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POROUS MEDIUM EQUATION TO HELE-SHAW FLOW WITH GENERAL INITIAL DENSITY

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Abstract. In this paper we study the "stiff pressure limit" of the porous medium equation, where the initial density is a bounded, integrable function with a sufficient decay at infinity. Our particular model, introduced by B. Perthame, F. Quirós, and J. L. Vázquez [*The Hele-Shaw asymptotics for mechanical models of tumor growth*, Arch. Ration. Mech. Anal. 212 (2014), 93–127] describes the growth of a tumor zone with a restriction on the maximal cell density. In a general context, this extends previous results of Gil–Quirós and Kim, who restrict the initial data to be the characteristic function of a compact set. In the limit a Hele-Shaw type problem is obtained, where the interface motion law reflects the acceleration effect of the presence of a positive cell density on the expansion of the maximal density (tumor) zone.

1. Introduction

In this paper we consider the degenerate diffusion equation

$$\rho_t - \operatorname{div}(\rho Dp) = \rho G(p) \qquad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.1}$$

with initial data

$$\rho(\cdot,0) = \rho_0,$$

where

$$p = P_m(\rho) := \frac{m}{m-1}\rho^{m-1}, \tag{1.2}$$

D denotes the spatial gradient, $G \in C^1(\mathbb{R})$ is a given function with $G' < 0$ and $G(p_M) = 0$ for some $p_M > 0$, and $n \in \mathbb{N}$. Equation (1.1) was introduced in [PQV] as a model problem which describes the growth of cancer cells, with a focus on the mechanical aspect of the cell density motion; for further developments see [PQTV, PTV, PV]. Here the pressure $p = p_m$ discourages the growth of the cell density $\rho = \rho_m$ over some critical density ρ_c, which is normalized here as 1. In [PQV] the convergence of the solution ρ_m of (1.1) and the corresponding pressure variable p_m were studied in the *stiff pressure limit*, i.e., as $m \to \infty$, in the setting of the weak solutions. In the model of a fluid flow, that is, when $G \equiv 0$ and (1.1) is the porous medium equation, m characterizes the compressibility of the fluid with $m \to \infty$ representing the incompressible limit. In this setting, [GQ1] and [K], respectively in weak and viscosity solutions frameworks, showed that the solutions with initial data restricted to a characteristic function of a set converge to the

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solution of the Hele-Shaw problem as $m \to \infty$. As for initial data which is not a characteristic function, [GQ2] shows that an initial layer could form in the limit $m \to \infty$ in general. We will discuss below the choice of initial data for ρ_m which avoids such initial layer formulation. With this choice of appropriate initial data, it is shown in [PQV] in the L^1 setting that ρ_m and p_m converge respectively to the limit functions ρ_∞ and p_∞, satisfying the following equations:

$$-\Delta p_\infty = G(p_\infty) \qquad \text{in } \Omega(t) := \{p_\infty(\cdot,t) > 0\} = \{\rho_\infty(\cdot,t) = 1\}, \tag{1.3}$$

$$(\rho_\infty)_t - \operatorname{div}(\rho_\infty Dp_\infty) = \rho_\infty G(p_\infty) \qquad \text{in } \mathbb{R}^n \times (0,\infty). \tag{1.4}$$

We mention that, even at a formal level, it is not clear how to derive from (1.3)–(1.4) the velocity law of the free boundary of the tumor region, $\partial\{\rho_\infty = 1\}$. In [PQV] it was conjectured that the normal velocity law

$$V = \frac{|Dp_\infty|}{1 - \min[1, \rho_0 e^{G(0)t}]} \quad \text{on } \partial\Omega(t) \tag{1.5}$$

holds for general solutions. This is what we prove, along with the uniform convergence of the density variable away from the boundary of the tumor region. Roughly speaking we will show the following (see Theorem 1.2 below for the precise statements):

(a) As $m \to \infty$, ρ_m locally uniformly converges to 1 inside $\Omega(t)$ and to $\rho_0 e^{G(0)t}$ outside $\overline{\Omega(t)}$,
(b) $\overline{\{\rho_\infty = 1\}}$ equals the closure of $\bigcup_{t>0}(\Omega(t) \times \{t\})$,
(c) the set $\Omega(t)$ evolves with the normal boundary velocity (1.5) (in the viscosity solutions sense).

Note that (a) and (b) above imply that $\rho_0 e^{G(0)t} \leq 1$ outside $\Omega(t)$, and hence the term $\min[1, \rho_0 e^{G(0)t}]$ in (1.5) at a boundary point $x \in \partial\Omega(t)$ is the outer limit of ρ_∞ from the complement of $\overline{\Omega(t)}$. Thus (1.5) coincides with the velocity law conjectured in [PQV]. See Theorem 1.2 for a more precise statement.

Note that (c) indicates that ρ is generically discontinuous across $\partial\Omega(t)$. Thus proving the convergence result requires keeping track of the pressure variable, which appears to be, at least when $\Omega(t)$ has a smooth boundary, continuous across $\Omega(t)$. In terms of the pressure, the equation (1.1) can be written as

$$p_t = (m-1)p\Delta p + |Dp|^2 + (m-1)pG(p). \tag{1.6}$$

Now to state our main result in precise terms, let us denote by ρ_m and p_m the (density and pressure) solutions of (1.1). We will show the convergence of p_m as $m \to \infty$ to the viscosity solution of the following free boundary problem:

$$\begin{cases} -\Delta p = G(p) & \text{in } \{p(\cdot,t) > 0\}, \\ V = g(\cdot,t)|Dp| & \text{on } \partial\{p(\cdot,t) > 0\}, \\ \{\rho^E \geq 1\} \subset \overline{\{p > 0\}}. \end{cases} \tag{FB}$$

Here $\rho^E(x,t) := \rho_0^E(x)e^{G(0)t}$ is the density in the "exterior" region with initial value ρ_0^E discussed below, and

$$g(x,t) := \begin{cases} \frac{1}{1-\rho^E(x,t)}, & \text{if } \rho^E(x,t) < 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

is the free boundary velocity coefficient.

As for the initial data for the free boundary problem (FB), it is sufficient to impose the initial shape of the tumor region Ω_0 and the initial cell density in the precancer zone ρ_0^E, that is,

$$\{p(\cdot,0)>0\}=\Omega_0,$$

and we shall assume that

$$\begin{aligned}&\Omega_0\subset\mathbb{R}^n \text{ open bounded}, \quad \partial\Omega_0\in C^{1,1},\\ &\rho_0^E\in L^1(\mathbb{R}^n)\cap C(\mathbb{R}^n) \text{ with } 0\le\rho_0^E<1 \text{ and } \rho_0^E\to 0 \text{ as } |x|\to\infty.\end{aligned}\tag{1.7}$$

Note that ρ_0^E is the initial density in the "exterior" region, that is, the region outside Ω_0, and is related to ρ_0 as

$$\rho_0=\chi_{\Omega_0}+\rho_0^E\chi_{\Omega_0^c}.$$

Initial data for ρ_m. In terms of the density variable, we would like to show that ρ_m converge to $\rho(\cdot,t):=\chi_{\Omega(t)}+\rho^E\chi_{\Omega(t)^c}$, where $\Omega(t)=\{p(\cdot,t)>0\}$. To this end we will show that the convergence holds locally uniformly for a "well-prepared" initial density $\rho_{0,m}$ approximating the initial density function $\rho_0:=\chi_{\Omega_0}+\rho_0^E\chi_{\Omega_0^c}$. Our approximation is constructed such that the corresponding solution ρ_m is increasing in time (see Lemma 4.1). As for general initial data $\rho_{0,m}$ approximating ρ_0, the convergence then will hold in the L^1 norm due to the convergence result for the specific $\rho_{0,m}$ (Theorem 1.2) as well as the L^1 contraction inequality for ρ_m (4.12). While we believe that the monotonicity of ρ_m is not an essential ingredient of the convergence proof in Section 4, it is not clear at the moment whether the uniform convergence result obtained in Theorem 1.2 holds for general choices of $\rho_{0,m}$ (see Corollary 4.9) in view of [GQ2].

To construct our specific approximation $\rho_{0,m}$, let us first assume that $\rho_{0,m}^E$ satisfies, for some $\delta>0$ which is independent of m,

$$\begin{aligned}&\rho_{0,m}^E\in L^1(\mathbb{R}^n)\cap C^{1,1}(\mathbb{R}^n),\quad 0\le\rho_{0,m}^E<1-\delta,\\ &\rho_{0,m}^E\to\rho_0^E\quad\text{locally uniformly as } m\to\infty,\\ &m(1-\delta/2)^m\|D^2\rho_{0,m}^E\|_\infty\to 0\quad\text{as } m\to\infty.\end{aligned}\tag{1.8}$$

Next suppose that

$$\rho_{0,m}:=\max\left(P_m^{-1}(p_0),\rho_{0,m}^E\right),\tag{1.9}$$

where P_m was introduced in (1.2), and p_0 is the unique smooth solution of

$$\begin{cases}-\Delta p_0=G(p_0) & \text{in } \Omega_0,\\ \quad p_0=0 & \text{on } \mathbb{R}^n\setminus\Omega_0.\end{cases}$$

As we shall see in Lemma 4.1, this will guarantee that ρ_m is monotone increasing in time. After we obtain the convergence result for this particular approximation of ρ_0, we can use the L^1 contraction property for solutions of (1.1) to address the case of general $\rho_{0,m}$ (see Corollary 4.9).

Remark 1.1. Given ρ_0^E satisfying (1.7), we can easily define $\rho_{0,m}^E=\rho_0^E*\eta_{1/m}$, where $\eta_{1/m}$ is the standard mollifier with radius $1/m$. Such initial data satisfies the assumptions (1.8). Indeed, we can easily estimate $\|D^2\rho_{0,m}^E\|_\infty\le\|\rho_0^E\|_{L^1}\|D^2\eta_{1/m}\|_\infty \le Cm^{n+2}$. Additionally, $\rho_{0,m}^E\le\max\rho_0^E<1-\delta$ for some small $\delta>0$ by (1.7). The rest of (1.8) is standard. These assumptions, similarly to the assumptions in [PQV], are required to prevent a jump singularity of ρ_m over time at $t=0$.

Let us now state the main result in this paper.

Theorem 1.2. *Let the pair ρ_m, p_m satisfy* (1.1)–(1.2) *with initial data $\rho_{0,m}$ satisfying* (1.8)–(1.9)*. Then the following hold:*

(a) *(Theorem* 2.17*) There is a unique viscosity solution p of* (FB) *with initial data Ω_0, ρ_0^E.*
(b) *(Lemma* 4.4(b)*) $\{\rho^E \geq 1\}$ is contained in the closure of $\{p > 0\}$.*
(c) *(Corollary* 4.8*) The pressure variable p_m locally uniformly converges to p as long as p is continuous.*
(d) *(Corollary* 4.8*) ρ_m locally uniformly converges to $\rho := \chi_{\{p>0\}} + \rho^E \chi_{\{p=0\}}$ away from $\partial \{p > 0\}$.*
(e) *(Corollary* 2.20*) Assuming that ρ_0^E is a Lipschitz continuous function, $\partial \{p > 0\}$ has zero Lebesgue measure in $\mathbb{R}^n \times [0, \infty)$.*
(f) *(Proposition* 5.2*) $\partial \{p(\cdot, t) > 0\}$ has finite perimeter as long as $\rho^E(\cdot, t) < 1$ on $\partial \{p(\cdot, t) > 0\}$.*

Note that the free boundary motion law in (FB) yields a generic discontinuity of ρ across $\partial \{p > 0\}$. Moreover, if a new component of the region $\{\rho^E(\cdot, t) \geq 1\}$ with a nonempty interior appears outside the tumor region $\{p(\cdot, s) > 0\} = \Omega(s)$, $s < t$, the pressure p develops a discontinuity in time as it immediately becomes positive in the interior of the new component. This phenomenon is known as nucleation in the literature of phase transitions. A similar discontinuity of p in time might occur due to a topological change of $\Omega(t)$, for instance, when a "bubble" closes up. For this reason the convergence of ρ_m and p_m as stated appears to be optimal.

Remark 1.3. Due to the fact that ρ may be nonzero outside $\{p > 0\}$, the set $\{p_m > 0\}$ will degenerate as $m \to \infty$ and will not converge to $\{p > 0\}$. But our result (Corollary 4.8) implies that for any $\varepsilon > 0$, the set $\{p_m > \varepsilon\}$ will be a subset of $\{p > 0\}$ for sufficiently large m.

As in [K] we will be using the notion of viscosity solutions, which is based on the comparison principle with appropriate choices of test functions. In our problem these will be radial functions in local neighborhoods with fixed boundaries. In the viscosity solutions theory, this corresponds to the usage of second-order polynomials as test functions for nonlinear elliptic equations (see for instance [CIL]). Therefore the first crucial step in the argument is to prove the above theorem in the radial case. When there is no surrounding density, i.e., when $\rho_0^E = 0$, we rely on *Barenblatt solutions*, a well-known family of radially symmetric, compactly supported solutions of the porous medium equation. Based on the convergence of these radial solutions we apply the viscosity solution approach to obtain the corresponding result in [K]. On the other hand, when ρ_0^E is nonzero, there are no such explicit solutions available in the radial setting. The other challenges we face are the possible jump-type discontinuity over time of the tumor set $\{p > 0\}$ due to the free boundary velocity becoming infinite in the law (1.5) when the density reaches one, as well as the source term $G(p)$, which each prevent the straightforward application of a comparison principle argument between subsolutions and supersolutions.

Formal derivation of the free boundary motion law. Before we finish this section let us present a formal computation indicating the free boundary velocity law (1.5). Let us write (1.1) as

$$\rho_t - \Delta(\rho^m) = \rho G(p).$$

It should be clear that ρ^m and the pressure variable $P_m(\rho)$ converge to the same limit p_∞ as $m \to \infty$. Let us also denote the limit density solution as ρ_∞, and suppose that ρ_∞ is discontinuous across the boundary of the set $\Omega(t) = \{p_\infty(\cdot, t) > 0\} = \{\rho_\infty(\cdot, t) = 1\}$. Again if we take the time derivative of the total mass at the formal level, denoting $p_\infty = p$, $\rho_\infty = \rho$ and ρ^+ and ρ^- as ρ_∞ inside and outside $\Omega(t)$, then we have

$$\begin{aligned}\int_{\mathbb{R}^n} \rho G(p) &= \frac{d}{dt}\int_{\mathbb{R}^n} \rho\, dx = \frac{d}{dt}\left[\int_{\Omega(t)} \rho\, dx + \int_{\mathbb{R}^n\setminus\Omega(t)} \rho\, dx\right] \\ &= \int_{\Omega(t)} (\rho^+)_t\, dx + \int_{\partial\Omega(t)} V(\rho^+ - \rho^-)\, dS + \int_{\mathbb{R}^n\setminus\Omega(t)} (\rho^-)_t\, dx \\ &= \int_{\Omega(t)} \Delta p + \int_{\partial\Omega(t)} V(\rho^+ - \rho^-) dS + \int_{\mathbb{R}^n} \rho G(p) \\ &= \int_{\partial\Omega(t)} [-|Dp| + V(\rho^+ - \rho^-)] dS + \int_{\mathbb{R}^n} \rho G(p).\end{aligned}$$

This computation indicates (1.5).

Outline. In Section 2 we will prove the comparison principle and uniqueness for the limiting free boundary problem (FB). The main results are Theorem 2.13 and Theorem 2.17. They extend the comparison and well-posedness results from [P] for the Hele-Shaw problem with a time-dependent free boundary velocity coefficient g. The main challenge is to allow for an infinite coefficient depending on time. This is handled by a shift in time using the fact that the coefficient is nondecreasing in time and possesses a certain regularity. In Section 3 we show the convergence in the radially symmetric setting with fixed boundary data. Let us mention that we rely on a compactness argument based on integral estimates to derive the convergence of the radial solutions in local neighborhoods. Direct derivation of convergence using barriers is an interesting open question at the moment. Our integral estimates are modified versions from [PQV] due to the presence of fixed boundaries. In Section 4 we prove the convergence result (Corollary 4.8) based on the comparison principle in Section 2 as well as the radial convergence result in Section 3. Lastly, in Section 5 we present an estimate on the perimeter of the set $\{p > 0\}$ based on geometric arguments.

Remark 1.4. Before completion of this paper we learned that similar results were shown by Mellet, Perthame and Quirós [MPQ] following a different approach. Their approach relies on integral estimates, while ours relies on pointwise arguments which yield uniform convergence results. We believe that both of our approaches have different merits for applications to different contexts.

2. Notion of solutions and the comparison principle

2.1. **Notation.** We will follow the notation from [P].

Let $E \subset \mathbb{R}^n$ for some $n \geq 1$. Then $USC(E)$ and $LSC(E)$ are respectively the sets of all upper semi-continuous and lower semi-continuous functions on E. For a locally bounded function u on E we define the semi-continuous envelopes

$u^{*,E} \in USC(\mathbb{R}^n)$ and $u_{*,E} \in LSC(\mathbb{R}^n)$ as

$$u^{*,E} := \inf \{v \in USC(\mathbb{R}^n) : v \geq u \text{ on } E\},$$
$$u_{*,E} := \sup \{v \in LSC(\mathbb{R}^n) : v \leq u \text{ on } E\}.$$

Note that $u^{*,E} : \mathbb{R}^n \to [-\infty, \infty)$ and $u_{*,E} : \mathbb{R}^n \to (-\infty, \infty]$ are finite on $\overline{E}$. We simply write u^* and u_* if the set E is understood from the context. The envelopes can also be expressed as

$$u^{*,E}(x) = \lim_{\delta \to 0} \sup \{u(y) : y \in E,\ |y - x| < \delta\} \quad \text{for } x \in \overline{E}, \qquad u_{*,E} = -(-u)^{*,E}.$$

Let us review the shorthand notation for the set of positive values of a given function $u : E \to \mathbb{R}$, defined on a set $E \subset \mathbb{R}^n \times \mathbb{R}$,

$$\Omega(u; E) := \{(x,t) \in E : u(x,t) > 0\}, \qquad \Omega^c(u; E) := \{(x,t) \in E : u(x,t) \leq 0\},$$

and $\overline{\Omega}(u; E) := \overline{\Omega(u; E)}$ for the closure. For $t \in \mathbb{R}$, the time-slices $\overline{\Omega}_t(u; E)$, $\Omega_t(u; E)$ and $\Omega^c_t(u; E)$ are defined in the obvious way, i.e.,

$$\overline{\Omega}_t(u; E) = \left\{x : (x,t) \in \overline{\Omega}(u; E)\right\}, \qquad \text{etc.}$$

We shall call the boundary of the positive set in E the *free boundary* of u and denote it $\Gamma(u; E)$, i.e.,

$$\Gamma(u; E) = (\partial \Omega(u; E)) \cap E.$$

If the set E is understood from the context, we shall simply write $\Omega(u)$, etc.

For given constant $\tau \in \mathbb{R}$ we will often abbreviate

$$\{t \leq \tau\} := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t \leq \tau\}, \qquad \text{etc.}$$

2.2. **Viscosity solutions.** We will consider a general problem for the introduction of the notion of viscosity solutions. To be more specific, we will define solutions of the problem

$$\begin{cases} F(D^2u, Du, u) = 0 & \text{in } \{u > 0\}, \\ \quad u_t - g\,|Du|^2 = 0 & \text{on } \partial\{u > 0\}, \end{cases} \tag{2.1}$$

where F is a general elliptic operator and the velocity coefficient g satisfies the assumption (2.2) below. We assume that F satisfies the following: There exist constants c_0, $c_1 \geq 0$ and $0 < \lambda \leq \Lambda$ such that

$$\begin{aligned} \mathcal{P}^-_{\lambda,\Lambda}(M - N) - c_1\,|p - q| - c_0\,|z - w| &\leq F(M,p,z) - F(N,q,w) \\ &\leq \mathcal{P}^+_{\lambda,\Lambda}(M - N) + c_1\,|p - q| + c_0\,|z - w|, \end{aligned}$$

for all symmetric $n \times n$-matrices M, N, and $p, q \in \mathbb{R}^n$, $z, w \in \mathbb{R}$, where $\mathcal{P}^\pm_{\lambda,\Lambda}$ are the Pucci extremal operators. This guarantees that F has the strong maximum principle and Hopf's lemma; see [A]. Then we need to assume that $F_u > 0$ and that for some $p_M > 0$,

$$F(0,0,0) < 0 \qquad \text{and} \qquad F(0,0,p_M) = 0.$$

Remark 2.1. In the case of (FB) we set $F(X,p,u) = -\operatorname{trace} X - G(u)$.

For the velocity coefficient $g : \mathbb{R}^n \times \mathbb{R} \to (0, \infty]$ we will assume that

$$g \text{ is continuous at every point of } \{g < \infty\}, \text{ and} \\ g(\hat{x}, \hat{t}) = \liminf_{(x,t)\to(\hat{x},\hat{t})} g(x,t) \text{ for all } (\hat{x}, \hat{t}). \tag{2.2}$$

As in the previous papers [CV, K, P], we define viscosity solutions in two ways: using barriers and using test functions. These two notions will be shown to be equivalent, but each has its advantages in certain arguments. We will use the notion using barriers, but we still include the notion via test functions to show the relation with the original definition in [K]. The main difference from [P] is to allow for $g = +\infty$.

Before proceeding with the definition of a viscosity solution, we first recall the definition of parabolic neighborhood and strict separation used in [P].

Definition 2.1 (Parabolic neighborhood and boundary)**.** A nonempty set $E \subset \mathbb{R}^n \times \mathbb{R}$ is called a *parabolic neighborhood* if $E = U \cap \{t \le \tau\}$ for some open set $U \subset \mathbb{R}^n \times \mathbb{R}$ and some $\tau \in \mathbb{R}$. We say that E is a parabolic neighborhood of $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ if $(x,t) \in E$. Let us define $\partial_P E := \overline{E} \setminus E$, the *parabolic boundary* of E.

Now we introduce an important concept in the theory, the notion of *strict separation.* We shall use the version introduced in [P], which differs slightly from the one introduced in [K].

Definition 2.2 (Strict separation)**.** Let $E \subset \mathbb{R}^n \times \mathbb{R}$ be a parabolic neighborhood, and $u, v : E \to \mathbb{R}$ be bounded functions on E, and let $K \subset \overline{E}$. We say that u and v are strictly separated on K with respect to E, and we write $u \prec_E v$ in K if

$$u^{*,E} < v_{*,E} \text{ in } K \cap \overline{\Omega}(u; E).$$

Remark 2.2. We do not require nonnegative functions above, since taking a semi-continuous envelope commutes with taking the positive part and $0 \le u^{*,E} = (u_+)^{*,E} = \left(u^{*,E}\right)_+$ on $\overline{\Omega}(u; E)$.

The following lemma was proved in [P].

Lemma 2.3 (Cf. [P, Lemma 2.14])**.** *Suppose that E is a bounded parabolic neighborhood and u, v are locally bounded functions on E. The set*

$$\Theta_{u,v;E} := \left\{\tau : u \prec_E v \text{ in } \overline{E} \cap \{t \le \tau\}\right\} \tag{2.3}$$

is open and $\Theta_{u,v;E} = (-\infty, T)$ for some $T \in (-\infty, \infty]$.

2.2.1. *Notion via barriers.* We build strict barriers for (2.1).

Definition 2.3. Let $U \subset \mathbb{R}^n \times \mathbb{R}$ be a nonempty open set and let $\phi \in C^{2,1}(U)$ be such that $D\phi \neq 0$ on $\Gamma(\phi; U)$. We say that ϕ is a *subbarrier* of (2.1) in U if there exists a positive constant $\delta > 0$ such that

(i) $F(D^2\phi, D\phi, \phi) < -\delta$ in $\Omega(\phi; U)$,
(ii) $\phi_t - g\,|D\phi|^2 < -\delta$ on $\Gamma(\phi; U)$.

A *superbarrier* is defined analogously by reversing the inequalities in (i)–(ii) and the sign in front of δ, and requiring additionally that $g < \infty$ on $\Omega^c(\phi; U)$.

Remark 2.4. Definition 2.3 does not assume $\phi \geq 0$; we can always take the positive part later, as needed. This does not play a role in the strict separation in Definition 2.2.

The definition of solutions on an arbitrary parabolic neighborhood $Q \subset \mathbb{R}^n \times \mathbb{R}$ follows.

Definition 2.4. We say that a locally bounded, nonnegative function $u : Q \to [0, \infty)$ is a *viscosity subsolution* of (2.1) on Q if for every bounded parabolic neighborhood $E \subset Q$, $E = U \cap \{t \leq \tau\}$ for some open set U and $\tau \in \mathbb{R}$, and every superbarrier ϕ on U such that $u \prec_E \phi$ on $\partial_P E$, we also have $u \prec_E \phi$ on $\overline{E}$.

Similarly, a locally bounded, nonnegative function $u : Q \to [0, \infty)$ is a *viscosity supersolution* of (2.1) if $\{g = \infty\} \cap Q \subset \overline{\Omega}(u_*; Q)$, and for every bounded parabolic neighborhood $E \subset Q$ and every subbarrier ϕ on U such that $\phi \prec_E u$ on $\partial_P E$, we also have $\phi \prec_E u$ on $\overline{E}$.

Finally, u is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution.

Remark 2.5. Since we require $\{g = \infty\} \subset \overline{\Omega}(v)$ for all viscosity supersolutions v, we also have to address the stability of this condition. That is,

$$\{g = \infty\} \subset \overline{\Omega}(\inf_{v \in \mathcal{A}} v)$$

whenever $\mathcal{A}$ is a family of viscosity supersolutions. We need that $\{g = \infty\} = \overline{\operatorname{int} \{g = \infty\}}$ for this. Then we use subsolutions of the elliptic problem in the interior of the positive phase; they give a uniform lower bound.

Remark 2.6. As is standard in the viscosity theory, it is enough to consider only simple cylinders with balls as their base as the parabolic neighborhoods E in Definition 2.4.

2.2.2. *Notion via test functions.* Similarly to the previous work in [K, P], we can give an equivalent definition of the notion of viscosity solutions via test functions. In the following definitions, Q is an arbitrary nonempty parabolic neighborhood.

Definition 2.5. We say that a locally bounded, nonnegative function $u : Q \to [0, \infty)$ is a *viscosity subsolution* of (2.1) on Q if

(i) *(continuous expansion)*

$$\overline{\Omega}(u; Q) \cap Q \cap \{t \leq \tau\} \subset \overline{\Omega(u; Q) \cap \{t < \tau\}} \cup \{g = \infty\} \quad \text{for every } \tau > 0;$$

(ii) *(maximum principle)*
for any $\phi \in C^{2,1}$ such that $u^* - \phi$ has a local maximum at $(\hat{x}, \hat{t}) \in Q \cap \overline{\Omega}(u; Q)$ in $\overline{\Omega}(u; Q) \cap \{t \leq \hat{t}\}$, we have:
(ii-1) if $u^*(\hat{x}, \hat{t}) > 0$, then $F(D^2\phi(\hat{x}, \hat{t}), D\phi(\hat{x}, \hat{t}), u^*(\hat{x}, \hat{t})) \leq 0$,
(ii-2) if $u^*(\hat{x}, \hat{t}) = 0$, then either $F(D^2\phi(\hat{x}, \hat{t}), D\phi(\hat{x}, \hat{t}), 0) \leq 0$ or $D\phi(\hat{x}, \hat{t}) = 0$ or $\phi_t(\hat{x}, \hat{t}) - g(\hat{x}, \hat{t}) \left|D\phi\right|^2 (\hat{x}, \hat{t}) \leq 0$.

Remark 2.7. Condition (i) in Definition 2.5 is necessary to prevent a scenario where a "bubble" closes instantly; more precisely, a subsolution cannot become instantly positive on an open set surrounded by a positive phase or cannot fill the whole space instantly, unless the expansion of the positive phase happens into the set $\{g = \infty\}$.

Definition 2.6. We say that a locally bounded, nonnegative function $u : Q \to [0, \infty)$ is a *viscosity supersolution* of (2.1) on Q if

(i) (support)
 (i-1) if $(\xi, \tau) \in \Omega(u_*; Q)$, then $(\xi, t) \in \Omega(u_*; Q)$ for all $(\xi, t) \in Q$, $t \geq \tau$,
 (i-2)
$$\{g = \infty\} \cap Q \subset \overline{\Omega}(u_*; Q);$$

(ii) (maximum principle) for any $\phi \in C^{2,1}$ such that $u_* - \phi$ has a local minimum at $(\hat{x}, \hat{t}) \in Q$ in $\{t \leq \hat{t}\}$, we have:
 (ii-1) if $u_*(\hat{x}, \hat{t}) > 0$, then $F(D^2\phi(\hat{x}, \hat{t}), D\phi(\hat{x}, \hat{t}), u_*(\hat{x}, \hat{t})) \geq 0$,
 (ii-2) if $u_*(\hat{x}, \hat{t}) = 0$, then either $F(D^2\phi(\hat{x}, \hat{t}), D\phi(\hat{x}, \hat{t}), 0) \geq 0$ or $D\phi(\hat{x}, \hat{t}) = 0$ or $g(\hat{x}, \hat{t}) < \infty$ and $\phi_t(\hat{x}, \hat{t}) - g(\hat{x}, \hat{t}) \left|D\phi\right|^2 (\hat{x}, \hat{t}) \geq 0$.

Remark 2.8. As was noted in [P], assumption Definition 2.6(i-1) is there only to make our life easier.

Remark 2.9. The closure in the condition Definition 2.6(i-2) cannot be removed since $\Omega(u_*; Q)$ is a (relatively) open set. If at a given time g becomes $+\infty$ on an open set outside $\Omega_t(u_*)$ in the previous times, then u_* is zero on this set.

Remark 2.10. As is standard in the theory of viscosity solutions, we can require that the test functions ϕ are smooth, even polynomials of at most second order in space and first order in time. For (ii-2) we can use only radially symmetric test functions.

The definition of a viscosity solution follows.

Definition 2.7. We say that a locally bounded, nonnegative function $u : Q \to [0, \infty)$ is a *viscosity solution* of (2.1) on Q if it is both a viscosity subsolution and a viscosity supersolution on Q.

2.3. **Equivalence of notions.** We now get a result similar to [P, Proposition 2.13].

Proposition 2.11. *The definitions of viscosity subsolutions (resp. supersolutions) in Definition* 2.5 *(resp.* 2.6*) and in Definition* 2.4 *are equivalent.*

Proof. The direction from Definition 2.5 follows the proof of [P, Proposition 2.13]. The only detail that we have to check is that the supports of a subsolution and a superbarrier stay ordered at the crossing time. Since the continuous expansion of a subsolution in Definition 2.5(i) is valid only in the set $\{g < \infty\}$, we need to use the fact that a superbarrier in Definition 2.3 satisfies $\Omega^c(\phi; U) \subset \{g < \infty\}$.

We do not have this issue with supersolutions, so the proof is standard.

The direction from Definition 2.4 to Definitions 2.5 and 2.6 is also standard. The continuous expansion Definition 2.5(i) can be verified by a comparison with radially symmetric barriers. The monotonicity of the support of a supersolution Definition 2.6(i-1), an open set at every time, can be shown by a comparison with a stationary subbarrier such as $\phi(x, t) = \alpha(c - |x|^2)_+$ for appropriate constants $\alpha, c > 0$. ◻

With this proposition, we will from now on use the two notions of subsolutions and supersolutions from Definition 2.4 and from Definition 2.5 and 2.6 interchangeably.

2.4. **Viscosity solution classes.**

Definition 2.8. For a given function g and a nonempty parabolic neighborhood $Q \subset \mathbb{R}^n \times \mathbb{R}$ and g satisfying (2.2) we define the following classes of solutions:

- $\overline{\mathcal{S}}(g,Q)$, the set of all viscosity supersolutions of the Hele-Shaw problem (2.1) on Q;
- $\underline{\mathcal{S}}(g,Q)$, the set of all viscosity subsolutions of (2.1) on Q;
- $\mathcal{S}(g,Q) = \overline{\mathcal{S}}(g,Q) \cap \underline{\mathcal{S}}(g,Q)$, the set of all viscosity solutions of (2.1) on Q.

2.5. **Basic properties of solutions.** A subsolution is a subsolution of the elliptic problem on the whole space, while a supersolution is a supersolution of the elliptic problem in its positive set only.

Proposition 2.12. *If $u \in \underline{\mathcal{S}}(g,Q)$ for some g and Q, then $x \mapsto u^*(x,\hat{t})$ is a standard viscosity solution of*

$$F(D^2\psi, D\psi, \psi) \leq 0$$

on $\left\{x : (x,\hat{t}) \in Q\right\}$ for every $\hat{t} \in \mathbb{R}$.

Similarly, if $u \in \overline{\mathcal{S}}(g,Q)$ for some g and Q, then $x \mapsto u_*(x,\hat{t})$ is a standard viscosity solution of

$$F(D^2\psi, D\psi, \psi) \geq 0$$

on $\Omega_{\hat{t}}(u_*, Q)$.

Proof. The proof is analogous to the proof of [KP, Lemma 3.3]. □

2.6. **Comparison principle.**

Theorem 2.13. *Let Q be a bounded parabolic neighborhood and let g_1 and g_2 be two velocity coefficients satisfying* (2.2) *for which there exists $\hat{r} > 0$ such that*

$$\overline{g}(x,t) := \sup_{\overline{B}_{\hat{r}}(x,t)\cap Q} g_1 \leq \inf_{\overline{B}_{\hat{r}}(x,t)\cap Q} g_2 =: \underline{g}(x,t) \qquad \text{for all } (x,t) \in Q. \tag{2.4}$$

If $u \in \underline{\mathcal{S}}(g_1,Q)$ and $v \in \overline{\mathcal{S}}(g_2,Q)$ such that $u \prec_Q v$ on $\partial_P Q$, then $u \prec_Q v$ in $\overline{Q}$.

2.7. **Proof of the comparison principle.** We can assume that $u \in USC(\overline{Q})$ and $v \in LSC(\overline{Q})$.

We would like to follow the proof of [P, Theorem 2.18]. We will use the assumption (2.4) to justify the use of sup- and inf-convolutions.

The structure of the proof is similar to the previous papers [K,KP,P], with minor modifications to allow for the unbounded velocity coefficient. We first regularize the free boundaries of u and v by means of the sup- and inf-convolutions over a set of particular shape to guarantee the interior/exterior ball property in both space and space-time. The set for inf-convolution is decreasing in time to add an additional perturbation by effectively increasing the free boundary velocity of the supersolution. Now, if the comparison fails, the regularized solutions must cross. We first show that due to the continuous expansion of the support of u and the fact that u and v are sub/supersolutions of the elliptic problem, this crossing must happen on the free boundary. At the first contact point, the boundaries are locally $C^{1,1}$ in space. Moreover, the velocity coefficient g_1 for the subsolution is bounded on the neighborhood of this point. At the regular contact point it is possible to define weak normal derivatives of the regularized solutions, which must be ordered by Hopf's lemma. Moreover, we can construct barriers to show that the free boundary velocity law is satisfied with these weak normal derivatives. An ordering of the free boundary velocities at the crossing point with the additional perturbation above then yields a contradiction. Therefore the comparison holds.

Let us define the crossing time

$$t_0 := \sup \Theta_{u,v;Q}, \tag{2.5}$$

using the set $\Theta_{u,v;Q}$ defined in (2.3). We observe that $u \prec_Q v$ in $\overline{Q}$ is equivalent to $t_0 = \infty$.

Let us therefore suppose that $t_0 < \infty$ and we will show that this leads to a contradiction.

2.7.1. *Regularization.* We shall use the standard sup/inf-convolutions to regularize the free boundaries at the contact point. We first introduce the open set $\Xi_r(x,t) \subset \mathbb{R}^n \times \mathbb{R}$ for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$ as

$$\Xi_r(x,t) = \left\{(y,s) : (|y-x| - r)_+^2 + |s-t|^2 < r^2\right\}.$$

Note that $\overline{\Xi}_r(x,t) \subset B_{\hat r}(x,t)$ if $2r < \hat r$.

Let $T > 0$ be such that $Q \subset \{t \le T\}$. For given $0 < r < \hat r/2$, $0 < \delta < \frac{r}{2T}$ we define

$$\begin{aligned} Z(x,t) &= \sup_{\overline{\Xi}_r(x,t)} u, \\ W(x,t) &= \inf_{\overline{\Xi}_{r-\delta t}(x,t)} v \end{aligned}$$

for $(x,t) \in Q_r$ with

$$Q_r := \left\{(x,t) \in Q : \overline{\Xi}_r(x,t) \subset Q\right\}.$$

Note that Q_r is a parabolic neighborhood.

The following lemma is standard.

Lemma 2.14. *For all $r, \delta > 0$ sufficiently small, $Z \in USC(Q_r)$, $W \in LSC(Q_r)$, and*

$$Z \prec_{Q_r} W \quad \text{on } \partial_P Q_r.$$

For every $(x,t) \in Q_r$ there exist $(x_u, t_u) \in \overline{\Xi}_r(x,t) \subset Q$ and $(x_v, t_v) \in \overline{\Xi}_{r-\delta t}(x,t)$ such that

$$u(x_u, t_u) = Z(x,t) \qquad \text{and} \qquad v(x_v, t_v) = W(x,t).$$

Moreover, $x \mapsto Z(x,t)$ is a subsolution of the elliptic problem on $\{x : (x,t) \in Q_r\}$, and $x \mapsto W(x,t)$ is a supersolution of the elliptic problem on $\Omega_t(W; Q_r)$.

The support of Z expands continuously in the sense

$$\overline{\Omega}(Z;Q_r) \cap Q_r \cap \{t \le \tau\} \subset \overline{\Omega(Z;Q_r) \cap \{t < \tau\}} \cup \{\overline{g} = \infty\} \qquad \text{for every } \tau > 0.$$

Similarly, the support of W is nondecreasing:

$$\text{if } (\xi,\tau) \in \Omega(W;Q_r), \text{ then } (\xi,t) \in \Omega(W;Q_r) \text{ for all } (\xi,t) \in Q_r,\ t \ge \tau,$$

and

$$\left\{\underline{g} = \infty\right\} \cap Q_r \subset \Omega(W; Q_r). \tag{2.6}$$

Remark 2.15. We can prove a stronger result that actually $Z \in \underline{\mathcal{S}}(\overline{g}; Q_r)$ and $W \in \overline{\mathcal{S}}(\underline{g}; Q_r)$, where $\overline{g}$ and $\underline{g}$ are sup/inf of g over $\overline{\Xi}_r$, but we actually never need this.

Proof. The semicontinuity and existence of points (x_u, t_u) and (x_v, t_v) is standard from the semicontinuity of u and v. We can choose $r < \hat{r}/2$ and $\delta < \frac{T}{2r}$ sufficiently small so that Z and W are strictly ordered on $\partial_P Q_r$ since u and v are strictly ordered on $\partial_P Q$.

To check that $x \mapsto Z(x,t)$ and $x \mapsto W(x,t)$ are a subsolution and a supersolution of the elliptic problem in $\{x : (x,t) \in Q_r\}$ and $\Omega_t(W; Q_r)$, respectively, for every $t \in \mathbb{R}$, we just need to recall that they are the supremum of subsolutions, respectively the infimum of supersolutions, of the elliptic problem due to Proposition 2.12.

The continuous expansion of Z follows from the continuous expansion of u. Indeed, if $(\xi, \tau) \in \overline{\Omega}(Z; Q_r) \cap Q_r$ and $\overline{g}(\xi, \tau) < \infty$, then $g_1 < \infty$ on $B_{\hat{r}}(\xi, \tau)$. Moreover, there exists $(\xi_u, t_u) \in \overline{\Omega}(u; Q) \cap \overline{\Xi}_r(\xi, \tau) \subset B_{\hat{r}}(\xi, \tau)$. By the continuous expansion of u, we have

$$(\xi_u, t_u) \in \overline{\Omega(u; Q) \cap \{t < t_u\}}.$$

By the definition of the sup-convolution, we conclude that

$$(\xi, \tau) \in \overline{\Omega(Z; Q_r) \cap \{t < \tau\}}.$$

To see that the support of W is nondecreasing, suppose that $(\xi, \tau) \in \Omega(W; Q_r)$. Then by definition $\overline{\Xi}_{r-\delta\tau}(\xi, \tau) \subset \Omega(v; Q)$. Since v is a supersolution, its support is nondecreasing, Definition 2.6(i-1), and therefore $\overline{\Xi}_{r-\delta t}(\xi, t) \subset \overline{\Xi}_{r-\delta\tau}(\xi, t) \subset \Omega(v; Q)$ for all $t \geq \tau$. We conclude that $(\xi, t) \in \Omega(W; Q_r)$ for all $t \geq \tau$.

Finally, if $(\xi, \tau) \in Q_r$ with $\underline{g}(\xi, \tau) = \infty$, then $g_2 = \infty$ on $\overline{B}_{\hat{r}}(\xi, \tau)$. By Definition 2.6(i-2) we have $B_{\hat{r}}(\xi, \tau) \subset \overline{\Omega}(v; Q)$. Therefore $B_\rho(\xi, \tau) \cap Q_r \subset \overline{\Omega}(W; Q_r)$ for small $\rho > 0$ such that $B_{\hat{r}-\rho}(\xi, \tau) \supset \overline{\Xi}_{r-\delta\tau}(\xi, \tau)$. ☐

2.7.2. *Contact.* Let us define the contact time

$$\hat{t} := \sup \Theta_{Z,W;Q_r} < t_0 < \infty,$$

where t_0 was introduced as the crossing time in (2.5). We will show that this leads to a contradiction.

Lemma 2.16. *$Z = W = 0$ on $\Omega^c_{\hat{t}}(W; Q_r)$ and $Z < W$ on $\Omega_{\hat{t}}(W; Q_r)$. In particular, $Z \leq W$ on $Q_r \cap \{t \leq \hat{t}\}$.*

Proof. Let us denote

$$z(x) := Z(x, \hat{t}), \qquad w(x) := W(x, \hat{t})$$

for $x \in D := \{x : (x, \hat{t}) = Q_r\}$. Recall that z and w are a subsolution and a supersolution, respectively, of the elliptic problem by Proposition 2.12. The set $V := \Omega_{\hat{t}}(W; Q_r)$ is open and has an exterior ball of radius $r/2$ at every point of its boundary. By (2.6), $\overline{g} \leq \underline{g} < \infty$ on $D \setminus V$. We know from the definition of the contact time that $\overline{\Omega}(Z; Q_r) \cap Q_r \cap \{t < \hat{t}\} \subset \Omega(W; Q_r)$. Let y be such that $B_{r/2}(y) \subset V^c$; we must have $z = 0$ on $B_{r/2}(y) \cap D$ by the continuous expansion of the support of Z and the monotonicity of the support of W in Lemma 2.14 and (2.6). z is a subsolution of the elliptic problem and therefore $z = 0$ on $\overline{B}_{r/2}(y) \cap D$. By covering $D \setminus V$ by such balls, we conclude that $z = 0$ on $D \setminus V$.

Let $\hat{x} \in \overline{V}$ such that $z(\hat{x}) \geq w(\hat{x})$. We only need to prove that $\hat{x} \in \partial V$, and the conclusion then follows. Let U be the connected component of V for which $\hat{x} \in U$.

We know that $z = 0$ on $\partial U \cap D$ from above, and therefore $z \leq w$ on ∂U. If $\overline{U} \times \{\hat{t}\} \cap \partial_P Q_r \neq 0$, then the strong maximum principle for the elliptic problem implies that $z < w$ on U, a contradiction.

If $\overline{U} \times \{\hat{t}\} \subset Q_r$, we have to give a different argument. Let $y \in U$ be a point of maximum of z on $\overline{U}$. Clearly $z(y) > 0$. By the interior ball property, there exists ξ such that $y \in B_r(\xi)$ and $z = z(y)$ on $B_r(\xi)$. Since $\psi = c$ for $c \geq p_M$ is a supersolution of the elliptic problem on U, the strong maximum principle implies $z(y) < p_M$. In particular, z is a strict subsolution of the elliptic problem on $B_r(\xi)$. We therefore cannot have $w \equiv z$ on $B_r(\xi)$. We conclude that $z < w$ on U by the strong maximum principle. □

We know from Lemma 2.3 that $\hat{t} \notin \Theta_{Z,W;Q_r}$ since $\Theta_{Z,W;Q_r}$ is open, and so Z and W are not strictly separated on $\overline{Q_r} \cap \{t \leq \hat{t}\}$ with respect to Q_r. Therefore due to Lemma 2.16 we can find

$$(\hat{x}, \hat{t}) \in \overline{\Omega}(Z; Q_r) \cap \Omega^c(W; Q_r).$$

Due to Lemma 2.14 there exist points

$$(x_u, t_u) \in \partial \Xi_r(\hat{x}, \hat{t}) \cap \partial \Omega(u; Q) \quad \text{and} \quad (x_v, t_v) \in \partial \Xi_{r-\delta\hat{t}}(\hat{x}, \hat{t}) \cap \partial \Omega(v; Q).$$

We have $\overline{\Xi}_r(\hat{x}, \hat{t}) \subset \Omega^c(u)$ and $\Xi_r(x_u, t_u) \cap Q_r \subset \Omega(Z)$. Since $Z \leq W$ on $Q_r \cap \{t \leq \hat{t}\}$, we have

$$\overline{\Xi}_{r-\delta t}(x, t) \subset \Omega(v) \qquad \text{for } (x, t) \in \Xi_r(x_u, t_u) \cap \{t \leq \hat{t}\}.$$

By ordering we have

$$\overline{\Xi}_r(x_u, t_u) \cap \overline{\Xi}_{r-\delta\hat{t}}(x_v, t_v) \cap \{t \leq \hat{t}\} = \{(\hat{x}, \hat{t})\}.$$

2.7.3. *Free boundary velocity.* Let $m_Z \in [-\infty, \infty]$ denote the normal velocity of $\partial \Xi_r(\hat{x}, \hat{t})$ at (x_u, t_u), which can be expressed as

$$m_Z = \frac{t_u - \hat{t}}{\sqrt{r^2 - \left(t_u - \hat{t}\right)^2}}.$$

Let us define the set

$$E := \bigcup_{\substack{(x,t) \in \Xi_r(x_u, t_u) \\ t \leq \hat{t}}} \Xi_{r-\delta t}(x, t).$$

Note that $E \subset \Omega(v)$ and $(x_v, t_v) \in \partial E$. Let m_W denote the normal velocity of the boundary of E at (x_v, t_v). Since $\Omega(v)$ is nondecreasing, we must have $m_W \geq 0$. But we can also estimate $m_Z - \delta \geq m_W$ and therefore

$$m_Z - \delta \geq m_W \geq 0.$$

We conclude in particular that $t_u > \hat{t} \geq t_v$.

2.7.4. *Gradients and velocities.* Since $\{g_2 = \infty\} \subset \overline{\Omega}(v)$ and $(x_v, t_v) \in \Omega^c(v)$, we must have $(x_v, t_v) \in \overline{\{g_2 < \infty\}}$. Since $(x_u, t_u), (x_v, t_v) \in \overline{\Xi}_r(\hat{x}, \hat{t}) \subset B_{\hat{r}}(\hat{x}, \hat{t})$, we can estimate

$$g_1(x_u, t_u) \leq \sup_{\overline{\Xi}_r(\hat{x},\hat{t})} g_1 \leq \sup_{\overline{B}_{\hat{r}}(\hat{x},\hat{t})} g_1 \leq \inf_{\overline{B}_{\hat{r}}(\hat{x},\hat{t})} g_2 < \infty.$$

Let ν be the unit outer normal to $\left\{x : (x, \hat{t}) \in \Xi_r(x_u, t_u)\right\}$. We can define the "weak gradients"

$$\alpha := \limsup_{h \to 0+} \frac{Z(\hat{x} - h\nu, \hat{t})}{h}, \qquad \beta := \liminf_{h \to 0+} \frac{W(\hat{x} - h\nu, \hat{t})}{h}.$$

Since $\hat{x}$ is a regular point of the boundary ∂U, weak Hopf's lemma implies $\alpha \leq \beta$, $\alpha < \infty$ and $\beta > 0$.

As $t_u > 0$, we have enough space to put a barrier above u as in [KP] in a neighborhood of (x_u, t_u) and prove that

$$m_Z \leq g_1(x_u, t_u)\alpha < \infty.$$

Therefore $m_W < \infty$. In particular, $t_v > \hat{t} - r + \delta\hat{t}$. Therefore we have enough space to put a barrier under v as in [KP] in a neighborhood of (x_v, t_v) and prove that

$$\infty > m_Z - \delta \geq m_W \geq g_2(x_v, t_v)\beta.$$

Note that a subbarrier does not need $g_2 < \infty$ in the complement of its support; see Definition 2.3. In particular, $g_2(x_v, t_v) < \infty$. Putting this all together, we have

$$m_Z \leq g_1(x_u, t_u)\alpha \leq g_2(x_v, t_v)\beta \leq m_Z - \delta,$$

a contradiction.

2.8. **Well-posedness of** (FB)**.** We have the following existence and uniqueness result for (FB).

Theorem 2.17 (Well-posedness)**.** *Suppose that $\Omega_0 \subset \mathbb{R}^n$ is a bounded open set with a $C^{1,1}$ boundary. Moreover, let $\rho_0^E \in C(\mathbb{R}^n)$ be a function such that $0 \leq \rho_0^E < 1$, $\lim_{|x|\to\infty} \rho_0^E(x) = 0$. Then there exists a unique viscosity solution u of* (FB) *with initial support Ω_0 and initial density ρ_0^E in the sense that u is a viscosity solution of* (2.1) *in $Q = \mathbb{R}^n \times (0, \infty)$ with*

$$g(x,t) := \frac{1}{1 - \min\left\{\rho_0^E(x) e^{tG(0)}, 1\right\}}, \tag{2.7}$$

where $g = \infty$ if the denominator is 0*, and u satisfies the initial condition as*

$$\left\{x : u^{*,Q}(x, 0) > 0\right\} = \{x : u_{*,Q}(x, 0) > 0\} = \Omega_0.$$

The solution is unique in the sense that if u and v are two viscosity solutions of (2.1) *with the same initial data, then*

$$u^{*,Q} = v^{*,Q}, \qquad u_{*,Q} = v_{*,Q}. \tag{2.8}$$

In the proof of the uniqueness in this theorem we also obtain the following version of the comparison principle.

Theorem 2.18. *Let Ω_0 and ρ_0^E be as in Theorem 2.17. Suppose that u is a viscosity subsolution and v is a viscosity supersolution of (2.1) in $Q = \mathbb{R}^n \times (0,\infty)$ with g as in (2.7), with the initial data*

$$\left\{x : u^{*,Q}(x,0) > 0\right\} \subset \Omega_0 \subset \{x : v_{*,Q}(x,0) > 0\}.$$

Then

$$u^{*,Q} \le v^{*,Q}, \qquad u_{*,Q} \le v_{*,Q}.$$

We now proceed with the proof of the well-posedness theorem. Let u and v be two solutions of (FB) on $Q = \mathbb{R}^n \times (0,\infty)$ with the given initial data. We want to prove that they must be equal in the sense of (2.8).

The basic idea is to perturb one of the solutions to create a strictly ordered pair and then apply the comparison principle. To apply Theorem 2.13, for $\alpha > 1$ and $\sigma > 0$ we consider the rescaled shifted function

$$w(x,t) = v(x, \alpha t + \sigma).$$

Clearly $w \in \overline{\mathcal{S}}(g_2; Q)$, where

$$g_2(x) = \alpha g(x, \alpha t + \sigma).$$

We want to show that we can find $\hat{r} > 0$ such that the assumptions of the comparison principle Theorem 2.13 are satisfied.

Lemma 2.19. *Suppose that g satisfies the assumptions (2.2), g is nondecreasing in time, and $\{g = \infty\}$ is the epigraph of a function $\tau : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that τ is continuous at every point in $\{\tau < \infty\}$. Then for every compact set $E \subset \mathbb{R}^n \times [0,\infty)$, $\alpha > 1$ and $\sigma > 0$ there exists $r > 0$ such that $\alpha g(x, \alpha t + s) \ge g(y,s)$ whenever $(x,t), (y,s) \in E$ and $|(x,t) - (y,s)| \le r$.*

Proof. Let us set $f(x,t) := \alpha g(x, \alpha t + \sigma)$.

1. We first show that we can find $K > 0$ such that

$$\delta_1 := \operatorname{dist}(\{g \ge K\}, \{f < \infty\} \cap E) > 0.$$

Indeed, suppose that $\delta_1 = 0$ for any $k \in \mathbb{N}$. Thus we can find sequences $(x_k, t_k) \in \{g \ge k\}$, $(y_k, s_k) \in \{f < \infty\} \cap E$ such that $|(x_k,t_k) - (y_k,s_k)| < \frac{1}{k}$. By compactness of E we can assume that up to a subsequence $(x_k,t_k) \to (\hat{x},\hat{t})$ and $(y_k,s_k) \to (\hat{x},\hat{t})$ for some $(\hat{x},\hat{t}) \in E$. In particular, $0 \le \hat{t} < \infty$. Since we have $g(\hat{x},\hat{t}) \ge \liminf_{k\to\infty} g(x_k,t_k) = \infty$ by (2.2), we deduce $\tau(\hat{x}) \le \hat{t}$. Furthermore, as $\alpha s_k + \sigma < \tau(y_k)$, continuity of τ yields

$$\alpha\hat{t} + \sigma \le \tau(\hat{x}) \le \hat{t},$$

a contradiction. Therefore we can choose $K > 0$ such that $\delta_1 > 0$.

2. Let $\delta_2 := \operatorname{dist}(\{f = \infty\}, \{f \le K\} \cap E)$, and observe that $\delta_2 > 0$ due to (2.2) and the compactness of E.

Since the set $Q := \{f \le K\} \cap E \subset \{g < K\}$ is compact, we can find a modulus of continuity ω of g on this set, and $m := \min_Q \min\{f, g\} > 0$. Let us find $\rho > 0$ such that $\omega(\rho) \le (\alpha - 1)m$. We set $r := \frac{1}{2}\min\{\delta_1, \delta_2, \rho\}$.

3. Now choose $(x,t), (y,s) \in E$ with $|(x,t) - (y,s)| \le r$. We now prove that $f(x,t) \ge g(y,s)$.

- If $f(x,t) = \infty$, then the conclusion is trivial.
- If $K \le f(x,t) < \infty$, then $g(y,s) < K$ and hence $f(x,t) \ge g(y,s)$.

- If $K \le g(y,s)$, then $f(x,t) = \infty$, and again the conclusion is trivial.
- Finally, if neither of the above is satisfied, we must have $f(x,t) \le K$ and $g(x,t) \le K$. Therefore we can estimate using the monotonicity in time and continuity that

$$\begin{aligned} f(x,t) &= \alpha g(x, \alpha t + \sigma) \ge \alpha g(x,t) \\ &= (\alpha - 1) g(x,t) + g(x,t) \ge (\alpha - 1)m + g(y,s) - \omega(r) \\ &\ge g(y,s). \end{aligned}$$

This finishes the proof. □

Proof of Theorem 2.17. **Uniqueness.** Let us first prove uniqueness. Suppose that u and v are two viscosity solutions satisfying the initial condition. For simplicity, in the following we write u instead of $u^{*,Q}$, and v instead of $v_{*,Q}$.

1. If u is a viscosity solution with initial condition Ω_0, a bounded set, we can compare it with a large radially symmetric superbarrier

$$W_T = \frac{G(0)}{n}(R^2 e^{16G(0)t/n} - |x|^2). \tag{2.9}$$

Indeed, since $\rho_0^E \to 0$ as $|x| \to \infty$, we can for any $T > 0$ find R sufficiently large such that $\rho^E(x,t) < \frac{1}{2}$ for $|x| \ge R$, $0 \le t \le T$ and $\Omega_0 \subset B_R(0)$. Then W_T is a superbarrier for $0 \le t \le T$ since $-\Delta W_T = 2G(0) > G(0) \ge G(W_T)$ in $\{W_T > 0\}$, while

$$\frac{\partial_t W_T}{|DW_T|^2} = 4 > 2 \ge \frac{1}{1 - \rho^E} = g \qquad \text{on } \partial\{W_T > 0\}.$$

The comparison with this superbarrier yields that $\Omega(u; \mathbb{R}^n \times [0,T]) \subset B_{Re^{8G(0)T/n}}(0) \times [0,T]$. Let us therefore define $Q_T = B_{2(Re^{8G(0)T/n})}(0) \times [0,T]$.

2. We apply the comparison principle on Q_T. Since Ω_0 has the interior ball condition, by comparison with radial subbarriers we can prove that $\Omega_0 \Subset \Omega_t(v)$ for $t > 0$. To see this, consider the function

$$w(x,t) = \alpha((ct + r)^2 - |x - x_0|^2).$$

For given $0 < r < 1$, we can first choose $0 < \alpha \ll 1$ such that $G(4\alpha) > 2n\alpha$ and then choose $0 < c \ll 1$ so that $c(c+r)/(2\alpha r^2) < 1$. Then for $0 \le t \le 1$ we have $w \le 4\alpha$ and therefore $G(w) \ge G(4\alpha) > -\Delta w$. Moreover,

$$\frac{w_t}{|Dw|^2} = \frac{2\alpha c(ct + r)}{4\alpha^2 |x - x_0|^2} \le \frac{c(ct + r)}{2\alpha r^2} < 1 \le g \qquad \text{on } \partial\{w > 0\}.$$

We see that w is a subbarrier for $0 \le t \le 1$. We conclude that if Ω_0 has an interior ball condition with radius $r > 0$, the free boundary of a solution must expand initially with velocity at least $c > 0$ given above.

3. Let us fix $\sigma > 0$. We can find an open set $U \subset \mathbb{R}^n$ with smooth boundary such that $\Omega_0 \Subset U \Subset \Omega_\sigma(v)$. $\Omega_t(u)$ cannot jump outside Ω_0 by the definition of a viscosity solution, and therefore $\overline{\Omega_t(u)} \subset U$ for all $t > 0$ sufficiently small. By the strong maximum principle for the elliptic problem, we obtain that the solution of the elliptic problem on U with data zero on ∂U is strictly smaller than the solution of the elliptic problem on $\Omega_\sigma(v)$. Since $x \mapsto u(x,t)$ is a subsolution of the elliptic problem on $\mathbb{R}^n$ for any $t > 0$, and $x \mapsto v(x,\sigma)$ is a supersolution of the elliptic problem on $\Omega_\sigma(v)$, we conclude that $u(\cdot, 0) < v(\cdot, \sigma)$ on $\Omega_\sigma(v)$.

Let us define $w(x,t) = v(x,(1+\sigma)t+\sigma)$ for some $\sigma > 0$. By the reasoning above, $u \prec_{Q_T} v$ on $\partial_P Q_T$. Lemma 2.19 implies that the functions $g_1 = g$ and $g_2(x,t) = (1+\sigma)g(x,(1+\sigma)t+\sigma)$ satisfy the assumptions of Theorem 2.13 on Q_T. Therefore $u \le w$ on $\mathbb{R}^n \times [0,T]$.

Now we send $\sigma \to 0+$ and recover

$$u_* \le (u^*)_* \le v_*.$$

By shifting u instead of v, that is, considering $u(x,(1+\sigma)^{-1}(t-\sigma))$ and then sending $\sigma \to 0+$, we also obtain

$$u^* \le (v_*)^* \le v^*.$$

By repeating the same argument with u and v interchanged, we obtain the uniqueness of solutions:

$$u^* = v^*, \qquad u_* = v_*.$$

Existence. Existence follows from the standard Perron-Ishii method. We first construct appropriate barriers.

1. Let Z_ρ for $\rho \ge 0$ be the unique solution of the elliptic problem in $\Omega_0 + B_\rho(0)$ with boundary value zero and zero outside $\Omega_0 + B_\rho$, where $B_0(0) = \{0\}$. Since $\Omega_0 \in C^{1,1}$, we see that $\Omega_0 + B_\rho(0) \in C^{1,1}$ for small $\rho > 0$ and therefore such Z_ρ exists. Clearly $U(x,t) = Z_0(x)$ is a viscosity subsolution of (2.1) in $\mathbb{R}^n \times (0,\infty)$.

On the other hand, for $a > 0$ and $\eta \in (0,1)$ let us define

$$V(x,t) = \begin{cases} Z_{at}(x), & 0 \le t \le \eta, \\ W_1(x,t), & \eta < t \le 1, \\ W_k(x,t), & k-1 < t \le k, \text{ iteratively } k = 2,3,\dots, \end{cases}$$

where W_k (with $k = T$) was defined in (2.9). Since $\rho_0^E < 1$ on $\partial\Omega_0$, g as defined in (2.7) is finite in a neighborhood of $\partial\Omega_0 \times \{0\}$. Therefore by continuity, we can find $\eta > 0$ sufficiently small and $a > 0$ large enough so that V is a viscosity supersolution.

Note that by continuity of U and V for all $t \ge 0$ small, we have

$$U^*(x,0) = U_*(x,0) = V^*(x,0) = V_*(x,0) = Z_0(x).$$

2. Now let u be the supremum of viscosity subsolutions w with initial data $w^{*,Q}(x,0) = U(x,0)$. Since U belongs to this class, we see that u is well-defined and $u \ge U$. Moreover, the comparison principle, with the perturbation above in the proof of uniqueness, yields

$$U^* \le u^* \le V^*, \qquad U_* \le u \le V_*.$$

In particular, u has the correct initial data. We only need to show that it is a solution. We use Definition 2.4. Let us show that u is a subsolution. If not, there exist a parabolic neighborhood and a superbarrier which u crosses, even though they are strictly ordered on the parabolic boundary. In this case, we can perturb the barrier at the crossing point (making it smaller) and deduce that one of the subsolutions must cross the perturbed barrier, leading to a contradiction.

Similarly, to show that it is a supersolution, we suppose that u crosses a subbarrier. If this happens, we can perturb the subbarrier, making it larger, and since the perturbed subbarrier is a viscosity subsolution, this makes u larger, contradicting the maximality of u. We therefore only need to check that $\overline{\{g = \infty\} \cap Q} \subset \overline{\Omega}(u_{*,Q}; Q)$. But by our assumption on ρ_0^E we have $\{g = \infty\} = \overline{\operatorname{int}\{g = \infty\}}$. Suppose that $\overline{B}_\rho(\xi) \times \{\tau\} \subset \operatorname{int}\{g = \infty\}$ for some (ξ, τ) and $\rho > 0$. Let z be the solution of the elliptic problem on $B_\rho(\xi)$, and 0 outside of $B_\rho(\xi)$. Then

$$Z(x,t) = \begin{cases} 0, & t < \tau, \\ z(x), & t \geq \tau, \end{cases}$$

is a viscosity subsolution. In particular, $u_* > 0$ in $B_\rho(\xi) \times \{t > \tau\}$. From this we conclude that $\operatorname{int}\{g = \infty\} \cap Q \subset \Omega(u_{*,Q}; Q)$, and thus $\{g = \infty\} \cap Q \subset \overline{\Omega}(u_{*,Q}; Q)$.

We have proved that u is the unique solution of (2.1) with g of the form (2.7) and initial support Ω_0. □

Corollary 2.20. *Suppose that p is the unique viscosity solution of* (FB) *from Theorem* 2.17 *and that additionally ρ_0^E is Lipschitz. Then $\partial\{p > 0\}$ has Lebesgue measure zero in $\mathbb{R}^n \times [0, \infty)$.*

Proof. We will show the following density estimate, which is sufficient to conclude: For any $T > 0$, there exists $k = k(T) \in (0, 1)$ such that for any space-time ball B_{n+1} with radius $r(T + 2)$, $r \in (0, 1)$, centered at $(x_0, t_0) \in \partial\{p > 0\} \cap \{t \leq T\}$, there is a space-time ball $\tilde{B}_{n+1}$ of radius kr which lies in both $\{p > 0\}$ and in B_{n+1}.

To show this, let us first prove the ordering

(2.10)
$$p_1(x,t) := \sup_{|x-y| \leq kr} p(x,t) \leq p_2(x,t) := p(x, (1 + r)t + r), \qquad x \in \mathbb{R}^n, t \in [0, T],$$

for $k = k(T)$. To see this, first note that the order holds at $t = 0$, due to step 2 in the uniqueness part of the proof of Theorem 2.17.

It is straightforward to check that p_1 is a viscosity subsolution of (FB) with modified normal velocity $V = |Dp_1| g_1$ with

$$g_1(x,t) := \frac{1}{1 - \min[1, \rho^E(x,t) + e^{G(0)t}\omega(kr)]},$$

where ω is the continuity mode of ρ_0^E, and p_2 is a viscosity supersolution of (FB) with normal velocity $V = |Dp_2| g_2$, where

$$g_2(x,t) := \frac{1 + r}{1 - \min[1, e^{G(0)r(1+t)}\rho^E(x,t)]}.$$

Now since ρ_0^E is Lipschitz, there is a constant $L \geq 1$ such that $\omega(s) \leq Ls$. We claim that if we choose $k = \frac{\min[G(0),1]}{2L} e^{-G(0)T} \in (0, \frac{1}{2})$, we have $g_1 \leq g_2$ for $x \in \mathbb{R}^n$, $t \in [0, T]$ and $r \in (0, 1)$. Then the comparison principle for (FB) yields (2.10).

To show the order $g_1 \leq g_2$ we fix $x \in \mathbb{R}^n$ and set $\gamma = G(0)$ and $\rho = \rho_0^E(x)$ to simplify the notation. By the Lipschitz estimate, we have

$$g_1(x,t) \leq \frac{1}{1 - \min[1, (\rho + Lkr)e^{\gamma t}]}.$$

First observe that if $g_1 = \infty$, then $g_2 = \infty$. Indeed, suppose that $g_1(x,t) = \infty$ for some $t \in [0, T]$. This implies $(\rho + Lkr)e^{\gamma t} \geq 1$ and thus

$$\rho e^{\gamma t} \geq 1 - Lkre^{\gamma t} \geq 1 - \tfrac{1}{2}\min(\gamma, 1)r.$$

This yields the estimate

$$\rho e^{\gamma t} e^{\gamma r(1+t)} \geq (1 - \tfrac{1}{2}\min(\gamma,1)r)e^{\min(\gamma,1)r} \quad \text{for } r \in [0,1].$$

Since the function $h(s) = (1 - \frac{s}{2})e^s$ is increasing for $s \in [0,1]$, the right-hand side above is greater than or equal to $h(0) = 1$, which implies $g_2 = \infty$.

Hence we only need to prove $g_1 \leq g_2$ when $\rho e^{\gamma((1+r)t+r)} < 1$, in which case also $(\rho + Lkr)e^{\gamma t} < 1$. Dividing $g_1 \leq g_2$ by $1+r$, we see that in this case $g_1 \leq g_2$ is equivalent to

$$1 + r - (1+r)(\rho + Lkr)e^{\gamma t} \geq 1 - \rho e^{\gamma t} e^{\gamma(1+t)r}$$

or, equivalently,

$$\rho e^{\gamma t} e^{\gamma(1+t)r} - (1+r)\rho e^{\gamma t} - (1+r)Lkre^{\gamma t} \geq -r. \tag{2.11}$$

Let us consider the first two terms, that is,

$$f(t) := \rho e^{\gamma t} e^{\gamma(1+t)r} - (1+r)\rho e^{\gamma t}.$$

Computing the derivative, we arrive at

$$f'(t) = \gamma(1+r)\rho e^{\gamma t}[e^{\gamma(1+t)r} - 1] \geq 0.$$

In particular, f is nondecreasing on $[0,T]$, and therefore

$$f(t) \geq f(0) = \rho(e^{\gamma r} - 1 - r) \geq \rho(\gamma - 1)r, \qquad t \in [0,T].$$

This and the choice of k allow us to estimate the left-hand side of (2.11) for $r \in [0,1]$, $t \in [0,T]$, $\rho \in [0,1]$ as

$$\begin{aligned}
\rho e^{\gamma t} e^{\gamma(1+t)r} - (1+r)\rho e^{\gamma t} - (1+r)Lkre^{\gamma t} &\geq f(0) - \min(\gamma,1)r \\
&\geq \rho(\gamma-1)r - \min(\gamma,1)r \\
&= r(\rho(\gamma-1) - \min(\gamma,1)) \\
&\geq -r.
\end{aligned}$$

This is nothing but (2.11), concluding the proof of $g_1 \leq g_2$, and by comparison the proof of (2.10).

Now to check our original claim, suppose $(x_0,t_0) \in \partial\{p > 0\} \cap \{t \leq T\}$. Let p_1 be as given in (2.10); then the spatial ball $\tilde{B}$ of radius kr and center x_0 lies in the positive set of p_1. Due to (2.10), $\tilde{B}$ also lies in the positive set of p at time $t_1 := (1+r)t_0 + r$. Due to the monotone increasing nature of p, we then end up with a space-time cylinder $B_{kr}(x_0) \times [t_1, t_1 + kr]$ lying in the positive set of p. Since $t_1 \leq t_0 + r(T+1)$, we can conclude that our density estimate holds. $\square$

3. Convergence in the local radial setting

Here we will introduce the notion of radial solutions and give the convergence proof of (1.1) to (FB) in this setting. To make local perturbations of general barriers to make first-order approximations in space and time, we need to consider radial barriers with fixed boundaries. The definition will follow the point of view of Section 2, which considers ρ outside the tumor region $\{p > 0\}$ as given a priori by $\rho^E(x,t) = \rho_0^E(x)e^{tG(0)}$.

Definition 3.1 (Radial solutions)**.** The pair (ϕ, ρ^E_ϕ) is a radial, classical solution of (FB) in the cylindrical domain $\{|x-x_0| \le R\} \times [t_1, t_0]$ or $\{|x-x_0| \ge R\} \times [t_1, t_0]$ if

$$\rho^E_\phi(x,t) = \rho^E_\phi(x,t_1) e^{(t-t_1)G(0)}, \qquad \rho^E_\phi(\cdot, t_1) \in C^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),$$

and

(a) $\phi(\cdot, t)$ is radial with respect to x_0 and is smooth in its positive phase, and ϕ is nondecreasing in time;
(b) ϕ solves (FB) in the classical sense with the free boundary motion law $V = \dfrac{|D\phi|}{1-\rho^E_\phi}$;
(c) $\phi(\cdot, t) > 0$ in $|x - x_0| = R$ for $t_1 \le t \le t_0$;
(d) $\rho^E_\phi(\cdot, t)$ is radial with respect to x_0, and $\rho^E_\phi < 1$ outside $\{\phi > 0\}$.
(e) In the case of the interior domain $\{|x-x_0| \le R\}$, there exists $0 < R_1 < R$ such that $\phi = 0$ for $|x| \le R_1$, $t_1 \le t \le t_0$.

The following lemma relates the radial solutions to the weak solutions of (1.1). We take $\Omega := \{|x-x_0| < R\}$ or $\Omega := \{|x-x_0| > R\}$, and for simplicity $t_1 = 0$, $t_0 = T > 0$. As is usual, $Q_T := \Omega \times (0,T)$.

Lemma 3.1. *If* (ϕ, ρ^E_ϕ) *is a radial, classical solution of* (FB) *in the cylindrical domain* Q_T*, then the pair* $(\chi_{\{\phi>0\}} + \chi_{\{\phi=0\}}\rho^E_\phi, \phi)$ *is the unique pair of functions* (ρ, p) *in* $L^\infty(Q_T)$*,* $\rho \in C([0,\infty]; L^1(\Omega))$*,* $p \in P_\infty(\rho)$*, satisfying*

(3.1)
$$\partial_t \rho = \Delta p + \rho G(p) \quad \text{in } \mathcal{D}'(Q_T), \quad \rho(0) = \chi_{\{\phi(\cdot,0)>0\}} + \chi_{\{\phi(\cdot,0)=0\}}\rho^E_\phi(\cdot, 0) \quad \text{in } L^1(\Omega),$$
$$p = \phi \quad \text{on } \partial\Omega \text{ in the sense of trace in } H^1(\Omega) \text{ for a.e. } t > 0,$$

such that

(1) $\rho, p \in L^\infty((0,T); L^1(\Omega))$*;*
(2) $\rho(t)$ *is uniformly compactly supported in* $t \in [0,T]$*;*
(3) $|Dp| \in L^2(Q_T)$*;*
(4) $\partial_t p \in \mathcal{M}(Q_T)$*,* $\partial_t \rho \in \mathcal{M}(Q_T)$*.*

Here P_∞ *is the Hele-Shaw monotone graph*

$$P_\infty(\rho) = \begin{cases} \{0\}, & 0 \le \rho < 1, \\ [0,\infty), & \rho = 1. \end{cases}$$

Proof. Let us first prove the uniqueness of the solutions of (3.1). The statement is analogous to [PQV, Theorem 2.4], with the extra boundary condition for p. The uniqueness proof in [PQV, Section 3] uses Hilbert's duality method with an idea from [C] to deal with the fact that P_∞ is not a Lipschitz function but a graph.

To apply this method, we need any two solutions (ρ_i, p_i) to satisfy

$$\int_{Q_T} (\rho_1 - \rho_2)\psi_t + (p_1 - p_2)\Delta\psi + (\rho_1 G(p_1) - \rho_2 G(p_2))\psi \, dx \, dt = 0$$

for all $\psi \in C^\infty(\overline{Q_T})$ with boundary data zero on $\partial\Omega$ and at $t = T$. For $\psi \in C^\infty_c(Q_T)$ this follows from (3.1). Then this can be extended to include ψ nonzero at $t = 0$ as in [V]. To extend this to all ψ whose support touches the boundary, we need to approximate $\Delta\psi$ by $\Delta\varphi$, $\varphi \in C^\infty_c(Q_T)$ in the correct norm (at least L^1 since $p_1 - p_2 \in L^\infty$). However, this is not possible since $D\psi \ne 0$ on the boundary in general. We therefore use the fact that $p \in L^1(0, T; H^1(\Omega))$ due to assumptions (1) and (3), and show first that

$$\int_{Q_T} (\rho_1 - \rho_2)\psi_t - D(p_1 - p_2) \cdot D\psi + (\rho_1 G(p_1) - \rho_2 G(p_2))\psi \, dx \, dt = 0,$$

by approximation and then integrate the second term by parts in space and use that $p_1 = p_2$ on $\partial\Omega$. Then we just follow [PQV, Section 3] since the rest does not see the boundary values.

To finish the proof, we have to show that $(\chi_{\{\phi>0\}} + \chi_{\{\phi=0\}}\rho^E_\phi, \phi)$ satisfies (3.1). Let us set $p = \phi$ and $\rho = \chi_{\{\phi>0\}} + \chi_{\{\phi=0\}}\rho^E_\phi$. We see that $p \in P^\infty(\rho)$, (ρ, p) has all the regularity required by the assumptions on (ρ^E_ϕ, ϕ) and has the correct initial and boundary data. We therefore only need to show that it satisfies (3.1) in the sense of distributions. Let $\varphi \in C^\infty_c(Q_T)$ be a test function. We will verify that

$$\int_{Q_T} \rho\varphi_t + p\Delta\varphi + \rho G(p)\varphi \, dx \, dt = 0.$$

Since the boundary $\partial\{p > 0\}$ is assumed to be smooth, its unit outer normal is $\frac{1}{\sqrt{1+V^2}}\left(-\frac{Dp}{|Dp|}, -V\right)$ where V is the normal velocity of $\partial\{p > 0\}$ at the given boundary point. Therefore it follows that

$$\begin{aligned}
\int_{Q_T} \rho\varphi_t &= -\int_{\{p=0\}} \rho_t\varphi - \int_{\partial\{p>0\}} (1-\rho)\varphi\frac{V}{\sqrt{1+V^2}} \, dS,\\
\int_{Q_T} p\Delta\varphi &= \int_{\{p>0\}} \varphi\Delta p + \int_{\partial\{p>0\}} |Dp|\,\varphi\frac{1}{\sqrt{1+V^2}} \, dS,\\
\int_{Q_T} \rho G(p)\varphi &= \int_{\{p>0\}} G(p)\varphi + \int_{\{p=0\}} \rho G(0)\varphi.
\end{aligned}$$

We see that the sum of these terms gives zero. □

Remark 3.2. Note that Lemma 3.1 does not address the existence of radial, classical solutions of (FB). However, we need their existence only for the construction of barriers in Section 4. For this purpose, it is enough to prescribe a smooth underlying external density ρ^E_ϕ, the initial radius of the free boundary at $t = t_1$ and the required *fixed* gradient of the solution on the free boundary. Then the evolution of the free boundary in time follows from the velocity law in (FB), yielding an ODE for its radius. From the classical theory, there exists a unique smooth solution for short time. Then, given the radius of the free boundary, for every t we can find the profile $\phi(\cdot, t)$ by solving the elliptic problem for the pressure in (FB) in radial coordinates, with boundary data 0 at the free boundary, and the prescribed gradient. The resulting second order ODE has a unique smooth solution for a small distance away from the free boundary. By the standard ODE theory ϕ is also smooth in time. This yields a radial, classical solution in the sense of Definition 3.1.

We now proceed with the proof of convergence of solutions in the radial setting. To avoid an initial layer in the limit $m \to \infty$, we need to match the initial data for the m-problems (1.1). In particular, we want to obtain locally uniform convergence of the pressure up to the initial time. However, this would not be possible if we solved (1.1) with initial data given in (3.1), independent of m. Therefore for a given radial solution (ϕ, ρ^E_ϕ) we define the initial data for ρ_m by first finding $\delta > 0$, independent of m, so that $\rho^E_\phi(x, t_1) < 1 - \delta$ on $\{\phi(\cdot, t) = 0\}$ and then setting

$$\rho_{0,m}(x) = \max\left[P_m^{-1}(\phi(x, t_1)), \min(1 - \delta, \rho^E_\phi(x, t_1))\right]. \tag{3.2}$$

Such δ can be found due to the assumption (d) in Definition 3.1 and the continuity of ρ^E_ϕ. Note that with the above choice of $\rho_{0,m}$ we have $\rho_{0,m} \to \chi_{\{\phi>0\}}(\cdot, t_1) + \chi_{\{\phi=0\}}(\cdot, t_1)\rho^E_\phi(\cdot, t_1)$ in L^1, $\|D\rho_{0,m}\|_{L^1} \le C$, and $p_{0,m} = P_m(\rho_{0,m}) \to \phi(\cdot, t_1)$ uniformly. Moreover ρ_m is nondecreasing in time when m is large enough according to the following lemma.

Lemma 3.3. *The solution ρ_m of* (1.1) *with initial data $\rho_{0,m}$ given in* (3.2) *and with the boundary data given in Theorem* 3.4 *below is nondecreasing in time for m sufficiently large depending only on $\|D\rho^E_0\|_\infty$, $\|\Delta\rho^E_0\|_\infty$.*

Proof. Note that due to the comparison principle and the fact that the boundary data is nondecreasing in time, we only need to show that ρ_m is nondecreasing at the initial time t_1. We assume that $t_1 = 0$ for simplicity. By the continuity of ρ^E_ϕ we can choose a compact set $K \subset \{\phi(\cdot, 0) > 0\}$ such that $\rho^E_\phi(\cdot, 0) < 1 - \delta$ on K^c. We can find m_0 so that $\phi(\cdot, 0) > P_m(1 - \delta)$ on K for $m \ge m_0$. In particular, when $m \ge m_0$, we see that $P_m(\rho_{0,m}(x)) = \phi(x, 0)$ or $\rho_{0,m}(x) = \rho^E_\phi(x, 0) < 1 - \delta$. We can therefore assume that $\rho^E_\phi(x, 0) < 1 - \delta$ in K without changing $\rho_{0,m}$ in (3.2).

First note that $u(x, t) := \phi(x, 0)$ is a (viscosity) subsolution of the pressure form (1.6) of (1.1) since $\Delta\phi + G(\phi) = 0$ in $\{\phi > 0\}$ at $t = 0$.

On the other hand, since we can assume $\rho^E_\phi < 1-\delta$ everywhere, $v(x, t) := \rho^E_\phi(x, 0)$ is a classical subsolution of (1.1). Indeed, $v_t = 0$, $\Delta(v^m) = m(m-1)v^{m-2}|D\rho^E_\phi|^2 + mv^{m-1}\Delta\rho^E_\phi > -v\frac{G(0)}{2}$ and $vG(P_m(v)) \ge v\frac{G(0)}{2}$ for sufficiently large m since $v^{m-3} \le (1 - \delta)^{m-3} \le \frac{1}{m^3}$ for large m.

We therefore conclude that, for large m, by the comparison principle $\rho_m \ge \max(P_m^{-1}(u), v)$ for $t \ge 0$, with equality at $t = 0$. Therefore ρ_m is nondecreasing in time. $\square$

Theorem 3.4. *For a given radial, classical solution (ϕ, ρ^E_ϕ) on $\{R_1 < |x - x_0| < R\} \times (t_1, t_0)$ or $\{|x - x_0| > R\} \times (t_1, t_0)$, $0 < R_1 < R$, the corresponding solutions p_m, ρ_m of* (1.1) *on the same domain with initial data $\rho_m(\cdot, t_1) = \rho_{0,m}$ at $t = t_1$ and boundary data $p_m = \phi$ on $|x| = R$ (and additionally $\rho_m = \rho^E_\phi$ on $|x| = R_1$ for the interior domain), satisfy the following: p_m uniformly converges to ϕ, and ρ_m locally uniformly converges to ρ^E_ϕ away from the support of ϕ.*

Proof. We consider the case of an exterior domain. The interior domain case is analogous. We will for simplicity assume that $x_0 = 0$, $t_1 = 0$ and $t_0 = T > 0$.

Estimates. Recall that p_m is nondecreasing by Lemma 3.3. We set

$$\kappa = \min\{\phi(x, t) : |x| = R, t \in [0, T]\} > 0.$$

By putting a subsolution under p_m, we can find $R_{1/2} > R$ such that $p_m(\cdot, t) \ge \kappa/2$ on $\Omega_{1/2} = \left\{x : R \le |x| \le R_{1/2}\right\}$ and $t \in [0, T]$.

We first derive the uniform $C^{1,\alpha}$ and $C^{2,\alpha}$ estimates for p_m on $\Omega_{1/2}$. Let us rescale in time. Note that $\tilde{p}_m(x, t) := p_m(x, \frac{t}{m-1})$ solves the equation

$$\partial_t\tilde{p}_m = \tilde{p}_m\Delta\tilde{p}_m + \frac{1}{m-1}|D\tilde{p}_m|^2 + \tilde{p}_mG(\tilde{p}_m).$$

Since $\tilde{p}_m$ is uniformly away from zero in $\Omega_{1/2}\times[0,(m-1)T]$ and uniformly bounded from above, this is a uniformly parabolic, quasilinear equation in the set considered above. Now we have a uniform $C^{1,\alpha}$ estimate up to the boundary for $\tilde{p}_m$, where the $C^{1,\alpha}$ norm depends only on the boundary data of $\tilde{p}_m$ as well as the initial data; see Theorem 4.7 and Theorem 5.3 in [L]. We also have uniform $C^{2,\alpha}$ interior estimates up to the initial boundary. In terms of p_m we lose the estimate in time, but we still have the estimate in space. Namely, for sufficiently small $\varepsilon > 0$ there exists a constant $C_T > 0$, independent of m, such that

$$\|p_m(\cdot,t)\|_{C^{1,\alpha}(\overline{\Omega_{1/2}})} + \|p_m(\cdot,t)\|_{C^{2,\alpha}(\{R+\varepsilon/2\le|x|\le R+2\varepsilon\})} \le C_T \quad \text{for every } 0\le t\le T.$$

This yields the bound

$$\left|D^2 p_m\right| + |Dp_m| \le C \quad \text{on } \{(x,t) : |x| = R+\varepsilon,\ t\ge 0\}. \tag{3.3}$$

Since the set $\{x : |x| = R\}$ is smooth, we can easily create barriers ϕ_1, ϕ_2 at the boundary that coincide with ϕ on the boundary and $\phi_1 \le \phi_2$. Moreover, ϕ_1 is a subsolution and ϕ_2 is a supersolution of

$$p_t = (m-1)p\Delta p + |Dp|^2 + (m-1)pG(p).$$

We conclude that

$$\phi_1 \le p_m \le \phi_2 \qquad \text{in a neighborhood of } \{|x| = R\}.$$

This will imply that the limit of p_m will have the correct boundary data.

Uniqueness. We shall prove that p_m and ρ_m converge to the unique solution of the problem in Lemma 3.1.

The main problem with fixed boundary data arises in the semiconvexity estimate for p_m, a variant of the Aronson–Bénilan estimate. Since the proof relies on the maximum principle for Δp_m, we need to handle the boundary value of this function. To accomplish this, we use the estimate (3.3).

Indeed, [PQV] derives that $w = \Delta p_m + G(p_m)$ is a solution of

$$\begin{aligned} w_t \ge{}& (m-1)p_m\Delta w + 2mDp_m\cdot Dw + (m-1)w^2 \\ &- (m-1)\left(G(p_m) - p_mG'(p_m)\right)w. \end{aligned} \tag{3.4}$$

All the arguments here can be made rigorous as explained in [V, Section 9.3]. Since $\min_{p\in[0,p_M]}\left(G(p) - pG'(p)\right) > 0$, $W(t) = -\frac{1}{(m-1)t}$ is a subsolution of (3.4).

Since on $\Gamma = \{(x,t) : |x| = R+\varepsilon,\ t\ge 0\}$ we have (3.3), we get

$$w = \Delta p_m + G(p_m) \ge \Delta p_m \ge -C \qquad \text{on } \Gamma \tag{3.5}$$

for some constant $C > 0$, independent of m. Let $T = \sup\{t > 0 : W(t) \le -C\} = \frac{C}{m-1}$. Thus $W(t)$ is a subsolution of (3.4) with boundary data $w(x,t) \ge W(t)$ on $\Gamma\cap\{t\le T\}$, and therefore $W(t) \le w(x,t)$ on $\{0\le t\le T\}$. By a bootstrap argument with a shift $W(t-\tau)$ for arbitrary $\tau > 0$, we can deduce that $w(x,t) \ge -C$ on $\{(x,t) : |x| > R+\varepsilon,\ t\ge T\}$.

With (3.5), we can recover all the uniform local L^1-estimates on $\partial_t\rho_m$, $D\rho_m$, $\partial_t p_m$, Dp_m from Section 2 of [PQV], including the L^1-continuity of $\rho_m(t)$ at $t = 0$. A standard argument implies that $\rho_m \to \chi_{\{\phi>0\}} + \chi_{\{\phi=0\}}\rho_\phi^E$ and $p_m \to \phi$ in $L^1_{\mathrm{loc}}(\Omega\times[0,\infty))$ by the uniqueness result (Lemma 3.1).

Lipschitz estimate. The functions p_m and ρ_m depend only on $r = |x|$ and t. In spherical coordinates, (3.5) reads

$$p_{rr} + \frac{n-1}{r}p_r + G(p) \geq \min\left(-\frac{1}{(m-1)t}, -C\right).$$

We observe that $p_{rr} + \frac{n-1}{r}p_r = r^{1-n}\frac{\partial}{\partial r}(r^{n-1}p_r)$. Therefore, for given fixed t and all m large so that $\frac{1}{(m-1)t} < C$ we have for $C_1 = C + G(0)$,

$$r^{1-n}\frac{\partial}{\partial r}(r^{n-1}p_r) \geq -C_1.$$

Integration yields

$$r_2^{n-1}p_r(r_2,t) - r_1^{n-1}p_r(r_1,t) \geq -\frac{C_1}{n}\left(r_2^n - r_1^n\right), \quad r_1 < r_2.$$

To get the lower bound on $p_r(r)$, $r > R+\varepsilon$, we use interior parabolic estimates (3.3) which yield $|p_r(R+\varepsilon,t)| \leq C$. Therefore

$$p_r(r,t) \geq -C\left(\frac{R+\varepsilon}{r}\right)^{n-1} - \frac{C_1 r}{n}\left(1 - \left(\frac{R+\varepsilon}{r}\right)^n\right), \quad r > R+\varepsilon.$$

To get the upper bound, we recall that $0 \leq p \leq p_M$. By the mean value theorem for any $r > R$ there exists $r_2 \in (r, r+1)$ with $|p_r(r_2,t)| \leq p_M$. Thus

$$\begin{aligned} p_r(r,t) &\leq \left(\frac{r_2}{r}\right)^{n-1} p_M + \frac{C_1 r}{n}\left(\left(\frac{r_2}{r}\right)^n - 1\right) \\ &\leq \left(\frac{r+1}{r}\right)^{n-1} p_M + \frac{C_1 r}{n}\left(\left(\frac{r+1}{r}\right)^n - 1\right). \end{aligned}$$

Therefore p_m is locally uniformly Lipschitz in space for every given time $t > 0$ as long as $m \geq C/t + 1$.

Uniform convergence of p_m to ϕ. Let us fix $K \subset \Omega$ compact and $T > 0$. From above we know that $p_m \to \phi$ in $L^1_{\mathrm{loc}}(\Omega \times [0,\infty))$. We can find a countable set $\{t_i\}_{i\in\mathbb{N}} \subset \{t \geq 0\}$ dense in $\{t \geq 0\}$ and a subsequence of p_m, still denoted by p_m, such that $p_m(t_i) \to \phi(t_i)$ in $L^1(K)$ for every t_i. We can choose $t_1 = 0$ since $p_m(\cdot,0) \to \phi(\cdot,0)$ uniformly by the choice of $\rho_{0,m}$ in (3.2). Due to the uniform Lipschitz bound, by taking a subsequence if necessary, we can assume that $p_m(\cdot,t_i) \to \phi(\cdot,t_i)$ uniformly on K for every t_i. Let us choose $\varepsilon > 0$. ϕ is uniformly continuous on $K \times [0,T]$, and so there exists $\delta > 0$ such that $|\phi(x,t) - \phi(x,s)| < \varepsilon$ for any $|t-s| < \delta$, $x \in K$. Find $N \in \mathbb{N}$ such that $\bigcup_{i=1}^N (t_i - \delta/4, t_i + \delta/4) \supset [0, T+\delta]$ and $M \in \mathbb{N}$ such that $\|p_m(\cdot,t_i) - \phi(\cdot,t_i)\|_\infty < \varepsilon$ for all $i = 1,\ldots,N$, $m \geq M$. Now let $t \in [0,T]$. We can find $1 \leq i,j \leq N$ such that $t_i \leq t \leq t_j$, $t_j - t_i < \delta$. Recall that $t \mapsto p_m(x,t)$ is nondecreasing. Thus for any $x \in K$ and $m \geq M$ we have

$$p_m(x,t) - \phi(x,t) \leq p_m(x,t_j) - \phi(x,t_j) + \phi(x,t_j) - \phi(x,t) < 2\varepsilon.$$

On the other hand,

$$p_m(x,t) - \phi(x,t) \geq p_m(x,t_i) - \phi(x,t_i) + \phi(x,t_i) - \phi(x,t) > -2\varepsilon.$$

We conclude that the subsequence $p_m \to \phi$ uniformly on $K \times [0,T]$. Since the limit is unique, the whole sequence must converge.

The uniform convergence of ρ_m. We have $\rho_m = P_m^{-1}(p_m)$. Let K be a compact subset of $\{\phi > 0\}$. By the uniform convergence of p_m, there exists $\varepsilon > 0$

with $p_m \geq \varepsilon$ on K for all m sufficiently large. Then for every $\delta > 0$ for all m large we have

$$\rho_m \geq \left(\frac{1}{2}\varepsilon\right)^{1/(m-1)} > 1 - \delta.$$

The upper bound follows from the uniform upper bound on p_m. Therefore $\rho_m \to 1$ locally uniformly in $\{\phi > 0\}$.

Lastly we would like to prove the local uniform convergence of ρ_m to ρ_ϕ^E outside $\overline{\{\phi > 0\}}$. Since $\{\phi(\cdot, t) = 0\}$ is nonincreasing and radially symmetric, it is enough to prove uniform convergence on sets $\{a \leq |x| \leq b\} \times [0, t_0] \subset \overline{\{\phi > 0\}}^c$ for some $0 < a < b$ and $t_0 > 0$. We will argue by iteration over small time intervals as follows:

Fix $0 < a < b$, $t_0 > 0$ and $\eta > 0$ such that $Q_\eta := \{a - 2\eta \leq |x| \leq b + 2\eta\} \times [0, t_0 + \eta] \subset \overline{\{\phi > 0\}}^c$. We can choose $\delta > 0$ such that $\rho_\phi^E < 1 - \delta$ on Q_η.

Now by the L^1-convergence of $\rho_m \to \rho_\phi^E$ on $\{\phi = 0\}$ and the radial symmetry, we can find $r_1 \in (a - 2\eta, a - \eta)$, $r_2 \in (b + \eta, b + 2\eta)$, $t_1 \in (t_0, t_0 + \eta)$ and a subsequence of ρ_m, still denoted by ρ_m, such that $\rho_m(x, t_1) \to \rho_\phi^E(x, t_1) < 1 - \delta$ as $m \to \infty$ for $|x| = r_i$, $i = 1, 2$. In particular, since ρ_m is nondecreasing in time, we have $\rho_m(x, t) \leq 1 - \delta$ for all $t \in [0, t_1]$, $|x| = r_i$ for all sufficiently large m along the subsequence.

Let us set the time γ that satisfies

$$e^{(G(0)+1)\gamma}(1 - \delta) = 1 - \tfrac{\delta}{2}. \tag{3.6}$$

Note that $\gamma = O(\delta)$. We will prove the uniform convergence of $\rho_m \to \rho_\phi^E$ along the subsequence on $\{a \leq |x| \leq b\} \times [0, \gamma]$ by a barrier argument as follows. Choose $\varepsilon \in (0, 1)$. At $t = 0$ we pick a radial, smooth function $\varphi_0(x)$ such that $\rho_\phi^E(\cdot, 0) < \varphi_0 \leq 1 - \delta$ for $|x| \in [r_1, r_2]$, $\varphi_0 = 1 - \delta$ at $|x| = r_i$, and $\varphi_0 < \rho_\phi^E(\cdot, 0) + \varepsilon$ for $|x| \in [a, b]$. Let $\varphi(x, t) = e^{(G(0)+\varepsilon)t}\varphi_0(x)$.

Note that from (3.6) we have $\varphi \leq 1 - \delta/2$ in $\Sigma := \{r_1 < |x| < r_2\} \times [0, \gamma]$, and thus $\varphi^m \leq \frac{1}{m^3}$ for large m. Due to this fact and that φ_0 is smooth, it follows that φ is a supersolution of (1.1) in Σ for sufficiently large m. Indeed, $\Delta(\varphi^m) = m(m-1)\varphi^{m-2}|D\varphi|^2 + m\varphi^{m-1}\Delta\varphi = o(m)\varphi$, $\varphi_t = (G(0) + \varepsilon)\varphi$ and $\varphi G(P_m(\varphi)) = \varphi(G(0) + o(m))$. Therefore $\varphi_t - \Delta(\varphi^m) - \varphi G(P_m(\varphi)) > 0$ for sufficiently large m. Since $\rho_m \leq \varphi$ on the parabolic boundary of Σ for m sufficiently large by the choice of the boundary values of φ, the comparison principle for (1.1) yields that $\rho_m \leq \varphi$ in Σ along the subsequence. In particular, $\rho_m \leq \rho_\phi^E + e^{(G(0)+\varepsilon)\gamma}\varepsilon$ on $\{a \leq |x| \leq b\} \times [0, \gamma]$ for sufficiently large m along a subsequence.

By constructing a similar barrier with boundary data zero for $|x| = r_i$, $0 \leq \varphi_0 \leq \rho_\phi^E$ for $|x| \in [r_1, r_2]$ and $\varphi_0 > \rho_\phi^E - \varepsilon$ for $|x| \in [a, b]$, and then setting $\varphi(x, t) = e^{(G(0)-\varepsilon)}\varphi_0(x)$, we can obtain an analogous bound for the (whole) sequence ρ_m from below. We conclude that ρ_m uniformly converges to ρ_ϕ^E on $\{a \leq |x| \leq b\} \times [0, \gamma]$ along a subsequence.

Now we can iterate the argument in time up to time $t = t_1$ in intervals of length γ, which yields the uniform convergence in $\{a \leq |x| \leq b\} \times [0, t_0]$. By uniqueness of the limit, the whole sequence ρ_m converges uniformly to ρ_ϕ^E on this set. This concludes the proof. $\square$

4. Convergence in the general setting

Now once we have Theorem 3.4, we next consider general, i.e., nonradial, solutions ρ_m of (1.1) and the corresponding pressure variable $p_m = P_m(\rho_m)$ with initial data $\rho_{0,m}$ given by (1.9) that approximate the initial data (1.7).

As we shall see in the lemma below, our choice of initial data $\rho_{0,m}$ will guarantee that ρ_m is monotonically increasing in time. After we obtain the convergence result for this particular approximation of ρ_0, we can use the L^1 contraction property for solutions of (1.1) to address the case of general $\rho_{0,m}$; see Corollary 4.9.

Lemma 4.1. *Suppose that ρ_m is the solution of* (1.1) *with initial data $\rho_{0,m}$ given by* (1.9)*. Then ρ_m increases in time for large enough m.*

Proof. Let us first consider $\tilde\rho_m(x,t) := \rho^E_{0,m}(x)\exp(tG(0)/2)$. Writing $\rho = \tilde\rho_m$ for the sake of brevity, we can estimate

$$\begin{aligned}\Delta(\rho^m) + \rho G(p) &= m(m-1)\rho^{m-2}|D\rho|^2 + m\rho^{m-1}\Delta\rho + \rho G(p)\\ &\geq \rho\left(m\rho^{m-2}\Delta\rho + G(p)\right).\end{aligned} \tag{4.1}$$

Due to our assumptions on the initial data in (1.8), there exists $t_0 > 0$ such that $e^{t_0 G(0)/2}(1-\delta) < 1-\delta/2$ and the right-hand side in (4.1) is greater than $\tilde\rho_m G(0)/2 = \partial_t\tilde\rho_m$ for $0 < t < t_0$ and $m \gg 1$, and therefore $\tilde\rho_m$ is a subsolution of (1.1) for a short time interval independent of m.

Additionally, $\hat\rho_m(x,t) := P_m^{-1}(p_0(x))$ is a stationary subsolution of (1.1). We have defined the nondecreasing-in-time functions $\tilde\rho_m$ and $\hat\rho_m$ in such a way that $\max(\tilde\rho_m(\cdot,0), \hat\rho_m(\cdot,0)) = \rho_{0,m}$. Since a maximum of two subsolutions is also a subsolution, we conclude that $\rho_m \geq \max(\tilde\rho_m, \hat\rho_m)$ for $0 \leq t < t_0$, with equality at $t = 0$. Therefore $\rho_m(\cdot,s) \geq \rho_m(\cdot,0)$ for any $0 < s < t_0$. By the comparison principle we have $\rho_m(\cdot,s) \geq \rho_m(\cdot,t)$ for any $s \geq t$. ☐

Recall that $p_m = P_m(\rho_m) := \frac{m}{m-1}\rho_m^{m-1}$. Our goal is to show their convergence to the solutions of (FB) as $m\to\infty$. To this end we first define the *semi-continuous limits* (also referred to as the *half-relaxed limits*) as $m\to\infty$ for a family of functions f_m as

$$\liminf_* f_m(x,t) := \lim_{r\to 0}\inf_{\substack{|y|+|s|\leq r\\ m\geq r^{-1}}} f_m(x+y,t+s)$$

and

$$\limsup^* f_m(x,t) := \lim_{r\to 0}\sup_{\substack{|y|+|s|\leq r\\ m\geq r^{-1}}} f_m(x+y,t+s).$$

Now let us consider the semi-continuous limits of ρ_m and p_m, i.e.,

$$\rho_1 := \liminf_* \rho_m, \quad p_1 := \liminf_* p_m$$

and

$$\rho_2 := \limsup^* \rho_m, \quad \tilde p_2 := \limsup^* p_m.$$

For technical reasons, it is useful to consider a regularization of ρ_2 as follows. For a given constant $\sigma > 0$ let us define

$$\rho_m^\sigma(x,t) := \sup_{|y-x|\leq\sigma}\rho_m(y,t), \qquad \rho^{\sigma,E}(x,t) := \sup_{|y-x|\leq\sigma}\rho^E(y,t).$$

Note that ρ^σ is a subsolution of (1.1). Now let us define

$$\rho_2^\sigma := \limsup^* \rho_m^\sigma.$$

Observe that ρ_1 is lower semi-continuous and ρ_2 and ρ_2^σ are upper semi-continuous. Let us also define the sets

$$\Omega_1(t) := \{p_1(\cdot,t) > 0\}, \quad \Omega_2(t) = \{\rho_2(\cdot,t) = 1\} \quad \text{and} \quad \Omega_2^\sigma(t) = \{\rho_2^\sigma(\cdot,t) = 1\},$$

and define $p_2^\sigma(\cdot,t)$ for each $t > 0$ as the smallest supersolution of $-\Delta u = G(u)$ with nonnegative Dirichlet boundary data in $\Omega_2^\sigma(t)$, that is,

$$p_2^\sigma(x,t) := \inf\{w(x) : w \in C^2(\mathbb{R}^n), -\Delta w > G(w) \text{ in a domain containing } \Omega_2^\sigma(t), \quad w > 0\}, \tag{4.2}$$

and we similarly define p_2 corresponding to the set $\Omega_2(t)$. p_2^σ is defined in addition to $\tilde{p}_2$ so that we can track the positive set of p_m. $\tilde{p}_2$ is not sufficient for this purpose since we do not know if p_m degenerates to zero as $m \to \infty$ inside the set $\{\rho_2 = 1\}$. We use the set $\Omega_2^\sigma(t)$ instead of $\Omega_2(t)$ to guarantee that the set is regular enough so that the positive set of $p_2^\sigma(\cdot,t)$ coincides with the reference set $\Omega_2^\sigma(t)$, as we see in the next lemma. The following lemma shows the relationship between the various sets, where the last equality is the only nontrivial relation, and explains the utility of p_2^σ.

Lemma 4.2. *For any $\sigma > 0$ we have*

$$\{p_1 > 0\} \subset \{\rho_1 = 1\} \subset \{\rho_2^\sigma = 1\} = \overline{\{p_2^\sigma > 0\}}.$$

In fact, we have

$$\Omega_2^\sigma(t) = \{\rho_2^\sigma(\cdot,t) = 1\} = \overline{\{p_2^\sigma(\cdot,t) > 0\}} \qquad \textit{for } t \geq 0.$$

Proof. Suppose that $\rho_1(x_0,t_0) < 1$ for some (x_0,t_0). Then there exist m_k, x_k, t_k, $m_k \to \infty$ and $(x_k,t_k) \to (x_0,t_0)$ as $k \to \infty$ such that $\rho_{m_k}(x_k,t_k) \to \rho_1(x_0,t_0) < 1$. But then $p_1(x_0,t_0) \leq \liminf_{k\to\infty} \frac{m_k}{m_k-1}\rho_{m_k}(x_k,t_k)^{m-1} = 0$. In particular, $\{p_1 > 0\} \subset \{\rho_1 = 1\}$. The second inclusion in the lemma is due to the fact that $\rho_1 \leq \rho_2 \leq \rho_2^\sigma$ for any $\sigma > 0$. Lastly, note that due to its definition $\Omega_2^\sigma(t)$ is closed and has the interior ball property with balls of radius σ. It now follows from the definition of p_2^σ that $\overline{\{p_2^\sigma(\cdot,t) > 0\}} = \Omega_2^\sigma(t)$. □

We also point out that due to Lemma 4.1, it follows that ρ_1 and ρ_2 are both nondecreasing in time.

Below we will show that

(a) $\tilde{p}_2 \leq p_2^\sigma$ (Lemma 4.5);
(b) p_1 and p_2^σ are respectively a supersolution of (FB) with ρ^E and a subsolution of (FB) with $\rho^{\sigma,E}$ (Theorem 4.6);
(c) $p_1(\cdot,0) = p_2(\cdot,0)$ is given by (1.3) with Ω_0 (Lemma 4.7).

Due to (b) and the stability property of the viscosity solutions of (FB), we have $(p_2)_* \leq p_1$. This and (a) yield the convergence results (see Corollary 4.8). We first show that $\Omega_2^\sigma(t)$ (and therefore $\Omega_2(t)$, $\Omega_1(t)$) is bounded.

Lemma 4.3. *$\Omega_2^\sigma(t)$ is bounded for any $t > 0$.*

Proof. By our assumption (1.7), ρ_0^E uniformly converges to zero as $|x| \to \infty$. Therefore for any $T > 0$, there exists $R > 0$ such that $\Omega_2^\sigma(0) \subset B_R(0)$ and

$$\rho^E(x,0) \leq \frac{1}{2}e^{-G(0)T} \text{ for } |x| > R.$$

Let us consider the radial solution ϕ of (FB) with initial support $\Omega_{\phi,0} = B_R(0)$ and with velocity coefficient $g := \frac{1}{1-\rho_\phi^E}$ given by

$$\rho_\phi^E(x,t) := \frac{1}{2}e^{G(0)(t-T)}.$$

Such a solution exists by the standard ODE theory and has a bounded support for all $0 \le t \le T$ by comparison with the barrier (2.9). If $\rho_{\phi,m}$ are the solutions of (1.1) with the initial data given in (3.2) (with appropriate cutoff of $\rho_\phi^E(\cdot,0)$ outside the support of $\phi(\cdot,T)$ so that $\rho_{\phi,m}(\cdot,0) \in L^1(\mathbb{R}^n)$), then $\rho_m \le \rho_{\phi,m}$ for $m \gg 1$ by the comparison principle for (1.1). Moreover, Theorem 3.4 yields that $\rho_{\phi,m}$ locally uniformly converge to ρ_ϕ outside the support of ϕ. Therefore it follows that $\Omega_2^\sigma(t) \subset \{\phi(\cdot,t) > 0\}$ for $0 \le t \le T$, which is bounded, and we conclude. $\square$

Next we prove the following lemma to match ρ_i's with ρ^E.

Lemma 4.4. *Let ρ_1, ρ_2 be as defined above. Then the following hold:*

(a) $\rho_2^\sigma \le 1$ *and* $\tilde{p}_2 \le p_M$ *for* $t \ge 0$*;*
(b) $\rho_1 \ge \min[1, \rho^E]$ *and* $\{\rho^E \ge 1\} \subset \overline{\{p_1 > 0\}}$*;*
(c) $\rho_2^\sigma \le \rho^{\sigma,E} < 1$ *outside* $\{\rho_2^\sigma = 1\}$*.*

Proof. 1. To show (a), recall the initial data $\rho_{0,m}$ from (1.9). By the comparison principle, $p_0 \le p_M$ since $G(p_M) = 0$. In particular, $p_{0,m} = P_m(\rho_{0,m}) \le p_M$ for $m \gg 1$. Therefore the comparison principle yields $p_m = P_m(\rho_m) \le p_M$ for all $m \gg 1$. In particular, $\rho_m \le P_m^{-1}(p_M) \to 1$ as $m \to \infty$. Therefore (a) follows.

2. To show the first part of (b), we fix $x_0 \in \mathbb{R}^n$. If $\rho_0^E(x_0) = 0$, clearly $\rho_1(x_0,t) \ge 0 = \rho^E(x_0,t)$, $t \ge 0$. Therefore we suppose that $\rho_0^E(x_0) > 0$ and we will show that for every fixed $0 < \varepsilon < \rho_0^E(x_0)$,

$$\rho_1(x_0,t) \ge \min[1-\varepsilon, e^{G(0)t}(\rho_0^E(x_0) - \varepsilon)] - \varepsilon, \qquad t \ge 0. \tag{4.3}$$

To show (4.3), we fix $r > 0$ so that $\rho_0^E \ge \rho_0^E(x_0) - \varepsilon/2$ on $B_r(x_0)$. By the uniform convergence of the initial data $\rho_{0,m}^E$ to ρ_0^E in (1.8), we have $\rho_{0,m}^E > \rho_0^E(x_0) - \varepsilon$ on $B_r(x_0)$ for m large enough. Now we consider the function

$$\phi(x,t) := a(t)\varphi(x) - \varepsilon t,$$

where $a(t) := \min[1-\varepsilon, e^{G(0)t}(\rho_0^E(x_0) - \varepsilon)]$, and $\varphi = \varphi_m$ is defined as

$$\varphi(x) := \left(\left(1 - |x-x_0|^2/r^2\right)_+\right)^{1/m}.$$

If m is sufficiently large, then ϕ satisfies in its positive set

$$\begin{aligned}\phi_t - \Delta(\phi^m) &= \phi_t - a^m\Delta(\varphi^m)\\ &\le G(0)\phi - \varepsilon + 2na^m/r^2\\ &\le G(0)\phi - \varepsilon/2 \le G(p_\phi)\phi,\end{aligned}$$

where $p_\phi := P_m(\phi) = \frac{m}{m-1}\phi^{m-1}$. Note that the first inequality holds by the definition of $a(t)$, the second one holds for m sufficiently large since $a(t) \le 1-\varepsilon$, and the last inequality holds for m large since $\phi \le a \le 1-\varepsilon$.

Thus ϕ is a subsolution of (1.1) with initial data $(\rho_0^E(x_0) - \varepsilon)\varphi$, and it follows from the comparison principle of (1.1) that $\phi \le \rho_m$ and thus $\phi \le \rho_1$, yielding (4.3). We conclude by sending $\varepsilon \to 0$.

3. Now let us prove the second part of (b) by modifying the subsolution barrier in the above step. Suppose $\rho^E(\cdot, t_0) \geq 1$ in $B_r(x_0)$ for some (x_0, t_0) and $0 < r < |2G'(0)|^{-1/2}$. Since ρ^E is nondecreasing in time, we have $\rho^E \geq 1$ on $\overline{B}_r(x_0)\times[t_0, \infty)$. Then from the first part we have $\rho_1 \geq 1$ in $B_r(x_0) \times [t_0, \infty)$, and thus for any $\delta > 0$ and for sufficiently large $m(\delta)$ we have

$$\rho_m \geq 1 - \delta \text{ in } \overline{B}_r(x_0) \times [t_0, t_1] \text{ for } m > m(\delta),$$

where $t_1 := t_0 + 2G(0)^{-1}\delta$.

Now let us construct the barrier $\phi(x, t) = a(t)\varphi(x)$ to compare with ρ_m in $B_r(x_0) \times [t_0, t_1]$, where $a(t) = e^{(G(0)-3\delta)(t-t_1)}$ and

$$\varphi(x) = \left[\frac{\delta}{2n}\left(r^2 - |x - x_0|^2\right) + (1 - \delta)^m\right]^{1/m}$$

so that we have $-\Delta(\varphi^m) = \delta$ and $\varphi \geq 1 - \delta$ in $B_r(x_0)$, with equality $\varphi = 1 - \delta$ on $\partial B_r(x_0)$. Also at initial time $t = t_0$, $1 - 2\delta \leq a(t_0) = e^{(G(0)-3\delta)(t_0-t_1)} < 1 - \delta$ for sufficiently small $\delta > 0$ since $t_0 - t_1 = -2G(0)^{-1}\delta$. Hence for small δ, we have $\phi \leq 1 - \delta \leq \rho_m$ at $t = t_0$ for all large m and $\phi \leq 1 - \delta$ on $\partial B_r(x_0) \times [t_0, t_1]$. Also $\phi \geq 1 - 3\delta \geq \frac{1}{2}$ in $B_r(x_0) \times [t_0, t_1]$ for δ small.

Then we can estimate

$$\begin{aligned}\phi_t - \Delta(\phi^m) &\leq \phi_t - \Delta(\varphi^m)\\ &\leq [G(0) - 3\delta]\phi + \delta\\ &\leq [G(0) - \delta]\phi \leq G(p_\phi)\phi,\end{aligned}$$

where $p_\phi := P_m(\phi)$. The first inequality holds due to the fact that $a(t) \leq 1$ and $-\Delta(\varphi^m) \geq 0$, and the last inequality holds for δ sufficiently small due to the fact that $\phi \geq \frac{1}{2}$ and $p_\phi G'(0) \geq \delta G'(0)r^2/n > -\delta/2$ for large m due to the choice of r at the beginning of step 3. Hence we conclude that $\phi \leq \rho_m$ in $B_r(x_0) \times [t_0, t_1]$ by the comparison principle for (1.1), which yields

$$\frac{\delta r^2}{8n} \leq \phi^m \leq p_m \text{ in } B_{r/2}(x_0) \times [t_0, t_1]$$

for $m > m(\delta)$. Thus

$$p_1(x_0, t_1) = p_1(x_0, t_0 + 2G(0)^{-1}\delta) > 0 \tag{4.4}$$

since r is independent of m. As (4.4) holds for arbitrarily small δ, it follows that $(x_0, t_0) \in \overline{\{p_1 > 0\}}$ and we can conclude.

4. Lastly, to show (c), we will show that for any given $\delta > 0$,

$$\rho_2^\sigma \leq \rho^{\sigma,E} \text{ on } \{\rho_2^\sigma < 1 - 2\delta\}. \tag{4.5}$$

We will show this iteratively over time intervals of fixed size $\gamma > 0$, where γ satisfies

$$e^{(G(0)+1)\gamma}(1 - \delta) = 1 - \delta/2. \tag{4.6}$$

Note that (4.5) holds for $t = 0$ by a standard argument via a comparison with suitable barriers. Suppose that (4.5) holds up to $t = T$, and let us choose (x_0, t_0) in $\{\rho_2^\sigma < 1-2\delta\}\cap\{T \leq t \leq T+\gamma\}$. Due to the upper-semi-continuity of ρ_2 and its monotonicity in time, there exists $r > 0$ such that $\rho_2^\sigma < 1 - \delta$ in $\overline{B}_{2r}(x_0) \times [T, t_0]$. Also note that, due to the first part of (b), we have $\min[\rho^{\sigma,E}(\cdot, T), 1] \leq \rho_1^\sigma(\cdot, T) \leq \rho_2^\sigma(\cdot, T) < 1 - \delta < 1$ on $\overline{B}_{2r}(x_0)$ and hence $\rho^{\sigma,E}(\cdot, t_0) = e^{G(0)(t_0-T)}\rho^{\sigma,E}(\cdot, T) \leq e^{G(0)\gamma}(1 - \delta) < 1 - \delta/2 < 1$ in $\overline{B}_{2r}(x_0)$.

Now based on these facts we will construct a supersolution barrier ϕ for (1.1) in $\Sigma := B_{2r}(x_0) \times [T, t_0]$ such that $\phi \le \rho^{\sigma,E}$ in $B_r(x_0) \times [T, t_0)$, concluding (4.5).

Let us choose $\varepsilon \in (0, 1-\delta)$ and let φ be a smooth function in $\overline{B}_{2r}(x_0)$, $\varphi \le 1-\delta$ such that $\varphi = 1-\delta$ on $\partial B_{2r}(x_0)$, $\max[\rho^{\sigma,E}(\cdot, T), \varepsilon] \le \varphi$ on $B_{2r}(x_0)$ and $\varphi \le \rho^{\sigma,E}(\cdot, T) + \varepsilon$ in $B_r(x_0)$. Now consider the function

$$\phi(x,t) := e^{(G(0)+\varepsilon)(t-T)}\varphi(x) \text{ in } \Sigma.$$

Note that from (4.6) we have $\phi \le 1-\delta/2$ in Σ, and thus $\phi^m \le \frac{1}{m^3}$ for large m. Due to this fact and that φ is smooth, it follows that ϕ is a supersolution of (1.1) in Σ for sufficiently large m. Indeed, $\Delta(\phi^m) = m(m-1)\phi^{m-2}|D\phi|^2 + m\phi^{m-1}\Delta\phi = o(m)\phi$, $\phi_t = (G(0)+\varepsilon)\phi$ and $\phi G(P_m(\phi)) = \phi(G(0) + o(m))$. Therefore $\phi_t - \Delta(\phi^m) - \phi G(P_m(\phi)) > 0$ for sufficiently large m. Since $\rho_2^\sigma < 1-\delta$ in $\overline{\Sigma}$, so is ρ_m^σ for sufficiently large m, and thus $\rho_m^\sigma \le \phi$ on the parabolic boundary of Σ. Hence the comparison principle for (1.1) yields that $\rho_m^\sigma \le \phi$ in Σ. By sending $\varepsilon \to 0$ we conclude that $\rho_2^\sigma \le \rho^{\sigma,E}$ at (x_0, t_0), proving (4.5) for the time interval $[T, T+\gamma]$. Now (4.5) follows by iterating our argument over time intervals of length γ. Lastly we finish the proof of (c) by sending $\delta \to 0$ in (4.5). ☐

Next let us prove that p_2^σ is bigger than the limit supremum of p_m.

Lemma 4.5. $\tilde{p}_2 \le p_2^\sigma$.

Proof. For any $\varepsilon > 0$ and $t_0 > 0$, take a smooth solution $w = w(x)$ of $-\Delta w \ge G(w) + \varepsilon$ with $w \ge \varepsilon$ in a domain U containing the closure of $\Omega_2^\sigma(t_0)$. We will show that $\tilde{p}_2(\cdot, t_0) \le w$. Then one can conclude by the definition of p_2^σ.

As $\{\rho_2^\sigma = 1\}$ is closed, there exists $\delta > 0$ such that $\Omega_2^\sigma(t) \subset U$ for $|t - t_0| \le \delta$. Due to Lemma 4.4(a) and the fact that $\tilde{p}_2 = 0$ on $\{\rho_2 < 1\} \supset \{\rho_2^\sigma < 1\}$, $\phi(x,t) := \frac{p_M}{\delta}(t_0 - t) + w(x)$ is above p_m on the parabolic boundary of $\Sigma := U \times [t_0 - \delta, t_0 + \frac{\varepsilon\delta}{2p_M}]$ for all $m \gg 1$.

Moreover, $\phi \ge \frac{\varepsilon}{2}$ on Σ and ϕ is a supersolution of (1.6) on Σ for sufficiently large m since

$$\phi_t + (m-1)\phi(-\Delta\phi - G(\phi)) - |D\phi|^2 \ge -\frac{p_M}{\delta} + \tfrac{1}{2}(m-1)\varepsilon^2 - |Dw|^2 \ge 0 \text{ for } m \gg 1.$$

Thus we conclude that $p_m \le \phi$ in Σ, which yields that $\tilde{p}_2 \le \phi$ in Σ, and hence $\tilde{p}_2(\cdot, t_0) \le w$ in U. The lemma follows since p_2^σ is the infimum of such w. ☐

Now we are ready to show our main claim:

Theorem 4.6. *p_1 and p_2^σ are respectively a supersolution of* (FB) *with* $g := \frac{1}{1-\rho^E}$ *and a subsolution of* (FB) *with* $g^\sigma := \frac{1}{1-\rho^{\sigma,E}}$.

First note that Lemma 4.4 will allow us to treat the limiting density outside the maximal density zone essentially as ρ^E.

Proof. 1. We will use Definition 2.4. Let us show the subsolution part first. Suppose p_2^σ is not a subsolution of (FB) with g^σ. This means that there is a *superbarrier* ϕ of (FB) with g^σ in $U := \{|x - z_0| < r\} \times (t_1, t_2]$ which crosses p_2^σ from above at $t = t_0 \in (t_1, t_2]$. In other words, we have

- $p_2^\sigma \prec \phi$ on the parabolic boundary of U;
- $p_2^\sigma \prec \phi$ in $U \cap \{t_1 \le t < t_0\}$;
- there exists $\hat{x} \in B_r(z_0) \cap \overline{\{p_2^\sigma(\cdot, t_0) > 0\}}$ such that $p_2^\sigma(\hat{x}, t_0) \ge \phi(\hat{x}, t_0)$.

Since ϕ is a superbarrier of (FB), we may choose $r > 0$ small such that there exists $\delta > 0$ such that $\rho^{\sigma,E} < 1 - 2\delta$ in U, $-\Delta\phi > G(\phi) + \delta$ on $\{\phi > 0\}$, and

$$V_\phi > \frac{|D\phi|}{1 - (\rho^{\sigma,E} + \delta)} \quad \text{on } \partial\{\phi > 0\}. \tag{4.7}$$

2. From its definition, p_2^σ cannot cross ϕ before its support crosses that of ϕ. It follows that $\chi_{\overline{\{p_2^\sigma > 0\}}}(\cdot, t_0)$ crosses $\chi_{\{\phi>0\}}$ at $t = t_0$, and thus along a subsequence $\rho_m^\sigma \geq \chi_{\{\phi>0\}} + (\rho^{\sigma,E} + \delta)\chi_{\{\phi=0\}}$ for the first time at (x_m, t_m) with $t_m \to t_1 \leq t_0$ as $m \to \infty$. Note that the crossing point exists since ρ_m is continuous in time, while $\chi_{\{\phi>0\}} + (\rho^{\sigma,E} + \delta)\chi_{\{\phi=0\}}$ is lower semi-continuous.

Let x_0 be a limit point of $\{x_m\}$. If $\phi(x_0, t_1) > 0$, then we have a contradiction since in that case it can be easily checked that ϕ is a supersolution of (1.6) in a neighborhood of (x_0, t_1) for sufficiently large m. Also due to Lemma 4.4(c) and the fact that, from Lemma 4.2,

$$\{\rho_2^\sigma(\cdot, t) = 1\} = \overline{\{p_2^\sigma(\cdot, t) > 0\}} \subset \{\phi(\cdot, t) > 0\} \text{ for } t < t_0,$$

the limit point (x_0, t_1) cannot be outside $\overline{\{\phi > 0\}}$. Hence (x_0, t_1) lies on $\partial\{\phi > 0\}$, and $t_1 = t_0$.

Relying on the continuity of ρ^E, let us choose $0 < r < \delta$ such that

$$\begin{aligned} \rho^{\sigma,E} &\leq \rho^{\sigma,E}(x_0, t_0) + \frac{\delta}{2} \leq \bar{\rho}^{\sigma,E}(t) \\ &:= \left(\rho^{\sigma,E}(x_0, t_0) + \frac{\delta}{2}\right) e^{G(0)(t - t_0 + r)} \quad \text{in } D := B_r(x_0) \times [t_0 - r, t_0 + r]. \end{aligned} \tag{4.8}$$

We now localize ϕ in D to a radial profile. Since $|D\phi| \neq 0$ on $\partial\{\phi > 0\}$, it follows from the regularity of ϕ that $\partial\{\phi(\cdot, t_0) > 0\}$ is a C^2 surface. Therefore we can choose r in the above definition of D small enough such that there is an exterior ball $B_{r/2}(y_0)$ in $\{\phi(\cdot, t_0) = 0\}$ touching x_0 on its boundary. By matching the first-order behavior of ϕ at (x_0, t_0), we can construct a new radial superbarrier $\varphi(x, t) = \varphi(|x - y_0|, t)$ of (FB) in D satisfying (4.7) such that $\{\varphi(\cdot, t_0) = 0\} = B_{r/2}(y_0)$ and $\phi \prec \varphi$ on the parabolic boundary of D. Then, replacing φ with $\varphi(\cdot, \cdot + \varepsilon)$ for sufficiently small $\varepsilon > 0$ if necessary, $p_m^\sigma = P_m(\rho_m^\sigma)$ crosses φ in D for large m. Due to Lemma 4.4(c) and (4.8),

$$\rho_m^\sigma < \chi_{\overline{\{\varphi>0\}}} + \bar{\rho}^{\sigma,E}\chi_{\overline{\{\varphi>0\}}^c} \text{ on the parabolic boundary of } D$$

for large m.

Now let p be the unique solution of (FB) on D, radially symmetric with respect to y_0, with initial support at $t = t_0 - r$ equal to that of φ, and with boundary data on $\partial B_r(x_0)$ equal to φ, and free boundary velocity coefficient given by $\bar{\rho}^{\sigma,E}(t)$. This $(p, \bar{\rho}^{\sigma,E})$ is an interior radial solution in the sense of Section 3. Now let $\tilde{\rho}_m$ be the corresponding solutions of (1.1) in D, with fixed Dirichlet boundary data $P_m^{-1}(\varphi)$ on $\partial B_r(x_0)$ and $\bar{\rho}^{\sigma,E}$ on $\partial B_{r/4}(x_0)$ with approximating initial data given as in (3.2) in Section 3. Note that, due to the comparison principle of (1.1), $\rho_m^\sigma \leq \tilde{\rho}_m$ in D. On the other hand, the solution $(p, \bar{\rho}^{\sigma,E})$ of (FB) in D satisfies $p \prec \varphi$ in D due to (4.7). Due to Theorem 3.4 $\limsup_{m\to\infty} \tilde{\rho}_m = \bar{\rho}^{\sigma,E} < \rho^{\sigma,E} + \delta$ outside the support of p in D, in particular in the zero set of φ in D. This contradicts the fact that ρ_m^σ crosses $\chi_{\{\varphi>0\}} + (\rho^{\sigma,E} + \delta)\chi_{\{\varphi=0\}}$ in D. We can now conclude.

3. For the supersolution part, first note that the requirement $\{\rho^E \geq 1\} \subset \overline{\{p_1 > 0\}}$ is satisfied by Lemma 4.4(b). Next suppose a *subbarrier* ϕ of (FB) crosses

p_1 from below in $\{|x - z_0| \geq r\} \times [t_1, t_2]$ at $t = t_0$. Parallel arguments as above using Lemma 4.4(b) would yield the conclusion. $\square$

Lastly, to apply the comparison principle for p_1 and p_2, we show that the initial data for ρ_i's and p_i's respectively coincide.

Lemma 4.7. *At $t = 0$ we have for $i = 1, 2$:*

(a) $\lim_{t\to 0^+} \rho_i(\cdot, t) = \rho_0 := \rho_0^E \chi_{\Omega_0^c} + \chi_{\Omega_0}$ *locally uniformly away from $\partial\Omega_0$;*
(b) $\lim_{t\to 0^+} p_i(\cdot, t) = p_0$ *uniformly,*

where p_0 is the unique solution of $-\Delta p = G(p)$ in Ω_0 with zero boundary data on $\partial\Omega_0$, extended by zero to Ω_0^c.

Proof. 1. Let us first show (*a*). First of all note that $\rho^E(\cdot, t)$ converges uniformly to ρ_0 as $t \to 0+$ away from $\Omega_0 = \{\rho_0 = 1\}$. Also note that, from their definition, $\Omega_2^\sigma(t)$ converges to $\Omega_2(t)$ in Hausdorff distance as $\sigma \to 0$. Hence by Lemma 4.4 we have

$$\rho^E = \rho_1 = \rho_2 \text{ outside } \{\rho_2 = 1\}. \tag{4.9}$$

Indeed, if $\rho_2(x, t) < 1$ for some (x, t), then for all $\sigma > 0$ small $\rho_2^\sigma(x, t) < 1$ by semi-continuity (or by the above convergence in Hausdorff distance). Therefore by Lemma 4.4(c) we have $\rho_2^\sigma \leq \rho^{\sigma,E}$. By definition, $\rho_1 \leq \rho_2^\sigma$, and by Lemma 4.4(b) $\rho^E \leq \rho_1$. Since $\rho^{\sigma,E} \to \rho^E$ as $\sigma \to 0$ by continuity, (4.9) follows.

Moreover, by Lemma 4.4 we have $\rho_1 \geq 1$ on Ω_0. Hence it is enough to show that

$$\{\rho_2 = 1\} \cap \{t = 0\} = \overline{\Omega_0} \times \{t = 0\}. \tag{4.10}$$

To this end we consider the domain

$$\Omega_\varepsilon := \{x : d(x, \Omega_0) \leq 3\varepsilon\}$$

for a given $\varepsilon > 0$, and choose a point $x_0 \in \partial\Omega_\varepsilon$. By our assumption there exists $\delta > 0$ depending on ε such that $\rho_0 \leq 1 - 2\delta$ in $B_{2\varepsilon}(x_0)$, and thus

$$\rho^E \leq 1 - \delta \text{ in } B_{2\varepsilon}(x_0) \times [0, t_1] \text{ for some } t_1 > 0. \tag{4.11}$$

Let us now consider the radial function $\phi(x, t)$ in $B_{2\varepsilon}(x_0) \setminus B_{\varepsilon(t)}(x_0)$ such that $\phi = 0$ on $\partial B_{\varepsilon(t)}(x_0)$, $\phi = 1$ on $\partial B_{2\varepsilon}(x_0)$ and

$$-\Delta\phi(x) = G(0) \text{ in } B_{2\varepsilon}(x_0) \setminus \overline{B}_{\varepsilon(t)}(x_0).$$

Note that we have $|D\phi| \leq M/\varepsilon$ on $\partial B_{\varepsilon(t)}(x_0)$ where M is independent of ε as long as $\varepsilon(t) \geq \varepsilon/2$. Combining this fact and (4.11), it follows that if we choose $\varepsilon(t) = (\varepsilon - \frac{M}{\varepsilon\delta} t)$ and $\rho_\phi^E(0) = 1 - 2\delta$, then (ϕ, ρ_ϕ^E) is a supersolution of (FB) in $B_{2\varepsilon}(x_0) \times [0, t_\varepsilon]$, where $t_\varepsilon = \min[\frac{\varepsilon^2\delta}{M}, t_1]$. This and Theorem 3.4 yield that

$$\rho_2 \leq \rho_\phi^E < 1 \text{ in } B_{\varepsilon/2}(x_0) \times [0, t_\varepsilon].$$

This concludes (4.10) and therefore (a).

2. Next we prove (b). To this end we need to ensure that p_m does not vanish inside Ω_0. Again this follows from Theorem 3.4, since at each interior point $x_0 \in \Omega_0$ with $B_r(x_0) \subset \Omega_0$ for some $r > 0$ we can consider a radial solution of (FB) with $\rho_\phi^E = 0$ and apply Theorem 3.4 to show that the corresponding solutions $\tilde{p}_m$ of (1.6) uniformly converge to ϕ. Now we can conclude since $p_m \geq \tilde{p}_m$ by the comparison principle of (1.6).

3. Now we are ready to prove (b). Fix $\varepsilon > 0$ and define

$$\Omega_f := \{x : \mathrm{dist}(x, \mathbb{R}^n \setminus \Omega_0) > \varepsilon\} \text{ and } \Omega_g := \Omega_\varepsilon = \{x : \mathrm{dist}(x, \Omega_0) \le \varepsilon\}.$$

In view of (a) and step 2, there exist $\delta = \delta(\varepsilon) > 0, t_0 = t_0(\varepsilon) > 0$ and M such that for $m > M$ and $0 \le t \le t_0$ the following holds: $p_m \le \delta$ on $\partial\Omega_g$, $p_m \ge \delta$ in Ω_f. Let us consider f and g defined by

$$-\Delta f = G(f) - \varepsilon \text{ in } \Omega_f \quad \text{and } f = \delta \text{ on } \partial\Omega_f$$

and

$$-\Delta g = G(g) + \varepsilon \text{ in } \Omega_g \quad \text{and } g = \delta \text{ on } \partial\Omega_g.$$

Let

$$\phi(x,t) := a(t)f(x) \text{ and } \psi(x,t) := b(t)g(x),$$

where

$$a(t) := \min[\delta e^{\frac{m}{2}\varepsilon t}, 1] \text{ and } b(t) := \max[\delta^{-1}e^{-\frac{m}{2}\varepsilon\delta t}, 1].$$

Note that the gradient of f is bounded from above in Ω_f. Using this fact, direct calculations then yield that ϕ and ψ are for sufficiently large m respectively a subsolution and a supersolution of (1.6) in $\Omega_f \times (0, t_0]$ and $\Omega_g \times (0, t_0]$. Thus the comparison principle for (1.6) and the choice of δ and t_0 yield

$$\psi \le p_m \text{ in } \Omega_g \times [0, t_0] \text{ and } p_m \le \phi \text{ in } \Omega_f \times [0, t_0].$$

Letting $m \to \infty$ and using arbitrarily small $\varepsilon > 0$, we conclude that the p_m's converge uniformly to the solution of the elliptic equation Ω_0 with zero boundary data. □

Theorem 4.6 and Lemma 4.7 together yield our main result:

Corollary 4.8. *Let p be the unique lower-semi-continuous viscosity solution of* (FB) *given by Theorem* 2.17. *Then the following hold as $m \to \infty$:*

(a) $\limsup^* p_m = p^*$ *and* $\liminf_* p_m = p_*$.

(b) ρ_m *locally uniformly converges to* $\rho := \chi_{\{p>0\}} + \rho^E \chi_{\{p=0\}}$ *away from* $\partial\{p > 0\}$.

Proof. From Theorem 4.6 and the stability property of viscosity solutions of (FB), it follows that $\bar{p} := (\liminf_{\sigma\to 0} p_2^\sigma)_*$ is a supersolution of (FB) with $g = \frac{1}{1-\rho^E}$. Due to Lemma 4.7(a) and the convergence of $\Omega_2^\sigma(0)$ to Ω_0 in Hausdorff distance, we conclude that $\bar{p}(\cdot, t)$ uniformly converges to $p_0(\cdot, 0)$ as $t \to 0$.

From the comparison principle Theorem 2.18 it follows that $\bar{p} \le p_1$. Since $p_2 \le p_2^\sigma$ for any $\sigma > 0$, it follows that $(p_2)_* \le \bar{p} \le p_1$. Since $p_1 \le p_2$ by definition, this means $(p_1)^* = (p_2)^*$ and $(p_1)_* = (p_2)_*$. This yields that $p = p_1 = (p_2)_*$ is a viscosity solution of (FB) with surrounding density ρ^E, and this yields (b). The convergence of ρ_m in the interior of $\{p > 0\}$ then follows from (b).

It remains to show that ρ_m converges to ρ^E away from $\{p > 0\}$. Note that due to Lemma 4.2

$$\overline{\{p > 0\}} = \overline{\{p_2 > 0\}} = \{\rho_2 = 1\}.$$

This and Lemma 4.4(c) yield that ${\limsup^*}_{m\to\infty} \rho_m = \rho_2 \le \rho^E$ away from $\overline{\{p > 0\}}$. Now we conclude by Lemma 4.4(b), which says ${\liminf_*}_{m\to\infty} \rho_m = \rho_1 \ge \min[1, \rho^E]$. □

Recall that an "almost" contraction property is available for any two solutions ρ_m, $\hat\rho_m$ of (1.1) from [PQV, (2.12)] in the form

$$\|\rho_m(t) - \hat\rho_m(t)\|_1 \leq e^{G(0)t}\|\rho_m(0) - \hat\rho_m(0)\|_1 \quad \text{for any } t > 0. \tag{4.12}$$

Using the above formula as well as the uniform convergence result obtained in Corollary 4.8 and Corollary 2.20, we have the following convergence result for general approximating initial data $\rho_{0,m}$:

Corollary 4.9. *Let $\rho_0 := \chi_{\Omega_0} + \rho_0^E\chi_{\Omega_0^c}$ with Ω_0, ρ_0^E as given in* (1.7)*, with Lipschitz continuous ρ_0^E. Suppose that $\rho_{0,m}$ converge to ρ_0 in $L^1(\mathbb{R}^n)$. Then the corresponding solution ρ_m of* (1.1) *with the initial data $\rho_{0,m}$ converges to ρ as given in Corollary* 4.8 *in the following sense:*

$$\|\rho_m(t) - \rho(t)\|_1 \to 0 \textit{ as } m \to \infty \textit{ for a.e. } t > 0.$$

Proof. Let p be the viscosity solution of (FB) with initial data Ω_0 and ρ_0^E, and define $\rho := \chi_{\{p>0\}} + \chi_{\{p=0\}}\rho^E$ as in Corollary 4.8. By Corollary 2.20 the set $\partial\{p > 0\}$ has measure zero and so $K_t := \{x : (x,t) \in \partial\{p > 0\}\}$ is of measure zero for a.e. $t > 0$.

Let us fix one such $t_0 > 0$ and $\varepsilon > 0$. We can find an open set $U \subset \mathbb{R}^n$, $K_{t_0} \subset U$, with measure $|U| < \varepsilon$.

For $R > 0$ let $\rho_{0,R}^E$ be a Lipschitz function that is a cutoff of ρ_0^E in the sense that $\rho_{0,R}^E = \rho_0^E$ for $|x| \leq R$, $\rho_{0,R}^E \leq \rho^E$ and $\rho_{0,R}^E = 0$ for $|x| \geq 2R$. By step 1 of the uniqueness part of the proof of Theorem 2.17, we have that $\{p > 0\} \cap [0, 2t_0] \subset B_R \times [0, 2t_0]$ for sufficiently large R.

Let us fix one such $R > 0$. It can be easily checked that p is consequently a viscosity solution of (FB) with initial data Ω_0, $\rho_{0,R}^E$ for $t \in [0, 2t_0]$. By making R larger if necessary, we can also require that

$$\|\rho_{0,R}^E - \rho_0^E\|_{L^1} < \varepsilon. \tag{4.13}$$

We now consider $\tilde\rho_{0,m}$ to be the initial data from (1.9), where $\rho_{0,m}^E$ in that formula is generated from $\rho_{0,R}^E$ with the use of Remark 1.1. Note that $\tilde\rho_{0,m} \to \tilde\rho_0 := \chi_{\Omega_0} + \chi_{\Omega_0^c}\rho_{0,R}^E$ in L^1 as $m \to \infty$. Let $\tilde\rho_m$ be the solution of (1.1) with initial data $\tilde\rho_{0,m}$.

Since the initial data for $\tilde\rho_m$ now has compact support, by a comparison with a Barenblatt type solution of the porous medium equation, we can find $R_1 > R$, independent of m, such that $\tilde\rho_m(x,t) = 0$ for $|x| \geq R_1$, $t \in [0, 2t_0]$ and all m.

Corollary 4.8(b) applies to $\tilde\rho_m \to \tilde\rho := \chi_{\{p>0\}} + \chi_{\{p=0\}}\rho_R^E$ locally uniformly away from $\partial\{p > 0\}$ for $t \in [0, 2t_0]$, with $\rho_R^E(x) = \rho_{0,R}^E(x)e^{G(0)t}$.

Now for sufficiently large m we have

$$\|\rho_{0,m} - \tilde\rho_{0,m}\|_{L^1} \leq \|\rho_{0,m} - \rho_0\|_{L^1} + \|\rho_0 - \tilde\rho_0\|_{L^1} + \|\tilde\rho_0 - \tilde\rho_{0,m}\|_{L^1} < 3\varepsilon,$$

where the second term can be bounded from (4.13) and the other two by the choice of the sequences $\rho_{0,m}$, $\tilde\rho_{0,m}$. The "almost" contraction (4.12) therefore yields

$$\|\rho_m(t_0) - \tilde\rho_m(t_0)\|_{L^1} < 3\varepsilon e^{G(0)t_0}. \tag{4.14}$$

Now we finally estimate

$$
\begin{aligned}
\|\tilde{\rho}_m(t_0) - \rho(t_0)\|_{L^1} &\le \|\tilde{\rho}_m(t_0) - \tilde{\rho}(t_0)\|_{L^1} + \|\tilde{\rho}(t_0) - \rho(t_0)\|_{L^1} \\
&\le \|\tilde{\rho}_m(t_0) - \tilde{\rho}(t_0)\|_{L^\infty(B_{R_1}\setminus U)}|B_{R_1}| + \|\tilde{\rho}_m(t_0) - \tilde{\rho}(t_0)\|_{L^1(U)} \\
&\quad + e^{G(0)t_0}\|\tilde{\rho}_0 - \rho_0\|_{L^1} \\
&\le \varepsilon + 2\varepsilon + \varepsilon e^{G(0)t_0},
\end{aligned} \tag{4.15}
$$

where the first term on the right-hand side can be estimated by ε for m sufficiently large by the uniform convergence in Corollary 4.8(b), and the second term is estimated by $|U| < \varepsilon$. With (4.14) and (4.15) we conclude since $\varepsilon > 0$ is arbitrary. $\square$

5. A BV estimate on the positivity set of the pressure

Here we show that $\partial\{p(\cdot,t) > 0\}$ has finite perimeter as long as ρ^E stays strictly less than 1 near $\partial\{p(\cdot,t) > 0\}$. The result already follows from the BV estimates in [PQV]; however our proof is based on geometric arguments and thus is of independent interest.

Lemma 5.1. *Let $\Omega_t(p) := \{p(\cdot,t) > 0\}$, where p is as given in Corollary 4.8, and assume that $\rho^E < 1$ on $\partial\Omega(t)$. Then for given $r > 0$, there exist sets $\Omega_{r,t}$ such that*

$$\Omega_{r,t} \subset \Omega_t(p) \text{ for each } t > 0$$

such that

(a) *$\Omega_{r,t}$ increases with respect to r;*
(b) *$\Omega_{r,t}$ has interior ball properties with radius r;*
(c) *$|\Omega_{r,t} - \Omega_t(p)| \le Ce^t r$.*

Proof. To prove this, take the initial positive set

$$\Omega_0^r := \{x : d(x, \Omega_0^c) \ge 2r\}$$

and consider the corresponding approximating solution $\rho_{m,r}$ of (1.1) with its limiting initial density

$$\rho_{0,r} := \chi_{\Omega_0^r} + \rho^E \chi_{(\Omega_0^r)^c}.$$

Let us now take $\Omega_2^\sigma(t)$ and p_2^σ as defined in (4.2) with $\rho_{m,r}$ instead of ρ_m. Let us choose now $\sigma = r$. Then due to Theorem 4.6, p_2^r is a subsolution of (FB) with g and $\Omega_2^r(0) = \{x : d(x, \Omega_0^r) \le r\} \subset \Omega_0$. Hence by the comparison principle of (FB) we have $p_2^r \le p$, and thus

$$\Omega_2^r(t) = \{p_2^r(\cdot,t) > 0\} \subset \Omega_t(p)$$

for all $t > 0$. (c) follows from the contraction inequality (4.12) applied to ρ_m and $\rho_{m,r}$ given in Corollary 4.8 in the limit $m \to \infty$. $\square$

Proposition 5.2. *Under the same assumptions as in Lemma 5.1, for any $r > 0$, $\Omega_{r,t}$ has uniformly bounded perimeter. As a consequence $\{p(\cdot,t) > 0\}$ is a set of finite perimeter.*

Proof. We consider $\Omega_t^n := \Omega_{r_n,t}$ with $r_n = 2^{-n}$. We claim that for $r \le r_n$ there is at most $C_d r^{1-d}$ balls of radius r covering the boundary of $\Omega_{r_n,t}$.

We will only show the claim for $r = r_n$. For smaller radius $r < r_n$, the claim holds due to Lemma 2.5 of [ACM]. We know that Ω_t^n increases with respect to n with

$$|\Omega_t^n - \Omega_t^{n+1}| \leq C r_n, \tag{5.1}$$

where C is independent of n. Moreover, from the construction above, in fact we have the following relation between Ω_t^n and Ω_t^{n+1}:

$$\{x : d(x, \Omega_t^n) \leq c r_{n+1}\} \subset \Omega_t^{n+1}, \tag{5.2}$$

where c is independent of the choice of n.

Now let us take an open covering $\mathcal{O}$ of the boundary of Ω_t^{n+1} consisting of balls of radius r_{n+1} with their centers on the boundary. Let us take out a family of disjoint balls in $\mathcal{O}$ obtained by Vitali's covering lemma. In each of these disjoint balls, at least one third of the ball is taken by the interior of Ω_t^{n+1} due to the interior ball property satisfied at the center of each ball. Also due to (5.2) at least a fixed portion of this interior is away from Ω_t^n. Now we conclude that if the number of the disjoint balls is N, then (5.1) yields that $N(r_{n+1})^d \leq C r_{n+1}$ or

$$N \leq C(r_{n+1})^{1-d}.$$

Hence we conclude. □

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