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CANONICAL BIMODULES AND DOMINANT DIMENSION

MING FANG, OTTO KERNER, AND KUNIO YAMAGATA

Dedicated to C. M. Ringel on the occasion of his 70th birthday

ABSTRACT. For a finite dimensional algebra A over a field k, the inherent A-bimodules which include A and its k-dual $\mathrm{D}(A)$, as well as those derived from them by iteratively taking their left or right A-duals or higher extensions, are crucial in many considerations. We study the properties of these bimodules, mainly of $\mathrm{Hom}_A(\mathrm{D}(A), A)$ (called the canonical A-bimodule), and utilize them to provide new characterizations of Morita algebras and the dominant dimension of A.

1. INTRODUCTION

Given a finite dimensional algebra A over a field k, there are two natural A-bimodules, namely the algebra A itself and its k-dual $\mathrm{D}(A)$. Without any doubt, they are of central importance in all aspects of the study of A. Derived from these two bimodules, there is the *canonical* A-bimodule $V := \mathrm{Hom}_A(\mathrm{D}(A), A)$, as well as many other bimodules, by taking extensions of $\mathrm{D}(A)$ and A as one-sided modules or iteratively taking the left or right A-duals of V, for example,

$$(\dagger) \qquad \mathrm{Ext}^i_A(\mathrm{D}(A), A), \quad \mathrm{Hom}_{A^{op}}(\mathrm{Hom}_A(\mathrm{D}(A), A), A), \quad \cdots$$

where all morphisms and extensions are taken in the category of left modules by identifying right A-modules with left A^{op}-modules. Some of these derived A-bimodules are of significant importance. The A-bimodule $\mathrm{Ext}^1_A(\mathrm{D}(A), A)$ is the key ingredient in Ringel's construction of preprojective algebras when A is hereditary ([18], see also [5, Proposition 3.1]), and recently $\mathrm{Ext}^i_A(\mathrm{D}(A), A)$ is used by Keller and Iyama to define higher preprojective algebras; see for example [1]. Our interest in the canonical A-bimodule V and the other derived A-bimodules is mainly motivated by [11, 12, 14]. In [11], the property $V \cong A$ as A-bimodules is proved to characterize *gendo-symmetric algebras*, a class of algebras that are endomorphism algebras of generators over symmetric algebras, and with this property the vanishing of $\mathrm{Ext}^i_A(\mathrm{D}(A), A)$ gives a cohomological characterization of the dominant dimension of A. In [14], more properties of V like the faithfulness, the double centralizer property and the isomorphism $V \cong A$ as one-sided A-modules are studied. These properties characterize a larger class of algebras called *Morita algebras* in [14, 27].

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In this paper we make an extensive study of the properties of the canonical bimodule V as well as the other bimodules from the series (†), and by relating these bimodules to the dominant dimension of the algebra A we obtain new characterizations of Morita algebras and dominant dimension. More precisely, we compare the left and right module structures on V (known as the *left right symmetry problem for a bimodule*) and obtain as our first main results several characterizations of the algebras whose canonical bimodules are projective or injective as one-sided modules. We compare the left and right A-duals of V and prove in section 4 that for an algebra A of dominant dimension at least two, the two A-duals are isomorphic as A-bimodules exactly when A is a Morita algebra, or equivalently

$$V \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A) \otimes_A V$$

as A-bimodules (Theorem 4.8). The double dual functors are left exact for algebras of dominant dimension at least two by [7, 8]. Studying their right derived functors, we deduce a new characterization of dominant dimension in terms of vanishing of these right derived functors (Theorem 4.1). Moreover, we obtain a restricted Grothendick spectral sequence whose E_2-page consists of the bimodules from the series (†) (Theorem 4.5). Applying this spectral sequence, we prove that for an algebra A (Theorem 4.7)

$$\operatorname{domdim} A \geq 2 \iff \mathrm{D}(A) \otimes_A V \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A) \text{ as } A\text{-bimodules.}$$

This result generalizes the main results of [11, 14] in full generality and exhibits the crucial role of the canonical bimodules in the theory of dominant dimension for the first time. As another application of the spectral sequence, we reprove the characterization of dominant dimension for gendo-symmetric algebras ([11, Proposition 3.3]) and generalize it to a characterization of dominant dimension for Morita algebras.

2. Preliminaries

Throughout, algebras are finite dimensional associative k-algebras, where k is an arbitrary field. All modules are finite dimensional left modules, and all morphisms operate on the left and are left module morphisms unless stated otherwise. Let A be an algebra. We denote by A^{op} the opposite algebra of A, and by A-mod the category of left A-modules. Thus A^{op}-mod is the category of right A-modules. Let $\mathrm{D} = \mathrm{Hom}_k(-, k)$ be the duality between A-mod and A^{op}-mod. For an A-module M, we denote by $\mathrm{add}(M)$ the full subcategory of A-mod consisting of direct summands of finite direct sums of M. We also denote by $\operatorname{proj.dim} M$ and $\operatorname{inj.dim} M$ the projective and injective dimensions of M respectively.

An algebra A is called basic if every indecomposable direct summand of the (left) regular module A is multiplicity-free. For $\theta \in \mathrm{Aut}_k(A)$ an automorphism of the algebra A and M an A-module, we denote by ${}_\theta M$ the A-module, which equals M as a k-vector space, and the A-module structure is defined by $a \cdot m = \theta(a)m$ for all $a \in A$ and $m \in {}_\theta M$. If M is a right A-module, M_θ is defined analogously. For an anti-automorphism τ of the algebra A and a right A-module N, we denote by ${}^\tau N$ the A-module, which equals N as a k-vector space, and the A-module structure is defined by $a \cdot n = n\tau(a)$ for $a \in A$ and $n \in {}^\tau N$. Similarly the right A-module N^τ is defined, provided N is a (left) A-module.

2.1. **Dominant dimension.** Dominant dimension was introduced by Nakayama in his study of complete homology theory and systematically studied later by Tachikawa, Morita, Müller and many others; see [7, 16, 17, 23–26] and also [9–12, 14, 27] for some recent developments.

Definition 2.1. Let A be an algebra. Let M be an A-module, and let $0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$ be a minimal injective resolution of M. The dominant dimension of M, denoted by $\operatorname{domdim} M$, is the largest number t or ∞ such that $I^0, \ldots, I^{t-1}$ are projective.

The dominant dimension of the (left) regular A-module is called the dominant dimension of the algebra A and is simply denoted by $\operatorname{domdim} A$, since $\operatorname{domdim} A = \operatorname{domdim} A^{op}$ [17, 26]. Note that if $\operatorname{domdim} A \geq 1$, then the injective hull of the regular module is faithful and projective. If $\operatorname{domdim} A \geq 2$, then any faithful projective injective A-module P has the *double centralizer property*, that is, $\operatorname{End}_R(P) \cong A$, where $R = \operatorname{End}_A(P)$. We remark that dominant dimension at least 2 has been used to characterize algebras of finite representation type [2] and to give computation-free proofs of Schur-Weyl type dualities in algebraic Lie theory [13]. Large dominant dimension is naturally related to self-orthogonality, which is crucial in higher Auslander theory developed by Iyama and the cover theory developed by Rouquier [10, 20]. In the following, we recall some known characterizations of dominant dimension; see also [4, 7, 10–12, 17, 26] and the references therein.

Let A be an algebra. The double dual functor is defined by

$$\Gamma = (\ )^{**} : \quad A\text{-mod} \longrightarrow A\text{-mod} \qquad M \mapsto \operatorname{Hom}_{A^{op}}(\operatorname{Hom}_A(M, A), A).$$

Here the right A-modules $\operatorname{Hom}_A(M, A)$ and A are regarded as A^{op}-modules naturally. Let $\xi : \operatorname{Id} \to \Gamma$ be the natural transformation such that $\xi_M(m)(f) = f(m)$ for any $m \in M$ and $f \in \operatorname{Hom}_A(M, A)$. M is called *torsionless* (respectively *reflexive*) if ξ_M is a monomorphism (respectively an isomorphism).

Theorem 2.2 (Colby-Fuller [7])**.** *Let A be an algebra. Then* $\operatorname{domdim} A \geq 1$ *if and only if the double dual functor* Γ *preserves monomorphisms;* $\operatorname{domdim} A \geq 2$ *if and only if the double dual functor* Γ *is left exact, and A is self-injective if and only if* Γ *is exact.*

Theorem 2.3 (Morita [16])**.** *Let A be an algebra with* $\operatorname{domdim} A \geq 2$*. An A-module M is torsionless if and only if* $\operatorname{domdim} M \geq 1$*, and M is reflexive if and only if* $\operatorname{domdim} M \geq 2$*.*

Theorem 2.4 (Müller [17])**.** *Let A be an algebra with* $\operatorname{domdim} A \geq 2$*, and let fA be a faithful projective injective module for some idempotent $f \in A$. Let $n \geq 2$ be a natural number. Then for any A-module M,* $\operatorname{domdim} M \geq n$ *if and only if* $M \cong \operatorname{Hom}_{fAf}(fA, fM)$ *canonically, and* $\operatorname{Ext}^i_{fAf}(fA, fM) = 0$ *for $1 \leq i \leq n-2$.*

With this homological characterization of dominant dimension, we deduce the following equivalent form of the double dual functor under the condition $\operatorname{domdim} A \geq 2$.

Proposition 2.5. *Let A be an algebra with* $\operatorname{domdim} A \geq 2$*. Let fA be a faithful projective injective right A-module for some idempotent $f \in A$. Let $\mathcal{G}$ be the endofunctor on* A-mod *such that* $\mathcal{G}(M) = \operatorname{Hom}_{fAf}(fA, fM)$ *for* $M \in A$-mod*, and let* $\eta : \operatorname{Id} \to \mathcal{G}$ *be the natural transformation such that* $\eta_M(m)(fa) = fam$ *for*

$a \in A, m \in M$. *Then there is a natural equivalence* $\theta : \Gamma \xrightarrow{\sim} \mathcal{G}$ *such that the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Id} \arrow[r, equal] \arrow[d, "\xi"'] & \mathrm{Id} \arrow[d, "\eta"] \\ \Gamma \arrow[r, "\theta", "\sim"'] & \mathcal{G}. \end{tikzcd}$$

Proof. We assume first that fAf is a basic algebra. Since fA is a projective injective right A-module, $\mathrm{D}(fA) \cong Ae$ as A-modules for some idempotent $e \in A$. Thus

$$fAf \cong \mathrm{End}_{A^{op}}(fA) \cong \mathrm{End}_A(\mathrm{D}(fA))^{op} \cong \mathrm{End}_A(Ae)^{op} \cong eAe$$

as algebras. Via these isomorphisms, the right eAe-module Ae becomes a right fAf-module, and $\mathrm{D}(fA) \cong Ae$ as (A, fAf)-bimodules. Since $\operatorname{domdim} A^{op} = \operatorname{domdim} A \geq 2$, Ae has the double centralizer property, that is, $A \cong \mathrm{End}_{(eAe)^{op}}(Ae)$ canonically. For any A-module M, let θ_M be the composite of the following isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{A^{op}}(\mathrm{Hom}_A(M,A),A) &\xrightarrow{\sim} \mathrm{Hom}_{A^{op}}(\mathrm{Hom}_A(M,A), \mathrm{Hom}_{(eAe)^{op}}(Ae,Ae)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{(eAe)^{op}}(\mathrm{Hom}_A(M,A) \otimes_A Ae, Ae) \\ &\xrightarrow{\sim} \mathrm{Hom}_{(eAe)^{op}}(\mathrm{Hom}_A(M,Ae), Ae) \\ &\xrightarrow{\sim} \mathrm{Hom}_{eAe}(\mathrm{D}(Ae), \mathrm{D}\,\mathrm{Hom}_A(M,Ae)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{fAf}(fA, fM), \end{aligned}$$

where the last isomorphism follows, since

$$\mathrm{D}\,\mathrm{Hom}_A(M,Ae) \cong \mathrm{D}\,\mathrm{Hom}_A(M,\mathrm{D}(fA)) \cong \mathrm{D}\,\mathrm{D}(fA \otimes_A M) \cong fM.$$

Since all isomorphisms in the construction of θ_M are canonical, thus independent of M, it follows that $\{\theta_M\}$ defines a natural equivalence θ from Γ to $\mathcal{G}$.

In order to show that $\eta_M = \theta_M \circ \xi_M$, we shall make these isomorphisms explicit, and for this, we fix an isomorphism $\tau : Ae \xrightarrow{\sim} \mathrm{D}(fA)$ of (A, fAf)-bimodules. Then for $\varphi \in \Gamma(M)$, $\alpha \in \mathrm{Hom}_A(M, Ae)$ and $a \in A$,

$$\tau(\alpha(\theta_M(\varphi)(fa)))(f) = \tau(fa\varphi(\alpha))(f).$$

Since $\theta_M(\varphi)$ is an fAf-module morphism and α is an A-module morphism, we have

$$\begin{aligned} \tau(fa\varphi(\alpha))(fxf) &= \tau(fxfa\varphi(\alpha))(f) \\ &= \tau(\alpha(fxf\theta_M(\varphi)(fa)))(f) \\ &= \tau(fxf\alpha(\theta_M(\varphi)(fa)))(f) = \tau(\alpha(\theta_M(\varphi)(fa)))(fxf) \end{aligned}$$

for any $x \in A$. Observe that $fa\varphi(\alpha), \alpha(\theta_M(\varphi)(fa)) \in fAe$, and τ induces the isomorphism $fAe \cong \mathrm{D}(fAf)$ as k-vector spaces. We obtain

$$\alpha(\theta_M(\varphi)(fa)) = fa\varphi(\alpha).$$

Hence, for $M \in A$-mod and $m \in M$, we have

$$\alpha(\theta_M(\xi_M(m))(fa)) = fa(\xi_M(m))(\alpha) = fa\alpha(m) = \alpha(fam) = \alpha(\eta_M(m)(fa))$$

for any $a \in A$ and $\alpha \in \mathrm{Hom}_A(M, Ae) \cong \mathrm{D}(fM)$ (see above). As a result, we have

$$\theta_M(\xi_M(m)) = \eta_M(m) \in \mathcal{G}(M).$$

That is, $\eta_M = \theta_M \circ \xi_M$, as desired.

In general, if fAf is not basic, we may choose an idempotent f_0 of A such that $f_0f = f_0 = ff_0$, f_0Af_0 is basic and fAf is Morita equivalent to f_0Af_0. Consequently, f_0A is a faithful projective and injective right A-module, and there is the canonical isomorphism of A-modules

$$\gamma_M : \mathrm{Hom}_{f_0Af_0}(f_0A, f_0M) \xrightarrow{\sim} \mathrm{Hom}_{fAf}(fA, fM),$$

for any $M \in A$-mod. Let $\mathcal{G}^0$ be the analogous endo-functor of A-mod and η^0_M : $\mathrm{Id} \to \mathcal{G}^0$ be the corresponding natural transformation, associated with f_0. Then $\{\gamma_M\}$ is a natural transformation from $\mathcal{G}^0$ to $\mathcal{G}$ such that $\eta_M = \gamma_M \circ \eta^0_M$. Let $\theta_M = \gamma_M \circ \theta^0_M$. By what we have proved above, $\eta^0_M = \theta^0_M \circ \xi_M$, it then follows that

$$\eta_M = \gamma_M \circ \eta^0_M = \gamma_M \circ \theta^0_M \circ \xi_M = \theta_M \circ \xi_M$$

as desired. □

2.2. **Morita algebras.** Morita algebras were first studied by Morita [15] as endomorphism rings of generators over self-injective algebras, though named and systematically studied later by Kerner and Yamagata in [14, 27]. The subclass of Morita algebras consisting of endomorphism rings of generators over symmetric algebras, called *gendo-symmetric* algebras, was introduced and studied independently by Fang and Koenig in [11, 12]; see also [9] for an application in algebraic Lie theory.

Definition 2.6. Let A be an algebra. An idempotent $e \in A$ is called *basic* if eAe is a basic algebra; e is called *self-dual* if $\mathrm{D}(Ae) \cong eA$ as right A-modules.

The following definition is based on [15, section 16] and [14, Theorem 2].

Definition 2.7. An algebra A is called a *Morita* algebra if one of the following equivalent conditions is satisfied:

(1) A is the endomorphism ring of a generator over a self-injective algebra.
(2) There is a self-dual idempotent $e \in A$ such that Ae is a faithful A-module, and $A \cong \mathrm{End}_{(eAe)^{op}}(Ae)$ canonically.
(3) $\mathrm{domdim}\, A \geq 2$ and $\mathrm{Hom}_A(\mathrm{D}(A), A)$ is a faithful (left) A-module.
(4) $A \cong \mathrm{End}_{A^{op}}(\mathrm{Hom}_A(\mathrm{D}(A), A))$ canonically.
(5) $A \cong \mathrm{End}_A(\mathrm{Hom}_A(\mathrm{D}(A), A))^{op}$ canonically.

In (2), the idempotent e being self-dual implies that $\mathrm{D}(eA) \cong Ae$ as A-modules. Hence Ae is a projective injective A-module, and $\mathrm{D}(eA)_\theta \cong Ae$ as (A, eAe)-bimodules for some automorphism θ of eAe which induces an eAe-bimodule isomorphism $\mathrm{D}(eAe)_\theta \cong eAe$. The self-injective algebra in (1) is Morita equivalent to eAe for some basic idempotent e described in (2).

Recall that an algebra A is said to be *Frobenius* if $\mathrm{D}(A) \cong A$ as A-modules or, equivalently, as right A-modules. An automorphism ν of A is called a *Nakayama automorphism* of A if $\mathrm{D}(A)_\nu \cong A$ as A-bimodules. Note that every Frobenius algebra A has a Nakayama automorphism (unique up to inner automorphisms [22, Corollary IV.3.5]) denoted by $\nu_{_A}$. An algebra A is said to be *symmetric* if $\mathrm{D}(A) \cong A$ as A-bimodules. Hence, a Frobenius algebra A is symmetric if and only if $\nu_{_A}$ is inner, and in this case, we may take the identity automorphism as a Nakayama automorphism. The following results seem to be well-known (cf. [21]).

Lemma 2.8. *Let e_0 be a basic idempotent in an algebra A such that A is Morita equivalent to e_0Ae_0. Let $e_0 = e_1 + \cdots + e_m$ be a decomposition of e_0 into pairwise* *orthogonal primitive idempotents. Then for any algebra automorphism ν of A, there exist an invertible element u in A and a permutation π on $\{1,\dots,m\}$, such that $u\nu(e_i)u^{-1}=e_{\pi(i)}$ for $i=1,\dots,m$. In particular, $u\nu(e_0)u^{-1}=e_0$.*

As an immediate consequence, we get

Corollary 2.9 ([21]). *Let A be a Frobenius algebra, and let e_0 be a basic idempotent of A such that A is Morita equivalent to e_0Ae_0. Then there is a Nakayama automorphism ν of A with $\nu(e_0)=e_0$.*

Proof (See also [21], p. 717). Let ν_0 be an arbitrary Nakayama automorphism of A. By Lemma 2.8 there is an invertible element u of A such that $u\nu_0(e_0)u^{-1}=e_0$. Let θ_u be the inner automorphism of A such that $\theta_u(x)=uxu^{-1}$ for $x\in A$. Let $\nu=\theta_u\circ\nu_0$. Then ν is a Nakayama automorphism of A and $\nu(e_0)=e_0$. $\square$

Theorem 2.10 (Morita [26]). *Let A be an algebra, let M be an A-module, and let $B=\mathrm{End}_A(M)$. Then M is a generator in A-*mod *if and only if M is projective in B-*mod *and $A\cong\mathrm{End}_B(M)$ canonically as algebras.*

Lemma 2.11 ([2,22,26]). *Let A be an algebra, and let M be an A-module. Let $M\cong M_1^{\oplus r_1}\oplus\cdots\oplus M_n^{\oplus r_n}$ be a direct sum decomposition of M into indecomposable direct summands, where r_i denotes the multiplicity of M_i in the decomposition. Let $B=\mathrm{End}_A(M)^{op}$ and $P_B(i)=\mathrm{Hom}_A(M,M_i)$. Then $\{P_B(i)\}_{i=1}^n$ forms a complete set of pairwise non-isomorphic indecomposable projective B-modules. Furthermore, let $L_B(i)$ be the simple head of $P_B(i)$ and $D_i=\mathrm{End}_B(L_B(i))$. Then $r_i=\dim_{D_i}L_B(i)$ for $1\le i\le n$.*

Corollary 2.12. *Let A be an algebra, and let $e\in A$ be a basic idempotent such that eAe is Morita equivalent to A. Let P be a projective A-module. If $\mathrm{End}_A(P)\cong A^{op}$ as algebras, then there exists an automorphism θ of eAe such that $\mathrm{Hom}_A(P,A)e\cong(Ae)_\theta$ as (A,eAe)-bimodules.*

Proof. Let $P=P_1^{\oplus r_1}\oplus\cdots\oplus P_n^{\oplus r_n}$ be a decomposition of P into indecomposable direct summands in A-mod. Let e_i be the composition of the projection $P\twoheadrightarrow P_i$ and the embedding $P_i\hookrightarrow P$, for each $1\le i\le n$. Then $\{e_1,\dots,e_n\}$ forms a set of primitive orthogonal idempotents in $\mathrm{End}_A(P)^{op}\cong A$. By Lemma 2.11, it follows that $e':=e_1+\cdots+e_n$ is a basic idempotent of A such that $e'Ae'$ is Morita equivalent to A. In particular, $Ae\cong Ae'$ as A-modules, and hence $eAe\cong e'Ae'$. Identifying these two algebras, we may assume without loss of generality that $e=e'=e_1+\cdots+e_n$. Since P_i is indecomposable projective in A-mod, it follows that $P_i\cong Ae_{\sigma(i)}$ for some permutation σ of $\{1,\dots,n\}$, and

$$Ae_i\cong\mathrm{Hom}_A(P,P)e_i\cong\mathrm{Hom}_A(P,P_i)\cong\mathrm{Hom}_A(P,Ae_{\sigma(i)})\cong\mathrm{Hom}_A(P,A)e_{\sigma(i)}$$

in A-mod. As a result, $Ae\cong\mathrm{Hom}_A(P,A)e$ as A-modules, and there exists an automorphism θ of eAe such that $(Ae)_\theta\cong\mathrm{Hom}_A(P,A)e$ as (A,eAe)-bimodules. $\square$

3. The canonical bimodule

As we have seen, given an algebra A, the A-bimodule $\mathrm{Hom}_A(\mathrm{D}(A),A)$ is not only natural in itself but also crucial in constructions of (higher) preprojective algebras ([1,18]) and characterizations of Morita algebras ([11,12,14,27]). In this section, we first, for simplicity, make a definition of the bimodule, then we study its behavior under Morita equivalences, the left right symmetry on projectivity and injectivity, and its left and right A-duals.

Definition 3.1. Let A be an algebra. The canonical bimodule associated to A is defined to be the A-bimodule

$$V := \mathrm{Hom}_A(\mathrm{D}(A), A).$$

As the first observation, the canonical A-bimodule and A^{op}-bimodule coincide, as the following lemma says.

Lemma 3.2. *Let A be an algebra. Then $V \cong \mathrm{Hom}_{A^{op}}(\mathrm{D}(A), A)$ as A-bimodules. Therefore, the canonical bimodules associated to A and A^{op} are isomorphic as A-bimodules.*

Proof. See [14, Lemma 1.7]. For convenience, here we write an explicit isomorphism $\iota : V \to \mathrm{Hom}_{A^{op}}(\mathrm{D}(A), A)$ of A-bimodules. By identifying A with $\mathrm{Hom}_k(\mathrm{D}(A), k)$ as A-bimodules, we define for $f \in V$ and $\delta_1, \delta_2 \in \mathrm{D}(A)$,

$$(\iota(f)(\delta_1))(\delta_2) := \delta_1(f(\delta_2)).$$

Since $\iota(f)(\delta_1 \cdot a)(\delta_2) = (\delta_1 \cdot a)(f(\delta_2)) = \delta_1(af(\delta_2)) = \delta_1(f(a \cdot \delta_2)) = (\iota(f)(\delta_1) \cdot a)(\delta_2)$ for any $a \in A$, it follows that $\iota(f)(\delta_1 \cdot a) = \iota(f)(\delta_1) \cdot a$; i.e., ι is well-defined. To see that ι is an A-bimodule morphism, we have

$$\begin{aligned}
\iota(a \cdot f)(\delta_1)(\delta_2) &= \delta_1((a \cdot f)(\delta_2)) = \delta_1(f(\delta_2 \cdot a)),\\
(a \cdot \iota(f))(\delta_1)(\delta_2) &= (a \cdot \iota(f)(\delta_1))(\delta_2) = \iota(f)(\delta_1)(\delta_2 \cdot a) = \delta_1(f(\delta_2 \cdot a)),\\
\iota(f \cdot a)(\delta_1)(\delta_2) &= \delta_1((f \cdot a)(\delta_2)) = \delta_1(f(\delta_2)a),\\
(\iota(f) \cdot a)(\delta_1)(\delta_2) &= \iota(f)(a\delta_1)(\delta_2) = (a\delta_1)(f(\delta_2)) = \delta_1(f(\delta_2)a).
\end{aligned}$$

That is $\iota(a \cdot f) = a \cdot \iota(f)$ and $\iota(f \cdot a) = \iota(f) \cdot a$. Therefore ι is an A-bimodule isomorphism since ι is trivially a k-vector space isomorphism. □

Proposition 3.3. *Let $V(A)$ and $V(B)$ be the canonical bimodules for the algebras A and B respectively. If $F : A\text{-mod} \xrightarrow{\sim} B\text{-mod}$ is a Morita equivalence, then the induced equivalence from $A \otimes_k A^{op}$-mod to $B \otimes_k B^{op}$-mod sends $V(A)$ to $V(B)$. In particular, $V(A)$ is projective (respectively, injective, a generator, a cogenerator) in* A-mod *if and only if so is $V(B)$ in* B-mod.

Proof. It suffices to show the case where $B = eAe$ for e an idempotent in A and the Morita equivalence F is given by $F(M) = eM$ for any $M \in A$-mod. The induced equivalence from the category of A-bimodules to the category of eAe-bimodules is then given by $W \mapsto eWe$ for any A-bimodule W. In particular, the image of $V(A)$ is

$$eV(A)e = e\,\mathrm{Hom}_A(\mathrm{D}(A), A)e \cong \mathrm{Hom}_A(\mathrm{D}(A)e, Ae) \cong \mathrm{Hom}_{eAe}(e\,\mathrm{D}(eA), eAe)$$

where the last isomorphism follows by the Morita equivalence F. Since $e\,\mathrm{D}(eA) \cong \mathrm{D}(eAe) \cong \mathrm{D}(B)$, we get $eV(A)e \cong V(B)$ as desired.

Note that the property of the idempotent e implies that $\mathrm{add}_A(W) = \mathrm{add}_A(We)$ for any A-bimodule W. Thus $V(A)$ is projective (respectively, injective, a generator, a cogenerator) in A-mod if and only if so is $V(A)e$ in A-mod, and via the equivalence F, if and only if so is $eV(A)e \cong V(B)$ in eAe-mod. □

3.1. **The left right symmetry on the bimodule structure.** Given a bimodule W over an algebra A, the left right symmetry problem means to compare the left and right A-module structures on W. As is well known, this is a hard problem in general, even for the A-bimodule A. The definition of the canonical A-bimodule V apparently depends on the one-sided module structure on A and $\mathrm{D}(A)$, but Lemma 3.2 tells us that V is independent of the left right balance of A. This seems to shed some light on the left right comparability of V itself. However, the following example indicates that one should not expect too much.

Example 1. Let k be any field, and let A be the k-algebra defined by the quiver

$$\begin{tikzcd} & & & \bullet\ 1 \\ \overset{5}{\bullet} \arrow[r, "\delta"] & \overset{4}{\bullet} \arrow[r, "\gamma"] & \overset{3}{\bullet} \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\ & & & \bullet\ 2 \end{tikzcd}$$

with relations $\alpha\gamma = 0, \beta\gamma = 0$ and $\gamma\delta = 0$. Then Ae_4 and Ae_5 are projective injective A-modules, and $Ae_4 = \mathrm{D}(e_3A), Ae_5 \cong \mathrm{D}(e_4A)$ in A-mod:

$$\begin{aligned} V = \mathrm{Hom}_A(\mathrm{D}(A), A) &= \mathrm{Hom}_A(\mathrm{D}(A), Ae_1 \oplus Ae_2 \oplus Ae_3 \oplus Ae_4 \oplus Ae_5) \\ &\cong \mathrm{Hom}_A(\mathrm{D}(A), Ae_4 \oplus Ae_5) \cong \mathrm{Hom}_{A^{op}}(\mathrm{D}(Ae_4) \oplus \mathrm{D}(Ae_5), A) \\ &\cong \mathrm{Hom}_{A^{op}}(e_3A \oplus e_4A, A) \cong Ae_3 \oplus Ae_4 \end{aligned}$$

as A-modules. On the other hand,

$$\begin{aligned} V &= \mathrm{Hom}_A(\mathrm{D}(A), A) \\ &= \mathrm{Hom}_A(\mathrm{D}(e_1A) \oplus \mathrm{D}(e_2A) \oplus \mathrm{D}(e_3A) \oplus \mathrm{D}(e_4A) \oplus \mathrm{D}(e_5A), A) \\ &\cong \mathrm{Hom}_A(Ae_4 \oplus Ae_5 \oplus \mathrm{D}(e_1A) \oplus \mathrm{D}(e_2A), A) \\ &\cong e_4A \oplus e_5A \oplus S_4' \oplus S_4' \end{aligned}$$

as right A-modules; here S_4' is the simple head of the projective right A-module e_4A. As a result, A is not a Morita algebra, and V is projective as an A-module, but not projective as a right A-module. We shall come back to this example in section 3.2.

Despite this example, the following results imply that for Morita algebras, the canonical bimodule V exhibits a certain kind of left right symmetry on the bimodule structure.

Proposition 3.4. *Let A be a Morita algebra, and let V be the canonical A-bimodule. Then the following assertions are equivalent.*

(1) *V is projective in A-*mod.
(2) *V is a generator in A-*mod.
(3) *V is projective in A^{op}-*mod.
(4) *V is a generator in A^{op}-*mod.

Proof. Since A is a Morita algebra, the canonical A-bimodule V has the property (Definition 2.7(4) and (5)) $A^{op} \cong \mathrm{End}_A(V)$ and $A \cong \mathrm{End}_{A^{op}}(V)$. By Theorem 2.10, it follows that (1) $\Leftrightarrow$ (4) and (2) $\Leftrightarrow$ (3). By Lemma 2.11, (1) $\Leftrightarrow$ (2) follows from $A^{op} \cong \mathrm{End}_A(V)$. Indeed, if $V = \bigoplus_{i=1}^m V_i^{\oplus r_i}$, with $V_1, \dots, V_m$ indecomposable and pairwise non-isomorphic, then $A^{op} \cong \mathrm{End}_A(V)$ implies that $\{P_i = \mathrm{Hom}_A(V, V_i) | 1 \leq i \leq m\}$ forms a complete set of pairwise non-isomorphic indecomposable projective A-modules. Hence m is the rank of the Grothendieck group $K_0(A)$ of A-mod. As a result, if V is projective, then each indecomposable projective A-module is a direct summand of V; hence V is a generator. If V is a generator, then V contains each of the m indecomposable projective A-modules as a direct summand, and it has no non-projective direct summands. $\square$

Proposition 3.5. *Let A be a Morita algebra, and let V be the canonical A-bimodule. Then the following assertions are equivalent.*

(1) *V is injective in* A-mod.
(2) *V is a cogenerator in* A-mod.
(3) *V is injective in* A^{op}-mod.
(4) *V is a cogenerator in* A^{op}-mod.
(5) *A is self-injective.*
(6) *A is Morita equivalent to a Frobenius algebra.*

Proof. Since A is a Morita algebra, by Definition 2.7(2) and (3), domdim $A \geq 2$ and there is a self-dual idempotent $e \in A$ such that Ae is a faithful projective injective A-module. Note that (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) follow by similar arguments in the proof of Proposition 3.4.

(1) $\Rightarrow$ (5) Ae being faithful in A-mod implies that there is an embedding $A \overset{u}{\hookrightarrow} (Ae)^{\oplus m}$ in A-mod for some m. Applying $\mathrm{Hom}_A(\mathrm{D}(A), -)$ to the morphism u and observing that

$$\mathrm{Hom}_A(\mathrm{D}(A), Ae) \cong \mathrm{Hom}_{A^{op}}(\mathrm{D}(Ae), A) \cong \mathrm{Hom}_{A^{op}}(eA, A) \cong Ae,$$

we get the embedding $V = \mathrm{Hom}_A(\mathrm{D}(A), A) \overset{u_*}{\hookrightarrow} (Ae)^{\oplus m}$ in A-mod. Now if V is injective in A-mod, then u_* splits. Thus, as a direct summand of $(Ae)^{\oplus m}$, the module V is projective and injective in A-mod. By Proposition 3.4, it then follows that V is a projective injective generator in A-mod and, in particular, that $A \in \mathrm{add}(V)$ is projective and injective. So A is self-injective.

(5) $\Rightarrow$ (6) is trivial.

(6) $\Rightarrow$ (1) Assume that A is Morita equivalent to B, where B is a Frobenius algebra. Then $\mathrm{D}(B) \cong B$ as (left) B-modules and

$$\mathrm{Hom}_B(\mathrm{D}(B), B) \cong \mathrm{Hom}_B(\mathrm{D}(B), \mathrm{D}(B)) \cong B$$

as (left) B-modules. In other words, the canonical B-bimodule $V(B)$ is injective in B-mod. Hence by Proposition 3.3, $V = V(A)$ is injective in A-mod. Note that A is self-injective if and only if A^{op} is self-injective. Thus (3) $\Leftrightarrow$ (5) follows from (1) $\Leftrightarrow$ (5) and Lemma 3.2. $\square$

We now formulate the main result on the left right symmetry of V, which may also be viewed as a generalization of Corollary 2.12 for Morita algebras. We recall that, for a self-dual idempotent e of an algebra A, there is an automorphism θ of the algebra eAe such that $\mathrm{D}(eAe)_\theta \cong eAe$ as eAe-bimodules, so eAe is a Frobenius algebra with $\theta = \nu_{eAe}$, a Nakayama automorphism of eAe; see [14, Lemma 2.4].

Theorem 3.6. *Let A be an algebra, and let V be the canonical A-bimodule. Then the following assertions are equivalent.*

(1) *V is projective in A-mod and $A^{op} \cong \mathrm{End}_A(V)$ canonically.*
(2) *V is projective in A^{op}-mod and $A \cong \mathrm{End}_{A^{op}}(V)$ canonically.*
(3) *A is a Morita algebra with a self-dual idempotent e such that Ae is a faithful A-module and $\mathrm{add}_{(eAe)^{op}}(Ae) = \mathrm{add}_{(eAe)^{op}}(Ae_{\nu_{eAe}})$.*
(4) *A is a Morita algebra with a self-dual idempotent e such that eA is a faithful right A-module and $\mathrm{add}_{eAe}(eA) = \mathrm{add}_{eAe}({}_{\nu_{eAe}}eA)$.*
(5) *$A \cong \mathrm{End}_{B^{op}}(M)$, where M is a faithful right module over a Frobenius algebra B such that $\mathrm{add}_{B^{op}}(M) = \mathrm{add}_{B^{op}}(M_{\nu_B})$.*
(6) *$A \cong \mathrm{End}_B(N)^{op}$, where N is a faithful left module over a Frobenius algebra B such that $\mathrm{add}_B(N) = \mathrm{add}_B({}_{\nu_B}N)$.*

Proof. If either of the canonical morphisms $A^{op} \to \mathrm{End}_A(V)$ and $A \to \mathrm{End}_{A^{op}}(V)$ is an isomorphism of algebras, then it follows by Definition 2.7(5) and (4) that A is a Morita algebra. Hence the equivalence (1) $\Leftrightarrow$ (2) follows by Proposition 3.4. With A replaced by its opposite algebra A^{op}, it only remains to show that (1) $\Rightarrow$ (3) $\Rightarrow$ (2) and (3) $\Leftrightarrow$ (5).

(1) $\Rightarrow$ (3) As noticed above, A is a Morita algebra. Let e be a self-dual idempotent of A such that Ae is a faithful A-module. Then eAe is a Frobenius algebra and $\mathrm{D}(eA)_\nu \cong Ae$ as (A, eAe)-bimodules for a Nakayama automorphism ν of eAe. Note that V being projective in A-mod implies that V is a progenerator in A^{op}-mod by Proposition 3.4. Thus we have $\mathrm{add}_{A^{op}}(V) = \mathrm{add}_{A^{op}}(A)$ and in particular $\mathrm{add}_{(eAe)^{op}}(Ve) = \mathrm{add}_{(eAe)^{op}}(Ae)$. On the other hand,

$$\begin{aligned} Ve = \mathrm{Hom}_A(\mathrm{D}(A), A)e &\cong \mathrm{Hom}_A(\mathrm{D}(A), Ae) \\ &\cong \mathrm{Hom}_{A^{op}}(\mathrm{D}(Ae), A) \cong \mathrm{Hom}_{A^{op}}({}_\nu eA, A) \cong Ae_\nu \end{aligned}$$

as right eAe-modules. Consequently,

$$\mathrm{add}_{(eAe)^{op}}(Ae) = \mathrm{add}_{(eAe)^{op}}(Ve) = \mathrm{add}_{(eAe)^{op}}(Ae_\nu).$$

(3) $\Rightarrow$ (2) By Definition 2.7 (2) and (4), $\mathrm{End}_{(eAe)^{op}}(Ae) \cong A \cong \mathrm{End}_{A^{op}}(V)$ canonically, as A-bimodules. Moreover, e being a self-dual idempotent implies that $\mathrm{D}(eA)_\nu \cong Ae$ as (A, eAe)-bimodules for a Nakayama automorphism ν of eAe. Now by Lemma 3.2,

$$\begin{aligned} V &\cong \mathrm{Hom}_{A^{op}}(\mathrm{D}(A), A) \cong \mathrm{Hom}_{A^{op}}(\mathrm{D}(A), \mathrm{End}_{(eAe)^{op}}(Ae)) \\ &\cong \mathrm{Hom}_{(eAe)^{op}}(\mathrm{D}(A)e, Ae) \cong \mathrm{Hom}_{(eAe)^{op}}(\mathrm{D}(eA), Ae) \cong \mathrm{Hom}_{(eAe)^{op}}(Ae, Ae_\nu) \end{aligned}$$

as A-bimodules. Together with $\mathrm{add}_{(eAe)^{op}}(Ae) = \mathrm{add}_{(eAe)^{op}}(Ae_\nu)$, we then get

$$V \cong \mathrm{Hom}_{(eAe)^{op}}(Ae, Ae_\nu) \in \mathrm{add}_{A^{op}}(\mathrm{Hom}_{(eAe)^{op}}(Ae, Ae)) = \mathrm{add}_{A^{op}}(A),$$

which implies that V is a projective right A-module.

(3) $\Rightarrow$ (5) is trivial by setting $B = eAe$ and $M = Ae$.

(5) $\Rightarrow$ (3) Assume that B is a Frobenius algebra and $A = \mathrm{End}_{B^{op}}(M)$ for a faithful right B-module M such that $\mathrm{add}_{B^{op}}(M) = \mathrm{add}_{B^{op}}(M_{\nu_B})$. If B is a basic algebra, then $M \cong B \oplus X$ in B^{op}-mod for some right B-module X, since B is injective and M is faithful in B^{op}-mod. Let e be the composition of the projection $M \twoheadrightarrow B$ and the embedding $B \hookrightarrow M$ in B^{op}-mod. Then e is an idempotent

in $A = \mathrm{End}_{B^{op}}(M)$, $B \cong eAe$ as algebras, $eA \cong \mathrm{Hom}_{B^{op}}(M, B)$ as (eAe, A)-bimodules, and $Ae \cong \mathrm{Hom}_{B^{op}}(B, M) \cong M$ as (A, eAe)-bimodules. As a result, eAe is a Frobenius algebra and

$$\mathrm{add}_{(eAe)^{op}}(Ae) = \mathrm{add}_{B^{op}}(M) = \mathrm{add}_{B^{op}}(M_{\nu_B}) = \mathrm{add}_{(eAe)^{op}}(Ae_{\nu_{eAe}}).$$

Since B is a Frobenius algebra, the isomorphism $\mathrm{D}(B) \cong B$ in B-mod yields the embedding $u : M \hookrightarrow B^{\oplus m}$ of right B-modules for some m, and

$$\mathrm{D}(eA) \cong \mathrm{D}\,\mathrm{Hom}_{B^{op}}(M, B) \cong M \otimes_B \mathrm{D}(B) \cong M \otimes_B B \cong M \cong Ae$$

as A-modules. Applying $\mathrm{Hom}_{B^{op}}(M, -)$ to u, we obtain an embedding $A \hookrightarrow (eA)^{\oplus m}$ of right A-modules. So e is a self-dual idempotent with Ae being a faithful A-module.

In general, let e_0 be a basic idempotent so that $B_0 = e_0 B e_0$ is Morita equivalent to B. By Corollary 2.9, we may choose the Nakayama automorphism ν_B so that $\nu_B(e_0) = e_0$. Then the restriction of ν_B to $e_0 B e_0$, denoted by ν, is a Nakayama automorphism of B_0, $M_0 = Me_0$ is a faithful right B_0-module with

$$A \cong \mathrm{End}_{B^{op}}(M) \cong \mathrm{End}_{B_0^{op}}(Me_0) = \mathrm{End}_{B_0^{op}}(M_0),$$

and $\mathrm{add}_{(B_0)^{op}}(M_0) = F(\mathrm{add}_{B^{op}}(M)) = F(\mathrm{add}_{B^{op}}(M_{\nu_B})) = \mathrm{add}_{B_0^{op}}(M_{0\nu})$. Here F denotes the equivalence from B-mod to B_0-mod represented by $- \otimes_B Be_0$. By what we have shown above for basic Frobenius algebras, we are done. $\square$

Corollary 3.7. *Let A be a basic Morita algebra, and let V be the canonical A-bimodule. Then the following statements are equivalent.*

(1) *V is projective as an A-module.*
(1') *V is projective as a right A-module.*
(2) *$V \cong A$ as A-modules.*
(2') *$V \cong A$ as right A-modules.*
(3) *V is a free A-module.*
(3') *V is a free right A-module.*

Proof. Since A is a Morita algebra, (1) $\Leftrightarrow$ (1') follows from Proposition 3.4. Therefore it suffices to show (1) $\Rightarrow$ (2), since (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are trivial. By Theorem 3.6, we have $A = \mathrm{End}_{B^{op}}(M)$ for some Frobenius algebra B and a faithful right B-module M such that $\mathrm{add}_{B^{op}}(M) = \mathrm{add}_{B^{op}}(M_\nu)$, where ν is a Nakayama automorphism of B. Note that A being a basic algebra implies that each indecomposable direct summand of the right B-module M is multiplicity-free. As a result, $M \cong M_\nu$ as right B-modules, and by [14, Theorem 3], $V \cong A$ as A-modules. $\square$

Remark. In Corollary 3.7, the Morita algebra A being basic is not an essential condition. Indeed, by Theorem 3.6 and [14, Theorem 3], let $A = \mathrm{End}_{B^{op}}(M)$ for some Frobenius algebra B and a faithful right B-module M with $\mathrm{add}_{B^{op}}(M) = \mathrm{add}_{B^{op}}(M_{\nu_B})$. Then V is a free A-module if and only if $M \cong M_{\nu_B}$ as right B-modules. The following example shows how this is applied to construct Morita algebras such that the canonical bimodule is projective but not free as one-sided modules.

Example 2. Let k be any field, and let B be the k-algebra defined by the quiver

$$\begin{tikzcd} 1 \bullet \arrow[r, bend left, "\alpha"] & \bullet 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

with relations $\beta\alpha = 0$ and $\alpha\beta = 0$. Then B is a Frobenius algebra, and for any Nakayama automorphism ν of B, $(S_1')_\nu \cong S_2'$ as right B-modules. Here S_1' and S_2' denote the simple heads of the projective right B-modules e_1B and e_2B respectively. Let $M = B \oplus S_1' \oplus S_1' \oplus S_2'$. Then $A = \mathrm{End}_{B^{op}}(M)$ is a Morita algebra with the canonical A-bimodule V being projective but not free as an A-module.

Besides Morita algebras, it seems hard to handle the left right symmetry problem for the canonical bimodules, as illustrated by Example 1. However, we have

Proposition 3.8. *Let A be an algebra, and let V be the canonical A-bimodule.*

(a) *If* $\mathrm{gl.dim}\, A \leq 2$*, then V is projective in both* A-mod *and* A^{op}-mod*.*
(b) *If A has a k-algebra anti-automorphism, then V is projective in* A-mod *if and only if so is V in* A^{op}-mod*.*

Proof. (a) Let $0 \to A \to I_0 \xrightarrow{u} I_1$ be an injective presentation of the regular A-module. Let $X = \mathrm{cok}(\mathrm{Hom}_A(\mathrm{D}(A), u))$. Then we have an exact sequence in A-mod,

$$0 \to V \to \mathrm{Hom}_A(\mathrm{D}(A), I_0) \to \mathrm{Hom}_A(\mathrm{D}(A), I_1) \to X \to 0.$$

Since $\mathrm{gl.dim}\, A \leq 2$ and $\mathrm{Hom}_A(\mathrm{D}(A), I_i)$ are projective A-modules for $i = 1, 2$, it follows that V must be a projective A-module. Similarly, V is a projective right A-module.

(b) Let $\tau : A \to A$ be a k-algebra anti-automorphism. Let $F : A\text{-mod} \to A^{op}\text{-mod}$ be the functor defined by (see section 2 for the notation)

$$F(M) = M^\tau$$

for any $M \in A$-mod. Then F is a Morita equivalence and it induces an equivalence from the category of A-bimodules to the category of A^{op}-bimodules, sending an A-bimodule W to the A^{op}-bimodule ${}^\tau W^\tau$. Here the A^{op}-bimodule structure on ${}^\tau W^\tau$ is given by

$$a^{op} \cdot w \cdot b^{op} = \tau(a) w \tau(b) \qquad \forall\, a^{op}, b^{op} \in A^{op} \text{ and } w \in {}^\tau W^\tau.$$

By Proposition 3.3, the canonical A^{op}-bimodule is isomorphic to ${}^\tau V^\tau$ and is isomorphic to V by Lemma 3.2. As a result, $V \cong {}^\tau V^\tau$ as A-bimodules. Note that ${}^\tau V^\tau = F(V)$ as right A-modules. We have $V \cong {}^\tau V^\tau = F(V)$ as right A-modules. Consequently, V is projective as an A-module if and only if $V \cong F(V)$ is projective as a right A-module. □

The following two examples serve as a complement to Theorem 3.6 and Corollary 3.7.

Example 3 (cf. [14, Example 3.7])**.** Let k be any field, and let A be the k-algebra defined by the quiver

$$\begin{tikzcd}
1 \arrow[dr, "\alpha"'] & & 3 \arrow[ll, "\gamma"'] \\
& 2 \arrow[ur, "\beta"'] &
\end{tikzcd}$$

with relations $\alpha\gamma = 0, \gamma\beta = 0$. Then $Ae_1 \cong \mathrm{D}(e_3A), Ae_3 = \mathrm{D}(e_1A)$ as A-modules, and for $e = e_1 + e_3$, Ae is a faithful A-module with $A \cong \mathrm{End}_{(eAe)^{op}}(Ae)$ canonically. By Definition 2.7, A is a Morita algebra. Now

$$V = \mathrm{Hom}_A(\mathrm{D}(A), A) \cong \mathrm{Hom}_A(\mathrm{D}(A), Ae \oplus Ae_2) \cong Ae \oplus S_1$$

as A-modules, and

$$V = \mathrm{Hom}_A(\mathrm{D}(A), A) \cong \mathrm{Hom}_A(Ae \oplus \mathrm{D}(e_2A), A) \cong eA \oplus S'_3$$

as right A-modules. Here S_1 and S'_3 denote the simple heads of Ae_1 and e_3A respectively.

Example 4. Let k be any field, and let A be the k-algebra defined by the quiver

$$\overset{1}{\bullet} \xrightarrow{\alpha} \overset{2}{\bullet} \xrightarrow{\beta} \overset{3}{\bullet}$$

with relation $\beta\alpha = 0$. Then $Ae_1 \cong \mathrm{D}(e_2A), Ae_2 \cong \mathrm{D}(e_3A)$ as A-modules. By Definition 2.7, A is not a Morita algebra. Now

$$V = \mathrm{Hom}_A(\mathrm{D}(A), A) = \mathrm{Hom}_A(\mathrm{D}(A), A(e_1 + e_2) \oplus Ae_3) \cong Ae_2 \oplus Ae_3$$

as A-modules, and as right A-modules,

$$V = \mathrm{Hom}_A(\mathrm{D}(A), A) = \mathrm{Hom}_A(Ae_1 \oplus Ae_2 \oplus \mathrm{D}(e_1A), A) \cong e_1A \oplus e_2A.$$

More generally, for any non-semisimple tilted algebra A, we have $\mathrm{gl.dim}\, A \leq 2$. Notice that the Ext-quiver of A contains no oriented cycles. It follows that A is not a Morita algebra and V is projective as both left and right A-modules by Proposition 3.8. Indeed, if A is a non-semisimple Morita algebra with Ae being a minimal faithful A-module, then eAe is a non-semisimple Frobenius algebra whose Ext-quiver must contain an oriented cycle. As a result, the Ext-quiver of A must also contain an oriented cycle, which is a contradiction.

3.2. **Projective dimension.** For simplicity, we denote by τ the Auslander-Reiten translation τ_A in A-mod or $\tau_{A^{op}}$ in A^{op}-mod whenever there is no confusion arising from the context (see [2] for more details on Auslander-Reiten translations).

Lemma 3.9. *Let A be an algebra, and let Y be a right A-module. The following statements are equivalent for any non-negative integer m.*

(1) $\mathrm{proj.dim}\, \mathrm{Hom}_{A^{op}}(Y, A) \leq m$ *in* A-mod.
(2) $\mathrm{inj.dim}\, \tau(Y) \leq m + 2$ *in* A^{op}-mod.

Proof. Let $P_1 \to P_0 \to Y \to 0$ be a minimal projective presentation of the right A-module Y. Consider the canonical exact sequence in A^{op}-mod ([22, Proposition III 5.3]):

$$0 \to \tau(Y) \to \mathcal{N}(P_1) \to \mathcal{N}(P_0) \to \mathcal{N}(Y) \to 0$$

where $\mathcal{N} = \mathrm{D}\,\mathrm{Hom}_{A^{op}}(-, A)$. Since $\mathrm{proj.dim}\, \mathrm{Hom}_{A^{op}}(Y, A) = \mathrm{inj.dim}\, \mathcal{N}(Y)$, it follows that $\mathrm{proj.dim}\, \mathrm{Hom}_{A^{op}}(Y, A) \leq m$ if and only if $\mathrm{inj.dim}\, \tau(Y) \leq \mathrm{inj.dim}\, \mathcal{N}(Y) + 2 \leq m + 2$. $\square$

Proposition 3.10. *Let A be an algebra, and let V be the canonical A-bimodule. The following statements are equivalent for any non-negative integer m.*

(1) $\mathrm{proj.dim}\, V \leq m$ *in* A-mod.
(2) $\mathrm{proj.dim}\, \tau^{-}(P) \leq m + 2$ *for any indecomposable projective A-module P.*
(3) $\mathrm{inj.dim}\, \tau(I) \leq m + 2$ *for any indecomposable injective right A-module I.*

Proof. (1) $\Leftrightarrow$ (3) follows from Lemma 3.2 and Lemma 3.9. (2) $\Leftrightarrow$ (3) follows from

$$\begin{aligned}\mathrm{proj.dim}\, \tau^{-1}(P) &= \mathrm{proj.dim}\, \mathrm{Hom}_{A^{op}}(\mathrm{D}(P), A) + 2 \\ &= \mathrm{inj.dim}\, \mathrm{D}\,\mathrm{Hom}_{A^{op}}(\mathrm{D}(P), A) + 2 = \mathrm{inj.dim}\, \tau(\mathrm{D}(P))\end{aligned}$$

for any projective A-module P. Since every indecomposable injective right A-module I is of the form $\mathrm{D}(P)$ for some indecomposable projective A-module P, we are done. $\square$

Corollary 3.11. *Let A be an algebra, and let V be the canonical A-bimodule. The following statements are equivalent.*

(1) *V is projective in A-mod.*
(2) $\mathrm{proj.dim}\,\tau^{-}(P) \leq 2$ *for any indecomposable projective A-module P.*
(2) $\mathrm{inj.dim}\,\tau(I) \leq 2$ *for any indecomposable injective right A-module I.*

In particular, V is projective in both A-mod and A^{op}-mod when $\mathrm{gl.dim}\,A \leq 2$.

This corollary is particularly useful in verifying the left right symmetry on projectivity for the canonical bimodules. In Example 1 of section 3.1, we have the Auslander-Reiten quiver of A-mod

$$\begin{tikzcd}
Ae_1 = S_1 \arrow[dr] \arrow[rr, dashed, no head] & & \mathrm{D}(e_2A) \arrow[dr] & & Ae_4 \arrow[dr] & & & \\
& Ae_3 \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & S_3 \arrow[ur] \arrow[rr, dashed, no head] & & S_4 \arrow[dr] \arrow[rr, dashed, no head] & & S_5 = \mathrm{D}(e_5A) \\
Ae_2 = S_2 \arrow[ur] \arrow[rr, dashed, no head] & & \mathrm{D}(e_1A) \arrow[ur] & & & & Ae_5 \arrow[ur] &
\end{tikzcd}$$

As a result, $\mathrm{proj.dim}\,\tau^{-1}(Ae_i) = 1$ for $i = 1, 2, 3$ and $\mathrm{inj.dim}\,\tau\,\mathrm{D}(e_iA) = 3$ for $i = 1, 2$ and $\mathrm{inj.dim}\,\tau\,\mathrm{D}(e_5A) = 1$. Since Ae_4 and Ae_5 are projective injective A-modules, by Corollary 3.11, V is a projective A-module but not a projective right A-module.

3.3. **Left and right A-duals.** The most important bimodule for an algebra A is A and its k-dual $\mathrm{D}(A)$, and a natural construction of bimodules is to take the left or the right A-duals. One interesting case seems to occur when A is a gendo-symmetric algebra [11], since the construction above from A and $\mathrm{D}(A)$ stabilizes at A and $\mathrm{D}(A)$. In [14, Theorem 3], a generalization of this case with an automorphism of A involved is obtained. In general, we have that by Lemma 3.2, the left and right A-duals of $\mathrm{D}(A)$ coincide. In this section, we shall go further to consider the left and right A-duals of the canonical A-bimodule V.

Lemma 3.12. *Let A be an algebra with* $\mathrm{domdim}\,A \geq 2$. *Then*

$$\mathrm{domdim}_{A^{op}} \mathrm{Hom}_A(V, A) \geq 2$$

and $\mathrm{domdim}_A \mathrm{Hom}_{A^{op}}(V, A) \geq 2$.

Proof. Since $\mathrm{domdim}\,A \geq 2$, there exists an idempotent e of A such that eA is a projective, injective and faithful right A-module, and $\mathrm{D}(eA) \cong Af$ as A-modules, for some idempotent f in A. By [17, Lemma 6], there is a minimal injective presentation of A in $A \otimes_k A^{op}$-mod:

$$0 \to A \to E^0 \to E^1$$

where $E^0, E^1 \in \mathrm{add}_{A\otimes_k A^{op}}(Af \otimes_k eA)$. Applying $\mathrm{Hom}_A(V, -)$ to the sequence, we obtain the following exact sequence in $A \otimes_k A^{op}$-mod (particularly in A^{op}-mod):

$$0 \to \mathrm{Hom}_A(V, A) \to \mathrm{Hom}_A(V, E^0) \to \mathrm{Hom}_A(V, E^1).$$

Note that $\mathrm{Hom}_A(V, Af \otimes_k eA) \cong \mathrm{Hom}_A(V, Af) \otimes_k eA \in \mathrm{add}_{A^{op}}(eA)$. We have

$$\mathrm{Hom}_A(V, E^i) \in \mathrm{add}_{A\otimes_k A^{op}}(\mathrm{Hom}_A(V, Af \otimes_k eA)) \subseteq \mathrm{add}_{A^{op}}(eA), \qquad i = 1, 2.$$

In particular, $\mathrm{domdim}_{\,A^{op}} \mathrm{Hom}_A(V, A) \geq 2$ as desired. With A replaced by its opposite algebra A^{op}, the arguments above together with Lemma 3.2 yield $\mathrm{domdim}_{\,A} \mathrm{Hom}_{A^{op}}(V, A) \geq 2$, since $\mathrm{domdim}\, A^{op} = \mathrm{domdim}\, A \geq 2$. □

Example 5. The converse of Lemma 3.12 is not true as the following simple example shows. Let k be any field and A be the k-algebra defined by the quiver $1 \longrightarrow 2$. Then $Ae_1 = \mathrm{D}(e_2A)$ as A-modules, $V = \mathrm{Hom}_A(\mathrm{D}(A), A) \cong e_1A$ and $V = \mathrm{Hom}_A(\mathrm{D}(A), A) \cong Ae_2$ as right and left A-modules respectively. Hence $\mathrm{Hom}_A(V, A) \cong e_2A$ as right A-modules, and $\mathrm{Hom}_{A^{op}}(V, A) \cong Ae_1$ as A-modules. As a result, $\mathrm{domdim}_{\,A^{op}} \mathrm{Hom}_A(V, A) = \infty = \mathrm{domdim}_{\,A} \mathrm{Hom}_{A^{op}}(V, A)$, while $\mathrm{domdim}\, A = 1$.

Proposition 3.13. *Let A be an algebra with* $\mathrm{domdim}\, A \geq 2$*, and let V be the canonical A-bimodule. Then* $\mathrm{domdim}_{\,A} \mathrm{Hom}_A(V, A) \geq 2$ *and* $\mathrm{domdim}_{\,A^{op}} \mathrm{Hom}_{A^{op}}(V, A) \geq 2$ *if and only if* $\mathrm{Hom}_A(V, A) \cong \mathrm{Hom}_{A^{op}}(V, A)$ *as A-bimodules.*

Proof. Following section 2.1, we denote by Γ the double dual functor for both A and A^{op} and by $\xi : \mathrm{Id} \to \Gamma$ the natural transformation by abuse of notation. We consider the following two canonical morphisms:

$$\begin{aligned} \alpha : \quad & \mathrm{D}(A) \longrightarrow \mathrm{Hom}_{A^{op}}(\mathrm{Hom}_A(\mathrm{D}(A), A), A), \\ \beta : \quad & \mathrm{D}(A) \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_{A^{op}}(\mathrm{D}(A), A), A). \end{aligned}$$

Here α and β are both defined as $\xi_{\mathrm{D}(A)}$ but with $\mathrm{D}(A)$ being regarded as an A-module and an A^{op}-module respectively. In particular, α is an A-module morphism, while β is an A^{op}-module morphism. For any $\delta \in \mathrm{D}(A)$ and $h \in \mathrm{Hom}_A(\mathrm{D}(A), A)$, $g \in \mathrm{Hom}_{A^{op}}(\mathrm{D}(A), A)$,

$$\begin{aligned} \alpha(\delta \cdot a)(h) &= h(\delta \cdot a) = (a \cdot h)(\delta), \\ (\alpha(\delta) \cdot a)(h) &= \alpha(\delta)(a \cdot h) = (a \cdot h)(\delta), \\ \beta(a \cdot \delta)(g) &= g(a \cdot \delta) = (g \cdot a)(\delta), \\ (a \cdot \beta(\delta))(g) &= \beta(\delta)(g \cdot a) = (g \cdot a)(\delta). \end{aligned}$$

It follows that both α and β are A-bimodule morphisms. Since $\mathrm{domdim}\, A \geq 2$, there exists an idempotent $f \in A$ such that fA is a faithful projective injective right A-module. By Proposition 2.5, there is the commutative diagram in A-mod:

(*)
$$\begin{tikzcd} \mathrm{D}(A) \arrow[r, equal] \arrow[d, "\alpha"'] & \mathrm{D}(A) \arrow[r, "\beta"] \arrow[d, "\eta_{\mathrm{D}(A)}"] & \mathrm{Hom}_A(V, A) \arrow[d, "\eta_{\mathrm{Hom}_A(V,A)}"] \\ \mathrm{Hom}_{A^{op}}(V, A) \arrow[r, "\theta", "\sim"'] & \mathcal{G}(\mathrm{D}(A)) \arrow[r, "\mathcal{G}(\beta)"] & \mathcal{G}(\mathrm{Hom}_A(V, A)) \end{tikzcd}$$

where $\mathcal{G}$ is the endo-functor of A-mod and $\eta : \mathrm{Id} \to \mathcal{G}$ is the natural transformation from Proposition 2.5.

If $\mathrm{domdim}_A \mathrm{Hom}_A(V, A) \geq 2$, then by Theorem 2.4, $\eta_{\mathrm{Hom}_A(V,A)}$ is an isomorphism, so that $\ker(\alpha) \subseteq \ker(\beta)$ by the commutative diagram (*) above. If

$$\mathrm{domdim}_{A^{op}} \mathrm{Hom}_{A^{op}}(V, A) \geq 2,$$

then replacing A by its opposite algebra A^{op}, the arguments above yield $\ker(\beta) \subseteq \ker(\alpha)$. As a result, if both $\operatorname{domdim}_A \operatorname{Hom}_A(V, A) \geq 2$ and $\operatorname{domdim}_{A^{op}} \operatorname{Hom}_{A^{op}}(V, A) \geq 2$ hold, we must have $\ker(\beta) = \ker(\alpha) = \ker(\eta_{\mathrm{D}(A)})$. Since $f\ker(\eta_{\mathrm{D}(A)}) = 0$ trivially, we have that

$$f\ker(\alpha) = f\ker(\beta) = 0$$

and hence the monomorphisms in the category of (fAf, A)-bimodules

$$f\,\mathrm{D}(A) \hookrightarrow f\operatorname{Hom}_{A^{op}}(V, A), \quad f\,\mathrm{D}(A) \hookrightarrow f\operatorname{Hom}_A(V, A).$$

Applying $\operatorname{Hom}_{fAf}(fA, -)$ to these monomorphisms, we obtain in the category of A-bimodules

$$\operatorname{Hom}_{A^{op}}(V, A) \cong \mathcal{G}(\mathrm{D}(A)) \hookrightarrow \mathcal{G}(\operatorname{Hom}_A(V, A)) \cong \operatorname{Hom}_A(V, A).$$

Here the last isomorphism follows since $\operatorname{domdim}_A \operatorname{Hom}_A(V, A) \geq 2$. Similarly, we have a monomorphism $\operatorname{Hom}_A(V, A) \hookrightarrow \operatorname{Hom}_{A^{op}}(V, A)$ as A-bimodules. As a consequence, we get $\operatorname{Hom}_A(V, A) \cong \operatorname{Hom}_{A^{op}}(V, A)$ as A-bimodules.

Conversely, assume $\operatorname{Hom}_A(V, A) \cong \operatorname{Hom}_{A^{op}}(V, A)$ as A-bimodules. By Lemma 3.12,

$$\operatorname{domdim}_A \operatorname{Hom}_A(V, A) = \operatorname{domdim}_A \operatorname{Hom}_{A^{op}}(V, A) \geq 2.$$

Similarly $\operatorname{domdim}_{A^{op}} \operatorname{Hom}_{A^{op}}(V, A) \geq 2$ as desired. □

Corollary 3.14. *Let A be a Morita algebra, and let V be the canonical A-bimodule. Then* $\operatorname{Hom}_A(V, A) \cong \operatorname{Hom}_{A^{op}}(V, A)$ *as A-bimodules.*

Proof. By Definition 2.7(2), there exists a basic self-dual idempotent f of A such that Af is a faithful injective A-module, and there exists an exact sequence in A-mod,

$$0 \to A \to I^0 \to I^1,$$

with $I^0, I^1 \in \operatorname{add}_A(Af)$. Applying $\operatorname{Hom}_A(V, -)$ to the sequence, we obtain

$$0 \to \operatorname{Hom}_A(V, A) \to \operatorname{Hom}_A(V, I^0) \to \operatorname{Hom}_A(V, I^1)$$

in A-mod. Note that $fV = f\operatorname{Hom}_A(\mathrm{D}(A), A) \cong \operatorname{Hom}_A(\mathrm{D}(fA), A) \cong \operatorname{Hom}_A(Af, A) \cong fA$ as right A-modules, and

$$\begin{aligned}\operatorname{Hom}_A(V, Af) &\cong \operatorname{Hom}_{A^{op}}(\mathrm{D}(Af), \mathrm{D}(V)) \cong \operatorname{Hom}_{A^{op}}(fA, \mathrm{D}(V)) \\ &\cong \mathrm{D}(fV) \cong \mathrm{D}(fA) \cong Af\end{aligned}$$

as A-modules. We have $\operatorname{Hom}_A(V, I^i) \in \operatorname{add}_A(Af)$ for $i = 1, 2$. Thus

$$\operatorname{domdim}_A \operatorname{Hom}_A(V, A) \geq 2.$$

Similarly we have $\operatorname{domdim}_{A^{op}} \operatorname{Hom}_{A^{op}}(V, A) \geq 2$. Applying Proposition 3.13, we obtain $\operatorname{Hom}_A(V, A) \cong \operatorname{Hom}_{A^{op}}(V, A)$ as A-bimodules. □

Remark. With the assumption $\operatorname{domdim} A \geq 2$, Corollary 3.14 will be completed in section 4 to provide a new characterization for Morita algebras.

4. Characterizing dominant dimension

Though the definition of dominant dimension (Definition 2.1) and Müller's characterization (Theorem 2.4) have nothing to do with the bimodules constructed from A and $\mathrm{D}(A)$, by taking left and right A-duals and extensions, Theorem 2.2 and [11] exhibit the use of these bimodules in characterizing dominant dimension. Note that by Theorem 2.2, one would expect that the larger the dominant dimension of A, the better the exactness of Γ is. The Nakayama conjecture says that if $\operatorname{domdim} A = \infty$, then Γ is exact. In the following, we first consider the right derived functor of Γ, since Γ is left exact when $\operatorname{domdim} A \geq 2$. Let $R^i\Gamma$ denote the i-th right derived functor of Γ.

Theorem 4.1. *Let A be an algebra with $\operatorname{domdim} A \geq 2$, and let M be an A-module. Let $n \geq 2$ be a non-negative integer. Then $\operatorname{domdim} M \geq n$ if and only if $\Gamma(M) \cong M$ canonically and $R^i\Gamma(M) = 0$ for $1 \leq i \leq n-2$.*

Proof. The proof is essentially the same as in [7, Theorem 2]. For the convenience of the reader, we include a proof below. If $\operatorname{domdim} M \geq n$, then by definition, in the minimal injective resolution of M,

$$0 \to M \to I_0 \xrightarrow{f_0} I_1 \xrightarrow{f_1} I_2 \to \cdots \to I_{n-1} \to I_n \to \cdots,$$

all I_i are projective for $0 \leq i \leq n-1$. Applying $\Gamma = (\ )^{**}$ to this exact sequence, we get

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\xi_M"'] & I_0 \arrow[r] \arrow[d, "\xi_{I_0}"'] & I_1 \arrow[r] \arrow[d, "\xi_{I_1}"'] & \cdots \arrow[r] & I_{n-1} \arrow[r] \arrow[d, "\xi_{I_{n-1}}"'] & I_n \arrow[r] \arrow[d, "\xi_{I_n}"'] & \cdots \\
0 \arrow[r] & \Gamma(M) \arrow[r] & \Gamma(I_0) \arrow[r] & \Gamma(I_1) \arrow[r] & \cdots \arrow[r] & \Gamma(I_{n-1}) \arrow[r] & \Gamma(I_n) \arrow[r] & \cdots .
\end{tikzcd}$$

By Morita's Theorem 2.3, ξ_M and ξ_{I_i} are isomorphisms for $0 \leq i \leq n-1$. As a result, $R^i\Gamma(M) = 0$ for $1 \leq i \leq n-2$.

Conversely, $\Gamma(M) \cong M$ canonically implies that M is reflexive. So by Morita's Theorem 2.3, $\operatorname{domdim} M \geq 2$. To show that $\operatorname{domdim} M \geq n$, we prove by induction that $I_0, \ldots, I_{n-1}$ above are projective A-modules, or alternatively $\xi_{I_0}, \ldots, \xi_{I_{n-1}}$ are isomorphisms by Morita's Theorem 2.3. Note that I_0, I_1 are projective since $\operatorname{domdim} M \geq 2$. Assume that $I_0, I_1, \ldots, I_t$ are projective A-modules for $t \leq n-2$, or equivalently $\xi_M, \xi_{I_0}, \ldots, \xi_{I_t}$ are isomorphisms. Since $R^i\Gamma(M) = 0$ for $1 \leq i \leq n-2$, it follows that the second row of the above commutative diagram is exact at degrees $0, 1, \ldots, n-2$. Therefore $\ker(\Gamma(f_i)) = \operatorname{Im}(\Gamma(f_{i-1}))$ for $1 \leq i \leq n-2$. If $K = \ker(\xi_{t+1}) \neq 0$, then $K \cap \operatorname{Im}(f_t) \neq 0$, since I_{t+1} is the injective hull of $\operatorname{Im}(f_t)$. For any non-zero element $z \in K \cap \operatorname{Im}(f_t)$, there exists $y \in I_t$ such that $z = f_t(y)$ and thus $\xi_{t+1}(z) = \Gamma(f_t) \circ \xi_{I_t}(y) = 0$. As a result,

$$\xi_{I_t}(y) \in \ker(\Gamma(f_t)) = \operatorname{Im}(\Gamma(f_{t-1}))$$

since $t \leq n-2$. Therefore, there exists $x \in \Gamma(I_{t-1})$ such that $\Gamma(f_{t-1})(x) = \xi_{I_t}(y)$. So

$$y = f_{t-1}(\xi^{-1}_{I_{t-1}}(x)), \quad z = f_t(y) = f_t(f_{t-1}(\xi^{-1}_{I_{t-1}}(x))) = 0,$$

which contradicts our choice of z. Altogether, $\xi_{I_{t+1}}$ must be a split monomorphism since I_{t+1} is injective. By Morita's Theorem 2.3, we deduce that I_{t+1} is projective as desired. $\square$

In order to compute $R^i\Gamma$, we need the technique of Grothendieck spectral sequences.

Lemma 4.2. *The following diagram commutes, up to natural equivalences:*

$$\begin{tikzcd}
A\text{-mod} \arrow[r, "(\,)^*"] \arrow[dr, "\mathrm{D}(A)\otimes_A -"'] & A^{op}\text{-mod} \arrow[r, "(\,)^*"] \arrow[d, "\mathrm{D}"] & A\text{-mod} \\
 & A\text{-mod} \arrow[ur, "\mathrm{Hom}_A(\mathrm{D}(A),-)"'] &
\end{tikzcd}$$

Proof. For any left A-module M, we have

$$\mathrm{Hom}_A(\mathrm{D}(A), \mathrm{D}(A) \otimes_A M) \cong \mathrm{Hom}_{A^{op}}(\mathrm{D}(\mathrm{D}(A) \otimes_A M), A) \cong \Gamma(M)$$

canonically, where the first isomorphism follows by the k-duality D and the second isomorphism follows by the tensor-hom adjunction. $\square$

Lemma 4.3. *Let $\mathscr{A} \xrightarrow{G} \mathscr{B} \xrightarrow{F} \mathscr{C}$ be additive covariant functors between abelian categories. Assume that $\mathscr{A}$ and $\mathscr{B}$ have enough injective objects and F is left exact. Let M be an object in $\mathscr{A}$, and let $0 \to M \to E^0 \to E^1 \to \cdots$ be an injective resolution of M in $\mathscr{A}$ such that $G(E^i)$ are right F-acyclic for $0 \leq i \leq n$. Then there exists a restricted first quadrant Grothendieck spectral sequence with the E_2 page given by*

$$E_2^{p,q} = R^pF(R^qG(M)) \underset{p}{\Longrightarrow} R^{p+q}(F \circ G)(M), \qquad 0 \leq p+q \leq n.$$

Proof. Following the standard arguments on spectral sequence (cf. [19, Theorem 10.47]), we consider an injective resolution of $G(E^i)$ for each $i \geq 0$, say $0 \to G(E^i) \to I^{i,0} \to I^{i,1} \to \cdots$. Applying F to the sequences, we obtain a first quadrant cohomological bicomplex $\{F(I^{i,j})\}$. The first type filtration of the total complex of $\{F(I^{i,j})\}$ gives rise to the spectral sequence

$${}^{\mathrm{I}}E_2^{p,q} = \begin{cases} 0, & 0 \leq p \leq n, q \neq 0; \\ R^p(F \circ G)(M), & 0 \leq p \leq n, q = 0; \\ *, & \text{else.} \end{cases}$$

The second type filtration of the total complex of $\{F(I^{i,j})\}$ gives rise to the spectral sequence

$${}^{\mathrm{II}}E_2^{p,q} = R^pF(R^qG(M)).$$

By standard arguments on convergence of spectral sequences, it follows that ${}^{\mathrm{II}}E_2^{p,q}$ converges to $R^{p+q}(F \circ G)(M)$ in the restricted region $0 \leq p+q \leq n$. $\square$

Lemma 4.4. *Let M be a right A-module. Then for any integers $n, q \geq 0$,*

$$R^q \operatorname{Tor}_n^A(M, -) \cong \operatorname{Ext}_A^q(\operatorname{Ext}_{A^{op}}^n(M, A), -)$$

where $R^q \operatorname{Tor}_n^A(M, -)$ denotes the right derived functor of the additive functor $\operatorname{Tor}_n^A(M, -)$. In particular, $R^q(\mathrm{D}(A) \otimes_A -)(N) \cong \operatorname{Ext}_A^q(\mathrm{Hom}_{A^{op}}(\mathrm{D}(A), A), N)$ for any left A-module N.

Proof. By [3, Theorem 2.8], there is an exact sequence of functors

$$0 \to \operatorname{Ext}^1_A(\mathrm{D}\,\Omega^n M, -) \to \operatorname{Tor}^A_n(M, -) \to \operatorname{Hom}_A(\operatorname{Ext}^n_{A^{op}}(M, A), -) \\ \to \operatorname{Ext}^2_A(\mathrm{D}\,\Omega^n M, -)$$

where $\Omega^n M$ denotes the n-th syzygy of M. So now we have that $\operatorname{Tor}^A_n(M, -)$ and $\operatorname{Hom}_A(\operatorname{Ext}^n_{A^{op}}(M, A), -)$ take the same values on injective modules. By definition of right derived functors of an additive functor, it follows that for any $q \geq 0$,

$$R^q \operatorname{Tor}^A_n(M, -) \cong \operatorname{Ext}^q_A(\operatorname{Ext}^n_{A^{op}}(M, A), -).$$

□

Theorem 4.5. *Let A be an algebra with* $\operatorname{domdim} A \geq 2$*, and let M be an A-module. Let $n \geq 2$ be a non-negative integer. If* $\operatorname{domdim} M \geq n$*, then we have the (restricted) first quadrant Grothendieck spectral sequence:*

$$E_2^{p,q} = \operatorname{Ext}^p_A(\mathrm{D}(A), \operatorname{Ext}^q_A(\operatorname{Hom}_{A^{op}}(\mathrm{D}(A), A), M)) \\ \underset{p}{\Longrightarrow} R^{p+q}\Gamma(M), \qquad 0 \leq p + q \leq n - 1.$$

Proof. Let $0 \to M \to I^0 \to I^1 \to \cdots$ be a minimal injective resolution of M. By definition, $\operatorname{domdim} M \geq n$ implies that $I^0, \ldots, I^{n-1}$ are projective. Hence $\mathrm{D}(A) \otimes_A I^i$ are injective, and particularly $\operatorname{Hom}_A(\mathrm{D}(A), -)$-acyclic for $0 \leq i \leq n-1$. By Lemma 4.4 and Lemma 4.3, we are done. □

At first sight, Theorem 4.5 is weak from the practical point of view, though it fits the idea of characterizing dominant dimension by using only certain bimodules. Next, we give three applications to demonstrate how it is applied.

4.1. **Dominant dimension at least 2.** Though the dominant dimension of an algebra may have values from 0 to ∞, the strength of the theory on dominant dimension only shines when the dominant dimension is at least 2, as we have seen from the fundamental results like Theorem 2.3 and Theorem 2.2. However, to see whether an algebra has dominant dimension at least 2, there seem to be no good ways except using the definition itself, or Theorem 2.4 by detecting projective injective modules and verifying the double centralizer property, or Theorem 2.2 by checking the left exactness of the double dual functor Γ. On the other hand, the theory of gendo-symmetric algebras and Morita algebras opens a new approach to the problem by investigating the canonical bimodules. Next, we follow this approach to give a characterization of dominant dimension at least 2 in terms of the bimodules, constructed from the left and right A-duals of A and $\mathrm{D}(A)$.

Proposition 4.6. *Let A be an algebra, and let V be the canonical A-bimodule. Then* $\operatorname{domdim} A \geq 1$ *if and only if there is an injective morphism of A-bimodules*

$$\Phi : A \to \operatorname{Hom}_A(\mathrm{D}(A), \operatorname{Hom}_A(V, A)).$$

Proof. If $\operatorname{domdim} A \geq 1$, let f be an idempotent in A such that fA is faithful and injective as an A^{op}-module. Let $\beta : \mathrm{D}(A) \to \operatorname{Hom}_A(\operatorname{Hom}_{A^{op}}(\mathrm{D}(A), A), A)$ be the canonical A-bimodule morphism from the proof of Proposition 3.13. We define an A-bimodule morphism

$$\Phi : A \to \operatorname{Hom}_A(\mathrm{D}(A), \operatorname{Hom}_A(\operatorname{Hom}_{A^{op}}(\mathrm{D}(A), A), A))$$

such that $\Phi(1) = \beta$. Note that $\mathrm{Hom}_{A^{op}}(\mathrm{D}(A), A) \cong V$ as A-bimodules by Lemma 3.2. We show next that Φ is a monomorphism. Indeed, if $\Phi(a) = 0$ for some $a \in A$, then for any $d \in \mathrm{D}(A)$ and $v \in V$, $0 = \Phi(a)(d)(v) = \Phi(1)(d)(v)a = v(d)a$. Let

$$T = \sum_{v \in \mathrm{Hom}_{A^{op}}(\mathrm{D}(A),A)} \mathrm{Im}(v).$$

Then $T \cdot a = 0$. Note that both $\mathrm{D}(A)$ and A contain fA as a direct summand; therefore T contains fA as a submodule. As a result, T is a faithful right A-module, and $T \cdot a = 0$ implies $a = 0$.

Conversely, if there is an A-bimodule monomorphism

$$\Phi : A \to \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Hom}_A(V, A)),$$

we show next that $\operatorname{domdim} A \geq 1$. Let $\varphi = \Phi(1)$ and let f be a basic idempotent of A such that $\mathrm{add}(\mathrm{top}(_AAf)) = \mathrm{add}(\mathrm{soc}(_AA))$ and $f = f_1 + \cdots + f_m$ is a sum of pairwise orthogonal primitive idempotents of A. Since Φ is a monomorphism, for any non-zero element $s = f_i s \in \mathrm{soc}(_AA)$ with $1 \leq i \leq m$, we have $\Phi(s) = \varphi \cdot s \neq 0$. So there exists $d_0 \in \mathrm{D}(A)$ and $v_0 \in V$ such that

$$(\varphi \cdot s)(d_0)(v_0) = \varphi(d_0)(v_0)s = \varphi(d_0)(v_0)f_i s \neq 0.$$

Then $\varphi_i := \varphi(d_0)f_i : V \to Af_i$ is a non-zero A-module morphism and $\varphi_i(v_0) = \varphi(d_0)(v_0)f_i$ which does not belong to $\mathrm{rad}(A)$. Otherwise, $0 = \varphi(d_0)(v_0)f_i \cdot s = \varphi(d_0)(v_0)s$ contradicts our choice of d_0 and v_0. As a consequence, φ_i is a split epimorphism. Let ψ_i be an A-module morphism from Af_i to V such that $\varphi_i \circ \psi_i = \mathrm{Id}$, and let $p_i = \psi_i(f_i) \in V$. Then we obtain a right A-module morphism $\rho_i : \mathrm{D}(A) \to f_iA$ which is defined by

$$\rho_i(d) = \varphi(d)(p_i), \qquad \forall\, d \in \mathrm{D}(A).$$

To see ρ_i is well-defined, we observe that $p_i = \psi_i(f_i) = \psi_i(f_i^2) = f_i \cdot p_i$ and hence

$$\rho_i(d) = \varphi(d)(p_i) = \varphi(d)(f_i \cdot p_i) = f_i\varphi(d)(p_i) = f_i\rho_i(d) \in f_iA,$$

as $\varphi(d)$ is an A-module morphism, and for any $a \in A$,

$$\begin{aligned}\rho_i(d \cdot a) &= \varphi(d \cdot a)(p_i) = (a \cdot \varphi)(d)(p_i) = \Phi(a)(d)(p_i)\\ &= (\varphi \cdot a)(d)(p_i) = \varphi(d)(p_i)a = \rho_i(d)a.\end{aligned}$$

Since $\rho_i(d_0 \cdot f_i) = \varphi(d_0 \cdot f_i)(p_i) = (\varphi(d_0)f_i)(p_i) = \varphi_i(p_i) = \varphi_i \circ \psi_i(f_i) = f_i$, we obtain that ρ_i is a split epimorphism, and thus f_iA are injective right A-modules for $1 \leq i \leq m$. As a result, $\mathrm{D}(fA) = \bigoplus_{i=1}^m \mathrm{D}(f_iA)$ is a projective injective A-module, which is also faithful since $\mathrm{soc}(_AA) \subset \mathrm{add}_A(\mathrm{top}(Af)) = \mathrm{add}_A(\mathrm{soc}(\mathrm{D}(fA)))$. □

Recall from the proof of Proposition 3.13 that there are canonical A-bimodule morphisms α and β.

Theorem 4.7. *Let A be an algebra, and let V be the canonical A-bimodule. The following statements are equivalent.*

(1) $\operatorname{domdim} A \geq 2$.
(2) *There is an A-bimodule isomorphism $A \cong \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Hom}_A(V, A))$.*
(2)' *The A-bimodule morphism $\Phi : A \to \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Hom}_A(V, A))$ defined by $\Phi(1) = \beta$ is an isomorphism.*
(3) *There is an A-bimodule isomorphism $A \cong \mathrm{Hom}_{A^{op}}(\mathrm{D}(A), \mathrm{Hom}_{A^{op}}(V, A))$.*

(3)' *The A-bimodule morphism $\Psi : A \to \mathrm{Hom}_{A^{op}}(\mathrm{D}(A), \mathrm{Hom}_{A^{op}}(V, A))$ defined by $\Psi(1) = \alpha$ is an isomorphism.*

(4) $\mathrm{D}(A) \otimes_A V \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)$ *as A-bimodules.*

Proof. Observe that (2)', (3)' $\Rightarrow$ (4) $\Rightarrow$ (2), (3) are trivial by the tensor-hom adjunction. Since $\operatorname{domdim} A = \operatorname{domdim} A^{op}$, it suffices to prove (1) $\Rightarrow$ (2)' and (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2)' Since $\operatorname{domdim} A \geq 2$, we have by Theorem 4.5 and Lemma 3.2 that

$$A \cong R^0\Gamma(A) \cong E_\infty^{0,0} \cong E_2^{0,0} \cong \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Hom}_A(V, A)).$$

On the other hand, since $\operatorname{domdim} A \geq 1$, by the proof of Proposition 4.6, Φ is an injective morphism. Altogether, Φ is an isomorphism.

(2) $\Rightarrow$ (1) Assume that there is an A-bimodule isomorphism

$$\Phi : A \longrightarrow \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Hom}_A(V, A)).$$

By Proposition 4.6, we have $\operatorname{domdim} A \geq 1$, and thus there exist idempotents e and f in A such that Ae is a projective injective and faithful A-module and $\mathrm{D}(Ae) \cong fA$ as right A-modules. Consider an exact sequence

$$0 \longrightarrow {}_AA \longrightarrow {}_A(Ae)^{\oplus m} \longrightarrow {}_A\,\mathrm{D}(A)^{\oplus m'} \quad (m, m' \geq 1).$$

Applying $\mathrm{Hom}_A(V, -)$ to the sequence, we have the exact sequence of A-modules

$$0 \longrightarrow \mathrm{Hom}_A(V, A) \longrightarrow \mathrm{Hom}_A(V, Ae)^{\oplus m} \longrightarrow \mathrm{Hom}_A(V, \mathrm{D}(A))^{\oplus m'}.$$

Applying $\mathrm{Hom}_A(\mathrm{D}(A), -)$ to this sequence, we have the exact sequence of A-modules

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Hom}_A(V, A)) &\longrightarrow \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Hom}_A(V, Ae))^{\oplus m} \\ &\longrightarrow \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Hom}_A(V, \mathrm{D}(A)))^{\oplus m'}. \end{aligned}$$

Note that there are isomorphisms of A-modules

$$\mathrm{Hom}_A(V, Ae) \cong \mathrm{Hom}_{A^{op}}(\mathrm{D}(Ae), \mathrm{D}(V)) \cong \mathrm{Hom}_{A^{op}}(fA, \mathrm{D}(V)) \cong \mathrm{D}(V)f = \mathrm{D}(fV)$$

and isomorphisms of right A-modules

$$fV = \mathrm{Hom}_A(\mathrm{D}(A)f, A) = \mathrm{Hom}_A(\mathrm{D}(fA), A) \cong \mathrm{Hom}_A(Ae, A) \cong eA,$$

so that $\mathrm{Hom}_A(V, Ae) \cong \mathrm{D}(eA)$ as A-modules. As a result, in A-mod,

$$\begin{aligned} \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Hom}_A(V, Ae)) &\cong \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{D}(eA)) = \mathrm{Hom}_{A^{op}}(eA, A) \cong Ae, \\ \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Hom}_A(V, \mathrm{D}(A))) &\cong \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{D}(V)) \cong \mathrm{Hom}_{A^{op}}(V, A), \end{aligned}$$

and by using the A-bimodule isomorphism Φ, we get an exact sequence

$$0 \longrightarrow A \longrightarrow (Ae)^{\oplus m} \longrightarrow \mathrm{Hom}_{A^{op}}(V, A)^{\oplus m'}.$$

Applying $\mathrm{Hom}_{A^{op}}(-, A)$ to an epimorphism $A^{\oplus m''} \to V$ in A^{op}-mod, for some $m'' > 0$, we know that $\mathrm{Hom}_{A^{op}}(V, A)$ is isomorphic to a submodule of the free A-module $A^{\oplus m''}$, and therefore we obtain an exact sequence in A-mod,

$$0 \longrightarrow A \longrightarrow (Ae)^{\oplus m} \longrightarrow (Ae)^{\oplus n},$$

for some n, which shows that $\operatorname{domdim} A \geq 2$. $\square$

Example 6. In Theorem 4.7, we cannot replace ${}_A\operatorname{Hom}_A(V, A)$ by ${}_A\operatorname{Hom}_{A^{op}}(V, A)$, as the following example indicates. Let k be any field and A be the k-algebra defined by the quiver

$$\begin{tikzcd} 1 \arrow[d, "\delta"'] & 2 \arrow[l, "\alpha"'] \\ 4 \arrow[r, "\gamma"'] & 3 \arrow[u, "\beta"'] \end{tikzcd}$$

with relations $\beta\gamma\delta = 0, \alpha\beta\gamma = 0, \delta\alpha = 0$. Let $e = e_1 + e_3 + e_4$ and $f = e_1 + e_2 + e_3$. It is straightforward to check that $\operatorname{domdim} A = 2$ and Ae is a faithful injective A-module with $\mathrm{D}(Ae) \cong fA$ as right A-modules. Moreover, $V \cong eA \oplus \operatorname{rad}(e_1A)$ as right A-modules and $\operatorname{Hom}_{A^{op}}(V, A) \cong Ae \oplus S_1$ as A-modules, where S_1 denotes the simple head of Ae_1. Consequently,

$${}_A\operatorname{Hom}_A(\mathrm{D}(A), \operatorname{Hom}_{A^{op}}(V, A)) \cong Af \oplus \operatorname{rad}(Ae_1) \ncong {}_AA.$$

4.2. **Characterizing Morita algebras.** The following result shows that the converse of Corollary 3.14 holds.

Theorem 4.8. *Let A be an algebra of dominant dimension at least 2. Then the following statements are equivalent.*

(1) *A is a Morita algebra.*
(2) $\operatorname{Hom}_A(V, A) \cong \operatorname{Hom}_{A^{op}}(V, A)$ *as A-bimodules.*
(2)' $\mathrm{D}(A) \otimes_A V \cong V \otimes_A \mathrm{D}(A)$ *as A-bimodules.*
(3) $\operatorname{domdim} \operatorname{Hom}_A(V, A) \geq 2$ *and* $\operatorname{domdim}_{A^{op}} \operatorname{Hom}_{A^{op}}(V, A) \geq 2$.

Proof. (1) $\Rightarrow$ (2) follows from Corollary 3.14, and (2) $\Leftrightarrow$ (3) follows from Proposition 3.13, and (2) $\Leftrightarrow$ (2)' follows by the tensor-hom adjunction. To finish the proof, it suffices to show (2) $\Rightarrow$ (1). Since $\operatorname{Hom}_A(V, A) \cong \operatorname{Hom}_{A^{op}}(V, A)$ as A-bimodules, it follows by Theorem 4.7 that

$$\begin{aligned} A &\cong \operatorname{Hom}_A(\mathrm{D}(A), \operatorname{Hom}_A(V, A)) \cong \operatorname{Hom}_A(\mathrm{D}(A), \operatorname{Hom}_{A^{op}}(V, A)) \\ &\xrightarrow{\pi} \operatorname{Hom}_{A^{op}}(V, \operatorname{Hom}_A(\mathrm{D}(A), A)) = \operatorname{Hom}_{A^{op}}(V, V) \end{aligned}$$

as A-bimodules. Here π is the isomorphism sending $f \in \operatorname{Hom}_A(\mathrm{D}(A), \operatorname{Hom}_{A^{op}}(V, A))$ to $\pi(f) \in \operatorname{Hom}_{A^{op}}(V, \operatorname{Hom}_A(\mathrm{D}(A), A))$, such that $\pi(f)(v)(\delta) = f(\delta)(v)$ for any $\delta \in \mathrm{D}(A)$ and $v \in V$. In particular, V is a faithful A-module, and hence it follows from Definition 2.7(3) that A is a Morita algebra. $\square$

Example 7. The following example shows that the assumption $\operatorname{domdim} A \geq 2$ in Theorem 4.8 is necessary. Let k be any field and A be the k-algebra given by quiver

$$\begin{tikzcd} 1 \arrow[d, "\delta"'] & 2 \arrow[l, "\alpha"'] \\ 4 \arrow[r, "\gamma"'] & 3 \arrow[u, "\beta"'] \end{tikzcd}$$

and relations $\alpha\beta = 0, \delta\alpha = 0$. Then it is straightforward to check that $\operatorname{domdim} A = 1$, and

$${}_AV \cong Ae_2 \oplus Ae_1 \oplus S_1 \oplus S_1, \qquad V_A \cong e_1A \oplus e_2A \oplus S_2' \oplus S_2',$$

and $\operatorname{Hom}_A(V, A) \cong \operatorname{Hom}_{A^{op}}(V, A) \cong V$ as A-bimodules. So $\operatorname{domdim}_A \operatorname{Hom}_A(V, A) = 2 = \operatorname{domdim}_{A^{op}} \operatorname{Hom}_{A^{op}}(V, A)$.

4.3. **Dominant dimension for Morita algebras.** Specializing Theorem 4.5 to Morita algebras, we obtain the following results.

Proposition 4.9. *Let A be an algebra with dominant dimension at least 2, and let V be the canonical A-bimodule. If V is projective as a right A-module, then for any A-module M, there is the first quadrant Grothendieck spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}_A^p(\mathrm{D}(A), \mathrm{Ext}_A^q(V, M)) \underset{p}{\Longrightarrow} R^{p+q}\Gamma(M), \quad \forall p, q \geq 0,$$

and a five term exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}_A^1(\mathrm{D}(A), \mathrm{Hom}_A(V, M)) \to R^1\Gamma(M) \to \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Ext}_A^1(V, M)) \\ \to \mathrm{Ext}_A^2(\mathrm{D}(A), \mathrm{Hom}_A(V, M)) \to R^2\Gamma(M). \end{aligned}$$

Proof. Note that $\mathrm{D}(V) \cong \mathrm{D}(A) \otimes_A \mathrm{D}(A)$ as A-bimodules. If V is projective as a right A-module, then $\mathrm{D}(A) \otimes_A \mathrm{D}(A)$ is injective as an A-module. Consequently, for any injective A-module I, $\mathrm{D}(A) \otimes_A I$ is again injective in A-mod, hence $\mathrm{Hom}_A(\mathrm{D}(A), -)$-acyclic. Now by Lemma 4.2, $\Gamma = \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{D}(A) \otimes_A -)$ and by Lemma 4.4

$$R^q(\mathrm{D}(A) \otimes_A -) \cong \mathrm{Ext}_A^q(\mathrm{Hom}_A(\mathrm{D}(A), A), -) = \mathrm{Ext}_A^q(V, -).$$

For any A-module M, we have by the Grothendieck spectral sequence [19] or Theorem 4.5,

$$E_2^{p,q} = \mathrm{Ext}_A^p(\mathrm{D}(A), \mathrm{Ext}_A^q(V, M)) \underset{p}{\Longrightarrow} R^{p+q}\Gamma(M) \qquad \forall p, q \geq 0,$$

and a five term exact sequence above as desired. □

Proposition 4.10. *Let A be an algebra of dominant dimension at least 2. If the canonical A-bimodule V is projective in both* A-mod *and* A^{op}-mod*, then for any A-module M,*

$$R^i\Gamma(M) \cong \mathrm{Ext}_A^i(\mathrm{D}(A), \mathrm{Hom}_A(V, M)), \qquad \textit{for all } i \geq 0.$$

In particular, $\Gamma(M) \cong \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Hom}_A(V, M))$, and for an integer $n \geq 2$, $\mathrm{domdim}\, M \geq n$ *if and only if* $\mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Hom}_A(V, M)) \cong M$ *canonically, and*

$$\mathrm{Ext}_A^i(\mathrm{D}(A), \mathrm{Hom}_A(V, M)) = 0, \qquad \textit{for } i = 1, 2, \ldots, n-2.$$

Proof. If V is projective as both a left and a right A-module, then by Proposition 4.9, we have the Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_A^p(\mathrm{D}(A), \mathrm{Ext}_A^q(V, M)) \underset{p}{\Longrightarrow} R^{p+q}\Gamma(M),$$

which degenerates to $\mathrm{Ext}_A^i(\mathrm{D}(A), \mathrm{Hom}_A(V, M)) \cong R^i\Gamma(M)$ for any A-module M and $i \geq 0$. In particular, $\Gamma(M) \cong \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Hom}_A(V, M))$ canonically. Note that by Theorem 4.1, $M \cong \Gamma(M)$ and $R^i\Gamma(M) = 0$ for $1 \leq i \leq n-2$ if and only if $\mathrm{domdim}\, M \geq n$. We are done. □

As an application, we obtain [11, Proposition 3.3] as a corollary.

Corollary 4.11. *If $\mathrm{Hom}_A(\mathrm{D}(A), A) \cong A$ as A-bimodules, then for any A-module M,*

$$R^i\Gamma(M) \cong \mathrm{Ext}_A^i(\mathrm{D}(A), M), \quad \textit{for all } i \geq 0.$$

In particular, $\Gamma(M) \cong \mathrm{Hom}_A(\mathrm{D}(A), M)$, and for an integer $n \geq 2$, $\mathrm{domdim}\, M \geq n$ *if and only if $\mathrm{Hom}_A(\mathrm{D}(A), M) \cong M$ and $\mathrm{Ext}_A^i(\mathrm{D}(A), M) = 0$ for $i = 1, 2, \ldots, n-2$.*

Proof. If $V \cong A$ as A-bimodules, then $\mathrm{Hom}_A(V, M) \cong M$ as A-module. Proposition 4.10 then specializes to the statement of the corollary. $\square$

In particular, we have the following proposition for Morita algebras.

Proposition 4.12. *Let A be a Morita algebra, and let $V = \mathrm{Hom}_A(\mathrm{D}(A), A)$. Let $n \geq 2$ be an integer, and let M be a left A-module. If V is projective as a left A-module, then* $\operatorname{domdim} M \geq n$ *if and only if $M \cong \Gamma(M)$ canonically and* $\mathrm{Ext}^i_A(\mathrm{D}(A), M) = 0$ *for* $1 \leq i \leq n-2$.

Proof. Let e_0 be an idempotent in A such that $A_0 := e_0Ae_0$ is a basic algebra and A_0 is Morita equivalent to A. If V is projective as a left A-module, then $V_0 := \mathrm{Hom}_{A_0}(\mathrm{D}(A_0), A_0)$ is projective as a left and a right A_0-module by Proposition 3.3 and Proposition 3.4. As a result, $V_0 \cong (A_0)_\sigma$ for some automorphism σ of A_0 by Corollary 3.7. Note that A being a Morita algebra implies $\operatorname{domdim} A \geq 2$ by Definition 2.7(3), and $\operatorname{domdim} M \geq n$ if and only if $\operatorname{domdim}_{A_0} e_0M \geq n$. Therefore, by Proposition 4.10, $\operatorname{domdim} M \geq n$ if and only if

$$e_0M \cong e_0\Gamma(M), \qquad \mathrm{Ext}^i_{A_0}(\mathrm{D}(A_0), e_0M) = 0 \quad \text{for} \quad 1 \leq i \leq n-2.$$

In fact, we have the canonical isomorphisms of A_0-modules:

$$\begin{aligned} e_0M \cong \Gamma(e_0M) &= \mathrm{Hom}_{A_0^{op}}(\mathrm{Hom}_{A_0}(e_0M, A_0), A_0) \\ &\cong \mathrm{Hom}_{A^{op}}(\mathrm{Hom}_{A_0}(e_0M, e_0A), e_0A) \\ &\cong \mathrm{Hom}_{A^{op}}(\mathrm{Hom}_A(M, A), e_0A) \cong e_0\Gamma(M), \end{aligned}$$

since A_0 is Morita equivalent to A, and the isomorphisms of k-vector spaces:

$$\begin{aligned} \mathrm{Ext}^i_{A_0}(\mathrm{D}(A_0), \mathrm{Hom}_{A_0}(V_0, e_0M)) &\cong \mathrm{Ext}^i_{A_0}(\mathrm{D}(A_0), \mathrm{Hom}_{A_0}((A_0)_\sigma, e_0M)) \\ &\cong \mathrm{Ext}^i_{A_0}(\mathrm{D}(A_0), {}_\sigma(e_0M)) \\ &\cong \mathrm{Ext}^i_{A_0}({}_{\sigma^{-1}}\mathrm{D}(A_0), e_0M) \cong \mathrm{Ext}^i_{A_0}(\mathrm{D}(A_0), e_0M). \end{aligned}$$

Here the last isomorphism follows from the fact that ${}_{\sigma^{-1}}\mathrm{D}(A_0)$ is a basic injective cogenerator in A_0-mod, so that ${}_{\sigma^{-1}}\mathrm{D}(A_0) \cong \mathrm{D}(A_0)$ as A_0-modules. Note that $\mathrm{Ext}^i_{A_0}(\mathrm{D}(A_0), e_0M) \cong \mathrm{Ext}^i_A(\mathrm{D}(A)e_0, M) \cong e_0\,\mathrm{Ext}^i_A(\mathrm{D}(A), M)$ for all i. It follows that $\mathrm{Ext}^i_{A_0}(\mathrm{D}(A_0), e_0M) = 0$ if and only if $\mathrm{Ext}^i_A(\mathrm{D}(A), M) = 0$ for all $i \geq 0$. Hence $\operatorname{domdim} M \geq n$ if and only if $M \cong \Gamma(M)$ canonically and $\mathrm{Ext}^i_A(\mathrm{D}(A), M) = 0$. $\square$

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Academy of Mathematics and Systems Science, Chinese Academy of Sciences, 100190 – and – School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, 100049, People's Republic of China
E-mail address: fming@amss.ac.cn

Mathematisches Institut, Heinrich-Heine-Universität, 40225, Düsseldorf, Germany
E-mail address: kerner@math.uni-duesseldorf.de

Institute of Engineering, Tokyo University of Agriculture and Technology, Tokyo 184-8588, Japan
E-mail address: yamagata@cc.tuat.ac.jp