In this appendix we follow the notations, conventions, equation numbers, section numbers, lemma, proposition and theorem numbers as used in the main article [18], unless otherwise is specifically mentioned (for instance, those equation numbers starting with ‘A’).

§ A.1 A proof of Proposition 2.24.

Proof of the necessary part (⇒). Here we can take ζₐ = ξₐ, and εₐ = λₐ, according to the analytic definition of simple blow-up point as in (2.1) and (2.2).

With the notations in (2.20), (2.26) is equivalent to

\[(A.1.1) \quad C^{-1} \cdot A_1(\mathcal{V}) \leq \mathcal{V}_i(\mathcal{Y}) \leq CA_1(\mathcal{V}) \quad \text{for} \quad |\mathcal{Y}| \leq \rho_o \lambda_m^{-1}.
\]

[Cf. (2.17) and (2.19).] It follows from (1.4) and \(A_1 = A_{1,0}\) that

\[(A.1.2) \quad \frac{1}{2^{n-2}} \cdot \frac{1}{|\mathcal{Y}|^{n-2}} \leq A_1(\mathcal{Y}) \leq \frac{1}{|\mathcal{Y}|^{n-2}} \quad \text{for} \quad |\mathcal{Y}| \geq 1 \quad [\text{i.e.} \quad \mathcal{Y} \notin B_o(1)].
\]

When \(i \gg 1\), \(|\lambda_m, \mathcal{Y}| \leq \rho_o \quad \implies \quad |\lambda_m, \mathcal{Y} + \xi_i| \leq \bar{\rho}_o\) for \(0 < \rho_o < \bar{\rho}_o\), From (2.19) and Proportionality Proposition 2.3 we obtain

\[(A.1.3) \quad \mathcal{V}_i(\mathcal{Y}) = \frac{v_i(\lambda_m, \mathcal{Y} + \xi_i)}{M_i} \leq \frac{1}{M_i^2} \cdot \frac{C_1}{|\lambda_m, \mathcal{Y} + \xi_i - \xi_i|^{n-2}} \leq \frac{1}{M_i^2} \cdot \frac{C_1}{\lambda_m \cdot |\mathcal{Y}|^{n-2}} \leq \frac{C_1}{|\mathcal{Y}|^{n-2}} \quad \text{for} \quad 0 < |\lambda_m, \mathcal{Y}| \leq \rho_o \quad \text{and} \quad i \gg 1.
\]
Here we use (2.4). As we already know that $\mathcal{V}_i(\mathcal{Y}) \rightarrow A_1(\mathcal{Y})$ uniformly for $\mathcal{Y} \in B_o(1)$, together with the above estimate and (A.1.2), we have

\begin{align}
(A.1.4) \quad \mathcal{V}_i(\mathcal{Y}) \leq C A_1(\mathcal{Y}) \quad \text{for } i \gg 1 \text{ and } |\mathcal{Y}| \leq \rho_o \lambda_{m_i}^{-1}.
\end{align}

As for the lower bound in (A.1.1), from Proportionality Proposition 2.3, we have

\begin{align}
(A.1.5) \quad M_i \cdot v_i(y) \geq \frac{a}{|y|^{n-2}} + h(y) - o(1) \quad \text{for } 0 < \rho_1 \leq |y| \leq \rho_o.
\end{align}

Here $o(1) \rightarrow 0$ when $i \rightarrow \infty$. Choosing $\rho_o$ to be small enough (correspondingly adjusting $\rho_1 < \rho_o$), we obtain

\begin{align}
(A.1.6) \quad M_i \cdot v_i(y) \geq \frac{2^{-1} \cdot a}{|y - \xi_{m_i}|^{n-2}} \quad \text{for } 0 < \rho_1 \leq |y - \xi_{m_i}| \leq \rho_o \text{ and } i \gg 1.
\end{align}

Repeat the argument in (A.1.3), we have

\begin{align}
(A.1.7) \quad \mathcal{V}_i(\mathcal{Y}) \geq \frac{C_2}{|\mathcal{Y}|^{n-2}} \quad \text{for } \rho_1 \lambda_{m_i}^{-1} \leq |\mathcal{Y}| \leq \rho_o \lambda_{m_i}^{-1} \text{ and } i \gg 1.
\end{align}

Again using the uniform convergence $\mathcal{V}_i(\mathcal{Y}) \rightarrow A_1(\mathcal{Y})$ for $\mathcal{Y} \in B_o(1)$, and (A.1.2), we obtain

\begin{align}
(A.1.8) \quad \mathcal{V}_i(\mathcal{Y}) \geq \frac{C_3}{|\mathcal{Y}|^{n-2}} \quad \text{for } |\mathcal{Y}| = 1.
\end{align}

Let us pay attention to the region $1 \leq |\mathcal{Y}| \leq \rho_o \lambda_{m_i}^{-1}$ and consider the function

\begin{align}
(A.1.9) \quad \left( \mathcal{V}_i(\mathcal{Y}) - \frac{C_4}{|\mathcal{Y}|^{n-2}} \right).
\end{align}

Choose $C_4 = \min \{ C_2, C_3 \}$. It follows from equation (2.18) that

\begin{align}
(A.1.10) \quad \Delta_o \left( \mathcal{V}_i(\mathcal{Y}) - \frac{C_4}{|\mathcal{Y}|^{n-2}} \right) = \Delta_o \mathcal{V}_i(\mathcal{Y}) < 0 \quad \text{for } 1 \leq |\mathcal{Y}| \leq \rho_o \lambda_{m_i}^{-1}.
\end{align}

For the boundary, we apply (A.1.7) and (A.1.8), then we use the maximum principle to obtain

\begin{align}
(A.1.11) \quad \mathcal{V}_i(\mathcal{Y}) \geq \frac{C_4}{|\mathcal{Y}|^{n-2}} \quad \text{for } 1 \leq |\mathcal{Y}| \leq \rho_o \lambda_{m_i}^{-1}.
\end{align}

As before, we already know that $\mathcal{V}_i(\mathcal{Y}) \rightarrow A_1(\mathcal{Y})$ for $\mathcal{Y} \in B_o(1)$. Combining with the above estimate and (A.1.2), we obtain

\begin{align}
\mathcal{V}_i(\mathcal{Y}) \geq C^{-1} \cdot A_1(\mathcal{Y}) \quad \text{for } i \gg 1 \text{ and } |\mathcal{Y}| \leq \rho_o \lambda_{m_i}^{-1}.
\end{align}
once we choose $C$ to be small enough. This complete the proof of $(\implies)$.

Proof of the sufficient part $(\iff)$. Assuming that we have (2.26), that is,

$$(A.1.12) \quad \frac{1}{C} \cdot \left( \frac{\epsilon_i}{\epsilon_i + |y - \zeta_i|^2} \right)^{\frac{n-2}{2}} \leq v_i(y) \leq C \cdot \left( \frac{\epsilon_i}{\epsilon_i + |y - \zeta_i|^2} \right)^{\frac{n-2}{2}}$$

for $|y - \zeta_i| \leq \rho_o$. It becomes clear that for $i \gg 1$, there is a point $\xi_i$ such that

$$(A.1.13) \quad v_i(\xi_i) = \max \left\{ v_i(y) \mid y \in B_{\rho_o}(\rho_o) \right\} \quad \text{and} \quad |\zeta_i - \xi_i| \leq B \cdot \epsilon_i.$$  

Here $B$ is a fixed positive number [one can take $B^2 = C \cdot \frac{4}{n-2} - 1$, where $C$ is the constant in (A.1.12)]. Moreover,

$$(A.1.14) \quad \lambda_i := \frac{1}{[v_i(\xi_i)]^{\frac{n-2}{2}}} \implies c^{-1} \cdot \epsilon_i \leq \lambda_i \leq c \cdot \epsilon_i \quad \text{for} \quad i \gg 1.$$  

Here $c \geq 1$ is a constant. In addition, via the triangle inequality $|y - \xi_i| \leq |y - \zeta_i| + |\zeta_i - \xi_i|$, and vice versa, we have

$$(A.1.15) \quad (A.1.13) \text{ and } (A.1.14) \implies \frac{1}{D} \leq \frac{\epsilon_i^2 + |y - \zeta_i|^2}{\lambda_i^2 + |y - \xi_i|^2} \leq D \quad \text{for} \quad y \in \mathbb{R}^n.$$  

Here we can take the constant $D = c^2 (B^2 + 1) = c^2 \cdot C \cdot \frac{4}{n-2}$. Thus

$$(A.1.16) \quad \frac{1}{C'} \cdot A_{\lambda_i, \xi_i}(y) \leq v_i(y) \leq C' \cdot A_{\lambda_i, \xi_i}(y) \quad \text{for} \quad |y - \xi_i| \leq \rho_1,$$

where $\rho_1 > 0$ is slightly less than $\rho_o$. Thus, without loss of generality, we may assume that

$$(A.1.17) \quad \zeta_i = \xi_i \quad \text{and} \quad \epsilon_i = \lambda_i \quad \text{for} \quad i \gg 1.$$  

Directly,

$$(A.1.18) \quad \left( \frac{\lambda_i}{\lambda_i^2 + |y - \zeta_i|^2} \right)^{\frac{n-2}{2}} \leq \frac{1}{|y - \zeta_i|^{\frac{n-2}{2}}} \implies v_i(y) \leq \frac{C}{|y - \zeta_i|^{\frac{n-2}{2}}}$$

for $0 < |y - \zeta_i| \leq \rho_1$. That is, 0 is an isolated blow-up point.

Next, consider the rescaled average

$$(A.1.19) \quad r \rightarrow \tilde{w}_i(r) := r^{\frac{n-2}{2}} \cdot \left[ \frac{\int_{B_{\tilde{\xi}_i}(r)} v_i \, dS}{\int_{B_{\tilde{\xi}_i}(r)} 1 \, dS} \right]$$
Via the change of variables $r = e^{-t}$, and (A.1.16), we have

\begin{equation}
(A.1.20) \quad \frac{1}{C'} \cdot \left[ \frac{1}{e^{(t-t_i)} + e^{-(t-t_i)}} \right]^{\frac{n-2}{2}} \leq \tilde{w}_i(t) \leq C' \cdot \left[ \frac{1}{e^{(t-t_i)} + e^{-(t-t_i)}} \right]^{\frac{n-2}{2}}
\end{equation}

for $r = e^{-t} \leq \rho \iff t \geq T_1$, where $T_1 := -\ln \rho_1$.

In the above

\begin{equation}
(A.1.21) \quad e^{-t_i} = \lambda_i \iff t_i = -\ln \lambda_i \quad \text{for } i = 1, 2, \ldots (t_i \to \infty \text{ as } i \to \infty).
\end{equation}

By performing a blow-up analysis as in Theorem 4.2 in [?] (cf. also §7c in [15]), and using (A.1.20), we obtain

\begin{equation}
(A.1.22) \quad \tilde{w}_i(t) = \bar{w}_i(t + t_i) \to \left[ \frac{1}{e^t + e^{-t}} \right]^{\frac{n-2}{2}} = \frac{1}{2^{\frac{n-2}{2}}} \cdot \left( \frac{1}{\cosh t} \right)^{\frac{n-2}{2}}.
\end{equation}

The convergence is in $C^2$-sense, uniform on any given bounded interval in $\mathbb{R}^d$. Directly,

\begin{equation}
(A.1.23) \quad \frac{d}{dt} \left( \frac{1}{\cosh t} \right)^{\frac{n-2}{2}} = 0 \quad \text{iff} \quad t = 0, \quad \frac{d^2}{dt^2} \left( \frac{1}{\cosh t} \right)^{\frac{n-2}{2}} \leq -c^2 < 0 \quad \text{for } |t| \leq \delta.
\end{equation}

Here $c$ is a constant which depends on the small number $\delta > 0$. It follows that $\tilde{w}_i$ has only one critical point in $[-\delta, \delta]$ for all $i \gg 1$. Likewise, the first statement in (A.1.23) shows that $\tilde{w}_i$ has no critical point in $[-T_2, T_2] \setminus [-\delta, \delta]$ for $i \gg 1$. Here $T_2 \in [-T_1, \infty)$ is a (fixed) large positive number. Any potential critical point in $[-(t_i - T_1), \infty) \setminus [-T_2, T_2]$ can be ruled out by using (A.1.20), together with Lemma 5.1 in [8] (cf. also Lemma 7.16 and Lemma 7.25 in [15], and the proof of Theorem 4.1 in [?]). Note that (A.1.20) implies

\begin{equation}
(A.1.24) \quad C^{-1} \cdot \left[ \frac{1}{\cosh t} \right]^{\frac{n-2}{2}} \leq \tilde{w}_i(t) \leq C \cdot \left[ \frac{1}{\cosh t} \right]^{\frac{n-2}{2}} \quad \text{for } t \in [-T_1, \infty).
\end{equation}

Any “small” critical value for $\tilde{w}_i$ comes from a local minimum, and according to (5.3) in [8] and Lemma 5.1 (loc. cit.), $\tilde{w}_i$ has to increase in either direction, which eventually contradicts (A.1.24). Via a translation back to $\bar{w}_i$ as defined in (A.1.19), it follows that, for each $i \gg 1$, $\tilde{w}_i(r)$ has only one critical point (around $r_i := e^{-t_i}$) for $r \in (0, \rho_1)$. This completes the checking that 0 is a simple blow-up point for \{v_i\}. □
§ A.2 Shifting to the maximal point.

From § A.1, in particular, (A.1.13) and (A.1.17), we can take \( \xi_{m_i} = \xi_i \) in (2.1) and (2.2) in the definition of simple blow-up points. Suppose there exists another sequence of points \( \{\tilde{\xi}_i\} \) which also satisfies (2.1) and (2.2). We show that (modulo a subsequence)

\[
\text{(A.2.1)} \quad |\tilde{\xi}_i - \xi_i| = o(\lambda_i)
\]

The proof, which requires only standard argument, can be readily recognized by people working on the area. For the benefit of general readers, we present the argument, and refer to available papers for selected technical details. Set

\[
\text{(A.2.2)} \quad \tilde{\lambda}_i = \frac{1}{[v_i(\tilde{\xi}_i)]^{\frac{n-2}{2}}} \implies \tilde{\lambda}_i \geq \lambda_i \left( = \frac{1}{[v_i(\xi_i)]^{\frac{n-2}{2}}} \right).
\]

To receive a contradiction, suppose

\[
\text{(A.2.3)} \quad \tilde{\lambda}_i^{-1} \cdot |\tilde{\xi}_i - \xi_i| \to \infty \quad \text{(modulo a subsequence)}.
\]

As in the proof of Lemma 3.10 in [15], both \( \{\tilde{\xi}_i\} \) and \( \{\xi_i\} \) can be used as in (2.17) to form (distinct) bubbling sequences (see [15]), contradicting that the blow-up is isolated (Proposition 3.32 in [15]). Hence there is a positive constant \( B \) such that

\[
\text{(A.2.4)} \quad \tilde{\lambda}_i^{-1} \cdot |\tilde{\xi}_i - \xi_i| \leq B \quad \text{for} \quad i \gg 1.
\]

As in § 2d (see also the proof of Lemma 3.10 in [15]),

\[
\text{(A.2.5)} \quad \tilde{V}_i(Y) := \frac{v_i(\tilde{\lambda}_i Y + \tilde{\xi}_i)}{v_i(\tilde{\xi}_i)} \to A_1(Y) \quad \text{in} \quad C^1\text{-sense}, \quad \text{uniformly for} \quad Y \in B_o(R).
\]

Once we take \( R \) to be large enough, the point \( Y_i \) defined by

\[
\text{(A.2.6)} \quad \tilde{\lambda}_i Y_i + \tilde{\xi}_i = \xi_i \quad \text{satisfies} \quad |Y_i| \leq B < R/2.
\]

Hence \( \tilde{V}_i \) has a critical point at \( Y = Y_i \) for \( i \gg 1 \). In particular, \( \triangledown \tilde{V}_i(Y_i) = 0 \) for \( i \gg 1 \). On the other hand,

\[
\text{(A.2.7)} \quad \min \{|\triangledown A_1(Y)| : \delta \leq |Y| \leq R\} \geq c_{\delta, R}^2 > 0.
\]

Here \( c_{\delta, R}^2 \) is a positive number depending on \( \delta \) and \( R \) (\( c_{\delta, R}^2 \to 0 \) as \( \delta \to 0^+ \)). The convergence in (A.2.5) together with (A.2.6) and (A.2.7) shows that \( |Y_i| \leq \delta \).

Moreover, we can let \( \delta \to 0 \) as \( i \to \infty \). Hence

\[
\text{(A.2.8)} \quad |\tilde{\xi}_i - \xi_i| = o(\tilde{\lambda}_i).
\]
Using again the convergence in (A.2.5) and (A.2.8), we have

\[ \lambda_i \leq \tilde{\lambda}_i \leq [1 + o(1)] \cdot \lambda_i \implies |\tilde{\xi}_i - \xi_i| = o(\lambda_i). \]

In the above, \( o(1) \to 0^+ \) as \( i \to \infty \).

§ A.3 Proof of Lemma 4.11.

The first conclusion in (i) follows directly from definition (4.10), and (ii) from the limitation \( j \leq k \). As for the second and third conclusions in (i), observe that when \( \ell \) is odd, \( \Delta^{(b_\ell)} P_\ell \) is a degree one polynomial. If it is not equivalent to zero, then \( \Delta^{(b_\ell)} P_\ell = \sum c_j Y_{j} \in F(\mathcal{P}_\ell) \) as claimed.

Similarly, when \( \ell \) is even, \( \Delta^{(b_\ell)} P_\ell = c \neq 0 \) is a number, then \( c \mathcal{P}^2 \in F(\mathcal{P}_\ell) \). Their difference is also in \( F(\mathcal{P}_\ell) \).

As for (iii), we first observe that, via direct calculation, we have

\[(A.3.1) \ Y \cdot \nabla Q_l = l \cdot Q_l \] for any homogeneous polynomial with degree \( l \).

For \( j \geq 1 \), using the product formula

\[ \Delta_o (f \cdot g) = f \cdot (\Delta_o g) + 2 \langle \nabla f, \nabla g \rangle + g \cdot (\Delta_o f), \]

we obtain

\[ (A.3.2) \ \Delta_o [(\mathcal{R}^2)^j \Delta_o^k \mathcal{P}_\ell] = (\mathcal{R}^2)^j \Delta_o^{k+1} \mathcal{P}_\ell + A_{\ell,j,k} \cdot (\mathcal{R}^2)^{j-1} \Delta_o^k \mathcal{P}_\ell, \]

\[ (A.3.3) \ (\mathcal{R}^2) \Delta_o [(\mathcal{R}^2)^j \Delta_o^k \mathcal{P}_\ell] = (\mathcal{R}^2)^{j+1} \Delta_o^{k+1} \mathcal{P}_\ell + A_{\ell,j,k} \cdot (\mathcal{R}^2)^j \Delta_o^k \mathcal{P}_\ell, \]

\[ (A.3.4) \ A_{\ell,j,k} = (2j) \cdot (2j + n - 2 + 2 \ell - 4k). \]

As \( j \leq k \implies (j + 1) \leq (k + 1) \), the terms which appear on the right hand side above belong to \( F(\mathcal{P}_\ell) \). \( \square \)
§ A.4 The case when $\Delta_{o}^{(h_{\ell})} \mathcal{P}_{\ell} \neq 0$.

For $\ell \leq n - 2$, where $n$ is even, we discuss how to eliminate the condition $\Delta_{o}^{(h_{\ell})} \mathcal{P}_{\ell} = 0$ by adding higher order (up to $n$-th order) terms. We start with

\begin{equation}
(1 + \mathcal{R}^{2}) \Delta_{o} (\mathcal{R}_{\ell}) = 2n [\mathcal{Y} \cdot \nabla (\mathcal{R}_{\ell})] + 2n (\mathcal{R}_{\ell})
\end{equation}

\begin{equation}
= \left( [1 + \mathcal{R}^{2}] \Delta_{o} - 2n \mathcal{R} \frac{\partial}{\partial \mathcal{R}} + 2n \right) \mathcal{R}_{\ell} = (\ell - 2) (\ell - n) \mathcal{R}_{\ell} + \ell (\ell + n - 2) \mathcal{R}_{\ell}^{2}.
\end{equation}

In particular, when $\ell = 2$ or $n$, we have

\begin{equation}
(1 + \mathcal{R}^{2}) \Delta_{o} (\mathcal{R}_{n}) = 2n [\mathcal{Y} \cdot \nabla (\mathcal{R}_{n})] + 2n (\mathcal{R}_{n}) = 0 \cdot \mathcal{R}_{n} + [2n (n - 1)] \mathcal{R}_{n}^{2},
\end{equation}

\begin{equation}
(1 + \mathcal{R}^{2}) \Delta_{o} (\mathcal{R}^{2}) = 2n [\mathcal{Y} \cdot \nabla (\mathcal{R}^{2})] + 2n (\mathcal{R}^{2}) = 0 \cdot \mathcal{R}^{2} + [2n].
\end{equation}

Consider finding a radial function $F(r)$ so that

\begin{equation}
(1 + \mathcal{R}^{2}) \Delta_{o} F(r) = 2n [\mathcal{Y} \cdot \nabla F(r)] + 2n F(r)
\end{equation}

\begin{equation}
= - [\Delta_{o}^{h_{\ell}} \mathcal{P}_{\ell}] \cdot \left\{ a_{o} + a_{1} \cdot (\mathcal{R}^{2}) + \cdots + a_{h_{\ell} - 1} \cdot (\mathcal{R}^{2})^{h_{\ell} - 1} + a_{h_{\ell}} \cdot (\mathcal{R}^{2})^{h_{\ell}} \right\}.
\end{equation}

(i) We start with using a $(\mathcal{R}^{2})$ term to cancel the constant term. By (A.4.3) above, we won’t introduce any new $(\mathcal{R}^{2})$ term.

(ii) The $(\mathcal{R}^{2})$-term in the right hand side can be canceled by introducing an $(\mathcal{R}^{2})^{2}$-term [using (A.4.1), and $\ell (\ell + n - 2) \neq 0$]. By doing so, a new $(\mathcal{R}^{2})^{2}$-term is introduced to the right hand side.

(iii) The combined $(\mathcal{R}^{2})^{2}$-term in the right hand side can be canceled by introducing an $(\mathcal{R}^{2})^{3}$-term. By doing so, an $(\mathcal{R}^{2})^{3}$-term is introduced to the right hand side. The process goes on until we reach the $\mathcal{R}^{n-2}$ term ($n \geq 4$ is even). Introducing a $\mathcal{R}^{n}$-term cancels the $\mathcal{R}^{n-2}$-term, and via (A.4.2), it does not re-introduce itself to the right hand side (that is, $\mathcal{R}^{n}$ is not present).
Diagram A.4.5. The cancelation order from bottom upward (when \( n \) is even).

In summary, when \( n \geq 4 \), \( \ell \) even with \( \ell \leq n - 2 \), we can find a polynomial
(A.4.6)

\[
F (r) := \left[ \Delta^h o \mathcal{P}_1 \right] \left\{ B_1 \cdot (R^2)^1 + B_2 \cdot (R^2)^2 + \cdots + B_k \cdot (R^2)^k + \cdots + B_{\frac{n}{2}} \cdot (R^2)^{\frac{n}{2}} \right\},
\]

which satisfies (A.4.4), where
(A.4.7) \( B_1 = \frac{-a_o}{2n}, \ B_2 = -\frac{a_1}{4(4 + n - 2)}, \ B_3 = -\frac{a_2 + (4 - 2)(4 - n) \cdot B_2}{6(6 + n - 2)}, \)

\[\cdots, \ B_k = -\frac{a_{k - 1} + \left[ 2(k - 1) - 2 \right] \cdot \left[ 2(k - 1) - n \right] \cdot B_{k - 1}}{(2k) \left[ (2k) + n - 2 \right]} \quad \text{for } 3 \leq k \leq \frac{n}{2}.\]

In particular,
(A.4.8) \( B_{\frac{n}{2}} = -\frac{a_{\frac{n}{2} - 1} + (n - 4)(-2) \cdot B_{\frac{n}{2} - 1}}{n \left[ 2(n - 1) \right]} \).
**Proposition A.4.9.** For \( n \geq 4 \) and \( \ell \leq n - 2 \), both being even, let \( P_\ell \) be a homogeneous polynomial of degree \( \ell \). Then equation (4.3) has a solution given by

\[
\sum_{0 \leq j \leq k \leq h \leq \ell - 1} C_{k}^{j} \cdot (R^2)^j \left[ \Delta_o^{(k)} P_\ell \right] + F(r)
\]

where \( F \) is given in (A.4.6), (A.4.7) and (A.4.8).

Although Proposition A.4.9 allows us to find a solution of equation (4.3) without the condition \( \Delta_o^{(h_{\ell})} P_\ell \equiv 0 \), the presence of an order \( n \) term (that is, \( R^n \)) hinders the application of the second order blow-up argument, cf. § 6 b.6.

**Remark on uniqueness.** Let \( \Gamma_a \) and \( \Gamma_b \) be two polynomial solutions to equation (4.3), with maximum degrees \( < n \). Via Theorem 4.16, [mindful of condition (4.18) which requires the maximum degrees \( < n \)], we have

\[
(A.4.10) \quad [\Gamma_a - \Gamma_b](\mathcal{Y}) = c_0 (1 - R^2) + \sum c_j \mathcal{Y}_j \quad (R = |\mathcal{Y}|).
\]

Thus when \( \ell < n \), the solution found in Proposition 4.49 are “unique” in the case of \((A.4.10)\). This cannot be extended to the solution found in Proposition A.4.9, even though \( \ell < n \), as the presence of \( R^n \)-term voids the application of Theorem 4.16. (For a generic \( n \), \( B_{\frac{n}{2}} \neq 0 \).)
§ A.5 Bounds on the Green function on \( B_o(a) \) with a point not too close to the boundary.

The Green’s function for \( \Delta_o \) on \( B_o(a) \) is given by

\[
G(y, \xi) = -\frac{1}{(n-2)\|S^{n-1}\|} \left[ \frac{1}{|y - \xi|^{n-2}} - \left( \frac{a}{|\xi|} \right)^{n-2} \frac{1}{|y - \xi^*|^{n-2}} \right],
\]

where \( \xi^* \) is the reflection of the point \( \xi \) upon the sphere \( \partial B_o(a) \), given by

\[
(\text{A.5.2}) \quad \xi^* = \frac{a^2}{|\xi|^2} \cdot \xi \quad \Rightarrow \quad |\xi^*| = \frac{a^2}{|\xi|} \quad \text{for} \quad \xi \in B_o(a) \setminus \{0\}.
\]

See for example [23]. Here \( \|S^{n-1}\| \) denotes the volume/measure of \( S^{n-1} \) with the standard metric. In order to obtain the bound in (6.40), we need only to consider the second term in (A.5.1). Let

\[
|\xi| = \gamma a \quad \Rightarrow \quad |\xi^*| = \frac{a}{\gamma} (> a), \quad \text{where} \quad \gamma \leq (1 - \delta).
\]

It follows that, for \( |\xi| \leq (1 - \delta) a \),

\[
(\text{A.5.3}) \quad \left( \frac{a}{|\xi|} \right)^{n-2} \cdot \frac{1}{|y - \xi^*|^{n-2}} \leq \left( \frac{1}{\gamma} \right)^{n-2} \cdot \frac{1}{|a - \frac{\gamma}{\gamma}|^{n-2}} = \left( \frac{1}{1 - \gamma} \right)^{n-2} \cdot \frac{1}{a^{n-2}}
\]

\[
\leq \frac{1}{\delta^{n-2}} \cdot \frac{2^{n-2}}{|y - \xi|^{n-2}} \quad \text{(since} \quad |\xi - y| \leq 2a).\]

We obtain

\[
|G(y, \xi)| \leq \left[ C_1 + \frac{C_2}{\delta^{n-2}} \right] \cdot \frac{1}{|y - \xi|^{n-2}} \quad \text{for} \quad |\xi| \leq (1 - \delta) a \quad \text{and} \quad y \in B_o(a) \setminus \{\xi\}.
\]

As for the bound in (6.41), we note that the term

\[
n \cdot \nabla_y G_i(y, \xi)
\]

is indeed the Poisson kernel for \( \Delta_o \) on \( B_o(a) \), which is given by (e.g. [23] pp. 116)

\[
(\text{A.5.4}) \quad \frac{1}{a \|S^{n-1}\|} \cdot \frac{a^2 - |\xi|^2}{|y - \xi|^n} \quad \text{for} \quad y \in \partial B_o(a)
\]

\[
\leq \frac{a^2}{a \|S^{n-1}\| \cdot (\delta a)^n} \quad \text{for} \quad |\xi| \leq (1 - \delta) a \quad \text{and} \quad |y| = a
\]

\[
\leq \frac{C'}{\delta^n} \cdot \frac{1}{a^{n-1}}.
\]
§ A.6 Balance and cancelation.

We first describe the classic balance formula for equation (1.2), due to Stanislav I. Pohozaev. See for examples [21] and [15].

**Theorem A.6.1.** Let $v$ and $K$ satisfy the general conditions in (1.27) and (1.28). We have the following.

**(I)** Global Pohozaev’s identity.

\[ \int_{\mathbb{R}^n} \langle y, \nabla K(y) \rangle [v(y)]^{\frac{2n}{n-2}} dy = 0. \]  

**(II)** Mezzo-scale Pohozaev’s identity. For a fixed number $\rho_0 > 0$, we have

\[ \int_{B_0(\rho_0)} \langle y, \nabla K(y) \rangle [v(y)]^{\frac{2n}{n-2}} dy = \frac{1}{c_n} \cdot \frac{2n}{n-2} \int_{\partial B_0(\rho_0)} \langle \tilde{V}, n \rangle dS, \]

\[ \tilde{V}(y) = \frac{n-2}{2} v(y) \nabla v(y) - \frac{\nabla v(y)}{2} y + [\langle y, \nabla v(y) \rangle] \nabla v(y) \]

\[ + \frac{n-2}{2n} \cdot \tilde{c}_n \cdot \left\{ [v(y)]^{\frac{2n}{n-2}} K(y) \right\} y. \]

In (A.6.3) and (A.6.4), $y$ is treated as a vector, and $n$ is the unit outward normal on $\partial B_0(\rho_0)$.

**§ A.6.a. Order of vanishing outside the blow-up points.** Observe that, via (A.1.2) and gradient estimate [?], we obtain

\[ \max_{\partial B_{\rho_0}(\rho_0)} \langle y, \nabla v_i(y) \rangle \cdot v_i = O_\lambda(n-2), \]

\[ \max_{\partial B_{\rho_0}(\rho_0)} |\nabla v_i|^2 = O_\lambda(n-2), \]

\[ \max_{\partial B_{\rho_0}(\rho_0)} |\nabla v_i| \cdot v_i = O_\lambda(n-2), \]

\[ \max_{\partial B_{\rho_0}(\rho_0)} [v(y)]^{\frac{2n}{n-2}} |K(y)| \cdot |v| = O_\lambda(2n) \]

\[ \text{(A.6.5)} \implies \int_{B_{\rho_0}(\rho_0)} \langle y, \nabla K(y) \rangle [v_i(y)]^{\frac{2n}{n-2}} dy = O_\lambda(n-2). \]

See also [15]. Whereas from (2.12) and (2.19), we have

\[ \left| \int_{\mathbb{R}^n \setminus \Omega} \langle y, \nabla K(y) \rangle [v_i(y)]^{\frac{2n}{n-2}} dy \right| = O_\lambda(n) \]

\[ \text{(A.6.6)} \implies \int_{\Omega} \langle y, \nabla K(y) \rangle [v_i(y)]^{\frac{2n}{n-2}} dy = O_\lambda(n). \]
In the discussion,
\begin{equation}
A.6.7 \quad \Omega = \bigcup_{j=0}^{k} B_{\hat{Y}_j}(\rho) \quad \left[ B_{\hat{Y}_j}(\rho) \cap B_{\hat{Y}_l}(\rho) = \emptyset \quad \text{for} \quad j \neq l \right],
\end{equation}
where (as usual) \( \{ \hat{Y}_o = 0, \ldots, \hat{Y}_k \} \) is the collection of all blow-up points.

\section{A.6.b. Linking the Pohozaev integral to the condition \( \Delta_o^{(h_\ell)} \mathbf{P}_\ell \equiv 0 \).

In the consideration of the integrals in the Pohozaev identities, we often encounter integral in the expression (A.6.9) below. We first record down the following observation.

\textbf{Lemma A.6.8.} For a homogeneous polynomial \( Q_\ell \) (defined on \( \mathbb{R}^n \)) of degree \( \ell \leq n - 1 \). If \( \ell \) is even, then the following equivalence holds.

\begin{equation}
A.6.9 \quad \int_{\mathbb{R}^n} Q_\ell (y) \cdot \left( \frac{1}{1 + |y|^2} \right)^n dy = 0 \iff \Delta_o^{(h_\ell)} Q_\ell = 0.
\end{equation}

(Recall that \( h_\ell = \ell/2 \) when \( \ell \) is even.)

\textbf{Proof.} We observe that, as \( \ell \leq n - 1 \), the integral in (A.6.9) is absolutely convergent. Keeping the notation \( y = (y_{i_1}, \ldots, y_{i_n}) \in \mathbb{R}^n \), consider a typical term in \( Q_\ell \):

\begin{equation}
A.6.10 \quad y_{i_1}^{\alpha_1} \cdot y_{i_2}^{\alpha_2} \cdots y_{i_n}^{\alpha_n}, \quad \text{where} \quad \alpha_j \geq 0 \quad \text{and} \quad \sum_{j=1}^{n} \alpha_j = \ell \leq n - 1.
\end{equation}

If one of the indices (say, \( \alpha_j \)) is an odd natural number, via symmetry, we have

\begin{equation}
A.6.11 \quad \int_{\mathbb{R}^n} \left[ y_{i_1}^{\alpha_1} \cdot y_{i_2}^{\alpha_2} \cdots y_{i_j}^{\alpha_j} \cdots y_{i_n}^{\alpha_n} \right] \cdot \left( \frac{1}{1 + |y|^2} \right)^n dy = 0 \quad (\alpha_j \text{ is odd}).
\end{equation}

Direct calculation also shows that in this situation

\[ \Delta_o^{(h_\ell)} \left[ y_{i_1}^{\alpha_1} \cdot y_{i_2}^{\alpha_2} \cdots y_{i_j}^{\alpha_j} \cdots y_{i_n}^{\alpha_n} \right] = 0 \quad (\alpha_j \text{ is odd}). \]

(Recall that \( \Delta_o^{(h_\ell)} Q_\ell \) is a number when \( \ell \) is even.) Thus we are left with the case where any one index in (A.6.10) is an even natural number or zero. Let us introduce the following notion: a multi-index

\[ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \quad \left( |\alpha| = \sum_{j=1}^{n} \alpha_j = \ell > 0 \right) \]
is even if each $\alpha_j$ ($1 \leq j \leq n$) is either an even natural number or zero. With respect to this, the simplest case that can happen to the integral in (A.6.11) is

\begin{equation}
J := \int_{\mathbb{R}^n} y_1^{2} \cdots y_n^{2} \left( \frac{1}{1+|y|^2} \right)^n dy.
\end{equation}

We seek to reduce other even multi-index cases to that in (A.6.12). As $\ell \leq n - 1$, we arrange in this way

$y_1^{k+2} \cdot y_2^{\alpha_2} \cdots y_{n-1}^{\alpha_{n-1}}$, where $\alpha = (k+2, \alpha_2, \ldots, \alpha_{n-1}, 0)$ is even.

Here $k \geq 2$ is an even number. Via symmetry, the ordering is not important when we compute the integral in (A.6.11). One obtains the following reduction formula.

\begin{equation}
\int_{\mathbb{R}^n} y_1^{k} \cdot y_2^{\alpha_2} \cdots y_{n-1}^{\alpha_{n-1}} \left( \frac{1}{1+|y|^2} \right)^n dy
= (k+1) \int_{\mathbb{R}^n} y_1^{k} \cdot y_2^{\alpha_2} \cdots y_{n-1}^{\alpha_{n-1}} \left( \frac{1}{1+|y|^2} \right)^n dy
\end{equation}

for $2 \leq k \leq n - 3$, by using Fubini’s theorem and integration by parts formula. See §A.8 below.

In view of (A.6.11) and (A.6.13), we introduce the following notation. For an integer $m \geq 0$, define

\begin{equation}
m!_{-2} = \begin{cases} 1 & \text{if } m = 0 \text{ or } 2; \\ 0 & \text{if } m \text{ is odd}; \\ (m-1)(m-3)(m-5) \cdots 3 \cdot 1 & \text{if } m \geq 4 \text{ is even}. \end{cases}
\end{equation}

Via the vanishing formula (A.6.11) and the reduction formula (A.6.13), we have

\begin{equation}
\int_{\mathbb{R}^n} y_1^{\alpha_1} \cdots y_n^{\alpha_n} \left( \frac{1}{1+|y|^2} \right)^n dy
= (\alpha_1)!_{-2} \times \cdots \times (\alpha_n)!_{-2} \cdot J.
\end{equation}

On the other side, calculation shows that

\begin{equation}
B := \Delta^{(h\ell)} \left[ y_1^2 \cdot y_2^2 \cdots y_{n-1}^2 \right] = \ell (\ell - 2) (\ell - 4) \cdots 2 \cdot 1.
\end{equation}

Claim. Let $\alpha_2, \ldots, \alpha_{n-1}$ be even natural numbers or zero, and

\begin{equation}
\ell = (k+2) + \alpha_2 + \cdots + \alpha_{n-1}, \text{ where } k \geq 2 \text{ is an even integer}.
\end{equation}

Then

\begin{equation}
\Delta^{(h\ell)} \left\{ y_1^{k+2} \cdot y_2^{\alpha_2} \cdots y_{n-1}^{\alpha_{n-1}} \right\} = (k+1) \cdot \Delta^{(h\ell)} \left\{ y_1^k \cdot y_2^2 \cdots y_{n-1}^{\alpha_{n-1}} \right\}.
\end{equation}
Refer to (A.6.17), set
\[ \lambda := \alpha_2 + \cdots + \alpha_{n-1}. \]

We demonstrate how to use induction on \( \lambda \) to prove the assertion in §A.8 in this e-Appendix. Thus using (A.6.18) repeatedly, we are led to
\[ \Delta^{(b_e)} \left[ y_1^{\alpha_1} \cdots y_n^{\alpha_n} \right] = (\alpha_1)!_{-2} \times \cdots \times (\alpha_n)!_{-2} \cdot B. \]

Using the linearity of the operations and symmetry, (A.6.15) and (A.6.20) yield
\[ \int_{\mathbb{R}^n} Q_\ell (y) \cdot \left( \frac{1}{1 + |y|^2} \right)^n \, dy = \frac{J}{B} \cdot [ \Delta^{(b_e)} Q_\ell ]. \]

In particular, we establish (A.6.9). \( \square \)

§ A.6. c. Change of center. With (6.57), let us write
\[ v_i (y) = A (y) + B (y) + C (y) \quad \text{for} \quad |y| \leq \rho_1, \]
where
\[ A (y) = \left( \frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2} \right)^{\frac{n-2}{2}}, \]
\[ B (y) = \lambda_i \cdot \left[ \lambda_\ell \cdot \Gamma_p \left( \frac{y - \xi_i}{\lambda_i} \right) \right] \cdot \left( \frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2} \right)^{\frac{n-2}{2}}, \]
\[ C (y) = \left[ v_i - A - B \right] (y) = o_{\lambda_i} \left( \ell - \frac{n - 2}{2} \right) + O_{\lambda_i} \left( \frac{n - 2}{2} \right). \]

For simplicity, we suppress the subindex \( i \) in \( A, B \) and \( C \). Because of the polynomial nature of \( \Gamma_p \), in general \( B \) is not rotationally symmetric, contrasting to \( A \). As this part of the discussion is used repeatedly in this article, we consider in general a homogeneous polynomial \( Q_\ell \) defined on \( \mathbb{R}^n \) with degree \( \ell \in [2, n-2] \). For \( \rho > 0 \), consider the integral
\[ \int_{B_\rho (\rho)} Q_\ell (y) \cdot [A (y)]^{\frac{2n}{n-2}} \, dy. \]

Here \( A \) is given in (A.6.23). We first observe that
\[ \left| \int_{B_\rho (\rho_2)} f (y) \cdot \left( \frac{\lambda_i}{\lambda_i^2 + |y|^2} \right)^n \, dy \right| - \int_{B_\ell (\rho_2)} f (y) \cdot \left( \frac{\lambda_i}{\lambda_i^2 + r^2} \right)^n \, dy \right| \]
\[ \leq \int_{B_\rho (\rho_2 + |\xi_i|) \setminus B_\rho (\rho_2 - |\xi_i|)} f (y) \cdot \left( \frac{\lambda_i}{\lambda_i^2 + r^2} \right)^n \, dy \quad = \quad O \left( |\xi_i| \cdot \lambda_i^n \right). \]
for $i \gg 1$. Here $f$ is a bounded continuous function defined on a slightly bigger ball, and $\xi_i$ continues to find its meaning in (1.13). In particular, $|\xi_i| \to 0$. It follows that

\[(A.6.27) \int_{B_{\rho}(\ell)} Q_{\ell}(y) \cdot [A(y)]^{\frac{2n}{\alpha}} dy = \int_{B_{\xi}(\rho)} Q_{\ell}(y) \cdot [A(y)]^{\frac{2n}{\alpha}} dy + o_{\lambda_i}(n).\]

Let us arrange

\[(A.6.28) Q_{\ell}(y) = Q_{\ell}(\xi_i + [y - \xi_i]) = Q_{\ell}(\xi_i) + M^Q(y - \xi_i) + Q_{\ell}(y - \xi_i).\]

Here the “intermediate” term $M^Q(y - \xi_i)$ can be further broken down into $\ell - 1$ terms based on the degree on $\xi_i$:

\[(A.6.29) M^Q(\xi_i; y - \xi_i) = \Xi^Q(\xi_i; y - \xi_i) = \sum_{|\alpha| = h} \frac{1}{\alpha!} \cdot \xi^\alpha D^{(h)}_{\alpha} Q_{\ell}(z)|_{z = (y - \xi_i)}\]

for $1 \leq h \leq \ell - 1$. We continue with

\[(A.6.30) \int_{B_{\rho}(\ell)} Q_{\ell}(y) \cdot [A(y)]^{\frac{2n}{\alpha}} dy = \int_{B_{\xi}(\rho)} Q_{\ell}(y) \left(\frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2}\right)^n dy\]

\[= \int_{B_{\xi}(\rho)} Q_{\ell}(y - \xi_i) \left(\frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2}\right)^n dy\]

\[+ \int_{B_{\xi}(\rho)} M^Q(\xi_i; y - \xi_i) \left(\frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2}\right)^n dy + \int_{B_{\xi}(\rho)} Q_{\ell}(\xi_i) \left(\frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2}\right)^n dy\]

\[= \int_{B_{\rho}(\rho)} Q_{\ell}(z) \left(\frac{\lambda_i}{\lambda_i^2 + |z|^2}\right)^n dz + \int_{B_{\rho}(\rho)} M^Q(\xi_i; z) \left(\frac{\lambda_i}{\lambda_i^2 + |z|^2}\right)^n dz + Q_{\ell}(\xi_i) \int_{B_{\rho}(\rho)} \left(\frac{\lambda_i}{\lambda_i^2 + |z|^2}\right)^n dz [z = (y - \xi_i)].\]

\[= \lambda_i^\ell \int_{B_{\rho}(\lambda_i^{-1}; \rho)} Q_{\ell}(y) \left(\frac{1}{1 + |y|^2}\right)^n dy + \]

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\[ + \lambda_i^\ell \left[ \sum_{h=1}^{\ell - 1} \int_{B_o(\lambda_i^{-1} \cdot \rho)} \Xi_h \left( \frac{\xi_i}{\lambda_i}, y \right) \left( \frac{1}{1 + |y|^2} \right)^n \, dy \right] + \]

\[ + \lambda_i^\ell \cdot \left[ Q_\ell \left( \frac{\xi_i}{\lambda_i} \right) \cdot \int_{B_o(\lambda_i^{-1} \cdot \rho)} \left( \frac{1}{1 + |y|^2} \right)^n \, dy \right] \quad (z \to \lambda_i \cdot y) . \]

### § A.6.c.1 Expressing the integrals on \( \mathbb{R}^n \)

Observe also that the above argument remains valid if we replace \( \rho \) by \( \infty \). Indeed, consider (in general) a homogeneous polynomial \( Q_k \) defined on \( \mathbb{R}^n \) with degree \( k \in [0, n - 2] \). For a sequence of positive numbers \( r_i \to \infty \):

\[
\left| \int_{\mathbb{R}^n \setminus B_o(r_i)} Q_k(y) \left( \frac{1}{1 + |y|^2} \right)^n \, dy \right| \leq C \int_{r_i}^{\infty} \frac{r^k}{r^{2n}} \cdot r^{n-1} \, dr \leq C \int_{r_i}^{\infty} \frac{1}{r^3} \, dr \leq \frac{C_2}{r_i^2},
\]

(A.6.32)

\[
\implies \int_{B_o(r_i)} Q_k(y) \left( \frac{1}{1 + |y|^2} \right)^n \, dy = \int_{\mathbb{R}^n} Q_k(y) \left( \frac{1}{1 + |y|^2} \right)^n \, dy + O(r_i^{-2}),
\]

for \( i \gg 1 \) and \( 0 \leq k \leq n - 2 \). Putting \( r_i = \lambda_i^{-1} \cdot \rho \), and combining (A.6.27), (A.6.31) and (A.6.32) with \( |\xi_i| \to 0 \), we obtain

\[
\int_{B_o(\rho)} Q_\ell(y) \cdot |A(y)|^{\frac{2n}{n-2}} \, dy = \lambda_i^\ell \int_{\mathbb{R}^n} Q_\ell(y) \left( \frac{1}{1 + |y|^2} \right)^n \, dy +
\]

\[
+ \lambda_i^\ell \left\{ \sum_{h=1}^{\ell - 1} \int_{\mathbb{R}^n} \Xi_h \left( \frac{\xi_i}{\lambda_i}, y \right) \cdot \left( \frac{1}{1 + |y|^2} \right)^n \, dy \right\} +
\]

\[
+ \lambda_i^\ell \cdot \left[ Q_\ell \left( \frac{\xi_i}{\lambda_i} \right) \cdot \int_{\mathbb{R}^n} \left( \frac{1}{1 + |y|^2} \right)^n \, dy \right] + O_{\lambda_i}(\ell + 2) + o_{\lambda_i}(n).
\]

### § A.6.d. \( \Delta^b_{\rho_o} P_\ell \equiv 0 \) when \( \ell \) is even and \( \ell < n - 2 \), or when \( \ell = n - 2 \) (\( n \) being even) and with only one simple blow-up point.

**Proposition A.6.34.** For \( n \geq 4 \), under the general conditions (1.6), (1.25), (1.26), assume that \( \{u_i\} \) has finite number of blow-up points, one at the south pole, but none at the north pole. Take the following conditions (i)–(iii) into account.

(i) \( 0 \) is a simple blow-up point for \( \{v_i\} \).

(ii) \( K \) is given by (1.8) in \( B_o(\rho_o) \), where \( 2 \leq \ell < n - 2 \).
(iii) The parameters $\lambda_i$ and $\xi_i$ corresponding to the simple blow-up point at 0
   \[\text{via (1.10) and (1.11)}\] satisfy (1.12), that is, $|\xi_i| = o(\lambda_i)$.

(iv) $\ell$ is even.

Then $\Delta_o^{(h)} P_\ell(y) = 0$ ($\Delta_o^{(h)} P_\ell$ is a number when $\ell$ is even). The same conclusion also holds when $\ell = n - 2$ with an additional assumption that 0 is the only blow-up point ($\ell$ is still required to be even).

**Proof.** The key is to combine the change of center formula (A.6.33) with the condition $|\xi_i| = o(\lambda_i)$, and observe the lower order terms. Other arguments actually proceed in similar fashion as those found in [15] and [21]. For the benefit of readers, we present the estimates in detail. From (A.6.5) and (A.6.6), we have

\[(A.6.35) \quad \int_{B_o(\rho_o)} \langle y, \nabla K(y) \rangle [v_i(y)]^{\frac{2n}{n-2}} dy = \begin{cases} O_{\lambda_i}(n-2), & \text{in general;} \\
O_{\lambda_i}(n), & \text{one blow-up point.} \end{cases}\]

Throughout this proof we assume that the positive constant $\rho_o > 0$ is chosen to be small enough. Let us pay attention to (2.21) and the number $R_i$ satisfying (2.20), together with the remark in §2f (on shifting the center, see also §A.2). Note that

\[(A.6.36) \quad \int_{B_o(\rho_o)} \langle y, \nabla K(y) \rangle [v_i(y)]^{\frac{2n}{n-2}} dy = \int_{B_o(\lambda_i R_i)} r \frac{\partial K}{\partial r} [A(y)]^{\frac{2n}{n-2}} dy \quad \cdots \quad (I)\]

\[+ \int_{B_o(\lambda_i R_i)} r \frac{\partial K}{\partial r} \left\{ [v_i(y)]^{\frac{2n}{n-2}} - [A(y)]^{\frac{2n}{n-2}} \right\} dy \quad \cdots \quad (II)\]

\[+ \int_{B_o(\lambda_i R_i) \setminus B_o(\lambda_i R_i)} r \frac{\partial K}{\partial r} [v_i(y)]^{\frac{2n}{n-2}} dy. \quad \cdots \quad (III)\]

(i) We begin with the core term (I). From (1.8), we have

\[(A.6.37) \quad r \cdot \frac{\partial [\tilde{c}_n K]}{\partial r} = \langle y, \nabla [\tilde{c}_n K] \rangle = \ell \times [-P_\ell(y)] + O\left(|y|^{\ell+1}\right)\]

for $y \in B_o(\rho_o)$. It follows that [recall that $A$ is given in (A.6.23)]

\[(A.6.38) \quad \int_{B_o(\lambda_i R_i)} r \frac{\partial K}{\partial r} [A(y)]^{\frac{2n}{n-2}} dy = -\frac{\ell}{c_n} \int_{B_o(\lambda_i R_i)} P_\ell \cdot [A(y)]^{\frac{2n}{n-2}} dy + \]

\[+ \int_{B_o(\lambda_i R_i)} O\left(|y|^{\ell+1}\right) \cdot [A(y)]^{\frac{2n}{n-2}} dy. \]

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\[
= -\frac{\ell}{c_n} \int_{B_o(\lambda R_i)} P_\ell \cdot [A(y)]^{\frac{2n}{n-2}} dy + O_{\lambda_i}(\ell + 1) \quad \text{for } \ell \leq n - 2
\]
[see (A.6.41); this only requires \(|\xi_i| = O(\lambda_i)\)\]

\[
= -\lambda_i^\ell \cdot \frac{\ell}{c_n} \cdot \int_{\mathbb{R}^n} P_\ell(y) \cdot \left(\frac{1}{1 + |y|^2}\right)^n dy + o_{\lambda_i}(\ell)
\]
[estimating as in (A.6.31)–(A.6.33), \(\uparrow\) using \(|\xi_i| = O(\lambda_i)\)↑]

replacing \(\rho_o\) by \(\lambda_i \cdot R_i\) in (A.6.31), and take \(r_i = R_i\) in (A.6.32)].

Here \(2 \leq \ell \leq n - 2\).

(ii) Now we turn to the term marked (II). Via inequality (3.14) and (2.26) in Proposition 2.24, we have

\[(A.6.39) \quad \int_{B_{\xi_i}(\lambda R_i)} \langle y, \nabla K(y) \rangle \left\{ [v_i(y)]^{\frac{2n}{n-2}} - [A(y)]^{\frac{2n}{n-2}} \right\} dy \]

\[
\leq C \cdot \frac{\varepsilon}{\lambda_i^{2\ell}} \cdot \frac{1}{\lambda_i^{n-2}} \cdot \int_{B_{\xi_i}(\lambda R_i)} r^\ell \, dy \quad (r = |y|)
\]

\[
\leq C \cdot \frac{\varepsilon}{\lambda_i^{2\ell}} \cdot \int_{B_o(\lambda (R_i + 1))} r^\ell \, dy \quad (\lambda_i \cdot |\xi_i| \to 0)
\]

\[
\leq C_1 \cdot \frac{\varepsilon}{\lambda_i^{2\ell}} \cdot [\lambda_i (R_i + 1)]^{\ell + n} = C_1 \cdot \lambda_i^{\ell} [\varepsilon (R_i + 1)^{\ell + n}] = o_{\lambda_i}(\ell).
\]

In the above we apply

\[
\varepsilon_i \cdot R_i^{2(n-1)} = o(1) \quad \text{[see (2.20)]}.
\]

(iii) As for the term marked (III), using the binomial expansion on \(|z| + |\xi_i|\)^\(\ell\), together with (2.5) or (2.26) (similar to Lemma 2.4 in [21]), we have

\[(A.6.40) \quad \int_{B_o(\rho_o) \setminus B_o(\lambda R_i)} \langle y, \nabla K(y) \rangle [v_i(y)]^{\frac{2n}{n-2}} dy \]

\[
\leq C_1 \cdot \int_{B_o(\rho_o) \setminus B_o(\lambda R_i)} |y|^{\ell} \left(\frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2}\right)^n dy
\]

\[
\leq C_2 \cdot \int_{B_{\xi_i}(\rho_o) \setminus B_{\xi_i}(\lambda (R_i - c))} |y|^{\ell} \left(\frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2}\right)^n dy
\]
Here $\rho'_o$ is slightly bigger than $\rho_o$, $c$ is a big enough constant (require $|\xi_i| = O(\lambda_i)$)

$$\leq C_3 \cdot \int_{B_\rho(\rho'_o) \setminus B_\rho(\lambda_i (R_i - c))} \left( \sum_{j=0}^{\ell} |z|^{\ell - j} \cdot |\xi_i|^j \right) \cdot \left( \frac{\lambda_i}{\lambda_i^2 + |z|^2} \right)^n \, dz \quad (y = z + \xi)$$

$$\leq C_4 \lambda_i^\ell \cdot \left[ \sum_{j=0}^{\ell} \int_{R_i - c}^{\epsilon} \left( \frac{1}{1 + r^2} \right)^n r^{\ell - j + (n-1)} \, dr \right] \quad \text{(polar coordinates and } r \to \lambda_i \cdot r\text{)}$$

$$= o(\lambda_i^\ell) \quad \text{for } 2 \leq \ell \leq n - 2$$

(needs only $|\xi| = O(\lambda_i)$; $R_i \to \infty$ and $\lambda_i \cdot R_i \to 0 \implies \lambda_i^{-1} \cdot \rho_o > R_i$).

Similarly,

$$\int_{B_\rho(\lambda_i R_i)} |y|^{\ell+1} \left( \frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2} \right)^n \, dy \leq C_1 \int_{B_\rho(\lambda_i (R_i + c))} |y|^{\ell+1} \left( \frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2} \right)^n \, dy$$

$$\leq C_2 \int_{B_\rho(\lambda_i (R_i + c))} \left( \sum_{j=0}^{\ell+1} |z|^{\ell+1-j} \cdot |\xi_i|^j \right) \cdot \left( \frac{\lambda_i}{\lambda_i^2 + |z|^2} \right)^n \, dz = O_{\lambda_i}(\ell + 1),$$

which requires only $|\xi_i| = O(\lambda_i)$. Here (as before) $y = z + \xi$, and we apply the change of variables $z \to \lambda_i \cdot z$. Using (A.6.35), (A.6.36), (A.6.38), (A.6.39) and (A.6.40) we obtain

$$\int_{B_\rho(\rho_o)} \langle y, \nabla K(y) \rangle \cdot [v_i(y)]^{2(n-2)} \, dy = \lambda_i^\ell \int_{\mathbb{R}^n} P_\ell(y) \left( \frac{1}{1 + |y|^2} \right)^n \, dy + o_{\lambda_i}(\ell)$$

(A.6.42) \quad \cdots \quad \implies \quad \int_{\mathbb{R}^n} P_\ell(y) \left( \frac{1}{1 + |y|^2} \right)^n \, dy = 0.$$

Here $2 \leq \ell \leq n - 2$. (A.6.42) together with Lemma A.6.8 ($\ell$ is even) imply that $\Delta_{bc}^{(o)} P_\ell(y) \equiv 0$. Let us end the proof with the remark that the condition $|\xi_i| = o(\lambda_i)$ is only ‘fully’ used in the last step in (A.6.38). \hfill \Box

We highlight that the smaller order term in (A.6.41) depends on convergence parameters $\varepsilon_i$ and $R_i$, as well as the condition $|\xi_i| = o(\lambda_i)$. In the next section, we apply the refined estimate (A.6.22) to discern out the layers of information hidden in $o_{\lambda_i}(\ell)$.
§ A.6.e. Isolating the key term with lowest order in \( \lambda_i \). Suppose that estimate (6.57) holds for \( \{v_i\} \) inside \( B_o(\rho_2) \), where \( \rho_2 > 0 \) is a constant. From (A.6.22)–(A.6.25), we obtain

\[
(A.6.43) \quad \int_{B_o(\rho_2)} \frac{r}{\partial r} \cdot [v_i(y)]^{\frac{2n}{\pi - 2}} dy = \int_{B_o(\rho_2)} \frac{r}{\partial r} [A(y)]^{\frac{2n}{\pi - 2}} dy +
\]

\[
+ \int_{B_o(\rho_2)} \frac{r}{\partial r} \left( [A + B + C](y)]^{\frac{2n}{\pi - 2}} - [A(y)]^{\frac{2n}{\pi - 2}} \right) dy.
\]

In order to estimate the last term in the above, we make use of the inequality

\[
(A.6.44) \quad \left| \int_{B_o(\rho_2)} \left| \frac{r}{\partial r} \right| \left[ (A + B + C)(y)]^{\frac{2n}{\pi - 2}} - [A(y)]^{\frac{2n}{\pi - 2}} \right] dy \right|
\]

\[
\leq \varepsilon \int_{B_o(\rho_2)} \left| \frac{r}{\partial r} \right| A^{\frac{2n}{\pi - 2}} dy + \frac{C_n}{\varepsilon^{\frac{2n}{\pi - 2}}} \int_{B_o(\rho_2)} \left| \frac{r}{\partial r} \right| \left( |B|^{\frac{2n}{\pi - 2}} + |C|^{\frac{2n}{\pi - 2}} \right) dy.
\]

Here \( \varepsilon > 0 \) is a given (small) number, and the dimensional constant \( C_n \) is independent on \( \varepsilon \). We demonstrate the argument toward (A.6.44) in §A.3 in the Appendix.

Remark A.6.45. Suppose that we seek to find \( \varepsilon > 0 \) so that

\[
\varepsilon \cdot \lambda_i^\ell + \frac{C_n}{\varepsilon^{\frac{2n}{\pi - 2}}} \cdot \lambda_i^{\ell + a} = \varepsilon \cdot \lambda_i^\ell + \frac{C_n}{\varepsilon^{\frac{2n}{\pi - 2}}} \cdot \lambda_i^{a - t} \cdot \lambda_i^{\ell + t}.
\]

That is, we want to re-distribute some order of \( \lambda_i \) to the first term so that in the end the two terms have the same order \( O_{\lambda_i}(\ell + t) \):

\[
\varepsilon = (\lambda_i)^t \implies \frac{C_n}{(\lambda_i)^\frac{2n}{n - 2} t} = \lambda_i^{a - t} \implies \frac{2n}{n - 2} \cdot t = (a - t) \implies t = \frac{n - 2}{3n - 2} \cdot a.
\]

‡ A.6.f Estimate on the leading order term. Recall (1.8) and (4.12).

\[
r \cdot \frac{\partial [\tilde{c}_n K]}{\partial r} = \langle y, \nabla [\tilde{c}_n K] \rangle = \ell \times [-P_\ell (y)] + O(|y|^{\ell + 1})
\]

\[
(A.6.46) \quad \cdots \implies \int_{B_o(\rho_2)} \frac{r}{\partial r} [A(y)]^{\frac{2n}{\pi - 2}} dy = -\frac{\ell}{C_n} \cdot \int_{B_o(\rho_2)} P_\ell \cdot [A(y)]^{\frac{2n}{\pi - 2}} dy +
\]

\[
+ \int_{B_o(\rho_2)} O(|y|^{\ell + 1}) \cdot [A(y)]^{\frac{2n}{\pi - 2}} dy.
\]

The first term in the right hand side of the last equation in (A.6.46) can be expanded by using (A.6.33). As for the second term, it can be estimated as in (A.6.41) (replac-
ing $\lambda_i R_i$ by $\rho_2$) showing that the term is of order $O_\lambda(\ell + 1)$. Hence we obtain the following.

**Lemma A.6.47.** Under the conditions in (2.63), (A.6.23), $\ell \in [2, n - 2]$, suppose that, for $i \gg 1$, $\xi_i = \lambda_i^{1+\eta_o} \cdot \vec{X}$, where $\vec{X} \in \mathbb{R}^n$ is a fixed vector. Then we have

$$
(A.6.48) \quad - \int_{B_o(\rho_2)} \frac{r}{\partial r} \cdot A^{\frac{2n}{n-2}} \, dy = \lambda_i^\ell \cdot \frac{\ell}{c_n} \int_{\mathbb{R}^n} P_\ell(y) \left( \frac{1}{1 + |y|^2} \right)^n \, dy + \\
+ \int_{\mathbb{R}^n} \left[ \lambda_i^{\ell+\eta_o} \cdot \Xi_1^P(\vec{X}, y) + \cdots + \lambda_i^{\ell+(\ell-1)\cdot\eta_o} \cdot \Xi_{\ell-1}^P(\vec{X}, y) \right] \cdot \left( \frac{1}{1 + |y|^2} \right)^n \, dy + \\
+ \lambda_i^{\ell+\ell \cdot \eta_o} \cdot P_\ell(\vec{X}) \int_{\mathbb{R}^n} \left( \frac{1}{1 + |y|^2} \right)^n \, dy + O_\lambda(\ell + 1).
$$

Here $\Xi_h^P$ is defined as in (A.6.29) and (A.6.30) by replacing $Q_\ell$ by $P_\ell$.

§ A.6. g. *Estimate on the term involving $B$.* We first shift the center in the term

$$
(A.6.49) \quad \int_{B_o(\rho_2)} \left| \frac{r}{\partial r} \cdot B(y) \right|^{\frac{2n}{n-2}} \, dy \leq C \int_{B_o(\rho_2)} |y|^\ell \cdot |B(y)|^{\frac{2n}{n-2}} \, dy
$$

$$
\leq C \int_{B_{\ell_i}(\rho_2)} |y|^\ell \cdot |B(y)|^{\frac{2n}{n-2}} \, dy + C_1 |\xi_i| \cdot \lambda_i^{\frac{n}{2} \cdot \frac{2n}{n-2}}
$$

[cf. (A.6.22), observe that $\lambda_i^\ell \cdot |\Gamma_P(\mathcal{Y})| \leq c$ in $B_o(\rho_2)$]

$$
\leq C \int_{B_{\ell_i}(\rho_2)} \left[ |y - \xi_i|^\ell + C_1 |\xi_i| \cdot |y - \xi_i|^{\ell-1} + \cdots + |\xi_i|^\ell \right] \cdot |B(y)|^{\frac{2n}{n-2}} \, dy
$$

$$
+ o_\lambda \left( \frac{n^2}{n - 2} + 1 \right) \quad [\text{recall that } |\xi_i| = o(\lambda_i)].
$$

In the above, we apply the triangle inequality and the binomial expansion as in

$$
|y|^\ell \leq (|y - \xi_i| + |\xi_i|)^\ell \quad (\ell \text{ a positive integer}).
$$

Introduce the change of variables

$$
(A.6.50) \quad |y - \xi_i| = \rho = \lambda_i \tan \theta \quad \Rightarrow \quad \tan \theta = \frac{|y - \xi_i|}{\lambda_i} \quad \Rightarrow \quad |\mathcal{Y}| = \tan \theta.
$$
Recall that in the expression (A.6.24) for $B$,
\[
|B(y)|^{2n/2} = \lambda_i^{(\ell+1)} \cdot \frac{2n}{n-2} \cdot |\gamma|^{2n/2} \cdot \left[ \frac{1}{\lambda_i (1 + \tan^2 \theta)} \right]^{n-2/2} \cdot \left( \gamma = \frac{y - \xi_i}{\lambda_i} \right).
\]

Moreover,
\[
(A.6.51)
|\Gamma_{\leq \ell} (\gamma)| \leq C \left[ R^2 + \cdots + R^\ell \right] \Rightarrow \begin{cases} 
|\Gamma_{\leq \ell} (\gamma)| \leq C_1 & \text{for } R = |\gamma| \leq 1; \\
|\Gamma_{\leq \ell} (\gamma)| \leq C_2 R^\ell & \text{for } |\gamma| \geq 1.
\end{cases}
\]
For $0 \leq l \leq \ell$, we have
\[
(A.6.52) \quad \int_{B_{\xi_i} (\rho_2)} |y - \xi_i|^l \cdot |B(y)|^{2n/2} \, dy
\]
\[
= \left( \int_{B_{\xi_i} (\lambda_i)} + \int_{B_{\xi_i} (\rho_2) \setminus B_{\xi_i} (\lambda_i)} \right) |y - \xi_i|^l \cdot |B(y)|^{2n/2} \, dy
\]
\[
\leq C_3 \int_0^{\lambda_i} \lambda_{i}^{\ell+1} \cdot \left( \lambda_i^{\ell+1-\frac{n}{2}} \right) \frac{2n}{n-2} [r^{n-1} \, dr]
\]
[where $r = |y - \xi_i|$; using first half in (A.6.40)]
\[
+ C_2 \int_{\arctan \frac{\rho}{2}}^{\arctan \frac{n}{2}} [\lambda_i \tan \theta]^l \cdot \lambda_i^{(\ell+1)} \cdot \frac{2n}{n-2} \cdot \frac{1}{\lambda_i (1 + \tan^2 \theta)} \times
\]
\[
\times \left\{ \lambda_i^{n} [\tan \theta]^{n-1} \sec^2 \theta \right\} d\theta
\]
\[
\leq O_{\lambda_i} \left( l + \ell \cdot \frac{2n}{n-2} \right) + \\
+ O_{\lambda_i} \left( l + \ell + 1 \cdot \frac{2n}{n-2} + n - \frac{n^2}{n-2} \right) \cdot \int_0^{\arctan \frac{n}{2}} \frac{[\cos \theta]^{2n/2}}{[\cos \theta]^{l+\frac{2n\ell}{n-2}+(n-1)+2}} \, d\theta
\]
\[
\leq O_{\lambda_i} \left( l + \frac{2n\ell}{n-2} \right) + O_{\lambda_i} \left( l + \frac{2n\ell}{n-2} \right) \cdot \int_0^{\arctan \frac{n}{2}} \frac{[\cos \theta]^{2n/2}}{[\cos \theta]^{l+\left( \frac{2n\ell}{n-2} - 4 \right) + 4 + (n+1)}} \, d\theta
\]
\[
\leq O_{\lambda_i} \left( l + \frac{2n\ell}{n-2} \right) + O_{\lambda_i} (l + 4) \cdot \int_0^{\arctan \frac{n}{2}} \frac{[\cos \theta]^{2n/2}}{[\cos \theta]^{l+n+5}} \, d\theta \quad [\text{see } \S A.6 g.2]
\]
\[ = o_{\lambda_i}(l + 4) + o_{\lambda_i}(l + 4) \quad \text{(as } \ell \geq 2) \]

\[ \text{note that } l + n + 5 \leq (n - 2) + n + 6 = 2n + 3 \leq \frac{2n^2}{n - 2} \].

Using (A.6.49), (A.6.52) and \(|\xi_i| = O(\lambda_i)|, we obtain the following.

**Lemma A.6.53.** Let \( B \) be given as in (A.6.24), \(|\xi_i| = O(\lambda_i)|, and \( K \) satisfies (1.8). Then for \( 2 \leq \ell \leq n - 2 \), we have

\[
\int_{B_o(\rho_o)} \left| r \frac{\partial K}{\partial r} \right| \cdot \left| B(y) \right|^{\frac{2n}{n - 2}} dy = o_{\lambda_i}(\ell + 4). \quad (A.6.54)
\]

**§ A.6. h. Estimate on the term involving \( C \).** As

\[
\ell \leq n - 2 \implies \ell + 1 - \frac{n}{2} \leq \frac{n - 2}{2}. \quad (A.6.55)
\]

From (A.6.25) and \(|\langle y, \nabla K \rangle| \leq c \) in \( B_o(\rho_o) \), we obtain

\[
\int_{B_o(\rho_o)} \left| r \frac{\partial K}{\partial r} \right| \cdot \left| C(y) \right|^{\frac{2n}{n - 2}} dy = O_{\lambda_i}\left( \left\lfloor \ell - \frac{n - 2}{2} \right\rfloor \cdot \frac{2n}{n - 2} \right). \quad (A.6.56)
\]

Observe that

\[
\left( \ell + 1 - \frac{n}{2} \right) \cdot \frac{2n}{n - 2} > \ell
\]

\[ \iff \ell > \frac{n(n - 2)}{n + 2} \implies \ell = (n - 2) \& n \geq 4; \text{ or } \ell = (n - 3) \& n > 6. \]

Moreover,

\[
\ell = n - 2 \implies \left( \ell + 1 - \frac{n}{2} \right) \cdot \frac{2n}{n - 2} = n, \quad (A.6.57)
\]

and when \( \ell = (n - 3) \) and \( n > 6 \),

\[
\left( (n - 3) + 1 - \frac{n}{2} \right) \cdot \frac{2n}{n - 2} - (n - 3) = \frac{n - 6}{n - 2} > 0. \quad (A.6.58)
\]

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§ A.6. i. Remaining estimates. Using $|\langle y, \nabla K \rangle| \leq C |y|^{\ell}$ and $|\xi_i| = o(\lambda_i)$ [actually we only need $|\xi_i| = O(\lambda_i)$], as in (A.6.33), we have

\[(A.6.59) \quad \varepsilon \int_{B_2(\rho_2)} \left| \frac{\partial K}{\partial r} \right| \cdot [A(y)]^{\frac{2n-2}{n-2}} dy = \varepsilon \cdot O(\lambda(\ell)) \quad \text{for } 2 \leq \ell \leq n - 2.\]

§ A.6. i.1. Estimate on the outside. Similar to (A.6.32), for $l \leq n - 1$, we have

\[(A.6.60) \quad \int_{\mathbb{R}^n \setminus B_2(\rho_2)} r^l \cdot \left( \frac{\lambda_i}{\lambda_i^2 + r^2} \right)^n dy \leq \int_{\mathbb{R}^n \setminus B_2(\rho_2)} r^l \cdot \left( \frac{\lambda_i}{r^2} \right)^n dy = \lambda_i^n \cdot \frac{1}{r^{2n-l-(n-1)}} dy = O(\lambda_i^n).\]

§ A.6. i.2. The angle. Let

\[(A.6.61) \quad \theta_{\lambda_i} := \arctan \frac{\rho_2}{\lambda_i} \quad \Rightarrow \quad \cos \theta_{\lambda_i} = \frac{1}{\sec \theta_{\lambda_i}} = \frac{1}{\sqrt{\sec^2 \theta_{\lambda_i}}} = \frac{1}{\sqrt{1 + \tan^2 \theta_{\lambda_i}}} = \frac{1}{\sqrt{1 + \frac{\rho_2^2}{\lambda_i^2}}} = O(\lambda_i).\]

§ A.6. j. Limitation on flexibility.

Theorem A.6.62. Assume the conditions in Main Theorem 1.14 and restrict $\ell$ to be either $n - 2$ or $n - 3$. Moreover, assume the following.

(i) When $\ell = n - 2$, we take it that $S$ is the only blow-up point.

(ii) When $\ell = n - 3$, we take it that $n > 6$, and it is possible to have other blow-up point(s).

Let $\Xi^P_\ell$ be given in (A.6.29) and (A.6.30) with $Q_\ell$ replaced by $P_\ell$. Assume also that

\[(A.6.63) \quad \xi_i = \lambda_i^{1+\eta_0} \cdot \tilde{X} \quad \text{for } i \gg 1 \text{ and a fixed } \tilde{X} \in \mathbb{R}^n.\]

Here $\eta_0 \leq \begin{cases} 2/(3n-2) & \text{when } \ell = n - 2; \\ n-6/(n-3)(3n-2) & \text{when } \ell = n - 3 \text{ and } n > 6. \end{cases}$

Then it is necessary that $P_\ell(\tilde{X}) = 0$, and

\[\int_{\mathbb{R}^n} \Xi^P_\ell (\tilde{X}, y) \left( \frac{1}{1 + |y|^2} \right)^n dy = \cdots = \int_{\mathbb{R}^n} \Xi^{P_{\ell-1}} (\tilde{X}, y) \left( \frac{1}{1 + |y|^2} \right)^n dy = 0.\]

\[(A.6.64) \quad - \int_{B_o(p_2)} \langle y, \nabla K(y) \rangle [v_i(y)]^{\frac{2n-2}{n}} dy = \lambda_i^\ell \cdot \frac{\ell}{c_n} \int_{\mathbb{R}^n} P_\ell(y) \left( \frac{1}{1 + |y|^2} \right)^n dy + \]

\[+ \int_{\mathbb{R}^n} \left[ \lambda_i^{\ell + \eta_o} \cdot \mathcal{E}_\ell \left( \tilde{X}, y \right) + \cdots + \lambda_i^{\ell + (\ell-1) \cdot \eta_o} \cdot \mathcal{E}_{\ell-1} \left( \tilde{X}, y \right) \right] \left( \frac{1}{1 + |y|^2} \right)^n dy + \]

\[+ \lambda_i^{\ell + \eta_o} \cdot P_\ell \left( \tilde{X} \right) \int_{\mathbb{R}^n} \left( \frac{1}{1 + |y|^2} \right)^n dy + O_{\lambda_i}(\ell + 1) + \]

\[+ \varepsilon \cdot O_{\lambda_i}(\ell) + \frac{C_n}{\varepsilon^{n-2}} \cdot \left\{ \begin{array}{ll}
O_{\lambda_i}(\ell + 2) & \text{for } \ell = n - 2; \\
O_{\lambda_i}(\ell + \frac{n-6}{n-2}) & \text{for } \ell = n - 3, \ n > 6.
\end{array} \right.\]

Referring to Remark A.6.45:

\[(A.6.65) \quad \text{when } \ell = n - 2, \ \ell \cdot \eta_o < 2 \cdot \frac{n-2}{3n-2} \iff \eta_o < \frac{2}{3n-2} \]

\[\left( a = 2, \ t = \frac{2(n-2)}{3n-2} < 1 \right); \]

\[(A.6.66) \quad \text{when } \ell = n - 3, \ \ell \cdot \eta_o < \frac{n-6}{n-2} \cdot \frac{n-2}{3n-2} \iff \eta_o < \frac{n-6}{(n-3)(3n-2)} \]

\[\left( a = \frac{n-6}{n-2}, \ t = \frac{n-2}{3n-2} \cdot \frac{n-6}{n-2} < 1 \right). \]

Combining with (A.6.35), we come to the conclusion of the theorem. \qed
§ A.7. \( \ell = n - 2 \) and multiple simple blow-up points – off-center cancelation.

We present the consideration on global cancelation/balance with finite number of blow-up points, say, at

\[
(\text{A.7.1}) \quad \tilde{Y}_0 = 0, \quad \tilde{Y}_1, \ldots, \tilde{Y}_k \quad (k \geq 1).
\]

[Cf. (2.7).] Throughout this section (§ A.7) we assume the general conditions (1.6), (1.25), (1.26), \( n > 6 \), and

\[
(\text{A.7.2}) \quad \tilde{Y}_j \text{ is a simple blow-up point and } \sum_{2 \leq l < n - 2} |\nabla^{(l)} K (\tilde{Y}_j)| = 0 \text{ for } 0 \leq j \leq k.
\]

Cf. (1.9) and the remark preceding it. For the simple blow-up point at 0, we keep the notations on \( \xi_i \) and \( \lambda_i \) as introduced in (1.10) and (1.11). Likewise (cf. Proposition 2.24 and § 2 f) we set

\[
(\text{A.7.3}) \quad \xi_{m_i} : v_i (\xi_{m_i}) = \max \left\{ v_i (y) \mid y \in \overline{B_{\tilde{Y}_m} (\rho_3)} \right\}, \quad \xi_{m_i} \to \tilde{Y}_m,
\]

\[
(\text{A.7.4}) \quad \lambda_{m_i} := \frac{1}{[v_i (\xi_{m_i})]^\frac{2}{n-2}} \text{ for } 1 \leq m \leq k.
\]

Here \( \rho_3 > 0 \) is a constant (small enough) so that Proposition 2.3 and Proposition 2.24 hold after a translation to each individual blow-up point, and

\[
B_{\tilde{Y}_m} (\rho_3) \cap B_{\tilde{Y}_j} (\rho_3) = \emptyset \quad \text{for } j \neq m.
\]

Via Proposition 2.24 and the Harnack inequality [15],

\[
(\text{A.7.5}) \quad \frac{1}{C} \cdot \lambda_{m_i}^{\frac{n-2}{2}} \leq \min_{|y - \tilde{Y}_m| = \rho_3} v_i (y) \leq \max_{|y - \tilde{Y}_m| = \rho_3} v_i (y) \leq C \lambda_{m_i}^{\frac{n-2}{2}}
\]

for \( 1 \leq m \leq k \) and \( i \gg 1 \). As there is no blow-up point which appears at the north pole [cf. (2.11)], apply the Harnack inequality [15] again on

\[
S^n \setminus \hat{P}^{-1} \left( \bigcup_{j=0}^{k} B_{\tilde{Y}_j} (\rho_3) \right)
\]

and obtain

\[
(\text{A.7.6}) \quad \frac{1}{C} \leq \frac{\lambda_{m_i}}{\lambda_i} \leq C \quad \text{for } 1 \leq m \leq k \text{ and } i \gg 1.
\]
It follows that, modulo a subsequence,

\[(A.7.7)\quad S_m := \lim_{i \to \infty} \frac{\lambda_m}{\lambda_i} \quad \text{is well-defined for } 1 \leq m \leq k.\]

Using the global formula (A.6.6), together with (A.7.5) and (A.7.6), we have

\[(A.7.8)\quad \int_{\Omega} \langle y, \nabla_y K(y) \rangle [v_i(y)]^{2n-2} \, dy = O_\lambda(n),\]

where \(\Omega = B_\alpha(\rho_3) \cup B_{\hat{Y}_1}(\rho_3) \cup \cdots \cup B_{\hat{Y}_k}(\rho_3).\)

\[\text{§ A.7. a. Off-origin blow-up point.} \quad \text{Consider the simple blow-up at } \hat{Y}_m, \text{ where } 0 < m \leq k. \text{ Via Taylor expansion,}\]

\[(A.7.9)\quad \tilde{c}_n \cdot K(y) = (\tilde{c}_n \cdot K)(\hat{Y}_m) + \sum_{|\alpha| = n-2} \frac{1}{\alpha!} [D_{\alpha}^{(n-2)}(\tilde{c}_n K)(\hat{Y}_m)] \cdot (y - \hat{Y}_m)\alpha
\]

\[+ O(|y - \hat{Y}_m|^{n-1}) \quad \text{for } |y - \hat{Y}_m| \leq \rho_3\]

\[= (\tilde{c}_n \cdot K)(\hat{Y}_m) + [-P_{n-2,m}(y)] + O(|y - \hat{Y}_m|^{n-1}).\]

Here \(P_{n-2,m}(y)\) is defined by the equation above [see also (A.7.15)]. For the sake of continuity, we keep the sign convention on \(\cdot \) \(P\), which we use in this article. Assume that

\[(A.7.10)\quad \Delta^{(h-2)}_{\alpha} P_{n-2, m}(\hat{Y}_m) \equiv 0 \quad \text{for } 0 \leq m \leq k,\]

and all the corresponding conditions as in Main Theorem (1.17) hold for each individual simple blow-up point, except \((\tilde{c}_n \cdot K)(\hat{Y}_m)\) may not be \(n(n-2)\). Thus the estimate contains a scaling factor

\[(A.7.11)\quad v_i(y) = \left[ \frac{n(n-2)}{(\tilde{c}_n K)(\hat{Y}_m)} \right]^{\frac{n-2}{2}} \cdot \left( \frac{\lambda_m}{\lambda_{m_i}} + \frac{1}{|y - \xi_{m_i}|^2} \right)^{\frac{n-2}{2}} + \]

\[+ \{ \text{expressions similar to those in (A.6.24) and (A.6.25)} \} \quad \text{for } y \in B_{\hat{Y}_m}(\rho_3).\]

[Here \(\rho_3\) is made smaller if necessary.]

We find the first derivative by using change of variables \(y = z + \hat{Y}_m,\)

\[(A.7.12)\quad \tilde{c}_n \cdot \langle y, \nabla_y K(y) \rangle = \tilde{c}_n \cdot \langle (y - \hat{Y}_m), \nabla_y K(y) \rangle + \tilde{c}_n \cdot \langle \hat{Y}_m, \nabla_y K(y) \rangle,\]

\[(A.7.13)\quad \langle (y - \hat{Y}_m), \nabla_y K(y) \rangle = \langle z, \nabla_y K|_{y = y + \hat{Y}_m} \rangle
\]

\[= \langle z, \nabla_z K|_{z = z + \hat{Y}_m} \rangle \quad \text{for } z = (y - \hat{Y}_m).\]
Here \( K_{\rightarrow} (z) = K (z + \hat{Y}_m) \).

Consider the second expression in the right hand side of (A.7.12). As in (A.7.13), we have

\[
(A.7.14) \quad \tilde{c}_n \cdot \left< \hat{Y}_m, \bigtriangledown_y K (y) \right> = \left< \hat{Y}_m, \bigtriangledown_z P_{n-2, m} (z) \right> + O (|y - \hat{Y}_m|^{n-2})
\]
\[
= \left< \hat{Y}_m, \bigtriangledown_z P_{n-2, m} \right>_z = y - \hat{Y}_m + O (|y - \hat{Y}_m|^{n-2}).
\]

Let

\[
\mathcal{L}_m (z) := \left< \hat{Y}_m, \bigtriangledown_z P_{n-2, m} \right>_z \quad \text{for} \quad 1 \leq m \leq k.
\]

\( \mathcal{L}_m \) is a homogeneous polynomial of degree \( n - 3 \). As in (A.7.13), from (A.7.9), we recognize

\[
(A.7.15) \quad P_{n-2, m} (z) = - \sum_{|\alpha| = n-2} \frac{1}{\alpha!} \left[ D_{(n-2)}^\alpha (\tilde{c}_n K) (\hat{Y}_m) \right] \cdot z^{\alpha}.
\]

Observe that, via (A.7.10), \( \Delta_{\alpha}^{n-2} P_{n-2, m} (z) \equiv 0 \).

In the following we assume that, for each \( m \) with \( 0 \leq m \leq k \), there is a positive number \( \eta_m \) such that

\[
(A.7.16) \quad \eta_m < \frac{n-6}{(n-3)(3n-2)} \quad \text{and} \quad \xi_{m_i} = \lambda_{m_i}^{\pm \eta_m} \bar{X}_m \quad \text{for} \quad i \gg 1,
\]

where \( \bar{X}_m \in \mathbb{R}^n \) is fixed. Consider the integral

\[
(A.7.17) \quad \tilde{c}_n \cdot \int_{B_{\hat{Y}_m} (\rho_3)} \left< y, \bigtriangledown_y K (y) \right> [v_i (y)]^{\frac{2n}{n-2}} dy
\]
\[
\quad + \sum_{|\alpha| = n-2} \frac{1}{\alpha!} \left[ D_{(n-2)}^\alpha (\tilde{c}_n K) (\hat{Y}_m) \right] \cdot z^{\alpha} \left< y, \bigtriangledown_z P_{n-2, m} \right>_z \quad \text{[using (2.26) & (2.27)]}
\]

\[
\quad = \tilde{c}_n \cdot \int_{B_{\hat{Y}_m} (\rho_3)} \left< z, \bigtriangledown z K_{\rightarrow} (z) \right>_z = (y - \hat{Y}_m) [v_i (y)]^{\frac{2n}{n-2}} dy + \]
\[
\quad + \int_{B_{\hat{Y}_m} (\rho_3)} \left< \hat{Y}_m, \bigtriangledown_z P_{n-2, m} \right>_z = (y - \hat{Y}_m) [v_i (y)]^{\frac{2n}{n-2}} dy + O_{\lambda_m} (n - 2)
\]
\[
\text{[as in (A.6.41) for the term } O (|y - \hat{Y}_m|^{n-2}) \text{]}
\]

\[
\quad = \tilde{c}_n \cdot \int_{B_{\hat{Y}_m} (\rho_3)} \left< z, \bigtriangledown z K (z) \right> [v_i (z + \hat{Y}_m)]^{\frac{2n}{n-2}} dz \quad (z = y - \hat{Y}_m)
\]
\[
\quad + \int_{B_{\hat{Y}_m} (\rho_3)} \mathcal{L}_m (z) [v_i (z + \hat{Y}_m)]^{\frac{2n}{n-2}} dz + O_{\lambda_m} (n - 2)
\]
\[
\begin{align*}
&= \lambda_m^{n-2} \left[ \frac{n(n-2)}{\hat{c}_n \cdot K(\hat{Y}_m)} \right] \frac{2}{\pi} \int_{\mathbb{R}^n} \mathbf{P}_{\rightarrow m}(z) \left( \frac{1}{1 + |z|^2} \right)^n \, dz + o_{\lambda_m}(n-2) + \\
&+ \lambda_m^{n-3} \left[ \frac{n(n-2)}{\hat{c}_n \cdot K(\hat{Y}_m)} \right] \frac{2}{\pi} \int_{\mathbb{R}^n} \mathcal{L}_m(z) \left( \frac{1}{1 + |z|^2} \right)^n \, dz + \\
&+ O_{\lambda_m}([n-3]+\eta_m) + \cdots + O_{\lambda_m}([n-3]+[n-4]\cdot\eta_m) + \\
&+ \lambda_m^{(n-3)+(n-3)\cdot\eta_m} \cdot \mathcal{L}_m(\hat{X}_m) \left[ \frac{n(n-2)}{\hat{c}_n \cdot K(\hat{Y}_m)} \right] \frac{2}{\pi} \int_{\mathbb{R}^n} \left( \frac{1}{1 + |z|^2} \right)^n \, dz + \\
&+ O_{\lambda_m}([n-3]+\frac{n-6}{3\mu-2})
\end{align*}
\]

[\overset{\uparrow}{\text{as in (A.6.34), Lemma A.6.47, (A.6.64)-(A.6.66) with } \ell = n-3, n > 6}].

In (A.7.17), we put priority on terms with lowest order in \( \lambda_m \). If \( n \) is even, then \( n-3 \) is odd, giving

\[(A.7.18) \quad \int_{\mathbb{R}^n} \mathcal{L}_m(z) \left( \frac{1}{1 + |z|^2} \right)^n \, dz = 0 .\]

In case \( n \) is odd, then \( n-3 \) is even, and \( h_{n-2} = \frac{n-3}{2} \). Thus the condition

\[
\Delta_o^{(h_{n-2})} \mathbf{P}_{m \rightarrow} \equiv 0 \iff \Delta_o^{\left(\frac{n-3}{2}\right)} \mathbf{P}_{m \rightarrow} \equiv 0
\]

\[
\implies \Delta_o^{\left(\frac{n-3}{2}\right)} \mathcal{L}_m(z) = \Delta_o^{\left(\frac{n-3}{2}\right)} \left( \hat{Y}_m, \nabla_z \mathbf{P}_{m \rightarrow}(z) \right)
\]

\[
= \left( \hat{Y}_m, \nabla_z \left[ \Delta_o^{\left(\frac{n-3}{2}\right)} \mathbf{P}_{n-2, m}(z) \right] \right) = 0 .
\]

Again we obtain (A.7.18) by using Lemma A.6.8.

When we add up the integrals and estimates from each simple blow-up point, for simplicity, we skip the terms with intermediate orders

\[
O_{\lambda_m}([n-3]+\eta_m) + \cdots + O_{\lambda_m}([n-3]+[n-4]\cdot\eta_m),
\]

and draw a conclusion from (A.7.8) and (A.7.17) that
\( L_m(\vec{X}_m) = \langle \hat{Y}_m, \nabla_z P_{n-2, m} (\vec{X}_m) \rangle = 0 \) (where \( \xi_m = \lambda_m^{1+\eta_m} \cdot \vec{X}_m \)),

(A.7.20) provided \((n-3) \cdot \eta_m < \frac{n-6}{3n-2}\) \((\text{observe that } \frac{n-6}{3n-2} < 1)\),

and there is no interference from other blow-up points, precisely:

(A.7.21)

\((n-3) \cdot \eta_m \neq h \cdot \eta_j \text{ for } j \neq m, 1 \leq h \leq n-3 \) \((h \text{ is a natural number})\).

In case some of the \( \eta_h = \eta_m \), together with (A.7.7), we obtain

(A.7.22)

\[
\left[ \frac{n(n-2)}{(\tilde{c}_n K)(\hat{Y}_m)} \right]^\frac{2}{n} \cdot S_m^{(n-3)+(n-3)\eta_m} \cdot \langle \hat{Y}_m, \nabla_z P_{n-2, m} (\vec{X}_m) \rangle \]

\[
+ \sum_{0 < h \leq k, h \neq m \text{ with } \eta_h = \eta_m} \left[ \frac{n(n-2)}{(\tilde{c}_n K)(\hat{Y}_h)} \right]^\frac{2}{n} \cdot S_h^{(n-3)+(n-3)\eta_h} \cdot \langle \hat{Y}_h, \nabla_z P_{n-2, h} (\vec{X}_h) \rangle = 0,
\]

where we set the condition

(A.7.23) \( \eta_j \neq \eta_m \implies \text{(A.7.21) holds} \).

We summarize the conditions assumed in the balance and cancelation formulas (A.7.19) and (A.7.22): besides the ones mentioned next to them \[\text{[they are (A.7.20), (A.7.21), and (A.7.23)]}\], and the conditions found in Main Theorem 1.14 for each blow-up point, plus (A.7.1)–(A.7.4), (A.7.10) and (A.7.16).
§ A.8. Verification of (A.6.13) and (A.6.18).

Refer to (A.6.13) for the notation we use.

\[ \int_{\mathbb{R}^n} (\text{terms without } y_1 \text{ & } y_n) \cdot y_{k+2}^n \cdot \left( \frac{1}{1+x^2} \right)^n dy \quad (\text{absolute convergence}) \]

\[ = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\text{terms without } y_1 \text{ & } y_n) \times \quad (\text{Fubini’s Theorem}) \]

\[ \times \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_{k+2}^n \cdot \left( \frac{1}{1 + [y_{1}^2 + \cdots + y_{n-1}^2] + y_1^2 + y_n^2} \right)^n dy_1 dy_n \right] dy_2 \cdots dy_{n-1} \]

\[ = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\text{terms without } y_1 \text{ & } y_n) \times \]

\[ \times \left[ \int_{0}^{2\pi} \int_{0}^{\infty} \rho^{k+2} \cdot \left( \frac{1}{1 + [y_{1}^2 + \cdots + y_{n-1}^2] + \rho^2} \right)^n \cdot \rho \, d\rho \, d\theta \right] dy_2 \cdots dy_{n-1} \]

(polar coordinates on \( \mathbb{R}^2 \), \( \rho^2 = y_1^2 + y_n^2 \), \( y_1 = \rho \cdot \sin \theta \), \( y_n = \rho \cdot \cos \theta \))

\[ = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\text{terms without } y_1 \text{ & } y_n) \times \]

\[ \times \left[ \int_{0}^{2\pi} \int_{0}^{\infty} \rho^{k+3} \cdot \left( \frac{1}{1 + [y_{1}^2 + \cdots + y_{n-1}^2] + \rho^2} \right)^n \cdot \left\{ \int_{0}^{2\pi} \sin^{k+2} \theta \, d\theta \right\} \right] dy_2 \cdots dy_{n-1} . \]

Here “(terms without \( y_1 \text{ & } y_n \))” is a polynomial on \( y_{1}, \cdots, y_{n-1} \), having sufficiently low degree so that the integral is absolutely convergent. Likewise,

\[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\text{terms without } y_1 \text{ & } y_n) \times \]

\[ \times \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_{1}^{k} y_{n}^{2} \left( \frac{1}{1 + [y_{1}^2 + \cdots + y_{n-1}^2] + y_1^2 + y_n^2} \right)^n dy_1 dy_n \right] dy_2 \cdots dy_{n-1} \]

\[ = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\text{terms without } y_1 \text{ & } y_n) \times \]

\[ \times \left[ \int_{0}^{2\pi} \int_{0}^{\infty} \rho^{k+3} \left( \frac{1}{1 + [y_{1}^2 + \cdots + y_{n-1}^2] + \rho^2} \right)^n \left\{ \int_{0}^{2\pi} (\sin^k \theta) (\cos^2 \theta) \, d\theta \right\} \right] dy_2 \cdots dy_{n-1} . \]
A direct calculation using integration by parts shows that
\[
\int_0^{2\pi} (\sin^{k+2} \theta) \, d\theta = - \int_0^{2\pi} (\sin^{k+1} \theta) \, d[\cos \theta] = (k+1) \int_0^{2\pi} (\sin^k \theta) (\cos^2 \theta) \, d\theta. 
\]
Hence we deduce (A.6.13).

To show (A.6.18), recall that
\[
\bowtie = \alpha_2 + \cdots + \alpha_{n-1}.
\]

We demonstrate how to use induction on \(\bowtie\) to prove the assertion. Recall that
\[\ell \in [0, n-2] \text{ is even.}\]

(I) When \(\bowtie = 0\), the term is a constant. We have
\[
\Delta_{\bowtie}^{(h \ell)} y_{1i}^{k+2} = (k + 2) (k + 1) \cdots 3 \cdot 2 \cdot 1;
\]
\[
\Delta_{\bowtie} \left\{ (k + 1) y_{1i}^k y_{n}^2 \right\} = (k + 1) k (k - 1) y_{1i}^{k-2} y_{n}^2 + 2 (k + 1) y_{1i}^k,
\]
\[
\Delta_{\bowtie}^{(2)} \left\{ (k + 1) y_{1i}^k y_{n}^2 \right\} = (k + 1) k (k - 1) (k - 2) (k - 3) y_{1i}^{k-4} y_{n}^2 + 2 \times 2 (k + 1) k (k - 1) y_{1i}^{k-2},
\]
\[
\Delta_{\bowtie}^{(h \ell - 2)} \left\{ (k + 1) y_{1i}^k y_{n}^2 \right\} = (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot 2 \cdot 1 \cdot y_{n}^2 + 2 [h_\ell - 2] (k + 1) k (k - 1) \cdots 5 \cdot y_{n}^1,
\]
\[
\Delta_{\bowtie}^{(h \ell - 1)} \left\{ (k + 1) y_{1i}^k y_{n}^2 \right\} = (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot 2 \cdot [y_{n}^2 + y_{n}^2] + 2 [h_\ell - 1] (k + 1) k (k - 1) \cdots 5 \cdot 4 \cdot 3 \cdot y_{n}^2,
\]
\[
\Delta_{\bowtie}^{h_\ell} \left\{ (k + 1) y_{1i}^k y_{n}^2 \right\} = (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot 2 \cdot 1 \cdot 4 + [\ell - 4] (k + 1) k (k - 1) \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot 2 \cdot 1 [\ell - 4 + 4]
\]
\[
= (k + 2) (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot 2 \cdot 1
\]
( as \(\ell = k + 2\) in this case).
Hence the case $\ell = 0$ is settled.

(II) As an induction hypothesis, suppose that

$$\Delta^{(h_\ell)}_o \left\{ y^{k+2}_{i_1} \cdot \left[ \text{\ldots degree} = \ell \ \ldots \right] \right\} = \Delta^{(h_\ell)}_o \left\{ (k + 1) y^k_{i_1} y^2_{i_n} \cdot \left[ \text{\ldots degree} = \ell \ \ldots \right] \right\}$$

holds for $\ell = (k + 2) + \ell$, where $k \geq 2$ (variable), but $\ell > 0$ (fixed). We continue to use the notations above and there is no $y_{i_1}$ or $y_{i_n}$ inside the homogeneous polynomial denoted by $\left[ \text{\ldots degree} = \ell \ \ldots \right]$. Let us go on to show

(A.8.1) 

$$\Delta^{(h_\ell)}_o \left\{ y^{k+2}_{i_1} \cdot \left[ \text{\ldots degree} = \ell + 2 \ \ldots \right] \right\} = \Delta^{(h_\ell)}_o \left\{ (k + 1) y^k_{i_1} y^2_{i_n} \cdot \left[ \text{\ldots degree} = \ell + 2 \ \ldots \right] \right\} ,$$

where $k \geq 2$ is even. Let us find the first Laplacians:

(A.8.2) 

$$\Delta_o \left\{ y^{k+2}_{i_1} \cdot \left[ \text{\ldots degree} = \ell + 2 \ \ldots \right] \right\}$$

$$= (k + 2) (k + 1) y^k_{i_1} \cdot \left[ \text{\ldots degree} = \ell + 2 \ \ldots \right]$$

$$+ y^{k+2}_{i_1} \cdot \left\{ \Delta_o \left[ \text{\ldots degree} = \ell + 2 \ \ldots \right] \right\}$$

$$= k (k + 1) y^k_{i_1} \cdot \left[ \text{\ldots degree} = \ell + 2 \ \ldots \right]$$

$$+ 2 (k + 1) y^k_{i_1} \cdot \left[ \text{\ldots degree} = \ell + 2 \ \ldots \right]$$

$$+ y^{k+2}_{i_1} \cdot \left\{ \Delta_o \left[ \text{\ldots degree} = \ell + 2 \ \ldots \right] \right\}$$

(← ↑ degree = \ell →);

$$\Delta_o \left\{ (k + 1) y^k_{i_1} y^2_{i_n} \cdot \left[ \text{\ldots degree} = \ell + 2 \ \ldots \right] \right\}$$

$$= (k + 1) k (k - 1) y^{k-2}_{i_1} y^2_{i_n} \cdot \left[ \text{\ldots degree} = \ell + 2 \ \ldots \right]$$

$$+ 2 (k + 1) y^k_{i_1} \cdot \left[ \text{\ldots degree} = \ell + 2 \ \ldots \right]$$

$$+ (k + 1) y^k_{i_1} y^2_{i_n} \cdot \left\{ \Delta_o \left[ \text{\ldots degree} = \ell + 2 \ \ldots \right] \right\}$$

(← ↑ degree = \ell →),

[ observe that $(k + 2) (k + 1) - 2 (k + 1) = k (k + 1)$].

Via the induction hypothesis, the last two terms in the respective expressions are equal. After simplification, to verify (A.8.1), it suffices to show that

$$\Delta^{(h_\ell - 1)}_o \left\{ y^k_{i_1} \cdot \left[ \text{\ldots degree} = \ell + 2 \ \ldots \right] \right\}$$

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\[
\Delta_o^{(h - \ell - 1)} \left\{ (k - 1) y_{i1}^{k-2} y_{i2}^2 \cdot [\ldots \text{degree} = \ell + 2 \ldots] \right\}.
\]

Applying \(\Delta_o\) on the terms
\[
\left\{ y_{i1}^k \cdot [\ldots \text{degree} = \ell + 2 \ldots] \right\} \text{ and } \left\{ (k - 1) y_{i1}^{k-2} y_{i2}^2 \cdot [\ldots \text{degree} = \ell + 2 \ldots] \right\},
\]
using similar calculation and cancelation as in (A.8.2), we come down gradually to verify
\[(A.8.3) \quad \Delta_o^{(h - \ell - k - \frac{2^2}{2})} \left\{ y_{i1}^4 \cdot [\ldots \text{degree} = \ell + 2 \ldots] \right\} = \Delta_o^{(h - \ell - k - \frac{2^2}{2})} \left\{ 3 y_{i1}^2 y_{i2}^2 \cdot [\ldots \text{degree} = \ell + 2 \ldots] \right\}.
\]

Note that
\[
\ell = (k + 2) + (\ell + 2) \implies \ell = \frac{k - 2}{2} = \frac{4 + (\ell + 2)}{2}.
\]

Apply the Laplacian on the two terms inside the brackets in (A.8.3) and obtain
\[
\Delta_o \left\{ y_{i1}^4 \cdot [\ldots \text{degree} = \ell + 2 \ldots] \right\} = 4 \cdot 3 y_{i1}^2 \cdot [\ldots \text{degree} = \ell + 2 \ldots] +
\]
\[
+ y_{i1}^4 \cdot \{ \Delta_o \cdot [\ldots \text{degree} = \ell + 2 \ldots] \} ; \quad (\leftarrow \uparrow \text{degree} = \ell \rightarrow)
\]
\[
\Delta_o \left\{ 3 y_{i1}^2 y_{i2}^2 \cdot [\ldots \text{degree} = \ell + 2 \ldots] \right\} = 3 \cdot 2 \cdot 1 \cdot [ y_{i1}^2 + y_{i2}^2 ] \cdot [\ldots \text{degree} = \ell + 2 \ldots] +
\]
\[
+ 3 y_{i1}^2 y_{i2}^2 \cdot \{ \Delta_o \cdot [\ldots \text{degree} = \ell + 2 \ldots] \} ; \quad (\leftarrow \uparrow \text{degree} = \ell \rightarrow).
\]

Again we apply the induction hypothesis to cancel the last term in each expression above. Apply the Laplacian again and obtain
\[(A.8.4) \quad \Delta_o \left\{ 4 \cdot 3 y_{i1}^2 \cdot [\ldots \text{degree} = \ell + 2 \ldots] \right\} = 4 \cdot 3 \cdot 2 \cdot [\ldots \text{degree} = \ell + 2 \ldots] + 4 \cdot 3 y_{i1}^2 \cdot \{ \Delta_o \cdot [\ldots \text{degree} = \ell + 2 \ldots] \} ;
\]
\[(A.8.5) \quad \Delta_o \left\{ 3 \cdot 2 \cdot \left[ y_{i1}^2 + y_{i2}^2 \right] \cdot [\ldots \text{degree} = \ell + 2 \ldots] \right\} = 3 \cdot 2 \cdot 4 \cdot [\ldots \text{degree} = \ell + 2 \ldots] +
\]
\[
+ 3 \cdot 2 \cdot \left[ y_{i1}^2 + y_{i2}^2 \right] \cdot \{ \Delta_o \cdot [\ldots \text{degree} = \ell + 2 \ldots] \}.
\]
As \( 4 \cdot 3 \cdot y^2_{l_i} \cdot \{ \Delta_o [\cdots \text{degree} = \ell + 2 \cdots] \} \)
\[ = 2 \cdot 3 \cdot y^2_{l_i} \cdot \{ \Delta_o [\cdots \text{degree} = \ell + 2 \cdots] \} + \]
\[ + 2 \cdot 3 \cdot y^2_{l_i} \cdot \{ \Delta_o [\cdots \text{degree} = \ell + 2 \cdots] \} , \]
\[ 3 \cdot 2 \cdot \left[ y^2_{l_i} + y^2_{n_i} \right] \cdot \{ \Delta_o [\cdots \text{degree} = \ell + 2 \cdots] \} \]
\[ = 3 \cdot 2 \cdot y^2_{l_i} \cdot \{ \Delta_o [\cdots \text{degree} = \ell + 2 \cdots] \} \]
\[ + 3 \cdot 2 \cdot y^2_{n_i} \cdot \{ \Delta_o [\cdots \text{degree} = \ell + 2 \cdots] \} , \]
and \( \Delta_o^{\left(\frac{2+(\ell+2)}{2}\right)} \left[ y^2_{l_i} \cdot \{ \Delta_o [\cdots \text{degree} = \ell + 2 \cdots] \} \right] \)
\[ = \Delta_o^{\left(\frac{2+(\ell+2)}{2}\right)} \left[ y^2_{l_i} \cdot \{ \Delta_o [\cdots \text{degree} = \ell + 2 \cdots] \} \right] (\leftarrow \text{equal to a number}), \]
we apply the remaining order of Laplacian on (A.8.4) and (A.8.5), yielding the same numbers. Hence we verify (A.8.3), and so (A.8.2). This completes the induction step.

\section*{§ A.9. Verification of (A.6.44).}

Let \( A, B \) and \( C \) be numbers such that

(A.9.1) \[ A > 0, \quad A + B + C > 0, \quad \text{and} \quad \beta = \frac{2n}{n-2} . \]

We show that for any number \( \varepsilon > 0 \) small enough, there exists a positive number \( \bar{C}_\beta \) so that

(A.9.2) \[ |(A + B + C)^\beta - A^\beta| \leq \varepsilon A^\beta + \frac{\bar{C}_\beta}{\varepsilon^{n-2}} \cdot (|B|^\beta + |C|^\beta) . \]

In this presentation, the proof of this statement relies on the following (\( \beta \) being relaxed).

\textbf{Lemma A.9.3.} \emph{Let \( \beta \geq 1 \) be given. There is a positive number \( c_o \) such that for any number \( \varepsilon \in (0, c_o) \), we have}

(A.9.4) \[ |(1 + t)^\beta - 1| \leq \varepsilon + \frac{C_\beta}{\varepsilon^\beta} \cdot |t|^\beta \quad \text{for} \quad t \in [-1, \infty) . \]

Here \( c_o \) and \( C_\beta \) do not depend on \( \varepsilon \) or \( t \).
Proof. Via Taylor expansion, there is a positive number $c_o (< 1)$ such that

$$|(1 + t)^\beta - 1| \leq (\beta + 1) |t| \quad \text{for} \quad |t| \leq \frac{c_o}{\beta + 1}.$$  

For $\varepsilon \in (0, c_o)$, consider the case

$$|t| \leq \frac{\varepsilon}{\beta + 1} \implies \frac{|(1 + t)^\beta - 1|}{\varepsilon} \leq \frac{(\beta + 1)|t|}{\varepsilon} \leq 1 \quad \text{[using (A.9.4)]}$$

$$\implies |(1 + t)^\beta - 1| \leq \varepsilon \implies \text{(A.9.4) holds for} \quad |t| \leq \frac{\varepsilon}{\beta + 1}. $$

When

$$1 \geq |t| \geq \frac{\varepsilon}{\beta + 1},$$

we find $C_\varepsilon$ to be large enough so that

$$\frac{2^\beta + 1}{C_\varepsilon \cdot \left[\frac{\varepsilon}{\beta + 1}\right]^\beta} \leq 1, \quad \text{that is,} \quad C_\varepsilon = \frac{1}{\varepsilon^\beta} \cdot (2^\beta + 1) (\beta + 1)^\beta.$$  

It follows that

$$|(1 + t)^\beta - 1| \leq (2^\beta + 1) \leq C_\varepsilon |t|^\beta \quad \text{for} \quad 1 \geq |t| \geq \frac{\varepsilon}{\beta + 1}. $$

When $t > 1$, from (A.9.6)

$$C_\varepsilon > 2^\beta \implies \frac{|(1 + t)^\beta - 1|}{C_\varepsilon t^\beta} \leq \frac{|(2t)^\beta|}{C_\varepsilon t^\beta} \leq 1$$

$$\implies |(1 + t)^\beta - 1| \leq C_\varepsilon t^\beta \quad \text{for} \quad t > 1.$$  

Combining the three cases, we have (A.9.4) with the choice of $C_\beta = (2^\beta + 1) (\beta + 1)^\beta$ as specified in (A.9.6).

\[ \square \]

Proof of (A.9.2). Using (A.8.4), we obtain

$$|(A + B + C)^\beta - A^\beta| = A^\beta \cdot \left|1 + \left[\frac{B + C}{A}\right]\right|^\beta - 1 \leq A^\beta \left(\varepsilon + \frac{C_\beta}{\varepsilon^\beta} \cdot \left|\frac{B + C}{A}\right|^\beta \right) \quad \text{\(A > 0 \& \ A + B + C > 0 \implies \frac{B + C}{A} \geq -1\)}$$

$$\leq A^\beta \left[\varepsilon + \frac{C_\beta}{\varepsilon^\beta} \cdot \left(\left|\frac{B}{A}\right| + \left|\frac{C}{A}\right|\right)^\beta \right] \leq A^\beta \cdot \left\{\varepsilon + \frac{C_\beta}{\varepsilon^\beta} \cdot \left[2^{\beta-1} \left(\left|\frac{B}{A}\right|^\beta + \left|\frac{C}{A}\right|^\beta\right)\right]\right\}$$

$$\leq \varepsilon A^\beta + 2^{\beta-1} \cdot \frac{C_\beta}{\varepsilon^\beta} \cdot \left[|B|^\beta + |C|^\beta\right] \quad \text{[using \((|B| + |C|)^\beta \leq 2^{\beta-1} (|B|^\beta + |C|^\beta)\).}$$
We can take $\bar{C}_\beta = 2^{\beta - 1} \cdot C_\beta$ to obtain (A.9.2), and take $\beta = \frac{2n}{n-2}$ to obtain 7.23. 

\[ \]

§ A.10. Linear approximation to $\left( A_1^{\frac{n+2}{n-2}} - \mathcal{V}_i^{\frac{n+2}{n-2}} \right)$ in case of simple blow-up.

The Taylor expansion

\[ (1 + t)^p = 1 + pt + O(t^2) \cdot (1 + \tau)^{p-2} \quad (\text{here } 0 \leq \tau \leq t) \]

tells us that

\[ a^p = [b + (a - b)]^p = b^p \left[ 1 + \frac{(a - b)}{b} \right]^p = b^p + p \cdot (a - b) b^{p-1} + O(1) (1 + \tau)^{p-2} (a - b)^2 \cdot b^{p-2} \]

for numbers with $a > b > 0$ and $p > 1$. It follows that

\[ A_1^{\frac{n+2}{n-2}} - b^{\frac{n+2}{n-2}} = \left[ \frac{n+2}{n-2} \right] (a - b) b^{\frac{4}{n-2}} + O(1) (1 + \tau)^{\frac{4}{n-2} - 1} (a - b)^2 \cdot b^{\frac{n+2}{n-2} - 2}. \]

Here

\[ 0 \leq \tau \leq \frac{a - b}{b}. \]

For $A_1(\mathcal{Y}) \geq \mathcal{V}_i(\mathcal{Y})$, (A.10.1) implies that

\[ A_1(\mathcal{Y}) - \mathcal{V}_i(\mathcal{Y}) \leq C A_1(\mathcal{Y}) \quad \text{for} \quad |\mathcal{Y}| \leq \rho_o \lambda_i^{-1}. \]

Recalling (2.26) in Proposition 2.24, and also § 2f, we have

\[ C^{-1} \cdot A_1(\mathcal{Y}) \leq \mathcal{V}_i(\mathcal{Y}) \leq C A_1(\mathcal{Y}) \quad \text{for} \quad |\mathcal{Y}| \leq \rho_o \lambda_i^{-1}. \]
It follows that

\[(A.10.4) \quad A_1 (\mathcal{Y}) \geq V_i (\mathcal{Y}) \implies [A_1 (\mathcal{Y})]^{\frac{n+2}{n-2}} - [V_i (\mathcal{Y})]^{\frac{n+2}{n-2}} = \left( \frac{n+2}{n-2} \right) [A_1 (\mathcal{Y})]^{\frac{4}{n-2}} \cdot [A_1 (\mathcal{Y}) - V_i (\mathcal{Y})] + O (1) [A_1 (\mathcal{Y}) - V_i (\mathcal{Y})]^2 \cdot [A_1 (\mathcal{Y})]^{\frac{4}{n-2}-1} .\]

For the opposite case

\[V_i (\mathcal{Y}) > A_1 (\mathcal{Y})\]

we obtain a similar expression as in (A.10.4), with the last term

\[(A.10.5) \quad [A_1 (\mathcal{Y})]^{\frac{4}{n-2}-1} \text{ changed to } [V_i (\mathcal{Y})]^{\frac{4}{n-2}-1} .\]

We obtain

\[
\frac{1}{C_2 \cdot A_1 (\mathcal{Y})} \leq \frac{1}{V_i (\mathcal{Y})} \leq \frac{C_2}{A_1 (\mathcal{Y})} \quad \text{for} \quad |\mathcal{Y}| \leq \rho_0 \lambda_i^{-1} .
\]

Thus the two terms in (A.10.5) have the same order. We come to the conclusion that

\[(A.10.6) \quad [A_1 (\mathcal{Y})]^{\frac{n+2}{n-2}} - [V_i (\mathcal{Y})]^{\frac{n+2}{n-2}} = \left( \frac{n+2}{n-2} \right) [A_1 (\mathcal{Y})]^{\frac{4}{n-2}} \cdot [A_1 (\mathcal{Y}) - V_i (\mathcal{Y})] + O (1) [A_1 (\mathcal{Y}) - V_i (\mathcal{Y})]^2 \cdot [A_1 (\mathcal{Y})]^{\frac{4}{n-2}-1} ,\]

which holds for $|\mathcal{Y}| \leq \rho_0 \lambda_i^{-1}$. (A.10.6) is independent on which value is bigger, $A_1 (\mathcal{Y})$ or $V_i (\mathcal{Y})$. 38
References


