g(x)}$. If we fix a Lagrangian decomposition of $V = U \oplus W$, then $A$ acts on $M := \text{Span}(\{t_b|b \in U\})$ by $u_x \cdot t_b = \omega(y,b)t_{x+b}$, and the associated representation corresponds to the usual Weil representation.

**Remark 4.5.** The Weil action defined in Example 4.4 can be seen as the canonical Weil representation because it is defined only in terms of the pair $(V,\omega)$.

4.4. The Weil representation of a symplectic module. Let $V$ be an abelian group. A skew-symmetric form on $V$ is a bicharacter $\omega : V \times V \rightarrow \mathbb{C}^*$ such that $\omega(v,v) = 1$ for all $v \in V$. We will denote by $\bigwedge^2 \hat{V}$ the abelian group of all skew-symmetric forms on $V$.

A symplectic module is a pair $(V,\omega)$, where $V$ is a finite abelian group and $\omega$ is a non-degenerate skew-symmetric form. We define $\text{Sp}(V,\omega)$ to be the symplectic group of $(V,\omega)$ by (12).
Example 4.6. Let $\mathbb{F}_p^n$ be the finite field with $p^n$ elements. Let $V$ be an $\mathbb{F}_p^n$-vector space and $\langle -, - \rangle : V \times V \rightarrow \mathbb{F}_p^n$ a symplectic $\mathbb{F}_p^n$-bilinear form. Using the trace map $\operatorname{Tr} : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$, $x \mapsto x + x^p + \cdots + x^{p-1}$ we define the skew-symmetric form

$$\omega : V \times V \rightarrow \mathbb{C}^*$$

$$(v, w) \mapsto e^{2\pi i \operatorname{Tr}(\langle v, w \rangle)/p}.$$

Proposition 4.7. The pair $(V, \omega)$ is a symplectic module, and the linear symplectic group $\operatorname{Sp}_k(V)$ is a subgroup of $\operatorname{Sp}(V, \omega)$.

Proof. It is clear that $\omega \in \wedge^2 \hat{V}$, so we will only see that $\omega$ is non-degenerate. Suppose that $\omega(v, w) = 1$ for all $w \in V$, so $\operatorname{Tr}(\langle v, w \rangle) = 0$ for all $y \in V$. Therefore, $\operatorname{Tr}(c(v, w)) = \operatorname{Tr}(\langle v, cw \rangle) = 0$ for all $c \in \mathbb{F}_p^n$. Since the bilinear form $\mathbb{F}_p^n \times \mathbb{F}_p^n \rightarrow \mathbb{F}_p^n, (a, b) \mapsto \operatorname{Tr}(ab)$ is non-degenerate, then $\langle v, w \rangle = 0$ for all $w \in V$. Finally, since $\langle -, - \rangle$ is non-degenerate $v = 0$, so $\omega$ is non-degenerate.

Let $(V, \omega)$ be a symplectic module. By Proposition 2.8 there exists $\alpha \in Z^2(V, \mathbb{C}^*)$ such that $\omega = \operatorname{Alt}(\alpha)$ and

$$\operatorname{Sp}(V, \omega) = \{g \in \operatorname{Aut}(V) | \exists \eta \in C^1(V, \mathbb{C}^*) : \frac{\alpha^g}{\alpha} = \delta(\eta)\}.$$ 

The tuple $(V, k, V, \alpha, \eta_\alpha)$ is a torsor datum. Hence, $A = \mathbb{C}_\alpha[V]$ is a simple $V$-Galois algebra and by Lemma 3.5

$$\operatorname{Aut}_V(A) = \hat{V}.$$ 

Let us fix for every $g \in \operatorname{Sp}(V, \omega)$ a cochain $\eta_\beta \in C^1(V, \mathbb{C}^*)$ such that $\frac{\alpha^g}{\alpha} = \delta(\eta_\beta)$.

The map $\alpha_g : A \rightarrow A^{(g)}, u_x \mapsto \eta_\beta(x)u_{g(x)}$ is an isomorphism of $V$-algebras. Thus, by Proposition 4.3 there is an associated crossed product group

$$\operatorname{ASp}(V, \omega) := \hat{V} \#_\theta \operatorname{Sp}(V, \omega).$$

Following [EG01] we call $\operatorname{ASp}(V, \omega)$ the affine pseudo-symplectic group.

The Weil representation of the affine pseudo-symplectic group $\operatorname{ASp}(V, \omega)$ is the projective representation associated to the Weil action of $\operatorname{ASp}(V, \omega)$ on the simple algebra $A = \mathbb{C}_\alpha[V]$; see Proposition 4.3.

4.5. The Weil representation associated to a quadratic module.

Definition 4.8. An abelian group $V$ isomorphic to $(\mathbb{Z}/n\mathbb{Z})^r$ will be called a homogeneous module. The number $r$ is called the rank and $n$ the exponent of $V$.

We will denote by $\mu_n$ the group of all roots of unity of order $n$ in $\mathbb{C}$.

A quadratic form on a homogeneous module $V$ of exponent $n$ is a map $q : V \rightarrow \mu_n$ such that $q(x) = q(x^{-1})$ and the map

$$\omega_q : V \times V \rightarrow \mathbb{C}^*$$

$$(x, y) \mapsto \frac{q(xy)}{q(x)q(y)}$$

is a bicharacter. The group of all quadratic forms on a homogeneous module $V$ will be denoted by $\operatorname{Quad}(V)$. 
Definition 4.9. A quadratic module is a pair \((V, q)\), where \(V\) is a finite homogeneous module and \(q\) is a quadratic form on \(V\) such that the associated skew-symmetric form \(\omega_q\) is non-degenerate.

Given a quadratic module \((V, q)\) the orthogonal group of \((V, q)\) is defined as
\[
O(V, q) = \{g \in \text{Aut}(V) | q(x) = q(g(x)), \forall x \in V\}.
\]

Example 4.10. Let \(k\) be a field and \(V\) a finite dimensional \(k\)-vector space. Recall that a (usual) quadratic form is a function \(q : V \rightarrow k\) such that \(q(ax + by) = a^2q(x) + abB(x, y) + b^2q(y)\) for all \(a, b \in k, x, y \in V\), where \(B\) is a \(k\)-bilinear form on \(V\). The form \(q\) is called non-defective if the bilinear form \(B\) is non-degenerate. Note that if \(k\) has characteristic different from 2, the quadratic form is totally determined by \(B\). A quadratic (linear) space is a pair \((V, q)\) where \(V\) is a linear space and \(q\) is a quadratic form whose associated bilinear form is non-degenerate. Given a linear quadratic module \((V, q)\) the linear orthogonal group of \((V, q)\) is defined as
\[
O_k(V, q) = \{g \in \text{GL}_k(V) | q(x) = q(g(x)), \forall x \in V\}.
\]

Let \(V\) be an \(\mathbb{F}_{p^n}\)-vector space and \(q : V \rightarrow \mathbb{F}_{p^n}\) a quadratic form. Let us define the quadratic form
\[
\widehat{q} : V \rightarrow \mu_p,
\]
\[
v \mapsto e^{\frac{2\pi i \text{Tr}(q(x))}{p}}.
\]

Proposition 4.11. If \(q\) is non-defective, then the pair \((V, \widehat{q})\) is a quadratic module of exponent \(p\) and the linear orthogonal group \(O_{\mathbb{F}_{p^n}}(V, q)\) is a subgroup of \(O(V, \widehat{q})\).

Proof. For all \(v \in V\),
\[
\widehat{q}(-v) = e^{\frac{2\pi i \text{Tr}(q(-v))}{p}} = e^{\frac{2\pi i (-1)^2 \text{Tr}(q(v))}{p}} = e^{\frac{2\pi i \text{Tr}(q(v))}{p}} = \widehat{q}(v).
\]
Let \(B\) be the \(\mathbb{F}_{p^n}\)-bilinear form associated to \(q\). Then
\[
\omega_{\widehat{q}}(x, y) = \frac{\widehat{q}(x + y)}{\widehat{q}(x)\widehat{q}(y)} = e^{\frac{2\pi i \text{Tr}(B(x, y))}{p}}
\]
for all \(x, y \in V\). Thus, \(\omega_{\widehat{q}}\) is a skew-symmetric form. It follows from Proposition 4.7 that \(\omega_{\widehat{q}}\) is non-degenerate. \(\square\)

For a homogeneous module \(V\) we denote by \(\text{Bil}(V)\) the abelian group of all bicharacters with values in \(\mathbb{C}^*\). The map \(\text{Tr} : \text{Bil}(V) \rightarrow \text{Quad}(V), \text{Tr}(b)(x) = b(x, x)\) defines a morphism of groups with kernel \(\bigwedge^2 \widehat{V}\).

Lemma 4.12. Let \(V\) be a finite homogeneous module. The sequence
\[
0 \rightarrow \bigwedge^2 \widehat{V} \rightarrow \text{Bil}(V) \rightarrow \text{Quad}(V) \rightarrow 1
\]
is exact.

Proof. It is clear that we only need to show that the trace map
\[
\text{Tr} : \text{Bil}(V) \rightarrow \text{Quad}(V)
\]
is surjective. If \(V = \langle g \rangle\) has rank one (that is, if \(V\) is a cyclic group of order \(n\)) and \(q \in \text{Quad}(V)\), then \(b : V \times V \rightarrow \mu_n, b(g^i, g^j) = q(g)^{ij}\) is a bilinear map with \(\text{Tr}(b) = q\). Let \(V = A \oplus C\) where \(C\) has rank one, let \(q \in \text{Quad}(V)\) and suppose that
for \(q|_A\) and \(q|_C\) we have \(b_A \in \text{Bil}(V)\), \(b_C \in \text{Bil}(C)\) with \(\text{Tr}(b_A) = q|_A\), \(\text{Tr}(b_C) = q|_C\). Then
\[
b : V \times V \to \mu_n,
a + c, a' + c' \mapsto b_A(a, a')b_C(c, c')\omega_q(a, c')
\]
is a bicharacter with \(\text{Tr}(b) = q\).

Let \(V\) be a finite homogeneous module of exponent \(n\). We define
\[
C^1_0(V, \mu_n) = \{\eta \in C^1(V, \mu_n) | \delta(\eta) \text{ is a bicharacter and } \eta(x^2) = \eta(x)^2, \forall x \in V\}
\]
and the morphism
\[
\delta : C^1_0(V, \mu_n) \to \bigwedge^2 \hat{V}
\]
\[
\eta \mapsto \delta(\eta).
\]

**Lemma 4.13.** The sequence
\[
0 \to \hat{V} \to C^1_0(V, \mu_n) \to \bigwedge^2 \hat{V} \to 1
\]
is exact.

**Proof.** Let \(b \in \bigwedge^2 \hat{V}\). Then, \(1 = b(xy, xy) = b(x, y)b(y, x)\), which implies that \(b(x, y) = b(y, x)^{-1}\) for all \(x, y \in V\).

If \(V\) is cyclic, then \(\bigwedge^2 \hat{V} = 0\) and \(\hat{V} = C^1_0(V, \mu_n)\). Let \(V = A \oplus C\), where \(C\) has rank one. Suppose that there exists \(\eta_A \in C^1_0(A, \mu_n)\) such that \(\delta(\eta_A) = b|_{A \times A}\). Let \(\hat{\eta}_A \in C^1_0(V, \mu_n)\), by \(\hat{\eta}_A(a + c) = \eta_A(a)\). Then \(b' : = b\delta(\hat{\eta}_A)^{-1} \in \bigwedge^2 \hat{V}\) and \(b'|_{A \times A} = 1\). We define \(s \in C^1(V, \mu_n)\) by \(s(a + c) = b(a, c)^{-1}\). A simple calculation shows that \(\delta(s) = b'\). Therefore, \(s \in C^1_0(V, \mu_n)\). Finally, for \(\gamma : = \hat{\eta}_A s \in C^1_0(V, \mu_n)\) we have \(\delta(\gamma) = \delta(s)\delta(\hat{\eta}_A) = b\).

**Corollary 4.14.** Let \((V, q)\) be a quadratic module and let \(b \in \text{Bil}(V)\) satisfy \(\text{Tr}(b) = q\). Then
\[
O(V, q) = \{g \in \text{Aut}(V) | \exists \eta \in C^1_0(V, \mu_n) : b^g/b = \delta(\eta)\}.
\]

**Proof.** It is clear that
\[
\{g \in \text{Aut}(V) | \exists \eta \in C^1_0(V, \mu_n) : b^g/b = \delta(\eta)\} \subseteq O(V, q).
\]

Let \(b \in \text{Bil}(V)\) be such that \(\text{Tr}(b) = q\) and \(g \in O(V, q)\). By Lemma 4.12 \(b^g/b \in \bigwedge^2 \hat{V}\), and by Lemma 4.13 there exists \(\eta \in C^1_0(V, \mu_n)\) such that \(\delta(\eta) = b^g/b\). Therefore,
\[
O(V, q) = \{g \in \text{Aut}(V) | \exists \eta \in C^1_0(V, \mu_n) : b^g/b = \delta(\eta)\}.
\]
is an isomorphism of a $V$-algebra. We denote the crossed product group associated to Proposition 4.3 as
\[ \text{Ps}(V, q) := \hat{V} \#_θ O(V, q) \]
and call it the pseudo-symplectic group; see [Wei64] and [Bla93].

The Weil representation of $\text{Ps}(V, q)$ is the projective representation associated to the Weil action of $\text{Ps}(V, q)$ on the simple algebra $A = \mathbb{C}_b[V]$.

**Remark 4.15.**
- If $(V, q)$ is a quadratic module with $V$ homogeneous of exponent $n$, then the Weil representation of $\text{Ps}(V, q)$ is defined over $\mathbb{Z}[\exp(\frac{2\pi i}{n})]$.
- The pseudo-symplectic group $\text{Ps}(V, q)$ is a proper subgroup of the affine pseudo-symplectic $\text{APs}(V, \omega_q)$, and the Weil representation of $\text{APs}(V, \omega_q)$ is an extension of the Weil representation of $\text{Ps}(V, q)$.

## 5. Examples of non-isomorphic isocategorical groups and Weil representations

The goal of this section is to construct some concrete examples of non-isomorphic isocategorical groups.

**Lemma 5.1.** Let $S$ be a finite abelian group and $(S, K, N, σ, γ)$ a torsor datum over $S$ with associated simple $S$-Galois algebra $B := A(K_σ[N], γ)$. Then
\[ \text{Aut}_S(B) \cong \hat{N} \oplus \text{Gal}(K|k) \]
and
\[ \text{St}(B) \cong \text{St}([σ, γ]) := \{ g \in \text{Aut}_N(S) : [(σ, γ)] = [(σ^g, γ^g)] \in H^2_{\text{Gal}(K|k)}(N, K^*) \}, \]
where $\text{Aut}_N(S) = \{ g \in \text{Aut}(S) : g|_N \in \text{Aut}(N) \}$.

**Proof.** The isomorphism $\text{Aut}_S(B) \cong \hat{N} \oplus \text{Gal}(K|k)$ follows from Lemma 3.5. The isomorphism $\text{St}(B) \cong \text{St}([σ, γ])$ follows from the proof of Theorem 3.9. \[\square\]

**Theorem 5.2.** Let $S$ be a finite abelian group and $(S, K, N, σ, γ)$ a torsor datum over $S$, with associated simple $S$-Galois algebra $B := A(K_σ[N], γ)$. Then the semidirect product group
\[ (\hat{N} \oplus \text{Gal}(K|k)) \rtimes \text{St}([σ, γ]) \]
and the crossed product group
\[ (\hat{N} \oplus \text{Gal}(K|k)) \#_θ \text{St}([σ, γ]) \]
(see Proposition-Definition 4.3) are isocategorical over $k$.

**Proof.** Follows from Theorem 3.9 and Proposition 4.2. \[\square\]

Let $(V, q)$ be a linear quadratic space over a finite field of characteristic $p$. Let $(V, \bar{q})$ be the quadratic module associated to Example 4.10. The *linear* pseudo-symplectic group is defined as
\[ \text{Ps}_k(V, q) := \hat{V} \#_θ O_k(V, q) \subset \hat{V} \#_θ O(V, \bar{q}) = \text{Ps}(V, \bar{q}), \]
and the Weil representation of $\text{Ps}_k(V, q)$ is by definition the restriction of the Weil representation of $\text{Ps}(V, \bar{q})$. 

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If \((V, \{-, -\})\) is a symplectic space over a finite field \(k\), by Example 4.6 there is an associated symplectic module \((V, \omega)\) and \(\text{Sp}_k(V) \subset \text{Sp}(V, \omega)\), where \(\text{Sp}_k(V)\) is the linear symplectic group.

The linear affine pseudo-symplectic group is defined as

\[
\text{APs}_k(V) := \hat{V} \#_{\#} \text{Sp}_k(V) \subset \text{APs}(V, \omega),
\]

and its Weil representation is the restriction of the Weil representation of \(\text{APs}(V, \omega)\).

We will denote by \(\Omega_k(V, q)\) the subgroup of index 2 in \(O_k(V, q)\), for which the Dickson invariant is zero; see [Gro02] for details and the basic properties of \(\Omega_k(V, q)\). The next proposition is a generalization of [Gri73, Theorem 1] to arbitrary finite fields of characteristic two.

**Proposition 5.3.** Let \(k\) be a finite field of characteristic two and \((V, q)\) a quadratic space over \(k\), where \(\dim_k(V) = 2n\) and \(n \geq 4\). Then the exact sequences

\[
0 \to \hat{V} \to \text{APs}_k(V) \to \text{Sp}_k(V, \omega_q) \to 1,
\]

\[
0 \to \hat{V} \to \text{P} \to \text{Sp}_k(V, q) \to O_k(V, q) \to 1,
\]

\[
0 \to \hat{V} \to \hat{V} \#_{\#} \Omega_k(V, q) \to \Omega_k(V, q) \to 1
\]

are non-split.

**Proof.** The proof consists of applying [Gri73, Theorem 0] to our exact sequences. We need to show that there exists a subgroup \(W = \{x, y, z, 0\} \subset \hat{V}\) such that \(x\) and \(y\) are singular (that is, \(q(x) = q(y) = 0\)), \(z = x + y\) is non-singular (that is, \(q(z) \neq 0\)) and there exists a subgroup \(G \subset \Omega_k(V, q) \subset O_k(V, q) \subset \text{Sp}_k(V, \omega_q)\) satisfying

1. \(G\) fixes \(z\),
2. \(G\) has an involution \(t\), with \(t(x) = y\),
3. \(G\) has no subgroup of index 2.

Let \((x, y) \subset V\) a hyperbolic pair, that is, \(q(x) = q(y) = 0\) and \(B_q(x, y) = 1\). Thus, \(z = x + y\) is non-singular. Let \(G = \text{Stab}_z(\Omega_k(V, q))\) and \(V = \langle z \rangle_k \oplus H\).

Then, the canonical map \(G \to O_k(H, q)\) is an isomorphism and since \(\dim_k(H)\) is odd, \(O_k(H, q) \cong \text{Sp}(2n - 2)\) (see [KL90, Proposition 4.1.7]). Therefore, we can find an involution \(t \in G\) such that \(t(x) = y\). Since \(n \geq 4\), \(\text{Sp}(2n - 2)\) is simple. Thus, \(G\) has no subgroups of index 2. \(\square\)

**Proposition 5.4.** Let \((V, q)\) be quadratic spaces over a finite field of characteristic two with \(\dim_k(V) = 2n\) and \(n \geq 4\). The affine symplectic group \(\text{ASp}_k(V)\) and the affine pseudo-symplectic group \(\text{APs}_k(V, \omega_q)\) are isocategorical over \(\mathbb{C}\) but not over \(\mathbb{R}\).

**Proof.** If we view the skew-symmetric form \(\omega_q\) as an element in \(Z^2(V, \mathbb{C}^*)\), then \(\text{St}([\sigma]) = \text{Sp}(V, \omega_q)\). Thus, by Theorem 5.2, the groups \(\text{ASp}_k(V)\) and \(\text{APs}_k(V, \omega_q)\) are isocategorical over the field of complex numbers.

Let \(S \triangleleft \text{ASp}(V)\) be an abelian normal subgroup. Since \(\text{Sp}(V)\) is a non-abelian simple group and \(V \triangleleft SV \triangleleft \text{ASp}(V)\), it follows that \(S/S \cap V \cong SV/V \triangleleft \text{Sp}(V)\), thus \(S \subset V\). Since \(S \triangleleft \text{ASp}(V)\), it follows that \(S = V\). Thus the unique normal abelian subgroup of \(\text{ASp}(V)\) is \(V\). Hence, a semi-real torsor datum over \(\text{ASp}(V)\) does not exist. Let \((V, \mu)\) exist, where \(\mu \in H^2(V, \mu_2)\) is a non-degenerate cohomology class. By Lemma 4.12 and Lemma 4.13, \(\text{St}([\mu]) = O_k(V, q)\) and the orthogonal group is a proper subgroup of the symplectic group. Therefore, the pair \((V, \mu)\) is not a real torsor. If follows that \(\text{ASp}(V)\) does not have real or semi-real torsor datum; hence
by Theorem 3.14 every group isocategorical to the ASp(V) group is isomorphic to ASp(V). □

**Proposition 5.5.** Let (V, q) be a quadratic linear space over a finite field of characteristic two. The following pairs of groups are concrete examples of non-isomorphic groups isocategorical over \(\mathbb{Q}\):

1. \(\tilde{V} \rtimes O_k(V, q)\) and \(\text{Ps}(V, q)\), where \(\dim_k(V) = 2n, n \geq 4\),
2. \(\tilde{V} \rtimes \Omega_k(V, q)\) and \(\tilde{V} \sharp \Omega_k(V, q)\), where \(\dim_k(V) = 2n, n \geq 4\),
3. \(\mathbb{F}_2^6 \rtimes P_2\) and \(\mathbb{F}_2^6 \# P_2\), where \(P_2\) is a Sylow 2-subgroup of \(O(\mathbb{F}_2^6, q)\).

**Proof.** By Theorem 5.2 we only need to check that the pairs of groups are non-isomorphic. By Proposition 5.3, the groups in (2) and (3) are non-isomorphic. Now, we need to see that the groups \(\mathbb{F}_2^6 \rtimes P_2\) and \(\mathbb{F}_2^6 \# P_2\) are non-isomorphic. This can be done easily using the function IsIsomorphicPGroup(G,R), included in the GAP package ANUPQ [GAP13]. □

**Remark 5.6.** For the construction of \(\mathbb{F}_2^6 \rtimes P_2\) and \(\mathbb{F}_2^6 \# P_2\) in GAP it is useful to know that the exact sequence

\[
1 \rightarrow \tilde{V} \rightarrow \text{Ps}(\mathbb{F}_2^6, V, q) \rightarrow O(\mathbb{F}_2^6, q_c) \rightarrow 1
\]

is isomorphic to the exact sequence

\[
1 \rightarrow \text{Inn}(E) \rightarrow \text{Aut}(E) \rightarrow \text{Out}(E) \rightarrow 1,
\]

where \(E\) is an extra-special 2-group and \(\text{Inn}(E), \text{Out}(E)\) are the groups of inner and outer automorphisms of \(E\) respectively; see [Gri73].

5.1. **Semi-real torsor data and real Weil representations associated to finite symplectic modules.** The aim of this section is to describe a systematic way of constructing torsor data with \(\text{Gal}(K/k)\) non-trivial.

Let \(N\) be a finite abelian group, \(k\) a field with a primitive root of unity of order the exponent of \(N\), \(\sigma \in Z^2(N, k^*)\) a non-degenerate 2-cocycle and \(k \subset K\) an abelian Galois field extension.

For the abelian group \(S := N \oplus \text{Gal}(K/k)\) we define the torsor datum \((S, K, N, \gamma_\sigma)\), where \(\gamma_\sigma : S \times N \rightarrow k^*\) is the pairing defined by \(\gamma|_{\text{Gal}(K/k) \times N} = 1\) and \(\gamma_\sigma|_{N \times N} = \text{Alt}(\sigma)\).

Let us define

\[
X = \{g \in \text{Aut}(N)|[\sigma] = [\sigma^g]\} \text{ as elements in } H^2(N, K^*)
\]

and

\[
Y = \{g \in \text{Aut}_N(S)|[(\sigma, \gamma_\sigma)]^g = [(\sigma, \gamma_\sigma)]\} \text{ as elements in } H^2_S(N, K^*)
\]

where \(\text{Aut}_N(S) = \{g \in \text{Aut}(S)|g|_N \in \text{Aut}(N)\}\).

**Theorem 5.7.** The restriction map

\[
r : Y \rightarrow X, g \mapsto g|_N
\]

defines an exact sequence of groups

\[
1 \rightarrow \text{Aut}(\text{Gal}(K/k)) \rightarrow Y \rightarrow X \rightarrow 1.
\]
The crossed product group isocategorical over \( (\text{see Proposition-Definition 4.3}) \) contains the pseudo-symplectic groups and it is

\[
\text{Proof. Let us first prove that the kernel of } r : Y \to X, g \mapsto g|_N \text{ is isomorphic to } \text{Aut}(\text{Gal}(K|k)). \text{ Let } g \in Y \text{ and } \eta_g : N \to K^* \text{ such that } (\sigma^g, \gamma^g) = \partial(\eta_g)(\sigma, \gamma_\sigma)\text{. If } g\mid_N = \text{id}_N, \text{ then } \eta_g : N \to K^* \text{ is a character and } \eta_g(x) \in k^* \text{ for all } x \in N. \text{ Therefore, } \gamma(g(a), x) = \gamma(a, x) \text{ for all } a \in \text{Gal}(K|k), x \in N. \text{ Thus, } \gamma(g(a)^{-1}, x) = 1 \text{ for all } a \in \text{Gal}(K|k), x \in N. \text{ Therefore, } g(a)^{-1} \in \text{Gal}(K|k), \text{ that is, } g(a) \in \text{Gal}(K|k) \text{ for all } a \in \text{Gal}(K|k). \]

Now we want to show that \( r \) is surjective. Let \( (\text{see Proposition-Definition 4.3}) \) be a symplectic module of exponent two, for example, the symplectic module associated to a symplectic linear space over finite field of characteristic two; \( \omega \) is a semi-real torsor datum over the group \( \text{Alt}(\text{Aut}(\text{Gal}(K|k))) \). Let \( g \in Y \) and \( \eta_g : N \to K^* \) such that \( \sigma^g/\sigma = \delta(\eta_g) \). For each \( a \in \text{Gal}(K|k) \), the map \( \eta_g(a\eta_g) \) is a character that does not depend on the choice of \( \eta_g \). Then, there exists a unique \( n(g,a) \in N \) such that \( \eta_g(a\eta_g) = \text{Alt}(\text{Alt}(\text{Aut}(\text{Gal}(K|k))) (n(g,a), g(x))) \) for all \( x \in N \).

Since

\[
\text{Alt}(\sigma)(n(g,a)n(g,b), g(x)) = (\eta_g/a(\eta_g))\eta_g/b(\eta_g)
\]

\[
= (\eta_g/a(\eta_g))\eta_g/ab(\eta_g)
\]

\[
= \eta_g/ab\eta_g
\]

\[
= \text{Alt}(\sigma)(n(g,ab), g(x))
\]

it follows that \( n(g,ab) = n(g,a)n(g,b) \), where we have used that \( \eta/\eta b(\eta) \in k^* \) for all \( x \in N, b \in A \). Define \( g' \in \text{Aut}_N(N \oplus A) \) by \( g'(x \oplus a) = g(x)n(g,a) \oplus a \). Then

\[
\frac{\gamma^{g'}(x \oplus a, y)}{\gamma(x \oplus a, y)} = \frac{\gamma(g'(x \oplus a), y)}{\gamma(x \oplus a, y)}
\]

\[
= \frac{\gamma(g(x)n(g,a) \oplus a, g(y))}{\gamma(x,y)\gamma(a,y)}
\]

\[
= \frac{\gamma(g(x), g(y))\gamma(n(g,a), g(y))\gamma(a, g(y))}{\gamma(x,y)\gamma(a,y)}
\]

\[
= \frac{\gamma(n(g,a), g(y))}{\eta_g(a\eta_g)(y)} = \frac{\eta_g(a\eta_g)(y)}{a\eta_g(y)}
\]

so \( g' \in Y \) and \( r(g') = g \).

**Remark 5.8.**

- The group \( Y \) only depends on \( [\sigma] \in H^2(N, K^*) \).
- If \( \text{Gal}(K|k) = \mathbb{Z}/2\mathbb{Z} \), then \( X \cong Y \).

Let \( (V, \omega) \) be a symplectic module of exponent two, for example, the symplectic module associated to a symplectic linear space over finite field of characteristic two; see Example 4.10. Since \( \omega \in Z^2(V, \mathbb{R}^*) \subset Z^2(V, \mathbb{C}^*) \), it follows by Theorem 5.7 that the tuple

\[
(\hat{V} \oplus \text{Gal}(\mathbb{C}|\mathbb{R}), \mathbb{C}, \omega, \gamma_\omega)
\]

is a semi-real torsor datum over the group

\[
(\hat{V} \oplus \text{Gal}(\mathbb{C}|\mathbb{R})) \rtimes \text{Sp}(V, \omega).
\]

The crossed product group

\[
(\hat{V} \oplus \text{Gal}(\mathbb{C}|\mathbb{R})) \rtimes \text{Sp}(V, \omega)
\]

(see Proposition-Definition 4.13) contains the pseudo-symplectic groups and it is isocategorical over \( \mathbb{Q} \) to \( (\hat{V} \oplus \text{Gal}(\mathbb{C}|\mathbb{R})) \rtimes \text{Sp}(V, \omega) \).
The group \( (\hat{V} \oplus \text{Gal}(\mathbb{C}|\mathbb{R}))^\# \text{Sp}(V, \omega) \) acts by algebra automorphisms on the rational simple algebra \( \mathbb{Q}_\omega[V] \). Once a simple module of \( \mathbb{Q}_\omega[V] \) is fixed, we have a rational projective representation of \( (\hat{V} \oplus \text{Gal}(\mathbb{C}|\mathbb{R}))^\# \text{Sp}(V, \omega) \) that extends the rational Weil representations of the pseudo-symplectic groups \( \text{Ps}(V, q) \).

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References


[DM82] Pierre Deligne and James S. Milne, Tannakian Categories, volume 900 of Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1982 (see MR0654325 (84m:14046)).


[EML53] Samuel Eilenberg and Saunders Mac Lane, On the groups of \( H(\Pi, n) \). I, Ann. of Math. (2) 58 (1953), 55–106. MR0056295


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