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PERMUTATION INVARIANT FUNCTIONALS OF LÉVY PROCESSES

F. BAUMGARTNER AND S. GEISS

Abstract. We study natural invariance properties of functionals defined on Lévy processes and show that they can be described by a simplified structure of the deterministic chaos kernels in Itô's chaos expansion. These structural properties of the kernels relate intrinsically to a measurability with respect to invariant σ-algebras. This makes it possible to apply deterministic functions to invariant functionals on Lévy processes while keeping the simplified structure of the kernels. This stability is crucial for applications. Examples are given as well.

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Introduction

In recent years, Itô's chaos expansion [13] for Lévy processes was applied to investigate various problems in stochastic analysis and stochastic process theory. For example, it was used to investigate quantitative properties of stochastic processes in continuous time or to prove covariance relations and inequalities, like the Poincaré inequality, for general Poisson processes; see [5, 8, 9, 12, 15]. Given a Lévy process $X = (X_t)_{t\in[0,1]}$ and letting $L_2(\mathcal{F}^X) := L_2(\Omega, \mathcal{F}^X, \mathbb{P})$, with $\mathcal{F}^X$ being the completion of $\sigma(X_t : t \in [0,1])$, the chaos expansion is an orthogonal decomposition

$$L_2(\mathcal{F}^X) = L_2 - \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

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where $F \in L_2(\mathcal{F}^X)$ is decomposed into

$$F = \sum_{n=0}^{\infty} I_n(f_n). \tag{0.1}$$

The functions $f_n \colon ((0,1] \times \mathbb{R})^n \to \mathbb{R}$ are symmetric and belong to

$$L_2^n = L_2(((0,1] \times \mathbb{R})^n, (\mathcal{B}((0,1]) \otimes \mathcal{B}(\mathbb{R}))^{\otimes n}, \mathrm{m}^{\otimes n}),$$

where m is a σ-finite measure derived from the Lévy measure ν of X, and in $f_n((t_1,x_1),\dots,(t_n,x_n))$ the variables $t_1,\dots,t_n$ represent the time and $x_1,\dots,x_n$ the state space. The expressions $I_n(f_n)$ are multiple integrals with respect to a random measure associated with the process $(X_t)_{t\in[0,1]}$. At first glance, the chaos expansion is a perfect tool to describe L_2-random variables by deterministic objects, the chaos kernels. In fact, various stochastic properties of F transfer to or can be seen by means of the kernel functions f_n. For example, measurability with respect to $\mathcal{F}_t^X$, the completion of $\sigma(X_s : s \in [0,t])$, can be checked by the support of the f_n. Malliavin differentiability or fractional Malliavin differentiability obtained by real interpolation can be formulated by moment conditions on the kernels [8]. Another example can be found in the initial paper of Itô [13], where the chaos expansion was introduced and used to investigate the spectral type of operators that are induced by a time shift of the underlying process (with the time domain $(-\infty,\infty)$). This is a first example to investigate Lévy-Wiener type spaces by the structure of the chaos kernels in the chaos representation.

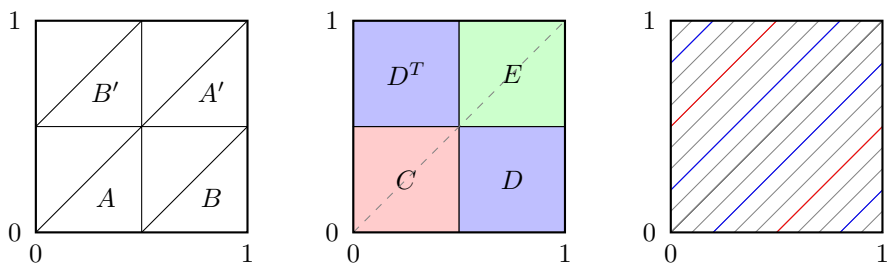
A general obstacle for the application of the chaos representation is the fact that the chaos kernels depend on an increasing number of coordinates. As a result, their structure gets involved and computations become difficult or sometimes impossible although one can represent the kernel functions in certain situations: using difference operators or Malliavin derivatives, kernel representations are obtained in [10] and [15] by iterated derivatives where differential properties of F are needed in the presence of the Brownian motion part (or see [26], where powers of increments of the Lévy process are considered). An account on involved combinatorial aspects of chaos decompositions and applications, including multiplication formulas, can be found in [18].

The aim of this paper is to restrict the chaos expansion (0.1) to $F \in \mathbf{H} \subseteq L_2(\mathcal{F}^X)$, where $\mathbf{H}$ is an appropriate closed linear subspace, and to make the expansion applicable in various situations while keeping essential properties of the chaos expansion. Applicable means that we reduce the complexity of the kernels by taking into account natural invariance properties induced by permutation groups, so that the kernels can be handled even if the dimension of the chaos gets large, in particular, an explicit computation of the kernels will not be needed. The results are required in recent developments of stochastic analysis and stochastic process theory. In Example 6 below we explain how our results were applied in [12] in the context of BSDEs. Summarizing, we have two goals: first, we want to present results that are needed in recent developments, second we continue the line of research from Itô [13].

To explain the invariance properties we have in mind we look at the three elementary examples

$$\begin{aligned} F_1 &:= \Phi_1\left(\int_{(0,1/2]} \varphi_t \mathrm{d}X_t, \int_{(1/2,1]} \varphi_{t-\frac{1}{2}} \mathrm{d}X_t\right), \\ F_2 &:= \Phi_2\big([X]_{1/2}, [X]_1 - [X]_{1/2}\big), \\ F_3 &:= \int_0^1 \int_0^t h(t-s)\mathrm{d}W_s \mathrm{d}W_t. \end{aligned}$$

Here, $\varphi\colon [0,1/2] \to \mathbb{R}$ is continuous, $\Phi_1\colon \mathbb{R}^2 \to \mathbb{R}$ symmetric, bounded and measurable, $\Phi_2\colon \mathbb{R}^2 \to \mathbb{R}$ bounded and measurable, but not necessarily symmetric, and $h\colon [0,1] \to \mathbb{R}$ is bounded and measurable with the symmetry $h(1/2-r) = h(1/2+r)$ for $r \in [0,1/2]$. Moreover, W is the normalized Brownian part of X and $[X]$ denotes the quadratic variation process of X; see [20, Section II.6]. The time variables of the corresponding kernels appearing in the second summand of the Itô chaos expansion of the random variables F_1, F_2, F_3 have symmetries that correspond to the pictures below:



Example F_1 Example F_2 Example F_3

In fact, there are two interacting symmetry groups: the general symmetry in (t_1, x_1) and (t_2, x_2), and the symmetries that come from Φ_1, the bracket process $([X]_t)_{t\in[0,1]}$ and from h.

Example F_1 is invariant with respect to an interchange of the Lévy process on $(0, \frac{1}{2}]$ with the process on $(\frac{1}{2}, 1]$ in the sense that $(X_t)_{t\in[0,1]}$ is replaced by

$$Y_t := \begin{cases} X_{t+1/2} - X_{1/2}, & t \in [0,1/2], \\ (X_1 - X_{1/2}) + X_{t-1/2}, & t \in (1/2, 1]. \end{cases}$$

Freezing the state variables (x_1, x_2) of the kernel leads to a symmetry in the time variables (t_1, t_2), where the areas A' resp. B' are copies of A resp. B obtained by a *shift*. The remaining parts are determined by the symmetry in (t_1, x_1) and (t_2, x_2).

Example F_2: Similarly as described above, the Lévy process can be exchanged on intervals within $(0, \frac{1}{2}]$ resp. $(\frac{1}{2}, 1]$. Later we show that this immediately results in the structure

$$\begin{aligned} f_2((t_1,x_1),(t_2,x_2)) = {} & \mathbb{1}_C(t_1,t_2) g_C(x_1,x_2) + \mathbb{1}_E(t_1,t_2) g_E(x_1,x_2) \\ & + \mathbb{1}_D(t_1,t_2) g_D(x_1,x_2) + \mathbb{1}_{D^T}(t_1,t_2) g_D(x_2,x_1), \end{aligned}$$

where the functions g_C and g_E appearing in the diagonal terms are already symmetric.

Example F_3: As we only consider the Brownian motion, there is no dependence of the kernel on the state variables x_1 and x_2. Directly, from the symmetries of h

one checks that the kernel is constant in time along the lines, whereas on lines with the same color the kernel takes the same values.

In this article, symmetries of this kind are the basis to restrict the chaos expansion to a subspace $\mathbf{H}$. Let us list some desired abstract properties of this restricted chaos expansion and describe how the structure of the paper is derived from their treatment:

(S) Stability: Given random variables $F_1, \dots, F_N \in \mathbf{H}$ and an appropriate bounded random functional $f\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$, such that $f(\cdot, x) \in \mathbf{H}$ for all $x \in \mathbb{R}^N$, we would like to guarantee that $f(F_1, \dots, F_N) \in \mathbf{H}$.

(C) Consistency: We consider three different stages of compatibility of $\mathbf{H}$ with the original chaos decomposition.

(C1) Are there closed linear subspaces $\mathbf{H}_n \subseteq \mathcal{H}_n$ such that

$$\mathbf{H} = L_2 - \bigoplus_{n=0}^{\infty} \mathbf{H}_n?$$

(C2) Can the subspaces $\mathbf{H}$ and $\mathbf{H}_n$ be obtained by measurability, i.e., are there σ-algebras $\mathcal{A}$ and $\mathcal{A}_n$ such that

$$\mathbf{H} = L_2(\Omega, \mathcal{A}, \mathbb{P}) \quad \text{and} \quad \mathbf{H}_n = I_n\Big(L_2\big(((0,1] \times \mathbb{R})^n, \mathcal{A}_n, \mathrm{m}^{\otimes n}\big)\Big)?$$

(C3) Can one realize $\mathcal{A}_n = \mathcal{A}_1^{\otimes n}$?

(G) Generating property of $\mathbf{H}_1$: Does one have that

$$\mathcal{A} = \sigma(F \in \mathbf{H}_1) \vee \{A \in \mathcal{F}^X : \mathbb{P}(A) = 0\}?$$

Chaos expansions based on multiple integrals with respect to a centered independently scattered random measure (also called a centered completely random measure) are usually proved under the condition that the control measure is non-atomic (see [18, Chapter 5.1]). Starting with a non-atomic control measure, in Theorem 4 below we also obtain chaos decompositions with control measures that are *not* non-atomic, but sharing the desirable basic properties (S), (C3), and (G) from above. This could open a way to transfer properties and results from the non-atomic case to the atomic one.

Before we proceed let us make some detailed comments on the above set of conditions:

Remark 1.

(1) Roughly speaking, property (C2) is stronger than the stability property (S): If the map $\omega \mapsto f(\omega, F_1(\omega), \dots, F_N(\omega))$ can be defined in a reasonable way, then the measurability will transfer automatically to the composition and implies $f(F_1, \dots, F_N) \in \mathbf{H}$ by (C2).

(2) The stability (S) excludes certain choices of $\mathbf{H}$ such as $\mathbf{H} = \mathcal{H}_n$ for some $n \geq 1$.

(3) The generating property (G) holds for Itô's chaos expansion as introduced above, and might be approached by orthogonal polynomials associated to certain Lévy processes (cf. [17, 19, 23]) in order to obtain other cases. For example, it holds for the Hermite expansion of the Gaussian space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \gamma_n)$ with γ_n being the standard Gaussian measure on $\mathbb{R}^n$, and for functionals $f(N_1)$, where $(N_t)_{t\in[0,1]}$ is a standard Poisson process by exploiting Charlier polynomials [19, Chapter 6].

(4) In general, condition (C2) does not imply (C3) nor (G): take for $\mathbf{H}'$ the space of random variables F that are invariant with respect to all dyadic permutations of the underlying Lévy process, and for $\mathbf{H}''$ the F that are invariant with respect to all dyadic periodic shifts of the underlying Lévy process. We have $\mathbf{H}' \subseteq \mathbf{H}''$ and in Section 6.1 we provide an example that $\mathbf{H}' \subsetneq \mathbf{H}''$. Because $g \in L_2((0,1])$ is a.s. constant if and only if g is a.s. invariant with respect to all periodic dyadic shifts, in both cases the first chaos coincides and equals

$$\mathbf{H}'_1 = \mathbf{H}''_1 = \{I_1(\mathbb{1}_{(0,1]}g_1) : \mathbb{1}_{(0,1]}g_1 \in L_2((0,1] \times \mathbb{R}, \mathrm{m})\},$$

where $(\mathbb{1}_{(0,1]}g_1)(t_1, x_1) = g_1(x_1)$. Using Theorem 4(2) below for $L = 1$ and $E_1 = (0,1]$ gives properties (C3) and (G) for $\mathbf{H}'$, so that (C3) and (G) cannot hold for $\mathbf{H}''$ as $\mathbf{H}' \subsetneq \mathbf{H}''$. This also means, although $g \in L_2((0,1])$ is a.s. constant whenever g is invariant with respect to all shifts, this phenomenon does not transfer to Itô's chaos representation.

Let us explain the structure of the paper along the above listed conditions. The answer to the problems of consistency (C1) and (C2) (and therefore the problem of stability (S)) is given in Section 3 by the following statement, which is part of Theorem 3.2 and Lemma A.2 (for notation see Sections 1 – 3).

Theorem 2. *For a group $\mathbb{G}$ of dyadic measure preserving maps*[1] $g\colon (0,1] \to (0,1]$ *and $F \in L_2(\mathcal{F}^X)$ the following assertions are equivalent:*

(1) *F is invariant with respect to all $\mathbb{G}$-induced permutations of the underlying Lévy process X.*

(2) *$F \in L_2(\Omega, \mathcal{H}_{\mathbb{G}}, \mathbb{P})$, where*

$$\mathcal{H}_{\mathbb{G}} := \sigma\Big(I_n(f_n) : f_n \text{ symmetric}, f_n = f_n \circ g[n] \text{ a.e.}, g \in \mathbb{G}, n \geq 1\Big) \vee \{A \in \mathcal{F}^X : \mathbb{P}(A) = 0\}$$

with $g[n]((t_1, x_1), \dots, (t_n, x_n)) := ((g(t_1), x_1), \dots, (g(t_n), x_n))$.

(3) *F has a chaos expansion with symmetric kernels*

$$f_n \in L_2(((0,1] \times \mathbb{R})^n, \mathcal{I}(\mathbb{G}[n]), \mathrm{m}^{\otimes n}),$$

where $\mathcal{I}(\mathbb{G}[n])$ is the invariant σ-algebra of the diagonal group $\mathbb{G}[n]$ on $((0,1] \times \mathbb{R})^n$ induced by $\mathbb{G}$.

That means that we have an orthogonal decomposition

$$L_2(\Omega, \mathcal{H}_{\mathbb{G}}, \mathbb{P}) = L_2 - \bigoplus_{n=0}^{\infty} I_n\Big(L_2(((0,1] \times \mathbb{R})^n, \mathcal{I}(\mathbb{G}[n]), \mathrm{m}^{\otimes n})\Big)$$

where for $n = 0$ we take the almost surely constant random variables. The examples F_1, F_2, and F_3 from the beginning fit into this theorem. In particular, for the case of shift invariant functionals, which corresponds to the setting in [13], we get the following example from Theorem 2.

[1]The concept of a dyadic measure preserving map g is defined in Definition 2.2 below.

Example 3. We call $g\colon (0,1] \to (0,1]$ a dyadic periodic shift if there is some integer $d \geq 1$ such that

$$g(t) = s_d(t) := \begin{cases} t + \frac{1}{2^d} & : t \in \left(\frac{k-1}{2^d}, \frac{k}{2^d}\right] \text{ and } 1 \leq k < 2^d, \\ t + \frac{1}{2^d} - 1 & : t \in \left(\frac{2^d-1}{2^d}, 1\right]. \end{cases}$$

A functional $F \in L_2(\mathcal{F}^X)$ is invariant with respect to all dyadic periodic shifts if and only if F is measurable with respect to

$$\mathcal{H}_{\text{shift}} := \sigma\Big(I_n(f_n) : f_n \text{ symmetric and } f_n = f_n \circ s_d[n],\ d, n \geq 1\Big) \\ \vee \{A \in \mathcal{F}^X : \mathbb{P}(A) = 0\},$$

where $s_d[n]$ is introduced in (3.1) below.

To handle conditions (C3) and (G) we introduce the concept of a *locally ergodic set* in Definition 4.1 below which yields a stronger invariance than for instance shift invariance. For the following, we let $\mathcal{O}((0,1])$ be the system of all unions of half-open dyadic intervals (including the empty set). For pairwise disjoint and non-empty $E_1, \ldots, E_L \in \mathcal{O}((0,1])$, we let

$$\mathcal{B}((0,1])_E := \mathcal{B}\Big((0,1] \setminus (E_1 \cup \cdots \cup E_L)\Big) \vee \sigma(E_1, \ldots, E_L).$$

As part of Theorem 5.3 below we prove

Theorem 4. *Let $E_1, \ldots, E_L \in \mathcal{O}((0,1])$ be pairwise disjoint and non-empty, and let $F \in L_2(\mathcal{F}^X)$.*

(1) *Let $\mathbb{G}$ be a group of dyadic permutations of $(0,1]$, let $E_1, \ldots, E_L$ be* locally ergodic *with respect to $\mathbb{G}$ and let the random variable F be invariant with respect to all permutations of the underlying Lévy process X induced by $\mathbb{G}$. Then there is a representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ with symmetric kernels $f_n\colon ((0,1] \times \mathbb{R})^n \to \mathbb{R}$ that are $(\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^{\otimes n}$-measurable.*

(2) *The following assertions are equivalent:*
   (a) *The random variable F is measurable with respect to*

$$\sigma\Big(I_1(f_1) : f_1 \in L_2^1 \text{ is } \mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}) \text{ measurable}\Big) \vee \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}.$$

   (b) *There are symmetric $(\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^{\otimes n}$-measurable $f_n \in L_2^n$ with $F = \sum_{n=0}^{\infty} I_n(f_n)$.*
   (c) *The random variable F is invariant with respect to all permutations of the underlying Lévy process induced by the groups $\mathbb{M}_{E_1}^{\text{dyad}}, \ldots, \mathbb{M}_{E_L}^{\text{dyad}}$, where $\mathbb{M}_{E_l}^{\text{dyad}}$ consists of all dyadic permutations that leave E_l^c invariant.*

Theorem 4 relies on the results of Section 4 that are proved in a wider setting and are applicable in other situations as well (like in [15]). In Section 6 we verify the following examples (including the introductory example F_2) to illustrate Theorem 4.

Example 5. Given a time net $0 \leq r_0 < \ldots < r_L \leq 1$, the examples

(1) $f([X]_{r_1} - [X]_{r_0}, \ldots, [X]_{r_L} - [X]_{r_{L-1}})$ and
(2) $f(S_{r_1}^{r_0}, \ldots, S_{r_L}^{r_{L-1}})$

admit invariances with respect to $\mathbb{M}^{\mathrm{dyad}}_{(r_{l-1},r_l]}$. In (1), the process $([X]_t)_{t\in[0,1]}$ is the *quadratic variation* of $(X_t)_{t\in[0,1]}$. In part (2) the process $(S^a_t)_{t\in[a,1]}$ is the *Doléans-Dade exponential* $\mathrm{d}S^a_t = S^a_{t-}\mathrm{d}X_t$ with initial condition $S^a_a = 1$ and chaos representation

$$S^a_t = 1 + \sum_{n=1}^{\infty} I_n\left(\frac{1}{n!}\mathbb{1}^{\otimes n}_{(a,t]}\right),$$

where $\mathbb{1}^{\otimes n}_{(a,t]}((t_1,x_1),\dots,(t_n,x_n)) := \mathbb{1}_{(a,t]}(t_1)\cdots\mathbb{1}_{(a,t]}(t_n)$ and $(X_t)_{t\in[0,1]}$ is assumed to be square-integrable and of mean zero.

We conclude with the example mentioned in the beginning:

Example 6. We describe the situation from [12] where the results of this paper were already applied. For this purpose we consider a Backward Stochastic Differential Equation (BSDE)

$$Y_t = F + \int_{(t,1]} f\left(s, Y_s, \int_{\mathbb{R}} Z_{s,x}h(x)\mathrm{d}\mu(x)\right)\mathrm{d}s - \int_{(t,1]\times\mathbb{R}} Z_{s,x}\mathrm{d}M(s,x) \text{ a.s.}, \quad t\in[0,1],$$

with $h \in L_2(\mathbb{R},\mu)$, where the random measure M and the Borel measure μ (both associated with X) are introduced in Section 1 below and f is an appropriate deterministic generator (for the precise setting see [12]). Given the initial data $F \in L_2(\mathcal{F}^X)$ and f, one looks for the solution processes $(Y_t)_{t\in[0,1]}$ and $(Z_{s,x})_{(s,x)\in[0,1]\times\mathbb{R}}$. To be able to control the variation of the BSDE, for example to upper bound $\|Y_t - Y_s\|_2$, the authors in [12] assume a time net $0 = r_1 < \cdots < r_L = 1$ such that the kernels f_n in the chaos expansion $F = \sum_{n=0}^{\infty} I_n(f_n)$ are constant on all cuboids

$$Q_{l_1,\dots,l_n} := (r_{l_1-1}, r_{l_1}] \times \cdots \times (r_{l_n-1}, r_{l_n}].$$

One main step in [12] consists in verifying in [12, Theorem 4.2] that the structure of the terminal condition F transfers to the solution processes Y and Z. This is done by a Picard iteration, where in [12, Lemma 4.3] Theorem 4 of this article is applied. In Section 6.4 below we outline the ideas behind this application in a more abstract and general way.

Outline of the paper. In Section 1 we provide some preliminaries for Lévy processes. The permutation operators acting on functionals of Lévy processes are introduced in Section 2. The first abstract set of general invariance properties is obtained in Section 3, which is based on the general concepts recalled in Appendix A. The main results concerning Lévy processes are presented in Section 5. They are directly derived from the results in the more general setting given in Section 4, where we consider diagonal groups. In Section 6 we discuss some examples, explain a relation to the chaotic expansion of Nualart and Schoutens based on the Teugels martingales, and finally return to Example 6 to discuss an application to Backward Stochastic Differential Equations in more detail.

Some notation. The space of bounded continuous functions on a metric space M is denoted by $\mathcal{C}_b(M)$, the set of positive integers by $\mathbb{N}$. Given an $L > 0$ and $\xi \in \mathbb{R}$, we shall use the truncation function $\psi_L(\xi) := \max\{-L, \min\{\xi, L\}\}$.

1. Preliminaries for Lévy processes

We recall some facts about Lévy processes; for more information the reader is referred, for example, to [1] and [21]. Let $X = (X_t)_{t\in[0,1]}$, $X_t\colon \Omega \to \mathbb{R}$, be a Lévy process, where all paths are right-continuous and have left-limits, $X_0 \equiv 0$, and where we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and that $\mathcal{F} = \sigma(X_t : t \in [0,1]) \vee \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$. To emphasize the minimality of $\mathcal{F}$ we write $\mathcal{F} = \mathcal{F}^X$. There are some places where stochastic integration is formally used. Here we assume that as filtration the augmentation of the natural filtration of X is taken. For $E \in \mathcal{B}((0,1] \times \mathbb{R})$ let

$$N(E) := \#\{t \in (0,1]\colon (t, \Delta X_t) \in E\}$$

be the Poisson random measure associated to X with values in $\{\infty, 0, 1, 2, \ldots\}$. Assuming $B \in \mathcal{B}(\mathbb{R})$ with $B \cap (-\varepsilon, \varepsilon) = \emptyset$ for some $\varepsilon > 0$, we set

$$\nu(B) := \mathbb{E}N((0,1] \times B)$$

and by $\varepsilon \to 0$ we obtain the Lévy measure ν on $\mathcal{B}(\mathbb{R})$ with $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} [x^2 \wedge 1] \mathrm{d}\nu(x) < \infty$. If $\sigma \geq 0$ is the parameter for the Brownian motion part of X, then we define the σ-finite measures

$$\begin{aligned} \mathrm{d}\mu(x) &:= \sigma^2 \mathrm{d}\delta_0(x) + x^2 \mathrm{d}\nu(x), \\ \mathrm{d}\mathrm{m}(t,x) &:= \mathrm{d}(\lambda \otimes \mu)(t,x), \end{aligned}$$

on $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}((0,1] \times \mathbb{R})$, respectively. The compensated Poisson random measure is defined by $\widetilde{N} := N - \lambda \otimes \nu$ on the ring of $E \in \mathcal{B}((0,1] \times \mathbb{R})$ with $\mathrm{m}(E) < \infty$. For such an E one introduces

$$M(E) := \sigma \left(\int\limits_{E \cap ((0,1] \times \{0\})} \mathrm{d}W_t \right) + \lim_{N \to \infty} \int\limits_{E \cap ((0,1] \times \{\frac{1}{N} < |x| < N\})} x \mathrm{d}\widetilde{N}(t,x), \tag{1.1}$$

where W is the Brownian motion part of X and the limit is taken in L_2. To recall Itô's chaos expansion [13], we let

$$L_2^n := L_2\big(((0,1] \times \mathbb{R})^n, \mathcal{B}(((0,1] \times \mathbb{R})^n), \mathrm{m}^{\otimes n}\big)$$

and define for pairwise disjoint $E_1, \ldots, E_n \in \mathcal{B}((0,1] \times \mathbb{R})$ with $\mathrm{m}(E_i) < \infty$ the multiple integral

$$I_n(f_n) := M(E_1) \cdots M(E_n)$$

if

$$f_n((t_1, x_1), \ldots, (t_n, x_n)) := \mathbb{1}_{E_1}(t_1, x_1) \cdots \mathbb{1}_{E_n}(t_n, x_n).$$

This extends by linearity and continuity to $I_n\colon L_2^n \to L_2(\mathcal{F}^X)$. For $n \neq m$ the integrals $I_n(f_n)$ and $I_m(f_m)$ are orthogonal for any kernels f_n and f_m. A kernel f_n is called *symmetric* provided that

$$f((t_1, x_1), \ldots, (t_n, x_n)) = f((t_{\pi(1)}, x_{\pi(1)}), \ldots, (t_{\pi(n)}, x_{\pi(n)}))$$

for all $(t_1, x_1), \ldots, (t_n, x_n)$ and $\pi \in \mathcal{S}_n$, where $\mathcal{S}_n$ is the set of all permutations acting on $\{1, \ldots, n\}$. The symmetrization of an $f_n \in L_2^n$ is given by

$$\tilde{f}_n((t_1, x_1), \ldots, (t_n, x_n)) := \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} f((t_{\pi(1)}, x_{\pi(1)}), \ldots, (t_{\pi(n)}, x_{\pi(n)}))$$

and shares the two important properties, $I_n(f_n) = I_n(\tilde{f}_n)$ a.s. and $\|I_n(\tilde{f}_n)\|_{L_2(\mathcal{F}^X)} = \sqrt{n!}\|\tilde{f}_n\|_{L_2^n}$. By Itô's orthogonal decomposition [13], for any $F \in L_2(\mathcal{F}^X)$ there exist unique symmetric kernels $f_n \in L_2^n$ such that

$$F = \sum_{n=0}^{\infty} I_n(f_n) \quad \text{in} \quad L_2(\mathcal{F}^X).$$

If $\mathcal{H}_n := I_n(L_2^n) \subseteq L_2(\mathcal{F}^X)$ and if $\widetilde{L}_2^n$ are the (equivalence classes of) symmetric functions in L_2^n, then

$$\begin{aligned} \mathcal{J}\colon \bigoplus_{n=0}^{\infty} \widetilde{L}_2^n &\longrightarrow L_2(\mathcal{F}^X) \cong \bigoplus_{n=0}^{\infty} \mathcal{H}_n \\ (f_n)_{n=0}^{\infty} &\mapsto \sum_{n=0}^{\infty} I_n(f_n), \end{aligned}$$

defines an isometric bijection, where $\bigoplus_{n=0}^{\infty} \mathcal{H}_n$ is the ℓ_2-product and $\bigoplus_{n=0}^{\infty} \widetilde{L}_2^n$ is equipped with the norm

$$\|(f_0, f_1, \ldots)\| := \left(\sum_{n=0}^{\infty} n! \|f_n\|^2 \right)^{\frac{1}{2}}.$$

2. Dyadic permutations and Lévy processes

In this section we investigate measure preserving transformations on $L_2(\mathcal{F}^X)$ and on the chaos decomposition induced by dyadic measure preserving maps $g\colon (0,1] \to (0,1]$. The final commutative diagram will be

$$\begin{array}{ccc} L_2(\mathcal{F}^X) & \xrightarrow{T_g} & L_2(\mathcal{F}^X) \\ \mathcal{J}\Big\uparrow & & \Big\uparrow\mathcal{J} \\ \bigoplus_{n=0}^{\infty} \widetilde{L}_2^n & \xrightarrow{S_{g^{-1}}} & \bigoplus_{n=0}^{\infty} \widetilde{L}_2^n \end{array}$$

and is verified in Theorem 2.8 below. This diagram transfers Lemma 1 of [13], where shift operations are considered, to our setting. The diagram is based on the fact that by the definition of Lévy processes, their increments are exchangeable. Later we investigate how this exchangeability transfers to certain functionals defined on the process X or more generally, to $L^2(\mathcal{F}^X)$-random variables. In order to shorten the presentation, given $0 \le a < b \le 1$ and $I := (a, b]$, we let $X_I := X_b - X_a$. The dyadic intervals we denote by

$$I_k^d := \left(\frac{k-1}{2^d}, \frac{k}{2^d} \right] \quad \text{for} \quad d \ge 0 \text{ and } k \in \{1, \ldots, 2^d\}.$$

2.1. Construction of T_g.

For an integer $d \ge 0$ we let

$$\mathcal{H}^{X,d} := \left\{ F \in L_2(\mathcal{F}^X) : F = f\big(X_{I_1^d}, \ldots, X_{I_{2^d}^d}\big), f \in \mathcal{C}_b(\mathbb{R}^{2^d}) \right\}$$

and

$$\mathcal{H}^X := \bigcup_{d \ge 0} \mathcal{H}^{X,d}.$$

All spaces $\mathcal{H}^{X,0} \subseteq \mathcal{H}^{X,1} \subseteq \cdots \subseteq \mathcal{H}^X$ are linear subspaces of $L_2(\mathcal{F}^X)$.

Lemma 2.1. *$\mathcal{H}^X$ is dense in $L_2(\mathcal{F}^X)$.*

Proof. It is known that $\{f(X_{(t_0,t_1]},\dots,X_{(t_{N-1},t_N]}) : 0 \le t_0 < \dots < t_N \le 1, f \in \mathcal{C}_b(\mathbb{R}^N), N \in \mathbb{N}\}$ is dense in $L_2(\mathcal{F}^X)$; cf. for example [13]. The right-continuity of $(X_t)_{t\in[0,1]}$ yields our assertion. □

Definition 2.2.

(1) For $d \ge 0$ and $\pi \in \mathcal{S}_{2^d}$ we define $g_\pi\colon (0,1] \to (0,1]$ by shifting I_k^d onto $I_{\pi(k)}^d$, i.e.,
$$g_\pi(t) := \frac{\pi(k)}{2^d} - \left(\frac{k}{2^d} - t\right) \quad \text{if} \quad t \in \left(\frac{k-1}{2^d}, \frac{k}{2^d}\right].$$
(2) We let $\mathbb{M}^{\mathrm{dyad}} := \{g_\pi\colon (0,1] \to (0,1] \,:\, \pi \in \mathcal{S}_{2^d}, d \ge 0\}$.
(3) We say that $g \in \mathbb{M}^{\mathrm{dyad}}$ is *represented by* $\pi \in \mathcal{S}_{2^d}$ for some $d \ge 0$ if $g = g_\pi$.
(4) For $g \in \mathbb{M}^{\mathrm{dyad}}$ we let $\deg(g) := \min d$, where the minimum is taken over all $d \ge 0$ such that g can be represented by some $\pi \in \mathcal{S}_{2^d}$.

Note that for $d \ge \deg(g)$ the map g can always be represented by some $\pi \in \mathcal{S}_{2^d}$ and that all $g \in \mathbb{M}^{\mathrm{dyad}}$ preserve the Lebesgue measure.

Definition 2.3. For $g \in \mathbb{M}^{\mathrm{dyad}}$, $d \ge \deg(g)$, and $\pi \in \mathcal{S}_{2^d}$ representing g, we define the operator $T_g\colon \mathcal{H}^{X,d} \to \mathcal{H}^{X,d}$ by
$$T_g f\big(X_{I_1^d},\dots,X_{I_{2^d}^d}\big) := f\big(X_{I_{\pi(1)}^d},\dots,X_{I_{\pi(2^d)}^d}\big) = f\big(X_{g(I_1^d)},\dots,X_{g(I_{2^d}^d)}\big),$$
where $g(I) := \{g(t) : t \in I\} \subseteq (0,1]$ for $I \subseteq (0,1]$.

Lemma 2.4.

(1) *For $d \ge \deg(g)$ the operator $T_g\colon \mathcal{H}^{X,d} \to \mathcal{H}^{X,d}$ is well-defined.*
(2) *For $e \ge d \ge \deg(g)$ the operators $T_g\colon \mathcal{H}^{X,d} \to \mathcal{H}^{X,d}$ and $T_g : \mathcal{H}^{X,e} \to \mathcal{H}^{X,e}$ are consistent in the sense that if $g = g_{\pi_e} = g_{\pi_d}$ with $\pi_e \in \mathcal{S}_{2^e}$ and $\pi_d \in \mathcal{S}_{2^d}$, then $T_{g_{\pi_e}}|_{\mathcal{H}^{X,d}\to\mathcal{H}^{X,d}} = T_{g_{\pi_d}}$.*
(3) *For $F \in \mathcal{H}^{X,d}$ with $d \ge \deg(g)$ the random variables F and T_gF have the same distribution. In particular, T_g is a linear isometry in $L_2(\mathcal{F}^X)$.*

Proof. (1) Assume that $f_1\big(X_{I_1^d},\dots,X_{I_{2^d}^d}\big) = f_2\big(X_{I_1^d},\dots,X_{I_{2^d}^d}\big)$ a.s. Because of the exchangeability of the increments of the Lévy process, the permuted vector of increments has the same distribution as the original vector. Therefore we have that $f_1\big(X_{I_{\pi(1)}^d},\dots,X_{I_{\pi(2^d)}^d}\big) = f_2\big(X_{I_{\pi(1)}^d},\dots,X_{I_{\pi(2^d)}^d}\big)$ a.s. and the equivalence classes coincide. Assertion (2) follows from the definition and assertion (3) follows by the same distributional argument as in (1). □

Because of Lemma 2.4 we can extend T_g to an L_2-isometry $T_g : \mathcal{H}^X \to \mathcal{H}^X$, and by Lemma 2.1, we obtain an isometry
$$T_g\colon L_2(\mathcal{F}^X) \to L_2(\mathcal{F}^X).$$
The operator T_g acts on the jump-part of X as follows:

Lemma 2.5. *Let $g \in \mathbb{M}^{\mathrm{dyad}}$, let N be the Poisson random measure associated to X, let $I = (a,b]$ with $0 \le a < b \le 1$ dyadic, and let $E = (c,d)$ with $-\infty < c < d < \infty$ and $0 \notin \overline{E}$. Then,*
$$T_g \int_{I\times E} x\mathrm{d}N(s,x) = \int_{g(I)\times E} x\mathrm{d}N(s,x) \ \ a.s. \tag{2.1}$$

Proof. The proof follows an idea of [10]. We show that for $L \in \mathbb{N}$ and the truncation $\psi_L(\xi) = \max\{-L, \min\{\xi, L\}\}$ it holds that

$$T_g\psi_L\left(\int_{I\times E} x\mathrm{d}N(s,x)\right) = \psi_L\left(\int_{g(I)\times E} x\mathrm{d}N(s,x)\right) \text{ a.s.}$$

Then the assertion follows from the fact that $\psi_L(F)$ converges in $L_2(\mathcal{F}^X)$ to F whenever $F \in L_2(\mathcal{F}^X)$. For $l \in \mathbb{N}$ with $2/l < d - c$ we define a continuous function h_l such that $h_l(x) = x$ on $[c + (1/l), d - (1/l)]$, $h_l(x) = 0$ if $x \notin [c, d]$ and on the remaining parts we take the linear interpolation. By construction, $\lim_{l\to\infty} h_l(x) = x\mathbb{1}_E(x)$ and $|h_l(x)| \le |x|\mathbb{1}_E(x)$. By definition,

$$T_g\psi_L\left(\sum_{\substack{k=1,\dots,2^n\\ a\cdot 2^n<k\le b\cdot 2^n}} h_l\left(X_{I_k^n}\right)\right) = \psi_L\left(\sum_{\substack{k=1,\dots,2^n\\ a\cdot 2^n<k\le b\cdot 2^n}} h_l\left(X_{g(I_k^n)}\right)\right) \text{ a.s.,}$$

where we assume that $n \ge \deg(g) \vee n_0$, with $n_0 \ge 0$ chosen such that a and b belong to the dyadic grid with mesh-size 2^{-n_0}. Using the fact that for a fixed càdlàg path $t \to \xi_t = X_t(\omega)$ and for any $\varepsilon > 0$ one finds a partition $0 = t_0 < \cdots < t_N = 1$ such that for all $t_{i-1} \le s < t < t_i$ one has that $|\xi_t - \xi_s| \le \varepsilon$ (see [4, Lemma 1, Chapter 3]), one concludes by $n \to \infty$ with dominated convergence that

$$T_g\psi_L\left(\sum_{t\in(a,b]} h_l(\Delta X_t)\right) = \psi_L\left(\sum_{t\in(a,b]} h_l(\Delta X_{g(t)})\right) \text{ a.s.}$$

Letting $l \to \infty$ and using again dominated convergence finally yields

$$T_g\psi_L\left(\sum_{t\in(a,b]} \Delta X_t\mathbb{1}_E(\Delta X_t)\right) = \psi_L\left(\sum_{t\in(a,b]} \Delta X_{g(t)}\mathbb{1}_E(\Delta X_{g(t)})\right) \text{ a.s.} \qquad \square$$

The Gaussian part of X is handled by the next lemma, which is proved in Appendix B.

Lemma 2.6. *Let $g \in \mathbb{M}^{\mathrm{dyad}}$ and let $(\sigma B_t)_{t\in[0,1]}$ be the Brownian motion part of X, where we assume that $\sigma > 0$ and that $t \in (0, 1]$ is dyadic. Then,*

$$T_gB_t = \int_{g((0,t])} \mathrm{d}B_s \quad \textit{a.s.}$$

2.2. **Construction of S_g.** For $g \in \mathbb{M}^{\mathrm{dyad}}$ we define the operator

$$S_g\colon \prod_{n=0}^{\infty} L_2^n \to \prod_{n=0}^{\infty} L_2^n \quad \text{by} \quad (f_n)_{n=0}^{\infty} \mapsto \big(S_{g,n}(f_n)\big)_{n=0}^{\infty}, \tag{2.2}$$

where $S_{g,n}\colon L_2^n \to L_2^n$ is given by

$$f_n\big((t_1,x_1),\dots,(t_n,x_n)\big) \mapsto f_n\big((g(t_1),x_1),\dots,(g(t_n),x_n)\big).$$

The distributions of f_n and $S_{g,n}f_n$ coincide, so that the operators $S_{g,n}$ and S_g are isometries. The next lemma shows that we can restrict ourselves to symmetric functionals in L_2^n when investigating S_g.

Lemma 2.7.

(1) *For* $f_n, h_n \in L_2^n$ *with* $I_n(f_n) = I_n(h_n)$ *one has* $I_n(S_{g,n} f_n) = I_n(S_{g,n} h_n)$.

(2) *For* $f_n \in L_2^n$ *one has* $I_n(S_{g,n} f_n) = I_n(S_{g,n} \widetilde{f_n})$.

Proof. (2) follows from (1) by the property $I_n(f_n) = I_n(\widetilde{f_n})$.

(1) We know that for the symmetrizations $\widetilde{f_n}$ and $\widetilde{h_n}$ we have $I_n(f_n) = I_n(h_n)$ a.e. if and only if $\widetilde{f_n} = \widetilde{h_n}$ a.e. Hence it suffices to show that $\widetilde{f_n} = \widetilde{h_n}$ a.e. implies $\widetilde{S_{g,n} f_n} = \widetilde{S_{g,n} h_n}$ a.e. Using the transformation $r = g(s)$, this follows from

$$\begin{aligned}
&\left(\widetilde{S_{g,n} f_n}\right)\big((s_1, x_1), \dots, (s_n, x_n)\big) \\
&\quad = \frac{1}{n!} \sum_{\varrho \in \mathcal{S}_n} f_n\big((g(s_{\varrho(1)}), x_{\varrho(1)}), \dots, (g(s_{\varrho(n)}), x_{\varrho(n)})\big) \\
&\quad = \frac{1}{n!} \sum_{\varrho \in \mathcal{S}_n} f_n\big((r_{\varrho(1)}, x_{\varrho(1)}), \dots, (r_{\varrho(n)}, x_{\varrho(n)})\big) \\
&\quad = \widetilde{f_n}\big((r_1, x_1), \dots, (r_n, x_n)\big) \\
&\quad = \widetilde{h_n}\big((r_1, x_1), \dots, (r_n, x_n)\big) \\
&\quad = \left(\widetilde{S_{g,n} h_n}\right)\big((s_1, x_1), \dots, (s_n, x_n)\big)
\end{aligned}$$

for every $((r_1, x_1), \dots, (r_n, x_n))$ for which $\widetilde{f_n}$ and $\widetilde{h_n}$ coincide. This concludes the proof. □

2.3. **The commutative diagram.**

Theorem 2.8. *For* $g \in \mathbb{M}^{\mathrm{dyad}}$ *the following diagram is commutative:*

$$\begin{tikzcd}
L_2(\mathcal{F}^X) \arrow[r, "T_g"] & L_2(\mathcal{F}^X) \\
\bigoplus_{n=0}^{\infty} \widetilde{L}_2^n \arrow[u, "\mathcal{J}"] \arrow[r, "S_{g^{-1}}"] & \bigoplus_{n=0}^{\infty} \widetilde{L}_2^n \arrow[u, "\mathcal{J}"']
\end{tikzcd} \tag{2.3}$$

Proof. As all linear combinations of

$$f_n((s_1, x_1), \dots, (s_n, x_n)) = \mathbb{1}_{(a_1, b_1] \times E_1}(s_1, x_1) \cdot \ldots \cdot \mathbb{1}_{(a_n, b_n] \times E_n}(s_n, x_n),$$

where the $(a_1, b_1], \dots, (a_n, b_n]$ are dyadic and pairwise disjoint and the E_i are of form $E_i = (c_i, d_i)$ with $c_i d_i > 0$ or $E_i = \{0\}$, are dense in L_2^n, and therefore the symmetrizations $\tilde{f}_n$ are dense in $\tilde{L}_2^n$, it suffices to show that $\mathcal{J} S_{g^{-1}}(0, \dots, 0, \tilde{f}_n, 0, \dots) = T_g \mathcal{J}(0, \dots, 0, \tilde{f}_n, 0, \dots)$ for all $n \in \mathbb{N}$. For this it is sufficient to check that $I_n S_{g^{-1},n} f_n = T_g I_n f_n$, which follows from Lemmas 2.5, 2.6, and B.1, where we use that the sets $g((a_i, b_i])$ are pairwise disjoint as well. □

Remark 2.9. There are formulas, the Stroock formulas, to compute the kernels $(f_n)_{n=0}^{\infty}$ from the chaos expansion $F = \sum_{n=0}^{\infty} I_n(f_n)$ as expected values of iterative applications of differential and difference operators to F (cf. [7, Theorem 3.3] and [15, Theorem 1.3]). This might be used in the proof of Theorem 2.8 as well. Our approach is slightly more direct and self-contained, and shows in a way that the

invariance properties, we consider, are not intrinsically connected to differentiability. On the other hand, the Stroock formulas might open the way to use the results of this article to link structural properties of F to structural properties of the Malliavin derivatives of F. Lemma 6.13 below goes in this direction.

3. Invariances for Lévy processes

Throughout this section we let $\mathbb{G} \subseteq \mathbb{M}^{\mathrm{dyad}}$ be a subgroup of the group of dyadic measure preserving maps. For $n \in \mathbb{N}$ we derive the group $\mathbb{G}[n]$ of the measure preserving $\big((0,1] \times \mathbb{R}\big)^n$-automorphisms

$$g[n] : \big((t_1, x_1), \dots, (t_n, x_n)\big) \mapsto \big((g(t_1), x_1), \dots, (g(t_n), x_n)\big) \text{ with } g \in \mathbb{G}. \tag{3.1}$$

Now we introduce the main concepts of invariance we are interested in.

Definition 3.1.

(1) **$\mathbf{H}_{\mathbb{G}}$-invariance**. An $F \in L_2(\mathcal{F}^X)$ is $\mathbb{G}$-invariant if $T_g F = F$ a.s. for all $g \in \mathbb{G}$. The set of all $\mathbb{G}$-invariant (equivalence classes of) random variables is denoted by $\mathbf{H}_{\mathbb{G}}$.

(2) **$\mathcal{H}_{\mathbb{G}}$-measurability**. A symmetric chaos kernel $f_n \colon ((0,1] \times \mathbb{R})^n \to \mathbb{R}$ is $\mathbb{G}[n]$-invariant if $f_n = f_n \circ g[n]$ a.e. for all $g \in \mathbb{G}$. We let

$$\mathcal{H}_{\mathbb{G}} := \sigma\big(I_n(f_n) : f_n \text{ is } \mathbb{G}[n]\text{-invariant}, n \in \mathbb{N}\big) \vee \mathcal{N}$$

with $\mathcal{N} := \{A \in \mathcal{F}^X : \mathbb{P}(A) = 0\}$.

(3) **$\mathbb{G}$-invariant chaos expansion**. An $F \in L_2(\mathcal{F}^X)$ has a $\mathbb{G}$-invariant chaos expansion if all chaos kernels f_n are symmetric and $\mathbb{G}[n]$-invariant.

The definition of $\mathcal{H}_{\mathbb{G}}$ can be understood in the way that we take particular representatives of $I_n(f_n)$ to define the σ-algebra. By adding the null-sets, all representatives become measurable with respect to $\mathcal{H}_{\mathbb{G}}$. The next theorem is the main result of this section:

Theorem 3.2. *For a group of dyadic measure preserving maps $\mathbb{G} \subseteq \mathbb{M}^{\mathrm{dyad}}$ and $F \in L_2(\mathcal{F}^X)$ the following assertions are equivalent:*

(1) $F \in \mathbf{H}_{\mathbb{G}}$.
(2) *F is measurable with respect to $\mathcal{H}_{\mathbb{G}}$.*
(3) *F has a $\mathbb{G}$-invariant chaos expansion.*
(4) *F has symmetric chaos kernels f_n, $n \in \mathbb{N}$, which are constant on the orbits of $\mathbb{G}[n]$ on $((0,1] \times \mathbb{R})^n$.*

Definition 3.3. If $F \in L_2(\mathcal{F}^X)$ satisfies one of the conditions of Theorem 3.2, then we will say that F is $\mathbb{G}$-invariant.

In order to prove Theorem 3.2 we start with the following lemma, which is verified in Appendix B.

Lemma 3.4. *Let $F_1, \dots, F_n \in \mathbf{H}_{\mathbb{G}}$ and $\varphi \colon \mathbb{R}^n \to \mathbb{R}$ be Borel measurable with $\varphi(F_1, \dots, F_n) \in L_2(\mathcal{F}^X)$. Then, $\varphi(F_1, \dots, F_n) \in \mathbf{H}_{\mathbb{G}}$.*

Proof of Theorem 3.2. (1) $\Longleftrightarrow$ (3) follows from Theorem 2.8 and the uniqueness of symmetric kernels in the chaos expansion.

(3) $\Longrightarrow$ (2) follows by definition and the completeness of $(\Omega, \mathcal{H}_{\mathbb{G}}, \mathbb{P})$.

(4) $\Longrightarrow$ (3) is a consequence of Lemma A.2.

(3) $\Longrightarrow$ (4) First we use Lemma A.4 to obtain a chaos kernel that is constant on the orbits. This new kernel will be symmetrized which keeps the property that the kernel is constant on the orbits.

(2) $\Longrightarrow$ (1) As $\mathbf{H}_{\mathbb{G}}$ is a closed subspace of $L_2(\mathcal{F}^X)$, it is sufficient to check that $\mathbb{1}_A \in \mathbf{H}_{\mathbb{G}}$ for all $A \in \mathcal{H}_{\mathbb{G}}$. Here it is sufficient to take A such that there exists a sequence $(I_{i_k}(f_{i_k}))_{k\in\mathbb{N}}$ with $(i_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ and $\mathbb{G}[i_k]$-invariant kernels f_{i_k} such that

$$A \in \mathcal{G} := \sigma\big(I_{i_k}(f_{i_k}) : k \in \mathbb{N}\big).$$

By martingale convergence, $\mathbb{1}_A$ can be approximated in L_2 by $\mathcal{G}_n$-measurable functions, where

$$\mathcal{G}_n := \sigma\big(I_{i_k}(f_{i_k}) : k \in \{1, \dots, n\}\big).$$

By Doob's factorization lemma (cf. [2, Lemma II.11.7]), there are Borel functions $\varphi_n \colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathbb{E}(\mathbb{1}_A|\mathcal{G}_n) = \varphi_n\big(I_{i_1}(f_{i_1}), \dots, I_{i_n}(f_{i_n})\big) \text{ a.s.},$$

so that

$$\lim_n \mathbb{E}|\mathbb{1}_A - \varphi_n\big(I_{i_1}(f_{i_1}), \dots, I_{i_n}(f_{i_n})\big)|^2 = 0.$$

Because of the equivalence (1) $\Longleftrightarrow$ (3) we have that $I_{i_k}(f_{i_k}) \in \mathbf{H}_{\mathbb{G}}$ for all $k \in \mathbb{N}$ and Lemma 3.4 implies that

$$\varphi_n\big(I_{i_1}(f_{i_1}), \dots, I_{i_n}(f_{i_n})\big) \in \mathbf{H}_{\mathbb{G}}.$$

Because $\mathbf{H}_{\mathbb{G}}$ is closed in $L_2(\mathcal{F}^X)$, we derive that $\mathbb{1}_A \in \mathbf{H}_{\mathbb{G}}$. □

4. Diagonal groups and locally ergodic sets

Let $(T, \mathcal{T}, \tau, (\mathcal{T}_N)_{N=0}^{\infty})$ be a filtered probability space such that there are refining partitions

$$T = T_{N,1} \cup \dots \cup T_{N,L_N}, \quad N = 0, 1, 2, \dots,$$

satisfying the following assumptions:

(1) $\mathcal{T}_N = \sigma(T_{N,1}, \dots, T_{N,L_N})$,
(2) $\tau(T_{N,l}) > 0$ for all (N, l),
(3) $\lim_{N\to\infty} \sup_{l=1,\dots,L_N} \tau(T_{N,l}) = 0$,
(4) $\mathcal{T} = \bigvee_{N=0}^{\infty} \mathcal{T}_N$.

We let $\mathcal{O}(T)$ be the system of countable unions of elements from $\bigcup_{N=0}^{\infty} \mathcal{T}_N$ (including the empty set). The system forms a topology, in particular a set $G \subseteq T$ is open provided that it is empty or for each $x \in G$ there is a $T_{N,l}$ with $x \in T_{N,l} \subseteq G$.

Finally, we suppose that there is a *countable* group $\mathbb{G}$ of bijective bi-measurable $g \colon T \to T$.

Definition 4.1.

(1) A set $E \subseteq T$ of positive measure is called *finite locally ergodic* with respect to $\mathbb{G}$ provided that there is an $N_E \geq 0$ such that $E \in \mathcal{T}_{N_E}$ and for all $A := T_{N,l} \cup T_{N,m} \subseteq E$ with $l \neq m$ and $N \geq N_E$ there is a subgroup $\mathbb{H} \subseteq \mathbb{G}$ such that
   (a) $g|_{A^c} = \mathrm{id}_{A^c}$ for all $g \in \mathbb{H}$,
   (b) the probability space $(A, \mathcal{I}(\mathbb{H}|_A), \tau_A)$ is trivial, i.e., contains only sets of measure one or zero, where $\mathbb{H}|_A$ is the restriction of $\mathbb{H}$ to A and τ_A the normalized restriction of τ to A.

(2) A set $E \subseteq T$ is called *locally ergodic* with respect to $\mathbb{G}$ provided that there is a sequence E^j of finite locally ergodic sets with respect to $\mathbb{G}$ such that

$$E^1 \subseteq E^2 \subseteq \cdots \subseteq E \quad \text{and} \quad E = \bigcup_{j=1}^{\infty} E^j.$$

Remark 4.2.

(1) By definition, locally ergodic sets belong to $\mathcal{O}(T)$.
(2) The local ergodicity is stable with respect to passing to open subsets: If $\emptyset \neq F \subseteq E$, where $F \in \mathcal{O}(T)$ and where E is locally ergodic, then F is locally ergodic.

Proof. Let us check (2). By definition, we find finite locally ergodic sets such that

$$E^1 \subseteq E^2 \subseteq \cdots \subseteq E \quad \text{and} \quad E = \bigcup_{j=1}^{\infty} E^j.$$

At the same time we find an increasing sequence $F^j \in \mathcal{T}_{N_j}$, $j \in \mathbb{N}$, such that $F = \bigcup_{j=1}^{\infty} F^j$. One obtains

$$F = F \cap E = \bigcup_{j=1}^{\infty} (F^j \cap E^j)$$

and that $F^j \cap E^j$ is finite locally ergodic because E^j is of this type and $F^j \cap E^j \subseteq E^j$. $\square$

Now we define our *diagonal group*: We fix $n \in \mathbb{N}$ and consider an auxiliary σ-finite measure space $(R, \mathcal{R}, \rho)$ with $\rho(R) > 0$ and the group $\mathbb{G}[n]$ that consists of all maps $g[n]\colon (T \times R)^n \to (T \times R)^n$ given by

$$\big((t_1, x_1), \ldots, (t_n, x_n)\big) \to \big((g(t_1), x_1), \ldots, (g(t_n), x_n)\big) \quad \text{with} \quad g \in \mathbb{G}.$$

To formulate our main result, we recall that $\mathcal{I}(\mathbb{G}[n])$ denotes the invariant σ-algebra with respect to the group $\mathbb{G}[n]$; see Definition A.1 below. For $A \in \mathcal{T}$ the trace-σ-algebra on A is denoted by $\mathcal{T}|_A$.

Theorem 4.3. *Let $n \in \mathbb{N}$, let $E_1, \ldots, E_L \in \mathcal{T}$ be pairwise disjoint and locally ergodic with respect to $\mathbb{G}$, and let*

$$\mathcal{T}_E := \mathcal{T}|_{T \setminus (\bigcup_{l=1}^{L} E_l)} \vee \sigma(E_1, \ldots, E_L)$$

and

$$\mathcal{N}_n := \{A \in (\mathcal{T} \otimes \mathcal{R})^{\otimes n} : (\tau \otimes \rho)^{\otimes n}(A) = 0\}.$$

Then $\mathcal{I}(\mathbb{G}[n]) \subseteq (\mathcal{T}_E \otimes \mathcal{R})^{\otimes n} \vee \mathcal{N}_n$.

Lemma 4.4. *Assume a probability space $(M, \mathcal{M}, m)$, a decreasing sequence of measurable sets $D_0 \supseteq D_1 \supseteq \ldots$, a sub-$\sigma$-algebra $\mathcal{I} \subseteq \mathcal{M}$ and*

$$\mathcal{G}_N := \mathcal{I} \vee \sigma(A_N \in \mathcal{M} \text{ with } A_N \subseteq D_N).$$

Assume that $m(D_N) \to 0$ as $N \to \infty$. Then

$$\bigcap_{N=0}^{\infty} (\mathcal{G}_N \vee \mathcal{N}) \subseteq \mathcal{I} \vee \mathcal{N} \quad \text{with} \quad \mathcal{N} := \{A \in \mathcal{M} : m(A) = 0\}.$$

Proof. The σ-algebra $\mathcal{G}_N$ consists of all

$$B_N = (I_N \cap D_N^c) \cup A_N$$

with $A_N \in \mathcal{M}$, $A_N \subseteq D_N$ and $I_N \in \mathcal{I}$. Therefore $B \in \bigcap_{N=0}^{\infty}(\mathcal{G}_N \vee \mathcal{N})$ gives $I_N \in \mathcal{I}$ and $A_N \in \mathcal{M}$ with $A_N \subseteq D_N$ such that

$$B_N := (I_N \cap D_N^c) \cup A_N \quad \text{satisfies} \quad B_N \Delta B \in \mathcal{N} \quad \text{for all} \quad N \geq 0.$$

Defining $C := \bigcup_{N=0}^{\infty}(B_N \Delta B) \in \mathcal{N}$, this implies *on* C^c that

$$B = B_N = (I_N \cap D_N^c) \cup A_N.$$

Let

$$I := \bigcup_{N=0}^{\infty} \bigcap_{k=N}^{\infty} I_k \in \mathcal{I}.$$

By construction, $I_N = B_N$ on D_N^c and $D_0^c \subseteq D_1^c \subseteq \cdots$. Therefore, $I \Delta B \subseteq D_N \cup C$ which implies $\mathbb{P}(I\Delta B) \leq \lim_N \mathbb{P}(D_N) = 0$ and proves the lemma. □

Proof of Theorem 4.3. We assume a partition $R = \bigcup_{j\in J} R_j$ with $\rho(R_j) \in (0,\infty)$. Choosing $\lambda_j \in (0,\infty)$ we can arrange that $\rho^0(A) := \sum_{j\in J} \lambda_j \rho(A \cap R_j)$ becomes a probability measure which has a strictly positive density with respect to ρ. As our statement only concerns null-sets we can replace ρ by ρ^0, or we can assume w.l.o.g. that ρ itself is a probability measure.

I. First we assume that $E_1, \ldots, E_L$ are finite locally ergodic. Let us fix a set $B \in \mathcal{I}(\mathbb{G}[n])$ of positive measure.

(a) We observe that $\bigvee_{N\geq 0}(\mathcal{T}_N \otimes \mathcal{R})^{\otimes n} = (\mathcal{T} \otimes \mathcal{R})^{\otimes n}$, so that martingale convergence yields

$$\lim_{N\to\infty} f_N = \mathbb{1}_B \quad (\tau \otimes \rho)^{\otimes n}\text{-a.s.},$$

where, for $(t_1, \ldots, t_n) \in Q^N_{l_1,\ldots,l_n} := T_{N,l_1} \times \cdots \times T_{N,l_n}$,

$$f_N((t_1,x_1),\ldots,(t_n,x_n)) := \int_{Q^N_{l_1,\ldots,l_n}} \mathbb{1}_B((s_1,x_1),\ldots,(s_n,x_n)) \frac{\mathrm{d}\tau(s_1)\cdots\mathrm{d}\tau(s_n)}{\tau^{\otimes n}(Q^N_{l_1,\ldots,l_n})}.$$

(b) For $N \geq 0$ we let

$$\Delta_N := \bigcup_{\substack{l_1,\ldots,l_n \in \{1,\ldots,L_N\} \\ \text{at least two } l_k \text{ coincide}}} Q^N_{l_1,\ldots,l_n}$$

which is empty for $n = 1$. For $n \geq 2$ the size of Δ_N can be upper bounded by

$$\tau^{\otimes n}(\Delta_N) \leq \binom{n}{2} \max_{l=1,\ldots,N} \tau(T_{N,l}) \quad \text{so that} \quad \lim_N \tau^{\otimes n}(\Delta_N) = 0.$$

Define

$$\mathcal{G}_N := (\mathcal{T}_E \otimes \mathcal{R})^{\otimes n} \vee \sigma\Big(D \times G : D \in \mathcal{T}^{\otimes n}, D \subseteq \Delta_N, G \in \mathcal{R}^{\otimes n}\Big)$$

with a slight abuse of notation concerning the order of components, which gives the σ-algebra $\mathcal{T}_E \otimes \mathcal{R}$ in the case $n = 1$. As $\Delta_0 \supseteq \Delta_1 \supseteq \cdots$ we have $\mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \cdots$.

(c) Let $N_0 := \max\{N_{E_1}, \ldots, N_{E_L}\} \geq 0$, where the N_{E_l} are taken from Definition 4.1 (1). The main observation of the proof is that f_N is $\mathcal{G}_N$-measurable for $N \geq N_0$. By definition, f_N is constant on all cuboids $Q^N_{l_1,\ldots,l_n}$. Assume two cuboids

$$Q^N_{l_1,\ldots,l_n} \quad \text{and} \quad Q^N_{m_1,l_2,\ldots,l_n}$$

such that $(l_1, \dots, l_n)$ are distinct, $(m_1, l_2, \dots, l_n)$ are distinct, $l_1 \neq m_1$, and that $T_{N,l_1}, T_{N,m_1} \subseteq E_l$, where $l \in \{1, \dots, L\}$ is now fixed. By assumption, there is a sub-group $\mathbb{H}$ of $\mathbb{G}$ such that for $A_l := T_{N,l_1} \cup T_{N,m_1}$ the probability space $(A_l, \mathcal{I}(\mathbb{H}|_{A_l}), \tau_{A_l})$ is trivial and $\mathbb{H}$ acts as an identity outside A_l. Because $B \in \mathcal{I}(\mathbb{G}[n])$ we have that

$$\mathbb{1}_B g[n] = \mathbb{1}_B \quad \text{for all} \quad g \in \mathbb{G},$$

so that, for all $g \in \mathbb{H}$,

$$\mathbb{1}_B((gt_1, x_1), (t_2, x_2), \dots, (t_n, x_n)) = \mathbb{1}_B((t_1, x_1), (t_2, x_2), \dots, (t_n, x_n))$$

on $(A_l \times R) \times (T_{N,l_2} \times R) \times \cdots \times (T_{N,l_n} \times R)$. This implies that the subset A_l of the section of B, taken at

$$(x_1, (t_2, x_2), \dots, (t_n, x_n)) \in R \times (T_{N,l_2} \times R) \times \cdots \times (T_{N,l_n} \times R), \tag{4.1}$$

is invariant with respect to $\mathbb{H}|_{A_l}$ and therefore the function

$$t_1 \to \mathbb{1}_B((t_1, x_1), (t_2, x_2), \dots, (t_n, x_n))$$

is almost surely constant on A_l under the condition (4.1). Consequently,

$$\begin{aligned}\int_{Q^N_{l_1,\dots,l_n}} &\mathbb{1}_A((t_1, x_1), (t_2, x_2), \dots, (t_n, x_n)) \frac{\mathrm{d}\tau(t_1) \cdots \mathrm{d}\tau(t_n)}{\tau^{\otimes n}(Q^N_{l_1,\dots,l_n})} \\ &= \int_{Q^N_{m_1,\dots,l_n}} \mathbb{1}_A((t_1, x_1), (t_2, x_2), \dots, (t_n, x_n)) \frac{\mathrm{d}\tau(t_1) \cdots \mathrm{d}\tau(t_n)}{\tau^{\otimes n}(Q^N_{m_1,\dots,l_n})}\end{aligned}$$

for $(x_1, \dots, x_n) \in R^n$. We can repeat the argument, where we replace the exchange of the first component of the cuboid by any other component. This implies that f_N is $\mathcal{G}_N$-measurable.

(d) From (c) we immediately get that f_M is $\mathcal{G}_N$-measurable for $M \geq N \geq N_0$. Therefore $\mathbb{1}_B$ is $\mathcal{G}_N \vee \mathcal{N}_n$-measurable for all $N \geq N_0$. Applying Lemma 4.4 we get that $\mathbb{1}_B$ is $(\mathcal{T}_E \otimes \mathcal{R})^{\otimes n} \vee \mathcal{N}_n$-measurable.

II. Now we assume general locally ergodic sets $E_1, \dots, E_L$. By definition, we find monotone sequences of finite locally ergodic sets $(E^j_l)_{j=1}^\infty$ with

$$\bigcup_{j=1}^\infty E^j_l = E_l.$$

We proved in step I that $\mathcal{I}(\mathbb{G}[n]) \subseteq (\mathcal{T}_{E^j} \otimes \mathcal{R})^{\otimes n} \vee \mathcal{N}_n$ with

$$\mathcal{T}_{E^j} := \mathcal{T}|_{T \setminus (\bigcup_{l=1}^L E^j_l)} \vee \sigma(E^j_1, \dots, E^j_L),$$

so that

$$\mathcal{I}(\mathbb{G}[n]) \subseteq \bigcap_{j=1}^\infty \Big((\mathcal{T}_{E^j} \otimes \mathcal{R})^{\otimes n} \vee \mathcal{N}_n \Big).$$

Observing

$$(\mathcal{T}_{E^j} \otimes \mathcal{R})^{\otimes n} \subseteq (\mathcal{T}_E \otimes \mathcal{R})^{\otimes n} \vee \sigma(A^j \in (\mathcal{T} \otimes \mathcal{R})^{\otimes n} : A^j \subseteq D^j)$$

with

$$D^j := \Bigg\{((t_1,x_1),\dots,(t_n,x_n)) \in (T\times R)^n : \\ t_k \in \bigcup_{l=1}^{L}(E_l \setminus E_l^j) \text{ for some } k\in\{1,\dots,n\}\Bigg\},$$

gives that

$$\mathcal{I}(\mathbb{G}[n]) \subseteq \bigcap_{j=1}^{\infty}\Big((\mathcal{T}_E\otimes\mathcal{R})^{\otimes n} \vee \sigma(A^j \in (\mathcal{T}\otimes\mathcal{R})^{\otimes n} : A^j \subseteq D^j) \vee \mathcal{N}_n\Big).$$

Finally, because of $D^1 \supseteq D^2 \supseteq \cdots$ and

$$(\tau\otimes\rho)^{\otimes n}(D^j) \le n\left[\sum_{l=1}^{L}\tau(E_l\setminus E_l^j)\right] \to 0 \text{ as } j\to\infty,$$

we can again apply Lemma 4.4. □

5. Reduced chaos expansions for Lévy processes

In this section we apply the results from Section 4 to Lévy processes. For this purpose we let

(1) $(T,\mathcal{T},\tau) := ((0,1],\mathcal{B}((0,1]),\lambda)$ with $\mathcal{T}_N = \mathcal{F}_N^{\mathrm{dyad}} := \sigma\left(\left(\frac{l-1}{2^N},\frac{l}{2^N}\right] : l = 1,\dots,2^N\right)$,
(2) $\mathbb{M}_E^{\mathrm{dyad}} := \left\{g\in\mathbb{M}^{\mathrm{dyad}} : g|_{E^c} = \mathrm{id}_{E^c}\right\}$ for $E\subseteq(0,1]$,
(3) $(R,\mathcal{R},\rho) := (\mathbb{R},\mathcal{B}(\mathbb{R}),\mu)$,
(4) and $\mathcal{N}_n$ be the null-sets in $((0,1]\times\mathbb{R})^n$ with respect to $(\lambda\otimes\mu)^{\otimes n}$.

Let us begin with a prototype of a locally ergodic set.

Lemma 5.1. *Let $E\in\mathcal{O}((0,1])$ be non-empty. Then E is locally ergodic with respect to $\mathbb{M}_E^{\mathrm{dyad}}$.*

Proof. It is enough to show the following: If $A\in\mathcal{F}_{N_0}^{\mathrm{dyad}}$ is a non-empty subset of E, then $(A,\mathcal{I}(\mathbb{M}_A^{\mathrm{dyad}}|_A),\lambda_A)$ is trivial. Take any $B\in\mathcal{I}(\mathbb{M}_A^{\mathrm{dyad}}|_A)$. Using the dyadic filtration restricted to A, where we start with the level N_0, we interpret $\mathbb{1}_B$ as closure of a martingale in $(A,\mathcal{B}((0,1])|_A,\lambda_A)$ along this filtration. By the invariance of B, the random variables, that form this martingale, are individually constant. Therefore we get a sequence of constants that converge to $\mathbb{1}_B$ in $L_2(A,\lambda_A)$ and λ_A-a.s. Hence $\mathbb{1}_B$ is a constant almost surely which implies the statement. □

Remark 5.2. One can find groups $\mathbb{G}$ such that for example $E=(0,1]$ is locally ergodic but $\mathbb{G}\subsetneq\mathbb{M}^{\mathrm{dyad}}$. Take for example all permutations that leave the first interval $(0,2^{-N}]$ invariant on each dyadic level N. It would be of interest to characterize those sub-groups $\mathbb{G}\subseteq\mathbb{M}^{\mathrm{dyad}}$ such that a given $E\in\mathcal{O}((0,1])$ gets locally ergodic.

Now we let $\mathbb{G}$ be a group like in Section 3. The main result is the following simplification of the chaos decomposition.

Theorem 5.3. *For pairwise disjoint $E_1, \dots, E_L \in \mathcal{O}((0,1])$, that are locally ergodic with respect to $\mathbb{G}$, and $F \in L_2(\mathcal{F}^X)$ consider the following conditions:*

(1) *F is $\mathbb{G}$-invariant.*
(2) *One has $F = \sum_{n=0}^{\infty} I_n(f_n)$ with symmetric, $\mathbb{G}[n]$-invariant, and $(\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^{\otimes n} \vee \mathcal{N}_n$-measurable f_n.*
(3) *One has $F = \sum_{n=0}^{\infty} I_n(f_n)$ with symmetric $(\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^{\otimes n}$-measurable f_n.*
(4) *F is invariant with respect to the group $\mathbb{H}$ generated by $\mathbb{M}^{\mathrm{dyad}}_{E_1}, \dots, \mathbb{M}^{\mathrm{dyad}}_{E_L}$.*
(5) *F is measurable with respect to*

$$\mathcal{A}_E := \sigma\big(I_1(f_1) : f_1 \in L_2^1 \text{ is } \mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R})\text{-measurable}\big) \vee \mathcal{N}.$$

Then it holds that (1) $\iff$ (2) $\implies$ (3) $\iff$ (4) $\iff$ (5). Moreover, for $\mathbb{G} = \mathbb{H}$ all assertions are equivalent.

Remark 5.4. The implication (4) $\implies$ (1) of Theorem 5.3 does not hold in general. In fact, assume $f : \mathbb{R}^2 \to \mathbb{R} \in \mathcal{C}_b(\mathbb{R}^2)$ such that

$$f(X_{\frac{1}{2}} - X_0, X_1 - X_{\frac{1}{2}}) = f(X_1 - X_{\frac{1}{2}}, X_{\frac{1}{2}} - X_0) \text{ a.s.}$$

is not satisfied. Define $F := f(X_{\frac{1}{2}} - X_0, X_1 - X_{\frac{1}{2}})$, let $\mathbb{H} \subseteq \mathbb{M}^{\mathrm{dyad}}$ be generated by $\mathbb{M}^{\mathrm{dyad}}_{(0,\frac{1}{2}]}$ and $\mathbb{M}^{\mathrm{dyad}}_{(\frac{1}{2},1]}$, and $\mathbb{G} \subseteq \mathbb{M}^{\mathrm{dyad}}$ by $\mathbb{H}$ and $h \in \mathbb{M}^{\mathrm{dyad}}$ exchanging the intervals $(0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$. The sets $(0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$ are locally ergodic with respect to $\mathbb{G}$, F is invariant with respect to $\mathbb{H}$, but F is not invariant with respect to $\mathbb{G}$.

For the proof of implication (3) $\Rightarrow$ (5) we need product formulas for multiple integrals; cf. [16, Theorem 3.5] and [18, Sections 6.4 and 6.5]. They require the definition of a *contraction* of chaos kernels as defined in (5.1) according to [16, Formula (21)] (cf. also [18, Definition 6.2.1]). As a preparation, we study the invariance properties of contractions:

Lemma 5.5. *Let $\mathbb{H}$ be a group of dyadic permutations of $(0,1]$. For $n, m \geq 1$, $0 \leq k \leq n \wedge m$, $0 \leq r \leq (n \wedge m) - k$, $f \in L_2^n$, and $f' \in L_2^m$ we define*

$$f \otimes_k^r f' : ((0,1] \times \mathbb{R})^{n-k-r} \times ((0,1] \times \mathbb{R})^{m-k-r} \times ((0,1] \times \mathbb{R})^r \to \mathbb{R}$$

by

$$\big(f \otimes_k^r f'\big)(\alpha, \beta, \gamma) = \Pi_x(\gamma) \int_{((0,1] \times \mathbb{R})^k} f(\alpha, \gamma, \rho) f'(\rho, \gamma, \beta) \mathrm{d}\mathrm{m}^{\otimes k}(\rho) \tag{5.1}$$

where $\Pi_x(\gamma)$ is the product of the x-coordinates of the vector γ and where we assume that

$$\int_{((0,1] \times \mathbb{R})^k} |f(\alpha, \gamma, \rho) f'(\rho, \gamma, \beta)| \mathrm{d}\mathrm{m}^{\otimes k}(\rho) < \infty$$

for all $(\alpha, \beta, \gamma) \in ((0,1] \times \mathbb{R})^{n+m-2k-r}$. If f is constant on the orbits of $\mathbb{H}[n]$ and f' is constant on the orbits of $\mathbb{H}[m]$, then $f \otimes_k^r f'$ is constant on the orbits of $\mathbb{H}[n + m - 2k - r]$.

Proof. For $g \in \mathbb{H}$ we simply obtain that

$$\begin{aligned}
&\big(f \otimes_k^r f'\big)(g[n-k-r]\alpha, g[m-k-r]\beta, g[r]\gamma) \\
= &\int_{((0,1]\times\mathbb{R})^k} f(g[n-k-r]\alpha, g[r]\gamma, \rho) f'(\rho, g[r]\gamma, g[m-k-r]\beta) \mathrm{dm}^{\otimes k}(\rho) \\
= &\int_{((0,1]\times\mathbb{R})^k} f(g[n-k-r]\alpha, g[r]\gamma, g[k]\rho) f'(g[k]\rho, g[r]\gamma, g[m-k-r]\beta) \mathrm{dm}^{\otimes k}(\rho) \\
= &\int_{((0,1]\times\mathbb{R})^k} f(\alpha, \gamma, \rho) f'(\rho, \gamma, \beta) \mathrm{dm}^{\otimes k}(\rho) \\
= &\big(f \otimes_k^r f'\big)(\alpha, \beta, \gamma).
\end{aligned}$$

□

For the proof of Theorem 5.3 we denote by $\big(f \hat{\otimes}_k^r f'\big)$ the symmetrization of $\big(f \otimes_k^r f'\big)$, and by $f_1 \hat{\otimes} \cdots \hat{\otimes} f_n$ the symmetrization of $f_1 \otimes \cdots \otimes f_n$.

Proof of Theorem 5.3. (2) $\Longrightarrow$ (1) follows from Theorem 3.2 and (1) $\Longrightarrow$ (2) from Theorems 3.2 and 4.3.

(2) $\Longrightarrow$ (3) We find an $f_n' = f_n$ a.e. that is $(\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^{\otimes n}$-measurable. By symmetrizing this f_n', we get a symmetric and $(\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^{\otimes n}$-measurable kernel.

(3) $\Longrightarrow$ (4) follows again from Theorem 3.2.

(4) $\Longrightarrow$ (3) By Theorem 3.2, we get symmetric kernels that are $\mathbb{H}[n]$-invariant. On the other side, Lemma 5.1 yields that $E_1, \ldots, E_L$ are locally ergodic with respect to $\mathbb{H}$ so that $\mathcal{I}(\mathbb{H}[n]) \subseteq (\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^{\otimes n} \vee \mathcal{N}_n$ by Theorem 4.3. One can finish as in (2) $\Longrightarrow$ (3).

(5) $\Longrightarrow$ (4) From Lemma 5.1 we know that $E_1, \ldots, E_L$ are locally ergodic with respect to $\mathbb{H}$. Next we observe that $I_1(f_1)$ is $\mathbb{H}$-invariant so that (using the arguments from (1) $\Longleftrightarrow$ (2) and the a.e. uniqueness of f_1) one can replace the f_1 by f_1' that is $\mathbb{H}[1]$-invariant. Therefore, F is $\mathcal{H}_{\mathbb{H}}$-measurable.

(3) $\Longrightarrow$ (5) Let $m \geq 1$ and $f_0, \ldots, f_m \in L_2^1$ be step-functions based on sets of type $A \times J$ with $A \in \{E_1, \ldots, E_L\}$ or $A \subseteq (E_1 \cup \cdots \cup E_L)^c$ is a Borel set and $J = (a, b]$ or $J = [-b, -a)$ with $0 < a < b < \infty$, or $J = \{0\}$. Then the f_i are constant on the orbits of $\mathbb{H}[1]$ and their integrability assures that we can we apply [16, Theorem 3.5] to get that

$$\begin{aligned}
&I_{m+1}(f_0 \hat{\otimes} f_1 \hat{\otimes} \cdots \hat{\otimes} f_m) \\
&= I_1(f_0) I_m(f_1 \hat{\otimes} \cdots \hat{\otimes} f_m) - m\big[I_m(f_0 \hat{\otimes}_0^1 (f_1 \hat{\otimes} \cdots \hat{\otimes} f_m)) + I_{m-1}(f_0 \hat{\otimes}_1^0 (f_1 \hat{\otimes} \cdots \hat{\otimes} f_m))\big].
\end{aligned}$$

Because of Lemma 5.5, all integrands occurring on the the right-hand hand side are constant on the orbits of $\mathbb{H}[1]$, $\mathbb{H}[m]$, $\mathbb{H}[m]$, and $\mathbb{H}[m-1]$, respectively. This implies that $I_{m+1}(f_0 \hat{\otimes} f_1 \hat{\otimes} \cdots \hat{\otimes} f_m)$ is measurable with respect to

$$\mathcal{H}_{\mathbb{H}}^{(m)} := \sigma\big(I_n(h_n) : h_n \text{ is } \mathbb{H}[n] - \text{invariant}, n \in \{1, \ldots, m\}\big) \vee \mathcal{N}.$$

Let h_{m+1} be symmetric and $(\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^{\otimes (m+1)}$-measurable. It is standard that finite linear combinations of tensor products $f_0 \hat{\otimes} f_1 \hat{\otimes} \cdots \hat{\otimes} f_m$ of the above form can be used to approximate h_{m+1} in L_2^{m+1}. Using that any L_2-convergent sequence contains a sequence that converges almost surely, we get that $I_{m+1}(h_{m+1})$

is already $\mathcal{H}_{\mathbb{H}}^{(m)}$-measurable. Induction over m and the identity $\mathcal{H}_{\mathbb{H}}^{(1)} = \mathcal{A}_E$ yield the implication (3) $\Longrightarrow$ (5).

Finally, the equivalence of the assertions in case $\mathbb{H} = \mathbb{G}$ is obvious as $E_1, \dots, E_L$ are locally ergodic with respect to $\mathbb{H}$ as already used above. □

Remark 5.6.

(1) If $E_1, \dots, E_L$ from Theorem 5.3 form a partition of $(0,1]$, then the symmetric kernels f_n in Theorem 5.3(3) are constant in the time variables on all cuboids $E_{l_1} \times \cdots \times E_{l_n}$ with $l_1, \dots, l_n \in \{1, \dots, L\}$.

(2) Given a system of $\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R})$-measurable $f_1^1, f_1^2, \dots$ such that

$$\mathcal{A}_E = \sigma\big(I_1(f_1^l) : l = 1, 2, \dots\big) \vee \mathcal{N},$$

Theorem 5.3(5) is equivalent to the fact that we find a functional $\Phi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, measurable with respect to the Borel σ-algebra on $\mathbb{R}^{\mathbb{N}}$ generated by the cylinder sets, such that

$$F = \Phi(I_1(f_1^1), I_1(f_1^2), \dots) \text{ a.s.}$$

This follows from a standard factorization due to Doob (see [2, Lemma II.11.7]). For example, for $E = (0,1]$, this leads to representations of F in terms of B_1 (the normalized Brownian part if present) and $N((0,1],(a,b))$ with $ab > 0$.

6. Examples and applications

6.1. **A negative example: Shift operators.** First we motivate the need of *locally ergodic* sets. We do this by considering the group generated by shifts, which is inspired by the work of Itô [13]. Assume that

$$F = I_2(f_2) \quad \text{where} \quad f_2((s,x),(t,y)) := g_2(|s-t|)h_2(x,y)$$

with a measurable function $g_2 : [0,1] \to \mathbb{R}$ such that $g_2(1/2 - s) = g_2(1/2 + s)$ for $s \in [0, 1/2]$ and a symmetric Borel function $h_2 : \mathbb{R}^2 \to \mathbb{R}$ such that $f_2 \in \tilde{L}_2^2$. It is straightforward to check that F is invariant with respect to all shifts $s_h : (0,1] \to (0,1]$, $0 < h < 1$, defined by $s_h(t) := t + h$ if $t + h \le 1$ and $s_h(t) := t + h - 1$ if $t + h > 1$. Obviously, the measure μ and the functions g_2 and h_2 can be chosen such that there is no symmetric $\tilde{f}_2((s,x),(t,y))$ not depending on (s,t), but with $f_2 = \tilde{f}_2$ a.e. (take for example μ as the Dirac measure in 1).

6.2. **Positive examples.** Our positive examples are based on Proposition 6.2 below for which we need the notion of *weak* $\mathbb{G}$-invariance:

Definition 6.1. Given a sub-group $\mathbb{G} \subseteq \mathbb{M}^{\text{dyad}}$, we say that an $\mathcal{F}^X$-measurable random variable $Z : \Omega \to \mathbb{R}$ is *weakly* $\mathbb{G}$-invariant provided that $f(Z)$ is $\mathbb{G}$-invariant for all $f \in \mathcal{C}_b(\mathbb{R})$.

$\mathbb{G}$-invariance implies *weak* $\mathbb{G}$-invariance by Lemma B.1, but the converse does not need to be true because of a possibly missing integrability. To consider our examples, let us fix a sequence of time-points

$$0 \le s_1 < t_1 \le \ldots \le s_L < t_L \le 1$$

together with the corresponding intervals $\widetilde{E}_l := (s_l, t_l]$ for the rest of this section. Similarly as before, we let

$$\mathcal{B}((0,1])_{\widetilde{E}} := \mathcal{B}\left((0,1] \setminus \bigcup_{l=1}^{L}(s_l,t_l]\right) \vee \sigma\left((s_1,t_1],\dots,(s_L,t_L]\right).$$

Proposition 6.2. *Assume $\mathcal{F}^X$-measurable and weakly $\mathbb{M}^{\mathrm{dyad}}_{(s_l,t_l]}$-invariant $Z_1,\dots,Z_N\colon \Omega \to \mathbb{R}$ for $l \in \{1,\dots,L\}$, and let $f\colon \mathbb{R}^N \to \mathbb{R}$ be a Borel function with $F = f(Z_1,\dots,Z_N) \in L_2(\mathcal{F}^X)$. Then, there are $(\mathcal{B}((0,1])_{\widetilde{E}} \otimes \mathcal{B}(\mathbb{R}))^{\otimes n}$-measurable and symmetric chaos kernels $\tilde{f}_n$ for F. In particular, they are constant on the cuboids*

$$\prod_{j=1}^{n}(s_{l_j}, t_{l_j}] \quad \textit{for} \quad l_1,\dots,l_n \in \{1,\dots,L\}.$$

Proof. The variables $\varphi(Z_k)$, $k = 1,\dots,N$, are $\mathbb{M}^{\mathrm{dyad}}_{(s_l,t_l]}$-invariant, where $\varphi(x) := \arctan(x)$. Letting $\psi(y) := \tan(y)$ for $y \in (-\pi/2, \pi/2)$ and $\psi(y) := 0$ otherwise, and using the change of variables $g(y_1,\dots,y_N) := f(\psi(y_1),\dots,\psi(y_N))$, Lemma 3.4 implies that $F = g(\varphi(Z_1),\dots,\varphi(Z_N))$ is $\mathbb{M}^{\mathrm{dyad}}_{(s_l,t_l]}$-invariant. The sets $E_l := (s_l,t_l]$ if t_l is dyadic, and $E_l := (s_l,t_l)$ otherwise, belong to $\mathcal{O}((0,1])$. According to Lemma 5.1 the set E_l is locally ergodic with respect to $\mathbb{M}^{\mathrm{dyad}}_{E_l}$ and therefore with respect to the group generated by $\mathbb{M}^{\mathrm{dyad}}_{E_1},\dots,\mathbb{M}^{\mathrm{dyad}}_{E_L}$. Furthermore, observing that $\mathbb{M}^{\mathrm{dyad}}_{E_l} = \mathbb{M}^{\mathrm{dyad}}_{(s_l,t_l)} = \mathbb{M}^{\mathrm{dyad}}_{(s_l,t_l]}$ if t_l is not dyadic, Theorem 5.3 gives the existence of symmetric kernels f_n that are $(\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^{\otimes n}$-measurable. Modifying the kernels on a null-set yields the assertion. □

6.2.1. *Doléans-Dade stochastic exponential.* We follow [8] and assume X to be L_2-integrable and of mean zero. For $0 \le a \le t \le 1$ we let

$$S_t^a := 1 + \sum_{n=1}^{\infty} \frac{I_n(\mathbb{1}^{\otimes n}_{(a,t]})}{n!},$$

where we can assume that all paths of $(S_t^a)_{t\in[a,1]}$ are càdlàg for any fixed $a \in [0,1]$. Then we get that

$$S_t^a = 1 + \int_{(a,t]} S_{u-}^a \mathrm{d}X_u \text{ a.s.} \quad \text{and} \quad S_t = S_t^a S_a \text{ a.s.} \quad \text{with} \quad S_t := S_t^0.$$

Therefore we get from the chaos representation of S_t^a:

Lemma 6.3. *Each random variable $S_{s_l}^{t_l}$ is $\mathbb{M}^{\mathrm{dyad}}_{(s_k,t_k]}$-invariant for $k = 1,\dots,L$.*

One could continue the investigation by using more general Doléans-Dade exponential formulas (see for example [20, Chapter II, Theorem 37]), which is not done here.

6.2.2. *Limit functionals.* Behind the next examples there is a common idea formulated in

Definition 6.4. For $0 \le s < t \le 1$ a random variable $Z\colon \Omega \to \mathbb{R}$ belongs to the class $C(s,t]$ provided that there exists a sequence $0 \le N_1 < N_2 < \dots$ of integers

and Borel functions $\Phi_k\colon \mathbb{R}^{M_k} \to \mathbb{R}$ such that

$$Z = \lim_{k\to\infty} Z^k := \lim_{k\to\infty} \Phi_k \Bigg(X_{\frac{a_k}{2^{N_k}}} - X_s, X_{\frac{a_k+1}{2^{N_k}}} - X_{\frac{a_k}{2^{N_k}}}, \dots, \\ X_{\frac{b_k}{2^{N_k}}} - X_{\frac{b_k-1}{2^{N_k}}}, X_t - X_{\frac{b_k}{2^{N_k}}} \Bigg) \quad \text{a.s.},$$

where $\frac{a_k}{2^{N_k}}$ is the smallest grid point greater than or equal to s and $\frac{b_k}{2^{N_k}}$ is the largest grid point smaller than or equal to t, $M_k := b_k - a_k + 2$, and the function Φ_k is symmetric in its arguments where the first and last coordinate are excluded.[2]

Proposition 6.5. *Let $Z_1, \dots, Z_L : \Omega \to \mathbb{R}$ be random variables such that Z_l belongs to the class $C(s_l, t_l]$ for $l = 1, \dots, L$, and let $f\colon \mathbb{R}^L \to \mathbb{R}$ be a Borel function with $F := f(Z_1, \dots, Z_L) \in L_2(\mathcal{F}^X)$. Then, there are $(\mathcal{B}((0,1])_{\widetilde{E}} \otimes \mathcal{B}(\mathbb{R}))^{\otimes n}$-measurable and symmetric chaos kernels $\tilde{f}_n$ for F. In particular, they are constant on the cuboids*

$$\prod_{j=1}^{n} (s_{l_j}, t_{l_j}] \quad \textit{for} \quad l_1, \dots, l_n \in \{1, \dots, L\}.$$

Proof. By Proposition 6.2, it is sufficient to show that $Z_1, \dots, Z_L$ are weakly $\mathbb{M}^{\text{dyad}}_{(s_l,t_l]}$-invariant for $l \in \{1, \dots, L\}$, i.e., that $\varphi(Z_m)$ is $\mathbb{M}^{\text{dyad}}_{(s_l,t_l]}$-invariant for $\varphi \in \mathcal{C}_b(\mathbb{R})$ and $m, l \in \{1, \dots, L\}$. Let $g \in \mathbb{M}^{\text{dyad}}_{(s_l,t_l]}$ (be not the identity). Then there exists an integer $M \geq 0$ such that g acts as a permutation of the dyadic intervals of length 2^{-M} and as an identity on $(s_l, t_l]^c$. Therefore, there exist integers $0 \leq a < b \leq 2^M$ such that

$$(s,t] := \left(\frac{a}{2^M}, \frac{b}{2^M} \right] \subseteq (s_l, t_l]$$

and g can be described by permuting dyadic intervals on $(s,t]$ of length 2^{-M}. By Definition 6.4, there is an approximation $Z_m = \lim_{k\to\infty} Z_m^k$ a.s. By construction, there is a $k_0 \geq 1$ such that for all $k \geq k_0$ one has that $\varphi(Z_m^k)$ is T_g-invariant (here one has to distinguish between the cases $m = l$ and $m \neq l$). By dominated convergence, $\lim_{k\to\infty} \varphi(Z_m^k) = \varphi(Z_m)$ in $L_2(\mathcal{F}^X)$ so that $\varphi(Z_m)$ is invariant with respect to T_g as well and the proof is complete. □

Example 6.6. For $0 \leq s < t \leq 1$ the following random variables belong to the class $C(s,t]$:

(1) $X_t - X_s$.
(2) $[X,X]_t - [X,X]_s$, where $[X,X]$ is the quadratic variation process of X.
(3) $\sup_{r\in(s,t]} |X_r - X_{r-}|$.

Proof. (1) is obvious.

(2) Here we first take $\Phi_k(x_1, \dots, x_{M_k}) := |x_1|^2 + \dots + |x_{M_k}|^2$ with $N_k = k \geq k_0$, use [20, Chapter II, Theorem 22] to get a sequence that converges in probability, and extract a sub-sequence that converges almost surely.

(3) Taking $\Phi_k(x_1, \dots, x_{M_k}) := \max\{|x_1|, \dots, |x_{M_k}|\}$ and $N_k := k$ with $k \geq k_0$ and the uniformity result for càdlàg paths [4, Chapter 3, Lemma 1] yields the assertion. □

[2]Here and in the following it is implicitly assumed that the partitions are taken always in a way that $\frac{a_k}{2^{N_k}} < \frac{b_k}{2^{N_k}}$ by choosing N_k large enough.

Remark 6.7. Combining Proposition 6.5 with Example 6.6(1) yields that the symmetric chaos kernels f_n of $F = f(X_{t_1} - X_{s_1}, \dots, X_{t_L} - X_{s_L})$ can be chosen to be constant on the cuboids

$$\prod_{j=1}^{n} (s_{l_j}, t_{l_j}] \quad \text{for} \quad l_1, \dots, l_n \in \{1, \dots, L\}.$$

This was used in [12] in the investigation of variational properties of Backward Stochastic Differential Equations driven by Lévy processes.

6.3. **An application to the chaotic representation property (CRP).** In this section we show how our results relate to the chaotic representation property (see [17,18,24] and for recent results [6]). For example, we consider the chaos expansion due to Nualart and Schoutens. The investigation to which extend more general expansions (like in [6] and particularly in their Section 6) can be treated is left to future research. For this subsection we assume that the Lévy measure satisfies $\int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda|x|)\mathrm{d}\nu(x) < \infty$ for some $\lambda, \varepsilon > 0$. Then [17, Theorem 3] (see also [24, Section 2.2]) gives an orthogonal decomposition

$$L_2(\mathcal{F}^X) = \mathbb{R} \oplus \left(\bigoplus_{n=1}^{\infty} \bigoplus_{i_1,\dots,i_n \geq 1} \mathcal{H}^{(i_1,\dots,i_n)} \right),$$

where the spaces $\mathcal{H}^{(i_1,\dots,i_n)}$ are the range of the n-fold iterated integrals $J_n^{i_1,\dots,i_n} : L_2(\Delta_n, \lambda_n) \to \mathcal{H}^{(i_1,\dots,i_n)}$ with respect to the martingales $H^{(i_1)}, \dots, H^{(i_n)}$, obtained by an orthogonalization of the Teugels martingales, with $\Delta_n := \{0 < t_1 < \dots < t_n \leq 1\}$ and λ_n being the Lebesgue measure on Δ_n. That means we have an expansion

$$F = \mathbb{E}F + \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n \geq 1} J^{(i_1,\dots,i_n)}(g_{i_1,\dots,i_n}) \quad \text{with} \quad g_{i_1,\dots,i_n} \in L_2(\Delta_n, \lambda_n). \tag{6.1}$$

In the following we explain that, although the spaces $\mathcal{H}^{(i_1,\dots,i_n)}$ are not invariant with respect to the operators T_g, the invariance properties still transfer.

(a) The spaces $\mathcal{H}^{(i_1,\dots,i_n)}$ are not invariant with respect to the operators T_g in general. To see this, let

$$F := J_n^{i_1,\dots,i_n}(\mathbb{1}_{I_1 \times \dots \times I_n})$$

with pairwise disjoint dyadic intervals $I_j := ((k_j - 1)/2^d, k_j/2^d] \subseteq (0, 1]$ with $1 \leq k_1 < \dots < k_n \leq 2^d$. Let $\pi \in \mathcal{S}_{2^d}$ and $g := g_\pi \in \mathbb{M}^{\mathrm{dyad}}$ be the corresponding measure preserving map. By [24, Proposition 7] (the statement is given without proof, a proof can be found in [6, Proposition 6.9]) and Theorem 2.8 of this article we derive that

$$T_g F = J_n^{i_{\sigma(1)},\dots,i_{\sigma(n)}} \left(\mathbb{1}_{g(I_{\sigma(1)}) \times \dots \times g(I_{\sigma(n)})} \right)$$

where $\sigma \colon \{1, \dots, n\} \to \{1, \dots, n\}$ is the permutation such that the family of intervals $(g(I_{\sigma(j)}))_{j=1}^n$ is ascending. Therefore, the chaos $\mathcal{H}^{(i_1,\dots,i_n)}$ is not stable with respect to T_g in general.

(b) The symmetries from the kernels we consider in Theorem 3.2 transfer to the kernels in (6.1) in the following sense: Assume a $\mathbb{G}$-invariant $F \in L_2(\mathcal{F}^X)$ and the corresponding symmetric and $\mathbb{G}$-invariant kernels $(f_n)_{n \in \mathbb{N}}$ as in Theorem 3.2(4). Let $p_1, p_2, \dots \in L_2(\mathbb{R}, \mu)$ be the orthogonal polynomials from [24, Section 2.1]

and $q_n := \|p_n\|^2_{L_2(\mathbb{R},\mu)}$. For integers $i_1,\dots,i_n \geq 1$ with $q_{i_1}\cdots q_{i_n} > 0$ we define $g_{i_1,\dots,i_n} \in L_2(\Delta_n,\lambda_n)$ to be

$$g_{i_1,\dots,i_n}(t_1,\dots,t_n) := n!\int_{\mathbb{R}^n} f_n((t_1,x_1),\dots,(t_n,x_n))\frac{p_{i_1}(x_1)\cdots p_{i_n}(x_n)}{q_{i_1}\cdots q_{i_n}}\mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n), \tag{6.2}$$

where we can assume that the above integral exists pointwise *for all* $(t_1,\dots,t_n) \in \Delta_n$ (consider

$$A_n := \{(t_1,\dots,t_n) \in (0,1]^n : \int_{\mathbb{R}^n} |f_n((t_1,x_1),\dots,(t_n,x_n))|^2\mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n) < \infty\},$$

which is symmetric with $\lambda^{\otimes n}(A_n) = 1$, and so that $A_n \times \mathbb{R}^n$ is $\mathbb{G}[n]$-invariant with a slight abuse of notation concerning the order of components; finally replace f_n by $f_n \mathbb{1}_{A_n\times\mathbb{R}^n}$). By [24, Proposition 7] we derive that

$$\sum_{n=1}^{\infty} I_n(f_n) = \sum_{n=1}^{\infty}\sum_{\substack{i_1,\dots,i_n\geq 1\\ q_{i_1}\cdots q_{i_n}>0}} J^{(i_1,\dots,i_n)}(g_{i_1,\dots,i_n}) \text{ a.s.}$$

and the properties of f_n in $(t_1,\dots,t_n)$ *on* Δ_n directly transfer to the kernels $g_{i_1,\dots,i_n}(t_1,\dots,t_n)$ from (6.1). Moreover, (6.2) and the symmetry of f_n imply that

$$g_{i_1,\dots,i_n}(g(t_{\sigma(1)}),\dots,g(t_{\sigma(n)})) = g_{i_{\sigma^{-1}(1)},\dots,i_{\sigma^{-1}(n)}}(t_1,\dots,t_n)$$

for $g \in \mathbb{M}^{\mathrm{dyad}}$ and the permutation $\sigma\colon \{1,\dots,n\} \to \{1,\dots,n\}$ that guarantees $g(t_{\sigma(1)}) < \cdots < g(t_{\sigma(n)})$.

6.4. **An application to Backward Stochastic Differential Equations (BSDEs).** An example of a BSDE driven by a Lévy process is a formal equation of the form

$$\begin{aligned} Y_t = F &+ \int_{(t,1]} f\left(s, Y_s, \left(\int_{\mathbb{R}} Z_{s,x}h_k(x)\mathrm{d}\mu(x)\right)_{k=1}^N\right)\mathrm{d}s \\ &- \int_{(t,1]\times\mathbb{R}} Z_{s,x}\mathrm{d}M(s,x) \text{ a.s.}, \quad t\in[0,1], \end{aligned} \tag{6.3}$$

where $h_1,\dots,h_N \in L_2(\mathbb{R},\mu)$ for some $N \in \mathbb{N}$, and further typical assumptions are $F \in L_2(\mathcal{F}^X)$ for the *terminal condition*, and for the *generator* $f\colon [0,1]\times\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ certain assumptions regarding adaptedness, that $\int_{(0,1]}|f(s,0,0)|\mathrm{d}s \in L_2(\mathcal{F}^X)$, and that f is Lipschitz in (y,z), uniformly in (s,ω). The initial data of the BSDE (6.3) are (F,f) and one seeks the solution processes (Y,Z) that consist of an adapted càdlàg process $Y = (Y_t)_{t\in[0,1]}$ and a predictable $Z = (Z_{s,x})_{(s,x)\in[0,1]\times\mathbb{R}}$, both satisfying certain integrability conditions. In order to solve BSDE (6.3) one might use Picard iterations. The aim of this section is to consider this Picard iteration separately and to demonstrate with this how the results and concepts of this paper contribute to the BSDE theory. In particular, we wish to emphasize that we express the properties in which we are interested in terms of measurability in parts (2) and (3) of Definition 6.8 below which allows us to compose random objects with these properties and the resulting objects automatically share the same property (see the proof of Theorem 6.12 below). As we detach the Picard iteration from the remaining BSDE theory we can keep our assumptions on the generator f below minimal so that our results might be applied for different types of BSDEs.

Setting. In the following we let $\mathcal{F}_t^X := \sigma(X_s : s \in [0,t]) \vee \{A \in \mathcal{F}^X : \mathbb{P}(A) = 0\}$ and obtain a right-continuous filtration $(\mathcal{F}_t^X)_{t\in[0,1]}$ with $\mathcal{F}^X = \mathcal{F}_1^X$. The symbol $\mathcal{P}$ denotes the σ-algebra of predictable events on $[0,1] \times \Omega$, i.e., $\mathcal{P}$ is generated by all adapted path-wise continuous processes. A process $Z = (Z_{s,x})_{(s,x)\in[0,1]\times\mathbb{R}}$ is called predictable if $Z\colon [0,1] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable. Our notion of invariance adapted to BSDEs reads as follows.

Definition 6.8. Let $Y\colon [0,1]\times\Omega \to \mathbb{R}$ be $\mathcal{B}([0,1])\otimes\mathcal{F}^X$-measurable, $Z\colon [0,1]\times\Omega\times\mathbb{R} \to \mathbb{R}$ be $\mathcal{B}([0,1]) \otimes \mathcal{F}^X \otimes \mathcal{B}(\mathbb{R})$-measurable, and $\mathbb{G}$ be a group of dyadic measure preserving maps $\mathbb{G} \subseteq \mathbb{M}^{\mathrm{dyad}}$.

(1) For $t \in [0,1)$ we let $\mathbb{G}_t := \{g \in \mathbb{G} : g(s) = s \text{ for all } s \in (t,1]\}$.
(2) We say that Y is $(\mathbb{G}_t)_{t\in(0,1)}$-invariant provided that for all $t \in (0,1)$ there is a $\mathcal{B}((t,1])\otimes\mathcal{H}_{\mathbb{G}_t}$-measurable $Y^{\mathbb{G}_t}\colon (t,1]\times\Omega \to \mathbb{R}$ with $\mathbb{E}\int_{(t,1]} |Y_s - Y_s^{\mathbb{G}_t}|\mathrm{d}s = 0$.
(3) We say that Z is $(\mathbb{G}_t)_{t\in(0,1)}$-invariant provided that for all $t \in (0,1)$ there is a $\mathcal{B}((t,1]) \otimes \mathcal{H}_{\mathbb{G}_t} \otimes \mathcal{B}(\mathbb{R})$-measurable $Z^{\mathbb{G}_t}\colon (t,1] \times \Omega \times \mathbb{R} \to \mathbb{R}$ with $\mathbb{E}\int_{(t,1]\times\mathbb{R}} |Z_{s,x} - Z_{s,x}^{\mathbb{G}_t}|\mathrm{d}\mathrm{m}(s,x) = 0$.

Of course, part (2) of the definition above is (in a sense) a special case of part (3). We obtain a family of subgroups $(\mathbb{G}_t)_{t\in(0,1)}$ of $\mathbb{G}$ with $\mathbb{G}_s \subseteq \mathbb{G}_t$ for $0 < s < t < 1$. Moreover, if $G \in L_2(\mathcal{F}^X)$ is $\mathbb{G}_t$-invariant, then $\mathbb{E}(G|\mathcal{F}_t^X)$ is $\mathbb{G}_t$-invariant as well since the conditional expectation corresponds to a restriction of the invariant kernel functions f_n to $f_n \mathbb{1}_{(0,t]^n}$ for $n \in \mathbb{N}$.

Picard scheme. We introduce operators $A_{F,f}$ and $B_{F,f}$ in (6.4) below that are used to solve BSDEs by Picard type iterations. For this purpose we fix the initial data of our BSDE and let the generator $f\colon [0,1] \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be such that

(1) $f(t,\omega,\cdot,\cdot)\colon \mathbb{R}^{1+N} \to \mathbb{R}$ is continuous for all $(t,\omega) \in [0,1] \times \Omega$,
(2) $f\colon [0,t] \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t^X \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^N)$-measurable for $t \in [0,1]$, and
(3) $F \in L_2(\mathcal{F}^X)$.

One could also investigate the case that f depends on its last coordinate on a sequence $(z_k)_{k=1}^\infty$ (for example Fourier coefficients with respect to an orthonormal basis $(h_k)_{k=1}^\infty$ in $L_2(\mathbb{R},\mu)$), but for simplicity we restricted ourselves to finite sequences.

Definition 6.9. Let $Y\colon [0,1] \times \Omega \to \mathbb{R}$ and $Z\colon [0,1] \times \Omega \times \mathbb{R} \to \mathbb{R}$. For $N \in \mathbb{N}$ and $h_1, \dots, h_N \in L_2(\mathbb{R},\mu)$ we say $(Y,Z) \in \mathcal{D}_f^{h_1,\dots,h_N}$ if

(1) the restriction of Y to $[0,t] \times \Omega \to \mathbb{R}$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t^X$-measurable for $t \in [0,1]$,
(2) the restriction of Z to $[0,t]\times\Omega\times\mathbb{R} \to \mathbb{R}$ is $\mathcal{B}([0,t])\otimes\mathcal{F}_t^X\otimes\mathcal{B}(\mathbb{R})$-measurable for $t \in [0,1]$ and $Z_{s,\cdot}(\omega) \in L_2(\mathbb{R},\mu)$ for all $(s,\omega) \in [0,1] \times \Omega$,
(3) $\int_{(0,1]} |f\big(s, Y_s, (\int_{\mathbb{R}} Z_{s,x} h_k(x)\mathrm{d}\mu(x))_{k=1}^N\big)|\mathrm{d}s \in L_2(\mathcal{F}^X)$.

Let $\mathcal{S}_2$ be the space of adapted càdlàg processes $Y = (Y_t)_{t\in[0,1]}$ with $\|Y\|_{\mathcal{S}_2} := \|\sup_{t\in[0,1]} |Y_t|\|_2 < \infty$ and

$$\mathcal{P}_2 := \Big\{ Z\colon [0,1]\times\Omega\times\mathbb{R}\to\mathbb{R} \text{ predictable with } \|Z\|^2_{\mathcal{P}_2} := \mathbb{E}\int_{(0,1]\times\mathbb{R}} |Z_{s,x}|^2 \mathrm{d}\mathrm{m}(s,x) < \infty \text{ and } Z_{s,\cdot}(\omega)\in L_2(\mathbb{R},\mu) \text{ for all } (s,\omega)\in[0,1]\times\Omega\Big\}.$$

For any predictable $Z\colon [0,1]\times\Omega\times\mathbb{R}\to\mathbb{R}$ with $\mathbb{E}\int_{(0,1]\times\mathbb{R}} |Z_{s,x}|^2\mathrm{d}\mathrm{m}(s,x) < \infty$ one can find a $Z'\in\mathcal{P}_2$ with $\mathbb{E}\int_{(0,1]\times\mathbb{R}} |Z_{s,x} - Z'_{s,x}|\mathrm{d}\mathrm{m}(s,x) = 0$, so that the last part of the definition of $\mathcal{P}_2$ is not a restriction for us.

Definition 6.10. We let $A_{F,f}\colon \mathcal{D}_f^{h_1,\dots,h_N}\to\mathcal{S}_2$ and $B_{F,f}\colon \mathcal{D}_f^{h_1,\dots,h_N}\to\mathcal{P}_2$ be given by

$$A_{F,f}(Y,Z) := (\overline{Y}_t)_{t\in[0,1]} \quad\text{and}\quad B_{F,f}(Y,Z) := (\overline{Z}_{t,x})_{(t,x)\in[0,1]\times\mathbb{R}}, \tag{6.4}$$

where

$$\overline{Y}_t := \mathbb{E}\left(F + \int_{(t,1]} f\left(s, Y_s, \left(\int_{\mathbb{R}} Z_{s,x} h_k(x)\mathrm{d}\mu(x)\right)_{k=1}^N\right)\mathrm{d}s \,\middle|\, \mathcal{F}_t^X\right),$$

and the process $\overline{Z}$ is determined by

$$\eta = \mathbb{E}\eta + \int_{(0,1]\times\mathbb{R}} \overline{Z}_{s,x}\mathrm{d}M(s,x)$$

with

$$\eta := F + \int_{(0,1]} f\left(s, Y_s, \left(\int_{\mathbb{R}} Z_{s,x} h_k(x)\mathrm{d}\mu(x)\right)_{k=1}^N\right)\mathrm{d}s.$$

By Definition 6.10 we mean that $A_{F,f}$ and $B_{F,f}$ map to the corresponding equivalence classes (in $\mathcal{S}_2$ elements of one class are indistinguishable, in $\mathcal{P}_2$ they coincide a.e. with respect to $\lambda\otimes\mathbb{P}\otimes\mu$) and we choose one element from each equivalence class in applications. In the BSDE-context iteratives of the operator $A_{F,f}$ usually converge to a *generalized non-linear conditional expectation* of the terminal condition F along the generator f, and iteratives of $B_{F,f}$ to a generalized *non-linear gradient* of F along the generator f.

Remark 6.11. (1) The càdlàg modification of $\overline{Y}$ can be obtained by observing

$$\begin{aligned}\mathbb{E}&\left(F + \int_{(t,1]} f\left(s, Y_s, \left(\int_{\mathbb{R}} Z_{s,x} h_k(x)\mathrm{d}\mu(x)\right)_{k=1}^N\right)\mathrm{d}s \,\middle|\, \mathcal{F}_t^X\right)\\ &= \mathbb{E}\left(F + \int_{(0,1]} f\left(s, Y_s, \left(\int_{\mathbb{R}} Z_{s,x} h_k(x)\mathrm{d}\mu(x)\right)_{k=1}^N\right)\mathrm{d}s \,\middle|\, \mathcal{F}_t^X\right)\\ &\qquad - \int_{(0,t]} f\left(s, Y_s, \left(\int_{\mathbb{R}} Z_{s,x} h_k(x)\mathrm{d}\mu(x)\right)_{k=1}^N\right)\mathrm{d}s \text{ a.s.}\end{aligned}$$

(2) To obtain $\overline{Z}$ (which is $\lambda\otimes\mathbb{P}\otimes\mu$-a.e. unique because of Itô's isometry) we use the representation property of the random measure M; cf. [1, Chapter 4].

Result. Our contribution is to show that the abstract Picard scheme is invariant with respect to $(\mathbb{G}_t)_{t\in(0,1)}$ provided that the terminal condition F is $\mathbb{G}$-invariant and the generator f is $(\mathbb{G}_t)_{t\in(0,1)}$-invariant:

Theorem 6.12. *Assume $(Y,Z)\in\mathcal{D}_f^{h_1,\dots,h_N}$ and $F\in L_2(\mathcal{F}^X)$ such that*

(i) *F is $\mathbb{G}$-invariant,*
(ii) *for all $t\in(0,1)$ and $(y,(z^k)_{k=1}^N)\in\mathbb{R}^{1+N}$ the restricted generator*

$$f(\cdot,\cdot,y,(z^k)_{k=1}^N):(t,1]\times\Omega\to\mathbb{R} \text{ is } \mathcal{B}((t,1])\otimes\mathcal{H}_{\mathbb{G}_t}\text{-measurable,}$$

(iii) *Y and Z are $(\mathbb{G}_t)_{t\in(0,1)}$-invariant.*

Then the following holds:

(1) *$A_{F,f}(Y,Z)_t$ is $\mathbb{G}_t$-invariant for all $t\in(0,1)$.*
(2) *$B_{F,f}(Y,Z)$ is $(\mathbb{G}_t)_{t\in(0,1)}$-invariant.*

In order to prove Theorem 6.12 we need the following lemma:

Lemma 6.13. *Let $t\in(0,1)$ and*

$$G=\int_{(t,1]\times\mathbb{R}} Z_{s,x}\mathrm{d}M(s,x)\in\mathbf{H}_{\mathbb{G}_t}$$

for some predictable Z with $\mathbb{E}\int_{(0,1]\times\mathbb{R}}|Z_{s,x}|^2\mathrm{d}\mathrm{m}(s,x)<\infty$. Then there is a $\mathcal{B}((t,1])\otimes\mathcal{H}_{\mathbb{G}_t}\otimes\mathcal{B}(\mathbb{R})$-measurable $Z^{\mathbb{G}_t}:(t,1]\times\Omega\times\mathbb{R}\to\mathbb{R}$ with $\mathbb{E}\int_{(t,1]\times\mathbb{R}}|Z_{s,x}-Z^{\mathbb{G}_t}_{s,x}|\mathrm{d}\mathrm{m}(s,x)=0$.

Proof. Assume that $G=\sum_{n=1}^\infty I_n(f_n)$, where the $(f_n)_{n\in\mathbb{N}}$ are symmetric chaos kernels that are constant on the orbits of $\mathbb{G}_t[n]$ on $((0,1]\times\mathbb{R})^n$ (see Theorem 3.2(4)). For an integer $N\geq 0$ define

$$I^t_{N,0}:=(0,t]\quad\text{and}\quad I^t_{N,k}:=\left(t+(1-t)\frac{k-1}{2^N},t+(1-t)\frac{k}{2^N}\right]$$

for $k=1,\dots,2^N$, and

$$J_{N,0}:=\{0\},\quad J_{N,l}:=\left[\frac{l-1}{2^N},\frac{l}{2^N}\right),\quad\text{and } J_{N,m}:=\left(\frac{m-1}{2^N},\frac{m}{2^N}\right]$$

for $l=0,-1,\dots$ and $m=1,2,\dots$. The corresponding σ-algebras are given by

$$\mathcal{G}^t_N:=\mathcal{B}((0,t])\vee\sigma\left(I^t_{N,k}:k=1,\dots,2^N\right)\quad\text{and}\quad\mathcal{S}_N:=\sigma\left(J_{N,l}:l\in\mathbb{Z}\right).$$

We have that $(\mathcal{B}((0,1])\otimes\mathcal{B}(\mathbb{R}))^{\otimes n}=\bigvee_{N\geq 0}(\mathcal{G}^t_N\otimes\mathcal{S}_N)^{\otimes n}$. For $P^t_{N,k}\colon L_2((0,1])\to L_2((0,1])$ and $Q_{N,l}\colon L_2(\mathbb{R},\mu)\to L_2(\mathbb{R},\mu)$ given by $P^t_{N,0}g:=\mathbb{1}_{I^t_{N,0}}g$, $Q_{N,0}h:=\mathbb{1}_{J_{N,0}}h(0)$,

$$P^t_{N,k}g:=\mathbb{1}_{I^t_{N,k}}\int_{I^t_{N,k}}g(s)\frac{\mathrm{d}s}{|I^t_{N,k}|},\quad\text{and}\quad Q_{N,l}h:=\mathbb{1}_{J_{N,l}}\int_{J_{N,l}}h(x)\frac{\mathrm{d}\mu(x)}{\mu(J_{N,l})},$$

where $k=1,\dots,2^N$ and $l\in\mathbb{Z}\setminus\{0\}$, and where we agree about $Q_{N,l}:=0$ if $\mu(J_{N,l})=0$, we define pointwise

$$\begin{aligned}&\mathbb{E}\left(f_n\,\middle|(\mathcal{G}^t_N\otimes\mathcal{S}_N)^{\otimes n}\right)\\&\quad:=\sum_{k_1=0}^{2^N}\sum_{l_1=-\infty}^{\infty}\cdots\sum_{k_n=0}^{2^N}\sum_{l_n=-\infty}^{\infty}\left[A^{t,N}_{\substack{k_1,\dots,k_n\\l_1,\dots,l_n}}f_n\right]:((0,1]\times\mathbb{R})^n\to\mathbb{R}\end{aligned}$$

with

$$A^{t,N}_{\substack{k_1,\dots,k_n\\ l_1,\dots,l_n}} := [P^t_{N,k_1}\otimes Q_{N,l_1}]\otimes\cdots\otimes[P^t_{N,k_n}\otimes Q_{N,l_n}].$$

Using the σ-finiteness of μ, dominated convergence in the sequence space ℓ_1, and martingale convergence, one sees that $\mathbb{E}\left(f_n|(\mathcal{G}^t_N\otimes\mathcal{S}_N)^{\otimes n}\right)$ converges to f_n in L^n_2 as $N\to\infty$. It is easy to verify (cf. [13]) that we can exclude the diagonal terms related to the time interval $(t,1]$ in the following sense: We can approximate f_n in L^n_2 by finite sums where each summand is of form

$$f^0_n := \sum_{\pi\in S_n} A^{t,N}_{\substack{k_{\pi(1)},\dots,k_{\pi(n)}\\ l_{\pi(1)},\dots,l_{\pi(n)}}} f_n$$

for some $0=k_1=\cdots=k_{n_0}<k_{n_0+1}<\cdots<k_n\le 2^N$ with $n_0\in\{0,\dots,n-1\}$ and $l_1,\dots,l_n\in\mathbb{Z}$. The case $n_0=0$ means that $k_1\ge 1$, the case $n_0=n$ can be excluded as it would imply $I_n(f^0_n)=0$ a.s. because $\mathbb{E}(G|\mathcal{F}^X_t)=0$ a.s. by our assumption. For $(s,x)\in(0,1]\times\mathbb{R}$ we set

$$Z^0_{s,x} := nI_{n-1}\left(f^0_n(\dots,(t^0,x^0))\mathbb{1}_{(0,T_{n-1}]^{n-1}}\right)\mathbb{1}_{I^t_{N,k_n}\times J_{N,l_n}}(s,x)$$

with $T_{n-1}:=t+(1-t)\frac{k_{n-1}}{2^N}$ and arbitrary $(t^0,x^0)\in I^t_{N,k_n}\times J_{N,l_n}$. By construction, $Z^0\colon(0,1]\times\Omega\times\mathbb{R}\to\mathbb{R}$ is predictable and the restriction $Z^0\colon(t,1]\times\Omega\times\mathbb{R}\to\mathbb{R}$ is $\mathcal{B}((t,1])\otimes\mathcal{H}_{\mathbb{G}_t}\otimes\mathcal{B}(\mathbb{R})$-measurable as $f^0_n(\dots,(t^0,x^0))\mathbb{1}_{(0,T_{n-1}]^{n-1}}$ is $\mathbb{G}_t[n-1]$-invariant by construction. Finally, we have that

$$I_n(f^0_n)=\int_{(t,1]\times\mathbb{R}} Z^0_{s,x}\mathrm{d}M(s,x)\text{ a.s.}$$

This can be extended to finite linear combinations that approximate f_n in L^n_2. The corresponding restrictions of Z-processes form a Cauchy sequence in

$$L_2\big((t,1]\times\Omega\times\mathbb{R},\mathcal{B}((t,1])\otimes\mathcal{H}_{\mathbb{G}_t}\otimes\mathcal{B}(\mathbb{R}),\lambda\otimes\mathbb{P}\otimes\mu\big)$$

and we find a $\mathcal{B}((t,1])\otimes\mathcal{H}_{\mathbb{G}_t}\otimes\mathcal{B}(\mathbb{R})$-measurable limit as well. □

Proof of Theorem 6.12. (1) We fix $t\in(0,1)$, replace $(Y_s)_{s\in(t,1]}$ by $(Y^{\mathbb{G}_t}_s)_{s\in(t,1]}$, $(Z_{s,x})_{(s,x)\in(t,1]\times\mathbb{R}}$ by $(Z^{\mathbb{G}_t}_{s,x})_{(s,x)\in(t,1]\times\mathbb{R}}$, and have

$$\overline{Y}_t=\mathbb{E}\left(F+\int_{(t,1]} f\left(s,Y^{\mathbb{G}_t}_s,\left(\int_{\mathbb{R}} Z^{\mathbb{G}_t}_{s,x}h_k(x)\mathrm{d}\mu(x)\right)_{k=1}^N\right)\mathrm{d}s\,\middle|\,\mathcal{F}_t\right)\text{ a.s.}$$

as well. By Fubini's theorem the processes $\int_{\mathbb{R}} Z^{\mathbb{G}_t}_{\cdot,x}h_k(x)\mathrm{d}\mu(x)\colon(t,1]\times\Omega\to\mathbb{R}$ are $\mathcal{B}((t,1])\otimes\mathcal{H}_{\mathbb{G}_t}$-measurable for $k=1,\dots,N$. Therefore,

$$\int_{(t,1]} f\left(s,Y^{\mathbb{G}_t}_s,\left(\int_{\mathbb{R}} Z^{\mathbb{G}_t}_{s,x}h_k(x)\mathrm{d}\mu(x)\right)_{k=1}^N\right)\mathrm{d}s$$

is $\mathcal{H}_{\mathbb{G}_t}$-measurable and finally $\overline{Y}_t$ is $\mathbb{G}_t$-invariant.

(2) It follows from the definition of $(\overline{Y},\overline{Z})$ that

$$\int_{(t,1]\times\mathbb{R}}\overline{Z}_{s,x}\mathrm{d}M(s,x)=F+\int_{(t,1]} f\left(s,Y_s,\left(\int_{\mathbb{R}} Z_{s,x}h_k(x)\mathrm{d}\mu(x)\right)_{k=1}^N\right)\mathrm{d}s-\overline{Y}_t\text{ a.s.}$$

Again we replace $(Y_s)_{s\in(t,1]}$ by $(Y^{\mathbb{G}_t}_s)_{s\in(t,1]}$, $(Z_{s,x})_{(s,x)\in(t,1]\times\mathbb{R}}$ by $(Z^{\mathbb{G}_t}_{s,x})_{(s,x)\in(t,1]\times\mathbb{R}}$, and deduce by step (1) that $\int_{(t,1]}\int_{\mathbb{R}}\overline{Z}_{s,x}\mathrm{d}M(s,x)$ is $\mathbb{G}_t$-invariant. We conclude by Lemma 6.13. □

Application. We fix the data (F, f) of a BSDE such that

(P_1) (F, f) satisfy conditions (1), (2), and (3) listed before Definition 6.9,

and a Picard scheme such that the following is satisfied:

(P_2) $(Y^k, Z^k) \in \mathcal{D}_f^{h_1,\dots,h_N}$ for $k = 0, 1, 2, \dots$.

(P_3) $Y^{k+1} = A_{F,f}(Y^k, Z^k)$ and $Z^{k+1} = B_{F,f}(Y^k, Z^k)$ for $k = 0, 1, 2, \dots$.

(P_4) There is a sub-sequence $0 \le k_1 < k_2 < \cdots$ such that $Y^{k_l} \to Y$ $\lambda \otimes \mathbb{P}$-a.e. and $Z^{k_l} \to Z$ $\lambda \otimes \mathbb{P} \otimes \mu$-a.e. as $l \to \infty$, where Y is adapted and càdlàg and $Z \in \mathcal{P}_2$.

(P_5) $\int_{(0,1]} |f\big(s, Y_s, (\int_{\mathbb{R}} Z_{s,x} h_k(x) \mathrm{d}\mu(x))_{k=1}^N\big)| \mathrm{d}s \in L_2(\mathcal{F}^X)$.

(P_6) The pair (Y, Z) satisfies BSDE (6.3).

If the initial data (F, f) of the BSDE satisfy the invariance conditions (i) and (ii) of Theorem 6.12 and if the initial processes (Y^0, Z^0) in the Picard scheme are $(\mathbb{G}_t)_{t\in(0,1)}$-invariant, then Y_t is $\mathbb{G}_t$-invariant for all $t \in [0, 1]$ and the Z-process is $(\mathbb{G}_t)_{t\in(0,1)}$-invariant. These invariances can be verified as follows:

(a) The Z-process: By Theorem 6.12 we know that all Z^k, $k \in \mathbb{N}$, are $(\mathbb{G}_t)_{t\in(0,1)}$-invariant. We fix some $t \in (0, 1)$ and get that the restrictions $Z^{k_l} \colon (t, 1] \times \Omega \times \mathbb{R} \to \mathbb{R}$ converge a.e. Replacing the restrictions by $(Z^{k_l})^{\mathbb{G}_t} \colon (t, 1] \times \Omega \times \mathbb{R} \to \mathbb{R}$ it follows that $(Z^{k_l})^{\mathbb{G}_t} \colon (t, 1] \times \Omega \times \mathbb{R} \to \mathbb{R}$ converge to $Z \colon (t, 1] \times \Omega \times \mathbb{R} \to \mathbb{R}$ a.e. Therefore we find a $Z^{\mathbb{G}_t} \colon (t, 1] \times \Omega \times \mathbb{R} \to \mathbb{R}$ which is $\mathcal{B}((t, 1]) \otimes \mathcal{H}_{\mathbb{G}_t} \otimes \mathcal{B}(\mathbb{R})$-measurable which coincides a.e. with the restriction of Z.

(b) The Y-process: By Theorem 6.12 and our assumptions, Y_t^k is $\mathbb{G}_t$-invariant for $k \ge 1$ and $t \in (0, 1)$. As $Y_1^k = F$ a.s., this extends to $t = 1$. By the càdlàg property of the Y^k the restrictions $Y^{k_l} \colon (t, 1] \times \Omega \to \mathbb{R}$ are $\mathcal{B}((t, 1]) \otimes \mathcal{H}_{\mathbb{G}_t}$-measurable. Therefore, by the arguments used in (a) we get that there is a $Y^{\mathbb{G}_t} \colon (t, 1] \times \Omega \to \mathbb{R}$ which is $\mathcal{B}((t, 1]) \otimes \mathcal{H}_{\mathbb{G}_t}$-measurable with $\mathbb{E} \int_{(t,1]} |Y_s - Y_s^{\mathbb{G}_t}| \mathrm{d}s = 0$. Consequently, Y and Z are $(\mathbb{G}_t)_{t\in(0,1)}$-invariant. Applying Theorem 6.12 to the BSDE (6.3) gives that Y_t is $\mathbb{G}_t$-invariant for $t \in (0, 1)$.

Remark 6.14. For $N = 1$ the above conditions (P_1)-(P_6) are fulfilled for example in [25, Lemma 2.4] (see also [11, Theorem 2.2 and pp. 34-35]) in the case of Lipschitz BSDEs. Moreover, in [11, 12] (F, f) and (Y^0, Z^0) have the suitable invariance properties with respect to $\mathbb{G}$ generated by $\mathbb{M}^{\mathrm{dyad}}_{(r_{l-1},r_l]}$, $l = 1, \dots, L$, for some fixed partition $0 = r_0 < r_1 < \cdots < r_L = 1$. In particular, the generator has the form

$$f(s, \omega, y, z) := f_0(s, X_s(\omega), y, z)$$

where $f_0 \colon [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is Lipschitz in (x, y, z) uniformly in s and continuous in (s, x, y, z). Therefore, $(f_0(s, X_s, y, z))_{s\in[0,1]}$ is a càdlàg process and for all $t \in (0, 1)$ and $(y, z) \in \mathbb{R}^2$ the restricted generator $f(\cdot, \cdot, y, z) : (t, 1] \times \Omega \to \mathbb{R}$ is $\mathcal{B}((t, 1]) \otimes \mathcal{H}_{\mathbb{G}_t}$-measurable.

Appendix A. Invariant sets

We recall concepts related to classical ergodic theory (see [14, Chapter 10] or [22, Chapter V]) and adapt them to our setting. The proofs of Lemmas A.2 and A.6 are omitted (for convenience they can be found in [3]) as the assertions are standard.

We assume a measurable space (S, Σ) and a group $\mathbb{A}$ of automorphisms of S, i.e., bijective bi-measurable functions $T \colon S \to S$.

Definition A.1. The *invariant* σ-algebra w.r.t. $\mathbb{A}$ is given by

$$\mathcal{I}(\mathbb{A}) := \{B \in \Sigma : B = T^{-1}(B) \text{ for all } T \in \mathbb{A}\}.$$

Lemma A.2. *For a function $\xi\colon S \to \mathbb{R}$ the following assertions are equivalent:*

(1) *ξ is $\mathcal{I}(\mathbb{A})$-measurable.*
(2) *ξ is Σ-measurable and constant on the orbits $\{Ts : T \in \mathbb{A}\}$, $s \in S$.*
(3) *ξ is Σ-measurable and $\xi \circ T = \xi$ for all $T \in \mathbb{A}$.*

Let (S, Σ, γ) be a σ-finite measure space with $\gamma(S) > 0$, $\mathbb{A}$ be a group of automorphisms acting on S, and

$$\overline{\mathcal{I}(\mathbb{A})} := \mathcal{I}(\mathbb{A}) \vee \mathcal{N} \quad \text{where} \quad \mathcal{N} := \{B \in \Sigma : \gamma(B) = 0\}.$$

The equivalence class of ξ w.r.t. to the γ-a.e.-equivalence is denoted by $[\xi]$.

Definition A.3. The measure γ is called *quasi-invariant* w.r.t. $\mathbb{A}$, if $\gamma(T^{-1}B) = 0$ for all $B \in \mathcal{N}$ and $T \in \mathbb{A}$.

Lemma A.4. *Let (S, Σ, γ) be a σ-finite measure space with $\gamma(S) > 0$ and $\mathbb{A}$ be a group of automorphisms acting on S. Then one has the following assertions:*

(1) *The operation $[\xi] \circ T := [\xi \circ T]$ is well-defined for all $T \in \mathbb{A}$ and Σ-measurable $\xi\colon S \to \mathbb{R}$ if and only if γ is quasi-invariant w.r.t. $\mathbb{A}$.*
(2) *Let γ be quasi-invariant w.r.t. $\mathbb{A}$ and $\mathbb{A}$ be countable. Then $[\xi] \circ T = [\xi]$ for all $T \in \mathbb{A}$ if and only if ξ is $\overline{\mathcal{I}(\mathbb{A})}$-measurable.*

Proof. (1) Assume that γ is quasi-invariant and that $\xi\colon S \to \mathbb{R}$ is Σ-measurable. Then for $\xi_1, \xi_2 \in [\xi]$ it holds that $\gamma(\xi_1 \neq \xi_2) = 0$ and the set

$$\{s : \xi_1(Ts) \neq \xi_2(Ts)\} = \{T^{-1}t : \xi_1(t) \neq \xi_2(t)\}$$

has measure zero as well so that the operator $[\xi] \mapsto [\xi] \circ T$ is well-defined. For the other implication let B be of measure zero and $\xi := \mathbb{1}_B$ so that $[\xi] = 0$. By assumption, $[\xi \circ T] = 0$ and

$$0 = \gamma(\{s : \mathbb{1}_B(Ts) \neq 0\}) = \gamma(T^{-1}(B)).$$

(2) If there exists an $\mathcal{I}(\mathbb{A})$-measurable $\xi_0 \in [\xi]$ it is obvious that the equivalence class is invariant by (1) and Lemma A.2. Conversely, let $[\xi \circ T] = [\xi]$ for all $T \in \mathbb{A}$. Define

$$S_0 := \big\{s \in S : \xi \circ T(s) = \xi(s) \text{ for all } T \in \mathbb{A}\big\} = \bigcap_{T \in \mathbb{A}} \{s \in S : \xi \circ T(s) = \xi(s)\},$$

which is a set of co-measure zero because $\mathbb{A}$ is countable. It is standard to check that $S_0 \in \mathcal{I}(\mathbb{A})$. Setting $\xi_0(s) := \xi(s)\mathbb{1}_{S_0}(s)$, we obtain from Lemma A.2 that ξ_0 is $\mathcal{I}(\mathbb{A})$-measurable and γ-a.e. equal to ξ. □

Definition A.5. Let $(S, \mathcal{I}, \gamma)$ be a σ-finite measure space with $\gamma(S) > 0$. A set $A \in \mathcal{I}$ with $\gamma(A) > 0$ is called *quasi-atom* provided that $B \subseteq A$ with $B \in \mathcal{I}$ implies that

$$\gamma(B) = 0 \quad \text{or} \quad \gamma(A \setminus B) = 0.$$

Lemma A.6. *Let $(S, \mathcal{I}, \gamma)$ be a σ-finite measure space with $\gamma(S) > 0$ and A, A_1, A_2 be quasi-atoms.*

(1) *If $B \in \mathcal{I}$ and $\gamma(A \Delta B) = 0$, then B is a quasi-atom.*
(2) *If $A_1 \subseteq A_2$, then $\gamma(A_2 \setminus A_1) = 0$.*
(3) *Either one has $\gamma(A_1 \cap A_2) = 0$ or $\gamma(A_1 \Delta A_2) = 0$.*
(4) *There exist countably many pairwise disjoint quasi-atoms $(A_i)_{i \in I}$ such that $S \setminus (\bigcup_{i \in I} A_i)$ does not contain any quasi-atom. For any quasi-atom A there is an $i \in I$ such that $\gamma(A \Delta A_i) = 0$.*

Lemma A.7. *Let (S, Σ, γ) be a σ-finite measure space with $\gamma(S) > 0$ and $\mathbb{A}$ be a group of automorphisms of S such that $(S, \mathcal{I}(\mathbb{A}), \gamma)$ is σ-finite. Assume that $(A_i)_{i\in I} \subseteq \mathcal{I}(\mathbb{A})$ is a countable collection of quasi-atoms like in Lemma* A.6(4). *Then for a function $\xi\colon S \to \mathbb{R}$ the following assertions are equivalent:*

(1) *ξ is $\overline{\mathcal{I}(\mathbb{A})}$-measurable.*
(2) *There exists a Σ-measurable η which is constant on the orbits and the quasi-atoms $(A_i)_{i\in I}$ and such that $\eta = \xi$ γ-a.e.*

Proof. (2) $\Longrightarrow$ (1) Using Lemma A.2 we get that η is $\mathcal{I}(\mathbb{A})$-measurable, so that ξ is $\overline{\mathcal{I}(\mathbb{A})}$-measurable.

(1) $\Longrightarrow$ (2) First we find an $\xi_0 \in [\xi]$ that is $\mathcal{I}(\mathbb{A})$-measurable. It can be easily seen that ξ_0 can be modified to an $\mathcal{I}(\mathbb{A})$-measurable random variable η satisfying the claimed properties. □

Appendix B. Some technical proofs

Lemma B.1. *Let $g \in \mathbb{M}^{\mathrm{dyad}}$, $F_1, \dots, F_n \in L_2(\mathcal{F}^X)$ and $f\colon \mathbb{R}^n \to \mathbb{R}$ be continuous such that $f(F_1, \dots, F_n) \in L_2(\mathcal{F}^X)$. Then $T_g f(F_1, \dots, F_n) = f(T_g F_1, \dots, T_g F_n)$ a.s.*

Proof. As in the proof of Lemma 2.5 it is enough to prove that

$$T_g \psi_L(f(F_1, \dots, F_n)) = \psi_L(f(T_g F_1, \dots, T_g F_n)) \text{ a.s.}$$

so that we can assume that $f \in \mathcal{C}_b(\mathbb{R}^n)$. By Lemma 2.1, we find $\mathcal{H}^X \ni F_{i,k} \to F_i$ in $L_2(\mathcal{F}^X)$ as $k \to \infty$. By a diagonal argument, we find a sub-sequence $(k_l)_{l=1}^\infty$ such that, for $l \to \infty$, $F_{i,k_l} \to F_i$ a.s. and $T_g F_{i,k_l} \to T_g F_i$ a.s. for $i = 1, \dots, n$. Therefore, as $l \to \infty$,

$$f(F_{1,k_l}, \dots, F_{n,k_l}) \to f(F_1, \dots, F_n)$$

and

$$f(T_g F_{1,k_l}, \dots, T_g F_{n,k_l}) \to f(T_g F_1, \dots, T_g F_n)$$

a.s. and therefore, by the boundedness of f, we have convergence in $L_2(\mathcal{F}^X)$. We conclude by

$$\begin{aligned} T_g f(F_1, \dots, F_n) &= \lim_{l\to\infty} T_g f(F_{1,k_l}, \dots, F_{n,k_l}) \\ &= \lim_{l\to\infty} f(T_g F_{1,k_l}, \dots, T_g F_{n,k_l}) = f(T_g F_1, \dots, T_g F_n), \end{aligned}$$

where the limits are taken in $L_2(\mathcal{F}^X)$. □

Proof of Lemma 2.6. From the Lévy-Itô decomposition [21, Theorem 19.2] we know that there is a set Ω_0 of measure one and a sequence $(\alpha_N)_{N=2}^\infty \subseteq \mathbb{R}$, such that for all $\omega \in \Omega_0$, $r \in [0,1]$, and $E_N := (-N, -\frac{1}{N}) \cup (\frac{1}{N}, N)$, one has

$$\sigma B_r(\omega) = X_r(\omega) - \lim_{\substack{N\to\infty \\ N\geq 2}} \left[\left(\int_{(0,r]\times E_N} x \mathrm{d}N(s,x) \right)(\omega) - \alpha_N r \right].$$

Using the truncations ψ_L, $L \in \mathbb{N}$, we get therefore

$$\sigma B_t = \lim_{L\to\infty} \psi_L \left(X_t - \lim_{\substack{N\to\infty \\ N\geq 2}} \left[\left(\int_{(0,t]\times E_N} x \mathrm{d}N(s,x) \right) - \alpha_N t \right] \right) \text{ a.s.,}$$

$$\sigma B_{g((0,t])} = \lim_{L\to\infty} \psi_L \left(X_{g((0,t])} - \lim_{\substack{N\to\infty \\ N\geq 2}} \left[\left(\int_{g((0,t])\times E_N} x \mathrm{d}N(s,x) \right) - \alpha_N t \right] \right) \text{ a.s.,}$$

where we assume that g is represented by some fixed permutation of dyadic intervals and $B_{g((0,t])}$ and $X_{g((0,t])}$ are obtained by finite differences over these intervals in the canonical way. Moreover, the term $\alpha_N t$ in the second equation appears due to the fact that g is measure preserving. Therefore, it is sufficient to prove that

$$T_g\psi_L\left(X_t - \lim_{\substack{N\to\infty\\ N\geq 2}}\left[\left(\int_{(0,t]\times E_N} x\mathrm{d}N(s,x)\right) - \alpha_N t\right]\right)$$
$$= \psi_L\left(X_{g((0,t])} - \lim_{\substack{N\to\infty\\ N\geq 2}}\left[\left(\int_{g((0,t])\times E_N} x\mathrm{d}N(s,x)\right) - \alpha_N t\right]\right) \text{ a.s.}$$

Because of the almost sure convergence in $N\to\infty$ it is sufficient to verify that

$$T_g\psi_L\left(X_t - \int_{(0,t]\times E_N} x\mathrm{d}N(s,x) + \alpha_N t\right)$$
$$= \psi_L\left(X_{g((0,t])} - \int_{g((0,t])\times E_N} x\mathrm{d}N(s,x) + \alpha_N t\right) \text{ a.s.}$$

for $N\geq 2$, or

$$T_g\psi_L\left(\psi_K(X_t) - \int_{(0,t]\times E_N} x\mathrm{d}N(s,x) + \alpha_N t\right)$$
$$= \psi_L\left(\psi_K(X_{g((0,t])}) - \int_{g((0,t])\times E_N} x\mathrm{d}N(s,x) + \alpha_N t\right) \text{ a.s.}$$

for $K, L\in\mathbb{N}$. As the integral terms belong to $L_2(\mathcal{F}^X)$, this follows from Lemmas 2.5 and B.1. □

Proof of Lemma 3.4. A Borel measurable function φ can be approximated by truncation by bounded Borel measurable functions $\varphi_L := \psi_L(\varphi)$, $L\in\mathbb{N}$, and $\varphi_L(F_1,\dots,F_n)\in\mathbf{H}_{\mathbb{G}}$ implies $\varphi(F_1,\dots,F_n)\in\mathbf{H}_{\mathbb{G}}$ by monotone convergence and the completeness of $\mathbf{H}_{\mathbb{G}}$ (which is easy to check as the operators $T_g\colon L_2(\mathcal{F}^X)\to L_2(\mathcal{F}^X)$ are isometries). Assuming that φ is bounded, we approximate φ pointwise by simple functions φ_k with $\|\varphi_k\|_\infty\leq\|\varphi\|_\infty$. It follows that $\varphi_k(F_1,\dots,F_n)\to\varphi(F_1,\dots,F_n)$ in $L_2(\mathcal{F}^X)$ by dominated convergence. Therefore, it is sufficient to check the statement for $\varphi=\mathbb{1}_B$ where B is a Borel set from $\mathbb{R}^n$. Using the outer regularity of the law of $(F_1,\dots,F_n)$ we can verify this in turn by using $\varphi\in\mathcal{C}_b(\mathbb{R}^n)$. But this case follows from Lemma B.1. □

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Department of Mathematics, University of Innsbruck, Technikerstrasse 13, A-6020 Innsbruck, Austria

E-mail address: florian.baumgartner@uibk.ac.at

Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland

E-mail address: stefan.geiss@jyu.fi