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INTEGRAL MENGER CURVATURE AND RECTIFIABILITY OF n-DIMENSIONAL BOREL SETS IN EUCLIDEAN N-SPACE

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ABSTRACT. In this paper we show that an n-dimensional Borel set in Euclidean N-space with finite integral Menger curvature is n-rectifiable, meaning that it can be covered by countably many images of Lipschitz continuous functions up to a null set in the sense of Hausdorff measure. This generalises Léger's rectifiability result for one-dimensional sets to arbitrary dimension and co-dimension. In addition, we characterise possible integrands and discuss examples known from the literature.

Intermediate results of independent interest include upper bounds of different versions of P. Jones's β-numbers in terms of integral Menger curvature without assuming lower Ahlfors regularity, in contrast to the results of Lerman and Whitehouse [Constr. Approx. 30 (2009), 325–360].

1. INTRODUCTION

For three points $x, y, z \in \mathbb{R}^N$, we denote by $c(x, y, z)$ the inverse of the radius of the circumcircle determined by these three points. This expression is called *Menger curvature* of x, y, z. For a Borel set $E \subset \mathbb{R}^N$, we define by

$$\mathcal{M}_2(E) := \int_E \int_E \int_E c^2(x, y, z)\ \mathrm{d}\mathcal{H}^1(x)\mathrm{d}\mathcal{H}^1(y)\mathrm{d}\mathcal{H}^1(z)$$

the *total Menger curvature* of E, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. In 1999, J.C. Léger proved the following theorem.

Theorem ([19]). *If $E \subset \mathbb{R}^N$ is some Borel set with $0 < \mathcal{H}^1(E) < \infty$ and $\mathcal{M}_2(E) < \infty$, then E is 1-rectifiable; i.e., there exists a countable family of Lipschitz functions $f_i : \mathbb{R} \to \mathbb{R}^N$ such that $\mathcal{H}^1(E \setminus \bigcup_i f_i(\mathbb{R})) = 0$.*

This result is an important step in the proof of Vitushkin's conjecture (for more details see [6, 36]), which states that a compact set with finite one-dimensional Hausdorff measure is removable for bounded analytic functions if and only if it is purely 1-unrectifiable, which means that every 1-rectifiable subset of this set has Hausdorff measure zero. A higher dimensional analogue of Vitushkin's conjecture is proven in [25] but without using a higher dimensional version of Léger's theorem, since in the higher dimensional setting there seems to be no connection between the n-dimensional Riesz transform and curvature (cf. introduction of [25]).

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There exist several generalisations of Léger's result. Hahlomaa proved in [12–14] that if X is a metric space and $\mathcal{M}_2(X) < \infty$,[1] then X is 1-rectifiable. Another version of this theorem dealing with sets of fractional Hausdorff dimension equal or less than $\frac{1}{2}$ is given by Lin and Mattila in [22].

In the present work, we generalise the result of Léger to arbitrary dimension and co-dimension, i.e., for n-dimensional subsets of $\mathbb{R}^N$ where $n \in \mathbb{N}$ satisfies $n < N$. In the case $n = N$ every $E \subset \mathbb{R}^N$ is n-rectifiable. On the one hand, it is quite clear which conclusion we want to obtain, namely that the set E is n-rectifiable, which means that there exists a countable family of Lipschitz functions $f_i : \mathbb{R}^n \to \mathbb{R}^N$ such that $\mathcal{H}^n(E \setminus \bigcup_i f_i(\mathbb{R}^n)) = 0$. On the other hand, it is by no means clear how to define integral Menger curvature for n-dimensional sets. Léger himself suggested an expression that turns out to be improper[2] for our proof, which is strongly inspired by Léger's own strategy (cf. section 3.2). We characterise possible integrands for our result in Definition 3.1, but for now let us start with an explicit example:

$$\mathcal{K}(x_0, \ldots, x_{n+1}) = \frac{\mathcal{H}^{n+1}(\Delta(x_0, \ldots, x_{n+1}))}{\prod_{0 \le i < j \le n+1} d(x_i, x_j)},$$

where the numerator denotes the $(n+1)$-dimensional volume of the simplex $(\Delta(x_0, \ldots, x_{n+1}))$ spanned by the vertices $x_0, \ldots, x_{n+1}$, and $d(x_i, x_j)$ is the distance between x_i and x_j. Using the law of sines, we obtain for $n = 1$:

$$\mathcal{K}(x_0, x_1, x_2) = \frac{\mathcal{H}^2(\Delta(x_0, x_1, x_2))}{d(x_0, x_1)d(x_0, x_2)d(x_1, x_2)} = \frac{1}{4} c(x_0, x_1, x_2).$$

Hence, $\mathcal{K}$ can be regarded as a generalisation of the original Menger curvature for higher dimensions. We set

$$\mathcal{M}_{\mathcal{K}^2}(E) := \int_E \cdots \int_E \mathcal{K}^2(x_0, \ldots, x_{n+1}) \, \mathrm{d}\mathcal{H}^n(x_0) \ldots \mathrm{d}\mathcal{H}^n(x_{n+1}). \tag{1.1}$$

Now we can state our main theorem for this specific integrand (see Theorem 3.5 for the general version).

Theorem 1.1. *If $E \subset \mathbb{R}^N$ is some Borel set with $\mathcal{M}_{\mathcal{K}^2}(E) < \infty$, then E is n-rectifiable.*

Let us briefly review a couple of results for the higher dimensional case. There exist well-known equivalent characterisations of n-rectifiability, for example, in terms of approximating tangent planes [23, Thm. 15.19], orthogonal projections [23, Thm. 18.1, Besicovitch-Federer projection theorem], and in terms of densities [23, Thm. 17.6 and Thm. 17.8 (Preiss's theorem)]. Recently Tolsa and Azzam proved in [35] and [2] a characterisation of n-rectifiability using the so-called β-numbers[3] defined for $k > 1$, $x \in \mathbb{R}^N$, $t > 0$, $p \ge 1$ by

$$\beta_{p;k;\mu}(x, t) := \inf_{P \in \mathcal{P}(N,n)} \left(\frac{1}{t^n} \int_{B(x,kt)} \left(\frac{d(y, P)}{t} \right)^p \mathrm{d}\mu(y) \right)^{\frac{1}{p}},$$

[1] Karl Menger [24] realized that $c(x, y, z)$ can be expressed purely in terms of mutual distances between the points; see [14] for the explicit expression.

[2] Hence, we agree with a remark made by Lerman and Whitehouse at the end of the introduction of [20].

[3] Introduced by P. W. Jones in [15] and [16].

where $\mathcal{P}(N,n)$ denotes the set of all n-dimensional planes in $\mathbb{R}^N$, $d(y,P)$ is the distance of y to the n-dimensional plane P and μ is a Borel measure on $\mathbb{R}^N$. They showed in particular that an $\mathcal{H}^n$-measurable set $E \subset \mathbb{R}^N$ with $\mathcal{H}^n(E) < \infty$ is n-rectifiable *if and only if*

$$\int_0^1 \beta_{2;1;\mathcal{H}^n|_E}(x,r)^2 \frac{\mathrm{d}r}{r} < \infty \qquad \text{for } \mathcal{H}^n\text{-a.e. } x \in E. \tag{1.2}$$

This result is remarkable in relation to our result since the β-numbers and even an expression similar to (1.2) play an important role in our proof. Nevertheless at the moment, we do not see how Tolsa's result could be used to shorten our proof of Theorem 1.1. There are further characterisations of rectifiability by Tolsa and Toro in [38] and [37].

Now we present some of our own intermediate results that finally lead to the proof of Theorem 1.1, but that might also be of independent interest. There is a connection between those β-numbers and integral Menger curvature (1.1). In section 4.2, we prove the following theorem (see Theorem 4.6 for a more general version).

Theorem 1.2. *Let μ be some arbitrary Borel measure on $\mathbb{R}^N$ with compact support such that there is a constant $C \geq 1$ with $\mu(B) \leq C(\operatorname{diam} B)^n$ for all balls $B \subset \mathbb{R}^N$, where* $\operatorname{diam} B$ *denotes the diameter of the ball B. Let $B(x,t)$ be a fixed ball with $\mu(B(x,t)) \geq \lambda t^n$ for some $\lambda > 0$ and let $k > 2$. Then there exist some constants $k_1 > 1$ and $C \geq 1$ such that*

$$\beta_{2;k}(x,t)^2 \leq \frac{C}{t^n} \int_{B(x,k_1 t)} \cdots \int_{B(x,k_1 t)} \chi_D(x_0,\dots,x_n) \\ \times \mathcal{K}^2(x_0,\dots,x_{n+1}) \,\mathrm{d}\mu(x_0)\dots\mathrm{d}\mu(x_{n+1}),$$

where χ_D denotes the characteristic function of the set

$$D = \{(x_0,\dots,x_{n+1}) \in B(x,k_1 t)^{n+2} | d(x_i,x_j) \geq \frac{t}{k_1}, i \neq j\}.$$

A measure μ is said to be n-dimensional Ahlfors regular if and only if there exists some constant $C \geq 1$ so that $\frac{1}{C}(\operatorname{diam} B)^n \leq \mu(B) \leq C(\operatorname{diam} B)^n$ for all balls B with centre on the support of μ. We mention that we do not have to assume for this theorem that the measure μ is n-dimensional Ahlfors regular. We only need the upper bound on $\mu(B)$ for each ball B and the condition $\mu(B(x,t)) \geq \lambda t^n$ for *one* specific ball $B(x,t)$.

Lerman and Whitehouse obtain a comparable result in [20, Thm. 1.1]. The main differences are that, on the one hand, they have to use an n-dimensional Ahlfors regular measure, but, on the other hand, they work in a real separable Hilbert space of possibly infinite dimension instead of $\mathbb{R}^N$. The higher dimensional Menger curvatures they used (see [20, introduction and section 6]) are examples of integrands that also fit in our more general setting.[4] This means that all of our results are valid if one uses their integrands instead of the initial $\mathcal{K}$ presented as an example above.

[4] A characterisation of all possible integrands for our result can be found at the beginning of section 3.1. In section 3.2, we discuss one of the integrands of Lerman and Whitehouse.

In addition to rectifiability, there is the notion of *uniform rectifiability*, which implies rectifiability. A set is uniformly rectifiable if it is Ahlfors regular[5] and if it fulfils a second condition in terms of β-numbers (cf. [5, Thm. 1.57, (1.59)]). In [20] and [21], Lerman and Whitehouse give an alternative characterisation of uniform rectifiability by proving that for an Ahlfors regular set this β-number term is comparable to a term expressed with integral Menger curvature. One of the two inequalities needed is given in [20, Thm. 1.3] and is similar to our following theorem, which is a consequence of Theorem 1.2 in connection with Fubini's theorem (see Theorem 4.7 for a more general version). We emphasise again that in our case the measure μ does not have to be Ahlfors regular.

Theorem 1.3. *Let μ, λ and k be as in the previous theorem. There exists a constant $C \geq 1$ such that*

$$\int \int_0^\infty \beta_{2;k}(x,t)^2 \chi_{\{\mu(B(x,t)) \geq \lambda t^n\}} \frac{\mathrm{d}t}{t} \mathrm{d}\mu(x) \leq C \mathcal{M}_{\mathcal{K}^2}(\mu).$$

In the last few years, several papers have appeared that work with integral Menger curvatures. Some deal with (one-dimensional) space curves and get higher regularity ($C^{1,\alpha}$) of the arc length parametrisation if the integral Menger curvature is finite, e.g. [29, 30]. Others handle higher dimensional objects [17, 18, 32], occasionally using versions of integral Menger curvatures similar to ours.[6] Remarkable are the results of Blatt and Kolasinski [3, 4]. They proved among other things that for $p > n(n+1)$ and some compact n-dimensional C^1 manifold Σ,

$$\int_\Sigma \cdots \int_\Sigma \left(\frac{\mathcal{H}^{n+1}(\Delta(x_0, \dots, x_{n+1}))}{\operatorname{diam}(\Delta(x_0, \dots, x_{n+1}))^{n+2}} \right)^p \mathrm{d}\mathcal{H}^n(x_0), \dots, \mathrm{d}\mathcal{H}^n(x_{n+1}) < \infty$$

is *equivalent* to having a local representation of σ as the graph of a function belonging to the Sobolev Slobodeckij space $W^{2-\frac{n(n+1)}{p},p}$. Finally, we mention that in [31, 33] Menger curvature energies are recently used as knot energies in geometric knot theory to avoid some of the drawbacks of self-repulsive potentials like the Möbius energy [10, 26].

Organisation of this work. In section 3, we give the precise formulation of our main result and discuss some examples of integrands known from several papers working with integral Menger curvatures. In section 4, we present some results for a Borel measure including the general versions of Theorems 1.2 and 1.3, namely Theorems 4.6 and 4.7. The following sections 5 to 8 give the proof of our main result. We remark that all statements in sections 6, 7 and 8, except section 7.1, depend on the construction given in section 6.

2. Preliminaries

2.1. **Basic notation and linear algebra facts.** Let $n, m, N \in \mathbb{N}$ with $1 \leq n < N$ and $1 \leq m < N$. If $E \subset \mathbb{R}^N$ is some subset of $\mathbb{R}^N$, we write $\overline{E}$ for its closure and $\mathring{E}$ for its interior. We set $d(x,y) := |x-y|$ where $x, y \in \mathbb{R}^N$ and $|\cdot|$ is the usual Euclidean norm. Furthermore, for $x \in \mathbb{R}^N$ and $E_1, E_2 \subset \mathbb{R}^N$, we set $d(x, E_2) = \inf_{y \in E_2} d(x,y)$, $d(E_1, E_2) = \inf_{z \in E_1} d(z, E_2)$ and $\#E$ means the number

[5] A set E is n-dimensional Ahlfors regular if and only if the restricted Hausdorff measure $\mathcal{H}^n \llcorner E$ is n-dimensional Ahlfors regular.

[6] Our main result does not work with their integrands, but most of the partial results are valid; cf. section 3.2.

of elements of E. By $B(x,r)$ we denote the closed ball in $\mathbb{R}^N$ with centre x and radius r, and we define by ω_n the n-dimensional volume of the n-dimensional unit ball. Let $G(N,m)$ be the Grassmannian, the space of all m-dimensional linear subspaces of $\mathbb{R}^N$ and $\mathcal{P}(N,m)$ the set of all m-dimensional affine subspaces of $\mathbb{R}^N$. For $P \in \mathcal{P}(N,m)$, we define π_P as the orthogonal projection on P. If $P \in \mathcal{P}(N,m)$, we have that $P - \pi_P(0) \in G(N,m)$; hence $P - \pi_P(0)$ is the linear subspace parallel to P. Furthermore, we set $\pi_P^\perp := \pi_{P-\pi_P(0)}^\perp := \pi_{(P-\pi_P(0))^\perp}$ where $\pi_{(P-\pi_P(0))^\perp}$ is the orthogonal projection on the orthogonal complement of $P - \pi_P(0)$.

Furthermore, for $A \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$, we set $A + x := \{y \in \mathbb{R}^n | y - x \in A\}$. By $\operatorname{span}(A)$, we denote the linear subspace of $\mathbb{R}^N$ spanned by the elements of A. If $A = \{o_1, \dots, o_m\}$ or $A = A_1 \cup A_2$, we may write $\operatorname{span}(o_1, \dots, o_m)$, resp. $\operatorname{span}(A_1, A_2)$, instead of $\operatorname{span}(A)$.

Remark 2.1. Let $b, a, a_i \in \mathbb{R}^N$, $\alpha_i \in \mathbb{R}$ for $i = 1, \dots, l$, $l \in \mathbb{N}$, with $b = a + \sum_{i=1}^l \alpha_i(a_i - a)$ and $P \in \mathcal{P}(N,m)$. Then we have $\pi_P(b) = \pi_P(a) + \sum_{i=1}^l \alpha_i \big[\pi_P(a_i) - \pi_P(a)\big]$ and $d(b,P) \le d(a,P) + \sum_{i=1}^l |\alpha_i| \left(d(a_i,P) + d(a,P)\right)$.

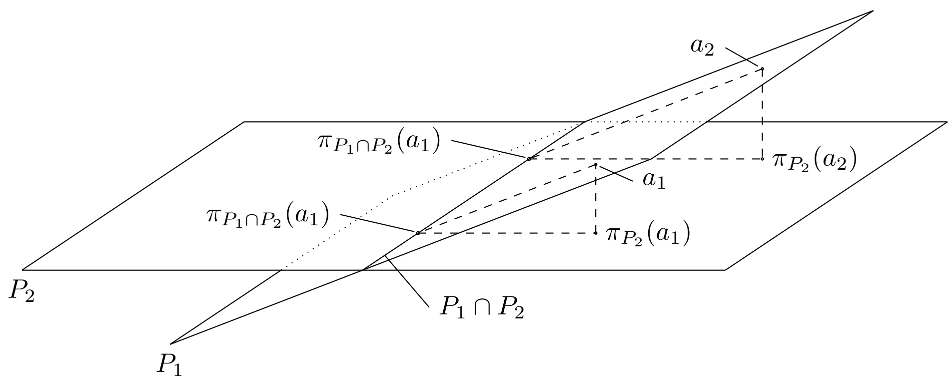

FIGURE 1. Illustration of Lemma 2.2: $\frac{|a_1 - \pi_{P_2}(a_1)|}{|a_1 - \pi_{P_1\cap P_2}(a_1)|} = \frac{|a_2 - \pi_{P_2}(a_2)|}{|a_2 - \pi_{P_1\cap P_2}(a_2)|}$

Lemma 2.2. *Let* $P_1, P_2 \in \mathcal{P}(N,m)$ *with* $\dim P_1 = \dim P_2 = m < N$ *and* $\dim(P_1 \cap P_2) = m - 1$. *For* $a_1, a_2 \in P_1 \setminus P_2$, *we have* $\frac{|a_1 - \pi_{P_2}(a_1)|}{|a_1 - \pi_{P_1\cap P_2}(a_1)|} = \frac{|a_2 - \pi_{P_2}(a_2)|}{|a_2 - \pi_{P_1\cap P_2}(a_2)|}$ *(see Figure* 1*).*

Proof. Translate the whole setting so that P_1, P_2 are linear subspaces. Then express a_1 by an orthonormal base of P_1 and compute that $\frac{|a_1 - \pi_{P_2}(a_1)|}{|a_1 - \pi_{P_1\cap P_2}(a_1)|}$ is independent of a_1. □

2.2. Simplices.

Definition 2.3. Let $x_i \in \mathbb{R}^N$ for $i = 0, 1, \dots, m$. We define $\Delta(x_0, \dots, x_m) = \Delta(\{x_0, \dots, x_m\})$ as the convex hull of the set $\{x_0, \dots, x_m\}$ and call it *simplex* or *m-simplex* if m is the Hausdorff dimension of $\Delta(x_0, \dots, x_m)$. If the vertices of $T = \Delta(x_0, \dots, x_m)$ are in some set $G \subset \mathbb{R}^N$, i.e., $x_0, \dots, x_m \in G$, we simply write $T = \Delta(x_0, \dots, x_m) \in G$. Note, however, that this new notation $T \in G$ does not mean $T \subset G$ unless G is convex.

With $\operatorname{aff}(E)$ we denote the smallest affine subspace of $\mathbb{R}^N$ that contains the set $E \subset \mathbb{R}^N$. If $E = \{x_0\}$, we set $\operatorname{aff}(E) = \{x_0\}$.

Definition 2.4. Let $T = \Delta(x_0, \dots, x_m) \in \mathbb{R}^N$. For $i, j \in \{0, 1, \dots, m\}$ we set

$$\begin{aligned} \mathfrak{fc}_i T &= \mathfrak{fc}_{x_i} T = \Delta(\{x_0, \dots, x_m\} \setminus \{x_i\}), \\ \mathfrak{fc}_{i,j} T &= \mathfrak{fc}_{x_i, x_j} T = \Delta(\{x_0, \dots, x_m\} \setminus \{x_i, x_j\}), \\ \mathfrak{h}_i T &= \mathfrak{h}_{x_i} T = d\big(x_i, \operatorname{aff}(\{x_0, \dots, x_m\} \setminus \{x_i\})\big). \end{aligned}$$

Definition 2.5. Let $T = \Delta(x_0, \dots, x_m)$ be an m-simplex in $\mathbb{R}^N$. If $\mathfrak{h}_i T \geq \sigma$ for all $i = 0, 1, \dots, m$, we call T an *(m, σ)-simplex.*

Definition 2.6. Let $T = \Delta(x_0, \dots, x_m)$ be an m-simplex in $\mathbb{R}^N$. By $\mathcal{H}^m(T)$ we denote the volume of T and we define the *normalized volume* $\mathfrak{v}(T) := m!\ \mathcal{H}^m(T)$ which is the volume of the parallelotope spanned by the simplex T (cf. [28]). We also have a characterisation of $\mathfrak{v}(T)$ by the Gram determinant

$$\mathfrak{v}(T) = \sqrt{\operatorname{Gram}(x_1 - x_0, \dots, x_m - x_0)},$$

where the Gram determinant of vectors $v_1, \dots, v_m \in \mathbb{R}^N$ is defined by

$$\operatorname{Gram}(v_1, \dots, v_m) := \det\big((v_1, \dots, v_m)^T (v_1, \dots, v_m)\big)\,.$$

Lemma 2.7. *Let $T = \Delta(x_0, \dots, x_m)$ be an m-simplex. We have $\frac{\mathfrak{h}_i T}{\mathfrak{h}_i \mathfrak{fc}_j T} = \frac{\mathfrak{h}_j T}{\mathfrak{h}_j \mathfrak{fc}_i T}$.*

Proof. We have $\frac{\mathfrak{h}_i(T)}{\mathfrak{h}_i(\mathfrak{fc}_j T)} = \frac{\mathfrak{v}(T)}{\mathfrak{h}_i(\mathfrak{fc}_j T)\ \mathfrak{v}(\mathfrak{fc}_i T)} = \frac{\mathfrak{h}_j(T)\ \mathfrak{v}(\mathfrak{fc}_j T)}{\mathfrak{h}_i(\mathfrak{fc}_j T)\ \mathfrak{h}_j(\mathfrak{fc}_i T)\ \mathfrak{v}(\mathfrak{fc}_{i,j} T)} = \frac{\mathfrak{h}_j(T)\ \mathfrak{v}(\mathfrak{fc}_j T)}{\mathfrak{h}_j(\mathfrak{fc}_i T)\ \mathfrak{v}(\mathfrak{fc}_j T)}$ $= \frac{\mathfrak{h}_j(T)}{\mathfrak{h}_j(\mathfrak{fc}_i T)}$. □

Lemma 2.8. *Let $0 < h < H$, $1 \leq m \leq N + 1$ and $y_0, x_i \in \mathbb{R}^N$, $i = 0, 1, \dots, m$. If $T_x = \Delta(x_0, \dots, x_m)$ is an (m, H)-simplex and $d(y_0, x_0) \leq h$, then $T_y = \Delta(y_0, x_1, \dots, x_m)$ is an $(m, H - h)$-simplex.*

Proof. We have $\mathfrak{h}_0 T_y \geq \mathfrak{h}_0 T_x - d(x_0, y_0) \geq H - h$. Now, we show that $\mathfrak{h}_1 T_y \geq H - h$. If $m = 1$, we have $\mathfrak{h}_1 T_y = d(y_0, x_1) = \mathfrak{h}_0 T_y$. So we can assume that $m \geq 2$ for the rest of this proof. We set $z_0 := \pi_{\operatorname{aff}(\mathfrak{fc}_1 T_y)}(x_0)$, $T_z := \Delta(z_0, x_1, \dots, x_m)$ and start with some intermediate results:

I. Due to $\mathfrak{h}_0 T_y \geq H - h > 0$, T_y is an m-simplex.

II. We have $d(x_0, z_0) = d(x_0, \operatorname{aff}(\mathfrak{fc}_1 T_y)) \leq d(x_0, y_0) \leq h$.

III. We have $z_0 = x_2 + r_0(y_0 - x_2) + \sum_{j=3}^m r_j(x_j - x_2)$ for some $r_i \in \mathbb{R}$, $i = 0, 3, \dots, m$ because $z_0 \in \operatorname{aff}(\mathfrak{fc}_1 T_y)$.

IV. With III, Remark 2.1 and because of $\pi_{\operatorname{aff}(\mathfrak{fc}_0 T_x)}(x_i) = x_i$ for $i = 2, \dots, m$ we get

$$\mathfrak{h}_0 T_z = |z_0 - \pi_{\operatorname{aff}(\mathfrak{fc}_0 T_x)}(z_0)| = |r_0 y_0 - r_0 \pi_{\operatorname{aff}(\mathfrak{fc}_0 T_x)}(y_0)| = r_0 \mathfrak{h}_0(T_y)$$

and analogously $\mathfrak{h}_0(\mathfrak{fc}_1 T_z) = r_0 \mathfrak{h}_0(\mathfrak{fc}_1 T_y)$.

V. It holds that $\pi_{\operatorname{aff}(\mathfrak{fc}_{0,1} T_x)}(z_0) = \pi_{\operatorname{aff}(\mathfrak{fc}_{0,1} T_x)}(x_0)$, and hence we obtain

$$\begin{aligned} \mathfrak{h}_0(\mathfrak{fc}_1 T_z) &= d(\pi_{\operatorname{aff}(\mathfrak{fc}_1 T_y)}(x_0), \pi_{\operatorname{aff}(\mathfrak{fc}_{0,1} T_x)}(z_0)) \\ &= d(\pi_{\operatorname{aff}(\mathfrak{fc}_1 T_y)}(x_0), \pi_{\operatorname{aff}(\mathfrak{fc}_1 T_y)}(\pi_{\operatorname{aff}(\mathfrak{fc}_{0,1} T_x)}(z_0))) \\ &\leq d(x_0, \pi_{\operatorname{aff}(\mathfrak{fc}_{0,1} T_x)}(z_0)) = \mathfrak{h}_0(\mathfrak{fc}_1 T_x). \end{aligned}$$

Now, with Lemma 2.7 ($i = 1$, $j = 0$, $T = T_y$), IV and V we deduce that

$$\mathfrak{h}_1 T_y \geq \mathfrak{h}_0 T_z \frac{\mathfrak{h}_1(\mathfrak{fc}_0 T_x)}{\mathfrak{h}_0(\mathfrak{fc}_1 T_x)} \geq (\mathfrak{h}_0 T_x - d(x_0, z_0))\, \frac{\mathfrak{h}_1(\mathfrak{fc}_0 T_x)}{\mathfrak{h}_0(\mathfrak{fc}_1 T_x)}.$$

If $\frac{\mathfrak{h}_1(\mathfrak{fc}_0 T_x)}{\mathfrak{h}_0(\mathfrak{fc}_1 T_x)} \geq 1$ this gives us directly $\mathfrak{h}_1 T_y \geq H - h$. In the other case, use Lemma 2.7 and II to obtain $\mathfrak{h}_1 T_y > \mathfrak{h}_1 T_x - d(x_0, z_0) \geq H - h$. Since, for $i = 2, \ldots, m$, the points x_i fulfil the same requirements as x_1, we are able to prove $\mathfrak{h}_i T_y \geq H - h$ for all $i = 1, \ldots, m$ in the same way. So, T_y is an $(m, H - h)$-simplex. □

Lemma 2.9. *Let $C > 0$, $1 \leq m \leq N$ and let $G \subset \mathbb{R}^N$ be a finite set so that for all $(m+1)$-simplices $S = \Delta(x_0, \ldots, x_{m+1}) \in G$, there exists some $i \in \{0, \ldots, m+1\}$ so that $\mathfrak{fc}_i(S)$ is no (m, C)-simplex.*

Then there exists some m-simplex $T_z = \Delta(z_0, \ldots, z_m) \in G$ so that for all $a \in G$, there exists some $i \in \{0, \ldots, m\}$ with $d(a, \operatorname{aff}(\mathfrak{fc}_i(T_z))) < 2C$.

Proof. Since G is finite, we are able to choose $T_z = \Delta(z_0, \ldots, z_m) \in G$ so that

$$\mathfrak{v}(T_z) = \max_{w_0, \ldots, w_m \in G} \mathfrak{v}(\Delta(w_0, \ldots, w_m)). \tag{2.3}$$

We can assume that T_z is an $(m, 2C)$-simplex; otherwise there would exist some $i \in \{0, \ldots, m\}$ with $\mathfrak{h}_i(T_z) < 2C$, and so for all $a \in G$ with (2.3) we would obtain $d(a, \operatorname{aff}(\mathfrak{fc}_i(T_z))) < 2C$.

Now, choose an arbitrary $y_0 \in G$. Set $S := \Delta(y_0, z_0, \ldots, z_m)$. The properties of G imply that one face of S is no (m, C)-simplex. Without loss of generality we assume that $T_y := \mathfrak{fc}_{z_0}(S)$ is not an (m, C)-simplex (but an m-simplex). So there exists some $i \in \{0, \ldots, m\}$ with $\mathfrak{h}_i(T_y) < C$. If $i = 0$, we are done. So let $i \neq 0$. We set $h := \pi_{\operatorname{aff}(\mathfrak{fc}_i T_y)}(z_i)$ and get $\pi_{\operatorname{aff}(\mathfrak{fc}_{0,i} T_y)}(h) = \pi_{\operatorname{aff}(\mathfrak{fc}_i T_y)}[\pi_{\operatorname{aff}(\mathfrak{fc}_{0,i} T_y)}(z_i)]$. This implies

$$d(h, \operatorname{aff}(\mathfrak{fc}_{0,i} T_y)) = d(\pi_{\operatorname{aff}(\mathfrak{fc}_i T_y)}(z_i), \pi_{\operatorname{aff}(\mathfrak{fc}_i T_y)}[\pi_{\operatorname{aff}(\mathfrak{fc}_{0,i} T_y)}(z_i)]) \leq \mathfrak{h}_i(\mathfrak{fc}_0 T_y). \tag{2.4}$$

Now, we use Lemma 2.2, with $a_1 = y_0$, $a_2 = h \in P_1 := \operatorname{aff}(\mathfrak{fc}_i(T_y))$, $P_2 := \operatorname{aff}(\mathfrak{fc}_i(T_z))$, $P_1 \cap P_2 = \operatorname{aff}(\mathfrak{fc}_{0,i}(T_y))$ and (2.4) to obtain

$$\mathfrak{h}_0(\mathfrak{fc}_i T_y) \leq \mathfrak{h}_i(\mathfrak{fc}_0 T_y) \frac{d(z_i, \operatorname{aff}(\mathfrak{fc}_i(T_z)))}{d(z_i, \operatorname{aff}(\mathfrak{fc}_i(T_z))) - d(z_i, h)}.$$

Now use (2.3) to get $d(y_0, \operatorname{aff}(\mathfrak{fc}_i(T_z))) \leq d(z_i, \operatorname{aff}(\mathfrak{fc}_i(T_z)))$ and deduce with $d(z_i, \operatorname{aff}(\mathfrak{fc}_i(T_z))) = \mathfrak{h}_i T_z \geq 2C$ and $d(z_i, h) = \mathfrak{h}_i(T_y) < C$ that $\mathfrak{h}_0(\mathfrak{fc}_i T_y) < 2\mathfrak{h}_i(\mathfrak{fc}_0 T_y)$. Finally, with Lemma 2.7, we have $d(y_0, \operatorname{aff}(\mathfrak{fc}_0(T_z))) = \mathfrak{h}_0(T_y) = \mathfrak{h}_i(T_y) \frac{\mathfrak{h}_0(\mathfrak{fc}_i T_y)}{\mathfrak{h}_i(\mathfrak{fc}_0 T_y)} < 2C$. □

Lemma 2.10. *Let $H > 0$ and $1 \leq m \leq N$, and let $D \subset \mathbb{R}^N$ be a bounded set. Assume that every simplex $S = \Delta(y_0, \ldots, y_m) \in D$ is not an (m, H)-simplex. Then there exists some $l \in \mathbb{N} \cup \{0\}$, $l \leq m - 1$ and $x_0, \ldots, x_l \in \overline{D}$ so that $\overline{D} \subset U_H(\operatorname{aff}(x_0, \ldots, x_l)) = \{x \in \mathbb{R}^N | d(x, \operatorname{aff}(x_0, \ldots, x_l)) \leq H\}$.*

Proof. We assume $\#D \geq 2$; otherwise the statement is trivial. Let $l \in \{0, \ldots, m-1\}$ be the largest value such that there exists an (l, H)-simplex in D. If $l = 0$, we have $\overline{D} \subset U_H(\operatorname{aff}(x_0)) = B(x_0, H)$ for an arbitrary $x_0 \in D$.

Now suppose $l \geq 1$. Since D is bounded, there exists $x_0, \ldots, x_l \in \bar{D}$ such that the volume $K := \mathfrak{v}(\triangle(x_0, \ldots, x_l))$ is maximal. For some arbitrary $x_{l+1} \in \bar{D}$ the definition of l and Lemma 2.8 imply that $\triangle(x_0, \ldots, x_l)$ is not an $(l+1, H)$-simplex. Hence there exists some $\tilde{l} \in \{0, \ldots, l+1\}$ so that $\mathfrak{h}_{\tilde{l}}(T) < H$. Furthermore we have $\mathfrak{v}(\mathfrak{fc}_{\tilde{l}}(T)) \leq K$ and $\mathfrak{v}(\mathfrak{fc}_{l+1}(T)) = K$, which implies $\mathfrak{h}_{l+1}(T) \leq H$. It follows that $\overline{D} \subset U_H(\operatorname{aff}(x_0, \ldots, x_l))$ because $x_{l+1} \in \overline{D}$ was arbitrarily chosen. □

Lemma 2.11. *Let $1 \le m \le N-1$, let B be a closed ball in $\mathbb{R}^N$ and let $F \subset B$ be an $\mathcal{H}^m$-measurable set with $\mathcal{H}^m(F) = \infty$. There exist a small constant $0 < \sigma = \sigma(F,B) \le \frac{\operatorname{diam} B}{2}$ and some $(m+1,(m+3)\sigma)$-simplex $T = \Delta(x_0,\dots,x_{m+1}) \in B$ with $\mathcal{H}^m(B(x_0,\sigma)\cap F) = \infty$ and $\mathcal{H}^m(B(x_i,\sigma)\cap F) > 0$ for all $i \in \{1,\dots,m+1\}$.*

Proof. We set $\mu := \mathcal{H}^m \llcorner F$. Since $\mu(B) = \infty$ there exists some $x_0 \in B$ with $\mu(B(x_0,h)) = \infty$ for all $h > 0$.

There exists some $c_1 > 0$ with $\mu(B \setminus \mathring{B}(x_0,c_1)) > 0$. With Lemma A.3, there exists some $x_1 \in B \setminus \mathring{B}(x_0,c_1)$ with $\mu(B(x_1,h)) > 0$ for all $h > 0$ and the simplex T_1 fulfils $\mathfrak{h}_1(T_1) = d(x_0,x_1) \ge c_1$.

Now we assume that we already have $c_l > 0$ and a simplex $T_l = \Delta(x_0,\dots,x_l) \in \mathbb{R}^N$ with $\mathfrak{h}_l(T_l) \ge c_l$ and $\mu(B(x_i,h)) > 0$ for all $i \in \{0,\dots,l\}$ and $h > 0$ where $l \le m$. So there exists some $0 < c_{l+1} < \frac{c_l}{2}$ with $\mu\left(\left(F \cap B\left(x_0,\frac{c_l}{2}\right)\right) \setminus \mathring{U}_{c_{l+1}}(\operatorname{aff}(x_0,\dots,x_l))\right)$ > 0 and, with Lemma A.3, there exists some $x_{l+1} \in F \subset B$ so that $T_{l+1} := \Delta(x_0,\dots,x_{l+1})$ fulfils $\mathfrak{h}_{l+1}(T_{l+1}) \ge c_{l+1}$ and $\mu(B(x_{l+1},h)) > 0$ for all $h > 0$.

Since $\mathfrak{h}_i(T_i) \ge C_i > 0$ for all $i \in \{1,\dots,m+1\}$ we obtain $\mathfrak{v}(T) > 0$, and hence there exists some constant $c > 0$ so that $T := T_{m+1}$ is an $(m+1,c)$-simplex.

To conclude the proof set $\sigma := \frac{c}{m+3}$. □

2.3. **Angles between affine subspaces.**

Definition 2.12. For $G_1, G_2 \in G(N,m)$, we define $\sphericalangle(G_1,G_2) := \|\pi_{G_1} - \pi_{G_2}\|$, where the right-hand side is the operator norm of the linear map $\pi_{G_1} - \pi_{G_2}$. For $P_1, P_2 \in \mathcal{P}(N,m)$, we define $\sphericalangle(P_1,P_2) := \sphericalangle(P_1 - \pi_{P_1}(0), P_2 - \pi_{P_2}(0))$.

Lemma 2.13. *Let $P_1, P_2 \in \mathcal{P}(N,m)$ with $\sphericalangle(P_1,P_2) < 1$ and $x, y \in P_1$. We have*

$$d(x,y) \le \frac{d(\pi_{P_2}(x),\pi_{P_2}(y))}{1-\sphericalangle(P_1,P_2)} \quad \text{and} \quad d(\pi_{P_2}^{\perp}(x),\pi_{P_2}^{\perp}(y)) \le \frac{\sphericalangle(P_1,P_2)}{1-\sphericalangle(P_1,P_2)} d(\pi_{P_2}(x),\pi_{P_2}(y)).$$

Proof. First assume that $P_1, P_2 \in G(N,m)$. With $z := \frac{x-y}{|x-y|} \in P_1$ and $\pi_{P_2}^{\perp}(z) + \pi_{P_2}(z) = z = \pi_{P_1}(z)$ we get $|\pi_{P_2}^{\perp}(x) - \pi_{P_2}^{\perp}(y)| = |x-y||\pi_{P_2}^{\perp}(z) + \pi_{P_2}(z) - \pi_{P_2}(z)| \le |x-y|\sphericalangle(P_1,P_2)$, This implies $d(x,y) \le d(\pi_{P_2}(x),\pi_{P_2}(y)) + d(x,y)\sphericalangle(P_1,P_2)$. These two estimates give the assertion in the case $P_1, P_2 \in G(N,m)$. Now choose $t_1 \in P_1$, $t_2 \in P_2$ and apply this result to $P_1 - t_1, P_2 - t_2 \in G(N,m)$. □

Corollary 2.14. *Let $P \in \mathcal{P}(N,m)$, $Q \in G(N,m)$ and $\sphericalangle(P,Q) < 1$. There exists some affine map $a : Q \to Q^{\perp}$ with $G(a) = P$, where $G(a)$ is the graph of the map a, and a is Lipschitz continuous with Lipschitz constant $\frac{\sphericalangle(P,Q)}{1-\sphericalangle(P,Q)}$.*

Proof. Set $a(y) = \pi_{P_2}^{\perp}(\pi_{P_2}^{-1}\big|_{P_1}(y))$ and use Lemma 2.13. □

Corollary 2.15. *Let $G_1, G_2 \in G(N,m)$ and let $o_1,\dots,o_m$ be an orthonormal basis of G_1. If $d(o_i,G_2) \le \tilde{\sigma} \le \tilde{\sigma}_1 := 10^{-1}(10^m+1)^{-1}$, then $\sphericalangle(G_1,G_2) \le 4m(10^m+1)\tilde{\sigma}$.*

Proof. For $i = 1,\dots,m$, set $h_i := \pi_{P_2}(o_i)$ and use Lemma 2.3 from [34]. □

For $x, y \in \mathbb{R}^N$, we set $\langle x,y\rangle$ to be the usual scalar product in $\mathbb{R}^N$.

Lemma 2.16. *Let $C, \hat{C} \ge 1$ and $t > 0$, and let $S = \Delta(y_0,\dots,y_m)$ be an $(m,\frac{t}{C})$-simplex with $S \subset B(x,\hat{C}t)$, $x \in \mathbb{R}^N$. There exist an orthonormal basis $(o_1,\dots,o_m)$*

of $\operatorname{span}(y_1 - y_0, \ldots, y_m - y_0)$ *and* $\gamma_{l,r} \in \mathbb{R}$ *so that for all* $1 \le l \le m$ *and* $1 \le r \le l$ *we have*

$$o_l := \sum_{r=1}^{l} \gamma_{l,r}(y_r - y_0) \qquad \textit{and} \qquad |\gamma_{l,r}| \le (2lC\hat{C})^l \frac{C}{t} \le (2mC\hat{C})^m \frac{C}{t}.$$

Proof. We set $z_i := y_i - y_0$ for all $i = 0, \ldots, m$, and $R := \Delta(z_0, \ldots, z_m) = S - y_0$. We obtain for all $i \in \{1, \ldots, m\}$ (S is an $(m, \frac{t}{C})$-simplex)

$$d(z_i, \operatorname{aff}(z_0, \ldots, z_{i-1})) \ge \mathfrak{h}_i(R) = \mathfrak{h}_i(S) \ge \tfrac{t}{C}. \tag{2.5}$$

Due to $\mathfrak{h}_i(R) \ge \frac{t}{C} > 0$, we have that $(z_1, \ldots, z_m)$ are linearly independent. So with the Gram-Schmidt process we are able to define some orthonormal basis of the m-dimensional linear subspace $\operatorname{span}(z_1, \ldots, z_m)$,

$$o_1 := \gamma_{1,1} z_1, \qquad\qquad o_{l+1} := \gamma_{l+1,l+1} z_{l+1} - \gamma_{l+1,l+1} \sum_{i=1}^{l} \langle z_{l+1}, o_i \rangle o_i,$$

where $\gamma_{1,1} := \frac{1}{|z_1|}$ and $\gamma_{l+1,l+1} := \frac{1}{d(z_{l+1}, \operatorname{aff}(z_0, \ldots, z_l))}$. Furthermore we define recursively

$$\gamma_{l+1,r} := -\sum_{i=r}^{l} \gamma_{l+1,l+1} \langle z_{l+1}, o_i \rangle \gamma_{i,r}$$

for $r \in \{1, \ldots, l\}$. Now we prove by induction that $\gamma_{l,r}$ fulfil the desired properties. We have $o_1 = \gamma_{1,1}(y_1 - y_0)$ and (2.5) implies $|\gamma_{1,1}| \le \frac{C}{t}$. Now let $1 \le l \le m$. We assume that, for all $i \in \{1, \ldots, l\}$, $j \in \{1, \ldots, i\}$, we have $o_i = \sum_{r=1}^{i} \gamma_{i,r} z_r$ and $|\gamma_{i,j}| \le (2lC\hat{C})^l \frac{C}{t}$. We obtain

$$o_{l+1} = \gamma_{l+1,l+1} z_{l+1} - \sum_{i=1}^{l} \sum_{r=1}^{i} \gamma_{l+1,l+1} \langle z_{l+1}, o_i \rangle \gamma_{i,r} z_r = \sum_{r=1}^{l+1} \gamma_{l+1,r} z_r.$$

If $r = l + 1$, (2.5) implies $|\gamma_{l+1,r}| \le \frac{C}{t}$, and if $1 \le r \le l$, we get with $|z_{l+1}| \le 2\hat{C}t$ that

$$|\gamma_{l+1,r}| \overset{(2.5)}{\le} \sum_{i=r}^{l} \frac{C}{t} |z_{l+1}| (2lC\hat{C})^l \frac{C}{t} < (2(l+1)C\hat{C})^{l+1} \frac{C}{t}.$$

□

Lemma 2.17. *Let* $C, \hat{C} \ge 1$, $t > 0$, $0 < \sigma \le \left(10(10^m + 1)mC(2mC\hat{C})^m\right)^{-1}$, *and* $P_1, P_2 \in \mathcal{P}(N, m)$, *and let* $S = \Delta(y_0, \ldots, y_m) \subset P_1$ *be an* $(m, \frac{t}{C})$*-simplex with* $S \subset B(x, \hat{C}t)$, $x \in \mathbb{R}^N$ *and* $d(y_i, P_2) \le t\sigma$ *for all* $i \in \{0, \ldots, m\}$. *It follows that*

$$\sphericalangle(P_1, P_2) \le 4m(10^m + 1)\left(2mC(2mC\hat{C})^m\right)\sigma.$$

Proof. Use Lemma 2.16 to get some orthonormal basis of $\operatorname{span}(y_1 - y_0, \ldots, y_m - y_0)$ and $\gamma_{l,r} \in \mathbb{R}$. We set $\hat{y}_0 := \pi_{P_2}(y_0)$ and we obtain for $1 \le l \le m$,

$$d(o_l, P_2 - \hat{y}_0) \le \sum_{r=1}^{l} |\gamma_{l,r}| (d(y_r, P_2) + d(y_0, P_2)) \le 2mC(2mC\hat{C})^m \sigma.$$

Setting $\tilde{\sigma} = 2mC(2mC\hat{C})^m \sigma \le \frac{1}{10(10^m+1)}$ the assertion follows with Corollary 2.15 ($G_1 = P_1 - y_0$, $G_2 = P_2 - \hat{y}_0$).

□

Lemma 2.18. *Let $\sigma > 0$, $t \geq 0$, $P_1, P_2 \in \mathcal{P}(N,m)$ with $\sphericalangle(P_1,P_2) \leq \sigma$ and assume that there exist $p_1 \in P_1$, $p_2 \in P_2$ with $d(p_1,p_2) \leq t\sigma$. Then $d(w,P_2) \leq \sigma(d(w,p_1)+t)$ holds for every $w \in P_1$.*

Proof. For $w \in P_1$, set $\tilde{w} := w - p_1 \in P_1 - p_1$. We obtain

$$d(w,P_2) \leq |\tilde{w}||\tfrac{\tilde{w}}{|\tilde{w}|} - \pi_{P_2-p_2}(\tfrac{\tilde{w}}{|\tilde{w}|})| + d(p_1,p_2) \leq |\tilde{w}|\sphericalangle(P_1-p_1,P_2-p_2)+t\sigma. \quad \square$$

3. Integral Menger curvature and rectifiability

3.1. **Main result.** Let $n, N \in \mathbb{N}$ with $1 \leq n < N$. We start with some definitions.

Definition 3.1 (Proper integrand)**.** Let $\mathcal{K} : \left(\mathbb{R}^N\right)^{n+2} \to [0,\infty)$ and $p > 1$. We say that $\mathcal{K}^p$ is a *proper integrand* if it fulfils the following four conditions:

- $\mathcal{K}$ is $(\mathcal{H}^n)^{n+2}$-measurable, where $(\mathcal{H}^n)^{n+2}$ denotes the $n+2$-times product measure of $\mathcal{H}^n$.
- There exist some constants $c = c(n,\mathcal{K},p) \geq 1$ and $l = l(n,\mathcal{K},p) \geq 1$ so that, for all $t > 0$, $C \geq 1$, $x \in \mathbb{R}^N$ and all $(n, \frac{t}{C})$-simplices $\Delta(x_0,\dots,x_n) \subset B(x,Ct)$, we have
$$\left(\frac{d(w,\operatorname{aff}(x_0,\dots,x_n))}{t}\right)^p \leq cC^l t^{n(n+1)}\mathcal{K}^p(x_0,\dots,x_n,w)$$
for all $w \in B(x,Ct)$.
- For all $t > 0$, we have $t^{n(n+1)}\mathcal{K}^p(tx_0,\dots,tx_{n+1}) = \mathcal{K}^p(x_0,\dots,x_{n+1})$.
- For every $b \in \mathbb{R}^N$, we have $\mathcal{K}(x_0+b,\dots,x_{n+1}+b) = \mathcal{K}(x_0,\dots,x_{n+1})$.

Remark 3.2. If instead of the first condition we have that $\mathcal{K}$ is $(\mu)^{n+2}$-measurable for some Borel measure μ on $\mathbb{R}^N$, we call $\mathcal{K}$ *μ-proper.*

Definition 3.3. (i) We call a Borel set $E \subset \mathbb{R}^N$ *purely n-unrectifiable* if for every Lipschitz continuous function $\gamma : \mathbb{R}^n \to \mathbb{R}^N$, we have $\mathcal{H}^n(E \cap \gamma(\mathbb{R}^n)) = 0$.

(ii) A Borel set $E \subset \mathbb{R}^N$ is *n-rectifiable* if there exists some countable family of Lipschitz continuous functions $\gamma_i : \mathbb{R}^n \to \mathbb{R}^N$ so that $\mathcal{H}^n(E \setminus \bigcup_{i=1}^{\infty}\gamma_i(\mathbb{R}^n)) = 0$.

Definition 3.4 (Integral Menger curvature)**.** Let $E \subset \mathbb{R}^N$ be a Borel set and μ be a Borel measure on $\mathbb{R}^N$. We define the *integral Menger curvature* of E and μ with integrand $\mathcal{K}^p$ by $\mathcal{M}_{\mathcal{K}^p}(E) := \mathcal{M}_{\mathcal{K}^p}(\mathcal{H}^N\big|_E)$ and

$$\mathcal{M}_{\mathcal{K}^p}(\mu) := \int\cdots\int \mathcal{K}^p(x_0,\dots,x_{n+1})\,\mathrm{d}\mu(x_0)\dots\mathrm{d}\mu(x_{n+1}).$$

Now we can state our main result.

Theorem 3.5. *Let $E \subset \mathbb{R}^N$ be a Borel set with $\mathcal{M}_{\mathcal{K}^2}(E) < \infty$, where $\mathcal{K}^2$ is some proper integrand. Then E is n-rectifiable.*

3.2. **Examples of proper integrands.** We start with flat simplices.

Definition 3.6. We define the $(\mathcal{H}^n)^{n+2}$-measurable set

$$X_0 := \left\{(x_0,\dots,x_{n+1}) \in (\mathbb{R}^N)^{n+2} \middle| \operatorname{Gram}(x_1-x_0,\dots,x_{n+1}-x_0) = 0\right\}$$

(the Gram determinant is defined in Definition 2.6), which is the set of all simplices with $n+2$ vertices in $\mathbb{R}^N$ which span at most an n-dimensional affine subspace.

The following lemma is helpful to prove that a given integrand fulfils the second condition of a proper integrand.

Lemma 3.7. *Let $t > 0$, $C \geq 1$, $x \in \mathbb{R}^N$, and $w \in B(x, Ct)$, and let $S = \Delta(x_0, \dots, x_n) \subset B(x, Ct)$ be some $(n, \frac{t}{C})$-simplex. Setting $S_w = \Delta(x_0, \dots, x_n, w)$, $A(S_w)$ as the surface area of the simplex S_w and choosing $i, j \in \{0, \dots, n\}$ with $j \neq i$ we have the following statements:*

- $\frac{t}{C} \leq d(x_i, x_j) \leq \operatorname{diam}(S_w) \leq 2Ct$,
- $d(x_i, w) \leq 2Ct$,
- $\frac{t^n}{C^n n!} \leq \mathcal{H}^n(S) \leq \frac{(2C)^n}{n!} t^n$,
- $\mathcal{H}^n(S) \leq A(S_w) \leq [(n+1)2C^2 + 1]\mathcal{H}^n(S)$,
- $d(w, \operatorname{aff}(x_0, \dots, x_n)) = n \frac{\mathcal{H}^{n+1}(S_w)}{\mathcal{H}^n(S)}$.

Proof. Since S is an $(n, \frac{t}{C})$-simplex, we have

$$\frac{t}{C} \leq \mathfrak{h}_i(S) \leq d(x_i, x_j) \leq \operatorname{diam}(S_w) = \max_{l,m \in \{0,\dots,n\}} \{d(x_l, x_m), d(x_l, w)\} \leq 2Ct, \tag{3.1}$$

and because of $x_i, w \in B(x, Ct)$, we get $d(x_i, w) \leq 2Ct$. Now, we conclude that $\mathcal{H}^n(S) = \frac{1}{n!} \prod_{l=0}^{n-1} d(x_l, \operatorname{aff}(x_{l+1}, \dots, x_n))$, which implies

$$\frac{t^n}{C^n n!} \overset{(3.1)}{\leq} \frac{1}{n!} \prod_{l=0}^{n-1} \mathfrak{h}_l(S) \leq \mathcal{H}^n(S) \leq \frac{1}{n!} \prod_{l=0}^{n-1} d(x_l, x_n) \overset{(3.1)}{\leq} \frac{(2C)^n}{n!} t^n.$$

Using $\mathfrak{h}_w(\mathfrak{fc}_i(S_w)) \leq d(w, x_j) \leq 2Ct$, we obtain

$$\begin{aligned} \mathcal{H}^n(\mathfrak{fc}_i(S_w)) &= \frac{1}{n} \mathfrak{h}_w(\mathfrak{fc}_i(S_w)) \mathcal{H}^{n-1}(\mathfrak{fc}_{i,w}(S_w)) \\ &\overset{(3.1)}{\leq} \frac{1}{n} 2C^2 \mathfrak{h}_i(S) \mathcal{H}^{n-1}(\mathfrak{fc}_i(S)) = 2C^2 \mathcal{H}^n(S), \end{aligned}$$

so that with $A(S_w) = \sum_{i=0}^n \mathcal{H}^n(\mathfrak{fc}_i S_w) + \mathcal{H}^n(\mathfrak{fc}_w S_w)$ and $\mathfrak{fc}_w(S_w) = S$, we get

$$\mathcal{H}^n(S) \leq A(S_w) \leq [(n+1)2C^2 + 1]\mathcal{H}^n(S).$$

Finally, using that $S = \mathfrak{fc}_w(S_w)$, we deduce that

$$d(w, \operatorname{aff}(x_0, \dots, x_n)) = \mathfrak{h}_w(S_w) = \frac{\mathfrak{h}_w(S_w) \cdot \mathcal{H}^n(\mathfrak{fc}_w(S_w))}{\mathcal{H}^n(S)} = \frac{n\mathcal{H}^{n+1}(S_w)}{\mathcal{H}^n(S)}. \qquad \square$$

Now we can state some examples of proper integrands. Use the previous lemma to verify the second condition. We define all following examples to be 0 on X_0 and will only give an explicit definition on $(\mathbb{R}^N)^{n+2} \setminus X_0$. We mention that our main result is only valid for all integrands which are proper for integrability exponent $p = 2$.

Proper integrands with exponent 2. We start with the one used in the introduction of this work. Let $x_0, \dots, x_{n+1} \in (\mathbb{R}^N)^{n+2} \setminus X_0$ and set

$$\mathcal{K}_1(x_0, \dots, x_{n+1}) := \frac{\mathcal{H}^{n+1}(\Delta(x_0, \dots, x_{n+1}))}{\Pi_{0 \leq i < j \leq n+1} d(x_i, x_j)};$$

then $\mathcal{K}_1^2$ is proper. The next proper integrand is used by Lerman and Whitehouse in [20, 21]:

$$\mathcal{K}_2^2(x_0, \dots, x_{n+1}) := \frac{1}{n+2} \cdot \frac{\operatorname{Vol}_{n+1}(\Delta(x_0, \dots, x_{n+1}))^2}{\operatorname{diam}(\Delta(x_0, \dots, x_{n+1}))^{n(n+1)}} \sum_{i=0}^{n+1} \frac{1}{\prod_{\substack{j=0 \\ j \neq i}}^{n+1} |x_j - x_i|^2},$$

where Vol_{n+1} is $(n+1)!$ times the volume of the simplex $\Delta(x_0,\dots,x_{n+1})$, which is equal to the volume of the parallelotope spanned by this simplex; cf. Definition 2.6. The following proper integrand, $\mathcal{K}_3^2$, is mentioned among others in [20, section 6]:

$$\mathcal{K}_3(x_0,\dots,x_{n+1}) := \frac{\mathcal{H}^{n+1}(\Delta(x_0,\dots,x_{n+1}))}{\operatorname{diam}\Delta(x_0,\dots,x_{n+1})^{\frac{(n+1)(n+2)}{2}}}.$$

Proper integrands with exponents different from 2. Now we present some integrands for integral Menger curvature used in several papers, where the scaling behaviour implies that our main result cannot be applied. Nevertheless, most of our partial results are valid also for these integrands. The first integrand we consider was introduced for $n=2, N=3$ in [32],

$$\mathcal{K}_4(x_0,\dots,x_{n+1}) := \frac{V(T)}{A(T)(\operatorname{diam}T)^2},$$

where $V(T)$ is the volume of the simplex $T=\Delta(x_0,\dots,x_{n+1})$ and $A(T)$ is the surface area of T. $\mathcal{K}_4^p$ is a proper integrand with $p=n(n+1)$. The next one, $\mathcal{K}_5^p$, is a proper integrand with $p=n(n+1)$ and is used, for example, in [4, 18]:

$$\mathcal{K}_5(x_0,\dots,x_{n+1}) := \frac{\mathcal{H}^{n+1}(\Delta(x_0,\dots,x_{n+1}))}{\operatorname{diam}(\Delta(x_0,\dots,x_{n+1}))^{n+2}}.$$

Finally, Léger suggested the following integrand in [19] for a higher dimensional analogue of his theorem. Unfortunately, we cannot confirm his suggestion. This one, $\mathcal{K}_6^p$, is a proper integrand with $p=(n+1)$ where

$$\mathcal{K}_6(x_0,\dots,x_{n+1}) := \frac{d(x_{n+1},\operatorname{aff}(x_0,\dots,x_n))}{d(x_{n+1},x_0)\dots d(x_{n+1},x_n)}.$$

Hence our main result does *not* apply for $n\neq 1$. For $n=1$ up to a factor of 2, this integrand gives the inverse of the circumcircle of the three points x_0,x_1,x_2.

4. β-numbers

In this section, let $C_0 \geq 10$ and let μ be a Borel measure on $\mathbb{R}^N$ with compact support F that is upper Ahlfors regular, i.e.,

(B) for every ball B we have $\mu(B) \leq C_0(\operatorname{diam}B)^n$.

If $B=B(x,r)$ is some ball in $\mathbb{R}^N$ with centre x and radius r and $t\in(0,\infty)$, then we set $tB := B(x,tr)$. Distinguish this notation from the case $t\Upsilon = \{tz | z\in\Upsilon\}$ where $\Upsilon\subset\mathbb{R}^N$ is some arbitrary set. Furthermore, in this and the following sections, we assume that every ball is closed. We need this to apply Vitali's and Besicovitch's covering theorems. By C, we denote a generic constant with a fixed value which may change from line to line.

4.1. **Measure quotient.**

Definition 4.1 (Measure quotient)**.** For a ball $B=B(x,t)$ with centre $x\in\mathbb{R}^N$, radius $t>0$ and a μ-measurable set $\Upsilon\subset\mathbb{R}^N$, we define the *measure quotient*

$$\delta(B\cap\Upsilon)=\delta_\mu(B\cap\Upsilon) := \frac{\mu(B(x,t)\cap\Upsilon)}{t^n}.$$

In most instances, we will use the special case $\Upsilon=\mathbb{R}^N$ and write $\delta(B)$ instead of $\delta(B\cap\mathbb{R}^N)$.

This measure quotient compares the amount of the support F contained in a ball with the size of this ball. The following lemma states that if we have a lower control on the measure quotient of some ball, then we can find a not too flat simplex contained in this ball, where at each vertex we have a small ball with a lower control on its quotient measure.

Lemma 4.2. *Let $0 < \lambda \leq 2^n$ and let $N_0 = N_0(N)$ be the constant from Besicovitch's covering theorem* [7, 1.5.2, Thm. 2] *depending only on the dimension N. There exist constants $C_1 := \frac{4 \cdot 120^n n^{n+1} N_0 C_0}{\lambda} > 3$ and $C_2 := \frac{2^{n+2} N_0 C_1^n}{\lambda} > 1$ so that for a given ball $B(x,t)$ and some μ-measureable set Υ with $\delta(B(x,t) \cap \Upsilon) \geq \lambda$, there exists some $T = \Delta(x_0, \ldots, x_{n+1}) \in F \cap B(x,t) \cap \Upsilon$ so that $\mathfrak{fc}_i(T)$ is an $(n, 10n\frac{t}{C_1})$-simplex and $\mu\left(B\left(x_i, \frac{t}{C_1}\right) \cap B(x,t) \cap \Upsilon\right) \geq \frac{t^n}{C_2}$ for all $i \in \{0, \ldots, n+1\}$.*

Proof. Let $B(x,t)$ be the ball with $\delta(B(x,t) \cap \Upsilon) \geq \lambda$ and $\mathcal{F} := \{B(y, \frac{t}{C_1}) | y \in F \cap B(x,t) \cap \Upsilon\}$. With Besicovitch's covering theorem [7, 1.5.2, Thm. 2] we get $N_0 = N_0(n)$ families $\mathcal{B}_m \subset \mathcal{F}$, $m = 1, \ldots, N_0$, of disjoint balls so that $F \cap B(x,t) \cap \Upsilon \subset \bigcup_{m=1}^{N_0} \dot{\bigcup}_{B \in \mathcal{B}_m} B$. We have

$$\lambda \leq \frac{1}{t^n} \mu \left(\bigcup_{m=1}^{N_0} \bigcup_{B \in \mathcal{B}_m} (B \cap B(x,t) \cap \Upsilon) \right) \leq \frac{1}{t^n} \sum_{m=1}^{N_0} \sum_{B \in \mathcal{B}_m} \mu(B \cap B(x,t) \cap \Upsilon),$$

and hence there exists a family $\mathcal{B}_m$ with

$$\sum_{B \in \mathcal{B}_m} \mu(B \cap B(x,t) \cap \Upsilon) \geq \frac{\lambda t^n}{N_0}. \tag{4.1}$$

We assume that for every $S = \Delta(y_0, \ldots, y_{n+1}) \in F \cap B(x,t) \cap \Upsilon$, there exists some $i \in \{0, \ldots, n+1\}$ so that either $\mathfrak{fc}_i(S)$ is not a $(n, 10n\frac{t}{C_1})$-simplex or $\mu(B(y_i, \frac{t}{C_1}) \cap B(x,t) \cap \Upsilon) < \frac{t^n}{C_2}$. We define $\mathcal{G} := \left\{ B \in \mathcal{B}_m \middle| \mu(B \cap B(x,t) \cap \Upsilon) \geq \frac{t^n}{C_2} \right\}$ and mention that $\mathcal{G}$ is a finite set since Lemma A.1 implies that $\#\mathcal{B}_m \leq (2C_1)^n$. With Lemma 2.9 (where we set G as the set of centres of balls in $\mathcal{G}$ and $C = 10n\frac{t}{C_1}$), we know that there exists some $T_z = \Delta(z_0, \ldots, z_n)$ so that for every ball $B(y, \frac{t}{C_1}) \in \mathcal{G}$, there exists some $i \in \{0, \ldots, n\}$ so that $d(y, \operatorname{aff}(\mathfrak{fc}_i(T_z))) \leq 20n\frac{t}{C_1}$. We define for $i \in \{0, \ldots, n\}$:

$$\begin{aligned} T_i &:= \operatorname{aff}(\mathfrak{fc}_i(T_z)) \cap B(\pi_{\operatorname{aff}(\mathfrak{fc}_i(T_z))}(x), 2t), \\ \mathcal{S}_i &:= \left\{ y \in \mathbb{R}^n | d(y, \operatorname{aff}(\mathfrak{fc}_i(T_z))) \leq \tfrac{30nt}{C_1}, \pi_{\operatorname{aff}(\mathfrak{fc}_i(T_z))}(y) \in T_i \right\}, \end{aligned}$$

and we know that $B \in \mathcal{G}$ implies $B \subset S_i$ for some $i \in \{0, \ldots, n\}$. With Lemma A.2 applied to $B(x,r) = T_i$, $s = \frac{4}{C_1}t < 2t = r$ and $m = n-1$, there exists a family $\mathcal{E}$ of disjoint closed balls with $\operatorname{diam} B = \frac{8}{C_1}t$ for all $B \in \mathcal{E}$, $T_i \subset \bigcup_{B \in \mathcal{E}} 5B$ and $\#\mathcal{E} \leq C_1^{n-1}$. Let $y \in S_i$. We have $d(y, \operatorname{aff}(\mathfrak{fc}_i(T_z))) \leq \frac{30n}{C_1}t$ and $\pi_{\operatorname{aff}(\mathfrak{fc}_i(T_z))}(y) \in T_i$. So, there exists some $B = B(z, \frac{4}{C_1}t) \in \mathcal{E}$ with $\pi_{\operatorname{aff}(\mathfrak{fc}_i(T))}(y) \in 5B$, and we have $d(y,z) \leq \frac{30n}{C_1}t + 5\frac{4}{C_1}t < \frac{60n}{C_1}t$. This proves $S_i \subset \bigcup_{B \in \mathcal{E}} 15nB$. We therefrom derive with (B) (see page 1196)

$$\begin{aligned}\mu(S_i) &\le \sum_{B\in\mathcal{E}} \mu\left(15nB\right) \overset{\text{(B)}}{\le} \sum_{B\in\mathcal{E}} C_0\left(15n \operatorname{diam} B\right)^n \\ &\le \#\mathcal{E} C_0 \frac{(120n)^n t^n}{C_1^n} \le (120n)^n C_0 \frac{t^n}{C_1}.\end{aligned} \tag{4.2}$$

We define for $i \in \{1,\dots,n\}$,

$$\mathcal{G}_0 := \{B \in \mathcal{G} | B \subset S_0\} \qquad \text{and} \qquad \mathcal{G}_i := \left\{B \in \mathcal{G} | B \subset S_i \text{ and } B \notin \bigcup_{j=0}^{i-1} \mathcal{G}_i\right\}$$

as a partition of $\mathcal{G}$ (compare the remark after the definition of $\mathcal{S}_i$). Now we have

$$\sum_{B\in\mathcal{G}} \mu(B\cap B(x,t)\cap\Upsilon) \le \sum_{i=0}^{n} \mu(S_i) \overset{(4.2)}{\le} n(120n)^n C_0 \frac{t^n}{C_1}.$$

Moreover, we have

$$\sum_{B\in\mathcal{B}_m\setminus\mathcal{G}} \mu(B\cap B(x,t)\cap\Upsilon) < \sum_{B\in\mathcal{B}_m\setminus\mathcal{G}} \frac{t^n}{C_2} \overset{\#\mathcal{B}_m\le(2C_1)^n}{\le} (2C_1)^n \frac{t^n}{C_2}.$$

All in all, we get with (4.1) and the definition of C_1 and C_2:

$$\lambda \le N_0 \frac{1}{t^n}\left(2^n t^n \frac{C_1^n}{C_2} + 120^n n^{n+1} t^n C_0 \frac{1}{C_1}\right) = N_0\left(2^n \frac{C_1^n}{C_2} + 120^n n^{n+1} C_0 \frac{1}{C_1}\right) \le \frac{\lambda}{2},$$

thus in contradiction to $\lambda > 0$. This completes the proof of Lemma 4.2. □

In most instances, we will use a weaker version of Lemma 4.2:

Corollary 4.3. *Let $0 < \lambda \le 2^n$. There exist constants $C_1 = C_1(N,n,C_0,\lambda) > 3$ and $C_2 = C_2(N,n,C_0,\lambda) > 1$ so that for a given ball $B(x,t)$ and some μ-measurable set Υ with $\delta(B(x,t)\cap\Upsilon) \ge \lambda$, there exists some $(n, 10n\frac{t}{C_1})$-simplex $T = \Delta(x_0,\dots,x_n) \in F\cap B(x,t)\cap\Upsilon$ so that $\mu\left(B\left(x_i, \frac{t}{C_1}\right)\cap B(x,t)\cap\Upsilon\right) \ge \frac{t^n}{C_2}$ for all $i \in \{0,\dots,n\}$.*

4.2. β-numbers and integral Menger curvature.

Definition 4.4 (β-numbers)**.** Let $k > 1$ be some fixed constant, let $x \in \mathbb{R}^N$, $t > 0$, $B = B(x,t)$, and $p \ge 1$, let $\mathcal{P}(N,n)$ be the set of all n-dimensional planes in $\mathbb{R}^N$, and let $P \in \mathcal{P}(N,n)$. We define

$$\begin{aligned}\beta_{p;k}^P(B) = \beta_{p;k}^P(x,t) = \beta_{p;k;\mu}^P(x,t) &:= \left(\frac{1}{t^n}\int_{B(x,kt)} \left(\frac{d(y,P)}{t}\right)^p \mathrm{d}\mu(y)\right)^{\frac{1}{p}}, \\ \beta_{p;k}(B) = \beta_{p;k}(x,t) = \beta_{p;k;\mu}(x,t) &:= \inf_{P\in\mathcal{P}(N,n)} \beta_{p;k}^P(x,t).\end{aligned}$$

The β-numbers measure how well the support of the measure μ can be approximated by some plane. A small β-number of some ball implies either a good approximation of the support by some plane or a low measure quotient δ (cf. Definition 4.1). Hence, since we are interested in good approximations by planes, we will use the β-numbers mainly for balls where we have some lower control on the measure quotient.

Definition 4.5 (Local version of $\mathcal{M}_{\mathcal{K}^p}$)**.** For $\kappa > 1$, $x \in \mathbb{R}^N$, $t > 0$, $p > 0$, we define

$$\mathcal{M}_{\mathcal{K}^p;\kappa}(x,t) := \int \cdots \int_{\mathcal{O}_\kappa(x,t)} \mathcal{K}^p(x_0,\dots,x_{n+1}) \mathrm{d}\mu(x_0)\dots\mathrm{d}\mu(x_{n+1}),$$

where $\mathcal{K}^p$ is a μ-proper integrand (cf. Definition 3.1) and

$$\mathcal{O}_\kappa(x,t)$$
$$:= \left\{(x_0,\dots,x_{n+1}) \in (B(x,\kappa t))^{n+2} \middle| d(a,b) \geq \frac{t}{\kappa}, \forall\, a,b \in \{x_0,\dots,x_{n+1}\}, a \neq b\right\}.$$

Theorem 4.6. *Let $\mathcal{K}^p$ be a symmetric μ-proper integrand and let $0 < \lambda < 2^n$, $k > 2$, $k_0 \geq 1$. There exist constants $k_1 = k_1(N,n,C_0,k,k_0,\lambda) > 1$ and $C = C(N,n,\mathcal{K},p,C_0,k,k_0,\lambda) \geq 1$ such that if $x \in \mathbb{R}^N$ and $t > 0$ with $\delta(B(x,t)) \geq \lambda$ for every $y \in B(x,k_0t)$, we have*

$$\beta_{p;k}(y,t)^p \leq C\frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^n} \leq C\frac{\mathcal{M}_{\mathcal{K}^p;k_1+k_0}(y,t)}{t^n}.$$

Proof. With Lemma 4.2 for $\Upsilon = \mathbb{R}^N$, there exists some $T = \Delta(x_0,\dots,x_{n+1}) \in F \cap B(x,t)$ so that $\mathfrak{fc}_i(T)$ is an $(n, 10n\frac{t}{C_1})$-simplex and $\mu\left(B\left(x_i,\frac{t}{C_1}\right) \cap B(x,t)\right) \geq \frac{t^n}{C_2}$ for all $i \in \{0,\dots,n+1\}$ where C_1, C_2 are the constants from Lemma 4.2 depending on the present constant $\lambda > 0$, the constant C_0 determined in (B) on page 1196, as well as N and n. We set $B_i := B\left(x_i,\frac{t}{C_1}\right)$, $k_1 := \max(C_1,(2+k+k_0)) > 1$ and go on with some intermediate results.

I. Let $z_i \in B_i$ for all $i \in \{0,\dots,n+1\}$, $w \in B(x,(k+k_0)t) \setminus \bigcup_{\substack{l=0\\ l\neq j}}^{n+1} 2B_l$ or $w \in 2B_j$ for some fixed $j \in \{0,\dots,n+1\}$. Since $\mathfrak{fc}_i(T)$ is an $(n,10n\frac{t}{C_1})$-simplex we obtain $(z_0,\dots,\hat{z}_j,\dots,z_{n+1},w) \in \mathcal{O}_{k_1}(x,t)$, where $(z_0,\dots,\hat{z}_j,\dots,z_{n+1},w)$ denotes the $(n+2)$-tuple $(z_0,\dots,z_{j-1},z_{j+1},\dots,z_{n+1},w)$.

II. Let $z_i \in B_i = B(x_i,\frac{t}{C_1})$ for all $i \in \{0,\dots,n+1\}$. Then Lemma 2.8 implies that $\mathfrak{fc}_i(\Delta(z_0,\dots,z_{n+1}))$ is an $\left(n,(9n-1)\frac{t}{C_1}\right)$-simplex for all $i \in \{0,\dots,n+1\}$.

III. Let $z_i \in B_i = B(x_i,\frac{t}{C_1})$ for all $i \in \{0,\dots,n+1\}$, $w \in B(x,(k+k_0)t)$. Since $\mathcal{K}^p$ is a μ-proper integrand with II there exists some constant $\tilde{C} = \tilde{C}(N,n,\mathcal{K},p,C_0,k,k_0,\lambda)$ so that for all $j \in \{0,\dots,n+1\}$, we have

$$\left(\frac{d(w,\operatorname{aff}(z_0,\dots,\hat{z}_j,\dots,z_{n+1}))}{t}\right)^p \leq \tilde{C}t^{n(n+1)}\mathcal{K}^p(z_0,\dots,\hat{z}_j,\dots,z_{n+1},w).$$

IV. There exist some constant $C = C(N,n,\mathcal{K},p,C_0,k,k_0,\lambda)$ and $z_i \in F \cap B_i \cap B(x,t)$, $i \in \{0,\dots,n+1\}$, so that for all $l \in \{0,\dots,n+1\}$, we have

(4.3)
$$\int \chi_{\{(z_0,\dots,\hat{z}_l,\dots,z_{n+1},w)\in\mathcal{O}_{k_1}(x,t)\}}\mathcal{K}^p(z_0,\dots,\hat{z}_l,\dots,z_{n+1},w)\mathrm{d}\mu(w) \leq C\,\frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^{(n+1)n}}$$

and with $P_{n+1} := \operatorname{aff}(z_0,\dots,z_n)$, we have

$$\left(\frac{d(z_{n+1},P_{n+1})}{t}\right)^p \leq C\,\frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^n}. \tag{4.4}$$

Proof of I–IV. For $E \subset \mathbb{R}^N$ with $\#E = m+1$, $E = \{e_0, \dots, e_m\}$, $0 \le m \le n$, we set

$$\mathcal{R}(E) := \int_{F^{n-m+1}} \chi_{\{(e_0,\dots,e_m,w_{m+1},\dots,w_{n+1}) \in \mathcal{O}_{k_1}(x,t)\}} \mathcal{K}^p(e_0,\dots,e_m,w_{m+1},\dots,w_{n+1}) \mathrm{d}\mu(w_{m+1}) \dots \mathrm{d}\mu(w_{n+1}).$$

The integrand $\mathcal{K}$ is symmetric; hence the value $\mathcal{R}(E)$ is well-defined because it does not depend on the numbering of the elements of E. In the following part, we use the convention that $\{0,\dots,-1\} = \emptyset$ and $\{z_0,\dots,z_{-1}\} = \emptyset$. At first, we show by an inductive construction that, for all $m \in \mathbb{N}$ with $0 \le m \le n+1$, there holds:

For all $j \in \{0,\dots,m\}$ and $i \in \{j,\dots,n+1\}$, there exist constants $C^{(j)} > 1$ and sets $Z_i^j \subset F \cap B_i \cap B(x,t)$. For all $l \in \{0,\dots,m-1\}$, there exist $z_l \in Z_l^l$ with

$$\mu(Z_i^j) > \frac{t^n}{2^{j+1}C_2}. \tag{4.5}$$

For all $u \in \{0,\dots,m\}$, for all $E \subset \{z_0,\dots,z_{u-1}\}$ and $z \in Z_r^u$, where $r \in \{u,\dots, n+1\}$, we have

$$\mathcal{R}(E \cup \{z\}) \le C^{(u)} \frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^{(\#E+1)n}}. \tag{4.6}$$

We start with $m = j = 0$ and choose the constant $C^{(0)} := 2C_2$, set $\Upsilon_i := F \cap B_i \cap B(x,t)$ and define for every $i \in \{0,\dots,n+1\}$,

$$Z_i^0 := \left\{ z \in \Upsilon_i \middle| \mathcal{R}(\{z\}) \le C^{(0)} \frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^n} \right\}. \tag{4.7}$$

We have $\mu(Z_i^0) \ge \mu(\Upsilon_i) - \mu(\Upsilon_i \setminus Z_i^0) > \frac{t^n}{2C_2}$ because $\mu(\Upsilon_i) \overset{\text{(ii)}}{\ge} \frac{t^n}{C_2}$, and with (4.7), Chebyshev's inequality and $\int \mathcal{R}(\{z\}) \mathrm{d}\mu(z) = \mathcal{M}_{\mathcal{K}^p;k_1}(x,t)$ we obtain $\mu(\Upsilon_i \setminus Z_i^0) < \frac{t^n}{C^{(0)}}$. If $u = 0$, $E \subset \{z_0,\dots,z_{-1}\} = \emptyset$ and $z \in Z_r^0$, where $r \in \{0,\dots,n+1\}$, the definition (4.7) implies (4.6) in this case.

Now we let $m \in \{0,\dots,n\}$ and we assume that for all $j \in \{0,\dots,m\}$ and $i \in \{j,\dots,n+1\}$, there exist constants $C^{(j)} > 1$ and sets $Z_i^j \subset F \cap B_i \cap B(x,t)$. For all $l \in \{0,\dots,m-1\}$ there exist $z_l \in Z_l^l$ with

$$\mu(Z_i^j) > \frac{t^n}{2^{j+1}C_2}. \tag{4.8}$$

For all $u \in \{0,\dots,m\}$, for all $E \subset \{z_0,\dots,z_{u-1}\}$ and $z \in Z_r^u$ where $r \in \{u,\dots, n+1\}$, we have

$$\mathcal{R}(E \cup \{z\}) \le C^{(u)} \frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^{(\#E+1)n}}. \tag{4.9}$$

Next we start with the inductive step. From the induction hypothesis, we already have the constants $C^{(j)}$ and the sets Z_i^j for $j \in \{0,\dots,m\}$ and $i \in \{j,\dots,n+1\}$ as well as $z_l \in Z_l^l$ for $l \in \{0,\dots,m-1\}$. Since $\mu(Z_m^m) > 0$, we can choose $z_m \in Z_m^m$. We define $C^{(m+1)} := 2^{2m+2} C^{(m)} C_2$ and, for $i \in \{m+1,\dots,n+1\}$, we define

$$Z_i^{m+1} := \bigcap_{\substack{E \subset \{z_0,\dots,z_m\} \\ z_m \in E}} \underbrace{\left\{ z \in Z_i^m \middle| \mathcal{R}(E \cup \{z\}) \le C^{(m+1)} \frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^{(\#E+1)n}} \right\}}_{=:D_{i,E}^m}. \tag{4.10}$$

We have $\mu(Z_i^{m+1}) \geq \mu(Z_i^m) - \mu\left(Z_i^m \setminus Z_i^{m+1}\right) \geq \frac{t^n}{2^{m+2}C_2}$ for all $i \in \{m+1, \dots, n+1\}$, because if $E \subset \{z_0, \dots, z_m\}$ with $z_m \in E$, we get, using (4.10), Chebyshev's inequality, $\int \mathcal{R}(E \cup \{z\}) \mathrm{d}\mu(z) = \mathcal{R}((E \setminus \{z_m\}) \cup \{z_m\})$ and (4.9) that

$$\mu\left(Z_i^m \setminus D_{i,E}^m\right) < \left(C^{(m+1)} \frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^{(\#E+1)n}}\right)^{-1} \mathcal{R}((E \setminus \{z_m\}) \cup \{z_m\}) = \frac{C^{(m)}}{C^{(m+1)}} t^n,$$

which implies

$$\mu(Z_i^m \setminus Z_i^{m+1}) \leq \sum_{\substack{E \subset \{z_0, \dots, z_m\} \\ z_m \in E}} \mu\left(Z_i^m \setminus D_{i,E}^m\right) < \frac{1}{2^{m+2}C_2} t^n.$$

Now let $u \in \{0, \dots, m+1\}$, $E \subset \{z_0, \dots, z_{u-1}\}$ and $z \in Z_r^u$ where $r \in \{u, \dots, n+1\}$. We have to show that (4.6) is valid. Due to the induction hypothesis and $z \in Z_r^{m+1} \subset Z_r^v$ for all $v \in \{0, \dots, m+1\}$, we only have to consider the case $u = m+1$ and $z_m \in E$. Then the inequality follows from (4.10). End of induction.

Now we construct z_{n+1}. We set $P_{n+1} := \operatorname{aff}(z_0, \dots, z_n)$, $\hat{C}^{(n+1)} := \tilde{C}\, C^{(n)} 2^{n+3} C_2$, where $\tilde{C}$ is the constant from III, and define

$$\hat{Z}_{n+1}^{n+1} := \left\{ z \in Z_{n+1}^{n+1} \,\middle|\, \left(\frac{d(z, P_{n+1})}{t}\right)^p \leq \hat{C}^{(n+1)} \frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^n} \right\}. \tag{4.11}$$

Next we show that $\mu\left(\hat{Z}_{n+1}^{n+1}\right) \geq \frac{t^n}{2^{n+3}C_2} > 0$. Let $u \in Z_{n+1}^{n+1} \setminus \hat{Z}_{n+1}^{n+1} \subset B_{n+1} \subset B(x, (k+k_0)t)$. With III applied on $w = u$ and $j = n+1$, we get

$$\left(\frac{d(u, P_{n+1})}{t}\right)^p \leq \tilde{C} t^{n(n+1)} \mathcal{K}^p(z_0, \dots, z_n, u). \tag{4.12}$$

Now we get with (4.11), Chebyshev's inequality and (4.12) that

$$\begin{aligned} \mu\left(Z_{n+1}^{n+1} \setminus \hat{Z}_{n+1}^{n+1}\right) \leq & \left(\hat{C}^{(n+1)} \frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^n}\right)^{-1} \tilde{C} t^{n(n+1)} \\ & \times \int_{Z_{n+1}^{n+1} \setminus \hat{Z}_{n+1}^{n+1}} \mathcal{K}^p(z_0, \dots, z_n, u) \mathrm{d}\mu(u). \end{aligned}$$

By using I we see that the integral on the RHS is equal to $\mathcal{R}(\{z_0, \dots, z_{n-1}\} \cup \{z_n\})$. Hence with (4.5) and (4.6) we obtain

$$\mu(\hat{Z}_{n+1}^{n+1}) \geq \mu(Z_{n+1}^{n+1}) - \mu(Z_{n+1}^{n+1} \setminus \hat{Z}_{n+1}^{n+1}) > 0,$$

and we are able to choose $z_{n+1} \in \hat{Z}_{n+1}^{n+1} \subset Z_{n+1}^{n+1}$. Let $l \in \{0, \dots, n+1\}$ and $E = \{z_0, \dots, z_{n+1}\} \setminus \{z_l\}$. Set $z := z_n$ if $l = n+1$ or $z := z_{n+1}$ otherwise. Now set $E' := E \setminus \{z\}$ and use (4.6) to obtain $\mathcal{R}(E) = \mathcal{R}(E' \cup \{z\}) \leq C^{(n+1)} \frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^{(n+1)n}}$.

All in all, there exists some constant $C = C(N, n, \mathcal{K}, p, C_0, k, k_0, \lambda)$ such that

$$\begin{aligned} & \int \chi_{\{(z_0, \dots, \hat{z}_l, \dots, z_{n+1}, w) \in \mathcal{O}_{k_1}(x,t)\}} \mathcal{K}^p(z_0, \dots, \hat{z}_l, \dots, z_{n+1}, w) \mathrm{d}\mu(w) \\ & \quad = \mathcal{R}(E) \leq C \frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^{(n+1)n}} \end{aligned}$$

for all $l \in \{0, \dots, n+1\}$. This ends the proof of IV. $\square$

With IV, there exist some $z_i \in F \cap B_i \cap B(x,t)$, $i \in \{0,\dots,n+1\}$, fulfilling (4.3) and (4.4). Let $w \in (F \cap B(x,(k+k_0)t)) \setminus \bigcup_{j=0}^n 2B_j$. Hence we get with III ($P_{n+1} = \operatorname{aff}(z_0,\dots,z_n)$), I and (4.3)

$$\begin{aligned} \int_{B(x,(k+k_0)t)\setminus\bigcup_{j=0}^n 2B_j} \left(\frac{d(w,P_{n+1})}{t}\right)^p \mathrm{d}\mu(w) \\ < C(N,n,\mathcal{K},p,C_0,k,k_0,\lambda)\mathcal{M}_{\mathcal{K}^p;k_1}(x,t). \end{aligned} \tag{4.13}$$

Now we prove this estimate on the ball $2B_j$, where $j \in \{0,\dots,n\}$. We define the plain $P_j := \operatorname{aff}(\{z_0,\dots,z_{n+1}\} \setminus \{z_j\})$ and get analogously with III, I and (4.3)

$$\int_{2B_j} \left(\frac{d(w,P_j)}{t}\right)^p \mathrm{d}\mu(w) < C(N,n,\mathcal{K},p,C_0,k,k_0,\lambda)\mathcal{M}_{\mathcal{K}^p;k_1}(x,t). \tag{4.14}$$

Now we have an estimate on the ball $2B_j$ but with plane P_j instead of P_{n+1}. If $z_{n+1} \in P_{n+1}$, we have $P_{n+1} = P_j$ for all $j \in \{0,\dots,n+1\}$, and hence we get estimate (4.14) for P_{n+1}. From now on, we assume that $z_{n+1} \notin P_{n+1}$. Let $w \in 2B_j$, set $w' := \pi_{P_j}(w)$, $w'' := \pi_{P_{n+1}}(w')$ and deduce by inserting the point w' with triangle inequality

$$d(w,P_{n+1})^p \le d(w,w'')^p \le 2^{p-1}\left(d(w,P_j)^p + d(w',P_{n+1})^p\right). \tag{4.15}$$

If $d(w',P_{n+1}) > 0$, i.e., $w' \notin P_{n+1}$, we gain with Lemma 2.2 ($P_1 = P_j$, $P_2 = P_{n+1}$, $a_1 = w'$, $a_2 = z_{n+1}$) where $P_{j,n+1} := P_j \cap P_{n+1}$:

$$d(w',P_{n+1}) = d(z_{n+1},P_{n+1})\frac{d(w',P_{j,n+1})}{d(z_{n+1},P_{j,n+1})}. \tag{4.16}$$

With $l \in \{0,\dots,n\}$, $l \neq j$ (k_1 is defined on page 1199), we get

$$d(w',P_{j,n+1}) \le d\big(w,P_{j,n+1}\big) \le d(w,x) + d(x,x_l) + d(x_l,z_l) \le k_1 t.$$

With II we get that $\mathfrak{fc}_j(\Delta(z_0,\dots,z_{n+1}))$ is an $(n,(9n-1)\frac{t}{C_1})$-simplex and we obtain

$$\left(\frac{d(w',P_{n+1})}{t}\right)^p \overset{(4.16)}{\le} \left(\frac{d(z_{n+1},P_{n+1})}{t}\frac{k_1 t C_1}{(9n-1)t}\right)^p \overset{(4.4)}{\le} C\frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^n} \tag{4.17}$$

where $C = C(N,n,\mathcal{K},p,C_0,k,k_0,\lambda)$. If $d(w',P_{n+1}) = 0$, this inequality is trivially true.

Finally, applying (4.14), (4.15), (4.17) and $\mu(2B_j) \overset{(B)}{\le} C_0(\operatorname{diam}(2B_j))^n \le C_0\left(\frac{4t}{C_1}\right)^n$ ((B) from page 1196), we obtain

$$\int_{2B_j} \left(\frac{d(w,P_{n+1})}{t}\right)^p \mathrm{d}\mu(w) \le C\left(N,n,\mathcal{K},p,C_0,k,k_0,\lambda\right)\mathcal{M}_{\mathcal{K}^p;k_1}(x,t).$$

Given that $B(y,kt) \subset B(x,(k+k_0)t)$, it follows with (4.13) that

$$\begin{aligned} \beta_{p;k}(y,t)^p &\le \frac{1}{t^n}\int_{B(x,(k+k_0)t)} \left(\frac{d(w,P_{n+1})}{t}\right)^p \mathrm{d}\mu(w) \\ &\le C(N,n,\mathcal{K},p,C_0,k,k_0,\lambda)\frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^n}. \end{aligned}$$

To obtain the main result of this theorem, the only thing left to show is $\mathcal{O}_{k_1}(x,t) \subset \mathcal{O}_{k_1+k_0}(y,t)$. Let $(z_0,\dots,z_{n+1}) \in \mathcal{O}_{k_1}(x,t)$. It follows that $z_0,\dots,z_{n+1} \in B(x,k_1 t)$ $\subset B(y,(k_0+k_1)t)$ and $d(z_i,z_j) \ge \frac{t}{k_1} \ge \frac{t}{k_1+k_0}$ with $i \neq j$ and $i,j = 0,\dots,n$. Thus $(z_0,\dots,z_{n+1}) \in \mathcal{O}_{k_1+k_0}(y,t)$. ∎

Theorem 4.7. *Let $0 < \lambda < 2^n$, $k > 2$, and $k_0 \geq 1$, and let $\mathcal{K}^p$ be some μ-proper symmetric integrand (see Definition* 3.1*). There exists a constant $C = C(N, n, \mathcal{K}, p, C_0, k, k_0, \lambda)$ such that*

$$\int\int_0^\infty \beta_{p;k}(x,t)^p \chi_{\{\tilde{\delta}_{k_0}(B(x,t))\geq\lambda\}} \frac{\mathrm{d}t}{t} \mathrm{d}\mu(x) \leq C\mathcal{M}_{\mathcal{K}^p}(\mu),$$

where $\tilde{\delta}_{k_0}(B(x,t)) := \sup_{y\in B(x,k_0 t)} \delta(B(y,t))$.

Proof. At first, we prove some intermediate results.

I. Let $x \in F$, $t > 0$ and $\tilde{\delta}_{k_0}(B(x,t)) \geq \lambda$. There exists some $z \in B(x, k_0 t)$ with $\delta(B(z,t)) \geq \frac{\lambda}{2}$. Now with Theorem 4.6 there exist some constants k_1 and C so that with $k_2 := k_1 + k_0$, we obtain $\beta_{p;k}(x,t)^p \leq C\frac{\mathcal{M}_{\mathcal{K}^p;k_2}(x,t)}{t^n}$.

II. Let $(x,t) \in \mathcal{D}_\kappa(u_0,\ldots,u_{n+1}) := \{(y,s) \in F \times (0,\infty) | (u_0,\ldots,u_{n+1}) \in \mathcal{O}_\kappa(y,s)\}$ where $u_0,\ldots,u_{n+1} \in F$. We have $(u_0,\ldots,u_{n+1}) \in \mathcal{O}_\kappa(x,t)$ and so $\frac{d(u_0,u_1)}{2\kappa} \leq t \leq \kappa d(u_0,u_1)$ as well as $x \in B(u_0, \kappa t)$.

III. With Fubini's theorem [7, 1.4, Thm. 1] and condition (B) from page 1196 we get

$$\int_F\int_0^\infty \chi_{\mathcal{D}_{k_2}(u_0,\ldots,u_{n+1})}(x,t)\frac{1}{t^n}\frac{\mathrm{d}t}{t}\mathrm{d}\mu(x) \overset{\mathrm{II}}{\leq} \int_{\frac{d(u_0,u_1)}{2k_2}}^{k_2 d(u_0,u_1)} \frac{1}{t^n}\int_{B(u_0,k_2 t)} 1 \,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t} \overset{(\mathrm{B})}{=} C.$$

Now we deduce with Fubini's theorem [7, 1.4, Thm. 1] that

$$\begin{aligned}&\int_F\int_0^\infty \beta_{p;k}(x,t)^p\chi_{\{\tilde{\delta}_{k_0}(B(x,t))\geq\lambda\}}\frac{\mathrm{d}t}{t}\mathrm{d}\mu(x)\\ &\overset{\mathrm{I}}{\leq} C\int_F\int_0^\infty\int\cdots\int_{\mathcal{O}_{k_2}(x,t)}\frac{\mathcal{K}^p(u_0,\ldots,u_{n+1})}{t^n}\mathrm{d}\mu(u_0)\ldots\mathrm{d}\mu(u_{n+1})\frac{\mathrm{d}t}{t}\mathrm{d}\mu(x)\\ &\overset{\mathrm{III}}{\leq} C\mathcal{M}_{\mathcal{K}^p}(\mu).\end{aligned}$$

□

Corollary 4.8. *Let $0 < \lambda < 2^n$, $k > 2$, and $k_0 \geq 1$, and let $\mathcal{K}^p$ be some symmetric μ-proper integrand (see Definition* 3.1*). There exists a constant $C = C(N, n, \mathcal{K}, p, C_0, k, k_0, \lambda)$ such that*

$$\int\int_0^\infty \beta_{1;k}(x,t)^p \chi_{\{\tilde{\delta}_{k_0}(B(x,t))\geq\lambda\}} \frac{\mathrm{d}t}{t} \mathrm{d}\mu(x) \leq C\mathcal{M}_{\mathcal{K}^p}(\mu).$$

Proof. This is a direct consequence of the previous theorem and Hölder's inequality. □

4.3. **β-numbers, approximating planes and angles.** The next lemma states that if two balls are close to each other and if each part of the support of μ contained in those balls is well approximated by some plane, then these planes have a small angle.

Lemma 4.9. *Let $x, y \in F$, $c \geq 1$, $\xi \geq 1$ and $t_x, t_y > 0$ with $c^{-1}t_y \leq t_x \leq ct_y$. Furthermore, let $k \geq 4c$ and $0 < \lambda < 2^n$ with $\delta(B(x,t_x)) \geq \lambda$, $\delta(B(y,t_y)) \geq \lambda$ and $d(x,y) \leq \frac{k}{2c}t_x$. Then there exist some constants $C_3 = C_3(N,n,C_0,\lambda,\xi,c) > 1$ and $\varepsilon_0 = \varepsilon_0(N,n,C_0,\lambda,\xi,c) > 0$ so that for all $\varepsilon < \varepsilon_0$ and all planes $P_1, P_2 \in \mathcal{P}(N,n)$ with $\beta_{1;k}^{P_1}(x,t_x) \leq \xi\varepsilon$ and $\beta_{1;k}^{P_2}(y,t_y) \leq \xi\varepsilon$ we get: For all $w \in P_1$, we have $d(w,P_2) \leq C_3\varepsilon(t_x + d(w,x))$, for all $w \in P_2$, we have $d(w,P_1) \leq C_3\varepsilon(t_x + d(w,x))$ and we have $\sphericalangle(P_1,P_2) \leq C_3\varepsilon$.*

Proof. Due to $\delta(B(x,t_x)) \geq \lambda$ and Corollary 4.3, there exist some constants $C_1 > 3$ and C_2 depending on N, n, C_0, λ, and some simplex $T = \Delta(x_0, \ldots, x_n) \in F \cap B(x,t_x)$ so that T is an $(n, 10n\frac{t_x}{C_1})$-simplex and $\mu(B(x_i, \frac{t_x}{C_1}) \cap B(x,t_x)) \geq \frac{t_x^n}{C_2}$ for all $i \in \{0,\ldots,n\}$. For $B_i := B(x_i, \frac{t_x}{C_1})$ and $i \in \{0,\ldots,n\}$, we have $\mu(B_i) \geq \mu(B_i \cap B(x,t_x)) \geq \frac{t_x^n}{C_2} \geq \frac{t_y^n}{c^n C_2}$. Since $B_i \cap B(x,t_x) \neq \emptyset$ and $k \geq 4c \geq 4$ we obtain $B_i \subset B(x, kt_x)$ and $B_i \subset B(y, kt_y)$. Now we see for $i \in \{0,\ldots,n\}$,

$$\begin{aligned}\frac{1}{\mu(B_i)} \int_{B_i} d(z,P_1) + d(z,P_2)\mathrm{d}\mu(z) &= C_2 t_x \beta_{1;k}^{P_1}(x,t_x) + c^n C_2 t_y \beta_{1;k}^{P_2}(y,t_y) \\ &\leq 2c^{n+1} C_2 xi t_x \varepsilon.\end{aligned}$$

With Chebyshev's inequality, there exists $z_i \in B_i$ so that

$$d(z_i, P_j) \leq d(z_i,P_1) + d(z_i,P_2) \leq 2c^{n+1} C_2 \xi t_x \varepsilon \tag{4.18}$$

for $i \in \{0,\ldots,n\}$ and $j = 1,2$. We set $y_i := \pi_{P_1}(z_i)$ and with

$$\varepsilon < \varepsilon_0 := \frac{1}{2c^{n+1}C_2\xi} \min\left\{ \tfrac{1}{C_1}, \left(10(10^n+1)\tfrac{C_1}{6}\left(2\tfrac{C_1}{3}\right)^n\right)^{-1}\right\}$$

we deduce that

$$d(y_i,x_i) \leq d(y_i,z_i) + d(z_i,x_i) \leq d(z_i,P_1) + \tfrac{t_x}{C_1} \leq 2c^{n+1}C_2\xi\ t_x\ \varepsilon + \tfrac{t_x}{C_1} \leq 2\tfrac{t_x}{C_1},$$

so, with Lemma 2.8, $S := \Delta(y_0,\ldots,y_n)$ is an $(n, 6n\frac{t_x}{C_1})$-simplex and $S \subset B(x, \frac{2t_x}{C_1} + t_x) \subset B(x, 2t_x)$. Furthermore, with (4.18) we have $d(y_i,P_2) \leq d(y_i,z_i) + d(z_i,P_2) \leq 2c^{n+1}C_2\xi t_x\varepsilon$. Now, with Lemma 2.17 ($C = \frac{C_1}{6n}$, $\hat{C} = 2$, $t = t_x$, $\sigma = 2c^{n+1}C_2\xi\varepsilon$, $m = n$) we obtain

$$\sphericalangle(P_1,P_2) \leq 4n(10^n+1)2\frac{C_1}{6}\left(2\frac{C_1}{3}\right)^n 2c^{n+1}C_2\xi\varepsilon = C(N,n,C_0,\lambda,\xi,c)\varepsilon.$$

Moreover, we have $d(y_0, \pi_{P_2}(z_0)) \leq d(z_0,P_1) + d(z_0,P_2) \overset{(4.18)}{\leq} 2c^{n+1}C_2\xi t_x\varepsilon$, so finally, with Lemma 2.18 ($\sigma = C\varepsilon$, $t = t_x$, $p_1 = y_0$, $p_2 = \pi_{P_2}(z_0)$), we get for $w \in P_1$ that $d(w,P_2) \leq C(d(w,y_0)+t_x)\varepsilon \leq C(d(w,x)+t_x)\varepsilon$ and for $w \in P_2$ we obtain $d(w,P_1) \leq C(d(w,\pi_{P_2}(z_0))+t_x) \leq C(d(w,x)+t_x)\varepsilon$, where $C = C(N,n,C_0,\lambda,\xi,c)$. □

The next lemma describes the distance from a plane to a ball if the plane approximates the support of μ contained in the ball.

Lemma 4.10. *Let $\sigma > 0$, $x \in \mathbb{R}^N$, $t > 0$ and $\lambda > 0$ with $\delta(B(x,t)) \geq \lambda$. If $P \in \mathcal{P}(N,n)$ with $\beta_{1;k}^P(x,t) \leq \sigma$, there exists some $y \in B(x,t) \cap F$ so that $d(y,P) \leq \frac{t}{\lambda}\sigma$. If additionally $\sigma \leq \lambda$, we have $B(x,2t) \cap P \neq \emptyset$.*

Proof. With the requirements, we get $\mu(B(x,t)) \geq t^n\lambda$, and so

$$\frac{1}{\mu(B(x,t))}\int_{B(x,t)} d(z,P)\mathrm{d}\mu(z) \leq \frac{t}{\lambda}\frac{1}{t^n}\int_{B(x,kt)} \frac{d(z,P)}{t}\mathrm{d}\mu(z) = \frac{t}{\lambda}\beta_{1;k}^P(x,t) \leq \frac{t}{\lambda}\sigma.$$

With Chebyshev's inequality, we get some $y \in B(x,t) \cap F$ with $d(y,P) \leq \frac{t}{\lambda}\sigma$. If $\sigma \leq \lambda$, it follows that $B(x,2t) \cap P \neq \emptyset$. □

5. Proof of the main result

At the end of this section (page 1207), we will give a proof of our main result Theorem 3.5 under the assumption that the forthcoming Theorem 5.4 is correct. We start with a few lemmas helpful for this proof.

5.1. **Reduction to a symmetric integrand.**

Lemma 5.1. *Let $\mathcal{K}^p$ be some proper integrand (see Definition 3.1). There exists some proper integrand $\tilde{\mathcal{K}}^p$, which is symmetric in all components and fulfils $\mathcal{M}_{\mathcal{K}^p}(E) = \mathcal{M}_{\tilde{\mathcal{K}}^p}(E)$ for all Borel sets E.*

Proof. We set $\tilde{\mathcal{K}}^p(x_0, \dots, x_{n+1}) := \frac{1}{\#S_{n+2}} \sum_{\phi \in S_{n+2}} \mathcal{K}^p(\phi(x_0, \dots, x_{n+1}))$, where S_{n+2} is the symmetric group of all permutations of $n+2$ symbols. Due to $\mathcal{K}^p \leq \#S_{n+2}\ \tilde{\mathcal{K}}^p$, the integrand $\tilde{\mathcal{K}}^p$ fulfils the conditions of a proper integrand. Now Fubini's theorem [7, 1.4, Thm. 1] implies $\mathcal{M}_{\tilde{\mathcal{K}}^p}(E) = \mathcal{M}_{\mathcal{K}^p}(E)$. □

5.2. **Reduction to finite, compact and more regular sets with small curvature.**

Lemma 5.2. *Let E be a Borel set with $\mathcal{M}_{\mathcal{K}^p}(E) < \infty$, where $\mathcal{K}^p$ is some proper integrand. Then we have $\mathcal{H}^n(E \cap B) < \infty$ for every ball B.*

Proof. Let B be some ball and set $F := E \cap B$. We prove the contraposition so we assume that $\mathcal{H}^n(F) = \infty$. With Lemma 2.11, there exists some constant $C > 0$ and some $(n+1, (n+3)C)$-simplex $T = \Delta(x_0, \dots, x_{n+1}) \in B$ with $\mathcal{H}^n(B(x_0, C) \cap F) = \infty$ and $\mathcal{H}^n(B(x_i, C) \cap F) > 0$ for all $i \in \{1, \dots, n+1\}$. With Lemma 2.8, we conclude that $S = \Delta(y_0, \dots, y_{n+1})$ is an $(n+1, C)$-simplex for all $y_i \in B(x_i, C)$, $i \in \{0, \dots, n+1\}$. For $t = C\sqrt{\frac{\operatorname{diam} B}{2C} + 1}$ and $\bar{C} = \sqrt{\frac{\operatorname{diam} B}{2C} + 1}$, we get $S \in B(x, t\bar{C})$, where x is the centre of the ball B and S is an $(n+1, \frac{t}{C})$-simplex. Hence we are in the right setting for using the second condition of a proper integrand. We obtain

$$\begin{aligned}\mathcal{M}_{\mathcal{K}^p}(E) &\geq \int_{B(x_{n+1}, C) \cap F} \cdots \int_{B(x_0, C) \cap F} \mathcal{K}^p(y_0, \dots, y_{n+1}) \mathrm{d}\mathcal{H}^n(y_0) \dots \mathrm{d}\mathcal{H}^n(y_{n+1}) \\ &= \infty.\end{aligned}$$

□

Lemma 5.3. *In this lemma, the integrand $\mathcal{K}$ of $\mathcal{M}_{\mathcal{K}^p}$ only needs to be an $(\mathcal{H}^n)^{n+2}$-integrable function. Let $p > 0$, $n < N$ and $E \subset \mathbb{R}^N$ be a Borel set with $0 < \mathcal{H}^n(E) < \infty$ and $\mathcal{M}_{\mathcal{K}^p}(E) < \infty$. For all $\zeta > 0$, there exists some compact $E^* \subset E$ with*

(i) $\mathcal{H}^n(E^*) > \frac{(\operatorname{diam} E^*)^n \omega_n}{2^{2n+1}}$,
(ii) $\forall x \in E^*, \forall t > 0,\ \mathcal{H}^n(E^* \cap B(x, t)) \leq 2\omega_n t^n$,
(iii) $\mathcal{M}_{\mathcal{K}^p}(E^*) \leq \zeta\ (\operatorname{diam} E^*)^n$,

where $\omega_n = \mathcal{H}^n(B(0,1))$ is the n-dimensional volume of the n-dimensional unit ball.

Proof. Due to $0 < \mathcal{H}^n(E) < \infty$ and [7, 2.3, Thm. 2], for $\mathcal{H}^n$-almost all $x \in E$ we have

$$\frac{1}{2^n} \leq \limsup_{t \to 0^+} \frac{\mathcal{H}^n(E \cap B(x, t))}{\omega_n t^n} \leq 1. \tag{5.1}$$

For $l \in \mathbb{N}$, we define the $\mathcal{H}^n$-measurable set

$$E_m := \left\{ x \in E \;\middle|\; \forall t \in \left(0, \frac{1}{m}\right), \mathcal{H}^n(E \cap B(x,t)) \le 2\omega_n t^n \right\}. \tag{5.2}$$

Due to $E_l \subset E_{l+1}$, [7, 1.1.1, Thm. 1(iii)] and (5.1) we get that

$$\lim_{l\to\infty} \mathcal{H}^n(E_l) = \mathcal{H}^n\left(\bigcup_{l=1}^\infty E_l\right) = \mathcal{H}^n(E).$$

Hence there exists some $m \in \mathbb{N}$ with $\mathcal{H}^n(E_m) \ge \frac{1}{2}\mathcal{H}^n(E)$ and $\mathcal{M}_{\mathcal{K}^p}(E_m) \le \mathcal{M}_{\mathcal{K}^p}(E) < \infty$. Define for $\tau > 0$,

$$\mathcal{I}(\tau) := \int_{A(\tau)} \mathcal{K}^p(x_0,\dots,x_{n+1}) \mathrm{d}\mathcal{H}^n(x_0)\dots\mathrm{d}\mathcal{H}^n(x_{n+1}), \tag{5.3}$$

where $A(\tau) := \left\{ (x_0,\dots,x_{n+1}) \in E_m^{n+2} \middle| d(x_0,x_i) < \tau \text{ for all } i \in \{1,\dots,n+1\} \right\}$. Using (5.2) we obtain $(\mathcal{H}^n)^{n+2}(A(\tau)) \to 0$ for $\tau \to 0$. With $\mathcal{M}_{\mathcal{K}^p}(E_m) < \infty$, we conclude that $\lim_{\tau\to 0}\mathcal{I}(\tau) = 0$, and so we are able to pick some $0 < \tau_0 \le \frac{1}{2m}$ with

$$\mathcal{I}(2\tau_0) \le \frac{\zeta \mathcal{H}^n(E_m)}{2\omega_n \cdot 2^{n+3}}. \tag{5.4}$$

We set

$$\mathcal{V} := \left\{ B(x,\tau) \middle| x \in E_m, 0 < \tau < \tau_0, \mathcal{H}^n(E_m \cap B(x,\tau)) \ge \frac{\tau^n \omega_n}{2^{n+1}} \right\}.$$

Since $0 < \mathcal{H}^n(E_m) < \infty$, we get (5.1) with E_m instead of E [7, 2.3, Thm. 2]. This implies $\inf\left\{\tau \middle| B(x,\tau) \in \mathcal{V}\right\} = 0$ for $\mathcal{H}^n$-almost every $x \in E_m$. According to [8, 1.3], $\mathcal{V}$ is a Vitali class. For every countable, disjoint subfamily $\{B_i\}_i$ of $\mathcal{V}$, we have $\sum_{i\in\mathbb{N}}(\operatorname{diam} B_i)^n \le \frac{2^{2n+1}}{\omega_n}\mathcal{H}^n(E_m) < \infty$. Applying Vitali's Covering Theorem [8, 1.3, Thm. 1.10], we get a countable subfamily of $\mathcal{V}$ with disjoint balls $B_i = B(x_i,\tau_i)$ fulfilling $\mathcal{H}^n\left(E_m \setminus \bigcup_{i\in\mathbb{N}} B_i\right) = 0$. Therefore, using (5.2), we have $\mathcal{H}^n(E_m) \le \sum_{i\in\mathbb{N}}\mathcal{H}^n(E_m \cap B_i) \le \sum_{i\in\mathbb{N}} 2\omega_n\tau_i^n$, so that

$$\sum_{i\in\mathbb{N}} \tau_i^n \ge \frac{\mathcal{H}^n(E_m)}{2\omega_n}. \tag{5.5}$$

Furthermore, with $(B_i \cap E_m)^{n+2} \subset A(2\tau_0) \cap B_i^{n+2}$, we obtain

$$\sum_{i\in\mathbb{N}} \mathcal{M}_{\mathcal{K}^p}(B_i \cap E_m) \overset{(5.3)}{\le} \mathcal{I}(2\tau_0) \overset{(5.4)}{\le} \frac{\zeta\mathcal{H}^n(E_m)}{2\omega_n \cdot 2^{n+3}}. \tag{5.6}$$

We define

$$I_b := \left\{ i \in \mathbb{N} \middle| \mathcal{M}_{\mathcal{K}^p}(B(x_i,\tau_i) \cap E_m) \ge \zeta \tfrac{\tau_i^n}{2^{n+2}} \right\},$$

and so

$$\sum_{i\in I_b} \mathcal{M}_{\mathcal{K}^p}(B(x_i,\tau_i)\cap E_m) \ge \zeta \frac{\sum_{i\in I_b}\tau_i^n}{2^{n+2}}.$$

We have $\sum_{i\in I_b}\tau_i^n \le \frac{\mathcal{H}^n(E_m)}{4\omega_n}$, since assuming the converse would imply

$$\sum_{i\in\mathbb{N}} \mathcal{M}_{\mathcal{K}^p}(B(x_i,\tau_i)\cap E_m) \overset{(5.6)}{<} \zeta\frac{\sum_{i\in I_b}\tau_i^n}{2^{n+2}} \le \sum_{i\in I_b}\mathcal{M}_{\mathcal{K}^p}(B(x_i,\tau_i)\cap E_m).$$

Using (5.5), we obtain $I_b \neq \mathbb{N}$. Now we choose some $i \in \mathbb{N} \setminus I_b$, and the regularity of the Hausdorff measure [8, 1.2, Thm. 1.6] implies the existence of some compact set $E^* \subset B(x_i, \tau_i) \cap E_m$ with

(i) $\mathcal{H}^n(E^*) > \frac{1}{2}\mathcal{H}^n(B(x_i,\tau_i) \cap E_m) \geq \frac{\tau_i^n \omega_n}{2^{n+1}} \geq \frac{(\operatorname{diam} E^*)^n \omega_n}{2^{2n+1}}$.

(ii) $\forall x \in E^*, \forall t > 0$, we have $\mathcal{H}^n(E^* \cap B(x,t)) \leq \mathcal{H}^n(B(x_i,\tau_i) \cap E_m \cap B(x,t)) \leq 2\omega_n t^n$, since if $t < \frac{1}{m}$ (5.2) implies $\mathcal{H}^n(E \cap B(x,t)) \leq 2\omega_n t^n$, and if $\tau_i < \frac{1}{m} < t$ (5.2) implies $\mathcal{H}^n(B(x_i,\tau_i) \cap E_m) \leq 2\omega_n t^n$.

(iii) $\mathcal{M}_{\mathcal{K}^p}(E^*) \leq \zeta \frac{\tau_i^n}{2^{n+2}} \leq \zeta (\operatorname{diam} E^*)^n$ since $i \notin I_b$ and for some ball B with $E^* \subset B$ and $\operatorname{diam} B = 2 \operatorname{diam} E^*$ we have $\frac{\tau_i^n}{2^{n+2}} \overset{\text{(i)}}{\leq} \frac{\mathcal{H}^n(E^* \cap B)}{2\omega_n} \overset{\text{(ii)}}{\leq} (\operatorname{diam} E^*)^n$. $\square$

Next, we present the crucial theorem of this work.

Theorem 5.4. *Let $\mathcal{K} : (\mathbb{R}^N)^{n+2} \to [0,\infty)$. There exists some $k > 2$ such that for every $C_0 \geq 10$, there exists some $\eta = \eta(N,n,\mathcal{K},C_0,k) \in (0, \omega_n 2^{-(2n+2)}]$ so that if μ is a Borel measure on $\mathbb{R}^N$ with compact support F such that $\mathcal{K}^2$ is a symmetric μ-proper integrand (cf. Definition* 3.1*) and μ fulfils*

(A) $\mu(B(0,5)) \geq 1$, $\mu(\mathbb{R}^N \setminus B(0,5)) = 0$,

(B) $\mu(B) \leq C_0 (\operatorname{diam} B)^n$ *for every ball* B,

(C) $\mathcal{M}_{\mathcal{K}^2}(\mu) \leq \eta$,

(D) $\beta^{P_0}_{1;k;\mu}(0,5) \leq \eta$ *for some plane* $P_0 \in \mathcal{P}(N,n)$ *with* $0 \in P_0$,

then there exists some Lipschitz function $A : P_0 \to P_0^\perp \subset \mathbb{R}^N$ so that the graph $G(A) \subset \mathbb{R}^N$ fulfils $\mu(G(A)) \geq \frac{99}{100}\mu(\mathbb{R}^N)$. ($P_0^\perp := \{x \in \mathbb{R}^N | x \cdot v = 0$ for all $v \in P_0\}$ denotes the orthogonal complement of P_0.)

First we show that, under the assumption that the previous theorem is correct, we can prove Theorem 3.5. The remaining proof of Theorem 5.4 is then given in the following sections 6, 7 and 8. We will use the notation $sE := \{x \in \mathbb{R}^N | s^{-1}x \in E\}$ for $s > 0$ and some set $E \subset \mathbb{R}^N$. Distinguish this notation from $sB(x,t) = B(x,st)$, where the centre stays unaffected and only the radius is scaled.

Proof of Theorem 3.5. Let $\mathcal{K}^2$ be some proper integrand (see Definition 3.1), let $E \subset \mathbb{R}^N$ be some Borel set with $\mathcal{M}_{\mathcal{K}^2}(E) < \infty$ and let $C_0 = 2^{2n+2}$. Furthermore, let $k > 2$ and $0 < \eta \leq \omega_n 2^{-(2n+2)}$ be the constants given by Theorem 5.4. Using Lemma 5.1, we can assume that $\mathcal{K}$ is symmetric.

We start with a countable covering of $\mathbb{R}^N$ with balls B_i so that $\mathbb{R}^N \subset \bigcup_{i\in\mathbb{N}} B_i$. We will show that for all $i \in \mathbb{N}$ the sets $E \cap B_i$ are n-rectifiable, which implies that E is n-rectifiable.

Let $i \in \mathbb{N}$ with $\mathcal{H}^n(E \cap B_i) > 0$. With Lemma 5.2, we conclude that $\mathcal{H}^n(E \cap B_i) < \infty$. Then, using [9, Thm. 3.3.13], we can decompose $E \cap B_i = E^i_{\mathrm{r}} \,\dot\cup\, E^i_{\mathrm{u}}$ into two disjoint subsets, where E^i_{r} is n-rectifiable and E^i_{u} is purely n-unrectifiable.

Now we assume that $E \cap B_i$ is not n-rectifiable, so $\mathcal{H}^n(E^i_{\mathrm{u}}) > 0$. The set E^i_{u} is a Borel set and fulfils $0 < \mathcal{H}^n(E^i_{\mathrm{u}}) \leq \mathcal{H}^N(E \cap B_i) < \infty$ and $\mathcal{M}_{\mathcal{K}^2}(E^i_{\mathrm{u}}) \leq \mathcal{M}_{\mathcal{K}^2}(E) < \infty$. Now we apply Lemma 5.3 with $\zeta = \eta \frac{1}{\hat{C}\tilde{C}}$ where the constants $\hat{C}$ and $\tilde{C}$ are given in this passage and we get some compact set $E^* \subset E^i_{\mathrm{u}}$ which fulfils condition (i), (ii) and (iii) from Lemma 5.3. We set $a := (\operatorname{diam} E^*)^{-1}$ and $\tilde{\mu} = \mathcal{H}^n \llcorner aE^*$. Let $\tilde{B}$ be a ball with $aE^* \subset \tilde{B}$ and $\operatorname{diam} \tilde{B} = 2$. Using (i), we get $\delta_{\tilde{\mu}}(\tilde{B}) \geq \frac{\omega_n}{2^{2n+1}}$. So, Theorem 4.6 ($p = 2$, $x = y \mathrel{\hat{=}}$ centre of $\tilde{B}$, $t = 1$, $\lambda = \frac{\omega_n}{2^{3n+1}}$, $k_0 = 1$)

implies $\beta_{2;k;\tilde{\mu}}(\tilde{B})^2 < \hat{C}\mathcal{M}_{\mathcal{K}^2}(\tilde{\mu}) \le \eta^2$, for some constant $\hat{C} = \hat{C}(N, n, \mathcal{K}, C_0, k) \ge 1$. Using Hölder's inequality there exists some n-dimensional plane $\tilde{P_0} \in \mathcal{P}(N, n)$ with $\beta^{\tilde{P_0}}_{1;k;\tilde{\mu}}(\tilde{B}) \le \eta$. Now we define a measure μ by $\mu(\cdot) := \frac{2^{2n+1}}{\omega_n}\tilde{\mu}(\ \cdot\ + \pi_{\tilde{P}_0}(b))$, where b is the centre of $\tilde{B}$. This is also a Borel measure with compact support, and Lemma 4.10 ($\sigma = \eta$, $B(x,t) = \tilde{B}$, $\lambda = \frac{\omega_n}{2^{2n+1}}$) implies that the support fulfils $F := aE^* - \pi_{\tilde{P}_0}(b) \subset B(0, 2)$. This measure fulfils condition (D) from Theorem 5.4 ($P_0 = \tilde{P}_0 - \pi_{\tilde{P}_0}(b)$), and (i) implies condition (A). To get condition (B) for some arbitrary ball, cover it by some ball with centre on F, double the diameter and apply (ii). Use $\mathcal{M}_{\mathcal{K}^2}(\mu) = \tilde{C}(n)a^n\ \mathcal{M}_{\mathcal{K}^2}(E^*)$ and (iii) to obtain (C). Finally we mention that $\mathcal{K}^2$ is μ-proper, since μ is an adapted version of $\mathcal{H}^n$. Hence we can apply Theorem 5.4 and after some scaling and translation we obtain some Lipschitz function which covers a part of positive Hausdorff measure of E^i_u which is in contrast to E^i_u being purely n-unrectifiable. Hence $E \cap B_i$ is n-rectifiable. $\square$

6. Construction of the Lipschitz graph

6.1. **Partition of the support of the measure μ.** Now we start with the proof of Theorem 5.4. Let $\mathcal{K} : (\mathbb{R}^N)^{n+2} \to [0, \infty)$ and let $C_0 \ge 10$ be some fixed constant. There is one step in the proof which only works for integrability exponent $p = 2$. ($p = 2$ is used in Lemma 8.11 so that the results of Theorem 7.3 and Theorem 7.17 fit together.) Since most of the proof can be given with fewer constraints to p, we start with $p \in (1, \infty)$ and restrict to $p = 2$ only if needed. Furthermore, let $k > 2$, $0 < \eta \le \omega_n 2^{-(2n+2)}$, $P_0 \in \mathcal{P}(N, n)$ with $0 \in P_0$ and μ be a Borel measure on $\mathbb{R}^N$ with compact support F such that $\mathcal{K}^p$ is a symmetric μ-proper integrand (cf. Definition 3.1) and

(A) $\mu(B(0, 5)) \ge 1,\ \mu(\mathbb{R}^N \setminus B(0, 5)) = 0$,
(B) $\mu(B) \le C_0\,(\operatorname{diam} B)^n$ for every ball B,
(C) $\mathcal{M}_{\mathcal{K}^p}(\mu) \le \eta$,
(D) $\beta^{P_0}_{1;k;\mu}(0, 5) \le \eta$.

In this section, we will prove that if k is large and η is small enough, we can construct some function $A : P_0 \to P_0^\perp$ which covers some part of the support F of μ. For this purpose, we will give a partition of the support of μ in four parts, $\operatorname{supp}(\mu) = \mathcal{Z}\dot{\cup}F_1\dot{\cup}F_2\dot{\cup}F_3$, and construct the function A so that the graph of A covers $\mathcal{Z}$, i.e., $\mathcal{Z} \subset G(A)$.

The following sections 7 and 8 will give a proof of $\mu(F_1 \cup F_2 \cup F_3) \le \frac{1}{100}$; hence with (A) we will obtain $\mu(G(A)) \ge \frac{99}{100}\mu(\mathbb{R}^N)$, which is the statement of Theorem 5.4.

From now on, we will only work with the fixed measure μ, so we can simplify the expressions by setting $\beta_{1;k} := \beta_{1;k;\mu}$ and $\delta(\cdot) := \delta_\mu(\cdot)$. Furthermore, we fix the constant

$$\delta := \min\left\{\frac{10^{-10}}{600^n N_0}, \frac{2}{50^n}\right\}, \tag{6.1}$$

where $N_0 = N_0(N)$ is the constant from Besicovitch's Covering Theorem [7, 1.5.2, Thm. 2].

Definition 6.1. Let $\alpha, \varepsilon > 0$. We define the set

$$S_{total}^{\varepsilon,\alpha} := \left\{ (x,t) \in F \times (0,50) \;\middle|\; \begin{array}{ll} \text{(i)} & \delta(B(x,t)) \geq \frac{1}{2}\delta, \\ \text{(ii)} & \beta_{1;k}(x,t) < 2\varepsilon, \\ \text{(iii)} & \exists\, P_{(x,t)} \in \mathcal{P}(N,n) \\ & \text{s.t.} \begin{cases} \beta_{1;k}^{P_{(x,t)}}(x,t) \leq 2\varepsilon \\ \text{and} \\ \sphericalangle(P_{(x,t)}, P_0) \leq \alpha \end{cases} \end{array} \right\}.$$

Having in mind that the definition of $S_{total}^{\varepsilon,\alpha}$ depends on the choice of ε and α, we will normally skip these and write S_{total} instead. In the same manner, we will handle the following definitions of H, h and S. For $x \in F$ we define

$$H(x) := \left\{ t \in (0,50) \;\middle|\; \exists\, y \in F,\ \exists\, \tau \text{ with } \frac{t}{4} \leq \tau \leq \frac{t}{3},\ d(x,y) < \frac{\tau}{3} \text{ and } (y,\tau) \notin S_{total} \right\},$$

$$h(x) := \sup(H(x) \cup \{0\}) \qquad \text{and} \qquad S := \{(x,t) \in S_{total} \mid t \geq h(x)\}.$$

Sometimes, we identify a ball $B = B(x,t)$ with the tuple (x,t) and write to simplify matters $B \in S$ instead of $(x,t) \in S$. In the same manner we use the notation $\beta_{1;k}(B)$.

Lemma 6.2. *Let $\alpha, \varepsilon > 0$. If $\eta \leq 2\varepsilon$, we have that $S_{total} \neq \emptyset$ and*

(i) *$F \times [40,50) \subset \{(x,t) \in F \times (0,50) | t \geq h(x)\} = S$.*

(ii) *If $(x,t) \in S$ and $t \leq t' < 50$, we have $(x,t') \in S$.*

Proof. (i) If $x \in F \subset B(0,5)$ and $10 \leq t < 50$, we have $F \subset B(x,t)$. Using (A), (D) and $P_{(x,t)} := P_0$ we get $(x,t) \in S_{total}$, which implies that $F \times [10,50) \subset S_{total}$. Now if $x \in F$ and $t \in [40,50)$ we deduce for arbitrary $y \in F$ and $\tau \in [\frac{t}{4}, \frac{t}{3}]$ that $(y,\tau) \in S_{total}$, which implies that $H(x) \subset (0,40)$, $h(x) \leq 40$ and hence the first inclusion. For the equality it is enough to prove that the central set is contained in S. Let $x \in F$ and $t \in (0,50)$ with $h(x) \leq t < 50$. Assume that $(x,t) \notin S$. Due to $h(x) \leq t$, we obtain $(x,t) \notin S_{total}$, which implies that $t < 10$. Hence with $y = x$ and $\tau = t$ we get $3t \in H(x)$. This implies $h(x) \geq 3t > t$ and hence a contradiction to $t \geq h(x)$. So, we obtain $(x,t) \in S$.

(ii) We have $x \in F$ and $h(x) \leq t \leq t' < 50$, so with (i) we conclude that $(x,t') \in S$. □

Remember that the function h depends on the set S_{total}, which depends on the choice of ε and α. Hence the sets defined in the following definition depend on α and ε as well.

Definition 6.3 (Partition of F)**.** Let $\alpha, \varepsilon > 0$. We define

$$\mathcal{Z} := \{x \in F \mid h(x) = 0\},$$

$$F_1 := \left\{ x \in F \setminus \mathcal{Z} \;\middle|\; \begin{array}{l} \exists y \in F, \exists \tau \in \left[\frac{h(x)}{5}, \frac{h(x)}{2}\right], \text{ with } d(x,y) \leq \frac{\tau}{2} \\ \text{and} \\ \delta(B(y,\tau)) \leq \delta \end{array} \right\},$$

$$F_2 := \left\{ x \in F \setminus (\mathcal{Z} \cup F_1) \;\middle|\; \begin{array}{l} \exists y \in F, \exists \tau \in \left[\frac{h(x)}{5}, \frac{h(x)}{2}\right], \text{ with } d(x,y) \leq \frac{\tau}{2} \\ \text{and} \\ \beta_{1;k}(y,\tau) \geq \varepsilon \end{array} \right\},$$

$$F_3 := \left\{ x \in F \setminus (\mathcal{Z} \cup F_1 \cup F_2) \;\middle|\; \begin{array}{l} \exists y \in F, \exists \tau \in \left[\frac{h(x)}{5}, \frac{h(x)}{2}\right], \text{ with } d(x,y) \le \frac{\tau}{2} \\ \text{and for all planes } P \in \mathcal{P}(N,n) \text{ with} \\ \beta_{1;k}^{P}(y,\tau) \le \varepsilon \text{ we have } \sphericalangle(P,P_0) \ge \frac{3}{4}\alpha \end{array} \right\}.$$

In this section, we prove that $\mathcal{Z}$ is rectifiable by constructing a function A such that the graph of A will cover $\mathcal{Z}$. This is done by inverting the orthogonal projection $\pi|_{\mathcal{Z}} : \mathcal{Z} \to P_0$. After that, to complete the proof, it remains to show that $\mathcal{Z}$ constitutes the major part of F. Right now, we can prove that $\mu(F_2) \le 10^{-6}$ (cf. section 8.3: F_2 is small) where the control of the other sets needs some more preparations.

Lemma 6.4. *Let $\alpha, \varepsilon > 0$. Definition* 6.3 *gives a partition of F, i.e., $F = \mathcal{Z} \mathbin{\dot\cup} F_1 \mathbin{\dot\cup} F_2 \mathbin{\dot\cup} F_3$.*

Proof. From the definition we see that the sets are disjoint. We show $F \setminus \mathcal{Z} \subset F_1 \cup F_2 \cup F_3$. Let $x \in F \setminus \mathcal{Z}$, so we have $h(x) > 0$. There exist some sequences $(y_l)_{l\in\mathbb{N}} \in F^{\mathbb{N}}$, $(t_l)_{l\in\mathbb{N}}$ and $(\tau_l)_{l\in\mathbb{N}}$ so that for all $l \in \mathbb{N}$, we have $0 < t_l \le h(x)$, $t_l \to h(x)$, $\frac{t_l}{4} \le \tau_l \le \frac{t_l}{3}$, $d(x, y_l) < \frac{\tau_l}{3}$ and $(y_l, \tau_l) \notin S_{total}$. Due to $\tau_l \le \frac{t_l}{3} \le \frac{h(x)}{3} \le \frac{50}{3}$, we have for every $l \in \mathbb{N}$ either $\delta(B(y_l,\tau_l)) = \frac{\mu(B(y_l,\tau_l))}{\tau_l^n} < \frac{1}{2}\delta$ or $\delta(B(y_l,\tau_l)) \ge \frac{1}{2}\delta$ and $\beta_{1;k}(y_l,\tau_l) \ge 2\varepsilon$ or $\delta(B(y_l,\tau_l)) \ge \frac{1}{2}\delta$ and $\beta_{1;k}(y_l,\tau_l) < 2\varepsilon$, and for every plane $P \in \mathcal{P}(N,n)$ with $\beta_{1;k}^{P}(y_l,\tau_l) \le 2\varepsilon$, we have $\sphericalangle(P,P_0) > \alpha$.

Choose l so large that $\frac{4h(x)}{5} \le t_l$. We obtain $\frac{h(x)}{5} \le \frac{t_l}{4} \le \tau_l \le \frac{t_l}{3} \le \frac{h(x)}{2}$. Furthermore, we have $y_l \in F$ and $d(x,y_l) \le \frac{\tau_l}{3} < \frac{\tau_l}{2}$. Since (y_l,τ_l) fulfils one of these three cases, it follows that $x \in F_1 \cup F_2 \cup F_3$. □

The following lemma is for later use (cf. Lemma 8.10 and Lemma 8.11).

Lemma 6.5. *Let $\alpha > 0$. There exists some constant $\bar\varepsilon = \bar\varepsilon(N,n,C_0,\alpha)$ so that if $\eta < 2\bar\varepsilon$ and $k \ge 2000$, there holds for all $\varepsilon \in [\frac{\eta}{2}, \bar\varepsilon)$: If $x \in F_3$ and $h(x) \le t \le \min\{100h(x), 49\}$, we get $\sphericalangle(P_{(x,t)}, P_0) > \frac{1}{2}\alpha$, where $P_{(x,t)}$ is the plane granted since $(x,t) \in S_{total}$ (cf. Definition* 6.1*).*

Proof. Let $\alpha > 0$ and $k \ge 400$. We set $\bar\varepsilon := \min\{\varepsilon_0, \varepsilon_0', \alpha(5C_3)^{-1}\}$, where ε_0, ε_0', C_3 and C_3' depend only on N, n and C_0 will be chosen during this proof. Furthermore, let $\eta \le 2\varepsilon < 2\bar\varepsilon$.

Since $x \in F_3$ and $x \notin (F_1 \cap F_2)$, there exists some $y \in F$, $\tau \in \left[\frac{h(x)}{5}, \frac{h(x)}{2}\right]$ and $\bar P \in \mathcal{P}(N,n)$ with $d(x,y) \le \frac{\tau}{2}$, $\beta_{1;k}^{\bar P}(y,\tau) \le \varepsilon$ and $\sphericalangle(\bar P, P_0) \ge \frac{3}{4}\alpha$. Furthermore $h(x) \le t$ implies $(x,t) \in S \subset S_{total}$ and hence $\delta(B(x,t)) \ge \frac{1}{2}\delta$ and $\beta_{1;k}^{P_{(x,t)}}(x,t) \le 2\varepsilon$. Now with Lemma 4.9 ($c = 500$, $\xi = 2$, $t_x = t$, $t_y = \tau$, $\lambda = \frac{\delta}{2}$), there exist some constants $C_3 = C_3(N,n,C_0) > 1$ and $\varepsilon_0 = \varepsilon_0(N,n,C_0) > 0$ so that $\sphericalangle(\bar P, P_{(x,t)}) \le C_3\varepsilon$. Due to $\sphericalangle(\bar P, P_0) \ge \frac{3}{4}\alpha$ and $\varepsilon < \frac{\alpha}{4C_3}$ this gives $\sphericalangle(P_{(x,t)}, P_0) > \frac{1}{2}\alpha$. □

6.2. **The distance to a well approximable ball.** We recall that the set S depends on the choice of α and ε. Hence the functions d and D defined in the next definition depend on α and ε as well. We introduce $\pi := \pi_{P_0} : \mathbb{R}^N \to P_0$, the orthogonal projection on P_0.

Definition 6.6 (The functions d and D)**.** Let $\alpha, \varepsilon > 0$. If $\eta \le 2\varepsilon$, we get with Lemma 6.2(i) that $S \ne \emptyset$. We define $d : \mathbb{R}^N \to [0,\infty)$ and $D : P_0 \to [0,\infty)$ with

$$d(x) := \inf_{(X,t)\in S} (d(X,x) + t), \qquad D(y) := \inf_{x \in \pi^{-1}(y)} d(x).$$

Let us call a ball $B(X,t)$ with $(X,t) \in S$ a good ball. Then the function d measures the distance from the given point x to the nearest good ball, using the furthermost point in the ball. This implies that a ball $B(x,d(x))$ always contains some good ball. The function D does something similar. Consider the projection of all good balls to the plane P_0. Then D measures the distance to the nearest projected good ball in the same sense as above (cf. the next lemma).

Lemma 6.7. *Let $\alpha, \varepsilon > 0$. If $\eta \le 2\varepsilon$ and $y \in P_0$ we have*

$$D(y) = \inf_{(X,t)\in S} (d(\pi(X), y) + t).$$

Proof. Due to $d(X,x) \ge d(\pi(X), \pi(x))$ we have $D(y) \ge \inf_{(X,t)\in S}(d(\pi(X), y) + t)$. Assume that $\lim_{l\to\infty}(d(\pi(X_l), y)+t_l) > \inf_{(X,t)\in S}(d(\pi(X), y)+t)$ for some sequence $(X_l, t_l) \in S$. Now there exists some $l \in \mathbb{N}$ so that

$$\begin{aligned} D(y) &> d\big(\pi(X_l) + X_l - \pi(X_l), y + X_l - \pi(X_l)\big) + t_l \\ &\ge \inf_{x\in\pi^{-1}(y)} d(X_l, x) + t_l \ge D(y), \end{aligned}$$

which is a contradiction. □

Lemma 6.8. *The functions d and D are Lipschitz functions with Lipschitz constant* 1.

Proof. Let $x, y \in \mathbb{R}^N$. We get with the triangle inequality $d(x) \le d(y) + d(x,y)$ and $d(y) \le d(x) + d(x,y)$. This implies $|d(x) - d(y)| \le d(x,y)$. Using the previous lemma, we can use the same argument for the function D. □

Lemma 6.9. *We have $\{x \in \mathbb{R}^N | d(x) < 1\} \subset B(0,6)$ and $d(x) \le 60$ for all $x \in B(0,5)$.*

Proof. Let $x \in \mathbb{R}^N$ with $\inf_{(X,t)\in S}(d(X,x) + t) = d(x) < 1$. Hence there exists some $X \in F \subset B(0,5)$ with $d(0,x) \le d(0,X) + d(X,x) < 6$. If $x \in B(0,5)$, we have $d(x) \le 10 + 50$. □

Lemma 6.10. *Let $\alpha, \varepsilon > 0$. If $\eta \le 2\varepsilon$, we have $d(x) \le h(x)$ for all $x \in F$ and*

$$\mathcal{Z} = \{x \in F | d(x) = 0\}, \quad \pi(\mathcal{Z}) = \{y \in P_0 \mid D(y) = 0\}.$$

Furthermore, both sets $\mathcal{Z}$ and $\pi(\mathcal{Z})$ are closed. We recall that π denotes the orthogonal projection on the plane P_0.

Proof. Let $x \in F$. With Lemma 6.2(i), we have $(x, h(x)) \in S$ and hence $d(x) \le h(x)$. This implies $\mathcal{Z} \subset \{x \in F | d(x) = 0\}$.

Now let $x \in F$ with $h(x) > 0$. We prove $d(x) > 0$. There exist some sequences $t_l \to h(x)$ and some sequence $(X_i, s_i) \in S$ with $d(X_i, x) + s_i \to d(x)$. If on the one hand there exists some subsequence with $X_i \to x$ we obtain for another subsequence $s_i \ge h(X_i) \ge t_i > 0$ for sufficiently large i and hence $d(x) > 0$. If on the other hand $d(X_i, x)$ has a positive lower bond, we conclude that $d(x) \ge \lim_{l\to\infty} d(X_l, x) > 0$.

Now we prove the second equality. If $y \in \pi(\mathcal{Z})$, there exists some $x_0 \in \mathcal{Z}$ with $\pi(x_0) = y$ and $d(x_0) = 0$. Now we get $0 \le D(y) \le d(x_0) = 0$.

If $y \in P_0$ with $D(y) = 0$, since d is continuous, we get with Lemma 6.9 that there exists some $a \in \pi^{-1}(y)$ with $d(a) = 0$. This implies $a \in F$ and hence $a \in \mathcal{Z}$. Thus $y \in \pi(\mathcal{Z})$.

According to Lemma 6.8, d and D are continuous, and hence these sets are closed. □

Lemma 6.11. *Let $0 < \alpha \leq \frac{1}{4}$. There exists some $\bar{\varepsilon} = \bar{\varepsilon}(N, n, C_0)$ so that if $\eta < 2\bar{\varepsilon}$ and $k \geq 4$ for all $\varepsilon \in [\frac{\eta}{2}, \bar{\varepsilon})$, there holds: For all $x, y \in F$ we have*

$$d(x,y) \leq 6(d(x) + d(y)) + 2d(\pi(x), \pi(y)),$$
$$d(\pi^\perp(x), \pi^\perp(y)) \leq 6(d(x) + d(y)) + 2\alpha d(\pi(x), \pi(y)).$$

Proof. Let $0 < \alpha < \frac{1}{4}$ and $k \geq 4$. During this proof, there occur several smallness conditions on ε. The minimum of those will give us the constant $\bar{\varepsilon}$. Let $\eta \leq 2\varepsilon < 2\bar{\varepsilon}$.

The first estimate is an immediate consequence of the second estimate. So we focus on this one. Due to $F \subset B(0,5)$ the LHS is always less than 10. Hence we can assume that $d(x)+d(y) < 2$. We choose some arbitrary $r_x \in (d(x), d(x)+1) \subset (0,3)$. There exists some $(X,t) \in S$ with $d(x) \leq d(X,x) + t < r_x$. According to Lemma 6.2(ii), it follows that $(X, r_x) \in S$. Analogously, for all $r_y \in (d(y), d(y)+1)$, we can choose some $Y \in F$ with $d(Y,y) < r_y$ and $(Y, r_y) \in S$. Now it is enough to prove $d(\pi^\perp(x), \pi^\perp(y)) \leq 6(r_x + r_y) + 2\alpha d(\pi(x), \pi(y))$ since $r_x \geq d(x)$ and $r_y \geq d(y)$ were arbitrarily chosen. We can assume that $d(X,Y) > 2(r_x + r_y)$ since otherwise $d(x,y) \leq d(x,X) + d(X,Y) + d(Y,y)$ immediately implies the desired estimate.

We define $B_1 := B(X, \frac{1}{2}d(X,Y))$ and $B_2 := B(Y, \frac{1}{2}d(X,Y))$. With Lemma 6.2(i) we obtain $B_1, B_2 \in S$. Let P_1 and P_2 be the associated planes to B_1 and B_2 (see Definition 6.1). With Lemma 4.9 ($x = X$, $y = Y$, $c = 1$, $\xi = 2$, $t_x = t_y = \frac{1}{2}d(X,Y)$, $\lambda = \frac{1}{2}\delta$) there exist some constants $C_3 = C_3(N,n,C_0) > 1$ and $\varepsilon_0 = \varepsilon_0(N,n,C_0) > 0$ so that if $\varepsilon < \varepsilon_0$ for $w \in P_1$, we obtain

$$d(w, P_2) \leq C_3(N,n,C_0,\delta)\varepsilon\left(\tfrac{1}{2}d(X,Y) + d(w,X)\right). \tag{6.2}$$

Let $B_1' := B(X, \frac{1}{2}\varepsilon^{\frac{1}{2n}}d(X,Y) + r_x)$ and $B_2' := B(Y, \frac{1}{2}\varepsilon^{\frac{1}{2n}}d(X,Y) + r_y)$. Lemma 6.2(i) implies that these balls are in S. Now we conclude using $\delta(B_i') \geq \frac{\delta}{2}$, $B_i' \subset kB_i$, and $\beta_{1;k}^{P_i}(B_i) \leq 2\varepsilon$ for $i \in \{1,2\}$ that

$$\frac{1}{\mu(B_i')}\int_{B_i'} \frac{d(X',P_i)}{d(X,Y)} \mathrm{d}\mu(X') \leq \frac{1}{\delta\varepsilon^{\frac{1}{2}}} \frac{1}{\left(\frac{1}{2}d(X,Y)\right)^n}\int_{kB_i} \frac{d(X',P_i)}{\frac{1}{2}d(X,Y)}\mathrm{d}\mu(X') \leq \frac{2}{\delta}\varepsilon^{\frac{1}{2}}.$$

With Chebyshev's inequality, we deduce that there exist some $X' \in B_1'$ and some $Y' \in B_2'$ so that $d(X', P_1) \leq \frac{2}{\delta}\varepsilon^{\frac{1}{2}}d(X,Y)$ and $d(Y',P_2) \leq \frac{2}{\delta}\varepsilon^{\frac{1}{2}}d(X,Y)$.

Now let $X_1' := \pi_{P_1}(X')$ be the orthogonal projection of X' on P_1, $Y_2' := \pi_{P_2}(Y')$ the orthogonal projection of Y' on P_2, and $X_{12}' := \pi_{P_2}(X_1')$ the orthogonal projection of X_1' on P_2. If ε is small enough, we have with $\varrho \in \{\pi, \pi^\perp\}$:

$$d(\varrho(X), \varrho(X')) \leq d(X,X') \leq \tfrac{1}{2}\varepsilon^{\frac{1}{2n}}d(X,Y) + r_x,$$
$$d(\varrho(Y), \varrho(Y')) \leq d(Y,Y') \leq \tfrac{1}{2}\varepsilon^{\frac{1}{2n}}d(X,Y) + r_y,$$
$$d(\varrho(X'), \varrho(X_1')) \leq d(X', X_1') = d(X', P_1) \leq \frac{2}{\delta}\varepsilon^{\frac{1}{2}}d(X,Y),$$
$$d(\varrho(Y'), \varrho(Y_2')) \leq d(Y', Y_2') = d(Y', P_2) \leq \frac{2}{\delta}\varepsilon^{\frac{1}{2}}d(X,Y),$$
$$d(\varrho(X_1'), \varrho(X_{12}')) \leq d(X_1', X_{12}') = d(X_1', P_2) \overset{(6.2)}{<} 2C_3\varepsilon d(X,Y).$$

According to Definition 6.1, we have $\sphericalangle(P_2, P_0) \leq \alpha$ and we get with Lemma 2.13 $(X'_{12}, Y'_2 \in P_2)$ using $\alpha \leq \frac{1}{4}$:

$$d(X'_{12}, Y'_2) \leq \frac{1}{1-\alpha} d(\pi(X'_{12}), \pi(Y'_2)) \leq 2d(\pi(X'_{12}), \pi(Y'_2)), \tag{6.3}$$

$$d(\pi^{\perp}(X'_{12}), \pi^{\perp}(Y'_2)) \leq \frac{\alpha}{1-\alpha} d(\pi(X'_{12}), \pi(Y'_2)) \leq \frac{4}{3}\alpha d(\pi(X'_{12}), \pi(Y'_2)). \tag{6.4}$$

Inserting the intermediate points X', X'_1, X'_{12}, Y'_2, Y' using triangle inequality twice and using the previous inequalities, there exists some constant C so that

$$\begin{aligned} d(X,Y) &\leq C\tfrac{1}{\delta}\varepsilon^{\frac{1}{2n}} d(X,Y) + r_x + r_y + 2d(\pi(X'_{12}), \pi(Y'_2)) \\ &\leq C\tfrac{1}{\delta}\varepsilon^{\frac{1}{2n}} d(X,Y) + 3(r_x + r_y) + 2d(\pi(X), \pi(Y)), \end{aligned}$$

and hence if ε fulfils $C\frac{1}{\delta}\varepsilon^{\frac{1}{2n}} \leq \frac{1}{2}$, we get

$$d(X,Y) \leq 6(r_x + r_y) + 4d(\pi(X), \pi(Y)). \tag{6.5}$$

As for $d(X,Y)$, we estimate $d\left(\pi^{\perp}(X), \pi^{\perp}(Y)\right)$ by repeated use of the triangle inequality and (6.4). With (6.5), we deduce that

$$\begin{aligned} & d\big(\pi^{\perp}(X), \pi^{\perp}(Y)\big) \\ \leq\ & C\tfrac{1}{\delta}\varepsilon^{\frac{1}{2n}} d(X,Y) + 3(r_x + r_y) + \tfrac{4}{3}\alpha d(\pi(X), \pi(Y)) \\ \overset{(6.5)}{\leq}\ & C\tfrac{1}{\delta}\varepsilon^{\frac{1}{2n}} [6(r_x + r_y) + 4d(\pi(X), \pi(Y))] + 3(r_x + r_y) + \tfrac{4}{3}\alpha d(\pi(X), \pi(Y)) \\ \leq\ & 4(r_x + r_y) + 2\alpha d(\pi(X), \pi(Y)). \end{aligned}$$

This implies using $d(\pi^{\perp}(x), \pi^{\perp}(X)) \leq d(x, X) \leq r_x$ and $d(\pi^{\perp}(Y), \pi^{\perp}(y)) \leq d(Y, y) \leq r_y$ that

$$\begin{aligned} d(\pi^{\perp}(x), \pi^{\perp}(y)) &\leq 5(r_x + r_y) + 2\alpha d(\pi(X), \pi(Y)) \\ &\leq 6(r_x + r_y) + 2\alpha d(\pi(x), \pi(y)). \end{aligned}$$ □

6.3. **A Whitney-type decomposition of $P_0 \setminus \pi(\mathcal{Z})$.** In this part, we show that $P_0 \setminus \pi(\mathcal{Z})$ can be decomposed as a union of disjoint cubes R_i, where the diameter of R_i is proportional to $D(x)$ for all $x \in R_i$. This result is a variant of the Whitney decomposition for open sets in $\mathbb{R}^n$; cf. [11, Appendix J].

Definition 6.12 (Dyadic primitive cells)**.**

1. We set $\mathcal{D}$ to be the set of all dyadic primitive cells on P_0. We recall that the plane P_0 is an n-dimensional linear subspace of $\mathbb{R}^N$.
2. Let $r \in (0, \infty)$ and Q be some cube in $\mathbb{R}^N$. By rQ, we denote the cube with the same centre and orientation as Q but r-times the diameter.

We mention that the function D depends on the choice of α and ε because D depends on the set $S \subset S^{\varepsilon,\alpha}_{total}$. Hence the family of cubes given by the following lemma depends on the choice of α and ε as well.

Lemma 6.13. *Let $\alpha, \varepsilon > 0$. If $\eta \leq 2\varepsilon$, then there exists a countable family of cubes $\{R_i\}_{i\in I} \subset \mathcal{D}$ such that*

(i) $10 \operatorname{diam} R_i \leq D(x) \leq 50 \operatorname{diam} R_i$ *for all $x \in 10R_i$,*
(ii) $P_0 \setminus \pi(\mathcal{Z}) = \bigcup_{i\in I} R_i = \bigcup_{i\in I} 2R_i$ *and cubes R_i have disjoint interior,*

(iii) *for every* $i, j \in I$ *with* $10R_i \cap 10R_j \neq \emptyset$, *we have* $\frac{1}{5} \operatorname{diam} R_j \leq \operatorname{diam} R_i \leq 5 \operatorname{diam} R_j$,

(iv) *for every* $i \in I$, *there are at most* 180^n *cells* R_j *with* $10R_i \cap 10R_j \neq \emptyset$.

Proof. For $z \in P_0$, $D(z) > 0$, we define $Q_z \in \mathcal{D}$ as the largest dyadic primitive cell that contains z and fulfils $\operatorname{diam} Q_z \leq \frac{1}{20} \inf_{u \in Q_z} D(u)$. For such a given z the cell Q_z exists because the function D is continuous and $D(z) > 0$. Hence if we choose a small enough dyadic primitive cell Q that contains z, we get $\operatorname{diam} Q \leq \frac{1}{20} \inf_{u \in Q} D(u)$. Due to the dyadic structure, there can only be one largest dyadic primitive cell that contains z and fulfils the upper condition. We choose $R_i \in \mathcal{D}$ such that $\{R_i | i \in I\} = \{Q_z \in \mathcal{D} | z \in P_0, D(z) > 0\}$ and $R_i = R_j$ is equivalent to $i = j$.

(i) Let $x \in 10R_i$ and $u \in R_i$. We get $20 \operatorname{diam} R_i \leq D(u) < D(x) + 10 \operatorname{diam} R_i$, and hence $10 \operatorname{diam} R_i \leq D(x)$. Let $J_i \in \mathcal{D}$ be the smallest cell in $\mathcal{D}$ with $R_i \subsetneq J_i$ and choose $u \in J_i$ so that $D(u) < 20 \operatorname{diam} J_i = 40 \operatorname{diam} R_i$. This is possible because otherwise R_i is not maximal relating to $\operatorname{diam} R_i \leq \frac{1}{20} \inf_{v \in R_i} D(v)$. We obtain $D(x) \leq D(u) + d(u, x) < 50 \operatorname{diam} R_i$.

(ii) If the interior of some cells R_i and R_j were not disjoint, because of the dyadic structure, one cell would be contained in the other. But then one of those would not be the maximal cell. Hence the R_i's have disjoint interior. For all $x \in 2R_i$, we obtain using (i) and Lemma 6.10 that $x \notin \pi(\mathcal{Z})$. Now let $x \in P_0 \setminus \pi(\mathcal{Z})$. With Lemma 6.10, we get $D(x) > 0$. So there exists the cube $Q_x \in \mathcal{D}$ with $x \in Q_x$ and hence $x \in \bigcup_{i \in I} R_i$.

(iii) If $10R_i \cap 10R_j \neq \emptyset$ we can apply (i) for some $x \in 10R_i \cap 10R_j$ and obtain the assertion.

(iv) Let $i \in I$ and R_j with $10R_i \cap 10R_j \neq \emptyset$. We conclude with (iii) that $d(R_i, R_j) \leq 30 \operatorname{diam} R_i$ and so $R_j \subset (1+30+5)R_i$. Furthermore, we have $\operatorname{diam} R_j \geq \frac{1}{5} \operatorname{diam} R_i$. Since the cells R_j are disjoint, there exist at most $\frac{\mathcal{H}^n(36R_i)}{\mathcal{H}^n(R_j)} \leq (180)^n$ cells R_j with $10R_i \cap 10R_j \neq \emptyset$. ☐

Now we set $U_{12} := B(0, 12) \cap P_0$ and $I_{12} := \{i \in I | R_i \cap U_{12} \neq \emptyset\}$.

Lemma 6.14. *Let* $\alpha, \varepsilon > 0$. *If* $\eta \leq 2\varepsilon$, *for every* $i \in I_{12}$, *there exists some ball* $B_i = B(X_i, t_i)$ *with* $(X_i, t_i) \in S$, $\operatorname{diam} R_i \leq \operatorname{diam} B_i \leq 200 \operatorname{diam} R_i$ *and* $d(\pi(B_i), R_i) \leq 100 \operatorname{diam} R_i$.

Proof. Let $i \in I_{12}$ and $x \in R_i$. Use Lemma 6.7, Lemma 6.10 and Lemma 6.13(i), (ii) to get some $(X, t) \in S$ with $d(\pi(X), x) + t \leq 2D(x) \leq 100 \operatorname{diam} R_i$. Choose $B_i := B(X_i, t_i) := B(X, r)$ with $r = \max\{t, \frac{\operatorname{diam} R_i}{2}\} \leq 100 \operatorname{diam} R_i$. Now we have $d(\pi(B_i), R_i) \leq 100 \operatorname{diam} R_i$ and $\operatorname{diam} R_i \leq \operatorname{diam} B_i \leq 200 \operatorname{diam} R_i$. We can show that $r < 50$, and hence with Lemma 6.2(ii), we get $(X, r) \in S$. ☐

6.4. **Construction of the function** A. We recall that $\pi := \pi_{P_0} : \mathbb{R}^N \to P_0$ is the orthogonal projection on P_0 and introduce $\pi^\perp := \pi^\perp_{P_0} : \mathbb{R}^N \to P_0^\perp$, the orthogonal projection on $P_0^\perp$, where $P_0^\perp := \{x \in \mathbb{R}^N | x \cdot v = 0 \text{ for all } v \in P_0\}$ is the orthogonal complement of P_0. To define the function A, we want to invert the projection $\pi|_{\mathcal{Z}}$ on $\mathcal{Z}$.

Lemma 6.15. *Let* $0 < \alpha \leq \frac{1}{4}$. *There exists some* $\bar{\varepsilon} = \bar{\varepsilon}(N, n, C_0)$ *so that if* $\eta < 2\bar{\varepsilon}$ *and* $k \geq 4$ *for all* $\varepsilon \in [\frac{\eta}{2}, \bar{\varepsilon})$, *the orthogonal projection* $\pi|_{\mathcal{Z}} : \mathcal{Z} \to P_0$ *is injective.*

Proof. The assertion follows directly from Lemma 6.10 and Lemma 6.11. ☐

Since $\pi|_{\mathcal{Z}} : \mathcal{Z} \to P_0$ is injective, we are able to define the desired Lipschitz function A on $\pi(\mathcal{Z})$ by

$$A(a) := \pi^{\perp}\left(\pi|_{\mathcal{Z}}^{-1}(a)\right)$$

where $a \in \pi(\mathcal{Z})$.

Lemma 6.16. *Under the conditions of the previous lemma, the map $A\big|_{\pi(\mathcal{Z})}$ is 2α-Lipschitz.*

Proof. Due to Lemma 6.15 for $a, b \in \pi(\mathcal{Z})$, there exist distinct $X, Y \in \mathcal{Z}$ with $\pi(X) = a$ and $\pi(Y) = b$. We have $A(a) = \pi^{\perp}(X)$, $A(b) = \pi^{\perp}(Y)$ and Lemma 6.10 implies that $d(X) = d(Y) = 0$. So, with Lemma 6.11, we get $d(A(a), A(b)) \leq 2\alpha d(a, b)$. □

Now we have a Lipschitz function A defined on $\pi(\mathcal{Z})$. By using Kirszbraun's theorem [9, Thm. 2.10.43], we would obtain a Lipschitz extension of A defined on P_0 with the same Lipschitz constant 2α, where the graph of the extension covers $\mathcal{Z}$. But until now, we do not know that $\mathcal{Z}$ is a major part of F. We cannot even be sure that $\mathcal{Z}$ is not a null set. So we do not use Kirszbraun's theorem here, but we will extend A by an explicit construction. This will help us to show that the other parts of F, in particular F_1, F_2, F_3, are quite small.

Definition 6.17. Let $\alpha, \varepsilon > 0$. If $\eta \leq 2\varepsilon$, for all $i \in I_{12}$, we set $P_i := P_{(X_i,t_i)}$, where $P_{(X_i,t_i)}$ is the n-dimensional plane, which is, in the sense of Definition 6.1, associated to the ball $B(X_i, t_i) = B_i$ given by Lemma 6.14.

Lemma 6.18. *Let $0 < \alpha \leq \frac{1}{2}$ and $\varepsilon > 0$. If $\eta \leq 2\varepsilon$, then for all $i \in I_{12}$, there exists some affine map $A_i : P_0 \to P_0^{\perp}$ with graph $G(A_i) = P_i$ and A_i is 2α-Lipschitz.*

Proof. Use $\sphericalangle(P_i, P_0) \leq \alpha \leq \frac{1}{2}$ (cf. definition of S_{total}) and apply Corollary 2.14. □

In the following, we use differentiable functions defined on subsets of P_0. For the definition of the derivative see section A.2 on page 1247.

Lemma 6.19. *Let $\alpha, \varepsilon > 0$. If $\eta \leq 2\varepsilon$, then there exists some partition of unity $\phi_i \in C^{\infty}(U_{12}, \mathbb{R})$, $i \in I_{12}$, with $0 \leq \phi_i \leq 1$ on U_{12}, $\phi_i \equiv 0$ on the exterior of $3R_i$ and $\sum_{i \in I_0} \phi_i(a) = 1$ for all $a \in U_{12}$. Furthermore there exists some constant $C = C(n)$ with $|\partial^{\omega} \phi_i(a)| \leq \frac{C(n)}{(\operatorname{diam} R_i)^{|\omega|}}$ where ω is some multi-index with $1 \leq |\omega| \leq 2$.*

Proof. For every $i \in I_{12}$, we choose some function $\tilde{\phi}_i \in \mathcal{C}^{\infty}(P_0, \mathbb{R})$ with $0 \leq \tilde{\phi}_i \leq 1$, $\tilde{\phi}_i \equiv 1$ on $2R_i$, $\tilde{\phi}_i \equiv 0$ on the exterior of $3R_i$, $|\partial^{\omega} \tilde{\phi}_i| \leq \frac{C}{\operatorname{diam} R_i}$ for all multi-indices ω with $|\omega| = 1$ and $|\partial^{\kappa} \tilde{\phi}_i| \leq \frac{C}{(\operatorname{diam} R_i)^2}$ for all multi-indices κ with $|\kappa| = 2$. Now on $V := \bigcup_{i \in I_{12}} 2R_i$, we can define the partition of unity $\phi_i(a) := \frac{\tilde{\phi}_i(a)}{\sum_{j \in I_{12}} \tilde{\phi}_j(a)}$. For all $a \in V$, there exists some $i \in I_{12}$ with $a \in 2R_i$ and hence $\sum_{j \in I_{12}} \tilde{\phi}_j(a) \geq 1$. Moreover, due to Lemma 6.13(iv), there are only finitely many $j \in I_{12}$ such that $\tilde{\phi}_j(a) \neq 0$. Due to the control we have on the derivatives of $\tilde{\phi}_i$, we obtain with Lemma 6.13(iv) the desired estimates of the derivatives of ϕ_i. □

Definition 6.20 (Definition of A on U_{12})**.** Let $\alpha, \varepsilon > 0$. If $\eta \leq 2\varepsilon$ and $k \geq 4$, we extend the function $A : \pi(\mathcal{Z}) \to P_0^{\perp} \subset \mathbb{R}^N$, $a \mapsto \pi^{\perp}\left(\pi|_{\mathcal{Z}}^{-1}(a)\right)$ (see page 1215) to the whole set U_{12} by setting for $a \in U_{12}$,

$$A(a) := \begin{cases} \pi^{\perp}\left(\pi|_{\mathcal{Z}}^{-1}(a)\right), & a \in \pi(\mathcal{Z}), \\ \sum_{i \in I_{12}} \phi_i(a) A_i(a), & a \in U_{12} \cap \bigcup_{i \in I_{12}} 2R_i. \end{cases}$$

With $\mathcal{Z} \subset F \subset B(0,5)$, we get $\pi(\mathcal{Z}) \subset U_{12}$ and, with Lemma 6.13(ii), we obtain $\bigcup_{i \in I_{12}} 2R_i \cap \pi(\mathcal{Z}) = \emptyset$; hence we have defined A on the whole set

$$U_{12} = (U_{12} \cap \bigcup_{i \in I_{12}} 2R_i) \,\dot{\cup}\, \pi(\mathcal{Z}).$$

6.5. A **is Lipschitz continuous.** In this section, we show that A is Lipschitz continuous. We start with some useful estimates.

Lemma 6.21. *Let $0 < \alpha \le \frac{1}{4}$. There exists some $\bar{k} \ge 4$ and some $\bar{\varepsilon} = \bar{\varepsilon}(N, n, C_0)$ so that if $k \ge \bar{k}$ and $\eta < 2\bar{\varepsilon}$ for all $\varepsilon \in [\frac{\eta}{2}, \bar{\varepsilon})$, there exist some constants $C > 1$ and $\bar{C} = \bar{C}(N, n, C_0) > 1$ so that for all $i, j \in I_{12}$ with $i \ne j$ and $10R_i \cap 10R_j \ne \emptyset$, we get*

(i) $d(B_i, B_j) \le C \operatorname{diam} R_j$,
(ii) $d(A_i(q), A_j(q)) \le \bar{C}\varepsilon \operatorname{diam} R_j$ *for all* $q \in 100R_j$,
(iii) *the Lipschitz constant of the map* $(A_i - A_j) : P_0 \to P_0^{\perp}$ *fulfils* $\operatorname{Lip}_{A_i - A_j} \le \bar{C}\varepsilon$,
(iv) $d(A(u), A_j(u)) \le \bar{C}\varepsilon \operatorname{diam} R_j$ *for all* $u \in 2R_j \cap U_{12}$.

Proof. Let $0 < \alpha \le \frac{1}{4}$. We set $\bar{\varepsilon} = \min\left\{\frac{\delta}{2}, \bar{\varepsilon}', \varepsilon_0\right\}$, where $\delta = \delta(N, n)$ is defined on page 1208, $\bar{\varepsilon}'$ is the upper bound for ε given by Lemma 6.11 and ε_0 is the constant from Lemma 4.9. Let $\eta < 2\bar{\varepsilon}$ and choose ε such that $\eta \le 2\varepsilon < 2\bar{\varepsilon}$.

(i) Let $B_i = B(X_i, t_i)$ and $B_j = B(X_j, t_j)$. Lemma 6.13 and Lemma 6.14 imply $d(\pi(X_i), \pi(X_j)) \le C \operatorname{diam} R_j$, and, using $(X_l, t_l) \in S$ we have $d(X_l) \le 500 \operatorname{diam} R_j$ for $l \in \{i, j\}$. Now Lemma 6.11 implies the assertion.

(ii) At first, we show for $q \in 100R_j$ that $d(A_i(q) + q, X_i) \le C \operatorname{diam} R_j$. Since $(X_i, t_i) \in S \subset S_{total}$, $\varepsilon \le \frac{\delta}{4}$, and using Lemma 4.10 ($\sigma = 2\varepsilon$, $x = X_i$, $t = t_i$, $\lambda = \frac{1}{2}\delta$, $P = P_i$) we get $B(X_i, 2t_i) \cap P_i \ne \emptyset$. Thus there exists some $a \in P_0$ with $A_i(a) + a \in B(X_i, 2t_i) \cap P_i$ and $a \in \pi(2B_i)$. Since A_i is 2α-Lipschitz and $\alpha < \frac{1}{2}$, using Lemmas 6.13 and 6.14 we obtain by inserting $A_i(a) + a$ with triangle inequality

$$d(A_i(q) + q, X_i) \le |A_i(q) - A_i(a)| + d(q, a) + \operatorname{diam} B_i \le C \operatorname{diam} R_j. \tag{6.6}$$

With Lemmas 6.13 and 6.14, there exists some constant $C > 2$ so that $\frac{1}{C} t_j \le t_i \le C t_j$. Moreover, we have $(X_i, t_i), (X_j, t_j) \in S \subset S_{total}$. With $k \ge \bar{k} := 2C^2 \ge 4C$, Lemma 4.9 ($x = X_j$, $y = X_i$, $c = C$, $\xi = 2$, $t_x = t_j$, $t_y = t_i$, $\lambda = \frac{\delta}{2}$) implies that there exist some $\varepsilon_0 > 0$ and some constant $C_3 = C_3(N, n, C_0) > 1$ so that, for $\varepsilon < \bar{\varepsilon} \le \varepsilon_0$ with the already shown (i), (6.6) and Lemma 6.14, we get

$$d(A_i(q) + q, P_j) \le C_3 \varepsilon \left(t_j + d(A_i(q) + q, X_j)\right) \le C\varepsilon \operatorname{diam} R_j. \tag{6.7}$$

Furthermore, there exists some $o \in P_0$ so that $A_j(o) + o = \pi_{P_j}(A_i(q) + q)$. Now we have $d(A_j(o) + o, A_j(q) + q) \leq 2d(o, q) \leq 2d(A_i(q) + q, A_j(o) + o)$ since A is 2α-Lipschitz, and hence with Lemma 6.13 and Lemma 6.14 we obtain for some $C = C(N, n, C_0)$:

$$d(A_i(q)+q, A_j(q)+q) \leq d(A_i(q)+q, P_j) + d(A_j(o)+o, A_j(q)+q) \overset{(6.7)}{\leq} C\varepsilon \operatorname{diam} R_j.$$

(iii) Without loss of generality, we assume that $\operatorname{diam} R_i \leq \operatorname{diam} R_j$. We have $B(y, 2\operatorname{diam} R_i) \cap P_0 \subset 20R_i \cap 20R_j$ for some $y \in 10R_i \cap 10R_j \neq \emptyset$. We choose arbitrary $a, b \in B(y, 2\operatorname{diam} R_i) \cap P_0$ with $d(a, b) \geq \operatorname{diam} R_i$. Now, with (ii), we get

$$|(A_i - A_j)(a) - (A_i - A_j)(b)| \leq C\varepsilon \operatorname{diam} R_i \leq C(N, n, C_0)\varepsilon d(a, b).$$

Since $A_i - A_j$ is an affine map, this implies $\operatorname{Lip}_{A_i - A_j} \leq C(N, n, C_0)\varepsilon$.

(iv) We get the estimate using Definition 6.20, $\sum_{l \in I_{12}} \phi_l(u) = 1$, Lemma 6.13(iv) and (ii) of the current lemma. □

Lemma 6.22. *Let $0 < \alpha \leq \frac{1}{4}$. There exist some $\bar{k} \geq 4$ and some $\bar{\varepsilon} = \bar{\varepsilon}(N, n, C_0, \alpha) < \alpha$ so that if $k \geq \bar{k}$ and $\eta < 2\bar{\varepsilon}$ for all $\varepsilon \in [\frac{\eta}{2}, \bar{\varepsilon})$, the function A is Lipschitz continuous on $2R_j \cap U_{12}$ for all $j \in I_{12}$ with Lipschitz constant 3α.*

Proof. Let $0 < \alpha \leq \frac{1}{4}$. We set $\bar{\varepsilon} := \min\left\{\bar{\varepsilon}', \frac{\alpha}{\tilde{C}}\right\}$, where $\bar{\varepsilon}'$ is the upper bound for ε given by Lemma 6.21 and $\tilde{C}(N, n, C_0)$ is some constant presented at the end of this proof. Let $\eta < 2\bar{\varepsilon}$ and choose $\varepsilon > 0$ such that $\eta \leq 2\varepsilon < 2\bar{\varepsilon}$. Let $a, b \in 2R_j \cap U_{12}$. We obtain

$$|A(a) - A(b)| \leq \sum_{i \in I_{12}} \phi_i(a)|A_i(a) - A_i(b)| + \sum_{i \in I_{12}} |\phi_i(a) - \phi_i(b)||A_i(b) - A_j(b)|.$$

If $\phi_i(a) - \phi_i(b) \neq 0$, we get $3R_i \cap 2R_j \neq \emptyset$ and so we can apply Lemma 6.13(iii), (iv) and Lemma 6.21(ii). Since $\varepsilon < \bar{\varepsilon} \leq \frac{\alpha}{\tilde{C}}$, we obtain with Lemma 6.18 and Lemma 6.19 that A is 3α Lipschitz. □

Lemma 6.23. *Under the conditions of the previous lemma for $a, b \in U_{12} \setminus \pi(\mathcal{Z})$ with $[a, b] \subset U_{12} \setminus \pi(\mathcal{Z})$, we have that $d(A(a), A(b)) \leq 3\alpha d(a, b)$.*

Proof. Lemma 6.13(ii) implies that for all $v \in [a, b]$, there exists some $j \in I_{12}$ with $v \in R_j$, and, with Lemma 6.13(i), we get $D(v) > 0$. Assume that the set $\tilde{I}_{12} := \{i \in I_{12} | R_i \cap [a, b] \neq \emptyset\}$ is infinite. The cubes R_i have disjoint interior, so there exists some sequence $(R_{i_l})_{l \in \mathbb{N}}$, $i_l \in \tilde{I}_{12}$, with $\operatorname{diam} R_{i_l} \to 0$. Hence there exists some sequence $(v_l)_{l \in \mathbb{N}}$ with $v_l \in R_{i_l} \cap [a, b]$, and, with Lemma 6.13(i), we obtain $D(v_l) \leq 50 \operatorname{diam} R_{i_l} \to 0$. Let $\overline{v} \in [a, b]$ be an accumulation point of $(v_l)_{l \in \mathbb{N}}$. Since D is continuous (Lemma 6.8), we deduce that $D(\overline{v}) = 0$, which is according to Lemma 6.10 equivalent to $\overline{v} \in \pi(\mathcal{Z})$. This is in contradiction to $[a, b] \subset P_0 \setminus \pi(\mathcal{Z})$, and so the set $\tilde{I}_{12}$ has to be finite. With Lemma 6.22 and $[a, b] \subset \bigcup_{i \in \tilde{I}_{12}} R_i$, we get $d(A(a), A(b)) \leq 3\alpha d(a, b)$. □

Now we show that A is Lipschitz continuous on U_{12} with some large Lipschitz constant. After that, using the continuity of A, we are able to prove that A is Lipschitz continuous with Lipschitz constant 3α.

Lemma 6.24. *Let $0 < \alpha \leq \frac{1}{4}$. There exists some $\bar{k} \geq 4$ and some $\bar{\varepsilon} = \bar{\varepsilon}(N, n, C_0, \alpha) < \alpha$ so that if $k \geq \bar{k}$ and $\eta < 2\bar{\varepsilon}$ for all $\varepsilon \in [\frac{\eta}{2}, \bar{\varepsilon})$, A is Lipschitz continuous on U_{12}.*

Proof. Let $0 < \alpha \leq \frac{1}{4}$, let $k \geq \bar{k} \geq 4$, where $\bar{k}$ is the constant from Lemma 6.22, and let $\bar{\varepsilon} = \bar{\varepsilon}(N, n, C_0, \alpha) \leq \frac{\delta}{4}$ be so small that we can apply Lemmas 6.11, 6.16, 6.21 and 6.23. Furthermore, let $\varepsilon > 0$ such that $\eta \leq 2\varepsilon < 2\bar{\varepsilon}$. Let $a, b \in U_{12}$ with $a \in \pi(\mathcal{Z})$ and $b \in 2R_j$ for some $j \in I_{12}$. We estimate $d(A(a), A(b)) \leq d(A(a) + a, X_j) + d(X_j, A(b) + b)$ where X_j is the centre of the ball $B_j = B(X_j, t_j)$ (see Lemma 6.14).

At first, we consider $d(A(a) + a, X_j)$. Since $A(a) + a \in \mathcal{Z}$, Lemma 6.10 implies $d(A(a) + a) = 0$. Moreover, with Lemma 6.14 and $(X_j, t_j) \in S$, we deduce that $d(X_j) \leq 100 \operatorname{diam} R_j$ and

$$d(\pi(A(a) + a), \pi(X_j)) \leq d(a, b) + d(b, \pi(X_j)) \leq d(a, b) + C \operatorname{diam} R_j.$$

Using those estimates, Lemma 6.11 implies $d(A(a) + a, X_j) \leq 2d(a, b) + C \operatorname{diam} R_j$.

Now we consider $d(X_j, A(b) + b)$. We have $(X_j, t_j) \in S \subset S_{total}$, and hence, with Lemma 4.10 using $\varepsilon < \bar{\varepsilon} \leq \frac{\delta}{4}$, there exists some $y \in B(X_j, 2t_j) \cap P_j$, where P_j is the associated plane to B_j (see Definition 6.17). Since $\sphericalangle(P_j, P_0) \leq \alpha \leq \frac{1}{4}$, we deduce from Lemma 2.13, Lemma 6.14 and Lemma 6.21(iv) that

$$\begin{aligned} d(X_j, A(b) + b) &\leq d(X_j, y) + d(y, A_j(b) + b) + d(A_j(b) + b, A(b) + b) \\ &\leq C(\operatorname{diam} R_j + d(a, b)). \end{aligned}$$

With Lemma 6.13, Lemma 6.10 and using that D is 1-Lipschitz (Lemma 6.8) we obtain $\operatorname{diam} R_j \leq D(b) - D(a) \leq d(a, b)$ and hence $d(A(a), A(b)) \leq Cd(a, b)$. Due to Lemma 6.16 and Lemma 6.23 it remains to handle the case were $a, b \notin \pi(\mathcal{Z})$ and $[a, b] \cap \pi(\mathcal{Z}) \neq \emptyset$. This follows immediately from the just proven case and the triangle inequality. $\square$

Lemma 6.25. *Under the conditions of Lemma* 6.24 *for some* $a \in \pi(\mathcal{Z})$, $i \in I_{12}$ *and* $b \in 2R_j$, *we get* $d(A(a), A(b)) \leq 3\alpha d(a, b)$.

Proof. We set $c := \inf_{x \in [a,b] \cap \pi(\mathcal{Z})} d(x, b)$. Due to Lemma 6.10, there exists some $v \in [a, b] \cap \pi(\mathcal{Z})$ with $d(v, b) = c$. Furthermore, there exists some sequence $(v_l)_l \subset [v, b]$ with $v_l \to v$ where $l \to \infty$. With Lemma 6.13, we deduce that $([v, b] \setminus \{v\}) \subset \bigcup_{j \in I_{12}} 2R_j$. For every $l \in \mathbb{N}$ we obtain with Lemma 6.23 $d(A(v), A(b)) \leq d(A(v), A(v_l)) + 3\alpha d(v, b)$, and, since A is continuous (Lemma 6.24) we conclude with $l \to \infty$ that $d(A(v), A(b)) \leq 3\alpha d(v, b)$. The assertion follows since we already know that A is 2α-Lipschitz on $\pi(\mathcal{Z})$. $\square$

Lemma 6.26. *Under the conditions of Lemma* 6.24 *we have* $d(A(a), A(b)) \leq 3\alpha d(a, b)$ *for* $a, b \in \bigcup_{j \in I_{12}} 2R_j \cap U_{12}$.

Proof. This is an immediate consequence of Lemma 6.22, Lemma 6.23 and Lemma 6.25. $\square$

Lemma 6.27. *Under the conditions of Lemma* 6.24, *the function* A *is Lipschitz continuous on* U_{12} *with Lipschitz constant* 3α.

Proof. This follows directly from the previous lemma and Lemma 6.16. $\square$

The following estimate is for later use.

Lemma 6.28. *Let* $0 < \alpha \leq \frac{1}{4}$. *There exists some* $\bar{k} \geq 4$ *and some* $\bar{\varepsilon} = \bar{\varepsilon}(N, n, C_0)$ *so that if* $k \geq \bar{k}$ *and* $\eta < 2\bar{\varepsilon}$ *for all* $\varepsilon \in [\frac{\eta}{2}, \bar{\varepsilon})$, *there exists some constant* $C = C(N, n, C_0)$ *so that for all* $j \in I_{12}$, $a \in 2R_j$ *and all multi-indices* κ *with* $|\kappa| = 2$ *we have* $\partial^\kappa A(a)| \leq \frac{C\varepsilon}{\operatorname{diam} R_j}$.

Proof. Choose $\bar{k}$ and $\bar{\varepsilon}$ as in Lemma 6.21. Let κ be some multi-index with $|\kappa| = 2$. For $i \in I_{12}$, the function A_i is an affine map, and hence for some suitable $l_1, l_2 \in \{1, \ldots, n\}$ we have

$$\partial^\kappa A = \partial^\kappa \Big(\sum_{i \in I_{12}} \phi_i A_i \Big) = \sum_{i \in I_{12}} (\partial^\kappa \phi_i) A_i + \sum_{i \in I_{12}} (\partial_{l_1} \phi_i \partial_{l_2} A_i + \partial_{l_2} \phi_i \partial_{l_1} A_i) . \tag{6.8}$$

Let $j \in I_{12}$ and $a \in 2R_j$. Lemma 6.13 implies that there exist at most 180^n cells R_i so that $\partial^\kappa \phi_i(a) \neq 0$ or $\partial^\omega \phi_i(a) \neq 0$, where ω is a multi-index with $|\omega| = 1$. So only finite sums occur in the following estimates. We have $\sum_{i \in I_{12}} \partial^\omega \phi_i = \partial^\omega \sum_{i \in I_{12}} \phi_i = \partial^\omega\ 1 = 0$ so that we get

$$\begin{aligned} |\partial^\kappa A| \overset{(6.8)}{\leq} & \sum_{i \in I_{12}} |\partial^\kappa \phi_i|\ |A_i - A_j| + \sum_{i \in I_{12}} |\partial_{l_1} \phi_i|\ |\partial_{l_2}(A_i - A_j)| \\ & + \sum_{i \in I_{12}} |\partial_{l_2} \phi_i|\ |\partial_{l_1}(A_i - A_j)|. \end{aligned}$$

To estimate these sums, we only have to consider the case when a is in the support of ϕ_i for some $i \in I_{12}$. This implies $3R_i \cap 2R_j \neq \emptyset$. Now use Lemma 6.21(ii), (iii), Lemma 6.19, and Lemma 6.13(iii), (iv) to obtain the assertion. □

7. γ-FUNCTIONS

In this section, we introduce the γ-function of some function $g : P_0 \to P_0^\perp$. This function measures how well g can be approximated in some ball by some affine function. The main results of this section are Theorem 7.3 and Theorem 7.17. We will use these statements in section 8.4 to prove that $\mu(F_3)$ is small.

Definition 7.1. Let $U \subset P_0$, $q \in U$ and $t > 0$ so that $B(q,t) \cap P_0 \subset U$. Furthermore, let $\mathcal{A} = \mathcal{A}(P_0, P_0^\perp)$ be the set of all affine functions $a : P_0 \to P_0^\perp$ and let $g : U \to P_0^\perp$ be some function. We define

$$\gamma_g(q,t) := \inf_{a \in \mathcal{A}} \frac{1}{t^n} \int_{B(q,t) \cap P_0} \frac{d(g(u), a(u))}{t} \mathrm{d}\mathcal{H}^n(u).$$

Lemma 7.2. *Let $U \subset P_0$, $q \in U$ and $t > 0$ so that $B(q,t) \cap P_0 \subset U$. Furthermore, let $g : U \to P_0^\perp$ be a Lipschitz continuous function such that the Lipschitz constant fulfils $60n(10^n + 1) \left(8n \frac{\omega_{n-1}}{\omega_n}\right)^{n+1} \leq \mathrm{Lip}_g^{-1}$, where ω_n denotes the n-dimensional volume of the n-dimensional unit ball. Then we have*

$$\gamma_g(q,t) \leq 3\ \tilde{\gamma}_g(q,t) := 3 \inf_{P \in \mathcal{P}(N,n)} \frac{1}{t^n} \int_{B(q,t) \cap P_0} \frac{d(u + g(u), P)}{t} \mathrm{d}\mathcal{H}^n(u),$$

where $\mathcal{P}(N,n)$ is the set of all n-dimensional affine planes in $\mathbb{R}^N$.

Proof. Let g be a Lipschitz continuous function with an appropriate Lipschitz constant. By using $a : u \to g(q) \in \mathcal{A}$ as a constant map and by using that g is 1-Lipschitz, we deduce that $\gamma_g(q,t) \leq \mathrm{Lip}_g\, \omega_n$. It follows, since for every $a \in \mathcal{A}$ the graph $G(a)$ of a is in $\mathcal{P}(N,n)$, that $\tilde{\gamma}_g(q,t) \leq \gamma_g(q,t) \leq \mathrm{Lip}_g\, \omega_n$. Let $0 < \xi < \mathrm{Lip}_g\, \omega_n$ and choose some $P \in \mathcal{P}(N,n)$ so that

$$\frac{1}{t^n} \int_{B(q,t) \cap P_0} \frac{d(u + g(u), P)}{t} \mathrm{d}\mathcal{H}^n(u) \leq \tilde{\gamma}_g(q,t) + \xi \leq 2\, \mathrm{Lip}_g\, \omega_n. \tag{7.1}$$

We set $D_1 := \{v \in B(q,t) \cap P_0 | d(v+g(v),P) \le 4\,\mathrm{Lip}_g\, t\}$, $D_2 := (B(q,t) \cap P_0) \setminus D_1$ and obtain using Chebyshev's inequality and (7.1)

$$\mathcal{H}^n(D_1) \ge \omega_n t^n - \mathcal{H}^n(D_2) \ge \frac{\omega_n}{2} t^n. \tag{7.2}$$

Assume that every simplex $\triangle(u_0,\dots,u_n) \in D_1$ is not an (n,H)-simplex, where $H = \frac{\omega_n}{4\omega_{n-1}} t$. With Lemma 2.10 ($m = n$, $D = D_1$), there exists some plane $\hat{P} \in \mathcal{P}(N, n-1)$ such that $D_1 \subset U_H(\hat{P}) \cap B(q,t) \cap P_0$. We get

$$\mathcal{H}^n(D_1) \le \mathcal{H}^n(U_H(\hat{P}) \cap B(q,t) \cap P_0) \le 2H\omega_{n-1} t^{n-1} = \frac{\omega_n}{2} t^n.$$

This is in contradiction to (7.2), so there exists some (n,H)-simplex $\triangle(u_0,\dots,u_n) \in D_1$. We set $\hat{P}_0 := P_0 + g(u_0)$, $y_i := u_i + g(u_0) \in \hat{P}_0$ for all $i \in \{0,\dots,n\}$ and $S := \Delta(y_0,\dots,y_n) \subset \hat{P}_0 \cap B(q+g(u_0),t)$. We recall that P is the plane satisfying (7.1). We obtain for all $i \in \{0,\dots,n\}$,

$$\begin{aligned} d(y_i,P) &\le d(u_i+g(u_0), u_i+g(u_i)) + d(u_i+g(u_i),P) \\ &\le \mathrm{Lip}_g\, d(u_0,u_i) + 4\,\mathrm{Lip}_g\, t \le 6\,\mathrm{Lip}_g\, t. \end{aligned}$$

With Lemma 2.17, $C = 4\frac{\omega_{n-1}}{\omega_n} > 1$,[7] $\hat{C} = 1$, $m = n$, $\sigma = 6\,\mathrm{Lip}_g$, $P_1 = \hat{P}_0$, $P_2 = P$ and $x = q+g(u_0)$, we get $\sphericalangle(P_0,P) = \sphericalangle(\hat{P}_0,P) < \frac{1}{2}$, and, with Corollary 2.14, there exists some affine map $\bar{a} : P_0 \to P_0^\perp$ with graph $G(\bar{a}) = P$. Now we obtain with Lemma 2.13 ($P_1 = P$, $P_2 = P_0$), $u,v \in P_0$ and $\sphericalangle(P_0,P) < \frac{1}{2}$ that

$$d(v+\bar{a}(v), u+\bar{a}(u)) \le 2d(\pi_{P_0}(v+\bar{a}(v)), \pi_{P_0}(u+g(u))). \tag{7.3}$$

This yields for $u \in B(q,t) \cap P_0$ and some suitable $v \in P_0$ with $v + \bar{a}(v) = \pi_P(u+g(u))$:

$$\begin{aligned} d(g(u),\bar{a}(u)) &\le d(u+g(u),P) + d(\pi_P(u+g(u)), u+\bar{a}(u)) \\ &\overset{(7.3)}{\le} d(u+g(u),P) + 2d(\pi_{P_0}(v+\bar{a}(v)), \pi_{P_0}(u+g(u))) \\ &= 3d(u+g(u),P). \end{aligned}$$

Finally, using $\bar{a} \in \mathcal{A}$ and the last estimate, we get $\gamma_g(q,t) \overset{(7.1)}{\le} 3(\tilde{\gamma}_g(q,t) + \xi)$, and $0 < \xi < \alpha\omega_n$ was arbitrarily chosen. $\square$

7.1. **γ-functions and affine approximation of Lipschitz functions.** In this and the following subsections, we use the notation $U_l := B(0,l) \cap P_0$ for $l \in \{6, 8, 10\}$.

Theorem 7.3. *Let $1 < p < \infty$ and let $g : P_0 \to P_0^\perp$ be a Lipschitz continuous function with Lipschitz constant Lip_g and compact support. For all $\theta > 0$, there exist some set $H_\theta \subset U_6$ and some constants $C = C(n,p)$ and $\hat{C} = \hat{C}(n,N)$ with*

$$\mathcal{H}^n(U_6 \setminus H_\theta) \le \frac{C}{\theta^{p(n+1)} \mathrm{Lip}_g^p} \int_{U_{10}} \left(\int_0^2 \gamma_g(x,t)^2 \frac{\mathrm{d}t}{t} \right)^{\frac{p}{2}} \mathrm{d}\mathcal{H}^n(x)$$

[7] As the volume of the unit sphere is strictly monotonously decreasing when the dimension $n \ge 5$ increases, we get $\frac{\omega_{n-1}}{\omega_n} > 1$ for all $n \ge 6$. With the factor 4 we have that $4\frac{\omega_{n-1}}{\omega_n} > 1$ for all $n \in \mathbb{N}$.

so that, for all $y \in P_0$, there exists some affine map $a_y : P_0 \to P_0^\perp$ so that if $r \leq \theta$ and $B(y,r) \cap H_\theta \neq \emptyset$, we have

$$\|g - a_y\|_{L^\infty(B(y,r)\cap P_0, P_0^\perp)} \leq \hat{C} r \theta \operatorname{Lip}_g,$$

where $\|\cdot\|_{L^\infty(E)}$ denotes the essential supremum on $E \subset P_0$ with respect to the $\mathcal{H}^n$-measure.

To prove this theorem, we need the following lemma. If ν is some map, we use the notation $\nu_t(x) := \frac{1}{t^n}\nu\left(\frac{x}{t}\right)$.

Lemma 7.4. *There exists some radial function $\nu \in C_0^\infty(P_0, \mathbb{R})$ with*

(1) $\operatorname{supp}(\nu) \subset B(0,1) \cap P_0$ *and* $\widehat{\nu}(0) = 0$,

(2) *for all $x \in P_0 \setminus \{0\}$ and $i \in \{1,\dots,n\}$, we have*

$$\int_0^\infty |\widehat{\nu}(tx)|^2 \frac{\mathrm{d}t}{t} = 1 \qquad \text{and} \qquad 0 < \int_0^\infty |\widehat{(\partial_i \nu)_t}(x)|^2 \ \frac{\mathrm{d}t}{t} < \infty, \tag{7.4}$$

(3) *for all $i \in \{1,\dots,n\}$, the function $\partial_i \nu$ has mean value zero and, for all $a \in \mathcal{A}(P_0, P_0^\perp)$ (affine functions), the function $a\nu$ has mean value zero as well.*

Proof. Let $\nu_1 : P_0 \to \mathbb{R}$ be some nonharmonic ($\Delta\nu_1 \neq 0$), radial C^∞ function with support in $B(0,1)\cap P_0$. We set $\nu_2 := \Delta\nu_1 \in C^\infty(P_0) \cap C_0^\infty(B(0,1)\cap P_0)$ and $0 < c_1 := \int_0^\infty |\widehat{\nu_2}(te)|^2 \frac{\mathrm{d}t}{t}$, where e is some normed vector in P_0. Since ν_1 is radial, ν_2 is radial as well. We have $|\widehat{\nu_2}(te)| = 4\pi^2 t^2 |\widehat{\nu_1}(te)|$ and hence

$$0 < c_1 = \int_0^\infty |\widehat{\nu_2}(te)|^2 \frac{\mathrm{d}t}{t} = 16\pi^4 \int_0^\infty t^3 |\widehat{\nu_1}(te)|^2 \mathrm{d}t < \infty$$

because ν_1 is in the Schwarz space and therefore $\widehat{\nu_1}$ as well [11, 2.2.15, 2.2.11 (11)]. The previous equality also implies $\widehat{\nu_2}(0) = 0$. Now we set $\nu := \sqrt{\frac{1}{c_1}}\nu_2$, which is a radial $C_0^\infty(P_0,\mathbb{R})$ function that fulfils (1). We have for all $x \in P_0 \setminus \{0\}$ (use substitution with $t = r\frac{1}{|x|}$ and the fact that $\widehat{\nu}$ is radial), $\int_0^\infty |\widehat{\nu}(tx)|^2 \frac{\mathrm{d}t}{t} = \int_0^\infty |\widehat{\nu}(re)|^2 \frac{\mathrm{d}r}{r} = 1$. In a similar way, we deduce for $i \in \{1,\dots,n\}$ (using $|(\phi^{-1}(tx))^\kappa| \leq |\phi^{-1}(tx)| = |tx|$ where κ is some multi-index with $|\kappa| = 1$) that

$$\begin{aligned} \int_0^\infty |\widehat{(\partial_i \nu)_t}(x)|^2 \ \frac{\mathrm{d}t}{t} &\leq |2\pi i|^2 \int_0^\infty |tx|^2 \, |\widehat{\nu}(tx)|^2 \ \frac{\mathrm{d}t}{t} \\ &= 4\pi^2 \int_0^\infty r \left| \widehat{\nu}\left(r\frac{x}{|x|}\right)\right|^2 \ \mathrm{d}r < \infty, \end{aligned}$$

where we use that the Fourier transform of a Schwartz function is a Schwartz function as well [11, 2.2.15]. The left-hand side of the previous inequality cannot be zero, because this would imply that $\partial_i \nu(x) = 0$ for all $x \in P_0$, which is in contradiction to $0 \neq \nu \in C_0^\infty(P_0,\mathbb{R})$. Hence ν fulfils (2). Using partial integration and $\Delta a = 0$ for all $a \in \mathcal{A}(P_0, P_0^\perp)$ implies that $\partial_i \nu$ and $a\nu$ have mean value zero. □

For some function $f : P_0 \to P_0^\perp$ and $x \in P_0$, we define the convolution of ν_t and f by

$$(\nu_t * f)(x) := \int_{P_0} \nu_t(x-y) f(y) \mathrm{d}\mathcal{H}^n(y).$$

Lemma 7.5 (Calderón's identity)**.** *Let ν be the function given by Lemma* 7.4 *and let $u \in P_0 \setminus \{0\}$ and $f \in L^2(P_0, P_0^\perp)$ or let $f \in \mathscr{S}'(P_0)$ be a tempered distribution and $u \in \mathscr{S}(P_0)$ (Schwartz space) with $u(0) = 0$. Then we have*

$$f(u) = \int_0^\infty (\nu_t * \nu_t * f)(u) \frac{\mathrm{d}t}{t}. \tag{7.5}$$

Léger calls this identity "Calderón's formula" [19, p. 862, §5. Calderón's formula and the size of F_3]. Grafakos presents a similar version called "Calderón reproducing formula" [11, p. 371, Exercise 5.2.2].

Proof. At first, let $f \in L^2(P_0, P_0^\perp)$ and $u \in P_0 \setminus \{0\}$. We have $\widehat{(\nu_t)}(u) = \widehat{\nu}(tu)$ and, with Fubini's theorem, we obtain

$$\left(\int_0^\infty (\nu_t * \nu_t * f)(u) \frac{\mathrm{d}t}{t}\right)^{\widehat{}} = \int_0^\infty \widehat{(\nu_t)}(u)\widehat{(\nu_t)}(u)\widehat{f}(u) \frac{\mathrm{d}t}{t} \overset{(7.4)}{=} \widehat{f}(u).$$

The Fourier inversion holds on $L^2(P_0, P_0^\perp)$ [11, 2.2.4. The Fourier transform on $L^1 + L^2$], which gives the statement. We use the same idea to get this result for tempered distributions. ☐

Proof of Theorem 7.3. Let $g \in C_0^{0,1}(P_0, P_0^\perp)$ and let ν be the function given by Lemma 7.4. We define

$$\begin{aligned} g_1(u) &:= \int_2^\infty (\nu_t * \nu_t * g)(u) \frac{\mathrm{d}t}{t} + \int_0^2 (\nu_t * (\chi_{P_0 \setminus U_{10}} \cdot (\nu_t * g)))(u) \frac{\mathrm{d}t}{t}, \\ g_2(u) &:= \int_0^2 (\nu_t * (\chi_{U_{10}} \cdot (\nu_t * g)))(u) \frac{\mathrm{d}t}{t}, \end{aligned}$$

and the previous lemma implies that $g = g_1 + g_2$. We recall the notation $U_l = B(0, l) \cap P_0$ for $l \in \{6, 8, 10\}$ and continue the proof of Theorem 7.3 with several lemmas.

Lemma 7.6. *$g_1 \in C^\infty(U_8)$ and there exists some constant $C = C(\nu)$ so that for all multi-indices κ with $|\kappa| \le 2$ we have $\|\partial^\kappa g_1\|_{L^\infty(U_8, P_0^\perp)} \le C \operatorname{Lip}_g$.*

g_2 is Lipschitz continuous on U_8 with Lipschitz constant $C(\nu) \operatorname{Lip}_g$.

Proof. For $x \in P_0$ we set

$$g_{11}(x) := \int_2^\infty (\nu_t * \nu_t * g)(x) \frac{\mathrm{d}t}{t}, \quad g_{12}(x) := \int_0^2 (\nu_t * (\chi_{P_0 \setminus U_{10}} \cdot (\nu_t * g)))(x) \frac{\mathrm{d}t}{t}$$

so that $g_1 = g_{11} + g_{12}$ and we set $\varphi(x) := \int_2^\infty (\nu_t * \nu_t)(x) \frac{\mathrm{d}t}{t}$. At first, we show some intermediate results:

I. $g_{12}(x) = 0$ for all $x \in U_8$, due to the support of ν_t and $\chi_{P_0 \setminus U_{10}} \cdot (\nu_t * g)$.

II. For every multi-index κ, there exists some constant $C = C(n, \nu, \kappa)$ such that $|\partial^\kappa \varphi| \le C$, where $\partial^\kappa \varphi(y) := \int_2^\infty \partial^\kappa (\nu_t * \nu_t)(y) \frac{\mathrm{d}t}{t}$. This is given by $\partial^\kappa(\nu_t(y)) = \frac{1}{t^{|\kappa|}} (\partial^\kappa \nu)_t(y)$, and $|\partial^\kappa (\nu_t * \nu_t)(y)| \le \|\nu\|_{L^\infty(P_0, \mathbb{R})} \|\partial^\kappa \nu\|_{L^\infty(P_0, \mathbb{R})} \frac{\omega_n}{t^{n+|\kappa|}}$.

III. For every multi-index κ, the function $\partial^\kappa \varphi$ has bounded support in $B(0, 4) \cap P_0$.

Proof of I–III. Let $0 < t \le 2$ and $x \in P_0 \setminus B(0, 4)$. We have $(\nu_t * \nu_t)(x) = 0$, which implies that $\int_0^2 (\nu_t * \nu_t)(x) \frac{\mathrm{d}t}{t} = 0$. Now we consider φ as a tempered distribution.

The convolution with δ_0, the Dirac mass at the origin, is an identity; hence we get with Calderón's identity (Lemma 7.5) for all $\eta \in \mathscr{S}(P_0)$ with $\eta(0) = 0$:

$$\begin{aligned}\varphi(\eta) = \varphi(\eta) - \delta_0(\eta) &= \left(\int_2^\infty (\nu_t * \nu_t)\frac{\mathrm{d}t}{t}\right)(\eta) - \left(\int_0^\infty (\nu_t * \nu_t)\frac{\mathrm{d}t}{t}\right)(\eta)\\ &= -\left(\int_0^2 (\nu_t * \nu_t)\frac{\mathrm{d}t}{t}\right)(\eta).\end{aligned}$$

Since this holds for arbitrary $\eta \in \mathscr{S}(P_0)$ with $\operatorname{supp}(\eta) \subset P_0 \setminus B(0,4)$, we conclude that for such η we have

$$\int_{P_0} \varphi(x)\eta(x)\mathrm{d}\mathcal{H}^n(x) = -\int_{P_0}\int_0^2 (\nu_t * \nu_t)(x)\frac{\mathrm{d}t}{t}\eta(x)\mathrm{d}\mathcal{H}^n(x) = 0$$

and hence $\operatorname{supp}(\varphi) \subset B(0,4) \cap P_0$. For the same kind of η, we get, using Fubini's theorem and partial integration,

$$\int_{P_0} \partial^\kappa \varphi(x)\eta(x)\mathrm{d}\mathcal{H}^n(x) = (-1)^{|\kappa|}\int_2^\infty \int_{P_0} (\nu_t * \nu_t)(x)\partial^\kappa\eta(x)\mathrm{d}\mathcal{H}^n(x)\frac{\mathrm{d}t}{t} = 0$$

since $\partial^\kappa \eta \in \mathscr{S}(P_0)$ with $\operatorname{supp}(\partial^\kappa\eta) \subset P_0 \setminus B(0,4)$. □

IV. $\varphi \in C_0^\infty(P_0)$.

Proof of IV. With II and III we conclude for every multi-index κ that $\partial^\kappa\varphi \in L^1(P_0, \mathbb{R})$. With Fubini's theorem and partial integration, we see that $\partial^\kappa\varphi$ is the weak derivative of φ; hence we have $\varphi \in W^{l,1}(P_0)$ for every $l \in \mathbb{N}$. The Sobolev imbedding theorem [1, Thm. 4.12] gives us $\varphi \in C^\infty(P_0)$ and, with III, we obtain $\varphi \in C_0^\infty(P_0)$. □

Now we have for all $x \in U_8$ with Fubini's theorem [7, 1.4, Thm. 1] $g_{11}(x) = (\varphi * g)(x)$. We know that $\varphi \in C_0^\infty(P_0)$ and $g \in C_0^{0,1}(P_0, P_0^\perp)$. Hence we have $g_{11} \in C_0^\infty(P_0)$, $g \in W^{1,\infty}(P_0)$ and for $i, j \in \{1, \dots, n\}$ we have $\partial_i g_{11} = \varphi * \partial_i g$ and $\partial_i\partial_j g_{11} = \partial_i\varphi * \partial_j g$. With the Minkowski inequality [11, Thm. 1.2.10] and IV, we obtain for $i, j \in \{1, \dots, n\}$:

$$\|\partial_i g_1\|_{L^\infty(U_8,\mathbb{R})} \stackrel{\mathrm{I}}{=} \|\partial_i g_{11}\|_{L^\infty(U_8,\mathbb{R})} \le \|\partial_i g\|_{L^\infty(U_8,\mathbb{R})}\|\varphi\|_{L^1(P_0)} \le C(\nu)\operatorname{Lip}_g,$$

$$\|\partial_i\partial_j g_1\|_{L^\infty(U_8,\mathbb{R})} \stackrel{\mathrm{I}}{=} \|\partial_i\partial_j g_{11}\|_{L^\infty(U_8,\mathbb{R})} \le \|\partial_i g\|_{L^\infty(U_8,\mathbb{R})}\|\partial_j\varphi\|_{L^1(P_0)} \le C(\nu)\operatorname{Lip}_g.$$

Now it is easy to see that g_2 is $C\operatorname{Lip}_g$-Lipschitz on U_8 because we have $g_2 = g - g_1$ and g as well as g_1 are $C\operatorname{Lip}_g$-Lipschitz on U_8. □

Remark 7.7. Under the assumption that

$$\int_{U_{10}}\left(\int_0^2 \gamma_g(x,t)^2\frac{\mathrm{d}t}{t}\right)^{\frac{p}{2}}\mathrm{d}\mathcal{H}^n(x) < \infty, \tag{7.6}$$

the next lemmas will prove that $g_2 \in W_0^{1,p}(P_0, P_0^\perp)$. We show for this purpose in Lemma 7.10 that $\partial_i g_2(x) := \int_0^2 \partial_i(\nu_t * (\chi_{U_{10}}(\nu_t * g)))(x)\frac{\mathrm{d}t}{t}$ is in $L^p(P_0, P_0^\perp)$. Using Fubini's theorem [7, 1.4, Thm. 1] and partial integration it turns out that $\partial_i g_2$ fulfils the condition of a weak derivative.

Lemma 7.8. *We have* $\operatorname{supp}(g_2) \subset B(0,12) \cap P_0$ *and* $\operatorname{supp}(\partial_i g_2) \subset B(0,12) \cap P_0$ *for all* $1 \le i \le n$.

Proof. If $0 < t < 2$ and $x \in P_0$, we have $\operatorname{supp}(\nu_t(x - \cdot)) \subset B(x, 2) \cap P_0$ and $\operatorname{supp}(\chi_{U_{10}}(\nu_t * g)) \subset B(0, 10) \cap P_0$. This implies $\operatorname{supp}(\nu_t * (\chi_{U_{10}}(\nu_t * g))) \subset B(0, 12) \cap P_0$, and hence we obtain $\operatorname{supp}(g_2) \subset B(0, 12)$ and $\operatorname{supp}(\partial_i g_2) \subset B(0, 12) \cap P_0$. □

Lemma 7.9. *Let $x \in U_{10}$ and $0 < t < 2$. We have* $\left|\frac{(\nu_t * g)(x)}{t}\right| \le \|\nu\|_{L^\infty(P_0,\mathbb{R})} \gamma_g(x, t)$.

Proof. If $a : P_0 \to P_0^\perp$ is an affine function, we get using Lemma 7.4(3) that $(\nu_t * a)(x) = 0$ and hence, with Lemma 7.4(1),

$$\begin{aligned}\left|\frac{(\nu_t * g)(x)}{t}\right| &= \left|\frac{(\nu_t * (g - a))(x)}{t}\right| \\ &\le \|\nu\|_{L^\infty(P_0,\mathbb{R})} \frac{1}{t^n} \int_{P_0 \cap B(x,t)} \left|\frac{g(y) - a(y)}{t}\right| \mathrm{d}\mathcal{H}^n(y).\end{aligned}$$

Since a was an arbitrary affine function, this implies the assertion. □

We have $p \in (1, \infty)$ and, for the proof of Theorem 7.3, we can assume (7.6).

Lemma 7.10. *We have $g_2 \in W_0^{1,p}(P_0, P_0^\perp)$ and there exists some constant $C = C(n, p, \nu)$ so that for all $i \in \{1, \dots, n\}$,*

$$\|\partial_i g_2\|^p_{L^p(P_0,P_0^\perp)} \le C \int_{U_{10}} \left(\int_0^2 \gamma_g(x, t)^2 \frac{\mathrm{d}t}{t}\right)^{\frac{p}{2}} \mathrm{d}\mathcal{H}^n(x),$$

where $\partial_i g_2(x) = \int_0^2 \partial_i(\nu_t * (\chi_{U_{10}}(\nu_t * g)))(x) \frac{\mathrm{d}t}{t}$.

Proof. We recall that $\partial_i g_2$ is the weak derivative of g_2 (cf. Remark 7.7). Due to [1, Cor. 6.31, An equivalent norm for $W_0^{m,p}(\Omega)$] and Lemma 7.8, we only have to consider $\|\partial_i g_2\|_{L^p(P_0)}$ for all $i \in \{0, \dots, n\}$ to get $g_2 \in W_0^{1,p}(P_0, P_0^\perp)$. For $x \in P_0$, we have $\partial_i \nu_t(x) = \partial_i t^{-n} \nu\left(\frac{x}{t}\right) = t^{-1}(\partial_i \nu)_t(x)$ and hence

$$\partial_i g_2(x) = \int_0^2 \partial_i(\nu_t * (\chi_{U_{10}}(\nu_t * g)))(x) \frac{\mathrm{d}t}{t} = \int_0^2 \left((\partial_i \nu)_t * \left(\chi_{U_{10}}\left(\frac{\nu_t * g}{t}\right)\right)\right)(x) \frac{\mathrm{d}t}{t}.$$

Using duality (cf. [1, The normed dual of $L^p(\Omega)$]) it suffices to consider the following. Let $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in L^{p'}(P_0)$ with $\|f\|_{L^{p'}(P_0)} = 1$. We get with Fubini's theorem [7, 1.4, Thm. 1] and Hölder's inequality

$$\begin{aligned}&\left|\int_{P_0} f(x)\ \partial_i g_2(x)\ \mathrm{d}\mathcal{H}^n(x)\right| \\ &\le \int_{P_0} \int_0^2 |((\partial_i \nu)_t * f)(y)|\ \left|\left(\chi_{U_{10}}\left(\frac{\nu_t * g}{t}\right)\right)(y)\right|\ \frac{\mathrm{d}t}{t} \mathrm{d}\mathcal{H}^n(y) \\ &\le \int_{P_0} \left(\int_0^2 |((\partial_i \nu)_t * f)(y)|^2\ \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \left(\int_0^2 \left|\left(\chi_{U_{10}}\left(\frac{\nu_t * g}{t}\right)\right)(y)\right|^2\ \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \mathrm{d}\mathcal{H}^n(y) \\ &\le \left\|\left(\int_0^\infty |(\partial_i \nu)_t * f|^2\ \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}\right\|_{L^{p'}(P_0)} \\ &\quad \times \left(\int_{P_0} \left(\int_0^2 \left|\left(\chi_{U_{10}}\left(\frac{\nu_t * g}{t}\right)\right)(y)\right|^2\ \frac{\mathrm{d}t}{t}\right)^{\frac{p}{2}} \mathrm{d}\mathcal{H}^n(y)\right)^{\frac{1}{p}}.\end{aligned}$$

There exists some constant $C = C(n, \nu)$ with $|\partial_i \nu(x)| + |\nabla \partial_i \nu(x)| \le C(1 + |x|)^{-n-1}$ because ν is a Schwartz function. Together with Lemma 7.4, all the requirements of

Lemma A.8 with $\phi = \partial_i \nu$ and $q = p'$ are fulfilled, which implies, since $\|f\|_{L^p(P_0)} = 1$, that the first factor of the RHS of the last estimate is some constant $C(n,p,\nu)$ independent of f. All in all, we obtain

$$\|\partial_i g_2\|_{L^p(P_0)} \leq C(n,p,\nu) \left(\int_{P_0} \left(\int_0^2 \left| \left(\chi_{U_{10}} \left(\frac{\nu_t * g}{t} \right) \right)(y) \right|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{p}{2}} \mathrm{d}\mathcal{H}^n(y) \right)^{\frac{1}{p}},$$

and with Lemma 7.9 the assertion holds. $\square$

Definition 7.11. Let B be a ball with centre in P_0 and let $f : P_0 \to P_0^\perp$ be some map. We define the average of f on B and some maximal function for $x \in P_0$:

$$\operatorname*{Avg}_B(f) := \frac{1}{(\operatorname{diam} B)^n} \int_{B \cap P_0} f \mathrm{d}\mathcal{H}^n,$$

$$N(f)(x) := \sup_{\substack{t \in (0,\infty), y \in P_0 \\ \text{with } d(y,x) \leq t}} \left\{ \frac{1}{2t} \operatorname*{Avg}_{B(y,t)} \left(\left| f - \operatorname*{Avg}_{B(y,t)}(f) \right| \right) \right\}.$$

Moreover we define the oscillation of f on B by

$$\operatorname{osc}_B(f) := \sup_{x \in B \cap P_0} |f(x) - \operatorname*{Avg}_B(f)|.$$

Lemma 7.12. *We have* $\|N(g_2)\|_{L^p(P_0,\mathbb{R})} \leq C\|Dg_2\|_{L^p(P_0,P_0^\perp)}$, *where* $C = C(n,p)$.

Proof. We recall that $g_2 \in W_0^{1,p}(P_0, P_0^\perp)$ (cf. Lemma 7.9) and conclude with Poincaré's inequality that $\operatorname{Avg}_B(|g_2 - \operatorname{Avg}_B(g_2)|) = C(n) \operatorname{diam} B \ \operatorname{Avg}_B(|Dg_2|)$ (if f is a matrix, we denote by $|f|$ a matrix norm), and hence we get for $x \in P_0$,

$$N(g_2)(x) \leq C(n) \sup_{\substack{t \in (0,\infty), y \in P_0 \\ \text{with } d(y,x) \leq t}} \operatorname*{Avg}_{B(y,t)}(|Dg_2|) = C(n) M(Dg_2)(x),$$

where $M(Dg_2)$ is the uncentred Hardy-Littlewood maximal function. Now, using [11, Thm. 2.1.6], we get the assertion. $\square$

Definition 7.13. Let $\theta > 0$. We define $H_\theta := \left\{ x \in U_6 | N(g_2)(x) \leq \theta^{n+1} \operatorname{Lip}_g \right\}$.

Lemma 7.14. *Let* $\theta > 0$. *There exists some constant* $C = C(n,p,\nu)$ *so that*

$$\mathcal{H}^n(U_6 \setminus H_\theta) \leq \frac{C}{\theta^{p(n+1)} \operatorname{Lip}_g^p} \int_{U_{10}} \left(\int_0^2 \gamma_g(x,t)^2 \frac{\mathrm{d}t}{t} \right)^{\frac{p}{2}} \mathrm{d}\mathcal{H}^n(x).$$

Proof. With Lemma 7.12, Lemma 7.10 and

$$\|Dg_2\|^p_{L^p(P_0,P_0^\perp)} \leq n^{p-1} \sum_{i=1}^n \|\partial_i g_2\|^p_{L^p(P_0,P_0^\perp)},$$

there exists some constant $C = C(n,p,\nu)$ with

$$\begin{aligned} \|N(g_2)\|^p_{L^p(P_0,P_0^\perp)} &\leq C sum_{i=1}^n \|\partial_i g_2\|^p_{L^p(P_0,P_0^\perp)} \\ &\leq C \int_{U_{10}} \left(\int_0^2 \gamma_g(x,t)^2 \frac{\mathrm{d}t}{t} \right)^{\frac{p}{2}} \mathrm{d}\mathcal{H}^n(x). \end{aligned}$$

Hence, using Chebyshev's inequality, we get the assertion. $\square$

Lemma 7.15. *Let B be a ball with centre in P_0. If $(B \cap P_0) \subset U_8$, then there exists some constant $C = C(N, n, \nu)$ with*

$$\operatorname{osc}_B(g_2) \le C \operatorname{diam} B \left(\frac{1}{\operatorname{diam} B} \operatorname*{Avg}_B \Big(|g_2 - \operatorname*{Avg}_B(g_2)| \Big) \right)^{\frac{1}{n+1}} \operatorname{Lip}_g^{\frac{n}{n+1}} .$$

Proof. Let $(B \cap P_0) \subset U_8$ and $\lambda := \operatorname{osc}_B(g_2)$. The function g_2 is Lipschitz continuous on U_8 with $\operatorname{Lip}_{g_2} = C(\nu) \operatorname{Lip}_g$ (see Lemma 7.6) and $B \cap P_0$ is closed. Hence there exists some $y \in B \cap P_0$ with $\lambda = |g_2(y) - \operatorname{Avg}_B g_2|$, and we get for $x \in B$ with $d(x, y) \le \frac{\lambda}{2 \operatorname{Lip}_{g_2}}$ using triangle inequality $|g_2(x) - \operatorname*{Avg}_B(g_2)| \ge \frac{\lambda}{2}$. Furthermore, using that g_2 is continuous on U_8 for all $l \in \{1, \dots, N\}$, there exists some $z_l \in B \cap P_0$, with $g_2^l(z_l) = \operatorname*{Avg}_B(g_2^l)$ (where $g_2^l(z_l) \in \mathbb{R}$ means the l-th component of $g_2(z_l) \in \mathbb{R}^N$). With $g_2^l(y) - \operatorname*{Avg}_B(g_2^l) \le \operatorname{Lip}_{g_2} d(y, z_l)$ for all $l \in \{1, \dots, N\}$ we get $\lambda^2 \le N \left(\operatorname{Lip}_{g_2} \operatorname{diam} B \right)^2$, which implies $\frac{\lambda}{\sqrt{N} \operatorname{Lip}_{g_2}} \le \operatorname{diam} B$. Since $y \in B$, there exists some ball $\hat{B} \subset B \cap B\left(y, \frac{\lambda}{2 \operatorname{Lip}_{g_2}} \right)$ with $\operatorname{diam} \hat{B} \ge \frac{\lambda}{2\sqrt{N} \operatorname{Lip}_{g_2}}$, and hence with $|g_2(x) - \operatorname{Avg}_B(g_2)| \ge \frac{\lambda}{2}$ we obtain $(\operatorname{diam} B)^n \operatorname*{Avg}_B |g_2(x) - \operatorname*{Avg}_B(g_2)| \ge \omega_n \big(\frac{\lambda}{4\sqrt{N} \operatorname{Lip}_{g_2}} \big)^n \frac{\lambda}{2}$. Using $\operatorname{Lip}_{g_2} = C(\nu) \operatorname{Lip}_g$, this implies the assertion. ☐

Lemma 7.16. *Let $\theta > 0$ and $y \in P_0$. There exists some constant $C = C(N, n, \nu)$ and some affine map $a_y : P_0 \to P_0^\perp$ so that if $r \le \theta$ and $B(y, r) \cap H_\theta \neq \emptyset$, we have $\|g - a_y\|_{L^\infty(B(y,r) \cap P_0, P_0^\perp)} \le C r \theta \operatorname{Lip}_g$.*

Proof. Let $y \in P_0$. If $\theta \ge 1$, we can choose $a_y(y') := g(y)$ as a constant and get the desired result directly from the Lipschitz condition. Now let $0 < \theta < 1$ and $y' \in B(y, r) \cap P_0$. We set $a_y(y') := g(y) + Dg_1(y) \phi^{-1}(y' - y)$. We have $d(y', U_6) \le d(y', H_\theta) \le d(y', y) + d(y, H_\theta) \le 2$. So we get $y', y \in U_8$. Using Taylor's theorem and Lemma 7.6 we obtain

$$\begin{aligned} |g_1(y') - [g_1(y) + Dg_1(y)\phi^{-1}(y' - y)]| &\le \sum\nolimits_{|\kappa|=2} \|\partial^\kappa g_1\|_{L^\infty(U_8)} |y' - y|^2 \\ &\le C(n, \nu) \operatorname{Lip}_g r^2. \end{aligned}$$

Since $r \le \theta < 1$, $B(y, r) \cap H_\theta \neq \emptyset$ and $H_\theta \subset U_6$, we obtain $B(y, r) \cap P_0 \subset U_8$, and we can apply Lemma 7.15. Together with the definition of H_θ this leads to $\operatorname{osc}_{B(y,r)} g_2 + \operatorname{Lip}_g r^2 \le C(N, n, \nu) r \theta \operatorname{Lip}_g$. Now by using $g = g_1 + g_2$ and $|g_2(y') - g_2(y)| \le 2 \operatorname{osc}_{B(y,r)} g_2$ we get for every $y' \in B(y, r) \cap P_0$ that

$$|g(y') - [g(y) + Dg_1(y)\phi^{-1}(y' - y)]| \le C(N, n, \nu) r \theta \operatorname{Lip}_g .$$

☐

Lemma 7.14 and Lemma 7.16 complete the proof of Theorem 7.3. ☐

7.2. **The γ-function of A and integral Menger curvature.** In this section, we prove the following Theorem 7.17. It states that we get a similar control on the γ-functions applied to our function A as we get in Corollary 4.8 on the β-numbers.

For $\alpha, \varepsilon > 0$, $\eta \le 2\varepsilon$ and $k \ge 4$, we defined A on U_{12} (cf. Definition 6.20). Since in this section we only apply the γ-functions to A, we set $\gamma(q, t) := \gamma_A(q, t)$ and we recall the notation $U_{10} := B(0, 10) \cap P_0$.

Theorem 7.17. *There exist some $\tilde{k} \geq 4$ and some $\tilde{\alpha} = \tilde{\alpha}(n) > 0$ so that, for all α with $0 < \alpha \leq \tilde{\alpha}$, there exists some $\tilde{\varepsilon} = \tilde{\varepsilon}(N, n, C_0, \alpha)$ so that, if $k \geq \tilde{k}$ and $\eta \leq \tilde{\varepsilon}^p$, we have for all $\varepsilon \in [\eta^{\frac{1}{p}}, \tilde{\varepsilon}]$ that there exists some constant $C = C(N, n, \mathcal{K}, p, C_0, k)$ so that*

$$\int_{U_{10}} \int_0^2 \gamma(q,t)^p \frac{\mathrm{d}t}{t} \mathrm{d}\mathcal{H}^n(q) \leq C\varepsilon^p + C\mathcal{M}_{\mathcal{K}^p}(\mu) \leq C\varepsilon^p.$$

Proof. Let $\bar{k} \geq 4$ be the maximum of all thresholds for k given in section 6 and let $\tilde{\alpha} = \tilde{\alpha}(n) \leq \frac{1}{4}$ be the upper bound for the Lipschitz constant given by Lemma 7.2. We set $\tilde{k} := \max\{\bar{k}, \tilde{C}+1, \hat{C}\}$ where the constants $\tilde{C}$ and $\hat{C}$ are fixed constants which will be set during this section.[8] Let $0 \leq \alpha \leq \tilde{\alpha}$. Let $\bar{\varepsilon} = \varepsilon(N, n, C_0, \alpha) \leq \alpha$ be the minimum of all thresholds for ε given in section 6. We set $\tilde{\varepsilon} := \min\{\bar{\varepsilon}, (2C'C_1)^{-1}\} < 1$,[9] and assume that $k \geq \tilde{k}$ and $\eta \leq \tilde{\varepsilon}^p$. Now let $\varepsilon > 0$ with $\eta \leq \varepsilon^p \leq \tilde{\varepsilon}^p$. For the rest of this section, we fix the parameters $k, \eta, \alpha, \varepsilon$ and mention that they meet all requirements of the lemmas in section 6.

We start the proof of Theorem 7.17 with several lemmas. First, we prove

Lemma 7.18. *There exists some constant $C = C(N, n, p, C_0)$ so that*

$$\sum_{i \in I_{12}} \int_{R_i \cap U_{10}} \int_0^{\frac{\operatorname{diam} R_i}{2}} \gamma(q,t)^p \frac{\mathrm{d}t}{t} \mathrm{d}\mathcal{H}^n(q) \leq C\varepsilon^p.$$

Proof. Let $i \in I_{12}$, $q \in R_i$, $0 < t < \frac{\operatorname{diam} R_i}{2}$ and $u \in B(q,t) \cap P_0 \subset 2R_i$. The function A is in $C^\infty(2R_i, P_0^\perp)$ (see the definition of A on page 1215). Taylor's theorem implies $\inf_{a \in \mathcal{A}} d(A(u), a(u)) \leq t^2 \frac{C(N,n,C_0)\varepsilon}{\operatorname{diam} R_i}$ since the infimum over all affine functions cancels out the linear part and the second order derivatives of the remainder can be estimated using Lemma 6.28. Now we have

$$\gamma(q,t) \leq \frac{\omega_n}{t} \sup_{u \in B(q,t) \cap P_0} \inf_{a \in \mathcal{A}} d(A(u), a(u)) \leq t \frac{C(N,n,C_0)\varepsilon}{\operatorname{diam} R_i}.$$

Hence, Lemma 6.13(ii) implies the statement. $\square$

The previous lemma implies that, due to Lemma 6.13(ii), it remains to handle the two terms in the following sum to prove Theorem 7.17. If $q_1 \in R_i$, we get with Lemma 6.13 that $\frac{D(q_1)}{100} \leq \frac{\operatorname{diam} R_i}{2}$ and if $q_2 \in \pi(\mathcal{Z})$, we obtain with Lemma 6.10 $D(q_2) = 0$. Hence we conclude using Lemma 6.13(ii) that

$$\begin{aligned} &\sum_{i \in I_{12}} \int_{R_i \cap U_{10}} \int_{\frac{\operatorname{diam} R_i}{2}}^2 \gamma(q,t)^p \frac{\mathrm{d}t}{t} \mathrm{d}\mathcal{H}^n(q) + \int_{\pi(\mathcal{Z}) \cap U_{10}} \int_0^2 \gamma(q,t)^p \frac{\mathrm{d}t}{t} \mathrm{d}\mathcal{H}^n(q) \\ &= \int_{U_{10}} \int_{\frac{D(q)}{100}}^2 \gamma(q,t)^p \frac{\mathrm{d}t}{t} \mathrm{d}\mathcal{H}^n(q). \end{aligned} \tag{7.7}$$

In the following, we prove some estimate for $\gamma(q,t)$ where $q \in U_{10}$ and $\frac{D(q)}{100} < t < 2$. To get this estimate, we need the next lemma.

[8] $\tilde{C}$ is given in Lemma 7.20; $\hat{C}$ is given in Lemma 7.24.V.

[9] C', C_1 are given in Lemma 7.23.

Lemma 7.19. *For all $q \in U_{10}$ and for all t with $\frac{D(q)}{100} < t < 2$, there exist some $\tilde{X} = \tilde{X}(q) \in F$ and some $T = T(t) > 0$ with*

$$(\tilde{X}, T) \in S, \qquad d(\pi(\tilde{X}), q) \leq T \qquad \text{and} \qquad 20t \leq T \leq 200t. \tag{7.8}$$

Proof. We have $D(q) = \inf_{(X,s)\in S}(d(\pi(X), q) + s)$, and hence there exists some $(\tilde{X}, \tilde{s}) \in S$ with $d(\pi(\tilde{X}), q) + \tilde{s} \leq D(q) + 100t \leq 200t$. We set $T := \min\{40, 200t\}$ which fulfils $20t \leq T \leq 200t$ as $t < 2$. Using Lemma 6.2(i), (ii) and $200t \geq \tilde{s}$, we obtain $(\tilde{X}, T) \in S$.

With $d(\pi(\tilde{X}), q) \leq d(\pi(\tilde{X}), 0) + d(0, q) \leq 5 + 10$ we get $d(\pi(\tilde{X}), q) \leq T$. □

Now let $q, t, \tilde{X}$ and T be as in Lemma 7.19. Furthermore, let $X \in B(\tilde{X}, 200t) \cap F$. We choose some n-dimensional plane called $\hat{P} = \hat{P}(q, t, X)$ with

$$\beta_{1;k}^{\hat{P}}(X, t) \leq 2\beta_{1;k}(X, t) \tag{7.9}$$

and define

$$\mathcal{I}(q, t) := \left\{ i \in I_{12} \middle| R_i \cap B(q, t) \neq \emptyset \right\}.$$

With Lemma 6.13, we have $(B(q, t) \cap P_0) \subset U_{12} \subset \pi(\mathcal{Z}) \cup \bigcup_{i\in I_{12}} R_i$. We set

$$K_0 := \int_{B(q,t)\cap\pi(\mathcal{Z})} \frac{d(u + A(u), \hat{P})}{t^{n+1}} \mathrm{d}\mathcal{H}^n(u),$$

$$K_i := \int_{B(q,t)\cap R_i} \frac{d(u + A(u), \hat{P})}{t^{n+1}} \mathrm{d}\mathcal{H}^n(u)$$

and get with Lemma 7.2 that

$$\gamma(q, t) \leq 3\ K_0 + 3 \sum_{i\in\mathcal{I}(q,t)} K_i. \tag{7.10}$$

First, we consider K_0.

Lemma 7.20. *There exists some constant $\tilde{C} > 1$ so that*

$$\int_{B(q,t)\cap\pi(\mathcal{Z})} d(u + A(u), \hat{P}) \mathrm{d}\mathcal{H}^n(u) \leq \int_{B(X,\tilde{C}t)\cap\mathcal{Z}} d(x, \hat{P}) \mathrm{d}\mathcal{H}^n(x).$$

Proof. Let $g : \pi(\mathcal{Z}) \to \mathcal{Z}, u \mapsto u + A(u)$. This function is bijective, continuous (A is 2α-Lipschitz on $\pi(Z)$) and $g^{-1} = \pi|_{\mathcal{Z}}$ is Lipschitz continuous with Lipschitz constant 1. With $f(x) = d(x, \hat{P})$ and $s = n$, we apply [27, Lem. A.1] and get

$$\int_{B(q,t)\cap\pi(\mathcal{Z})} d(u + A(u), \hat{P}) \mathrm{d}\mathcal{H}^n(u) \leq \int_{g(B(q,t)\cap\pi(\mathcal{Z}))} d(x, \hat{P}) \mathrm{d}\mathcal{H}^n(x).$$

Now it remains to show that there exists some constant C so that $g(B(q, t)\cap\pi(\mathcal{Z})) \subset B(X, Ct) \cap \mathcal{Z}$. Let $x \in g(B(q, t) \cap \pi(\mathcal{Z}))$. This implies $x \in \mathcal{Z}$ and so, using Lemma 6.10, we get $d(x) = 0$. With (7.8), we conclude that $d(\tilde{X}) \leq d(\tilde{X}, \tilde{X}) + T \leq 200t$, and we obtain with (7.8) $d(\pi(x), \pi(\tilde{X})) \leq 201t$. So, with Lemma 6.11, we have $d(x, \tilde{X}) \leq 1602t$. We deduce with $\tilde{C} = 1802$ that $d(x, X) \leq d(x, \tilde{X}) + d(\tilde{X}, X) \leq \tilde{C}t$ and so $g(B(q, t) \cap \pi(\mathcal{Z})) \subset B(X, \tilde{C}t) \cap \mathcal{Z}$. □

Lemma 7.21. *There exists some constant $C = C(N, n, C_0) > 1$ so that*

$$\int_{B(X,\tilde{C}t)\cap\mathcal{Z}} d(x, \hat{P}) \mathrm{d}\mathcal{H}^n(x) \leq C \int_{B(X,(\tilde{C}+1)t)} d(x, \hat{P}) \mathrm{d}\mu(x).$$

Proof. First, we prove for an arbitrary ball B with centre in $\mathcal{Z}$,

$$\mathcal{H}^n(\mathcal{Z}\cap B)\le C(N,n,C_0)\mu(B). \tag{7.11}$$

With [7, Dfn. 2.1], we get $\mathcal{H}^n(\mathcal{Z}\cap B)=\lim_{\tau\to 0}\mathcal{H}^n_\tau(\mathcal{Z}\cap B)$. Let $0<\tau_0<\min\left\{\frac{\operatorname{diam} B}{2},50\right\}$. We define $\mathcal{F}:=\{B(x,s)|x\in\mathcal{Z}\cap B, s\le\tau_0\}$. With Besicovitch's covering theorem [7, 1.5.2, Thm. 2], there exist $N_0=N_0(N)$ countable families $\mathcal{F}_j\subset\mathcal{F}$, $j=1,\dots,N_0$, of disjoint balls where the union of all those balls covers $\mathcal{Z}\cap B$. For every ball $\tilde{B}=B(x,s)\in\mathcal{F}_j$, we have $x\in\mathcal{Z}$ and hence, using the definition of $\mathcal{Z}$ (see page 1209), we deduce that $h(x)=0$. With $h(x)=0<s<50$ and Lemma 6.2(i), we get $(x,s)\in S\subset S_{total}$ and so $\left(\frac{\operatorname{diam}\tilde{B}}{2}\right)^n\le 2\frac{\mu(\tilde{B})}{\delta}$. The centre of B is also in $\mathcal{Z}$, and hence, analogously, we conclude that $\left(\frac{\operatorname{diam} B}{2}\right)^n\le 2\frac{\mu(B)}{\delta}$. With (B) from page 1208, we get $\mu(2B)\le 4^nC_0\frac{2}{\delta}\mu(B)$. Since $x\in B$ and $s\le\tau_0<\frac{\operatorname{diam} B}{2}$, we obtain $\tilde{B}=B(x,s)\subset 2B$. Now, by definition of $\mathcal{H}^n_{\tau_0}$ [7, Dfn. 2.1] and because $\delta=\delta(N,n)$ (see (6.1)), we deduce that

$$\mathcal{H}^n_{\tau_0}(\mathcal{Z}\cap B)\le 2\sum_{j=1}^{N_0}\sum_{\tilde{B}\in\mathcal{F}_j}\omega_n\frac{\mu(\tilde{B})}{\delta}\le 2\frac{\omega_n}{\delta}\sum_{j=1}^{N_0}\mu(2B)\le C(N,n,C_0)\mu(B).$$

So, with $\tau_0\to 0$, the inequality (7.11) is proven.

Let $\tilde{C}$ be the constant from Lemma 7.20. For an arbitrary $0<\sigma\le t$, we define

$$\mathcal{G}_\sigma:=\left\{B(x,s)\middle| x\in\mathcal{Z}\cap B(X,\tilde{C}t), s\le\sigma\right\}.$$

With Besicovitch's covering theorem [7, 1.5.2, Thm. 2], there exist $N_0=N_0(N)$ families $\mathcal{G}_{\sigma,j}\subset\mathcal{G}_\sigma$ of disjoint balls, where $j=1,\dots,N_0$, and those balls cover $\mathcal{Z}\cap B(X,\tilde{C}t)$. We denote by p_B the centre of the ball B and conclude that

$$\begin{aligned}\int_{\mathcal{Z}\cap B(X,\tilde{C}t)} d(x,\hat{P})\mathrm{d}\mathcal{H}^n(x) &\le \sum_{j=1}^{N_0}\sum_{B\in\mathcal{G}_{\sigma,j}}\int_{\mathcal{Z}\cap B}\sigma+d(p_B,\hat{P})\mathrm{d}\mathcal{H}^n(x)\\ &\overset{(7.11)}{\le} C(N,n,C_0)\sum_{j=1}^{N_0}\sum_{B\in\mathcal{G}_{\sigma,j}}\int_B\left(\sigma+d(p_B,\hat{P})\right)\mathrm{d}\mu(x)\\ &\le C(N,n,C_0)\left(\mu(B(X,(\tilde{C}+1)t))2\sigma+\int_{B(X,(\tilde{C}+1)t)}d(x,\hat{P})\mathrm{d}\mu(x)\right).\end{aligned}$$

With $\sigma\to 0$, the assertion holds. □

With Lemma 7.20 and Lemma 7.21, we get for K_0 using that $k\ge\tilde{k}\ge\tilde{C}+1$, where $\tilde{k}$ is defined on page 1227,

$$K_0 \le C(N,n,C_0)\,\beta^{\hat{P}}_{1;k}(X,t)\overset{(7.9)}{\le} C(N,n,C_0)\,\beta_{1;k}(X,t). \tag{7.12}$$

To estimate K_i, we need the following lemma.

Lemma 7.22. *There exists some constant $C_4=C_4(N,n,C_0)>1$ so that, for all $i\in I_{12}$ and $u\in R_i$, we have $d(\pi_{P_i}(u+A(u)),B_i)\le C_4\operatorname{diam} R_i$. We recall that P_i is the n-dimensional plane, which is, in the sense of Definition* 6.1*, associated to the ball $B(X_i,t_i)=B_i$ given by Lemma* 6.14 *(cf. Definition* 6.17*).*

Proof. For every $i \in I_{12} \subset I$, we have with Lemma 6.14 that $B_i = B(X_i, t_i)$ and $(X_i, t_i) \in S \subset S_{total}$. Hence we can use Lemma 4.10 ($\sigma = 2\varepsilon$, $x = X_i$, $t = t_i$, $\lambda = \frac{\delta}{2}$, $P = P_i$) to get some $y \in 2B_i \cap P_i$, where $P_i = P_{(X_i,t_i)}$. We obtain with Lemma 2.13 ($P_1 = P_j$, $P_2 = P_0$), $\alpha \le \tilde{\alpha} < \frac{1}{2}$ ($\tilde{\alpha}$ is defined on page 1227) and Lemma 6.14:

$$d(u + A_i(u), y) \le \frac{1}{1-\alpha} d(u, \pi(y)) < 2[d(u, \pi(X_i)) + d(\pi(X_i), \pi(y))] \le C \operatorname{diam} R_i.$$

Moreover, with Lemma 6.21(iv) and $\varepsilon \le \tilde{\varepsilon} \le 1$ ($\tilde{\varepsilon}$ is defined on page 1227), we get

$$d(\pi_{P_i}(u + A(u)), u + A_i(u)) \le d(u + A(u), u + A_i(u)) \le C \operatorname{diam} R_i$$

for some $C = C(N, n, C_0)$. Using these estimates, $u + A_i(u) = \pi_{P_i}(u + A_i(u))$ and triangle inequality, we obtain the assertion. □

Now, with Lemma 7.22 and K_i from (7.10), we obtain for $i \in \mathcal{I}(q,t) \subset I_{12}$:

$$\begin{aligned} K_i \quad &\le \quad \frac{1}{t^n} \int_{B(q,t)\cap R_i} \frac{d(u + A(u), P_i)}{t} \mathrm{d}\mathcal{H}^n(u) \\ &\qquad + \frac{1}{t^n} \sup \left\{ \frac{d(\pi_{P_i}(v + A(v)), \hat{P})}{t} \Big| v \in B(q,t) \cap R_i \right\} \mathcal{H}^n(B(q,t) \cap R_i) \\ &\overset{\text{L. 7.22}}{\le} \frac{1}{t^n} \int_{B(q,t)\cap R_i} \frac{d(u + A(u), P_i)}{t} \mathrm{d}\mathcal{H}^n(u) \\ &\qquad + \omega_n \left(\frac{\operatorname{diam} R_i}{t} \right)^n \sup \left\{ \frac{d(w, \hat{P})}{t} \Big| w \in P_i, d(w, B_i) \le C_4 \operatorname{diam} R_i \right\}. \end{aligned} \tag{7.13}$$

Since P_i is the graph of A_i, we get for any $u \in B(q,t) \cap R_i$ with Lemma 6.21(iv) that there exists some $\bar{C} = \bar{C}(N, n, C_0)$ with

$$d(u + A(u), P_i) \le d(u + A(u), u + A_i(u)) = d(A(u), A_i(u)) \le \bar{C}\varepsilon \operatorname{diam} R_i,$$

and so, using Lemma A.4,

$$\frac{1}{t^n} \int_{B(q,t)\cap R_i} \frac{d(u + A(u), P_i)}{t} \mathrm{d}\mathcal{H}^n(u) \le \varepsilon\, C(N, n, C_0) \left(\frac{\operatorname{diam} R_i}{t} \right)^{n+1}. \tag{7.14}$$

Lemma 7.23. *There exists some constant $C = C(N, n, C_0)$ so that for all $i \in \mathcal{I}(q,t)$,*

$$\begin{aligned} &\sup \left\{ \frac{d(w, \hat{P})}{t} \Big| \begin{matrix} w \in P_i, \\ d(w, B_i) \le C_4 \operatorname{diam} R_i \end{matrix} \right\} \\ &\qquad \le \frac{C}{t} \left[\varepsilon \operatorname{diam} R_i + \left(\frac{1}{(\operatorname{diam} R_i)^n} \int_{2B_i} d(z, \hat{P})^{\frac{1}{3}} \mathrm{d}\mu(z) \right)^3 \right]. \end{aligned}$$

Proof. Let $i \in \mathcal{I}(q,t)$. Due to the construction of $B_i = B(X_i, t_i)$ (see Lemma 6.14), we have $(X_i, t_i) \in S \subset S_{total}$ and so $\delta(X_i, t_i) \ge \frac{\delta}{2}$. With Corollary 4.3 ($\lambda = \frac{\delta}{2}$, $B(x,t) = B(X_i, t_i)$, $\Upsilon = \mathbb{R}^N$), there exist constants $C_1 = C_1(N, n, C_0) > 3$, $C_2 = C_2(N, n, C_0) > 1$ and some $(n, 10n\frac{t_i}{C_1})$-simplex $T = \Delta(x_0, \dots, x_n) \in F \cap B_i$ with

$$\mu\left(B\left(x_\kappa, \tfrac{t_i}{C_1}\right) \cap B_i\right) \ge \tfrac{t_i^n}{C_2} \quad \text{and} \quad B\left(x_\kappa, \tfrac{t_i}{C_1}\right) \subset 2B_i \subset kB_i = B(X_i, kt_i), \tag{7.15}$$

for all $\kappa = 0, \dots, n$, and we used that $C_1 > 3$ and $k \geq \tilde{k} \geq 2$ ($\tilde{k}$ is defined on page 1227). We set $C' := 400C_2$, $\tilde{B}_\kappa := B\left(x_\kappa, \frac{t_i}{C_1}\right)$ and define for all $\kappa = 0, \dots, n$,

$$Z_\kappa := \left\{ z \in \tilde{B}_\kappa \cap F \middle| d(z, P_i) \leq C' \varepsilon \operatorname{diam} R_i \right\}. \tag{7.16}$$

We have $(X_i, t_i) \in S_{total}$ and hence $\beta_{1;k}^{P_i}(X_i, t_i) \leq 2\varepsilon$. Using this and Lemma 6.14, we obtain with Chebyshev's inequality

$$\mu(\tilde{B}_\kappa \setminus Z_\kappa) < \frac{t_i^{n+1}}{C' \varepsilon \operatorname{diam} R_i} \beta_{1;k}^{P_i}(X_i, t_i) \leq \frac{t_i^{n+1}}{C' \varepsilon t_i} \frac{100}{} 2\varepsilon = \frac{t_i^n}{2C_2}.$$

Using Lemma 6.14 again, we get

(7.17)

$$\mu(Z_\kappa) \quad \geq \quad \mu(\tilde{B}_\kappa) - \mu(\tilde{B}_\kappa \setminus Z_\kappa) \overset{(7.15)}{\geq} \frac{t_i^n}{C_2} - \frac{t_i^n}{2C_2} \quad = \quad \frac{t_i^n}{2C_2} \quad \geq \quad \frac{\operatorname{diam} R_i^n}{2^{n+1} C_2} > 0.$$

For all $\kappa \in \{0, \dots, n\}$, let $z_\kappa \in Z_\kappa \subset \tilde{B}_\kappa$ and set $y_\kappa := \pi_{P_i}(z_\kappa)$. Since $\varepsilon \leq \tilde{\varepsilon} \leq \frac{1}{2C' C_1}$ ($\tilde{\varepsilon}$ was chosen on page 1227), we deduce that

$$d(y_\kappa, x_\kappa) \leq d(y_\kappa, z_\kappa) + d(z_\kappa, x_\kappa) \leq d(z_\kappa, P_i) + \frac{t_i}{C_1} \overset{(7.16)}{\leq} C' \varepsilon \operatorname{diam} R_i + \frac{t_i}{C_1} \leq 2\frac{t_i}{C_1}.$$

Due to Lemma 2.8, the simplex $S = \Delta(y_0, \dots, y_n)$ is an $(n, 6n\frac{t_i}{C_1})$-simplex, and, using the triangle inequality, we obtain $S \subset 2B_i$. Now, with Lemma 2.16 ($C = \frac{C_1}{6n}$, $\hat{C} = 2$, $t = t_i$, $m = n$, $x = X_i$) there exists some orthonormal basis $(o_1, \dots, o_n)$ of $P_i - y_0$ and there exist $\gamma_{l,r} \in \mathbb{R}$ with $o_l = \sum_{r=1}^{l} \gamma_{l,r}(y_r - y_0)$ and $|\gamma_{l,r}| \leq \left(\frac{2C_1}{3}\right)^n \frac{C_1}{6nt_i}$ for all $1 \leq l \leq n$ and $1 \leq r \leq l$.

Now let $w \in P_i$ with $d(w, B_i) \leq C_4 \operatorname{diam} R_i$. We obtain

$$w - y_0 = \sum_{\kappa=1}^{n} \langle w - y_0, o_\kappa \rangle o_\kappa = \sum_{\kappa=1}^{n} \langle w - y_0, o_\kappa \rangle \sum_{r=1}^{\kappa} \gamma_{\kappa,r}(y_r - y_0), \tag{7.18}$$

and so, with Remark 2.1 ($b = w$, $P = \hat{P}$) and $|w - y_0| \leq d(w, B_i) + \operatorname{diam} B_i + d(B_i, y_0) \leq Ct_i$, we get

$$\begin{aligned} d(w, \hat{P}) &\overset{(7.18)}{\leq} nCC_1^{n+1} \sum_{r=1}^{n} \left(d(y_r, z_r) + d(z_r, \hat{P}) \right) \\ &\overset{(7.16)}{\leq} n^2 CC_1^{n+1} C' \varepsilon \operatorname{diam} R_i + nCC_1^{n+1} \sum_{r=0}^{n} d(z_r, \hat{P}). \end{aligned} \tag{7.19}$$

The previous results are valid for arbitrary $z_\kappa \in Z_\kappa$; hence we get

$$\begin{aligned} & d(w, \hat{P}) - n^2 CC_1^{n+1} C' \varepsilon \operatorname{diam} R_i \\ \overset{(7.19)}{\leq} & \left(\frac{1}{\prod_{r=0}^{n} \mu(Z_r)} \int_{Z_0} \cdots \int_{Z_n} \left(nCC_1^{n+1} \sum_{r=0}^{n} d(z_r, \hat{P}) \right)^{\frac{1}{3}} \mathrm{d}\mu(z_n) \dots \mathrm{d}\mu(z_0) \right)^3 \\ \leq & \quad nCC_1^{n+1} \left(\sum_{r=0}^{n} \frac{1}{\mu(Z_r)} \int_{Z_r} d(z_r, \hat{P})^{\frac{1}{3}} \, \mathrm{d}\mu(z_r) \right)^3 \\ \overset{(7.17)(7.15)}{\leq} & \quad nCC_1^{n+1} \left(\frac{2^{n+1} C_2}{\operatorname{diam} R_i^n} \int_{2B_i} d(z, \hat{P})^{\frac{1}{3}} \mathrm{d}\mu(z) \right)^3, \end{aligned}$$

where we used that the sets Z_r are disjoint. Since $w \in P_i$ was arbitrarily chosen with $d(w, B_i) \leq C_4 \operatorname{diam} R_i$, we get the statement. $\square$

Lemma 7.24. *There exists some constant $C = C(n, C_0)$ so that*
$$\sum_{i\in\mathcal{I}(q,t)} \left(\frac{\operatorname{diam} R_i}{t}\right)^n \frac{1}{t}\left(\frac{1}{(\operatorname{diam} R_i)^n}\int_{2B_i} d(z,\hat{P})^{\frac{1}{3}}\mathrm{d}\mu(z)\right)^3 \leq C\beta_{1;k}(X,t).$$

Proof. Let $i \in \mathcal{I}(q,t)$ ($\mathcal{I}(q,t)$ is defined on page 1228) and $x \in 2B_i$. We define
$$J(i) := \left\{ j \in \mathcal{I}(q,t) \middle| \operatorname{diam} B_j \leq \operatorname{diam} B_i, 2B_i \cap 2B_j \neq \emptyset \right\}$$
and
$$\Xi_i(x) := \sum_{j\in J(i)} \chi_{2B_j}(x).$$
First, we prove some intermediate results:

I. For all $i \in \mathcal{I}(q,t)$, we have $\int_{2B_i} \Xi_i(x)\mathrm{d}\mu(x) \leq C(n,C_0)(\operatorname{diam} R_i)^n$. This implies that $\Xi_i(x) < \infty$ for μ-almost all $x \in 2B_i$.

Proof of I. Let $i \in \mathcal{I}(q,t)$ and $j \in J(i)$. With Lemma 6.14 applied to j and the definition of $J(i)$, we deduce that $\operatorname{diam} R_j \leq 200 \operatorname{diam} R_i$. Using Lemma 6.14 and $j \in J(i)$, we get $d(R_i, R_j) \leq C \operatorname{diam} R_i$. This implies for some large enough constant $C > 1$ that $R_j \subset CR_i$. Since the cubes $\mathring{R}_j$ are disjoint (see Lemma 6.13(ii)), we get with Lemma A.4:
$$\sum_{j\in J(i)} (\operatorname{diam} R_j)^n = \sum_{j\in J(i)} (\sqrt{n})^n \mathcal{H}^n(R_j) \leq (\sqrt{n})^n \mathcal{H}^n\big(CR_i\big) = C(n)(\operatorname{diam} R_i)^n.$$
In the following, we apply Fatou's lemma [7, 1.3, Thm. 1] to interchange the integration with the summation. With (B) from page 1208 and Lemma 6.14, we obtain
$$\begin{aligned}\int_{2B_i} \Xi_i(x)\mathrm{d}\mu(x) &\leq \sum_{j\in J(i)} \mu(2B_j) \overset{(\mathrm{B})}{\leq} C(n,C_0)\sum_{j\in J(i)} (\operatorname{diam} R_j)^n \\ &\leq C(n,C_0)(\operatorname{diam} R_i)^n.\end{aligned}$$
$\square$

II. Let $x\in\mathbb{R}^N$ and $m\in\mathbb{N}$. There exists some $C=C(n)>1$ with $\sum_{\substack{i\in\mathcal{I}(q,t)\\ \Xi_i(x)=m}} \chi_{2B_i}(x) \leq C$.

Proof of II. Let $l, o \in \mathcal{I}(q,t)$ with $x \in 2B_l \cap 2B_o$ and $\Xi_l(x) = m = \Xi_o(x)$. Without loss of generality, we have $\operatorname{diam} B_l \leq \operatorname{diam} B_o$.

Assume that $\operatorname{diam} B_l < \operatorname{diam} B_o$. We define $J(l,x) := \left\{\iota \in J(l)\middle| x \in 2B_\iota\right\}$. Let $j \in J(l,x)$. By definition of $J(l)$, we get $\operatorname{diam} B_j \leq \operatorname{diam} B_l < \operatorname{diam} B_o$ and $x \in 2B_j$. Since $x \in 2B_o$, it follows that $2B_o \cap 2B_j \neq \emptyset$ and, because $\operatorname{diam} B_j < \operatorname{diam} B_o$, we get $j \in J(o,x)$. Furthermore, we have $o \in J(o,x)$, but $o \notin J(l,x)$ because by our assumption we have $\operatorname{diam} B_l < \operatorname{diam} B_o$. So we get $J(l,x) \subsetneq J(o,x)$. Now we obtain a contradiction:
$$m = \Xi_l(x) = \sum_{j\in J(l)} \chi_{2B_j}(x) = \sum_{j\in J(l,x)} \chi_{2B_j}(x) < \sum_{j\in J(o,x)} \chi_{2B_j}(x) = \Xi_o(x) = m.$$
Hence there exists some $\lambda = \lambda(x,m) \in (0,\infty)$ so that, for $l \in \mathcal{I}(q,t)$ with $x \in 2B_l$ and $\Xi_l(x) = m$, we have $\operatorname{diam} B_l = \lambda$, and, we obtain with Lemma 6.14 that $\lambda \leq 200 \operatorname{diam} R_l \leq 200\lambda$ and $d(R_l, \pi(B_l)) \leq 100\lambda$. Using $d(R_l, \pi(x)) \leq d(R_l, \pi(B_l)) + 2\operatorname{diam} B_l \leq 102\lambda$, we get $R_l \subset B(\pi(x), 103\lambda) \cap P_0$. With Lemma A.4, we have

$\mathcal{H}^n(R_l) \geq (\sqrt{n})^{-n}(\frac{1}{200}\lambda)^n$, and according to Lemma 6.13(ii) the cubes R_l have disjoint interior. This implies that there exists some constant $C(n)$ so that there are at most $C(n)$ indices $l \in \mathcal{I}(q,t)$ with $\Xi_l(x) = m$ and $x \in 2B_l$. This implies the assertion. □

III. We have $i \in J(i)$ and so $\Xi_i(x) \neq 0$ for all $x \in 2B_i$. Hence, with $x \in \mathbb{R}^N$, the term

$$\chi_{2B_i}(x)\Xi_i(x)^{-2} := \begin{cases} \Xi_i(x)^{-2} & \text{if } x \in 2B_i, \\ 0 & \text{otherwise} \end{cases}$$

is well-defined. Now there exists some constant $C(n)$ so that, for all $x \in \mathbb{R}^N$, we get

$$\sum_{i\in\mathcal{I}(q,t)} \chi_{2B_i}(x)\Xi_i(x)^{-2} = \sum_{m=1}^{\infty} \sum_{\substack{i\in\mathcal{I}(q,t)\\ \Xi_i(x)=m}} \chi_{2B_i}(x)\frac{1}{m^2} \overset{\mathrm{II}}{\leq} C(n).$$

IV. Let $i \in \mathcal{I}(q,t)$. Since $i \in J(i)$, we have $\Xi_i(x) \neq 0$ for $x \in 2B_i$. We obtain with Hölder's inequality

$$\left[\frac{1}{(\operatorname{diam} R_i)^n} \int\limits_{2B_i} d(z,\hat{P})^{\frac{1}{3}}\Xi_i(z)^{\frac{-2}{3}}\Xi_i(z)^{\frac{2}{3}}\mathrm{d}\mu(z)\right]^3$$
$$\overset{\mathrm{I}}{\leq} C(n,C_0)\frac{1}{(\operatorname{diam} R_i)^n} \int\limits_{2B_i} d(z,\hat{P})\Xi_i(z)^{-2}\mathrm{d}\mu(z).$$

V. We have

$$\frac{1}{t^{n+1}}\int_{\bigcup_{i\in\mathcal{I}(q,t)} 2B_i} d(z,\hat{P})\mathrm{d}\mu(z) \leq 2\beta_{1;k}(X,t),$$

where $X \in B(\tilde{X}(q), 200t)$ (cf. page 1228).

Proof of III–V. At first, we prove that there exists some constant $\hat{C} > 1$ so that for $i \in \mathcal{I}(q,t)$ we have $2B_i \subset B(X, \hat{C}t)$. Let $i \in \mathcal{I}(q,t)$. By definition of $\mathcal{I}(q,t)$ (see page 1228), we obtain $R_i \cap B(q,t) \neq \emptyset$. Let $\tilde{u} \in R_i \cap B(q,t)$. Since $\frac{D(q)}{100} < t$ (see page 1228), we get, using the triangle inequality, $D(\tilde{u}) \leq D(q) + d(q,\tilde{u}) < 101t$. It follows with Lemma 6.13(i) that

$$\operatorname{diam} R_i \leq \tfrac{1}{10}D(\tilde{u}) < 11t. \tag{7.20}$$

With Lemma 6.14 and (7.8), we get ($X \in B(\tilde{X}, 200t)$; see page 1228)

$$\begin{aligned} d(\pi(B_i),\pi(X)) &\leq d(\pi(B_i),\tilde{u}) + d(\tilde{u},q) + d(q,\pi(\tilde{X})) + d(\pi(\tilde{X}),\pi(X)) \\ &\overset{(7.8)}{\leq} d(\pi(B_i),R_i) + \operatorname{diam} R_i + t + 200t + d(\tilde{X},X) \overset{(7.20)}{\leq} Ct. \end{aligned} \tag{7.21}$$

Now let $x \in 2B_i = B(X_i, 2t_i)$. Since $(X_i,t_i) \in S$, using Lemma 6.14 and (7.20), we get $d(x) < 4400t$. Due to $X \in B(\tilde{X}, 200t) \cap F$ and (7.8), we deduce that $d(X) \leq 400t$. With Lemma 6.14 and estimates (7.20) and (7.21), we obtain with triangle inequality $d(\pi(x),\pi(X)) \leq Ct$. Now there exists some constant $\hat{C} > 1$ so that we get with Lemma 6.11 $d(x,X) \leq \hat{C}t$. All in all we have proven that, for all $i \in \mathcal{I}(q,t)$, we have $2B_i \subset B(X,\hat{C}t)$. Since $k \geq \tilde{k} \geq \hat{C}$ (see page 1227), we get the assertion with condition (7.9) from page 1228. □

Now, Lemma 7.24 can be proven by applying IV, III, and V and using the monotone convergence theorem [7, 1.3, Thm. 2] to interchange the summation and the integration. $\square$

Now we can give some estimate for $\gamma(q,t)$, where $q \in U_{10}$ and $\frac{D(q)}{100} < t < 2$. Using the inequalities (7.10), (7.12), (7.13), (7.14), Lemma 7.23 and Lemma 7.24, we get using $T \leq 200t$ (cf. Lemma 7.19) for every $X \in B(\tilde{X},T) \cap F \subset B(\tilde{X},200t) \cap F$:

$$\gamma(q,t) \leq C(N,n,C_0)\ \beta_{1;k}(X,t) + C(N,n,C_0)\ \varepsilon \sum_{i\in\mathcal{I}(q,t)} \left(\frac{\operatorname{diam} R_i}{t}\right)^{n+1}.$$

With Lemma 7.19, we get $(\tilde{X},T) \in S \subset S_{total}$ and $20t \leq T \leq 200t$. Using this, the previous estimate, the definition of $\delta = \delta(n)$ on page 1208 and (B) from page 1208, we get

$$\begin{aligned}\gamma(q,t)^p \ &\leq \ \frac{2}{\delta T^n}\int_{B(\tilde{X},T)} \gamma(q,t)^p \mathrm{d}\mu(X)\\ &\leq \ C\frac{1}{t^n}\int_{B(\tilde{X},200t)} \beta_{1;k}(X,t)^p \mathrm{d}\mu(X) + C\varepsilon^p \left(\sum_{i\in\mathcal{I}(q,t)}\left(\frac{\operatorname{diam} R_i}{t}\right)^{n+1}\right)^p,\end{aligned}$$

where $C = C(N,n,p,C_0)$. We recall that for every $q \in U_{10}$ there exists some $\tilde{X} = \tilde{X}(q)$ (cf. Lemma 7.19) such that the previous inequality is valid. This implies

$$\int_{U_{10}}\int_{\frac{D(q)}{100}}^{2} \gamma(q,t)^p \frac{\mathrm{d}t}{t}\mathrm{d}\mathcal{H}^n(q) \leq C(N,n,p,C_0)\ a + C(N,n,p,C_0)\ \varepsilon^p\ b, \tag{7.22}$$

where

$$a := \int_{U_{10}}\int_{\frac{D(q)}{100}}^{2} \frac{1}{t^n}\int_{B(\tilde{X}(q),200t)} \beta_{1;k}(X,t)^p \mathrm{d}\mu(X)\frac{\mathrm{d}t}{t}\mathrm{d}\mathcal{H}^n(q),$$

$$b := \int_{U_{10}}\int_{\frac{D(q)}{100}}^{2} \left(\sum_{i\in\mathcal{I}(q,t)}\left(\frac{\operatorname{diam} R_i}{t}\right)^{n+1}\right)^p \frac{\mathrm{d}t}{t}\mathrm{d}\mathcal{H}^n(q).$$

To estimate a and b, we need the following lemma.

Lemma 7.25. *Let $q \in U_{10}$, $\frac{D(q)}{100} \leq t \leq 2$ and $X \in B(\tilde{X}(q),200t) \cap F$, where $\tilde{X}(q)$ is given by Lemma 7.19. Then $d(\pi(X),q) \leq 400t$ and there exists some $\tilde{\lambda} = \tilde{\lambda}(N,n,C_0) > 0$ so that, with $k_0 = 401$, we have $\tilde{\delta}_{k_0}(B(X,t)) = \sup_{y\in B(X,k_0 t)} \frac{\mu(B(y,t))}{t^n} \geq \tilde{\lambda}$, where $\tilde{\delta}_{k_0}(B(X,t))$ was defined on page 1196. Furthermore, there holds for all $i \in \mathcal{I}(q,t)$ that*

$$d(q,R_i) \leq t, \qquad \operatorname{diam} R_i < 11t, \tag{7.23}$$

and there exists some constant $C = C(n)$ with

$$\sum_{i\in\mathcal{I}(q,t)}\left(\frac{\operatorname{diam} R_i}{t}\right)^{n+1} \leq C, \qquad \sum_{i\in I_{12}} (\operatorname{diam} R_i)^n \leq C. \tag{7.24}$$

Proof. Let $q \in U_{10}$, $\frac{D(q)}{100} \leq t \leq 2$ and $X \in B(\tilde{X}(q),200t) \cap F$. We have $d(X,\tilde{X}(q)) \leq 200t$ and, with (7.8), we get $d(\pi(\tilde{X}(q)),q) \leq 200t$. This implies $d(\pi(X),q) \leq 400t$ by using triangle inequality. With (7.8), we obtain $(\tilde{X}(q),T) \in S \subset S_{total}$ and, by definition of S_{total}, we conclude that $\delta(B(\tilde{X}(q),T)) \geq \frac{\delta}{2}$. We have $B(\tilde{X}(q),T) \subset B(X,400t)$ and hence with (7.8) we get $\delta(B(X,400t)) \geq \frac{\delta}{2\cdot 20^n}$. Applying Corollary 4.3(ii) with $\lambda = \frac{\delta}{2\cdot 20^n}$ on $B(X,400t)$, we get constants $C_1 = C_1(N,n,C_0)$, $C_2 = C_2(N,n,C_0)$ and in particular one ball $B(x,s)$ with $s = \frac{400t}{C_1}$ and

$$\mu(B(x,s) \cap B(X,400t)) \geq \frac{(400t)^n}{C_2}. \tag{7.25}$$

We have $\delta \leq \frac{2}{50^n}$ (cf. (6.1)), and Lemma 4.2 gives us $C_1 > 400$. This yields $s < t$. From (7.25), we get $B(x,s) \cap B(X,400t) \neq \emptyset$, which implies $d(x,X) < 401t$, and with (7.25) we get $\sup_{y\in B(X,401t)} \delta(B(y,t)) \geq \frac{400^n}{C_2} =: \tilde{\lambda}$. Let $i \in \mathcal{I}(q,t)$. Due to the definition of $\mathcal{I}(q,t)$ (see page 1228), we have $d(q,R_i) \leq t$ and we can choose some $\tilde{u} \in R_i \cap B(q,t)$. With Lemma 6.13(i), we obtain $10 \operatorname{diam} R_i \leq (D(q) + d(q,\tilde{u})) < 11t$. The intervals R_i have disjoint interior (see Lemma 6.13(ii)) and, from $R_i \cap B(q,t) \neq \emptyset$ for all $i \in \mathcal{I}(q,t)$, we get $R_i \subset B(q,12t)$. With Lemma A.4, this implies

$$\begin{aligned}\sum_{i\in\mathcal{I}(q,t)} \left(\frac{\operatorname{diam} R_i}{t}\right)^{n+1} &\overset{(7.23)}{<} \frac{11}{t^n} \sum_{i\in\mathcal{I}(q,t)} (\operatorname{diam} R_i)^n \\ &= \frac{11}{t^n} \sum_{i\in\mathcal{I}(q,t)} (\sqrt{n})^n \mathcal{H}^n(R_i) = C(n).\end{aligned}$$

Now let $i \in I_{12}$. We have $R_i \cap B(0,12) \neq \emptyset$. If $(Y,r) \in S \subset S_{total}$, we get $Y \in F \subset B(0,5)$ (cf. (A) on page 1208) and hence we obtain $d(\pi(Y),0) \leq 5$ as well as $r \leq 50$. With $\tilde{v} \in R_i \cap B(0,12)$ and Lemma 6.13(i), we get

$$\operatorname{diam} R_i \leq \frac{1}{10} D(\tilde{v}) = \frac{1}{10} \inf_{(Y,r)\in S} (d(\pi(Y),\tilde{v}) + r) \leq \frac{1}{10}(5 + 12 + 50) < 7.$$

Hence, for all $i \in I_{12}$, we have $R_i \subset B(0,19)$, and the cubes R_i have disjoint interior (cf. Lemma 6.13(ii)). With Lemma A.4, we deduce that $\sum_{i\in I_{12}} (\operatorname{diam} R_i)^n = C(n)$. $\square$

To control the terms a and b we will use Fubini's theorem [7, 1.4, Thm. 1] in the following abbreviated by (F). Now, using Lemma 7.25 and Corollary 4.8 ($\lambda = \tilde{\lambda}$, $k_0 = 401$), we conclude that

$$\begin{aligned} a &\overset{(F)}{\leq} \int_F \int_0^2 \frac{1}{t^n} \int_{U_{10}} \chi_{\{d(\pi(X),q)\leq 400t\}} \mathrm{d}\mathcal{H}^n(q)\ \chi_{\left\{\tilde{\delta}_{k_0}(B(X,t))\geq\tilde{\lambda}\right\}} \beta_{1;k}(X,t)^p \ \frac{\mathrm{d}t}{t} \mathrm{d}\mu(X) \\ &\leq C(N,n,\mathcal{K},p,C_0,k)\ \mathcal{M}_{\mathcal{K}^p}(\mu).\end{aligned}$$

Now we consider the integral b. We use Fatou's lemma [7, 1.3, Thm. 1] to interchange the summation with the integration:

$$\begin{aligned} b &\overset{(7.24)(7.23)}{\leq} C \int_{U_{10}} \int_{\frac{D(q)}{100}}^2 \sum_{i\in I_{12}} \chi_{\left\{t>\frac{\operatorname{diam} R_i}{11}, d(q,R_i)\leq t\right\}} \left(\frac{\operatorname{diam} R_i}{t}\right)^{n+1} \frac{\mathrm{d}t}{t} \mathrm{d}\mathcal{H}^n(q) \\ &\overset{(F)}{\leq} C \sum_{i\in I_{12}} (\operatorname{diam} R_i)^{n+1} \int_{\frac{\operatorname{diam} R_i}{11}}^\infty \int_{U_{10}} \chi_{\{d(q,R_i)\leq t\}} \mathrm{d}\mathcal{H}^n(q) \frac{\mathrm{d}t}{t^{n+2}} \\ &\overset{(7.24)}{\leq} C(n,p).\end{aligned}$$

Due to Lemma 6.13(ii) the proof of Theorem 7.17 is completed by applying Lemma 7.18, (7.7) and (C) from page 1208 because $\mathcal{M}_{\mathcal{K}^p}(\mu) \overset{\text{(C)}}{\leq} \eta < \varepsilon^p$ (see pages 1208 and 1227). $\square$

8. $\mathcal{Z}$ is not too small

Our aim is to prove Theorem 5.4. In Definition 6.3, we defined a partition of the support F of our measure μ in four parts, namely $\mathcal{Z}$, F_1, F_2, F_3. Then, in section 6.4, we constructed some function A, the graph Γ of which covers the set $\mathcal{Z}$. To get our main result, we need to know that we covered a major part of F. In this last part of the proof of Theorem 5.4, we show that the μ-measure of F_1, F_2, F_3 is quite small. In particular, we deduce that $\mu(F_1 \cup F_2 \cup F_3) \leq \frac{1}{100}$. As stated at the beginning of section 6.1, this completes the proof of Theorem 5.4.

8.1. **Most of F is close to the graph of A.** With $K := 2\,(104 \cdot 10 \cdot 6 + 214)$, we define the set G by

$$\begin{aligned}&\{x \in F \setminus \mathcal{Z} \mid \forall i \in I_{12} \text{ with } \pi(x) \in 3R_i, \text{ we have } x \notin KB_i\}\\ &\quad \cup \{x \in F \setminus \mathcal{Z} \mid \pi(x) \in \pi(\mathcal{Z})\}\,.\end{aligned}$$

At first, we show that the μ-measure of G is small.

Lemma 8.1. *Let $0 < \alpha \leq \frac{1}{280}$. There exist some $\tilde{\varepsilon} = \tilde{\varepsilon}(N, n, C_0, \alpha)$ so that, if $\eta < 2\tilde{\varepsilon}$ and $k \geq 4$, there exists some constant $C = C(N, n, \mathcal{K}, p, C_0)$ so that, for all $\varepsilon \in [\frac{\eta}{2}, \tilde{\varepsilon})$, we have*

$$\mu(G) \leq C\mathcal{M}_{\mathcal{K}^p}(\mu) \overset{(C)}{\leq} C\eta,$$

where the condition (C) *was given on page* 1208.

Proof. Let $0 < \alpha \leq \frac{1}{280}$ and $\tilde{\varepsilon} := \min\{\bar{\varepsilon}, \frac{\alpha}{\bar{C}}\}$ where $\bar{\varepsilon}$ is given by Lemma 6.11 and $\bar{C} = \bar{C}(N, n, C_0)$ is a fixed constant defined in this proof on page 1237. Furthermore let $\eta < 2\tilde{\varepsilon}$, $k \geq 4$ and $\eta \leq 2\varepsilon < 2\tilde{\varepsilon}$.

Let $x \in G$. If $x \in G \setminus \pi^{-1}(\pi(\mathcal{Z})) \subset F \subset B(0, 5)$, with Lemma 6.13(ii), there exists some $i \in I_{12}$ with $\pi(x) \in R_i \subset 2R_i$. Let X_i be the centre of B_i (cf. Lemma 6.14). We set

$$X(x) := \begin{cases} X_i & \text{if } x \in G \setminus \pi^{-1}(\pi(\mathcal{Z})), \\ \pi(x) + A(\pi(x)) & \text{if } x \in G \cap \pi^{-1}(\pi(\mathcal{Z})). \end{cases}$$

Claim 1. For all $x \in G$ and $X = X(x)$ defined as above, we have

$$d(x, X) < 7d(x), \quad d(\pi(x), \pi(X)) \leq \tfrac{d(x)}{10}, \quad \tfrac{d(x)}{2} \leq d(X, x), \quad \left(X, \tfrac{d(x)}{10}\right) \in S. \tag{8.1}$$

Proof of Claim 1.

1. Case: $x \in G \setminus \pi^{-1}(\pi(\mathcal{Z}))$. Due to the definition of G and $\pi(x) \in 2R_i \subset 3R_i$, we have $x \notin KB_i$. By adding some $q \in R_i$ with the triangle inequality and using Lemma 6.14 we get $d(\pi(x), \pi(X_i)) \leq 104 \operatorname{diam} B_i$. With Lemma 6.14, we know $\left(X_i, \frac{\operatorname{diam} B_i}{2}\right) \in S$ and hence we get $d(X_i) < \operatorname{diam} B_i$. Using $x \notin KB_i$ and Lemma 6.11, we get $K \cdot \frac{\operatorname{diam} B_i}{2} < d(x, X_i) < 6d(x) + 214 \operatorname{diam} B_i$, which yields by definition of K (cf. the beginning of this subsection) $104 \operatorname{diam} B_i < \frac{d(x)}{10}$. From the previous two estimates, we get $d(x, X_i) < 7d(x)$; i.e., the first inequality holds in this case. Furthermore, we have the second one since $d(\pi(x), \pi(X_i)) \leq 104 \operatorname{diam} B_i < \frac{d(x)}{10}$.

We have $\left(X_i, \frac{\operatorname{diam} B_i}{2}\right) \in S$, so we get $d(x) \le d(X_i, x) + \frac{\operatorname{diam} B_i}{2} < d(X_i, x) + \frac{d(x)}{2}$, and hence the third inequality holds in this case. Due to Lemma 6.9, we have $\frac{\operatorname{diam} B_i}{2} < \frac{d(x)}{10} < \frac{60}{10} < 50$, so that with Lemma 6.2(ii) we deduce that $\left(X, \frac{d(x)}{10}\right) \in S$.

2. Case: $x \in G \cap \pi^{-1}(\pi(\mathcal{Z}))$. We have $\pi(x) \in \pi(\mathcal{Z})$ and hence $X = \pi(x) + A(\pi(x)) \in \mathcal{Z}$ (cf. Definition 6.20). By definition of $\mathcal{Z}$ and Lemma 6.2(i), we obtain $(X, \sigma) \in S$ for all $\sigma \in (0, 50)$ and hence $\frac{d(x)}{2} \le d(X, x) + \sigma$, which establishes the third estimate. Moreover, we have $d(\pi(X), \pi(x)) = d(\pi(x), \pi(x)) = 0$. Using Lemma 6.10, we obtain $d(X) = 0$ and hence we get with Lemma 6.11 $d(x, X) \le 6d(x)$. Furthermore, we have with Lemma 6.9 that $\frac{d(x)}{10} \le 6 < 50$ so that by definition of $\mathcal{Z}$, we get $\left(X, \frac{d(x)}{10}\right) \in S$. Claim 1 is proved.

Let $P_x := P_{\left(X, \frac{d(x)}{10}\right)}$ be the plane associated to $B(X, \frac{d(x)}{10})$ (cf. Definition 6.1). We define

$$\Upsilon := \left\{ u \in B\left(X, \tfrac{d(x)}{10}\right) \middle| d(u, P_x) \le \tfrac{8}{\delta} \tfrac{d(x)}{10} \varepsilon \right\}. \tag{8.2}$$

Due to Definition 6.1 we have $\beta_{1;k}^{P_x}\left(X, \frac{d(x)}{10}\right) \le 2\varepsilon$ and hence we get using Chebyshev's inequality

$$\mu\left(B\left(X, \tfrac{d(x)}{10}\right) \setminus \Upsilon\right) \le \tfrac{\delta}{8\varepsilon} \left(\tfrac{d(x)}{10}\right)^n \beta_{1;k}^{P_x}\left(X, \tfrac{d(x)}{10}\right) \le \tfrac{\delta}{4} \left(\tfrac{d(x)}{10}\right)^n.$$

Since $\Upsilon \subset B\left(X, \frac{d(x)}{10}\right)$ and $\delta\big(B\big(X, \frac{d(x)}{10}\big)\big) \ge \frac{1}{2}\delta$ (cf. Definition 6.1 of S_{total}), we obtain

$$\mu\left(B\left(X, \tfrac{d(x)}{10}\right) \cap \Upsilon\right) \ge \mu\left(B\left(X, \tfrac{d(x)}{10}\right)\right) - \mu\left(B\left(X, \tfrac{d(x)}{10}\right) \setminus \Upsilon\right) \ge \tfrac{\delta}{4} \left(\tfrac{d(x)}{10}\right)^n.$$

With Corollary 4.3 ($\lambda = \frac{\delta}{4}$, $t = \frac{d(x)}{10}$), there exist constants $C_1 = C_1(N, n, C_0)$, $C_2 = C_2(N, n, C_0)$ and an $\left(n, 10n\frac{d(x)}{10C_1}\right)$-simplex $T = \Delta(x_0, \dots, x_n) \in F \cap B\left(X, \frac{d(x)}{10}\right) \cap \Upsilon$ so that for all $j \in \{0, \dots, n\}$,

$$\mu\left(B\left(x_j, \tfrac{d(x)}{10C_1}\right) \cap B\left(X, \tfrac{d(x)}{10}\right) \cap \Upsilon\right) \ge \left(\tfrac{d(x)}{10}\right)^n \tfrac{1}{C_2}. \tag{8.3}$$

Let $y_j \in B\left(x_j, \frac{d(x)}{10C_1}\right) \cap \Upsilon$ for all $j \in \{0, \dots, n\}$. By applying Lemma 2.8 $(n+1)$ times, we find that $\Delta(y_0, \dots, y_n)$ is an $\left(n, 8n\frac{d(x)}{10C_1}\right)$-simplex.

Claim 2. For all $x \in G$, we have $d(x, \operatorname{aff}(y_0, \dots, y_n)) \ge \frac{d(x)}{4}$.

Proof of Claim 2. Let $P_y := \operatorname{aff}(y_0, \dots, y_n)$ be the plane through $y_0, \dots, y_n$. Applying Lemma 2.17 ($C = \frac{C_1}{8n}$, $\hat{C} = 1$, $t = \frac{d(x)}{10}$, $\sigma = \frac{8}{\delta}\varepsilon$, $P_1 = P_y$, $P_2 = P_x$, $S = \Delta(y_0, \dots, y_n)$, $x = X$, $m = n$) yields $\sphericalangle(P_y, P_x) \le \alpha$, where we use that $\varepsilon \le \tilde{\varepsilon} \le \frac{\alpha}{\tilde{C}}$ and $\tilde{C}$ is given by Lemma 2.17 . So, with Definition 6.1, we obtain $\sphericalangle(P_y, P_0) \le 2\alpha$. Let $\hat{P}_y \in \mathcal{P}(N, n)$ be the n-dimensional plane parallel to P_y with $X \in \hat{P}_y$, and $\hat{P}_0 \in \mathcal{P}(N, n)$ be the plane parallel to P_0 with $X \in \hat{P}_0$. We have $\alpha \le \frac{1}{280}$, and hence

$$d(\pi_{\hat{P}_y}(x), \pi_{\hat{P}_0}(x)) = |\pi_{\hat{P}_y - X}(x - X) - \pi_{\hat{P}_0 - X}(x - X)| \le d(x, X) \sphericalangle(\hat{P}_y, \hat{P}_0) \overset{(8.1)}{<} \frac{d(x)}{20}.$$

Furthermore, with (8.1), we get $d(\pi_{\hat{P}_0}(x), X) = d(\pi(x), \pi(X)) \le \frac{d(x)}{10}$. Using the triangle inequality, the previous two estimates imply $d(\pi_{\hat{P}_y}(x), X) \le \frac{d(x)}{20} + \frac{d(x)}{10}$.

Since $y_0 \in \Upsilon \subset B(X, \frac{d(x)}{10})$ we have $d(P_y, \hat{P}_y) = d(X, P_y) \leq d(X, y_0) \leq \frac{d(x)}{10}$, and hence

$$\tfrac{d(x)}{2} \overset{(8.1)}{\leq} d(x, P_y) + d(P_y, \hat{P}_y) + d(\pi_{\hat{P}_y}(x), X) \leq d(x, P_y) + \tfrac{d(x)}{4},$$

and we gain $d(x, P_y) \geq \frac{d(x)}{4}$. Claim 2 is proved.

With (8.1) and $d(y_j, X) \leq d(y_j, x_j) + d(x_j, X) \leq \frac{d(x)}{10C_1} + \frac{d(x)}{10}$, we obtain $y_0, \ldots y_n$, $x \in B(X, 7d(x))$, which is a subset of $B(X, \frac{C_1}{8n}\frac{d(x)}{10})$, where we used the explicit characterisation of C_1 given in Lemma 4.2. Due to the second property of a μ-proper integrand (see Definition 3.1), there exists some $\tilde{C} = \tilde{C}(N, n, \mathcal{K}, p, C_0) \geq 1$ so that we get with Claim 2

$$\begin{aligned} \mathcal{K}^p(y_0, \ldots, y_n, x) &\geq \tilde{C}^{-1}\left(\frac{d(x)}{10}\right)^{-n(n+1)}\left(\frac{d(x, \operatorname{aff}(y_0, \ldots, y_n))}{\frac{d(x)}{10}}\right)^p \\ &> \tilde{C}^{-1}\left(\frac{d(x)}{10}\right)^{-n(n+1)}. \end{aligned}$$

This estimate holds for all $y_i \in B\big(x_i, \frac{d(x)}{10C_1}\big) \cap \Upsilon$. By restricting the integration to the balls $B\big(x_i, \frac{d(x)}{10C_1}\big)$ and using the previous estimate as well as estimate (8.3), we get

$$\int \cdots \int \mathcal{K}^p(y_0, \ldots, y_n, x)\mathrm{d}\mu(y_0) \ldots \mathrm{d}\mu(y_n) \geq \tilde{C}^{-1}C_2^{-(n+1)}.$$

We have proven the previous inequality for all $x \in G$, so finally we deduce with (C) from page 1208 that there exists some constant $C = C(N, n, \mathcal{K}, p, C_0)$ so that

$$\mu(G) \leq \tilde{C}C_2^{(n+1)} \int_G \int \cdots \int \mathcal{K}^p(y_0, \ldots, y_n, x)\mathrm{d}\mu(y_0) \ldots \mathrm{d}\mu(y_n)\mathrm{d}\mu(x) \overset{(C)}{\leq} C\eta. \qquad \square$$

Lemma 8.2. *Let $\alpha, \varepsilon > 0$. If $\eta \leq 2\varepsilon$, we have $(20K)^{-1}d(x) \leq D(\pi(x)) \leq d(x)$ for all $x \in F \setminus G$, where K is the constant defined on page 1236 at the beginning of this subsection.*

Proof. Let $x \in F \setminus G$. We have $D(\pi(x)) = \inf_{y \in \pi^{-1}(\pi(x))} d(y) \leq d(x)$. If $x \in \mathcal{Z}$, Lemma 6.10 implies $d(x) = 0$, so the statement is trivial. Now we assume $x \notin \mathcal{Z}$. Since $x \notin G \cup \mathcal{Z}$, by definition of G, there exists some $i \in I_{12}$ with $\pi(x) \in 3R_i$ and $x \in KB_i$. We have $B_i = B(X_i, t_i)$ where $(X_i, t_i) \in S$ (see Lemma 6.14) and $K > 1$ (see page 1236), so we obtain $d(x) \leq d(X_i, x) + t_i < K \operatorname{diam} B_i$. Now, with Lemma 6.13(i) and 6.14, we deduce that $D(\pi(x)) \geq \frac{1}{20K}d(x)$. $\square$

Lemma 8.3. *Let $0 < \alpha \leq \frac{1}{4}$. There exists some $\bar{\varepsilon} = \bar{\varepsilon}(N, n, C_0)$ and some $\tilde{k} \geq 4$ so that, if $\eta < 2\bar{\varepsilon}$ and $k \geq \tilde{k}$, for all $\varepsilon \in [\frac{\eta}{2}, \bar{\varepsilon})$ we have that the following is true. There exists some constant $C = C(n)$ so that, for all $x \in F$ with $t \geq \frac{d(x)}{10}$, we have*

$$\int_{B(x,t) \setminus G} d\big(u, \pi(u) + A(\pi(u))\big)\mathrm{d}\mu(u) \leq C\varepsilon t^{n+1}.$$

Proof. Let $0 < \alpha \leq \frac{1}{4}$. We choose some ε with $\eta \leq 2\varepsilon < 2\bar{\varepsilon}$ and some $k \geq \tilde{k} := \max\{\bar{k}, \tilde{C}\}$, where $\bar{\varepsilon}$ and $\bar{k}$ are given by Lemma 6.21 and $\tilde{C}$ is a fixed constant introduced in step VI of this proof. Let $x \in F$ and $t \geq \frac{d(x)}{10}$. We define

$$I(x,t) := \left\{ i \in I_{12} | (3R_i \times P_0^\perp) \cap B(x,t) \cap (F \setminus G) \neq \emptyset \right\}$$

where $3R_i \times P_0^\perp := \{x \in \mathbb{R}^N | \pi(x) \in 3R_i\}$. At first, we prove some intermediate results:

I. Due to the definition of G we have

$$(B(x,t) \cap F) \setminus (G \cup \mathcal{Z}) \subset \bigcup_{i \in I(x,t)} (3R_i \times P_0^\perp) \cap KB_i.$$

II. Let $u \in 3R_i \times P_0^\perp$. Using Lemma 6.13(iv) implies that $\sum_{j \in I_{12}} \phi_j(\pi(u))$ is a finite sum.

III. Let $i \in I(x,t)$ and $j \in I_{12}$. We define $\phi_{i,j}$ to be 0 if $3R_i$ and $3R_j$ are disjoint and 1 if they are not disjoint. We have $\phi_j(\pi(u)) \leq 1 = \phi_{i,j}$ for all $u \in (3R_i \times P_0^\perp) \cap KB_i$, since if $\phi_j(\pi(u)) \neq 0$ the definition of ϕ_j (see page 1215) gives us $\pi(u) \in 3R_j$ and because $\pi(u) \in 3R_i$, we deduce that $3R_i \cap 3R_j \neq \emptyset$.

IV. If $\phi_{i,j} \neq 0$, we can apply Lemma 6.13(iii) and Lemma 6.21(i). Hence, using Lemma 6.14, the size of B_i as well as the distance of B_i to B_j are comparable to the size of B_j. Consequently, there exists some constant $\tilde{C}$ so that $KB_i \subset \tilde{C}B_j \subset kB_j$.

V. If $u \in kB_j$, we have $|\pi^\perp(u) - A_j(\pi(u))| < 2d(u, P_j)$. We recall that P_j is the graph of the affine map A_j (cf. Definition 6.17 and Lemma 6.18).

Proof of I–V. We set $\hat{P}_0 := P_0 + A_j(\pi(u))$ and $v := \pi(u) + A_j(\pi(u)) = \pi_{\hat{P}_0}(u)$. We get

$$|\pi_{P_j}(u) - v| = |\pi_{P_j - v}(u-v) - \pi_{\hat{P}_0 - v}(u-v)| \leq |u-v| \sphericalangle(P_j, P_0).$$

Using this and $\sphericalangle(P_j, P_0) \leq \alpha < \frac{1}{2}$ (cf. Definition 6.17) we obtain $|u - v| < d(u, P_j) + \frac{1}{2}|u - v|$ and hence $|\pi^\perp(u) - A_j(\pi(u))| = |u - v| < 2d(u, P_j)$. ☐

If $u \in \mathcal{Z}$, the definition of A (see page 1215) yields $d(u, \pi(u) + A(\pi(u))) = 0$. Using Lemma 6.19 and Definition 6.20, we get

$$\begin{aligned}\int_{B(x,t)\setminus G} & d(u, \pi(u) + A(\pi(u)))\mathrm{d}\mu(u) \\ & \leq \int_{B(x,t)\setminus(G\cup\mathcal{Z})} \sum_{j \in I_{12}} \phi_j(\pi(u)) \left|\pi^\perp(u) - A_j(\pi(u))\right| \mathrm{d}\mu(u).\end{aligned}$$

Using I to V we obtain

$$\int_{B(x,t)\setminus G} d(u, \pi(u) + A(\pi(u)))\mathrm{d}\mu(u) \leq 2 \sum_{i \in I(x,t)} \sum_{j \in I_{12}} \phi_{i,j} t_j^{n+1} \frac{1}{t_j^n} \int_{kB_j} \frac{d(u, P_j)}{t_j} \mathrm{d}\mu(u).$$

Now we get the statement by using the following five results.

VI. Lemma 6.21 and the definition of S_{total} imply $\beta_{1;k}^{P_j}(B_j) \leq 2\varepsilon$.

VII. Let $i \in I(x,t)$ and $j \in I_{12}$. If $\phi_{i,j} \neq 0$, we conclude that $3R_i \cap 3R_j \neq \emptyset$. Hence, with Lemma 6.13(iii) and Lemma 6.14, we deduce that $2t_j = \operatorname{diam} B_j \leq 1000 \operatorname{diam} R_i$.

VIII. For $i \in I(x,t)$, we have with Lemma 6.13(iv) that $\sum_{j \in I_{12}} \phi_{i,j} \leq (180)^n$.

IX. For $i \in I(x,t)$, there exists some $y \in B(x,t) \cap (F \setminus G)$ with $\pi(y) \in 3R_i$. We obtain with Lemma 6.13, Lemma 8.2 and our assumption $t \geq \frac{d(x)}{10}$ that $10 \operatorname{diam} R_i \leq d(x) + d(x,y) \leq 11t$.

X. Let $i \in I(x,t)$. With IX we obtain $\operatorname{diam} R_i < 2t$ and, because $(3R_i \times P_0^{\perp}) \cap B(x,t) \neq \emptyset$, we get $R_i \subset B(\pi(x), t + \operatorname{diam} 3R_i) \cap P_0 \subset B(\pi(x), 7t) \cap P_0$. Moreover, with Lemma 6.13(ii), the primitive cells R_i have disjoint interior and hence we get with Lemma A.4 (we recall that ω_n denotes the volume of the n-dimensional unit sphere)

$$\sum_{i \in I(x,t)} (\operatorname{diam} R_i)^n \leq \sqrt{n}^n \mathcal{H}^n(B(\pi(x), 7t) \cap P_0) = \sqrt{n}^n \omega_n (7t)^n. \qquad \square$$

Definition 8.4. We define $\tilde{F} := \left\{ x \in F \setminus G \mid d(x, \pi(x) + A(\pi(x))) \leq \varepsilon^{\frac{1}{2}} d(x) \right\}$.

Theorem 8.5. *Let $0 < \alpha \leq \frac{1}{4}$. There exist some $\hat{\varepsilon} = \hat{\varepsilon}(N, n, C_0) \leq \frac{1}{4}$ and some $\tilde{k} \geq 4$ so that if $\eta < 2\hat{\varepsilon}$ and $k \geq \tilde{k}$, there exists some constant $C_5 = C_5(N, n, \mathcal{K}, p, C_0)$ so that, for all $\varepsilon \in [\frac{\eta}{2}, \hat{\varepsilon})$, we have $\mu(F \setminus \tilde{F}) \leq C_5 \varepsilon^{\frac{1}{2}}$.*

Proof. Let $0 < \alpha \leq \frac{1}{4}$. We choose some ε with $\eta \leq 2\varepsilon < 2\hat{\varepsilon} := \min\{2\tilde{\varepsilon}, 2\bar{\varepsilon}, \frac{1}{2}\}$ and some $k \geq \tilde{k}$ where $\tilde{\varepsilon}$ is given by Lemma 8.1 and $\bar{\varepsilon}$ and $\tilde{k}$ are given by Lemma 8.3.

At first, we prove some intermediate results:

I. We have $\mathcal{Z} \subset \tilde{F}$ because for $x \in \mathcal{Z}$ the definition of A on $\mathcal{Z}$ (see Definition 6.20) implies that $d(x, \pi(x) + A(\pi(x))) = d(x, x) = 0$.

II. If $x \in F \setminus (\tilde{F} \cup G)$, we conclude with I that $x \notin \mathcal{Z}$ and, with Lemma 6.10, we deduce that $d(x) \neq 0$. So $\mathcal{G} = \left\{ B\left(x, \frac{d(x)}{10}\right) \middle| x \in F \setminus (\tilde{F} \cup G) \right\}$ is a set of nondegenerate balls. For $x \in F \subset B(0,5)$, we have $d(x) \leq 60$ (see Lemma 6.9) so that we can apply Besicovitch's covering theorem [7, 1.5.2, Thm. 2] to $\mathcal{G}$ and get $N_0 = N_0(N)$ families $\mathcal{B}_m \subset \mathcal{G}, m = 1, \ldots, N_0$, of disjoint balls with $F \setminus (\tilde{F} \cup G) \subset \bigcup_{m=1}^{N_0} \bigcup_{B \in \mathcal{B}_m} B$.

III. Since d is 1-Lipschitz (Lemma 6.8), for all $u \in B\big(x, \frac{d(x)}{10}\big)$, $d(x) - d(u) \leq d(x,u) \leq \frac{d(x)}{10}$ and hence $\frac{1}{d(u)} \leq \frac{10}{9} \frac{1}{d(x)} < \frac{2}{d(x)}$.

IV. Let $1 \leq m \leq N_0$ and let $B_x = B(x, \frac{d(x)}{10})$ and $B_y = B(y, \frac{d(y)}{10})$ be two balls in $\mathcal{B}_m$. Then we either have

a) $\pi\left(\frac{1}{40K} B_x\right) \cap \pi\left(\frac{1}{40K} B_y\right) = \emptyset$ or

b) if $2d(x) \geq d(y)$, then $B_y \subset 200 B_x$ and $\operatorname{diam} B_y > (40K)^{-1} \operatorname{diam} B_x$,

where K is the constant from page 1236.

Proof of I–IV. Let $\pi\left(\frac{1}{40K} B_x\right) \cap \pi\big(\frac{1}{40K} B_y\big) \neq \emptyset$ and $2d(x) \geq d(y)$. We deduce with Lemma 6.11 $d(x,y) < 19 d(x)$, which implies $B_y \subset B\big(x, 19d(x) + \frac{d(y)}{10}\big) = 200 B_x$. With Lemma 8.2, we get $\frac{d(x)}{20K} \leq D(\pi(y)) + d(\pi(x), \pi(y)) < d(y) + \frac{d(x)}{40K}$, and hence $d(y) > (40K)^{-1} d(x)$. All in all, we have proven that either case a) or case b) is true. $\square$

V. There exists some constant $C = C(n)$ so that $\sum_{B \in \mathcal{B}_m} (\operatorname{diam} B)^n \leq C$ for all $1 \leq m \leq N_0$.

Proof of V. Let $1 \leq m \leq N_0$. We recursively construct for every m some sequence of numbers, some sequence of balls and some sequence of sets. At first, we define the initial elements. Let $d_m^1 := \sup_{B \in \mathcal{B}_m} \operatorname{diam} B$. We have $d_m^1 < \infty$ because, for all $x \in F \subset B(0,5)$, we have with Lemma 6.9 that $d(x) \leq 60$. Now we choose

$B_m^1 \in \mathcal{B}_m$ with $\operatorname{diam} B_m^1 \geq \frac{d_m^1}{2}$ and define

$$\mathcal{B}_m^1 := \left\{ B \in \mathcal{B}_m \middle| \pi\left(\tfrac{1}{40K} B_m^1\right) \cap \pi\left(\tfrac{1}{40K} B\right) \neq \emptyset \right\}.$$

We continue these sequences recursively. We set $d_m^{i+1} = \sup_{B' \in \mathcal{B}_m \setminus \bigcup_{j=1}^i \mathcal{B}_m^j} \operatorname{diam} B'$, choose $B_m^{i+1} \in \mathcal{B}_m \setminus \bigcup_{j=1}^i \mathcal{B}_m^j$ with $\operatorname{diam} B_m^{i+1} \geq \frac{d_m^{i+1}}{2}$ and define

$$\mathcal{B}_m^{i+1} := \left\{ B \in \mathcal{B}_m \setminus \bigcup_{j=1}^{i} \mathcal{B}_m^j \middle| \pi\left(\tfrac{1}{40K} B_m^{i+1}\right) \cap \pi\left(\tfrac{1}{40K} B\right) \neq \emptyset \right\}.$$

If there exists some $l \in \mathbb{N}$ so that eventually $\mathcal{B}_m \setminus \bigcup_{j=1}^l \mathcal{B}_m^j = \emptyset$, we set $\mathcal{B}_m^i := \emptyset$ for all $i \geq l$ and interrupt the sequences (d_m^i) and (B_m^i). We have the following results:

(i) For all $l \in \mathbb{N}$ and $B_m^l = B\big(x_m^l, \frac{d(x_m^l)}{10}\big)$, we have with Lemma 6.9 and $x_m^l \in F \subset B(0,5)$ that $\frac{d(x_m^l)}{10} \leq 6$. Hence we get $B_m^l \subset B(0,11)$.

(ii) For all $1 \leq m \leq N_0$, we have $\bigcup_{i=1}^\infty \mathcal{B}_m^i = \mathcal{B}_m$.

Proof of (i) *and* (ii). If there exist only finitely many d_m^l, the construction implies $\mathcal{B}_m \subset \bigcup_{j=1}^\infty \mathcal{B}_m^j$.
Now we assume that there exist infinitely many d_m^l. Since $\mathcal{B}_m$ is a family of disjoint balls, the set $\{B_m^l | l \in \mathbb{N}\}$ is also a family of disjoint balls. Due to (i), all of those balls are contained in $B(0,11)$. If there exists some $c > 0$ with $\operatorname{diam} B_m^l > c$ for all $l \in \mathbb{N}$, there cannot be infinitely many such balls. Hence we deduce that $\operatorname{diam} B_m^l \to 0$ if $l \to \infty$. Let $B \in \mathcal{B}_m$. If $B \notin \bigcup_{i=1}^\infty \mathcal{B}_m^i$, we obtain $2 \operatorname{diam} B_m^l \geq d_m^l \geq \operatorname{diam} B$ for all $l \in \mathbb{N}$ where we used the definition of d_m^l. This is in contradiction to $\operatorname{diam} B_m^l \to 0$. So we get $B \in \bigcup_{i=1}^\infty \mathcal{B}_m^i$. All in all, we have proven $\bigcup_{i=1}^\infty \mathcal{B}_m^i \supset \mathcal{B}_m$. The inverse inclusion follows by definition of $\mathcal{B}_m^i$. $\square$

(iii) Let $1 \leq m \leq N_0$, $l \in \mathbb{N}$ and $B_y = B\left(y, \frac{d(y)}{10}\right) \in \mathcal{B}_m^l$, $B_m^l = B\left(x_m^l, \frac{d(x_m^l)}{10}\right) \in \mathcal{B}_m^l$. We have $\pi\left(\frac{1}{40K} B_m^l\right) \cap \pi\left(\frac{1}{40K} B_y\right) \neq \emptyset$ and $2d(x_m^l) = 10 \operatorname{diam} B_m^l \geq 10 \frac{d_m^l}{2} \geq 10 \frac{\operatorname{diam} B_y}{2} = d(y)$. Hence IV implies $B_y \subset 200 B_m^l$ and $\operatorname{diam} B_y > (40K)^{-1} \operatorname{diam} B_m^l$. The balls in $\mathcal{B}_m^l$ are disjoint, so, with Lemma A.1 ($s = \frac{\operatorname{diam} B_m^l}{80K}$, $r = 200 \frac{\operatorname{diam} B_m^l}{2}$), we deduce that $\# \mathcal{B}_m^l \leq (200 \cdot 80K)^N$.

(iv) $\{\frac{1}{40K} B_m^l\}_{l \in \mathbb{N}}$ is a family of disjoint balls, and with (i) we get $\pi\left(\frac{1}{40K} B_m^l\right) \subset \pi(B(0,11))$ for all $l \in \mathbb{N}$. Hence we obtain

$$\sum_{l=1}^\infty \left(\operatorname{diam} \pi\left(\tfrac{1}{40K} B_m^l\right)\right)^n \leq \frac{2^n}{\omega_n} \mathcal{H}^n\left(\pi\left(B(0,11)\right)\right) = 22^n.$$

Now we are able to prove V by using (ii), (iii) and (iv):

$$\sum_{B \in \mathcal{B}_m} (\operatorname{diam} B)^n \leq \sum_{l=1}^\infty \sum_{B \in \mathcal{B}_m^l} \left(d_m^l\right)^n = C(n) \sum_{l=1}^\infty \left(\operatorname{diam} \pi\left(\tfrac{1}{40K} B_m^l\right)\right)^n \leq C(n). \quad \square$$

Finally, we can finish the proof of Theorem 8.5. Let p_B denote the centre of some ball B. Using the definition of $\tilde{F}$ and Lemma 8.3, there exists some constant

$C = C(n)$ so that we obtain

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\mu(F\setminus(\tilde{F}\cup G)) &< \int_{F\setminus(\tilde{F}\cup G)} \frac{d(u,\pi(u)+A(\pi(u)))}{d(u)}\mathrm{d}\mu(u)\\ &\overset{\mathrm{II}}{\leq} \sum_{m=1}^{N_0}\sum_{B\in\mathcal{B}_m}\int_{B\setminus(\tilde{F}\cup G)} \frac{d(u,\pi(u)+A(\pi(u)))}{d(u)}\mathrm{d}\mu(u)\\ &\overset{\mathrm{III}}{<} \sum_{m=1}^{N_0}\sum_{B\in\mathcal{B}_m}\frac{2}{d(p_B)}C\varepsilon\left(\frac{\operatorname{diam} B}{2}\right)^{n+1} \overset{\mathrm{V}}{\leq} C(N,n)\varepsilon.\end{aligned}$$

This leads to $\mu(F\setminus(\tilde{F}\cup G)) \leq C(N,n)\varepsilon^{\frac{1}{2}}$. With $\eta < 2\varepsilon \leq \varepsilon^{\frac{1}{2}}$ and Lemma 8.1 the assertion holds. $\square$

8.2. F_1 **is small.** Now we are able to estimate $\mu(F_1)$. We recall that η and k are fixed constants (cf. the first lines of section 6.1) and that F_1 depends on the choice of $\alpha, \varepsilon > 0$ (cf. Definition 6.3).

Theorem 8.6. *Let* $0 < \alpha \leq \frac{1}{4}$. *There exist some* $\varepsilon^* = \varepsilon^*(N,n,C_0)$ *and some* $\tilde{k} \geq 4$ *so that if* $\eta < 2\varepsilon^*$ *and* $k \geq \tilde{k}$, *for all* $\varepsilon \in [\frac{\eta}{2}, \varepsilon^*)$, *we have* $\mu(F_1) < 10^{-6}$.

Proof. Let $0 < \alpha \leq \frac{1}{4}$ and let $\hat{\varepsilon}$, C_5 and $\tilde{k}$ be the constants given by Theorem 8.5. We set $\varepsilon^* := \min\left\{\hat{\varepsilon}, \frac{10^{-14}}{C_5^2}\right\}$ and choose some $k \geq \tilde{k}$ and some $\varepsilon \in [\frac{\eta}{2}, \varepsilon^*)$. First, we prove some intermediate results:

I. Let $\mathcal{G} = \left\{B\big(x, \frac{h(x)}{10}\big)\middle| x \in F_1 \cap \tilde{F}\right\}$. This is a set of nondegenerate balls because $\mathcal{Z}\cap F_1 = \emptyset$ and, by definition of $h(\cdot)$ (see page 1208), we get $h(x) \leq 50$ for all $x \in F$. With Besicovitch's covering theorem [7, 1.5.2, Thm. 2], there exist $N_0 = N_0(N)$ families $\mathcal{B}_m \subset \mathcal{G}$, $m = 1, \dots, N_0$, containing countably many disjoint balls with $F_1 \cap \tilde{F} \subset \bigcup_{m=1}^{N_0}\bigcup_{B\in\mathcal{B}_m} B$.

II. Let $1 \leq m \leq N_0$ and $B = B\big(x, \frac{h(x)}{10}\big)$ where $x \in F_1 \cap \tilde{F}$. Due to the definition of F_1, there exist some $y \in F$ and some $\tau \in [\frac{h(x)}{5}, \frac{h(x)}{2}]$ with $d(x,y) \leq \frac{\tau}{2}$ and $\delta(B(y,\tau)) \leq \delta$. For any $z \in B$, we get $d(z,y) \leq \frac{h(x)}{10} + \frac{\tau}{2} \leq \tau$. Hence we obtain $B \subset B(y,\tau)$ and conclude that $\mu(B) \leq \delta\tau^n < 3^n\delta(\operatorname{diam} B)^n$.

III. For all $1 \leq m \leq N_0$, we have $\sum_{B\in\mathcal{B}_m}(\operatorname{diam} B)^n \leq 192^n$.

Proof of I–III. We define the function $\tilde{A} : U_{12} \to \mathbb{R}^N, u \mapsto u + A(u)$, where $U_{12} = B(0,12)\cap P_0$. $\tilde{A}$ is Lipschitz continuous with Lipschitz constant less than 2 because A is defined on U_{12} (see page 1216), 3α-Lipschitz continuous (see Lemma 6.27) and $\alpha \leq \frac{1}{4}$. Let $B = B\left(x, \frac{h(x)}{10}\right) \in \mathcal{B}_m$. We have $F \subset B(0,5)$ (see (A) on page 1208), and so $\pi(F) \subset P_0 \cap B(0,5)$ because π is the orthogonal projection on P_0 and $0 \in P_0$. With the definition of $\tilde{F}$, Lemma 6.10 and $\varepsilon^{\frac{1}{2}} < \frac{1}{20}$, we obtain $d(x,x_0) < \frac{h(x)}{20}$ where $x_0 := \tilde{A}(\pi(x))$. Let $z \in \pi\left(B\left(x_0, \frac{h(x)}{40}\right)\right) \subset U_{12}$. Using the triangle inequality with the point $\tilde{A}(\pi(x_0)) = x_0$ and where $\tilde{A}$ is 2-Lipschitz, we get $d(\tilde{A}(z), x) \leq \frac{h(x)}{10}$. This implies $\tilde{A}(\pi(B(x_0, \frac{h(x)}{40}))) \subset B \cap \tilde{A}(U_{12})$, and hence we gain

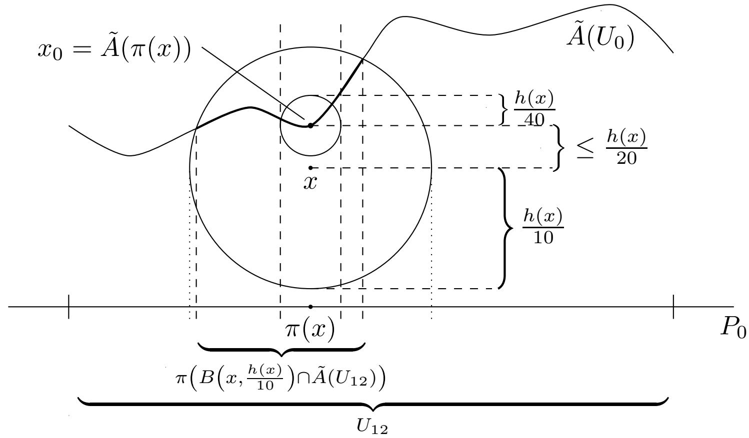

FIGURE 2. $\pi\left(B\left(x_0, \frac{h(x)}{40}\right)\right) \subset \pi\left(B\left(x, \frac{h(x)}{10}\right) \cap \tilde{A}(U_{12})\right)$

$\pi\left(B\left(x_0, \frac{h(x)}{40}\right)\right) \subset \pi\left(B \cap \tilde{A}(U_{12})\right)$ (see Figure 2). Now we have with [7, 2.4.1, Thm. 1]

$$\text{(8.4)} \quad \frac{\omega_n}{8^n}(\operatorname{diam} B)^n = \omega_n\left(\frac{h(x)}{40}\right)^n = \mathcal{H}^n\left(\pi\left(B\left(x_0, \frac{h(x)}{40}\right)\right)\right) \le \mathcal{H}^n(B \cap \tilde{A}(U_{12})).$$

The balls in $\mathcal{B}_m$ are disjoint, so we conclude using [7, 2.4.1, Thm. 1] for the last estimate

$$\sum_{B\in\mathcal{B}_m}(\operatorname{diam} B)^n \overset{(8.4)}{\le} \frac{8^n}{\omega_n}\sum_{B\in\mathcal{B}_m}\mathcal{H}^n(B\cap\tilde{A}(U_{12})) \le \frac{8^n}{\omega_n}\mathcal{H}^n(\tilde{A}(U_{12})) \le 192^n. \qquad \square$$

Now we have $\mu(F_1 \cap \tilde{F}) \overset{\mathrm{I}}{\le} \sum_{m=1}^{N_0}\sum_{B\in\mathcal{B}_m}\mu(B) \overset{\mathrm{II,\,III}}{\le} \delta N_0 \cdot 576^n$. Since $\delta \le \frac{10^{-10}}{600^n N_0}$ (see (6.1)) and $\varepsilon^{\frac{1}{2}} < \frac{10^{-7}}{C_5}$, we deduce together with Theorem 8.5 that $\mu(F_1) < 10^{-6}$. $\square$

8.3. F_2 **is small.** We recall that $0 < \eta \le 2^{-(n+1)}$ and $k \ge 1$ are fixed constants (cf. the first lines of section 6.1) and that F_2 depends on the choice of $\alpha, \varepsilon > 0$ (cf. Definition 6.3).

Theorem 8.7. *Let $\alpha, \varepsilon > 0$. There exists some constant $C = C(N, n, \mathcal{K}, p, C_0, k)$ so that if $\eta \le \frac{\varepsilon^p}{C}10^{-6}$, we have $\mu(F_2) \le 10^{-6}$.*

Proof. Let $x \in F_2$ and $t \in (h(x), 2h(x))$. It follows that $x \notin F_1 \cup \mathcal{Z}$, and hence, for all $y \in F$ and for all $\tau \in \left[\frac{h(x)}{5}, \frac{h(x)}{2}\right]$ with $d(x,y) \le \frac{\tau}{2}$, we obtain $\delta(B(y,\tau)) > \delta$. So, in particular, we get $\delta\big(B\big(x, \frac{h(x)}{2}\big)\big) > \delta$ for $x = y$ and $\tau = \frac{h(x)}{2}$. If $k_0 = 1$, this implies $\tilde{\delta}_{k_0}(B(x,t)) \ge \delta(B(x,t)) > \frac{\delta}{4^n}$, where we used $\frac{h(x)}{2} < t < 2h(x)$. Let (y,τ) be as in the definition of F_2. Then we have $d(x,y) + \tau < 2\tau \le h(x) < t$ and hence $B(y,\tau) \subset B(x,t)$. We conclude that $\beta_{1;k}(x,t) \ge \left(\frac{\tau}{t}\right)^{n+1}\beta_{1;k}(y,\tau) \ge \frac{\varepsilon}{10^{n+1}}$. Now, with Corollary 4.8 ($\lambda = \frac{\delta}{4^n}$, $k_0 = 1$), there exists some constant

$C = C(N, n, \mathcal{K}, p, C_0, k)$ so that

$$\begin{aligned}\mathcal{M}_{\mathcal{K}^p}(\mu) &\geq \frac{1}{C}\int_{F_2}\int_{h(x)}^{2h(x)} \beta_{1;k}(x,t)^p \chi_{\{\tilde{\delta}_{k_0}(B(x,t))\geq \frac{\delta}{4^n}\}} \frac{\mathrm{d}t}{t}\mathrm{d}\mu(x) \\ &\geq \frac{1}{C}\int_{F_2}\int_{h(x)}^{2h(x)} \left(\frac{\varepsilon}{10^{n+1}}\right)^p \frac{\mathrm{d}t}{t}\mathrm{d}\mu(x) \geq \frac{1}{C}\left(\frac{\varepsilon}{10^{n+1}}\right)^p \mu(F_2)\ln(2).\end{aligned}$$

Finally, using the previous inequality, condition (C) from page 1208 and $\eta \leq \frac{\ln(2)}{10^{p(n+1)}C}\varepsilon^p 10^{-6}$, we get the assertion. ☐

8.4. F_3 **is small.** We mention for review that $\tilde{F}$ is defined on page 1240 and set

$$\tilde{\tilde{F}} := \left\{ x \in \tilde{F} \;\middle|\; \mu(\tilde{F}\cap B(x,t)) \geq \frac{99}{100}\mu(F\cap B(x,t)) \text{ for all } t\in(0,2)\right\}.$$

Lemma 8.8. *Let $0 < \alpha \leq \frac{1}{4}$. There exists some $\hat{\varepsilon} = \hat{\varepsilon}(N, n, C_0) \leq \frac{1}{4}$ and some $\tilde{k} \geq 4$ so that if $\eta < 2\hat{\varepsilon}$ and $k \geq \tilde{k}$, there exists some constant $C = C(N, n, \mathcal{K}, p, C_0)$ so that, for all $\varepsilon \in [\frac{\eta}{2}, \hat{\varepsilon})$, we have $\mu(F\setminus\tilde{\tilde{F}}) \leq C\varepsilon^{\frac{1}{2}}$.*

Proof. Let $0 < \alpha \leq \frac{1}{4}$ and choose $\hat{\varepsilon}$, $\tilde{k}$ to be the constants given by Theorem 8.5 and let $k \geq \tilde{k}$, $\eta \leq 2\varepsilon < 2\hat{\varepsilon}$. Due to Theorem 8.5, we only have to consider $\mu(\tilde{F}\setminus\tilde{\tilde{F}})$. For all $x \in \tilde{F}\setminus\tilde{\tilde{F}}$ using the definition of $\tilde{F}$, there exists some $t_x \in (0,2)$ with $\mu(\tilde{F}\cap B(x,t_x)) \leq 99\mu((F\setminus\tilde{F})\cap B(x,t_x))$. Hence $\tilde{F}\setminus\tilde{\tilde{F}}$ is covered by balls $B(x,t_x)$ with centre in $\tilde{F}\setminus\tilde{\tilde{F}}$. So with Besicovitch's covering theorem [7, 1.5.2, Thm. 2] there exist $N_0 = N_0(N)$ families $\mathcal{B}_m$, $m = 1,\dots,N_0$, of disjoint balls $B(x,t_x)$ so that

$$\mu(\tilde{F}\setminus\tilde{\tilde{F}}) \leq \sum_{m=1}^{N_0}\sum_{B\in\mathcal{B}_m}\mu(\tilde{F}\cap B) \leq 99\sum_{m=1}^{N_0}\sum_{B\in\mathcal{B}_m}\mu((F\setminus\tilde{F})\cap B) \leq 99N_0\,\mu(F\setminus\tilde{F}),$$

and with Theorem 8.5 the assertion holds. ☐

Lemma 8.9. *Let $\theta, \alpha > 0$. There exist some constant $C = C(N, n, C_0, \theta) > 1$ and some constant $\varepsilon_0 = \varepsilon_0(N, n, C_0, \theta) > 0$ so that if $\eta < 2\varepsilon_0$ and $k \geq 4$, we have for all $\varepsilon \in [\frac{\eta}{2}, \varepsilon_0)$ that the following is true. If $(x,t) \in S$ and $100t \geq \theta$, then we have $\sphericalangle(P_{(x,t)}, P_0) \leq C\varepsilon$.*

Proof. Let $\theta, \alpha > 0$, $k \geq 4$ and $\eta < 2\varepsilon < 2\varepsilon_0$ where the constant ε_0 is given by Lemma 4.9. Let $t \geq \frac{\theta}{100}$ and $(x,t) \in S$. We get with (A) and (D) (see page 1208) $\beta_{1;k}^{P_0}(x,t) \leq \left(\frac{500}{\theta}\right)^{n+1} 2\varepsilon$. Furthermore, we have with Definition 6.1 that $\beta_{1;k}^{P_{(x,t)}}(x,t) \leq 2\varepsilon$ and with $(x,t) \in S \subset S_{total}$ we obtain $\delta(B(x,t)) \geq \frac{\delta}{2}$. Now, with Lemma 4.9 ($y = x$, $c = 1$, $\xi = 2\left(\frac{500}{\theta}\right)^{n+1}$, $t_x = t_y = t$, $\lambda = \frac{\delta}{2}$), there exists some constant $C_3 = C_3(N, n, C_0, \theta)$ so that $\sphericalangle(P_{(x,t)}, P_0) \leq C_3\varepsilon$. ☐

Lemma 8.10. *Let $\theta, \alpha > 0$. If $k \geq 400$, there exists some constant $\varepsilon^* = \varepsilon^*(N, n, C_0, \alpha, \theta)$ so that if $\eta < 2\varepsilon^*$, we have for all $\varepsilon \in [\frac{\eta}{2}, \varepsilon^*)$ that for all $x \in F_3$ we have $h(x) < \frac{\theta}{100}$.*

Proof. Let $\theta, \alpha > 0$ and $k \geq 400$. We set $\varepsilon^* := \min\{\bar{\varepsilon}, \varepsilon_0, \frac{\alpha}{2C}\}$ where $\bar{\varepsilon}$ is given by Lemma 6.5 and ε_0 as well as C are given by Lemma 8.9. Let $\eta \leq 2\varepsilon < 2\varepsilon^*$ and

$x \in F_3$. With Lemma 6.2(i), we have $(x, h(x)) \in S$ and, with Lemma 6.5, we get $\sphericalangle(P_{(x,h(x))}, P_0) > \frac{1}{2}\alpha$. Hence we obtain $h(x) < \frac{\theta}{100}$ with Lemma 8.9. □

Lemma 8.11. *Let $p = 2$. There exist some $\hat{k} \geq 400$, some $\tilde{\alpha} = \tilde{\alpha}(n) > 0$ and some $\hat{\theta} = \hat{\theta}(N, n, C_0) \in (0, 1)$ so that for all $\alpha \in (0, \tilde{\alpha}]$ and $\theta \in (0, \hat{\theta}]$ there exists some $\hat{\varepsilon} = \hat{\varepsilon}(N, n, C_0, \alpha, \theta)$ so that if $k \geq \hat{k}$ and $\eta < \hat{\varepsilon}^2$, we have for all $\varepsilon \in [\sqrt{\eta}, \hat{\varepsilon})$ that there exist some set $H_\theta \subset U_6$ and some constant $C = C(N, n, \mathcal{K}, C_0, k)$ with $\mathcal{H}^n(U_6 \setminus H_\theta) < C\left(\frac{\varepsilon}{\theta^{n+1}\alpha}\right)^2$ and, for all $x \in F_3 \cap \tilde{\tilde{F}}$, we have $d(\pi(x), H_\theta) > h(x)$.*

Proof. Let $\tilde{k}$ and $\tilde{\alpha}(n)$ be the thresholds given by Theorem 7.17 and let $\hat{C} = \hat{C}(N, n)$ be the constant given by Theorem 7.3. Moreover, let $C_1 = C_1(N, n, C_0)$ and $C_2 = C_2(N, n, C_0)$ be the constants given by Corollary 4.3 applied with $\lambda = \frac{\delta}{4}$, and let $\delta = \delta(N, n)$ be the value fixed on page 1208. We set $\hat{\theta} := \frac{1}{400}\left[18n(10^n + 1)\left(\frac{C_1}{4}\right)^{n+1}\hat{C}\right]^{-1}$ and choose $\theta \in (0, \hat{\theta}]$. Let $\alpha \in (0, \tilde{\alpha}]$, and let $\bar{\varepsilon}_1 = \bar{\varepsilon}(N, n, C_0, \alpha)$, $\bar{\varepsilon}_2 = \bar{\varepsilon}(N, n, C_0, \alpha)$, $\tilde{\varepsilon} = \tilde{\varepsilon}(N, n, C_0, \alpha)$, $\varepsilon_0 = \varepsilon_0(N, n, C_0, \theta)$, and $\varepsilon^* = \varepsilon^*(N, n, C_0, \alpha, \theta)$ be the thresholds given by Lemmas 6.5, 6.24, Theorem 7.17, Lemma 8.9 and Lemma 8.10 respectively. Finally, let C be the constant from Lemma 8.9. We set

$$\hat{\varepsilon} := \min\left\{\bar{\varepsilon}_1, \bar{\varepsilon}_2, \tilde{\varepsilon}, \varepsilon_0, \varepsilon^*, (\hat{C}\theta\alpha)^2, \frac{\alpha}{400}\left[4n(10^n + 1)\left(\frac{C_1}{4}\right)^{n+1} 2C_2\right]^{-1}, \frac{\alpha}{100C}\right\}$$

and assume that $k \geq \hat{k} := \max\{\tilde{k}, 400\}$ and $\eta \leq \hat{\varepsilon}^2$. Now let $\varepsilon > 0$ with $\eta \leq \varepsilon^2 < \hat{\varepsilon}^2$.

Until now, we defined the map A only on $U_{12} = B(0, 12) \cap P_0$ (see page 1216). Furthermore, we have shown that A is Lipschitz continuous with Lipschitz constant 3α (see Lemma 6.27). With Lemma A.5, an application of Kirszbraun's theorem, there exists an extension $\tilde{A} : P_0 \to \mathbb{R}^N$ of A with compact support, the same Lipschitz constant 3α and $A = \tilde{A}$ on U_{12}. If one wants to omit Zorn's lemma, used for the proof of Lemma A.5, one can get the same result with a slightly larger Lipschitz constant (see the remark after Lemma A.5 for details). We denote this extension of A also by A.

Using Theorem 7.3 with $g = A$, $p = 2$ and Theorem 7.17, there exist some set $H_\theta \subset U_6$ and some constant $C = C(N, n, \mathcal{K}, C_0, k)$ with $\mathcal{H}^n(U_6 \setminus H_\theta) \leq \frac{C(n)}{\theta^{2(n+1)}\operatorname{Lip}_A^2}C\varepsilon^2$. Furthermore, we get for all $y \in P_0$ some affine map $a_y : P_0 \to P_0^\perp$ so that if $r \leq \theta$ and $B(y, r) \cap H_\theta \neq \emptyset$, we have $\|A - a_y\|_{L^\infty(B(y,r)\cap P_0, P_0^\perp)} \leq \hat{C}r\theta \operatorname{Lip}_A$. We recall that $\operatorname{Lip}_A = 3\alpha$ (cf. Lemma 6.27). For $x \in F_3 \cap \tilde{\tilde{F}} \subset F_3 \cap \tilde{F}$, we have with the previous lemma that $h(x) < \frac{\theta}{100}$. Let $t \in [h(x), \frac{\theta}{100}]$. If $x' \in B(x, 2t) \cap \tilde{F}$, we obtain with Lemma 6.10 and the definition of $\tilde{F}$: $d(x', \pi(x') + A(\pi(x'))) \leq \varepsilon^{\frac{1}{2}}(d(x) + d(x, x')) \leq 3\varepsilon^{\frac{1}{2}}t$. Let $P_{\pi(x)}$ denote the n-dimensional plane, which is the graph of the affine map $a_{\pi(x)}$. Now we assume, contrary to the statement of this lemma, that $d(\pi(x), H_\theta) \leq h(x)$. This implies $\pi(B(x, 2t)) \cap H_\theta \neq \emptyset$, and so we have $d(\pi(x') + A(\pi(x')), P_{\pi(x)}) \leq \|A - a_{\pi(x)}\|_{L^\infty(B(\pi(x),2t)\cap P_0, P_0^\perp)} \leq 6\hat{C}\theta\alpha t$ for all $x' \in B(x, 2t) \cap \tilde{F}$. We combine those estimates and obtain, using $3\varepsilon^{\frac{1}{2}} \leq 3\hat{C}\theta\alpha$,

$$d(x', P_{\pi(x)}) \leq d(x', \pi(x') + A(\pi(x'))) + d(\pi(x') + A(\pi(x')), P_{\pi(x)}) \leq 9\hat{C}\theta\alpha t. \tag{8.5}$$

Since $h(x) \leq t$, we get $(x,t) \in S \subset S_{total}$ with Lemma 6.2(i) so that we have $\delta(B(x,t)) \geq \frac{\delta}{2}$. If $x \in \tilde{\tilde{F}}$, this estimate and the definition of $\tilde{\tilde{F}}$ imply $\delta(\tilde{F} \cap B(x,t)) > \frac{1}{4}\delta$.

Now we apply Corollary 4.3 ($\Upsilon = \tilde{F}$, $\lambda = \frac{\delta}{4}$), and so there exist constants $C_1(N,n,C_0)$, $C_2(N,n,C_0)$ and an $(n, 10n\frac{t}{C_1})$-simplex $T = \Delta(x_0,\dots,x_n) \in F \cap B(x,t) \cap \tilde{F}$ so that $\mu(\tilde{B}_i) \geq \frac{t^n}{C_2}$ for all $i \in \{0,\dots,n\}$ where $\tilde{B}_i := B\left(x_i, \frac{t}{C_1}\right) \cap B(x,t) \cap \tilde{F}$. With $(x,t) \in S \subset S_{total}$, we get for all $i \in \{0,\dots,n\}$,

$$\frac{1}{\mu(\tilde{B}_i)} \int_{\tilde{B}_i} d(z, P_{(x,t)}) \mathrm{d}\mu(z) \leq C_2 t \beta_{1;k}^{P_{(x,t)}}(x,t) \leq 2C_2 t\varepsilon.$$

This implies for all $i \in \{0,\dots,n\}$ the existence of $y_i \in \tilde{B}_i$ with $d(y_i, P_{(x,t)}) \leq 2C_2 t\varepsilon$. With Lemma 2.8, we deduce that $S := \Delta(y_0,\dots,y_n) \subset B(x,t)$ is an $(n, 8n\frac{t}{C_1})$-simplex. Next, we apply Lemma 2.17 ($m = n$, $C = \frac{C_1}{8n}$, $\hat{C} = 1$, $\sigma = 2C_2\varepsilon$) and get $\sphericalangle(P_{(x,t)}, P_{y_0,\dots,y_n}) \leq \frac{\alpha}{400}$. We have $y_i \in \tilde{B}_i \subset B(x,2t) \cap \tilde{F}$, and hence we get with (8.5) and Lemma 2.17 ($C = \frac{C_1}{8n}$, $\hat{C} = 1$, $\sigma = 9\hat{C}\theta\alpha$) $\sphericalangle(P_{y_0,\dots,y_n}, P_{\pi(x)}) \leq \frac{\alpha}{400}$. We combine those two angle estimates and conclude that $\sphericalangle(P_{(x,t)}, P_{\pi(x)}) \leq \frac{\alpha}{200}$, which is true for all $x \in F_3 \cap \tilde{\tilde{F}}$ with $d(\pi(x), H_\theta) \leq h(x)$ and all $t \in [h(x), \frac{\theta}{100}]$. Now we use this result for $t = h(x)$ and for $t = \frac{\theta}{100}$ and obtain $\sphericalangle(P_{(x,h(x))}, P_{(x,\frac{\theta}{100})}) \leq \frac{\alpha}{100}$. Together with Lemma 8.9 we get $\sphericalangle(P_{(x,h(x))}, P_0) \leq \frac{\alpha}{50}$. This is in contradiction to Lemma 6.5; hence our assumption that $d(\pi(x), H_\theta) \leq h(x)$ is invalid for all $x \in F_3 \cap \tilde{\tilde{F}}$. $\square$

Theorem 8.12. *Let $p = 2$. There exist some constants $\bar{\bar{k}} \geq 4$, $0 < \bar{\bar{\alpha}} = \bar{\bar{\alpha}}(n) < \frac{1}{6}$ and $0 < \bar{\bar{\theta}} = \bar{\bar{\theta}}(N,n,C_0)$ so that, for all $\alpha \in (0, \bar{\bar{\alpha}}]$ and all $\theta \in (0, \bar{\bar{\theta}}]$, there exists some $0 < \bar{\bar{\varepsilon}} = \bar{\bar{\varepsilon}}(N,n,C_0,\alpha,\theta) < \frac{1}{8}$ so that if $k \geq \bar{\bar{k}}$ and $\eta < \bar{\bar{\varepsilon}}^2$, we obtain for all $\varepsilon \in [\sqrt{\eta}, \bar{\bar{\varepsilon}})$:*

$$\mu(F_3) \leq 10^{-6}.$$

Proof. Let $\bar{\bar{k}}$ be the maximum and $\bar{\bar{\alpha}} < \frac{1}{6}$ be the minimum of all thresholds for k and α given by Lemmas 6.27, 8.8, 8.10 and 8.11. Furthermore, we set $\bar{\bar{\theta}} := \hat{\theta}$, where $\hat{\theta} = \hat{\theta}(N,n,C_0)$ is given by Lemma 8.11. Let $0 < \alpha \leq \bar{\bar{\alpha}}$ and $0 < \theta \leq \bar{\bar{\theta}}$. We define $\bar{\bar{\varepsilon}} = \bar{\bar{\varepsilon}}(N,n,C_0,\alpha,\theta)$ as the minimum of $\frac{1}{16}$, a small constant depending on $N,n,\mathcal{K},C_0,\alpha,\theta$ given by the last lines of this proof, and of all upper bounds for ε stated in Lemmas 6.27, 8.8, 8.10 and 8.11. Let $k \geq \bar{\bar{k}}$ and $\eta \leq \varepsilon^2 < \bar{\bar{\varepsilon}}^2$. We have $\mu(F_3) \leq \mu(F_3 \cap \tilde{\tilde{F}}) + \mu(F_3 \setminus \tilde{\tilde{F}})$. With Lemma 8.8 ($p = 2$), there exists some constant $C = C(N,n,\mathcal{K},C_0)$ so that $\mu(F_3 \setminus \tilde{\tilde{F}}) \leq \mu(F \setminus \tilde{\tilde{F}}) \leq C\varepsilon^{\frac{1}{2}}$. Hence we only have to consider $\mu(F_3 \cap \tilde{\tilde{F}})$. We set $\mathcal{G} := \left\{ B(x, 2h(x)) | x \in (F_3 \cap \tilde{\tilde{F}}) \right\}$. This is a set of nondegenerate balls because $x \in F_3 \subset F \setminus \mathcal{Z}$. Furthermore, we have $h(x) \leq 50$ for all $x \in F$ (see the definition of h on page 1208). With Besicovitch's covering theorem [7, 1.5.2, Thm. 2] there exist N_0 families $\mathcal{B}_l \subset \mathcal{G}$, $l = 1,\dots,N_0$, of disjoint balls such that we conclude with property (B) from page 1208 that

$$\mu(F_3 \cap \tilde{\tilde{F}}) \leq \sum_{l=1}^{N_0} \sum_{B \in \mathcal{B}_l} \mu(B \cap \tilde{\tilde{F}}) \overset{\text{(B)}}{\leq} C_0 \sum_{l=1}^{N_0} \sum_{B \in \mathcal{B}_l} (\operatorname{diam} B)^n.$$

Let $1 \le l \le N_0$ and let $B_1 = B(x_1, 2h(x_1)), B_2 = B(x_2, 2h(x_2)) \in \mathcal{B}_l$ with $B_1 \ne B_2$. Since the balls in $\mathcal{B}_l$ are disjoint, we deduce $2h(x_1) + 2h(x_2) \le d(x_1, x_2)$ and, because of the definition of $\tilde{F}$ and Lemma 6.10, we get $d(x_i, \pi(x_i) + A(\pi(x_i))) \le \varepsilon^{\frac{1}{2}} d(x_i) \le \varepsilon^{\frac{1}{2}} h(x_i)$ for $i = 1, 2$. Since $\varepsilon^{\frac{1}{2}} < \frac{1}{4}$, $\alpha < \frac{1}{6}$ and A is 3α Lipschitz continuous, the former two estimates imply $h(x_1) + h(x_2) < d(\pi(x_1), \pi(x_2))$. Thus $\pi(\frac{1}{2}B_1)$ and $\pi(\frac{1}{2}B_2)$ are disjoint. We have $x_i \in \left(\tilde{\tilde{F}} \cap F_3\right) \subset F \subset B(0,5)$ for $i = 1, 2$. With Lemma 8.10, we conclude that $h(x_i) \le \frac{\theta}{100} < \frac{1}{2}$. This implies $\pi(\frac{1}{2}B_i) \subset U_6$. Using Lemma 8.11, there exists some set $H_\theta \subset U_6$ and some constant $C = C(N, n, \mathcal{K}, C_0, k)$ with $\mathcal{H}^n(U_6 \setminus H_\theta) < C \left(\frac{\varepsilon}{\theta^{n+1}\alpha}\right)^2$ so that $d(\pi(x), H_\theta) > h(x)$ for all $x \in F_3 \cap \tilde{\tilde{F}}$, in particular for $x = x_i$. We conclude that $\pi(\frac{1}{2}B_i) \cap H_\theta = \emptyset$, and hence

$$\begin{aligned}\sum_{B \in \mathcal{B}_l} (\operatorname{diam} B)^n &= 4^n \sum_{B \in \mathcal{B}_l} \left(\tfrac{1}{2} \operatorname{diam} \pi\left(\tfrac{1}{2}B\right)\right)^n \\ &= 4^n \sum_{B \in \mathcal{B}_l} \frac{1}{\omega_n} \mathcal{H}^n\left(\pi\left(\tfrac{1}{2}B\right)\right) \le \frac{4^n}{\omega_n} \mathcal{H}^n(U_6 \setminus H_\theta).\end{aligned}$$

Now we obtain

$$\mu(F_3 \cap \tilde{\tilde{F}}) \le C_0 N_0 \frac{4^n}{\omega_n} \mathcal{H}^n(U_6 \setminus H_\theta) \le C \left(\frac{\varepsilon}{\theta^{n+1}\alpha}\right)^2,$$

and we have already shown that $\mu(F_3 \setminus \tilde{\tilde{F}}) \le C\varepsilon^{\frac{1}{2}}$. Using $\varepsilon < \bar{\bar{\varepsilon}}$, we finally get $\mu(F_3) < 10^{-6}$. □

Appendix A

A.1. **Measure theoretical statements.** The following lemmas are stated without proof.

Lemma A.1. *Let $\mathcal{E}$ be a set of disjoint balls (open or closed) in $\mathbb{R}^N$ with radius equal or larger than $s \in (0, \infty)$ and $B \subset B(x, r)$ for all $B \in \mathcal{E}$. Then $\mathcal{E}$ is a finite set with $\#\mathcal{E} \le \left(\frac{r}{s}\right)^N$.*

Lemma A.2. *Let $s > 0$ and $B(x, r)$ be an open or closed ball in $\mathbb{R}^m$ with $s < r$. There exists some family $\mathcal{E}$ of disjoint closed balls with* $\operatorname{diam} B = 2s$ *for all $B \in \mathcal{E}$, $B(x, r) \subset \bigcup_{B \in \mathcal{E}} 5B$ and $\#\mathcal{E} \le \left(\frac{2r}{s}\right)^m$.*

Lemma A.3. *Let $A \subset \mathbb{R}^N$ be a closed set with $\nu(A) > 0$, where ν is some outer measure on $\mathbb{R}^n$. There exists some $x \in A$ so that $\nu(B(x, h)) > 0$ for all $h > 0$.*

Lemma A.4. *Let R be an n-dimensional cube in $\mathbb{R}^N$. Then* $(\operatorname{diam} R)^n = (\sqrt{n})^n \mathcal{H}^n(R)$.

Lemma A.5. *Let $K \subset \mathbb{R}^m$ be a bounded set and let $f : K \to \mathbb{R}^N$ be a Lipschitz function. Then f has a Lipschitz extension $g : \mathbb{R}^m \to \mathbb{R}^N$ with compact support and the same Lipschitz constant.*

A.2. **Differentiation and Fourier transform on a linear subspace.** Let $P_0 \in G(N, n)$ be an n-dimensional linear subspace of $\mathbb{R}^N$ and let $f : P_0 \to R$ be some function, where $R \in \{\mathbb{R}, \mathbb{R}^N\}$. In this section, we explain what we mean by differentiating this function. Furthermore, we define the Fourier transform of f and give some basic properties. Let $\phi : \mathbb{R}^n \to P_0$ be a fixed isometric isomorphism. We set $\tilde{f} : \mathbb{R}^n \to R$, $\tilde{f}(x) = f(\phi(x)) = (f \circ \phi)(x)$.

Definition A.6. Let $l \in \mathbb{N} \cup \{0\}$. We say $f \in C^l(P_0, R)$ iff $\tilde{f} \in C^l(\mathbb{R}^n, R)$ (l-times continuously differentiable). If $l \geq 1$ for all $i \in \{1, \dots, n\}$, we set $\partial_i f := D_i \tilde{f} \circ \phi^{-1} = D_i(f \circ \phi) \circ \phi^{-1}$, $\Delta f := \sum_{j=1}^n \partial_j \partial_j f$, $Df := (\partial_1 f, \dots, \partial_n f)$, and if $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_n)$ is a multi-index, we set $\partial^\kappa f := \partial_1^{\kappa_1} \partial_2^{\kappa_2} \dots \partial_n^{\kappa_n} f$ and $|\kappa| = \kappa_1 + \dots + \kappa_n$.

Now we define the Fourier transform for some function $f \in \mathscr{S}(P_0)$, where $\mathscr{S}(P_0)$ is the Schwartz space of rapidly decreasing functions $f : P_0 \to \mathbb{C}$; cf. [11, 2.2.1. The class of Schwartz functions]. We will get the same results as for some function $f \in \mathscr{S}(\mathbb{R}^n)$.

Definition A.7 (Fourier transform)**.** Let $y \in P_0$ and $f \in \mathscr{S}(P_0)$. We set

$$\widehat{f}(y) := \widehat{(f \circ \phi)}(\phi^{-1}(y)) = \int_{\mathbb{R}^n} f(\phi(z)) e^{-2\pi i \phi^{-1}(y) \cdot z} \mathrm{d}\mathcal{L}^n(z).$$

If $f : P_0 \to \mathbb{C}^N$ with $f_i \in \mathscr{S}(P_0)$, i.e., every component of f is a Schwartz function, then we write $f \in \mathscr{S}(P_0, \mathbb{C}^N)$. We define the Fourier transform of some function $f \in \mathscr{S}(P_0, \mathbb{C}^N)$ by $\widehat{f} := (\widehat{f}_1, \dots, \widehat{f}_N)$, and if $f, g \in \mathscr{S}(P_0)$ we define the convolution of f and g by $(g * f)(w) = \int_{P_0} g(w - v) f(v) \mathrm{d}\mathcal{H}^n(v)$.

A.3. **Littlewood-Paley theorem.**

Lemma A.8 (Continuous version of the Littlewood-Paley theorem)**.** *Let ϕ be an integrable $C^1(\mathbb{R}^n; \mathbb{R})$ function with mean value zero fulfilling $|\phi(x)| + |\nabla\phi(x)| \leq C(1 + |x|)^{-n-1}$ and $0 < \int_0^\infty |\widehat{(\phi_t)}(x)|^2 \frac{\mathrm{d}t}{t} < \infty$, where $\phi_t(x) = \frac{1}{t^n}\phi(\frac{x}{t})$. For all $q \in (1, \infty)$, there exists some constant $C = C(n, q, \phi)$ such that, for all $f \in L^q(\mathbb{R}^n; \mathbb{R}^N)$, we have*

$$\left\| \left(\int_0^\infty |\phi_t * f|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \right\|_{L^q(\mathbb{R}^n;\mathbb{R})} \leq C \|f\|_{L^q(\mathbb{R}^n;\mathbb{R}^N)}.$$

Proof. The proof is analogous to the proof of the Littlewood-Paley theorem [11, Thm. 5.1.2]. ☐

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