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STRICTLY CONVEX CENTRAL CONFIGURATIONS OF THE PLANAR FIVE-BODY PROBLEM

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Abstract. In this paper we investigate strictly convex central configurations of the planar five-body problem, and prove some necessary conditions for such configurations. In particular, given such a central configuration with multiplier λ and total mass M, we show that all exterior edges are less than $r_0 = (M/\lambda)^{1/3}$, at most two interior edges are less than or equal to r_0, and its subsystem with four masses cannot be a central configuration. We also obtain some other necessary conditions for strictly convex central configurations with five bodies, and show numerical examples of strictly convex central configurations with five bodies that have either one or two interior edges less than or equal to r_0. Our work develops some formulae in a classic work by W. L. Williams in 1938 and we rectify some unsupported assumptions there.

1. Introduction

Central configurations are special configurations for the n-body problem characterized by the property that gravitational acceleration at each mass point is a constant multiple of the vector from the mass point to the mass center. They generate self-similar solutions for the n-body problem. In the planar case, they also generate relative equilibrium solutions which move like rigid bodies, for which reason they were also known as permanent configurations. There are many reasons why such configurations and solutions are of special importance, and why many fundamental questions about them are still open. Readers are referred to [1, 2, 4, 16, 19] for introductions, bibliographies, recent advances, and open questions.

A problem which attracted considerable attention is the finiteness conjecture, also known as the Chazy-Wintner conjecture, a problem included in Smale's list of problems for the 21st century [22]. It asserts that for any choice of n positive masses, there are only finitely many similarity classes of central configurations. The conjecture was completely resolved for $n = 4$ [13] and for $n = 5$ except for masses in some codimension two variety [4]. The finiteness conjecture is wide open for $n \geq 6$.

A problem closely related to the finiteness conjecture is the determination of possible shapes for central configurations. It is well known that for any choices of n masses the n-body problem has $n!/2$ similarity classes of collinear central configurations [7, 20]. When $n = 3$, a celebrated discovery by Lagrange [14] states

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that a noncollinear configuration is central if and only if it is an equilateral triangle. With only one possible shape, the number of their similarity classes is obviously finite. Although the finiteness conjecture is now settled for $n = 4$, there are still many unsolved problems regarding shapes of four-body central configurations [2]. Much less is known for $n \geq 5$.

In 1932 W. MacMillan and W. Bartky [15] studied shapes of central configurations for the planar four-body problem using mutual distances as primary variables. Mutual distances are ideal and natural variables for studying shapes since they eliminate dependence on translations and rotations from original coordinates. It is known that central configurations for the n-body problem, which has $\binom{n}{2}$ mutual distances, are zeros of a system of $\binom{n}{2}$ algebraic equations (called Albouy-Chenciner equations) in mutual distances [3]. Presumably there are finitely many zeros for this system for arbitrary positive masses, then the finiteness conjecture would be settled. One result in [15] states that there exists a convex central configuration for any given four masses and cyclic order. An elegant and much shorter proof was given by Z. Xia [24]. Existence of concave central configurations was posed in [15] and settled by M. Hampton [11]. Another simpler proof was again found by Z. Xia [25]. The work of MacMillan-Bartky [15] was generalized to the planar five-body problem by W. L. Williams [23] in 1938, and to the spatial five-body problem by D. Schmidt [21] in 1988.

The work of W. L. Williams [23] contains some valuable formulae but several conclusions in there are problematic, due to an unsupported assumption which were explicitly or implicitly invoked multiple times in that paper. Although the paper contains several inspirational observations, certain parts of the paper are difficult to follow and no further development on shapes of five-body central configurations have been published on the grounds of this paper. We will discuss some of Williams' formulae and rectify the unsupported assumption in section 3.

To a very large extent, the shapes that five-body central configurations can be remains a mysterious question. There are a few things we know. For example, a general criterion known as the perpendicular bisector theorem [16] can be used to rule out certain possibilities. Our goal is to better understand this question, and in this paper we focus on strictly convex configurations.

The main results in these paper are as follows. Let r_0 be the cubic root of the ratio of total mass and the multiplier. It is a canonical variable whose ratios with mutual distances $r_{ij} = |q_i - q_j|$ determine the shape of the configuration (see section 2 for details). Given a strictly convex central configuration with five bodies in the plane with vertices in cyclic order $(q_1, q_2, q_3, q_4, q_5)$, we find

1. if $S_{ij} = 1/r_{ij}^3 - 1/r_0^3$, then $S_{12}S_{34} > S_{13}S_{24} > S_{14}S_{23}$ (Theorem 5.1);
2. all exterior edges are less than r_0 (Theorem 5.2);
3. at most two interior edges are less than or equal to r_0 (Theorem 6.1);
4. if two interior edges are less than or equal to r_0, then they cross each other (Theorem 6.1);
5. its subsystem with four masses cannot be a central configuration (Theorem 7.1).

Apart from these, we also show some numerical examples of symmetric central configurations of five bodies that have either one or two interior edges less than or equal to r_0 (section 8).

2. Basic concepts, equations, notation, and assumptions

The Newtonian n-body problem concerns the motion of n mass points $m_1, \cdots, m_n$ moving in space in accordance with Newton's law of gravitation:

$$\ddot{q}_k = \sum_{i\neq k} \frac{m_i(q_i - q_k)}{|q_i - q_k|^3}, \quad k = 1, 2, \cdots, n. \tag{2.1}$$

Here $q_k \in \mathbb{R}^d$ $(1 \leq d \leq 3)$ is the position of mass $m_k > 0$. The position vector $q = (q_1, \cdots, q_n) \in (\mathbb{R}^d)^n$ is often referred to as the *configuration* of the system, and vectors $\{q_k\}$ are *vertices* of the configuration q. The system (2.1) is smooth except on the collision set Δ:

$$\Delta = \{q \in (\mathbb{R}^d)^n : q_i = q_j \text{ for some } i \neq j\}.$$

Let $M = m_1 + \cdots + m_n$ be the total mass and let

$$c_q = \frac{1}{M}(m_1 q_1 + \cdots + m_n q_n)$$

be the mass center. A noncollision configuration $q = (q_1, \cdots, q_n) \in (\mathbb{R}^d)^n \setminus \Delta$ is called a *central configuration* for the system of masses $(m_1, \cdots, m_n)$ if there exists some positive constant λ, called the *multiplier*, such that

$$-\lambda(q_k - c_q) = \sum_{i\neq k} \frac{m_i(q_i - q_k)}{|q_i - q_k|^3}, \quad k = 1, 2, \cdots, n. \tag{2.2}$$

The definition of central configuration can be extended to cases with some zero masses but we shall only consider positive masses throughout this paper.

Below we briefly describe a few features and equivalent formulations of central configurations which will be helpful here. We refer to [1, 19] for further discussions.

Central configurations are clearly invariant under similarity transformations; i.e., compositions of translations, rotations, and scalings. Translations and rotations do not affect the multiplier, while the scaling $q \mapsto cq$ changes the multiplier from λ to λ/c^3. Central configurations within the same similarity class are considered equivalent. A central configuration for masses $m = (m_1, \cdots, m_n)$ with multiplier λ is also a central configuration for masses αm, $\alpha > 0$, with multiplier $\alpha\lambda$.

By restricting the mass center to the origin, (2.2) can be alternatively written

$$-\frac{\lambda}{2}\nabla\mathcal{I}(q) = \nabla U(q), \tag{2.3}$$

where the moment of inertia $\mathcal{I}$ and potential U are given by

$$\mathcal{I}(q) := \sum_{k=1}^{n} m_k|q_k|^2 = \frac{1}{M}\sum_{i<j} m_i m_j |q_i - q_j|^2, \quad U(q) := \sum_{i<j} \frac{m_i m_j}{|q_i - q_j|}.$$

This justifies the choice of terminology "multiplier" for λ as it is the Lagrange multiplier of $-U$ constrained to constant levels of $\mathcal{I}/2$. Equation (2.3) can be equivalently expressed as

$$\sum_{i\neq k} m_i \left(\frac{1}{|q_i - q_k|^3} - \frac{\lambda}{M}\right)(q_i - q_k) = 0, \quad k = 1, 2, \cdots, n. \tag{2.4}$$

Another equivalent formulation for (2.2) is

$$-\lambda(q_i - q_j) = A_i - A_j \quad \text{for any } i \neq j, \tag{2.5}$$

where

$$A_k \;=\; \sum_{i\neq k} \frac{m_i(q_i-q_k)}{|q_i-q_k|^3} \tag{2.6}$$

is the gravitational acceleration of mass m_k. Clearly (2.2) implies (2.5). That (2.5) implies (2.2) follows from

$$MA_i \;=\; \sum_{j=1}^{N} m_j(A_i-A_j) \;=\; -\lambda \sum_{j=1}^{N} m_j(q_i-q_j) \;=\; -\lambda M(q_i-c_q).$$

Due to homogeneity of $\mathcal{I}$ and U, from (2.3) it is easy to see that $\lambda = U(q)/\mathcal{I}(q)$. The term $\mathcal{I}/U$ depends not only on mutual distances $r_{ij} = |q_i - q_j|$ but also on masses $m = (m_1, \cdots, m_n)$. It is homogeneous in $r = (r_{ij})_{i<j}$ of degree -3 and homogeneous in m of degree 1, so the set of dimension-less "shape" variables $u = (u_{ij})_{i<j}$ given by

$$u_{ij} \;=\; r_{ij}\left(\frac{\lambda}{M}\right)^{\frac{1}{3}}$$

reveals the shape of the central configuration, and is independent of the total mass and size. With this in mind, it is natural to introduce a variable r_0 in place of λ/M and consider their ratios with r_{ij}'s:

$$r_0 = \left(\frac{\lambda}{M}\right)^{-\frac{1}{3}} = \left(\frac{\sum\limits_{i<j} m_i m_j r_{ij}^2}{\sum\limits_{i<j} m_i m_j / r_{ij}}\right)^{\frac{1}{3}}, \quad u_{ij} = \frac{r_{ij}}{r_0}. \tag{2.7}$$

In terms of mutual distances r_{ij} and r_0, the system (2.3) can be written

$$\sum_{i\neq k} m_i \left(\frac{1}{r_{ik}^3} - \frac{1}{r_0^3}\right)(q_i - q_k) \;=\; 0, \quad k = 1, 2, \cdots, n. \tag{2.8}$$

Given n masses $(m_1, \cdots, m_n)$, let $q = (q_1, \cdots, q_n) \in (\mathbb{R}^d)^n$ be its configuration and $\mathrm{Conv}(q)$ be the convex hull of vertices $\{q_k\}$. Assume $\mathrm{Conv}(q)$ is d-dimensional; otherwise we may choose an affine equivalent coordinate system for which $\mathrm{Conv}(q)$ lies in a subspace having the same dimension as $\mathrm{Conv}(q)$, and then work on this subspace. We say the configuration q is *convex* if each q_k is on the boundary of $\mathrm{Conv}(q)$ and it is *strictly convex* if no q_k is in the convex hull of other vertices. *Concave* configurations are configurations which are not strictly convex. *Strictly concave* configurations are configurations which are not convex.

The objective of this paper is to understand strictly convex planar central configurations with five bodies, in which case their vertices have a well-defined cyclic order. We say vertices (or masses) are *ordered counterclockwise* (*clockwise*) if, by starting with q_1, $(q_1, \cdots, q_n)$ is the order of their appearance when traversed along $\partial\mathrm{Conv}(q)$ counterclockwise (clockwise). To understand shapes of central configurations, it is sufficient to fix the orientation since central configurations are clearly invariant under reflections.

For convenience, and without loss of generality, throughout this paper we accept the following assumptions and terminologies.

Assumptions/Terminologies:

(1) **Vertices of any convex central configuration** $(q_1, \cdots, q_5)$ **are assumed to be ordered counterclockwise.**
(2) **Indices (subscripts) are considered elements in** $\mathbb{Z}_5$**.**
For example, $q_6 = q_1$, $q_7 = q_2$, $r_{56} = r_{15}$, $S_{79} = S_{24}$, etc.
When there are n masses, these indices are elements in $\mathbb{Z}_n$.
(3) **Edges** refer to line segments connecting two different vertices of a polygon. For a convex pentagon with vertices $(q_1, \cdots, q_5)$ ordered counterclockwise, edges of the form $\overline{q_i q_{i+1}}$ are called **exterior edges**, and edges of the form $\overline{q_i q_j}$ are **diagonals** or **interior edges** when $|i - j| \geq 2$.

3. Some remarks on Williams' formulae

In this section we discuss some formulae by W. L. Williams [23] and a few corollaries. These formulae are identities for planar five-body central configurations. At the end of this section we discuss the previously mentioned unsupported assumption in [23].

We incorporate two types of Williams' formulae, the first type relates mass ratios to oriented areas of triangles, the second type are mass-independent equations which relate mutual distances and oriented areas of triangles. In order to make those formulae more accessible, to avoid ambiguities on signs of oriented areas, and to make those consequences clearer, below we quickly review and prove Williams' formulae. The derivation and notation follow those in [23].

By (2.8), $q = (q_1, \cdots, q_5) \in (\mathbb{R}^2)^5 \setminus \Delta$ is a central configuration for masses $m_1, \cdots, m_5$ with multiplier λ if and only if

$$\sum_{i \neq k} m_i \left(\frac{1}{r_{ik}^3} - \frac{1}{r_0^3} \right) (q_i - q_k) = 0, \quad k = 1, 2, \cdots, 5, \tag{3.1}$$

where $r_{ik} = |q_i - q_k|$ and r_0 is as in (2.7). Taking the wedge product with $(q_j - q_k)/2$, we find

$$\sum_{i \neq k, j} m_i S_{ik} \Delta_{ikj} = 0, \quad j, k = 1, 2, \cdots, 5, \ j \neq k, \tag{3.2}$$

where

$$S_{ik} := \frac{1}{r_{ik}^3} - \frac{1}{r_0^3}, \quad \Delta_{ikj} := (q_j - q_k) \wedge (q_i - q_k)/2. \tag{3.3}$$

For $(k, j) = (i + 3, i + 4)$ and $(i + 4, i + 3)$, $i \in \{1, 2, \cdots, 5\}$, we obtain

$$m_i S_{i,i+3} \Delta_{i,i+3,i+4} + m_{i+1} S_{i+1,i+3} \Delta_{i+1,i+3,i+4} + m_{i+2} S_{i+2,i+3} \Delta_{i+2,i+3,i+4} = 0,$$
$$m_i S_{i,i+4} \Delta_{i,i+4,i+3} + m_{i+1} S_{i+1,i+4} \Delta_{i+1,i+4,i+3} + m_{i+2} S_{i+2,i+4} \Delta_{i+2,i+4,i+3} = 0.$$

Given $\{i, j, r, k, l\} = \{1, 2, 3, 4, 5\}$ with $r < k < l$, denote by $D_{ij} := \Delta_{rkl}$ the oriented area of the triangle with vertices (q_r, q_k, q_l). Then each $D_{ij} = D_{ji}$ is strictly positive (since the pentagon is strictly convex and vertices are ordered counterclockwise) and the identities above can be written

$$m_i S_{i,i+3} D_{i+1,i+2} + m_{i+1} S_{i+1,i+3} D_{i,i+2} + m_{i+2} S_{i+2,i+3} D_{i,i+1} = 0, \tag{3.4}$$

$$m_i S_{i,i+4} D_{i+1,i+2} + m_{i+1} S_{i+1,i+4} D_{i,i+2} + m_{i+2} S_{i+2,i+4} D_{i,i+1} = 0. \tag{3.5}$$

Eliminating m_{i+2} by calculating (3.4) $\times S_{i+2,i+4}$ − (3.5) $\times S_{i+2,i+3}$, we find

$$\begin{aligned} 0 &= (m_i S_{i,i+3} D_{i+1,i+2} + m_{i+1} S_{i+1,i+3} D_{i,i+2}) S_{i+2,i+4} \\ &\quad -(m_i S_{i,i+4} D_{i+1,i+2} + m_{i+1} S_{i+1,i+4} D_{i,i+2}) S_{i+2,i+3} \\ &= (S_{i+2,i+4} S_{i,i+3} - S_{i+2,i+3} S_{i,i+4}) D_{i+1,i+2} m_i \\ &\quad +(S_{i+1,i+3} S_{i+2,i+4} - S_{i+1,i+4} S_{i+2,i+3}) D_{i,i+2} m_{i+1}. \end{aligned}$$

Therefore

$$\frac{m_i}{m_{i+1}} = \frac{(S_{i+1,i+3} S_{i+2,i+4} - S_{i+1,i+4} S_{i+2,i+3}) D_{i,i+2}}{(S_{i+2,i+3} S_{i,i+4} - S_{i+2,i+4} S_{i,i+3}) D_{i+1,i+2}}, \quad i = 1, 2, \cdots, 5. \tag{3.6}$$

Similarly, (3.4) $\times S_{i+1,i+4}$ − (3.5) $\times S_{i+1,i+3}$ eliminates m_{i+1} and yields

$$\frac{m_i}{m_{i+2}} = \frac{(S_{i+1,i+3} S_{i+2,i+4} - S_{i+1,i+4} S_{i+2,i+3}) D_{i,i+1}}{(S_{i,i+3} S_{i+1,i+4} - S_{i,i+4} S_{i+1,i+3}) D_{i+1,i+2}}, \quad i = 1, 2, \cdots, 5. \tag{3.7}$$

Identities (3.6) and (3.7) form the first set of Williams' formulae, from which we obtain

$$1 = \frac{m_i}{m_{i+2}} \frac{m_{i+2}}{m_{i+4}} \frac{m_{i+4}}{m_i} = \frac{(S_{i+1,i+3} S_{i+2,i+4} - S_{i+1,i+4} S_{i+2,i+3}) D_{i+2,i+3} D_{i+4,i+1}}{(S_{i+1,i+2} S_{i+3,i+4} - S_{i+1,i+3} S_{i+2,i+4}) D_{i+1,i+2} D_{i+3,i+4}}.$$

Substituting i by $i+4$, we obtain the second set of Williams' formulae:

$$\frac{S_{i,i+3} S_{i+1,i+2} - S_{i,i+2} S_{i+1,i+3}}{S_{i,i+2} S_{i+1,i+3} - S_{i,i+1} S_{i+2,i+3}} = \frac{D_{i+2,i+3} D_{i,i+1}}{D_{i+1,i+2} D_{i,i+3}}, \quad i = 1, 2, \cdots, 5. \tag{3.8}$$

By normalizing the r_{ij}'s into dimensionless variables u_{ij} in (2.8) we see that (3.8) is indeed mass-independent. We will see later (Proposition 4.4) that the denominators appearing in Williams' formulae (3.6), (3.7), (3.8) are nonzero.

Now we introduce the following symbols to replace the S_{ij}'s in Williams' formulae:

$$\begin{aligned} P_i &= S_{i,i+2} S_{i+1,i+3} - S_{i,i+3} S_{i+1,i+2}, \\ Q_i &= S_{i,i+1} S_{i+2,i+3} - S_{i,i+2} S_{i+1,i+3}, \quad i \in \{1, 2, 3, 4, 5\}. \end{aligned} \tag{3.9}$$

Then Williams' formulae (3.6), (3.7), (3.8) can be written

$$\begin{aligned} m_i Q_{i+2} D_{i+1,i+2} &= m_{i+1} P_{i+1} D_{i,i+2}, \\ m_i P_{i+3} D_{i+1,i+2} &= m_{i+2} P_{i+1} D_{i,i+1}, \\ P_i D_{i+1,i+2} D_{i,i+3} &= Q_i D_{i+2,i+3} D_{i,i+1}, \quad i \in \{1, 2, 3, 4, 5\}. \end{aligned} \tag{3.10}$$

As a corollary of formulae (3.10), we obtain the following equivalent conditions for these P_i and Q_i.

Proposition 3.1. *Let $q \in (\mathbb{R}^2)^5 \setminus \Delta$ be a strictly convex central configuration given in* (3.1). *Assume vertices of the resulting pentagon are ordered counterclockwise. Let P_i, Q_i, S_{ij} be as in* (3.9), (3.3). *Then the following statements are equivalent:*

(a) *P_i is positive (resp. zero, negative) for some $i \in \{1, \cdots, 5\}$;*
(b) *P_i is positive (resp. zero, negative) for every $i \in \{1, \cdots, 5\}$;*
(c) *Q_i is positive (resp. zero, negative) for some $i \in \{1, \cdots, 5\}$;*
(d) *Q_i is positive (resp. zero, negative) for every $i \in \{1, \cdots, 5\}$.*

Proof. For a strictly convex pentagon, each D_{ij} is strictly positive. It follows immediately from the last identity of (3.10) that P_i and Q_i have the same sign, and $P_i = 0$ if and only if $Q_i = 0$. Therefore (b) and (d) are equivalent.

The second identity of (3.10) tells us that P_{i+1} and P_{i+3} have the same sign, and $P_{i+1} = 0$ if and only if $P_{i+3} = 0$. Since it is valid for each $i \in \{1, 2, 3, 4, 5\}$, we find that one P_i is positive (resp. zero, negative) implies every P_i is positive (resp. zero, negative). Thus (a) and (b) are equivalent.

The first identity of (3.10) shows that P_{i+1} and Q_{i+2} have the same sign, and $P_{i+1} = 0$ if and only if $Q_{i+2} = 0$. This proves that (c) is equivalent to (a). □

An immediate corollary of Proposition 3.1 is

Corollary 3.2. *Let $q \in (\mathbb{R}^2)^5 \setminus \Delta$ and let S_{ij} be as in Proposition* 3.1*. Then exactly one of the following holds:*

$$S_{i,i+1}S_{i+2,i+3} > S_{i,i+2}S_{i+1,i+3} > S_{i,i+3}S_{i+1,i+2} \quad \textit{for every } i, \tag{3.11}$$

$$S_{i,i+1}S_{i+2,i+3} < S_{i,i+2}S_{i+1,i+3} < S_{i,i+3}S_{i+1,i+2} \quad \textit{for every } i, \tag{3.12}$$

$$S_{i,i+1}S_{i+2,i+3} = S_{i,i+2}S_{i+1,i+3} = S_{i,i+3}S_{i+1,i+2} \quad \textit{for every } i. \tag{3.13}$$

This is a conclusion drawn from Williams' formulae. In the next two sections we will show that only (3.11) can happen; i.e., every P_i and Q_i is positive. This poses a very concrete geometric constraint on strictly convex central configurations of the planar five-body problem.

Another simple corollary of the formulae in this section will be useful later:

Proposition 3.3. *Let $q = (q_1, \cdots, q_5) \in (\mathbb{R}^2)^5 \setminus \Delta$ be a strictly convex central configuration given in* (3.1) *with their vertices ordered counterclockwise. Let S_{ij} be as in* (3.3)*. For any i, either $S_{i,i+1}$, $S_{i,i+2}$, $S_{i,i+3}$ are all zero, or at least two of them are nonzero and have opposite signs. Similarly, either $S_{i,i+4}$, $S_{i,i+3}$, $S_{i,i+2}$ are all zero, or at least two of them are nonzero and have opposite signs.*

Proof. Note that we are dealing with strictly convex configurations, those D_{ij} in (3.4), (3.5) are positive. Those S_{ij} in (3.4) (or (3.5)) are either all zero or include two nonzero terms with opposite signs. Replace i in (3.4), (3.5) by $i+2$, $i+1$, respectively; then the proposition is an immediate corollary of (3.4), (3.5). □

We mentioned in the introductory section that there is an unsupported assumption in Williams' work [23] that has been invoked explicitly or implicitly multiple times in that paper. This unsupported assumption is [23, p. 570]

$$r_{12}, r_{23}, r_{34}, r_{15}, r_{45} \le r_0 \le r_{13}, r_{14}, r_{24}, r_{25}, r_{35}. \tag{3.14}$$

It appears to be motivated by MacMillan-Bartky's work for the four-bodies problem [15, §14] where a similar condition was proved to hold. More precisely, if (q_1, q_2, q_3, q_4) is a convex central configuration for the planar four-body problem, then

$$r_{12}, r_{23}, r_{34}, r_{14} \le r_0 \le r_{13}, r_{24}. \tag{3.15}$$

The r_0 in (3.15) is, of course, defined using multiplier and total mass of the four-body system as in (2.7). According to Williams, the assumption (3.14) was introduced to "assure masses are all positive", a statement which was briefly explained in [23] without details because a complete proof would be "too long to write down".

Assumption (3.14) trivializes certain propositions. For example, it follows trivially from (3.14) that (3.12) fails to hold, and $0 < S_{i,i+1}$ for every i. These conclusions are indeed correct (Theorem 5.1, Theorem 5.2 in our paper) but without (3.14) they are by no means obvious. In [23, pp. 565–570] there are related assertions which appear to be independent of (3.14) but we find certain parts difficult to follow. We will see later (section 8) that there are examples which violate (3.14).

4. Some applications of the perpendicular bisector theorem

In this section we will rule out the possibility (3.13) in Corollary 3.2. This is done by repeated applications of two lemmas, both of which are corollaries of a beautiful theorem known as the perpendicular bisector theorem. These lemmas will also be used in the next two sections.

The perpendicular bisector theorem was attributed to C. Conley by R. Moeckel in [16], where the theorem and its proof appeared in the literature for the first time. A simpler proof can be found in A. Albouy [1].

Given two points p_1, p_2 on the plane, their perpendicular bisector and the line through p_1 and p_2 determine two open double cones. Each open double cone consists of a pair of open cones which are symmetric with respect to the midpoint of p_1, p_2. We call them *open double cones determined by* p_1 *and* p_2.

Theorem 4.1 (Perpendicular bisector theorem)**.** *Let q_i and q_j be two points of a planar central configuration, and let C_1, C_2 be the two open double cones determined by q_i and q_j. If one of C_1, C_2 has nonempty intersection with the central configuration, then so does the other open double cone.*

The perpendicular bisector theorem is very useful in restricting possible shapes of planar central configurations. The next two lemmas for planar convex central configurations are simple corollaries of this theorem.

Lemma 4.2. *If masses $m_1, \cdots, m_n$ of a convex central configuration $(q_1, \cdots, q_n) \in (\mathbb{R}^2)^n$ are ordered counterclockwise and $n \geq 4$, then*

$$r_{i,i+2} > r_{i,i+1},\ r_{i+1,i+2} \quad \textit{for } i = 1, 2, \cdots, n$$
$$(\textit{i.e.}, \quad S_{i,i+2} < S_{i,i+1},\ S_{i+1,i+2} \quad \textit{for } i = 1, 2, \cdots, n).$$

Proof. Suppose not. Assume $r_{i,i+2} \leq r_{i,i+1}$. Consider the open double cones determined by q_{i+1} and q_{i+2}. One of the open double cones intersects the central configuration but the other does not. This contradicts the perpendicular bisector theorem. The other case $r_{i,i+2} \leq r_{i+1,i+2}$ is similar. □

Lemma 4.3. *If masses $m_1, \cdots, m_n$ of a convex central configuration $(q_1, \cdots, q_n) \in (\mathbb{R}^2)^n$ are ordered counterclockwise and $n \geq 5$, then*

$$r_{i,i+1} \geq r_{i+1,i+2} \quad \textit{implies} \quad r_{i,i+3} > r_{i+2,i+3}$$
$$(\textit{i.e.}, \quad S_{i,i+1} \leq S_{i+1,i+2} \quad \textit{implies} \quad S_{i,i+3} < S_{i+2,i+3}).$$

Similarly,

$$r_{i+2,i+3} \geq r_{i+1,i+2} \quad \textit{implies} \quad r_{i,i+3} > r_{i,i+1}$$
$$(\textit{i.e.}, \quad S_{i+2,i+3} \leq S_{i+1,i+2} \quad \textit{implies} \quad S_{i,i+3} < S_{i,i+1}).$$

Proof. We only consider $r_{i,i+1} \geq r_{i+1,i+2}$ since the other case is similar. Consider the two open double cones determined by q_i and q_{i+2}. If $r_{i,i+3} \leq r_{i+2,i+3}$, then one of the double cone contains q_{i+4} while the other has empty intersection with the central configuration. This contradicts the perpendicular bisector theorem. $\square$

Lemmas 4.2 and 4.3 concern triangles and quadrilaterals, respectively, of a planar convex central configuration whose edges include precisely one interior edge. Lemma 4.2 says that, for triangles, the interior edge is strictly greater than the two exterior edges. Lemma 4.3 says that, for quadrilaterals, the interior edge is strictly greater than one of its neighboring exterior edges if the other neighboring exterior edge is no less than the remaining exterior edge. With the assistance of Lemmas 4.2 and 4.3, we can sharpen the result in Proposition 3.1:

Proposition 4.4. *Let $q = (q_1, \cdots, q_5) \in (\mathbb{R}^2)^5 \setminus \Delta$ be a strictly convex central configuration given in* (3.1). *Assume vertices of the resulting pentagon are ordered counterclockwise. Let P_i, Q_i, S_{ij} be as in* (3.9), (3.3). *Then every P_i, Q_i is nonzero, and exactly one of* (3.11), (3.12) *holds.*

Proof. By Proposition 3.1, every P_i, Q_i is nonzero if and only if exactly one of (3.11), (3.12) holds. To prove by contradiction, we assume that equalities (3.13) in Corollary 3.2 hold.

Take the largest diagonal $\mathbf{d}$ and the largest exterior edge $\mathbf{e}$. Either they have no common vertex, or they have one common vertex $\mathbf{v}$. We will show that neither case is possible.

First observe that $|\mathbf{d}| > |\mathbf{e}|$, where $|\cdot|$ means the length. This follows easily from Lemma 4.2, since $\mathbf{e}$ is shorter than some diagonal, and $\mathbf{d}$ is the longest diagonal.

Assume $\mathbf{d}$ and $\mathbf{e}$ have no common vertex. Without loss of generality, let $\mathbf{d} = \overline{q_1q_4}$, $\mathbf{e} = \overline{q_2q_3}$; then $|\mathbf{d}| = r_{14}$, $|\mathbf{e}| = r_{23}$. By the observation and assumptions above, we have $r_{14} > r_{23} \geq r_{12}, r_{34}$. This implies $S_{14} < S_{23} \leq S_{12}$, S_{34}. By Lemma 4.2, $r_{13}, r_{24} > r_{23}$, so that $S_{13}, S_{24} < S_{23}$. Since $\mathbf{d} = \overline{q_1q_4}$ is the longest diagonal, $r_{14} \geq r_{13}$, r_{24}. Putting these observations together, we find

$$S_{14} \leq S_{13},\, S_{24} < S_{23} \leq S_{12},\, S_{34}. \tag{4.1}$$

From (3.13) we know

$$S_{12}S_{34} = S_{13}S_{24} = S_{14}S_{23}. \tag{4.2}$$

It is easy to see that none of these S_{ij} $(1 \leq i \neq j \leq 4)$ is zero: if S_{12} or $S_{34} = 0$, then $S_{13}, S_{24} < 0$, which clearly contradicts (4.2). Clearly these S_{ij} cannot have the same sign, so $S_{14} < 0$ and at least one of S_{12}, S_{34} is positive. By (4.1), (4.2) we have $S_{23} < 0$, for otherwise we would have $S_{14}S_{23} < 0 < S_{12}S_{34}$. By (4.1), (4.2) we also have $0 < S_{12}$, S_{34}, for otherwise we would have $S_{12}S_{34} < 0 < S_{14}S_{23}$. To summarize, we have proved

$$S_{14} \leq S_{13},\, S_{24} < S_{23} < 0 < S_{12},\, S_{34}.$$

Now consider the effect of the fifth body. If both S_{15}, S_{45} were positive, then $S_{15}S_{24} < 0 < S_{12}S_{45}$, which contradicts (3.13). If $S_{15} < 0$, then $0 < S_{35}$ (since $S_{12}S_{35} = S_{23}S_{15}$), and so $r_{35} < r_{15}$, which contradicts Lemma 4.3. Likewise, if $S_{45} < 0$, then $0 < S_{25}$ (since $S_{23}S_{45} = S_{34}S_{25}$) and so $r_{45} > r_{25}$, which again contradicts Lemma 4.3. Therefore at least one of S_{15}, S_{45} is zero. If $S_{15} = 0$, then S_{14}, $S_{25} < 0$ (since r_{14}, $r_{25} > r_{15}$, by Lemma 4.2), but this contradicts the identity $S_{14}S_{25} = S_{15}S_{24}$ in (3.13). The case $S_{45} = 0$ is similar; it implies S_{14}, $S_{35} < 0$ but that contradicts the identity $S_{14}S_{35} = S_{13}S_{45}$ in (3.13). We have now ruled out all possibilities, so it is impossible that $\mathbf{d}$ and $\mathbf{e}$ have no common vertex.

Next we deal with the case that **d** and **e** have one common vertex **v**. It is sufficient to consider the following two cases: (i) $\mathbf{e} = \overline{q_1q_2}$, $\mathbf{d} = \overline{q_2q_4}$; (ii) $\mathbf{e} = \overline{q_2q_3}$, $\mathbf{d} = \overline{q_2q_4}$.

In case (i), since $r_{24} \geq r_{13} > r_{12} \geq r_{23}, r_{34}$, we find

$$S_{24} \leq S_{13} < S_{12} \leq S_{23},\ S_{34}.$$

Since $S_{13}S_{24} = S_{12}S_{34}$, none of these four S_{ij} (i.e., S_{13}, S_{24}, S_{12}, S_{34}) is zero. The inequalities above show that these four S_{ij} cannot have the same sign, so $S_{24} < 0 < S_{34}$. In order that $S_{13}S_{24} = S_{12}S_{34}$, we must have $S_{13} < 0 < S_{12}$. To summarize, we have proved

$$S_{24} \leq S_{13} < 0 < S_{12} \leq S_{23},\ S_{34}.$$

We have assumed that $\overline{q_1q_2}$ is the longest exterior edge, so

$$S_{24} \leq S_{13} < 0 < S_{12} \leq S_{23},\ S_{34},\ S_{45},\ S_{15}.$$

But then $S_{13}S_{45} < 0 < S_{34}S_{15}$, contradicting (3.13).

In case (ii) we have $r_{24} \geq r_{13} > r_{23} \geq r_{12}, r_{34}$, and so

$$S_{24} \leq S_{13} < S_{23} \leq S_{12},\ S_{34}.$$

Using $S_{13}S_{24} = S_{12}S_{34}$, we find these four S_{ij} are nonzero, and they cannot have the same sign. Therefore we must have

$$S_{24} \leq S_{13} < 0 < S_{12},\ S_{34}.$$

To determine the sign of S_{23}, we use inequalities $S_{24} \leq S_{14}$ (since $r_{24} \geq r_{14}$) and $S_{13} < S_{23}$. Since $0 < S_{13}S_{24} = S_{23}S_{14}$, S_{23} and S_{14} must both be positive. Now since $\overline{q_2q_3}$ is the longest exterior edge, we conclude that

$$S_{24} \leq S_{13} < 0 < S_{23} \leq S_{12},\ S_{34},\ S_{45},\ S_{15}.$$

This implies $S_{13}S_{45} < 0 < S_{34}S_{15}$, which again contradicts (3.13). ☐

5. Geometric constraints in terms of S_{ij}

In this section we provide some geometric constraints for strictly convex central configurations of the planar five-body problem in terms of the S_{ij}'s defined in (3.3). We have already observed one such constraint in Proposition 3.3. A much less trivial one states that only one option in Corollary 3.2 holds:

Theorem 5.1. *Let $q = (q_1, \cdots, q_5) \in (\mathbb{R}^2)^5 \setminus \Delta$ be a strictly convex central configuration given in* (3.1)*, and let S_{ij} be as in* (3.3)*. Assume vertices are ordered counterclockwise. Then*

$$S_{i,i+1}S_{i+2,i+3} \ > \ S_{i,i+2}S_{i+1,i+3} \ > \ S_{i,i+3}S_{i+1,i+2} \quad \textit{for every } i.$$

Proof. By Proposition 4.4, we know that in Corollary 3.2 either (3.11) or (3.12) holds. We need to show that only (3.11) holds. Suppose **not**, then there would be some strictly convex central configuration, with their vertices ordered counterclockwise, such that inequalities in (3.12) hold:

$$\begin{aligned} S_{12}S_{34} &< S_{13}S_{24} < S_{14}S_{23}, \\ S_{23}S_{45} &< S_{24}S_{35} < S_{25}S_{34}, \\ S_{34}S_{15} &< S_{14}S_{35} < S_{13}S_{45}, \\ S_{45}S_{12} &< S_{14}S_{25} < S_{24}S_{15}, \\ S_{15}S_{23} &< S_{25}S_{13} < S_{12}S_{35}. \end{aligned} \tag{5.1}$$

Under assumption (5.1), we prove the following statements and then use them to get a contradiction:

(a) There exists some i such that $S_{i,i+1} < 0$.
(b) There exists some i such that $S_{i,i+1}, S_{i+2,i+3} < 0$.
(c) There exists a unique i such that $0 < S_{i,i+2}$.

Suppose (a) does not hold; then $0 \leq S_{i,i+1}$ for each i. In this case the first column in (5.1) is constituted by nonnegative terms, so the third column in (5.1) is constituted by positive terms. These positive terms are of the form $S_{i,i+1}S_{i+2,i+4}$, so we must have $0 < S_{i,i+1}, S_{i+2,i+4}$. Since i is arbitrary, it follows that every S_{ij} is positive, but this contradicts both (3.4) and (3.5).[1] Therefore (a) holds.

Without loss of generality, by (a) we may assume $S_{12} < 0$. Then $S_{13}, S_{25} < 0$ since $r_{12} < r_{13}, r_{25}$ (by Lemma 4.2). Observe that $S_{i,i+1} < 0$ for some $i \neq 1$; in fact, we can find such an $S_{i,i+1}$ among S_{23}, S_{34}, S_{45}. Suppose not; i.e., $0 \leq S_{23}, S_{34}, S_{45}$. By the second line of (5.1),

$$0 \ \leq \ S_{23}S_{45} \ < \ S_{25}S_{34}.$$

This contradicts the fact that $S_{25} < 0$. Therefore at least one of S_{23}, S_{34}, S_{45} is negative. Now statement (b) follows if one of S_{34}, S_{45} is negative (corresponding to $i = 1$ or 4 in (b)), or if both S_{15}, S_{23} are negative (corresponding to $i = 5$ in (b)). Suppose neither of S_{34}, S_{45} is negative. From the observation above,

$$S_{23} < 0 \leq S_{34}, S_{45}.$$

By the third line of (5.1),

$$S_{34}S_{15} \ < \ S_{13}S_{45} \ \leq \ 0.$$

This implies $S_{15} < 0$ as desired.

Now we prove (c). Let's begin with the existence part. If $S_{i,i+2} \leq 0$ for every i, then by (3.4) or (3.5) we would have $S_{i,i+1} \geq 0$ for every i, but this contradicts (a). Now, knowing that $0 < S_{i,i+2}$ for some i, we know from Lemma 4.2 that $0 < S_{i,i+1}, S_{i+1,i+2}$. If there were another $j \neq i$ such that $0 < S_{j,j+2}$, then $0 < S_{j,j+1}, S_{j+1,j+2}$, and so by Lemma 4.2 there would be three connecting exterior edges with their $S_{k,k+1}$ positive, but this contradicts (b). Thus, the i making $0 < S_{i,i+2}$ is unique.

Finally, we use (b), (c) to prove the theorem by showing a contradiction with (5.1). By (b), we may assume without loss of generality that $S_{12}, S_{34} < 0$. Then

$$S_{25}, S_{13}, S_{24}, S_{35} \ < \ 0 \ < \ S_{14}.$$

The first inequality follows from Lemma 4.2 and the second one follows from (c). By Lemma 4.3, we must have $S_{23} < S_{12}\,(< 0)$, for otherwise we would get the contradictory inequality $S_{14} < S_{34}\,(< 0)$. We conclude from these discussions that $S_{14}S_{23} < 0 < S_{12}S_{34}$, but this contradicts the first line of (5.1). □

According to Theorem 5.1, inequalities in (3.11) hold for a strictly convex central configuration of the planar five-body problem. It will be convenient to write out

[1] Alternatively, one may consider the product $\mathbf{p}_k$ of entries in the k-column of (5.1). It is easy to see that $\mathbf{p}_1 < \mathbf{p}_3$ contradicts $\mathbf{p}_2 < \mathbf{p}_3$.

all these correct inequalities for later purposes:

$$
\begin{array}{rcccl}
S_{14}S_{23} & < & S_{13}S_{24} & < & S_{12}S_{34}, \\
S_{25}S_{34} & < & S_{24}S_{35} & < & S_{23}S_{45}, \\
S_{13}S_{45} & < & S_{14}S_{35} & < & S_{34}S_{15}, \\
S_{24}S_{15} & < & S_{14}S_{25} & < & S_{45}S_{12}, \\
S_{12}S_{35} & < & S_{25}S_{13} & < & S_{15}S_{23}.
\end{array} \tag{5.2}
$$

Another geometric constraint for strictly convex central configurations of the planar five-body problem is that the third column of (5.2) consists of positive terms. This would be obvious if we can show that every $S_{i,i+1}$ is positive.

Theorem 5.2. *Let $q = (q_1, \cdots, q_5) \in (\mathbb{R}^2)^5 \setminus \Delta$ be a strictly convex central configuration given in* (3.1)*, and let S_{ij} be as in* (3.3)*. Assume vertices are ordered counterclockwise. Then*

$$0 < S_{i,i+1} \quad \textit{for every } i.$$

Proof. First we prove that the third column of (5.2) consists of positive terms:

$$0 < S_{i,i+1}S_{i+2,i+3} \quad \text{for every } i. \tag{5.3}$$

Suppose otherwise; without loss of generality, assume $S_{12}S_{34} \leq 0$. There are two cases. The first case is:

$$S_{12} \leq 0 \leq S_{34}.$$

In the sequel we determine signs of some S_{ij} by using Lemma 4.2 and (5.2):

$$
\begin{array}{rcl}
 & S_{13} < S_{12} \leq 0 \leq S_{34} & \text{(by Lemma 4.2)} \\
\Longrightarrow & S_{13} < S_{12} \leq 0 < S_{24} < S_{34} & \text{(by first line of (5.2), Lemma 4.2)} \\
\Longrightarrow & S_{13} < S_{12} \leq 0 < S_{24} < S_{34},\, S_{23} & \text{(by Lemma 4.2)} \\
\Longrightarrow & S_{14} < 0 & \text{(by first line of (5.2)).}
\end{array}
$$

In the last two lines, inequalities on the left sides that involve $\{S_{12}, S_{13}, S_{14}\}$ violate Proposition 3.3.

The other case is similar:

$$
\begin{array}{rcl}
 & S_{34} \leq 0 \leq S_{12} & \\
\Longrightarrow & S_{24} < S_{34} \leq 0 \leq S_{12} & \text{(by Lemma 4.2)} \\
\Longrightarrow & S_{24} < S_{34} \leq 0 < S_{13} < S_{12} & \text{(by first line of (5.2), Lemma 4.2)} \\
\Longrightarrow & S_{24} < S_{34} \leq 0 < S_{13} < S_{12},\, S_{23} & \text{(by Lemma 4.2)} \\
\Longrightarrow & S_{14} < 0 & \text{(by first line of (5.2)).}
\end{array}
$$

As in the previous case, inequalities in the last two lines that involve $\{S_{14}, S_{24}, S_{34}\}$ contradict Proposition 3.3. This proves (5.3).

According to (5.3), $S_{i,i+1}$ and $S_{i+1,i+2}$ are nonzero and have the same sign. Since i is arbitrary, this implies that every $S_{i,i+1}$ is nonzero and all of them have the same sign. If they were all negative, by Lemma 4.2 every S_{ij} would be negative, which again contradicts Proposition 3.3. Therefore they are all positive. $\square$

6. A classification for strictly convex central configurations

This section is devoted to proving the following classification theorem for strictly convex central configurations of the planar five-body problem.

Theorem 6.1. *For any strictly convex five-body central configuration, all exterior edges are less than r_0. The number of interior edges larger than r_0 is either* 5, 4, *or* 3. *If there are precisely three interior edges larger than r_0, then the other two interior edges must cross each other.*

Proof. That all exterior edges are less than r_0 is simply a reiteration of Theorem 5.2. According to Theorem 5.1, the inequalities in (5.2) hold. These inequalities will be repeatedly used in the proof.

First we observe that $S_{i,i+2} < 0$ for some i. This is a simple consequence of Lemma 4.2, for if $0 \leq S_{i,i+2}$ for every i, we would have $0 \leq S_{i,i+2} < S_{i,i+1}$ for every i, and that contradicts Proposition 3.3.

Assume $S_{i,i+2} < 0$. We claim that:

(a) $S_{i+1,i+3} < 0$ or $S_{i,i+3} < 0$;
(b) $S_{i+1,i+4} < 0$ or $S_{i+2,i+4} < 0$.

Without loss of generality, it is sufficient to consider the case $i = 1$; i.e., $S_{13} < 0$.

To prove (a), we need to show that either $S_{24} < 0$ or $S_{14} < 0$. Suppose $0 \leq S_{24}$. By Lemma 4.2, this implies $0 < S_{23}$. By the first line of (5.2),

$$S_{14}S_{23} \;<\; S_{13}S_{24} \;\leq\; 0.$$

This implies $S_{14} < 0$ as claimed.

To prove (b), we need to show that either $S_{25} < 0$ or $S_{35} < 0$. The proof is similar to (a). Suppose $0 \leq S_{25}$. By Lemma 4.2, this implies $0 < S_{12}$. By the last line of (5.2),

$$S_{12}S_{35} \;<\; S_{25}S_{13} \;\leq\; 0.$$

This implies $S_{35} < 0$ as claimed.

Following (a), (b), still assuming that $S_{13} < 0$, there are four possibilities:

(i) $S_{13}, S_{24}, S_{25} \;<\; 0,$ S_{14}, S_{35} undetermined;
(ii) $S_{13}, S_{24}, S_{35} \;<\; 0,$ S_{14}, S_{25} undetermined;
(iii) $S_{13}, S_{14}, S_{25} \;<\; 0,$ S_{24}, S_{35} undetermined;
(iv) $S_{13}, S_{14}, S_{35} \;<\; 0,$ S_{24}, S_{25} undetermined.

In any case, there are at least three interior edges with their $S_{i,i+2} < 0$, so the number of interior edges larger than r_0 is at least 3. Actually, in case (iv), by the second line in (5.2) and by Theorem 5.2, one of S_{24}, S_{25} must be negative:

$$S_{25}\underbrace{S_{34}}_{(+)} \;<\; S_{24}\underbrace{S_{35}}_{(-)} \quad\Longrightarrow\quad \text{if } S_{25} \geq 0, \text{ then } S_{24} < 0.$$

Therefore, if there are precisely three interior edges with their $S_{i,i+2} < 0$, then one of (i), (ii), (iii) holds, with those two undetermined $S_{i,i+2}$, $S_{j,j+2}$ greater than or equal to zero. In these three cases, those two undetermined $S_{i,i+2}$, $S_{j,j+2}$ have no index in common, with only five possible indices their corresponding interior edges must cross each other. This proves that if two undetermined S_{ij} are nonnegative, or equivalently that two interior edges are less than or equal to r_0, then they must cross each other. □

Theorem 6.1 is sharp in the sense that there are examples of strictly convex five-body central configurations with either one or two interior edges less than or equal to r_0. Some of these examples will be shown in section 8.

7. An application to piling up central configurations

It is possible to add a mass to a central configuration with n bodies which makes the new system a central configuration with $n+1$ bodies. A rather obvious way is to place the additional mass at the mass center c of the system when each mass is equally distant from c, and this new system is not convex. There are less obvious ways. For instance, it is possible to add a mass to a co-circular central configuration with four bodies which makes the new system a nonplanar convex central configuration with five bodies, and it is impossible to do so if the original planar configuration with four bodies is not co-circular [1, Proposition 5]. This increases the dimension of the central configuration.

The procedure of adding one mass to a central configuration so as to make it another central configuration is what we mean by *piling up central configurations.* The piled up central configuration with at least four masses is a special case of *stacked central configurations* defined by M. Hampton in [12]; they are central configurations in which some subset of three or more points form a central configuration.

The following application of results in previous sections addresses the question about piling up convex central configurations, without increasing the dimension of the configuration: for convex central configurations with four bodies in the plane, is it possible to add another mass on the same plane which makes it a strictly convex central configuration? A negative answer has been obtained by A. C. Fernandes and L. F. Mello [8].[2] Here we provide an alternative (and less direct) proof.

Theorem 7.1. *Let $q=(q_1,\cdots,q_5)\in(\mathbb{R}^2)^5\setminus\Delta$ be a strictly convex central configuration for the planar five-body problem with positive masses $(m_1,\cdots,m_5)$. Then (q_1,q_2,q_3,q_4) is not a central configuration for the four-body problem with masses (m_1,m_2,m_3,m_4).*

Proof. We follow notation in previous sections, such as multiplier λ, total mass M, r_0, mutual distances r_{ij}, and S_{ij} for the system q with five bodies. We will prove the theorem by contradiction.

Suppose $\hat{q}=(q_1,q_2,q_3,q_4)$ is a central configuration for masses (m_1,m_2,m_3,m_4). We add a tilde sign $\sim$ to symbols for the four-body subsystem. That is, $\tilde{\lambda}$ is the multiplier of the subsystem, $\tilde{M}=M-m_5$, and

$$\tilde{r}_0 = \left(\frac{\tilde{\lambda}}{\tilde{M}}\right)^{-\frac{1}{3}}, \quad \tilde{S}_{ik} = \frac{1}{r_{ik}^3}-\frac{1}{\tilde{r}_0^3}.$$

[2] In [8] a stronger statement was made: the only possible way to pile up a convex central configuration with four bodies in the plane is to have four equal masses at vertices of a square and the additional mass placed at the mass center. However, the proof there for the concave case is incorrect, so the above mentioned assertion is still open. One ambiguity appears on page 296 of [8], where the authors claim the equivalence of (2) and (3). It should be emphasized that they are equivalent only when the mass center is at the origin. An error appears on page 302 of [8], where the authors claim to have a quadratic polynomial in m_5, but coefficients of that "quadratic polynomial" involve a term r_5 which depends on m_5. It depends on m_5 because, by assuming equivalence of (2) and (3), r_5 must be given by $r_5=-\frac{1}{m_5}(m_1r_1+m_2r_2+m_3r_3+m_4r_4)$.

By strict convexity of q, $\hat{q}$ must be a Dziobek configuration [6].[3] A well-known property of Dziobek configurations is that (see [6], [15], [18, §3])

$$\tilde{S}_{ij}\tilde{S}_{kl} \;=\; \tilde{S}_{ik}\tilde{S}_{jl} \quad \text{provided } \{i,j,k,l\} = \{1,2,3,4\}.$$

That is,

$$\tilde{S}_{12}\tilde{S}_{34} \;=\; \tilde{S}_{13}\tilde{S}_{24} \;=\; \tilde{S}_{14}\tilde{S}_{23}. \tag{7.1}$$

As a convex central configuration with four bodies, by MacMillan-Bartky's inequalities (3.15) (with r_0 in (3.15) replaced by $\tilde{r}_0$),

$$\tilde{S}_{13},\, \tilde{S}_{24} \;\le\; 0 \;\le\; \tilde{S}_{12},\, \tilde{S}_{23},\, \tilde{S}_{34},\, \tilde{S}_{14}. \tag{7.2}$$

With all $r_{ij} > 0$ fixed, consider the following linear polynomial in w:

$$\begin{aligned} p(w) \;&=\; \left(\frac{1}{r_{12}^3} - w\right)\left(\frac{1}{r_{34}^3} - w\right) - \left(\frac{1}{r_{23}^3} - w\right)\left(\frac{1}{r_{14}^3} - w\right) \\ &=\; \left(\frac{1}{r_{12}^3 r_{34}^3} - \frac{1}{r_{23}^3 r_{14}^3}\right) + \left(\frac{1}{r_{23}^3} + \frac{1}{r_{14}^3} - \frac{1}{r_{12}^3} - \frac{1}{r_{34}^3}\right) w. \end{aligned}$$

By Theorem 5.1 and (7.1),

$$p\left(\frac{1}{\tilde{r}_0^3}\right) \;=\; \tilde{S}_{12}\tilde{S}_{34} - \tilde{S}_{23}\tilde{S}_{14} \;=\; 0 \;<\; S_{12}S_{34} - S_{23}S_{14} \;=\; p\left(\frac{1}{r_0^3}\right). \tag{7.3}$$

Assume that the coefficient of w in $p(w)$ is nonpositive. In this case $p(w)$ is decreasing in w, so by (7.3) we find

$$\frac{1}{r_0^3} \;<\; \frac{1}{\tilde{r}_0^3}, \quad \tilde{S}_{ij} \;<\; S_{ij} \;\text{ for every } 1 \le i \ne j \le 4.$$

Then by (7.2),

$$0 \;<\; S_{12},\, S_{23},\, S_{34},\, S_{14}.$$

By Theorem 5.1 again,

$$\underbrace{S_{12}}_{(+)}\,\underbrace{S_{34}}_{(+)} \;>\; S_{13}S_{24} \;>\; \underbrace{S_{14}}_{(+)}\,\underbrace{S_{23}}_{(+)}.$$

Therefore S_{13} and S_{24} have the same sign, and they cannot be zero.

By Theorem 6.1, there are at least three interior edges larger than r_0, so at least three $S_{i,i+2}$'s are negative. Knowing that $0 < S_{14}$, we are forced to conclude that $S_{13},\ S_{24} < 0$, and hence $\tilde{S}_{13},\ \tilde{S}_{24} < 0$. Therefore, by (7.1), "$\le$" in (7.2) can be replaced by "$<$".

From discussions above, we find

$$\begin{aligned} \tilde{S}_{13} < S_{13} < 0 < \tilde{S}_{14} < S_{14}, \\ \tilde{S}_{24} < S_{24} < 0 < \tilde{S}_{23} < S_{23}. \end{aligned}$$

Then

$$S_{13}S_{24} \;<\; \tilde{S}_{13}\tilde{S}_{24} \;=\; \tilde{S}_{14}\tilde{S}_{23} \;<\; S_{14}S_{23}.$$

This contradicts Theorem 5.1 or the first line of (5.2).

[3] A Dziobek configuration for the n-body problem is a central configuration with n mass points such that the convex hull of these mass points is an $(n-2)$-dimensional simplex.

We have proved that the coefficient of w in $p(w)$ is positive. In this case $p(w)$ is strictly increasing in w, so by (7.3) we find

$$\frac{1}{\tilde{r}_0^3} < \frac{1}{r_0^3}, \quad S_{ij} < \tilde{S}_{ij} \text{ for every } 1 \le i \ne j \le 4.$$

Then by Theorem 5.2 we have $0 < S_{12}, S_{34}$, and by (7.1) and (7.2) we find

$$S_{13} < \tilde{S}_{13} < 0 < S_{12} < \tilde{S}_{12},$$
$$S_{24} < \tilde{S}_{24} < 0 < S_{34} < \tilde{S}_{34}.$$

Then

$$S_{12}S_{34} < \tilde{S}_{12}\tilde{S}_{34} = \tilde{S}_{13}\tilde{S}_{24} < S_{13}S_{24}.$$

Again, this contradicts Theorem 5.1 or the first line of (5.2). Thus the subsystem $\hat{q}$ cannot be a central configuration for masses (m_1, m_2, m_3, m_4). □

We remark here that a convex central configuration for the planar four-body problem with positive masses cannot have a three-body subsystem which forms an equilateral triangle. This follows easily from the perpendicular bisector theorem. It would be interesting to know if this phenomenon is valid for more general "piling" of convex central configurations; i.e., whether "four" and "three" above could be replaced by arbitrary "n" and "$n-1$", under the condition that the original and new configurations have the same dimension.

8. Numerical examples with one axis of symmetry

Theorem 6.1 tells us that, for any strictly convex central configuration of the planar five-body problem, the number n_e of interior edges larger than r_0 is at least 3. The regular pentagon with equal masses is a case with $n_e = 5$. In this section we show numerical examples with $n_e = 3,\ 4$.

Consider pentagons with two pairs of equal masses and one axis of symmetry:

$$(m_1, m_2, m_3, m_4, m_5) = (1, \mu, \mu, 1, \nu),$$
$$q_1 = (\alpha, \beta),\ \ q_2 = (-0.5, \gamma),\ \ q_3 = (0.5, \gamma),\ \ q_4 = (-\alpha, \beta),\ \ q_5 = (0, \zeta).$$

In this case we only need to compare r_{13}, r_{14}, and r_{25} with r_0. The position q_5 of m_5 is determined by setting the mass center at the origin: $\zeta = -(2\beta + 2\mu\gamma)/\nu$. Figure 1 shows two numerical examples of such central configurations. Numerical data accurate to the 16th decimal places are given in Table 1.

The example on the left side of Figure 1 has $r_{13} = r_{24} < r_0 < r_{25} < r_{14}$, so that $n_e = 3$. It is a central configuration with one dominant mass. The problem with one dominant mass is also known as the $(1+n)$-body problem. If all inferior masses are equal, as in [10] and [17, §4.3], our numerical computations show that all interior edges are larger than r_0. In our example $m_2 = m_3$ are much smaller than $m_1 = m_4$.

The example on the right side of Figure 1 has $r_{14} < r_0 < r_{25} < r_{13}$, so that $n_e = 4$. It is a perturbation of an example in [9, §4] and [5, §2], in which convex but not strictly convex central configurations were considered. We perturb it to a strictly convex central configuration by reducing masses $m_2 = m_3 = \mu$. It is interesting to note that, although interior edges $r_{25} = r_{35}$ must be greater than r_0 (according to Theorem 6.1), they approach r_0 as μ increases to a threshold value $\mu_0 \approx 11.23156073$ (see [5, Table 1]). This is the critical case where the central configurations cease to be strictly convex.

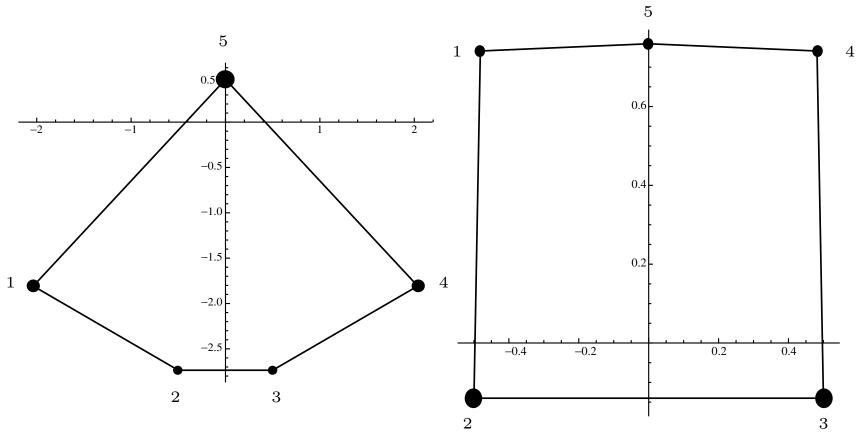

FIGURE 1. Some strictly convex central configurations with five bodies.

TABLE 1. Numerical data for examples in Figure 1.

Example on the left side of Figure 1:

α	=	-2.0364473961007020,	r_0	=	3.0937834761596283,
β	=	-1.8075177021856360,	r_{13}	=	2.7009869413818559,
γ	=	-2.7358307228469223,	r_{14}	=	4.0728947922014040,
ζ	=	0.4709367693510041,	r_{25}	=	3.2455134800239820,
(μ, ν)	=	(0.2,10).			

Example on the right side of Figure 1:

α	=	-0.4816548712069194,	r_0	=	1.0214672502517865,
β	=	0.7409158174373895,	r_{13}	=	1.3190064694925061,
γ	=	-0.1400675992049511,	r_{14}	=	0.9633097424138388,
ζ	=	0.7592499524044385,	r_{25}	=	1.0289665002480437,
(μ, ν)	=	(8,1).			

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