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EXTENDED DE FINETTI THEOREMS FOR BOOLEAN INDEPENDENCE AND MONOTONE INDEPENDENCE

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ABSTRACT. We construct several new spaces of quantum sequences and their quantum families of maps in the sense of Sołtan. The noncommutative distributional symmetries associated with these quantum maps are noncommutative versions of spreadability and partial exchangeability. Then, we study simple relations between these symmetries. We will focus on studying two kinds of noncommutative distributional symmetries: monotone spreadability and boolean spreadability. We provide an example of a spreadable sequence of random variables for which the usual unilateral shift is an unbounded map. As a result, it is natural to study bilateral sequences of random objects, which are indexed by integers, rather than unilateral sequences. At the end of the paper, we will show Ryll-Nardzewski type theorems for monotone independence and boolean independence: Roughly speaking, an infinite bilateral sequence of random variables is monotonically (boolean) spreadable if and only if the variables are identically distributed and monotone (boolean) with respect to the conditional expectation onto its tail algebra. For an infinite sequence of noncommutative random variables, boolean spreadability is equivalent to boolean exchangeability.

1. INTRODUCTION

The characterization of random objects with distributional symmetries is an important object in modern probability, and the recent text of Kallenberg [14] provides a comprehensive treatment of distributional symmetries in classical probability. A finite sequence of random variables $(\xi_1, \xi_2, \ldots, \xi_n)$ is said to be exchangeable if

$$(\xi_1, \ldots, \xi_n) \stackrel{d}{=} (\xi_{\sigma(1)}, \ldots, \xi_{\sigma(n)}) \quad \forall \sigma \in S_n,$$

where S_n is the permutation group of n elements and $\stackrel{d}{=}$ means that the joint distribution of the two sequences are the same. Compared with exchangeability, there is a weaker condition of spreadability: $(\xi_1, \ldots, \xi_n)$ is said to be spreadable if for any $k < n$, we have

$$(\xi_1, \ldots, \xi_k) \stackrel{d}{=} (\xi_{l_1}, \ldots, \xi_{l_k}) \quad \forall 1 \leq l_1 < l_2 < \cdots < l_k \leq n.$$

An infinite sequence of random variables is said to be exchangeable or spreadable if all its finite subsequences have this property. In the study of distributional symmetries in classical probability, one of the most important results is de Finetti's

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theorem, which states that an infinite sequence of random variables whose joint distribution is invariant under all finite permutations is conditionally independent and identically distributed. Later, in [23], Ryll-Nardzewski showed that de Finetti's theorem holds under the weaker condition of spreadability. Therefore, for infinite sequences of random variables in classical probability, spreadability is equivalent to exchangeability.

Recently, Köstler [15] studied three kinds of distributional symmetries, which are stationarity, contractability and exchangeability, in noncommutative probability. It was shown that exchangeability and spreadability do not characterize any universal independent relation in his framework. In addition, for infinite sequences, exchangeability is strictly stronger than spreadability in noncommutative probability. It should be pointed out that the framework in his paper is a W^*-probability space with a faithful state. In this paper, we will consider our problems in a more general framework.

In the 1980's, Voiculescu developed his free probability theory and introduced a universal independence relation, namely free independence, via reduced free products of unital C^*-algebras [30]. For more details on free probability, the reader is referred to the monograph [31]. One can see that there is a deep parallel between classical probability and free probability. Recently, in [16], Köstler and Speicher extended this parallel to the aspect of distributional symmetries. In their work, by strengthening classical exchangeability to quantum exchangeability, they proved a de Finetti type theorem for free independence; i.e. for an infinite sequence of random variables, quantum exchangeability is equivalent to the fact that the random variables are identically distributed and free with respect to a conditional expectation onto their tail algebra. The notion of quantum exchangeability is given by invariance conditions associated with quantum permutation groups $A_s(n)$ of Wang [32]. This noncommutative de Finetti type theorem is an instance where free independence plays in the noncommutative world the same role as classical independence plays in the commutative world. It naturally raises a motivation for further study of noncommutative symmetries that "any result in classical probability should have an extension in free probability." For applications of this philosophy, see [2], [5], [6]. In particular, in [5] Curran introduced a quantum version of spreadability for free independence. It was shown that quantum spreadability is weaker than quantum exchangeability for finite sequences and is a characterization of free independence for infinite sequences. More specifically, in a W^*-probability space with a tracial faithful state, for an infinite sequence of random variables, quantum spreadability is equivalent to the fact that the random variables are identically distributed and free with respect to a conditional expectation onto their tail algebra. In other words, for infinite sequences, quantum spreadability is equivalent to quantum exchangeability in tracial W^*-probability spaces. Another remarkable application of quantum exchangeability was given by Freslon and Weber [10]. They characterize Voiculescu's bi-freeness [29] via certain invariance conditions associated with Wang's quantum groups $A_s(n)$.

In [27], Speicher and Woroudi introduced another independence relation, which is called boolean independence. It was shown that boolean independence is related to the full free product of algebras [4] and the boolean product is the unique nonunital universal product in noncommutative probability [26]. The study of distributional symmetries for boolean independence was started in [17]. We constructed a family of quantum semigroups in analogy with Wang's quantum permutation groups and defined their coactions on joint distributions of sequences. It was shown that the distributional symmetries associated with those coactions can be used to characterize boolean independence in a proper framework. For more details about boolean independence and universal products, see [26]. It inspires us to study more distributional symmetries for boolean independence under the philosophy "any result in classical probability and free probability should have an extension for boolean independence". In analogy with easy quantum groups in [1], "easy" boolean semigroups and their de Finetti type theorems are studied by Tomohiro [11] and the author [18]. To apply our philosophy further, it is natural to find an extended de Finetti type theorem for boolean independence. Specifically, we need to find a "noncommutative version of spreadability" for boolean independence and prove an extended de Finetti type theorem associated with noncommutative spreadability.

The main purpose of this paper is to study noncommutative versions of spreadability and extended de Finetti type theorems associated with them.

Some other objects come into consideration when we study spreadable sequences of random objects. It was shown in [20] that there are two other universal products in noncommutative probability if people do not require the universal construction to be commutative. We call the two universal products monotone and anti-monotone products. As tensor products, free products and boolean products, we can define monotone and anti-monotone independence associated with monotone and anti-monotone products. Monotone independence and anti-monotone independence are essentially the same but with different orders; i.e. if a is monotone with b, then b is anti-monotone with a. For more details of monotone independence, the reader is referred to [19], [22]. It is well known that a sequence of monotone random variables is not exchangeable but spreadable. Therefore, there should be a noncommutative spreadability which can characterize conditionally monotone independence.

The first few sections are devoted to defining noncommutative distributional symmetries in analogy with spreadability and partial exchangeability. Recall that in [2], [5], noncommutative distributional symmetries are defined via invariance conditions associated with certain quantum structures. For instance, Curran's quantum spreadability is described by a family of quantum increasing sequences and their quantum family of maps in the sense of Sołtan. The family of quantum increasing sequences are universal C^*-algebras $A_i(n,k)$ generated by the entries of an $n \times k$ matrix which satisfy certain relations R. Following the idea in [17], to construct a boolean type of space of increasing sequences $B_i(n,k)$, we replace the unit partition condition in R by an invariant projection condition. Recall that in [8], Franz studied relations between freeness, monotone independence and boolean independence via Bożejko, Marek and Speicher's two-state free products [3]. In his construction, a monotone product is something "between" a free product and a boolean product. Thereby, we construct the noncommutative spreadability for monotone independence by modifying quantum spreadability and our boolean spreadability. We will study simple relations between those distributional symmetries, i.e. which one is stronger.

As in the situation for boolean independence, there is no nontrivial pair of monotonically independent random variables in W^*-probability spaces with faithful states. Therefore, the framework we use in this paper is a W^*-probability space with a nondegenerated normal state which gives a faithful GNS representation of the probability space. In this framework, we will see that spreadability is too weak to ensure the existence of a conditional expectation. Recall that in W^*-probability spaces with faithful states, we can define a normal shift on a unilateral infinite sequence of spreadable random variables. Here, "unilateral" means the sequence is indexed by natural numbers $\mathbb{N}$. An important property of this shift is that its norm is one. Therefore, given an operator, we can construct a WOT convergent sequence of bounded variables via shifts. This is the key step to constructing a normal conditional expectation in previous works. But, in W^*-probability spaces with nondegenerated normal states, the unilateral shift of spreadable random variables is not necessarily norm one. An example is provided in the beginning of section 6. Actually, the sequence of random variables is monotonically spreadable, which is an invariance condition stronger than classical spreadability. Therefore, we cannot construct a conditional expectation for unilateral sequences via shifts under the condition of spreadability. To fix this issue, we will consider bilateral sequences of random variables instead of unilateral sequences. "Bilateral" means that the sequences are indexed by integers $\mathbb{Z}$. In this framework, we will see that the shift of spreadable random variables is norm one so that we can define a conditional expectation via shifts by following Köstler's construction. Notice that the index set $\mathbb{Z}$ has two infinities, i.e. the positive infinity and the negative infinity. Therefore, we will have two tail algebras with respect to the two infinities and will consequently define two conditional expectations. We denote by E^+ the conditional expectation which shifts indices to positive infinity and E^- the conditional expectation which shifts indices to negative infinity. We will see that the two tail algebras are subsets of fixed points of the shift and the conditional expectations may not be extended normally to the whole algebra. In general, the two tail algebras are different, and the conditional expectation may have different properties. For noncommutative spreadability for monotone independence, we have the following:

Theorem 1.1. *Let $(\mathcal{A}, \phi)$ be a nondegenerated W^*-probability space and $(x_i)_{i\in\mathbb{Z}}$ a bilateral infinite sequence of selfadjoint random variables which generate $\mathcal{A}$ as a von Neumann algebra. Let $\mathcal{A}_k^+$ be the WOT closure of the nonunital algebra generated by $\{x_i | i \geq k\}$. Then, the following are equivalent:*

(a) *The joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable.*
(b) *For all $k \in \mathbb{Z}$, there exists a ϕ-preserving conditional expectation $E_k : \mathcal{A}_k^+ \to \mathcal{A}_{tail}^+$ such that the sequence $(x_i)_{i\geq k}$ is identically distributed and monotonically independent with respect to E_k. Moreover, $E_k|_{\mathcal{A}_{k'}} = E_{k'}$ whenever $k \geq k'$.*

In general, we cannot extend E^+ to the whole algebra $\mathcal{A}$, but we have the following.

Proposition 1.2. *Let $(\mathcal{A}, \phi)$ be a nondegenerated W^*-probability space and $(x_i)_{i\in\mathbb{Z}}$ a bilateral infinite sequence of selfadjoint random variables which generate $\mathcal{A}$ as a von Neumann algebra. If the joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable, then E^- can be extended to the whole algebra $\mathcal{A}$ normally.*

We will see that boolean spreadability implies monotone spreadability and anti-monotone spreadability. Therefore, both E^+ and E^- can be extended normally to the whole algebra $\mathcal{A}$. Moreover, for boolean spreadable sequences, $E^+ = E^-$ and the two algebras are identical. In summary, we have

Theorem 1.3. *Let $(\mathcal{A}, \phi)$ be a nondegenerated W^*-probability space and $(x_i)_{i\in\mathbb{Z}}$ a bilateral infinite sequence of selfadjoint random variables which generate $\mathcal{A}$ as a von Neumann algebra. Then, the following are equivalent:*

(a) *The joint distribution of $(x_i)_{i\in\mathbb{N}}$ is boolean spreadable.*
(b) *The sequence $(x_i)_{i\in\mathbb{Z}}$ is identically distributed and boolean independent with respect to the ϕ-preserving conditional expectation E onto the nonunital tail algebra of $(x_i)_{i\in\mathbb{Z}}$.*

The paper is organized as follows: In section 2, we will introduce preliminaries and notation from noncommutative probability and recall Wang's quantum permutation groups and boolean quantum semigroups. In section 3, we briefly review distributional symmetries for finite sequences of random variables in classical probability and we restate these symmetries in words of quantum maps. Then, we introduce noncommutative versions of these symmetries and their quantum maps. At the end of this section, we will define quantum spreadability, monotone spreadability and boolean spreadability for bilateral infinite sequences of random variables. In section 4, we will study simple relations between our noncommutative symmetries. In particular, we will show boolean exchangeability is strictly stronger than boolean spreadability. Therefore, operator-valued boolean independent random variables are boolean spreadable. In section 5, we will introduce an equivalence relation on the set of sequences of indices. With the help of the equivalence relation, we will show that operator-valued monotone independent sequences of random variables are monotonically spreadable. In section 6, we provide an example that a monotonically spreadable unilateral sequence of bounded random variables is unbounded. Therefore, we cannot define a conditional expectation for unilateral spreadable sequences via shifts in a W^*-probability space with a nondegenerated normal state. Then we will turn to studying bilateral sequences of random variables. We will introduce tail algebras associated with positive infinity and negative infinity and study elementary properties of conditional expectations associated with the two tail algebras. In section 7, we will study properties of conditional expectations under the assumption that our bilateral sequences are monotonically spreadable. In section 8, we will prove a Ryll-Nardzewski type theorem for monotone independence. In section 9, we will prove a Ryll-Nardzewski type theorem for boolean independence.

2. Preliminaries and examples

We recall some necessary definitions and notions from noncommutative probability. For further details, see [16], [21], [31], [22].

Definition 2.1. A noncommutative probability space $(\mathcal{A}, \phi)$ consists of a unital algebra $\mathcal{A}$ and a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ such that $\phi(1_\mathcal{A}) = 1$. $(\mathcal{A}, \phi)$ is called a $*$-probability space if $\mathcal{A}$ is a $*$-algebra and $\phi(xx^*) \geq 0$ for all $x \in \mathcal{A}$. $(\mathcal{A}, \phi)$ is called a W^*- probability space if $\mathcal{A}$ is a W^*-algebra and ϕ is a normal state on it. We will not assume that ϕ is faithful. The elements of $\mathcal{A}$ are called random variables. Let $x \in \mathcal{A}$ be a random variable. Then, the distribution of x is the linear functional μ_x on $\mathbb{C}[X]$ (the algebra of complex polynomials in one variable) defined by $\mu_x(P) = \phi(P(x))$.

Definition 2.2. Let $\mathcal{A}$ be a W^*-algebra. A normal state ϕ on $\mathcal{A}$ is said to be nondegenerated if $x = 0$ whenever $\phi(axb) = 0$ for all $a, b \in \mathcal{A}$.

By Proposition 7.1.15 in [12], the GNS representation of $\mathcal{A}$ associated to ϕ is normal and faithful if ϕ is a nondegenerated normal state. In some lectures, nondegenerated normal states are also called normal GNS-faithful states. In this paper, we will work with W^*-probability space with a nondegenerated normal state, since there is no nontrivial pair of boolean or monotonically independent random variables in W^*-probability spaces with faithful states. See [17].

Definition 2.3. Let I be an index set. The algebra of noncommutative polynomials in $|I|$ variables, $\mathbb{C}\langle X_i | i \in I\rangle$, is the linear span of 1 and noncommutative monomials of the form $X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_n}^{k_n}$ with $i_1 \neq i_2 \neq \cdots \neq i_n \in I$ and all k_j's are positive integers. For convenience, we will denote by $\mathbb{C}\langle X_i | i \in I\rangle_0$ the set of noncommutative polynomials without a constant term. Let $(x_i)_{i \in I}$ be a family of random variables in a noncommutative probability space $(\mathcal{A}, \phi)$. Their joint distribution is the linear functional $\mu : \mathbb{C}\langle X_i | i \in I\rangle \to \mathbb{C}$ defined by

$$\mu(X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_n}^{k_n}) = \phi(x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}),$$

and $\mu(1) = 1$.

In general, joint distributions depend on the order of random variables, e.g. $\mu_{x,y}$ may not equal $\mu_{y,x}$. According to our notation, $\mu_{x,y}(X_1X_2) = \phi(xy)$ whereas $\mu_{y,x}(X_1X_2) = \phi(yx)$. In this paper, our index set I is always an ordered set with order "$>$" e.g. $\mathbb{N}$, $\mathbb{Z}$.

Definition 2.4. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. A family of (not necessarily unital) subalgebras $\{\mathcal{A}_i | i \in I\}$ of $\mathcal{A}$ is said to be boolean independent if

$$\phi(x_1 x_2 \cdots x_n) = \phi(x_1)\phi(x_2) \cdots \phi(x_n)$$

whenever $x_k \in \mathcal{A}_{i_k}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$. The family of subalgebras $\{\mathcal{A}_i | i \in I\}$ is said to be monotonically independent if

$$\phi(x_1 \cdots x_{k-1} x_k x_{k+1} \cdots x_n) = \phi(x_k)\phi(x_1 \cdots x_{k-1} x_{k+1} \cdots x_n)$$

whenever $x_j \in \mathcal{A}_{i_j}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$ and $i_{k-1} < i_k > i_{k+1}$. A set of random variables $\{x_i \in \mathcal{A} | i \in I\}$ is said to be boolean (monotonically) independent if the family of nonunital subalgebras $\mathcal{A}_i$, which are generated by x_i's respectively, is boolean (monotonically) independent.

One should refer to [9] for more details on boolean products and monotone products of random variables. In general, the framework for boolean independence and monotone independence is a nonunital algebra. Thereby, we will use the following version of operator valued probability spaces:

Definition 2.5 (Operator valued probability space)**.** An operator valued probability space $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ consists of an algebra $\mathcal{A}$, a subalgebra $\mathcal{B}$ of $\mathcal{A}$ and a $\mathcal{B} - \mathcal{B}$ bimodule linear map $E : \mathcal{A} \to \mathcal{B}$, i.e.

$$E[b_1 a b_2] = b_1 E[a] b_2, \ \ E[b] = b,$$

for all $b_1, b_2, b \in \mathcal{B}$ and $a \in \mathcal{A}$. According to the definition in [28], we call E a conditional expectation from $\mathcal{A}$ to $\mathcal{B}$ if E is onto, i.e. $E[\mathcal{A}] = \mathcal{B}$. The elements of $\mathcal{A}$ are called random variables.

Remark 2.6. In free probability theory, $\mathcal{A}$ and $\mathcal{B}$ are assumed to be unital and share the same unit.

Definition 2.7. For an algebra $\mathcal{B}$, we denote by $\mathcal{B}\langle X\rangle$ the algebra which is freely generated by $\mathcal{B}$ and an indeterminate X. Let 1_X be the identity of $\mathbb{C}\langle X\rangle$. Then $\mathcal{B}\langle X\rangle$ is the set of linear combinations of noncommutative monomials $b_0Xb_1Xb_2 \cdots b_{n-1}Xb_n$ where $b_k \in \mathcal{B} \cup \{\mathbb{C}1_X\}$ and $n \geq 0$. The elements in $\mathcal{B}\langle X\rangle$ are called $\mathcal{B}$-polynomials. In addition, $\mathcal{B}\langle X\rangle_0$ denotes the subalgebra of $\mathcal{B}\langle X\rangle$ which does not contain a constant term in $\mathcal{B}$, i.e. the linear span of the noncommutative monomials $b_0Xb_1Xb_2 \cdots b_{n-1}Xb_n$ where $b_k \in \mathcal{B} \cup \{\mathbb{C}1_X\}$ and $n \geq 1$.

Now, we define the operator valued versions of noncommutative independences:

Definition 2.8. Let $\{x_i\}_{i\in I}$ be a family of random variables in an operator valued probability space $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are not necessarily unital. $\{x_i\}_{i\in I}$ are said to be boolean independent over $\mathcal{B}$ if

$$E[p_1(x_{i_1})p_2(x_{i_2}) \cdots p_n(x_{i_n})] = E[p_1(x_{i_1})]E[p_2(x_{i_2})] \cdots E[p_n(x_{i_n})]$$

whenever $i_1, \cdots, i_n \in I$, $i_1 \neq i_2 \neq \cdots \neq i_n$ and $p_1, \cdots, p_n \in \mathcal{B}\langle X\rangle_0$. $\{x_i\}_{i\in I}$ are said to be monotonically independent over $\mathcal{B}$ if

$$\begin{aligned} & E[p_1(x_{i_1}) \cdots p_{k-1}(x_{i_{k-1}})p_k(x_{i_k})p_{k+1}(x_{i_{k+1}}) \cdots p_n(x_{i_n})] \\ = \; & E[p_1(x_{i_1}) \cdots p_{k-1}(x_{i_{k-1}})E[p_k(x_{i_k})]p_{k+1}(x_{i_{k+1}}) \cdots p_n(x_{i_n})] \end{aligned}$$

whenever $i_1, \cdots, i_n \in I$, $i_1 \neq i_2 \neq \cdots \neq i_n$, $i_{k-1} < i_k > i_{k+1}$ and $p_1, \cdots, p_n \in \mathcal{B}\langle X\rangle_0$.

Notice that there is another natural order "$<$" on I, i.e. $a < b$ if $b > a$. Therefore, we can define another noncommutative independence relation. $\{x_i\}_{i\in I}$ are said to be anti-monotonically independent with respect to E and index order "$>$" if $\{x_i\}_{i\in I}$ are monotonically independent with respect to E and index order "$<$". See more details in [20].

2.1. **Noncommutative distributional symmetries.** Recall that, in [33], Wang introduced the following quantum analogue of permutation groups.

Definition 2.9. $A_s(n)$ is defined as the universal unital C^*-algebra generated by elements $(u_{i,j})_{i,j=1,\ldots,n}$ such that we have the following:

- Each $u_{i,j}$ is an orthogonal projection; i.e. $u_{i,j}^* = u_{i,j} = u_{i,j}^2$ for all $i, j = 1, \ldots, n$.
- The elements in each row and column of $u = (u_{i,j})_{i,j=1,\ldots,n}$ form a partition of unit, i.e. are orthogonal and sum up to 1: for each $i = 1, \ldots, n$ and $k \neq l$ we have

$$u_{i,k}u_{i,l} = 0 \quad \text{and} \quad u_{k,i}u_{l,i} = 0,$$

and for each $i = 1, \ldots, n$ we have

$$\sum_{k=1}^{n} u_{i,k} = 1 = \sum_{k=1}^{n} u_{k,i}.$$

$A_s(n)$ is a compact quantum group in the sense of Woronowicz [34], with comultiplication, counit and antipode given by the formulas

$$\Delta(u_{i,j}) = \sum_{k=1}^{n} u_{i,k} \otimes u_{k,j};$$
$$\epsilon(u_{i,j}) = \delta_{i,j};$$
$$S(u_{i,j}) = u_{j,i}.$$

It was shown that quantum permutation groups can be used to characterize conditionally free independence [16].

In [17], we modified the universal conditions of Wang's quantum permutation groups: By replacing the condition associated with partitions of the unit by a condition associated with an invariant projection, we get the following universal algebras.

Quantum semigroups ($B_s(n)$, Δ). The algebra $B_s(n)$ is defined as the universal unital C^*-algebra generated by elements $u_{i,j}$ $(i, j = 1, \ldots, n)$ and a projection $\mathbf{P}$ such that we have

- each $u_{i,j}$ is an orthogonal projection, i.e. $u^*_{i,j} = u_{i,j} = u^2_{i,j}$ for all $i, j = 1, \ldots, n$.
- $u_{i,k}u_{i,l} = 0$ and $u_{k,i}u_{l,i} = 0$, whenever $k \neq l$.
- For all $1 \leq i \leq n$, $\mathbf{P} = \sum_{k=1}^{n} u_{k,i}\mathbf{P}$.

There is a natural comultiplication $\Delta : B_s(n) \to B_s(n) \otimes_{min} B_s(n)$ defined by

$$\Delta(u_{i,j}) = \sum_{k=1}^{n} u_{i,k} \otimes u_{k,j}, \quad \Delta(\mathbf{P}) = \mathbf{P} \otimes \mathbf{P}, \quad \Delta(I) = I \otimes I,$$

where I is the identity of $B_s(n)$ and $\otimes_{min}$ stands for the reduced C^*-tensor product. The existence of these maps is guaranteed by universal properties of $B_s(n)$. Therefore, $(B_s(n), \Delta)$'s are quantum semigroups in the sense of Sołtan [25]. These quantum structures can conditionally characterize boolean independence; see more details in [17].

3. Distributional symmetries for finite sequences of random variables

In this section, we will review two kinds of distributional symmetries, spreadability and partial exchangeability, in classical probability. In [14], we see that the distributional symmetries can be defined for either finite sequences or infinite sequences. Moreover, each kind of distributional symmetry for infinite sequences of random objects is determined by distributional symmetries on all its finite subsequences. For example, an infinite sequence of random variables is exchangeable if and only if all its finite subsequences are exchangeable. We will present distributional symmetries for finite sequences and then introduce their counterparts in the noncommutative case. In the first subsection, we recall notions of spreadability and partial exchangeability in classical probability and rephrase these notions in words of quantum maps. In the second subsection, we will introduce counterparts of spreadability and partial exchangeability in the noncommutative case. Even though there are many interesting properties of partial exchangeability, we are not going to study it too much here because the main problem we are concerned with is extended de Finetti type theorems for noncommutative spreadable sequences.

3.1. **Spreadability and partial exchangeability.** Recall that in [13], a finite sequence of random variables $(x_1, \ldots, x_n)$ is said to be spreadable if for any $k < n$, we have

$$(x_1, \ldots, x_k) \stackrel{d}{=} (x_{l_1}, \ldots, x_{l_k}), \quad l_1 < l_2 < \cdots < l_k. \tag{1}$$

For fixed natural numbers $n > k$, it is mentioned in [5] that the above relation can be described in words of quantum family of maps in the sense of Sołtan [24]: Consider the space $I_{k,n}$ of increasing sequences $\mathcal{I} = (1 \le i_1 < \cdots < i_k \le n)$. For $1 \le a \le n$, $1 \le b \le k$, define $f_{a,b} : I_{k,n} \to \mathbb{C}$ by

$$f_{a,b}(\mathcal{I}) = \begin{cases} 1, & i_b = a, \\ 0, & \text{otherwise.} \end{cases}$$

If we consider $I_{n,k}$ as a discrete space, then the functions $f_{i,j}$ generate $C(I_{n,k})$ by the Stone-Weierstrass theorem. Let $\mathbb{C}[X_1, \ldots, X_m]$ be the set of commutative polynomials in m variables. The algebra $C(I_{n,k})$ together with an algebraic homomorphism $\alpha : \mathbb{C}[X_1, \ldots, X_k] \to \mathbb{C}[X_1, \ldots, X_n] \otimes C(I_{k,n})$ is defined by

$$\alpha : X_j = \sum_{i=1}^{n} X_i \otimes f_{i,j}, \quad \alpha(1) = 1 \otimes 1_{C(I_{k,n})},$$

which defines a quantum family of maps from $\{1, \ldots, k\}$ to $\{1, \ldots, n\}$.

Equation (1) can be rephrased in the following way: For fixed natural numbers $n > k$,

$$\mu_{x_1,\ldots,x_k}(p) 1_{C(I_{n,k})} = (\mu_{x_1,\ldots,x_n} \otimes id_{C(I_{n,k})})(\alpha(p)) \tag{2}$$

for all $p \in \mathbb{C}[x_1, \ldots, x_k]$, where $\mu_{x_1,\ldots,x_n}$ is the joint distribution of $(x_1, \ldots, x_n)$.

For completeness, we provide a sketch of the proof here: Suppose equation (1) holds. Let $p = X_{j_1}^{i_1} \cdots X_{j_m}^{i_m}$ be a monomial in $\mathbb{C}[X_1, \ldots, X_k]$ such that $1 \le j_1 < j_2 < \cdots < j_m \le k$ and $i_1, \ldots, i_m$ are positive integers. Let $\mathcal{I} = (1 \le l_1 < \cdots < l_k \le n)$ be a point in $I_{k,n}$. Then, the $\mathcal{I}$-th component of $\mu_{x_1,\ldots,x_k}(p) 1_{C(I_{n,k})}$ is $E[x_{j_1}^{i_1} \cdots x_{j_m}^{i_m}]$. On the other hand, the $\mathcal{I}$-th component of $\mu_{x_1,\ldots,x_n} \otimes id_{C(I_{n,k})}(\alpha(p))$ is

$$\sum_{s_1,\ldots,s_m=1}^{n} E[x_{s_1}^{i_1} \cdots x_{s_m}^{i_m}](f_{s_1,j_1} \cdots f_{s_m,j_m})(\mathcal{I}).$$

According to the definition of $f_{i,j}$, $(f_{s_1,j_1} \cdots f_{s_m,j_m})(\mathcal{I})$ does not vanish only if $s_t = l_{j_t}$ for all $1 \le t \le m$. Therefore,

$$\sum_{s_1,\ldots,s_m=1}^{n} E[x_{s_1}^{i_1} \cdots x_{s_l}^{i_l}](f_{s_1,j_1} \cdots f_{s_m,j_m})(\mathcal{I}) = E[x_{l_{j_1}}^{i_1} \cdots x_{l_{j_m}}^{i_m}].$$

Since $1 \le j_1 < j_2 < \cdots < j_m \le k$ and $\mathcal{I}$ is an increasing sequence, we have $1 \le l_{j_1} < \cdots < l_{j_m} \le n$. Hence, the $\mathcal{I}$-components of the two sides of equation (2) are equal to each other. Since $\mathcal{I}$ is arbitrary, equation (2) holds. By checking the

$\mathcal{I}$-th component of equation (2), we can also show that (2) implies (1). We will say that $(\xi_1, \dots, \xi_n)$ is (n,k)-spreadable if $(x_1, \dots, x_n)$ satisfies equation (2).

Remark 3.1. We see that the above (n,k)-spreadability describes limited relations between the mixed moments of $(x_1, \dots, x_n)$. For fixed n, k, the (n,k)-spreadability gives no information about mixed moments which involve $k+1$ variables. For example, let $n = 4$, $k = 2$ and assume that $(x_1, \dots, x_4)$ is a $(4,2)$-spreadable sequence. According to equation (1), we know nothing about the relation between $E[x_1x_2x_3]$ and $E[x_2x_3x_4]$. We will call this kind of distributional symmetry partial symmetries because they just provide information for part of mixed moments but not all.

By using the idea of partial symmetries, we can define another family of distributional symmetries which is stronger than (n,k)-spreadability but weaker than exchangeability.

Definition 3.2. For fixed natural numbers $n > k$, we say a sequence of random variables $(x_1, \dots, x_n)$ is (n,k)-exchangeable if

$$(x_1, \dots, x_k) \overset{d}{=} (x_{\sigma(1)}, \dots, x_{\sigma(k)}) \quad \forall \sigma \in S_n,$$

where S_n is the permutation group of n elements.

This kind of distributional symmetry is called partial exchangeability. See [7] for more details. As well as (n,k)-spreadability, we can rephrase partial exchangeability in terms of quantum family of maps: Consider the space $E_{n,k}$ of length k sequences $\{\mathcal{I} = (i_1, \dots, i_k) | 1 \le i_1, \dots, i_k \le n, i_j \ne i_{j'} \text{ for } j \ne j'\}$. For $1 \le a \le n$, $1 \le b \le k$, define $g_{a,b} : I_{n,k} \to \mathbb{C}$ by

$$g_{a,b}(\mathcal{I}) = \begin{cases} 1, & i_b = a, \\ 0, & \text{otherwise.} \end{cases}$$

Given two different sequences $\mathcal{I} = (i_1, \dots, i_k)$ and $\mathcal{I}' = (i'_1, \dots, i'_k)$, there must exist a number j such that $i_j \ne i'_j$. Then, we have that $g_{i,i_j}(\mathcal{I}) = 1 \ne 0 = g_{i,i'_j}(\mathcal{I})$. Therefore, the set of functions $\{g_{i,j} | i = 1, \dots, n; j = 1, \dots, k\}$ separates $E_{n,k}$. According to the Stone-Weierstrass theorem, the functions $g_{i,j}$ generate $C(E_{n,k})$. Again, we can define a homomorphism $\alpha' : \mathbb{C}[X_1, \dots, X_k] \to \mathbb{C}[X_1, \dots, X_n] \otimes C(E_{n,k})$ by the following formulas:

$$\alpha' : X_j = \sum_{i=1}^{n} X_i \otimes g_{i,j}, \quad \alpha'(1) = 1_{C(I_{k,n})}.$$

Lemma 3.3. *Let $\mu_{x_1,\dots,x_n}$ be the joint distribution of $x_1, \dots, x_n$. Then*

$$\mu_{x_1,\dots,x_k}(p)1_{C(I_{n,k})} = (\mu_{x_1,\dots,x_n} \otimes id_{C(I_{n,k})})(\alpha(p))$$

for all $p \in \mathbb{C}[X_1, \dots, X_k]$ if and only if $x_1, \dots, x_n$ is (n,k)-exchangeable.

The proof is similar to the proof of (n,k)-spreadability; we just need to check the values at all components of $E_{n,k}$.

3.2. **Noncommutative analogue of partial symmetries.** Now, we introduce noncommutative versions of spreadability and partial exchangeability. The pioneering work was done by Curran [5]. He defined a quantum version of $C(I_{n,k})$ in analogy with Wang's quantum permutation groups as follows:

Definition 3.4. For $k, n \in \mathbb{N}$ with $k \leq n$, the quantum increasing space $A(n, k)$ is the universal unital C^*-algebra generated by elements $\{u_{i,j} | 1 \leq i \leq n, 1 \leq j \leq k\}$ such that:

1. Each $u_{i,j}$ is an orthogonal projection: $u_{i,j} = u_{i,j}^* = u_{i,j}^2$ for all $i = 1, \ldots, n; j = 1, \ldots, k$.
2. Each column of the rectangular matrix $u = (u_{i,j})_{i=1,\ldots,n;j=1,\ldots,k}$ forms a partition of unity: for $1 \leq j \leq k$ we have $\sum_{i=1}^n u_{i,j} = 1$.
3. Increasing sequence condition: $u_{i,j}u_{i',j'} = 0$ if $j < j'$ and $i \geq i'$.

Remark 3.5. Our notation is different from Curran's; we use $A_i(n, k)$ instead of his $A_i(k, n)$ for our convenience.

For any natural numbers $k < n$, in analogy with coactions of $A_s(n)$, there is a unital $*$-homomorphism $\alpha_{n,k} : \mathbb{C}\langle X_1, \ldots, X_k\rangle \to \mathbb{C}\langle X_1, \ldots, X_n\rangle \otimes A_i(n, k)$ determined by

$$\alpha_{n,k}(X_j) = \sum_{i=1}^{n} X_i \otimes u_{i,j}.$$

The quantum spreadability of random variables is defined as the following:

Definition 3.6. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. A finite ordered sequence of random variables $(x_i)_{i=1,\ldots,n}$ in $\mathcal{A}$ is said to be $A_i(n, k)$-spreadable if their joint distribution $\mu_{x_1,\ldots,x_n}$ satisfies

$$\mu_{x_1,\ldots,x_k}(p)1_{A_i(n,k)} = (\mu_{x_1,\ldots,x_n} \otimes id_{A_i(n,k)})(\alpha_{n,k}(p)),$$

for all $p \in \mathbb{C}\langle X_1, \ldots, X_k\rangle$. $(x_i)_{i=1,\ldots,n}$ is said to be quantum spreadable if $(x_i)_{i=1,\ldots,n}$ is $A_i(n, k)$-spreadable for all $k = 1, \ldots, n - 1$.

Remark 3.7. In [5], Curran studied sequences of C^*-homomorphisms which are more general than random variables. For consistency, we state his definitions in words of random variables. It is routine to extend our work to the framework of sequences of C^*-homomorphisms.

Recall that in [17], by replacing the condition associated with partitions of the unity of Wang's quantum permutation groups, we defined a family of quantum semigroups with invariant projections. With a natural family of coactions, we defined invariance conditions which can characterize conditional boolean independence. Here, we can modify Curran's quantum increasing spaces in the same way.

Definition 3.8. For $k, n \in \mathbb{N}$ with $k \leq n$, the noncommutative increasing space $B_i(n, k)$ is the unital universal C^*-algebra generated by elements $\{u_{i,j} | 1 \leq i \leq n, 1 \leq j \leq k\}$ and an invariant projection $\mathbf{P}$ such that:

1. Each $u_{i,j}$ is an orthogonal projection: $u_{i,j} = (u_{i,j})^* = (u_{i,j})^2$ for all $i = 1, \ldots, n; j = 1, \ldots, k$.
2. For $1 \leq j \leq k$ we have $\sum_{i=1}^n u_{i,j}\mathbf{P} = \mathbf{P}$.
3. Increasing sequence condition: $u_{i,j}u_{i',j'} = 0$ if $j < j'$ and $i \geq i'$.

Similarly as for $A_i(n,k)$, there is a unital $*$-homomorphism $\alpha^{(b)}_{n,k} : \mathbb{C}\langle X_1, \dots, X_k\rangle \to \mathbb{C}\langle X_1, \dots, X_n\rangle \otimes B_i(n,k)$ determined by

$$\alpha^{(b)}_{n,k}(x_j) = \sum_{i=1}^{n} x_i \otimes u_{i,j}.$$

As boolean exchangeability was defined in [17], we have

Definition 3.9. A finite ordered sequence of random variables $(x_i)_{i=1,\dots,n}$ in $(\mathcal{A}, \phi)$ is said to be $B_i(n,k)$-spreadable if their joint distribution $\mu_{x_1,\dots,x_n}$ satisfies

$$\mu_{x_1,\dots,x_k}(p)\mathbf{P} = \mathbf{P}(\mu_{x_1,\dots,x_n} \otimes id_{B_i(n,k)}(\alpha^{(b)}_{n,k})(p))\mathbf{P}$$

for all $p \in \mathbb{C}\langle X_1, \dots, X_k\rangle$. $(x_i)_{i=1,\dots,n}$ is said to be boolean spreadable if $(x_i)_{i=1,\dots,n}$ is $B_i(n,k)$-spreadable for all $k = 1, \dots, n-1$.

We will see that $B_i(k,n)$ is an increasing space of boolean type, because we can derive an extended de Finetti type theorem for boolean independence.

Recall that, in [8], Franz showed some relations between free independence, monotone independence and boolean independence via Bożejko, Marek and Speicher's two-states free products [3]. We can see that a monotone product is "between" a free product and a boolean product. From this viewpoint of Franz's work, we may hope to define a kind of "spreadability" for monotone independence by modifying quantum spreadability and boolean spreadability. Notice that there are at least two ways to get quotient algebras of $B_i(k,n)$ such that the $\mathbf{P}$-invariance condition of the quotient algebras is equivalent quantum spreadability:

1. Require $\mathbf{P}$ to be the unit of the algebra.
2. Let $P_j = \sum_{i=1}^{n} u_{i,j}$, and require $P_{j'}u_{ij} = u_{ij}P_{j'}$ for all $1 \le j, j' \le k$ and $1 \le i \le n$.

To define our monotone increasing spaces, we will modify the second condition a little:

Definition 3.10. For fixed $n, k \in \mathbb{N}$ and $k < n$, a monotone increasing sequence space $M_i(n,k)$ is the universal unital C^*-algebra generated by elements $\{u_{i,j}\}_{i=1,\dots,n;j=1,\dots,k}$ such that:

1. Each $u_{i,j}$ is an orthogonal projection.
2. Monotone condition: Let $P_j = \sum_{i=1}^{n} u_{i,j}$, $P_j u_{i'j'} = u_{i'j'}$ if $j' \le j$.
3. $\sum_{i=1}^{n} u_{i,j} P_1 = P_1$ for all $1 \le j \le k$.
4. Increasing condition: $u_{i,j}u_{i',j'} = 0$ if $j < j'$ and $i \ge i'$.

We see that P_1 plays the role as the invariant projection $\mathbf{P}$ in the boolean case. For consistency, we denote P_1 by $\mathbf{P}$. Then, we can define a $\mathbf{P}$-invariance condition associated with $M_i(n,k)$ in analogy with $B_i(n,k)$: For fixed $n, k \in \mathbb{N}$ and $k < n$, there is a unique unital $*$-homomorphism $\alpha^{(m)}_{n,k} : \mathbb{C}\langle X_1, \dots, X_k\rangle \to \mathbb{C}\langle X_1, \dots, X_n\rangle \otimes M_i(n,k)$ such that

$$\alpha^{(m)}_{n,k}(X_j) = \sum_{i=1}^{n} X_i \otimes u_{i,j}.$$

The existence of such a homomorphism is given by the universality of $\mathbb{C}\langle X_1, \dots, X_k\rangle$.

Definition 3.11. A finite ordered sequence of random variables $(x_i)_{i=1,\dots,n}$ in $(\mathcal{A},\phi)$ is said to be $M_i(n,k)$-invariant if their joint distribution $\mu_{x_1,\dots,x_n}$ satisfies

$$\mu_{x_1,\dots,x_k}(p)\mathbf{P} = \mathbf{P}(\mu_{x_1,\dots,x_n} \otimes id_{M_i(n,k)})(\alpha_{n,k}^{(m)}(p))\mathbf{P}$$

for all $p \in \mathbb{C}\langle X_1,\dots,X_k\rangle$. $(x_i)_{i=1,\dots,n}$ is said to be monotonically spreadable if it is $M_i(n,k)$-invariant for all $k = 1,\dots,n-1$.

We will see that these invariance conditions can characterize conditionally monotone independence in a proper framework.

As in Remark 2.3 in [5], a first question to our definitions is whether $A_i(n,k)$, $B_i(n,k)$, $M_i(n,k)$ exist. In [5], Curran has showed several nontrivial representations of $A_i(n,k)$. In the following, we provide a family of representations of $A_i(n,k)$, $B_i(n,k)$, $M_i(n,k)$ for $n > k$. Fix natural numbers $n > k$, let $l_1,\dots,l_k \in \mathbb{N}$ such that

$$l_1 + \cdots + l_k = n,$$

and consider the following matrix:

$$\begin{pmatrix} P_{1,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_{l_1,1} & 0 & \cdots & 0 \\ 0 & P_{1,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & P_{l_2,2} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_{l_k,k} \end{pmatrix}.$$

We see that the entries of the matrix satisfy the increasing condition of spaces of increasing sequences. By choosing proper projections $P_{i,j}$, we will get representations for our universal algebras:

We denote by $\mathcal{H}_j$ an l_j-dimensional Hilbert spaces with orthonormal basis $\{e_i^{(j)} | i = 1,\dots,l_i\}$. Let I_j be the unit of the algebra $B(\mathcal{H}_j)$, $P_{e_i^{(j)}}$ be the one dimensional orthogonal projection onto $\mathbb{C}e_i^{(j)}$, P_j be the one dimensional projection onto $\mathbb{C}\sum_{i=1}^{l_j} e_i^{(j)}$. Then, some representations of $A_i(n,k)$, $B_i(n,k)$, $M_i(n,k)$ can be constructed in the following way:

A representation of $A_i(n,k)$. For each $1 \le j \le k$, the algebra generated by $\{P_{e_j^i} | i = 1,\dots,l_j\}$ is isomorphic to $C^*(\mathbb{Z}_{l_j})$. The reduced free product $*_{j=1}^k \mathbb{Z}_{l_i}$ is a quotient algebra of $A_i(n,k)$. One can define a C^*-homomorphism π from $A_i(n,k)$ to $*_{j=1}^k C^*(\mathbb{Z}_{l_j})$ such that

$$\pi(u_{i,j}) = \begin{cases} \text{the image of } P_{e_{i'}^{(j)}} \text{ in } *_{j=1}^k C^*(\mathbb{Z}_{l_j}) & \text{if} \quad 0 < i' = i - \sum\limits_{l=m}^{j-1} l_m \le l_j, \\ 0 & \text{otherwise.} \end{cases}$$

A representation of $B_i(n,k)$. One can define a C^*-homomorphism π from $B_i(n,k)$ into $B(\bigotimes_{i=1}^k \mathcal{H}_i)$ such that

$$\pi(u_{i,j}) = \begin{cases} \bigotimes_{m_1=1}^{i-1} P_{m_1} \otimes P_{e_{i'}^{(j)}} \bigotimes_{m_2=i+1}^{k} P_{m_2} & \text{if} \quad 0 < i' = i - \sum_{l=m}^{j-1} l_m \leq l_j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\pi(\mathbf{P}) = \bigotimes_{j=1}^{k} P_j.$$

A representation of $M_i(n,k)$. One can define a C^*-homomorphism π from $M_i(n,k)$ into $B(\bigotimes_{i=1}^k \mathcal{H}_i)$:

$$\pi(u_{i,j}) = \begin{cases} \bigotimes_{m_1=1}^{i-1} I_{m_1} \otimes P_{e_{i'}^{(j)}} \bigotimes_{m_2=i+1}^{k} P_{m_2} & \text{if} \quad 0 < i' = i - \sum_{l=m}^{i-1} l_m \leq l_j, \\ 0 & \text{otherwise.} \end{cases}$$

The existence of these homomorphisms is given by the universal conditions for $A_i(n,k)$, $B_i(n,k)$ and $M_i(n,k)$ respectively. The case of $M_i(n,k)$ plays an important role in our work; we summarize it as the following proposition.

Proposition 3.12. *For fixed natural numbers $n > k$, let $l_1, \dots, l_k \in \mathbb{N}$ such that $l_1 + \cdots + l_k = n$. Let $\mathcal{H}_i$ be l_i-dimensional Hilbert spaces with orthonormal basis $\{e_j^{(i)} | j = 1, \dots, l_i\}$, let I_{l_i} be the unit of the algebra $B(\mathcal{H}_{l_i})$, let $P_{e_j^{(l_i)}}$ be the one-dimensional orthogonal projection onto $\mathbb{C}e_j^{(l_i)}$, and let P_i be the one-dimensional projection onto $\mathbb{C}\sum_j e_j^{(l_i)}$. Then, there is a C^*-homomorphism $\pi : M_i(n,k) \to B(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k)$ defined as follows:*

$$\pi(u_{i,j}) = \begin{cases} \bigotimes_{m_1=1}^{i-1} I_{l_{m_1}} \otimes P_{e_{j'}^{(l_i)}} \bigotimes_{m_2=i+1}^{k} P_{l_{m_2}} & \textit{if} \quad 0 < j' = j - \sum_{l=m}^{i-1} l_m \leq l_i, \\ 0 & \textit{otherwise.} \end{cases}$$

In addition, we need the following property.

Lemma 3.13. *Given natural numbers $n_1, n_2, n, k \in \mathbb{N}$ such that $n > k$. Let $(u_{i,j})_{i=1,\dots,n;j=1,\dots,k}$ be the standard generators of $M_i(n,k)$, and let $(u'_{i,j})_{i=1,\dots,n+n_1+n_2;j=1,\dots,k+n_1+n_2}$ be the standard generators of $M_i(n+n_1+n_2, k+n_1+n_2)$. Then, there exists a C^*-homomorphism $\pi : M_i(n+n_1+n_2, k+n_1+n_2) \to M_i(n,k)$ such that*

$$\pi(u'_{i,j}) = \begin{cases} \delta_{i,j}\mathbf{P} & \textit{if} \quad 1 \leq i \leq n_1, \\ u_{i-n_1, j-n_1} & \textit{if} \quad n_1 + 1 \leq i \leq n + n_1, \ n_1 \leq j \leq n_1 + k, \\ 0 & \textit{if} \quad n_1 + 1 \leq i \leq n + n_1, \ j \leq n_1 \ \textit{or}\ j > n_1 + k, \\ \delta_{i-n, j-k} I & \textit{if} \quad i \geq n + n_1 + 1, \end{cases}$$

where $\mathbf{P} = P_1 = \sum_{i=1}^n u_{i,1}$ and I is the identity of $M_i(n,k)$.

Proof. We can see that the matrix form of $(\pi(u'_{i,j}))_{i=1,\dots,n+n_1+n_2;j=1,\dots,k+n_1+n_2}$ is

$$\begin{pmatrix} \mathbf{P} & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{P} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & u_{1,1} & \cdots & u_{1,k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{n,1} & \cdots & u_{n,k} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & I & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & I \end{pmatrix}.$$

It is easy to check that the coordinates of the above matrix satisfy the universal conditions of $M_i(n+n_1+n_2, k+n_1+n_2)$. The proof is complete. □

In analogy with the (n,k)-partial exchangeability, we can define noncommutative versions of partial exchangeability for free independence and boolean independence.

Definition 3.14. For $k, n \in \mathbb{N}$ with $k \leq n$, the quantum space $A_l(n,k)$ is the universal unital C^*-algebra generated by elements $\{u_{ij} | 1 \leq i \leq n, 1 \leq j \leq k\}$ such that:

1. Each u_{ij} is an orthogonal projection: $u_{ij} = u_{ij}^* = u_{ij}^2$.
2. Each column of the rectangular matrix $u = (u_{ij})$ forms a partition of unity: for $1 \leq j \leq k$ we have $\sum_{i=1}^n u_{ij} = 1$.

Remark 3.15. $A_i(n,k)$ is a quotient algebra of $A_l(n,k)$, because the definition of $A_i(n,k)$ has one more restriction than $A_l(n,k)$. $A_l(n,n)$ is exactly Wang's quantum permutation group $A_s(n)$.

There is a well defined unital algebraic homomorphism

$$\alpha_{n,k}^{(fp)} : \mathbb{C}\langle X_1, \dots, X_k\rangle \to \mathbb{C}\langle X_1, \dots, X_n\rangle \otimes A_l(n,k)$$

such that

$$\alpha_{n,k}^{(fp)} X_j = \sum_{i=1}^n X_i \otimes u_{i,j}$$

where $1 \leq j \leq k$.

Definition 3.16. Let $x_1, \dots, x_n \in (\mathcal{A}, \phi)$ be a sequence of n-noncommutative random variables, and let $k \leq n$ be a positive integer. We say the sequence is (n,k)-quantum exchangeable if

$$\mu_{x_1,\dots,x_k}(p) = \mu_{x_1,\dots,x_n} \otimes id_{A_l(n,k)}(\alpha_{n,k}^{(fp)}(p)),$$

for all $p \in \mathbb{C}\langle X_1, \dots, X_k\rangle$, where $\mu_{x_1,\dots,x_j}$ is the joint distribution of $x_1, \dots, x_j$ with respect to ϕ for $j = k, n$.

By modifying the second universal condition of $A_l(n,k)$, we can define a boolean version of partial exchangeability.

Definition 3.17. For natural numbers $k \leq n$, $B_l(n,k)$ is the nonunital universal C^*-algebra generated by the elements $\{u_{i,j}\}_{i=1,\ldots,n;j=1,\ldots,k}$ and an orthogonal projection $\mathbf{P}$, such that:

1. $u_{i,j}$ is an orthogonal projection, i.e. $u_{i,j} = u_{i,j}^* = u_{i,j}^2$.
2. $\sum_{i=1}^n u_{i,j}\mathbf{P} = \mathbf{P}$ for all $1 \leq j \leq k$.

Remark 3.18. $B_l(n,n)$ is exactly the boolean exchangeable quantum semigroup $B_s(n)$.

There is a well defined unital algebraic homomorphism

$$\alpha_{n,k}^{(bp)} : \mathbb{C}\langle X_1, \ldots, X_k\rangle \to \mathbb{C}\langle X_1, \ldots, X_n\rangle \otimes B_l(n,k)$$

such that

$$\alpha_{n,k}^{(bp)} X_j = \sum_{i=1}^n X_i \otimes u_{i,j},$$

where $1 \leq j \leq k$.

Definition 3.19. Let $x_1, \ldots, x_n \in (\mathcal{A}, \phi)$ be a sequence of n-noncommutative random variables, and let $k \leq n$ be a positive integer. We say the sequence is (n,k)-boolean exchangeable if

$$\mu_{x_1,\ldots,x_k}(p)\mathbf{P} = \mathbf{P}(\mu_{x_1,\ldots,x_n} \otimes id_{B_l(n,k)})(\alpha_{n,k}^{(bp)}(p))\mathbf{P}$$

for all $p \in \mathbb{C}\langle X_1, \ldots, X_k\rangle$, where $\mu_{x_1,\ldots,x_j}$ is the joint distribution of $x_1, \ldots, x_j$ with respect to ϕ.

Proposition 3.20. *Let $(x_1, \ldots, x_{n+1})$ be a monotonically spreadable sequence of random variables in $(\mathcal{A}, \phi)$. Then, all its subsequences are monotonically spreadable.*

Proof. By induction, it suffices to show that the subsequence $(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_{n+1})$ is monotonically spreadable for all $1 \leq l \leq n$. If we denote $(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_{n+1})$ by $(y_1, \ldots, y_n)$, then we need to show that $(y_1, \ldots, y_n)$ is $M_i(n,k)$-spreadable for all $k < n$.

Fix $k < n$; let $\{u_{i,j}\}_{i=1,\ldots,n;j=1,\ldots,k}$ be the set of generators of $M_i(n,k)$ and let $\{P_{i,j}\}_{i=1,\ldots,n+1;j=1,\ldots,k+1}$ be an $n+1$ by $k+1$ matrix with entries in $M_i(n,k)$ such that

$$P_{i,j} = \begin{cases} u_{i,j} & \text{if } 1 \leq i,j < l, \\ u_{i-1,j} & \text{if } 1 \leq j < l,\ i \geq l, \\ u_{i,j-1} & \text{if } 1 \leq i < l,\ j \geq l, \\ u_{i-1,j-1} & \text{if } i,j \geq l, \\ 0 & \text{otherwise.} \end{cases}$$

It is routine to check that the set $\{P_{i,j}\}_{i=1,\ldots,n+1;j=1,\ldots,k+1}$ satisfies the universal conditions of $M_i(n+1,k+1)$. Thus, there exists a C^*-homomorphism $\psi : M_i(n+1,k+1) \to M_i(n,k)$ such that

$$\psi(u'_{i,j}) = P_{i,j},$$

where $\{u'_{i,j}\}$ is the set of generators of $M_i(n+1,k+1)$. For convenience, we will use the following notation:

$$\sigma(i) = \begin{cases} i & \text{if } 1 \leq i < l, \\ i+1 & \text{if } i \geq l. \end{cases}$$

Then, $P_{\sigma(i),\sigma(j)} = u_{i,j}$ and $y_i = x_{\sigma(i)}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, k+1$. For all monomial $X_{j_1} \cdots X_{j_m} \in \mathbb{C}\langle X_1, \ldots, X_k\rangle$, let $P_1' = \sum_{i=1}^n u_{i,1}'$ and $\mathbf{P}$ be the invariant projection of $M_i(n,k)$. We have

$$\psi(P_1') = \sum_{i=1}^{n} u_{i,1} = P_1 = \mathbf{P}$$

and

$$\begin{aligned}
&\mu_{y_1,\ldots,y_n}(X_{j_1}\cdots X_{j_m})\mathbf{P}\\
&= \mathbf{P}\mu_{x_1,\ldots,x_{n+1}}(X_{\sigma(j_1)}\cdots X_{\sigma(j_m)})\psi(P_1')\mathbf{P}\\
&= \mathbf{P}\psi(\mu_{x_1,\ldots,x_{n+1}}(X_{\sigma(j_1)}\cdots X_{\sigma(j_m)})P_1')\mathbf{P}\\
&= \mathbf{P}\psi((\mu_{x_1,\ldots,x_{n+1}}\otimes id_{M_i(n+1,k+1)})(\sum_{i_1,\ldots,i_m=1}^{n+1} X_{i_1}\cdots X_{i_m}\otimes u'_{i_1,\sigma(j_1)}\cdots u'_{i_m,\sigma(j_m)}))\mathbf{P}.
\end{aligned}$$

Notice that $u'_{l,\sigma(j)} = 0$ since $\sigma(j)$ never equals l, so it follows that:

$$\begin{aligned}
&\mu_{y_1,\ldots,y_n}(X_{j_1}\cdots X_{j_m})\mathbf{P}\\
&= \mathbf{P}\psi((\mu_{x_1,\ldots,x_{n+1}}\otimes id_{M_i(n+1,k+1)})(\textstyle\sum_{i_1,\ldots,i_m=1}^{n} X_{\sigma(i_1)}\cdots X_{\sigma(i_m)}\\
&\qquad\qquad \otimes u'_{\sigma(i_1),\sigma(j_1)}\cdots u'_{\sigma(i_m),\sigma(j_m)}))\mathbf{P}\\
&= \mathbf{P}\sum_{i_1,\ldots,i_m=1}^{n} \mu_{x_1,\ldots,x_{n+1}}(X_{\sigma(i_1)}\cdots X_{\sigma(i_m)})\psi(u'_{\sigma(i_1),\sigma(j_1)}\cdots u'_{\sigma(i_m),\sigma(j_m)})\mathbf{P}\\
&= \sum_{i_1,\ldots,i_m=1}^{n} \mu_{y_1,\ldots,y_n}(X_{i_1}\cdots X_{i_m})\mathbf{P}u_{i_1,j_1}\cdots u_{i_m,j_m}\mathbf{P},
\end{aligned}$$

which completes the proof. □

Now, we define our noncommutative distributional symmetries for infinite sequences. In this paper, our infinite ordered index set I would be either $\mathbb{N}$ or $\mathbb{Z}$.

Definition 3.21. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space, I be an ordered index set and $(x_i)_{i\in I}$ a sequence of random variables in $\mathcal{A}$. $(x_i)_{i\in I}$ is said to be monotonically (boolean) spreadable if all its finite ordered subsequences $(x_{i_1}, \ldots, x_{i_l})$ are monotonically (boolean) spreadable.

Proposition 3.22. *Let $(\mathcal{A}, \phi)$ be a noncommutative probability space and $(x_i)_{i\in\mathbb{Z}}$ be a sequence of random variables in $\mathcal{A}$. Then, $(x_i)_{i\in\mathbb{Z}}$ is monotonically (quantum, boolean) spreadable if and only if $(x_i)_{i=-n,-n+1,\ldots,n-1,n}$ is monotonically (quantum, boolean) spreadable for all n.*

Proof. It is sufficient to prove "⇐". Given a subsequence $(x_{i_1}, \ldots, x_{i_l})$ of $(x_i)_{i\in\mathbb{Z}}$, there exists an n such that $-n < i_1, \ldots, i_l < n$. Since $(x_i)_{i=-n,-n+1,\ldots,n-1,n}$ is monotonically spreadable, by Proposition 3.20, we have that $(x_{i_1}, \ldots, x_{i_l})$ is monotonically spreadable. The same for quantum spreadability and boolean spreadability. □

4. Relations between noncommutative probabilistic symmetries

In this section, we will study some simple relations between noncommutative distributional symmetries introduced in the previous section.

It is well known that every C^*-algebra admits a faithful representation. Fix $n, k \in \mathbb{N}$ such that $1 \leq k \leq n-1$. Let Φ be a faithful representation of $B_l(n,k)$ into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. For convenience, we denote $\Phi(u_{i,j})$ by $u_{i,j}$ and $\Phi(\mathbf{P})$ by $\mathbf{P}$.

According to the definition of $B_l(k,n)$, $u_{i,j}$ and $\mathbf{P}$ are orthogonal projections in $B(\mathcal{H})$. Let $Q_i = \sum_{j=1}^{k} u_{i,j}$ for $1 \leq i \leq n$. In [12], we know that the set $P(\mathcal{H})$ of orthogonal projections on $\mathcal{H}$ is a lattice with respect to the usual order $\leq$ on the set of selfadjoint operators, i.e. two selfadjoint operators A and B, $A \leq B$ iff $B - A$ is a positive operator.

Now, we need the following notation in our construction. Given two projections E and F, we denote by $E \vee F$ the minimal orthogonal projection in $P(\mathcal{H})$, such that $E \vee F$ is greater than or equal to E and F. $E \vee F$ is well defined and unique; we call it the supremum of E and F. It is easy to see that $(E \vee F)E = E$ and $(E \vee F)F = F$.

We define a sequence of orthogonal projections $\{P'_i\}_{i=1,\ldots,n}$ in $P(\mathcal{H})$ as follows:

$$P'_1 = I - Q_1,$$

$$P'_i = I - P'_1 \vee \cdots \vee P'_{i-1} \vee Q_i$$

for $2 \leq i \leq n$.

To proceed with our work, we need the following well known lemma:

Lemma 4.1. *Given a nonzero vector $v \in \mathcal{H}$, E and F are two orthogonal projections on $\mathcal{H}$. If $(E \vee F)x = x$ and $Ex = 0$, then $Fx = x$.*

According to the construction of $\{P'_i\}_{1 \leq i \leq n}$, we have

$$P'_i P'_j = \delta_{i,j} P'_i$$

and

$$P'_i u_{i,j} = 0$$

for all $1 \leq i \leq n$ and $1 \leq j \leq k$.

Lemma 4.2. *$\sum_{i=1}^{n} P'_i = I$, where I is the identity in $B(\mathcal{H})$.*

Proof. Since the orthogonal projections P'_i are orthogonal to each other, $\sum_{i=1}^{n} P'_i$ is an orthogonal projection which is less than or equal to the identity I. If $\sum_{i=1}^{n} P'_i < I$, then there exists a nonzero vector $v \in \mathcal{H}$ such that

$$\sum_{i=1}^{n} P'_i v = 0.$$

Then, we have

$$0 = P'_i x = (I - P'_1 \vee \cdots \vee P'_{i-1} \vee Q_i)v$$

or, say,

$$(P'_1 \vee \cdots \vee P'_{i-1} \vee Q_i)v = v$$

for all i. Since $P'_m v = 0$ for all $1 \le m \le i-1$, by Lemma 4.1, $Q_i x = x$. Then, we have

$$\begin{aligned} nv &= \sum_{i=1}^{n} Q_i v \\ &= \sum_{i=1}^{n}\sum_{j=1}^{k} u_{i,j} v \\ &= \sum_{j=1}^{k}(\sum_{i=1}^{n} u_{i,j} v), \end{aligned}$$

which implies that n is in the spectrum of $\sum_{j=1}^{k}\sum_{i=1}^{n} u_{i,j}$. Notice that to every $1 \le j \le k$, $\sum_{i=1}^{n} u_{i,j} \le I$ since they are orthogonal projections and orthogonal to each other. Therefore,

$$0 \le \sum_{j=1}^{k}\sum_{i=1}^{n} u_{i,j} \le \sum_{j=1}^{k} I \le kI.$$

This contradicts the implication above. The proof is complete. □

Corollary 4.3. $\sum_{i=1}^{n} P'_i \mathbf{P} = \mathbf{P}$.

Now, we show some relations between partial distributional symmetries. The above construction can be applied to quantum partial exchangeability.

Proposition 4.4. *Let $(\mathcal{A}, \phi)$ be a noncommutative probability space, and let $(x_i)_{i=1,\ldots,n}$ be a finite ordered sequence of random variables in $\mathcal{A}$. For fixed $n > k$, the joint distribution $\mu_{x_1,\ldots,x_n}$ is $A_l(n,k)$-invariant if it is $A_l(n,k+1)$-invariant.*

Proof. Let $\{u_{ij} | 1 \le i \le n, 1 \le j \le k\}$ be the set of standard generators of $A_l(n,k)$, and let Φ be a faithful representation of $A_l(n,k)$ into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. With the above construction, we can define $\{u'_{i,j}\}_{i=1,\ldots,n;j=1,\ldots,k+1}$ as follows:

$$u'_{i,j} = \begin{cases} \Phi(u_{i,j}) & \text{if } \ j \le k, \\ P'_i & \text{if } \ j = k+1. \end{cases}$$

By Lemma 4.2, $\{u'_{i,j}\}_{i=1,\ldots,n;j=1,\ldots,k+1}$ satisfies the universal conditions for $A_l(n,k+1)$. Let $\{u''_{ij} | 1 \le i \le n, 1 \le j \le k+1\}$ be the set of standard generators of $A_l(n,k+1)$. Then, there exists a C^*-homomorphism $\Phi' : A_l(n,k+1) \to B(\mathcal{H})$ such that

$$\Phi'(u''_{ij}) = u'_{i,j}.$$

Therefore, $\Phi^{-1}\Phi'$ defines a unital C^*-homomorphism

$$\Phi^{-1}\Phi' : C^* - alg\{u'_{i,j} | 1 \le i \le n, 1 \le j \le k\} \to A_l(n,k)$$

such that

$$\Phi^{-1}\Phi'(u'_{i,j}) = u_{i,j}$$

for all $1 \le i \le n, 1 \le j \le k$.

If $\mu_{x_1,\ldots,x_n}$ is $A_l(n,k+1)$-invariant, then

$$\mu_{x_1,\ldots,x_{k+1}}(p)1_{A_l(n,k+1)} = (\mu_{x_1,\ldots,x_k} \otimes id_{A_l(n,k+1)})(\alpha^{(fp)}_{n,k+1}(p))$$

for all $p \in \mathbb{C}\langle X_1, \ldots, X_{k+1}\rangle$. Let $p = X_{j_1} \cdots X_{j_l} \in \mathbb{C}\langle X_1, \ldots, X_k\rangle$. Then, we have

$$\begin{aligned}
& \mu_{x_1,\ldots,x_k}(p)1_{A(n,k)} \\
& \Phi^{-1}\Phi'(\mu_{x_1,\ldots,x_{k+1}}(p)1_{A(n,k+1)}) \\
= \; & \Phi^{-1}\Phi'((\mu_{x_1,\ldots,x_n} \otimes id_{A_l(n,k+1)})(\alpha^{(fp)}_{n,k+1}(X_{j_1} \cdots X_{j_l}))) \\
= \; & \Phi^{-1}\Phi'((\mu_{x_1,\ldots,x_n} \otimes id_{A_l(n,k+1)})(\sum_{i_1,\ldots,i_l}^{n} X_{i_1} \cdots X_{i_l} \otimes u'_{i_1,j_1} \cdots u'_{i_l,j_l})) \\
= \; & (\mu_{x_1,\ldots,x_n} \otimes id_{A_l(n,k)})(\sum_{i_1,\ldots,i_l}^{n} X_{i_1} \cdots X_{i_l} \otimes u_{i_1,j_1} \cdots u_{i_l,j_l}) \\
= \; & (\mu_{x_1,\ldots,x_n} \otimes id_{A_l(n,k)})(\alpha^{(fp)}_{n,k}(p)).
\end{aligned}$$

Since p is an arbitrary monomial, the proof is complete. ☐

Similarly, by comparing universal conditions, we have

Corollary 4.5. *$\mu_{x_1,\ldots,x_n}$ is $B_l(n,k)$-invariant if it is $B_l(n,k+1)$-invariant*

Lemma 4.6. *$\mu_{x_1,\ldots,x_n}$ is (n,k)-quantum spreadable if it is $A_l(n,k)$-invariant.*

Proof. Let $\{u_{i,j}\}_{i=1,\ldots,n;j=1,\ldots,k}$ be generators of $A_i(n,k)$ and $\{u'_{i,j}\}_{i=1,\ldots,n;j=1,\ldots,k}$ be generators of $A_l(n,k)$. Then, there is a well defined C^*-homomorphism $\beta : A_l(n,k) \to A_i(n,k)$ such that $\beta(u'_{i,j}) = u_{i,j}$. The existence of β is given by the universality of $A_l(n,k)$. Since $\mu_{x_1,\ldots,x_n}$ is $A_l(n,k)$-invariant, for all monomials $p = X_{i_1} \cdots X_{i_m} \in \mathbb{C}\langle X_1, \ldots, X_k\rangle$, we have

$$\begin{aligned}
\mu_{x_1,\ldots,x_k}(p)1_{A_l(n,k)} &= (\mu_{x_1,\ldots,x_n} \otimes id_{A_l(n,k)})(\alpha^{(fp)}_{n,k}(p)) \\
&= \sum_{j_1,\ldots,j_m} \phi(x_{j_1} \cdots x_{j_m}) u'_{j_1,i_1} \cdots u'_{j_m,i_m}.
\end{aligned}$$

Applying β on both sides of the above equation, we have

$$\begin{aligned}
\mu_{x_1,\ldots,x_k}(p)1_{A_i(n,k)} &= \sum_{j_1,\ldots,j_m} \phi(x_{j_1} \cdots x_{j_m}) u_{j_1,i_1} \cdots u_{j_m,i_m} \\
&= (\mu_{x_1,\ldots,x_n} \otimes id_{A_i(n,k)})(\alpha_{n,k}(p)).
\end{aligned}$$

The proof is complete. ☐

Similarly, we have

Corollary 4.7. *$\mu_{x_1,\ldots,x_n}$ is (n,k)-boolean spreadable if it is $B_l(n,k)$-invariant.*

Corollary 4.8. *$(x_1, \ldots, x_n)$ is boolean spreadable if it is boolean exchangeable. $(x_1, \ldots, x_n)$ is quantum spreadable if it is quantum exchangeable.*

Let $\{u_{i,j}\}_{i=1,\ldots,n;j=1,\ldots,k} \cup \{\mathbf{P}\}$ be generators of $B_i(n,k)$, $\{u'_{i,j}\}_{i=1,\ldots,n;j=1,\ldots,k}$ be generators of $M_i(n,k)$ and $\{u''_{i,j}\}_{i=1,\ldots,n;j=1,\ldots,k}$ be generators of $A_i(n,k)$. By comparing universal conditions of $B_i(n,k)$, $M_i(n,k)$ and $A_i(n,k)$, we have two well defined C^*-homomorphisms

$$\Phi : B_i(n,k) \to M_i(n,k)$$

and

$$\Psi : M_i(n,k) \to A_i(n,k)$$

such that

$$\Phi(u_{i,j}) = u'_{i,j},\ \forall i = 1, \ldots, n; j = 1, \ldots, k,$$

$$\Phi(\mathbf{P}) = \sum_{i=1}^{n} u'_{i,1}$$

and

$$\Psi(u'_{i,j}) = u''_{i,j},\ \forall i = 1, \ldots, n; j = 1, \ldots, k.$$

By using a similar proof to Lemma 4.6, we have

Corollary 4.9. *$\mu_{x_1,\ldots,x_n}$ is $M_i(n,k)$-invariant if it is $B_i(n,k)$-invariant.*

Corollary 4.10. *$\mu_{x_1,\ldots,x_n}$ is $A_i(n,k)$-invariant if it is $M_i(n,k)$-invariant.*

In summary, for fixed $n, k \in \mathbb{N}$ such that $k < n$, we have the following diagrams:

$$\begin{array}{ccccc} B(n,n)_{\text{inv}} & \longrightarrow & B_l(n,k)_{\text{inv}} & \longrightarrow & B_i(n,k)_{\text{inv}} \\ \Big\downarrow & & \Big\downarrow & & \downarrow \\ & & & & M_i(n,k)_{\text{inv}} \\ & & & & \downarrow \\ A(n,n)_{\text{inv}} & \longrightarrow & A_l(n,k)_{\text{inv}} & \longrightarrow & A_i(n,k)_{\text{inv}} \end{array}$$

and

$$\begin{array}{ccc} \text{Boolean exchangeability} & \longrightarrow & \text{Boolean spreadability} \\ \Big\downarrow & & \downarrow \\ & & \text{Monotone spreadability} \\ & & \downarrow \\ \text{Quantum exchangeability} & \longrightarrow & \text{Quantum spreadability.} \end{array}$$

The arrow "condition (a) $\to$ condition (b)" means that condition (a) implies condition (b).

5. Monotonically equivalent sequences

In order to study monotone spreadability, we need to find some relations between mixed moments of monotonically spreadable sequences of random variables. In this section, we will introduce an equivalence relation, which has a deep relation with monotone spreadability, on finite sequences of ordered indices.

Definition 5.1. Given two pairs of integers (a, b), (c, d), we say these two pairs have the same order if $a - b, c - d$ are both positive or negative or 0.

For example, $(1, 2)$ and $(3, 5)$ have the same order, but $(1, 2)$ and $(5, 3)$ do not have the same order.

Definition 5.2. Let $\mathbb{Z}$ be the set of integers with the natural order "$>$" and $\mathbb{Z}^L = \mathbb{Z}\times\cdots\times\mathbb{Z}$ be the set of finite sequences of length L. We define a partial relation $\sim_m$ on $\mathbb{Z}^L$: Given two sequences of indices $\mathcal{I} = \{i_1, \ldots, i_L\}, \mathcal{J} = \{j_1, \ldots, j_L\} \in \mathbb{Z}^L$. If for all $1 \le l_1 < l_2 \le L$ such that $i_{l_3} > \max\{i_{l_1}, i_{l_2}\}$ for all $l_1 < l_3 < l_2$, (i_{l_1}, i_{l_2}) and (j_{l_1}, j_{l_2}) have the same order, then we denote $\mathcal{I} \sim_m \mathcal{J}$.

Example. $(5, 3, 4) \sim_m (5, 3, 5)$, but $(5, 6, 4) \not\sim_m (5, 6, 5)$. It follows from the definition that (i_l, i_{l+1}) and (j_l, j_{l+1}) have the same order for all $1 \le l < L$ if $\mathcal{I} \sim_m \mathcal{J}$.

Remark 5.3. In general, the relation can be defined on any ordered set, not only $\mathbb{Z}$. We will show this partial relation is actually an equivalence relation.

To show that $\sim_m$ is an equivalence relation, we need to show that the relation $\sim_m$ is reflexive, symmetric and transitive.

Reflexivity. First, reflexivity is obvious, because a pair (i_{l_1}, i_{l_2}) always has the same order as itself.

Lemma 5.4 (Symmetry)**.** *Let $\mathcal{I} = \{i_1, \ldots, i_L\}, \mathcal{J} = \{j_1, \ldots, j_L\} \in \mathbb{Z}^L$ such that $\mathcal{I} \sim_m \mathcal{J}$. Then, we have $\mathcal{J} \sim_m \mathcal{I}$.*

Proof. Suppose that $\mathcal{J} \not\sim_m \mathcal{I}$. Then, there exist two natural numbers $1 \le l_1 < l_2 \le L$ such that

$$j_{l_3} > \max\{j_{l_1}, j_{l_2}\}$$

for all $l_1 < l_3 < l_2$, but (j_{l_1}, j_{l_2}) and (i_{l_1}, i_{l_2}) do not have the same order. Fixing l_1, we choose the smallest l_2 which satisfies the above property. Notice that $\mathcal{I} \sim_m \mathcal{J}$, (j_{l_1}, j_{l_1+1}) and (i_{l_1}, i_{l_1+1}) have the same order; then

$$l_2 \ne l_1 + 1.$$

According to our assumption, we have

$$j_{l_3'} > \max\{j_{l_1}, j_{l_2}\}$$

for $l_1 < l_3' < l_2$.

Suppose that there exists an l_3'' between l_1 and l_2 such that

$$i_{l_3''} \le \max\{i_{l_1}, i_{l_2}\}.$$

Without loss of generality, we assume that

$$i_{l_1} \ge i_{l_2};$$

then

$$i_{l_3''} \le i_{l_1}.$$

Again, among these l_3'', we choose the smallest one. Then, we have $i_l > i_{l_1} \ge i_{l_3''}$ for

$$l_1 < l < l_3''.$$

Since $\mathcal{I} \sim_m \mathcal{J}$, $(i_{l_1}, i_{l_3''})$ and $(j_{l_1}, j_{l_3''})$ must have the same order, but $i_{l_1} \ge i_{l_3''}$ and $i_{l_1} < j_{l_3''}$. This contradicts the existence of our l_3''. Hence, $i_{l_3'} > \max\{i_{l_1}, i_{l_2}\}$ for all $l_1 < l_3' < l_2$. It follows that (i_{l_1}, i_{l_2}) and (j_{l_1}, j_{l_2}) have the same order. But, it contradicts our original assumption. Therefore, $\mathcal{J} \sim_m \mathcal{I}$. □

Lemma 5.5. *Given two sequences* $\mathcal{I} = \{i_1, \ldots, i_L\}, \mathcal{J} = \{j_1, \ldots, j_L\} \in \mathbb{Z}^L$ *such that* $\mathcal{I} \sim_m \mathcal{J}$*, let* $1 \leq l_1 < l_2 \leq L$ *such that* $i_{l_3} > \max\{i_{l_1}, i_{l_2}\}$ *for all* $l_1 < l_3 < l_2$*. Then, we have*

$$j_{l_3} > \max\{j_{l_1}, j_{l_2}\}$$

for all $l_1 < l_3 < l_2$*.*

Proof. If the statement is false, then there exists l_3 between l_1 and l_2 such that

$$j_{l_3} \leq \max\{j_{l_1}, j_{l_2}\}.$$

Suppose $j_{l_1} \geq j_{l_2}$; then

$$j_{l_3} \leq j_{l_1}.$$

Among all these l_3, we take the smallest one. Then, we have

$$j_{l_4} > \max\{j_{l_1}, j_{l_3}\}$$

for all $l_1 < l_4 < l_3$. By Lemma 5.4, $\mathcal{J} \sim_m \mathcal{I}$ since $\mathcal{I} \sim_m \mathcal{J}$. Therefore, (j_{l_1}, j_{l_3}) and (i_{l_1}, i_{l_3}) must have the same order, which means

$$i_{l_1} \geq i_{l_3}.$$

This is a contradiction. If we assume that $j_{l_1} < j_{l_2}$, then we just need to consider the largest one among those l_3 and we will get the same contradiction. The proof is complete. □

Lemma 5.6 (Transitivity)**.** *Given three sequences* $\mathcal{I}=\{i_1, \ldots, i_L\}, \mathcal{J}=\{j_1, \ldots, j_L\}, \mathcal{Q}=\{q_1, \ldots, q_L\} \in \mathbb{Z}^L$ *such that* $\mathcal{I} \sim_m \mathcal{J}$ *and* $\mathcal{J} \sim_m \mathcal{Q}$*, we have* $\mathcal{I} \sim_m \mathcal{J}$*.*

Proof. Given $1 \leq l_1 < l_2 \leq L$ such that

$$i_{l_3} > \max\{i_{l_1}, i_{l_2}\}$$

for all $l_1 < l_3 < l_2$. By Lemma 5.5, we have

$$j_{l_3} > \max\{j_{l_1}, j_{l_2}\}$$

for all $l_1 < l_3 < l_2$. It follows the definition that (i_{l_1}, i_{l_2}), (j_{l_1}, j_{l_2}) have the same order and that (j_{l_1}, j_{l_2}), (q_{l_1}, q_{l_2}) have the same order. Therefore, (i_{l_1}, i_{l_2}), (q_{l_1}, q_{l_2}) have the same order. Since l_1, l_2 are arbitrary, the proof is complete. □

So now we have shown that the relation $\sim_m$ is reflexive, symmetric and transitive.

Proposition 5.7. $\sim_m$ *is an equivalence relation on* $\mathbb{Z}^L$*.*

As we mentioned before, $\mathbb{Z}$ can be replaced by any ordered set I. When there is no confusion, we always use $\sim_m$ to denote the monotone equivalence relation on I^L for ordered set I and positive integers L. For example, I can be $[n] = \{1, \ldots, n\}$.

Definition 5.8. Let $\mathcal{I} = (i_1, \ldots, i_L)$ be a sequence of ordered indices. An ordered subsequence $(i_{l'_1}, \ldots, i_{l'_2})$ of $\mathcal{I}$ is called an interval if the sequence contains all the elements $i_{l'_3}$ whose position l'_3 is between l'_1 and l'_2. An interval $(i_{l'_1}, \ldots, i_{l'_2})$ of $\mathcal{I}$ is called a crest if $i_{l'_1} = i_{l'_1+1} \cdots = i_{l'_2} > \max\{i_{l'_1-1}, i_{l'_2+1}\}$. In addition, we always assume that $i_0 < i_1$ and $i_L > i_{L+1}$ even though i_0, i_{L+1} are not in $\mathcal{I}$.

Example. $(1, 2, 3, 4)$ has one crest of length 1, namely (4). $(1, 2, 1, 3, 4, 4, 3, 5)$ has 3 crests $(2), (4, 4), (5)$, and (2) is the first peak of the sequence. $(1, 1, 1, 1, 1)$ has one crest $(1, 1, 1, 1, 1)$, which is the sequence itself.

Lemma 5.9. *Given* $\mathcal{I} = (i_1, \ldots, i_L) \in \mathbb{Z}^L$*,* $\mathcal{I}$ *has at least one crest.*

Proof. Since $\mathcal{I}$ consists of finite elements, it has a maximal one, i.e. i_l such that $i_l \geq i_{l'}$ for $1 \leq l' \leq L$. It is obvious that i_l must be contained in an interval $(i_{l_1'}, \ldots, i_{l_2'})$ such that

$$i_{l_1'} = i_{l_1'+1} \cdots = i_{l_2'} = i_{l'}$$

and

$$i_{l'} > \max\{i_{l_1'-1}, i_{l_2'+1}\}.$$

Therefore, $\mathcal{I}$ contains a crest. □

Lemma 5.10. *Given two index sequences $\mathcal{I}, \mathcal{J} \in \mathbb{Z}^L$ such that $\mathcal{I} \sim_m \mathcal{J}$. If $(i_{l_1'}, \ldots, i_{l_2'})$ is a crest of $\mathcal{I}$, then $(j_{l_1'}, \ldots, j_{l_2'})$ is a crest of $\mathcal{J}$.*

Proof. Since $\mathcal{I} \sim_m \mathcal{J}$, all consecutive pairs (i_l, i_{l+1}) and (j_l, j_{l+1}) have the same order. According to the definition, we have

$$i_{l_1'-1} < i_{l_1'} = i_{l_1'+1} \cdots = i_{l_2'} > j_{l_2'+1}.$$

If follows that

$$j_{l_1'-1} < j_{l_1'} = j_{l_1'+1} \cdots = j_{l_2'} > j_{l_2'+1};$$

thus $(j_{l_1'}, \ldots, j_{l_2'})$ is a crest of $\mathcal{J}$. □

Now, we will introduce some $\sim_m$ preserving operations on index sequences. The first operation is to remove a crest from a sequence. Let $(i_{l_1'}, \ldots, i_{l_2'})$ be an interval of $\mathcal{I} = (i_1, \ldots, i_L)$. We denote by $\mathcal{I} \setminus (i_{l_1'}, \ldots, i_{l_2'})$ the new sequence $(i_1, \ldots, i_{l_1'-1}, i_{l_2'+1}, \ldots, i_L)$. We denote the empty set by $\emptyset = \mathcal{I} \setminus \mathcal{I}$ and we assume that $\emptyset \sim_m \emptyset$.

Lemma 5.11. *Let $\mathcal{I} = (i_1, \ldots, i_L), \mathcal{J} = (j_1, \ldots, j_L) \in \mathbb{Z}^L$ such that $\mathcal{I} \sim_m \mathcal{J}$. If $(i_{l_1'}, \ldots, i_{l_2'})$ is a crest of $\mathcal{I}$ and $(j_{l_1'}, \ldots, j_{l_2'})$ is a crest of $\mathcal{J}$, then*

$$\mathcal{I} \setminus (i_{l_1'}, \ldots, i_{l_2'}) \sim_m \mathcal{J} \setminus (j_{l_1'}, \ldots, j_{l_2'}).$$

Proof. If $\mathcal{I} \setminus (i_{l_1'}, \ldots, i_{l_2'})$ is empty, then $\mathcal{J} \setminus (j_{l_1'}, \ldots, j_{l_2'})$ must be empty because the lengths of $\mathcal{I}$, $\mathcal{J}$ are the same. The statement is true in this situation. If $\mathcal{I} \setminus (i_{l_1'}, \ldots, i_{l_2'})$ is nonempty, then $\mathcal{I}$ can be written as

$$(i_1, \ldots, i_{l_1'}, \ldots, i_{l_2'}, \ldots, i_L)$$

and

$$\mathcal{I} \setminus (i_{l_1'}, \ldots, i_{l_2'}) = (i_1, \ldots, i_{l_1'-1}, i_{l_2'+1}, \ldots, i_L) = (i_1', \ldots, i_{l_1'-1}', i_{l_1'}', \ldots, i_{L-l_2'+l_1'-1}')$$

and

$$\mathcal{J} \setminus (j_{l_1'}, \ldots, j_{l_2'}) = (j_1, \ldots, j_{l_1'-1}, j_{l_2'+1}, \ldots, j_L) = (j_1', \ldots, j_{l_1'-1}', j_{l_1'}', \ldots, j_{L-l_2'+l_1'-1}').$$

For any indices $1 \leq l_1 < l_2 < L - l_2' + l_1' - 1$ such that $i_{l_3} > \max\{i_{l_1}', i_{l_2}'\}$ for all $l_1 < l_3 < l_2$:

If $l_1, l_2 \leq l_1' - 1$ or $l_1, l_2 \geq l_1'$, then $(i_{l_1}', \ldots, i_{l_2}')$ is an interval of $\mathcal{I}$. Since $\mathcal{I} \sim_m \mathcal{J}$, (i_{l_1}', i_{l_2}') and (j_{l_1}', j_{l_2}') have the same order.

If $l_1 < l_1' \leq l_2$, then $i_{l_2}' = i_{l_2+l_2'-l_1'+1}$. We have

$$i_{l_3} > i_{l_1'-1} \geq \max\{i_{l_1}', i_{l_2}'\}$$

for all $l_1' \leq l_3 \leq l_2'$. It follows that

$$i_{l_3} > \max\{i_{l_1}, i_{l_2}\}$$

for all $l_1 < l_3 < l_2 + l_2' - l_1' + 1$. Therefore, $(i_{l_1}, i_{l_2+l_2'-l_1'+1})$ and $(j_{l_1}, j_{l_2+l_2'-l_1'+1})$ have the same order, which shows that (i'_{l_1}, i'_{l_2}) and (j'_{l_1}, j'_{l_2}) have the same order. The proof is complete. $\square$

Similarly, as in the previous proof, by checking the definition of $\sim_m$, we have

Lemma 5.12. *Let $\mathcal{I} = (i_1, \ldots, i_L) \in \mathbb{Z}^L$ and $(i_{l_1'}, \ldots, i_{l_2'})$ be a crest of $\mathcal{I}$. Then we have*

$$\mathcal{I} = (i_1, \ldots, i_L) \sim_m (i_1, \ldots, i_{l_1'-1}, i_{l_1'} + K, \ldots, i_{l_2'} + K, i_{l_2'+1}, \ldots, i_l)$$

for any integer K such that $i_{l_1'} + K > \max\{i_{l_1'-1}, i_{l_2'+1}\}$.

Now, we study some relations between $M_i(n, k)$ and $\sim_m$:

Proposition 5.13. *Given two sequences $\mathcal{I}=\{i_1, \ldots, i_L\} \in [k]^L, \mathcal{J}=\{j_1, \ldots, j_L\} \in [n]^L$, let $\{u_{i,j}\}_{i=1,\ldots,n;j=1,\ldots,k}$ be the set of standard generators of $M_i(n, k)$. Then we have*

$$\sum_{(q_1,\ldots,q_L)\sim_m \mathcal{J}} u_{q_1,i_1} \cdots u_{q_L,i_L} \mathbf{P} = \begin{cases} \mathbf{P} & \text{if } \mathcal{J} \sim_m \mathcal{I}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will prove the proposition by induction.

When $L = 1$, the statement is obviously true.

Suppose the statement is true for all $L \leq L'$. Let us consider the case $L = L' + 1$. Let $(i_{l_1'}, \ldots, i_{l_2'})$ be a crest of $\mathcal{I}$.

Case 1. If $(j_{l_1'}, \ldots, j_{l_2'})$ is not a crest of $\mathcal{J}$, then $\mathcal{I} \not\sim_m \mathcal{J}$ and one of the following cases happens:

1. There exists an index $j_{l_3'}$ of $\mathcal{J}$ such that $j_{l_3'} \neq j_{l_3'+1}$ for some $l_1' \leq l_3' < l_2'$.
2. $j_{l_1'} \leq j_{l_1'-1}$.
3. $j_{l_2'} \leq j_{l_2'+1}$.

But, for all $\mathcal{Q} = (q_1, \ldots, q_L) \sim_m \mathcal{J}$, we have:

1. $(q_{l_3'}, q_{l_3'-1})$ and $(j_{l_3'}, j_{l_3'-1})$ have the same order.
2. $(q_{l_1'}, q_{l_1'-1})$ and $(j_{l_1'}, j_{l_1'-1})$ have the same order.
3. $(q_{l_2'}, q_{l_2'+1})$ and $(j_{l_2'}, j_{l_2'+1})$ have the same order.

Therefore, we have at least one of the following:

1. $q_{l_3'} \neq q_{l_3'-1}$ and $i_{l_3'} = i_{l_3'-1}$ for some $l_1' \leq l_3' < l_2'$.
2. $q_{l_1'} \leq q_{l_1'-1}$ and $i_{l_1'} > i_{l_1'-1}$.
3. $q_{l_2'} \leq q_{l_2'+1}$ and $i_{l_2'} > i_{l_2'+1}$.

According to the definition of $M_i(n, k)$, we have one of the following equations:

1. $u_{q_{l_3'}, i_{l_3'}} u_{q_{l_3'+1}, i_{l_3'+1}} = 0$ for some $l_1' \leq l_3' < l_2'$.
2. $u_{q_{l_1'-1}, i_{l_1'-1}} u_{q_{l_1'}, i_{l_1'}} = 0$.
3. $u_{q_{l_2'}, i_{l_2'}} u_{q_{l_2'+1}, i_{l_2'+1}} = 0$.

In this case, we always have

$$\sum_{(q_1,\ldots,q_L)\sim_m \mathcal{J}} u_{q_1,i_1} \cdots u_{q_L,i_L} \mathbf{P} = 0.$$

Case 2. If $(j_{l_1'}, \ldots, j_{l_2'})$ is a crest of $\mathcal{J}$, then $(q_{l_1'}, \ldots, q_{l_2'})$ is a crest of $\mathcal{Q}$. Therefore,

$$u_{q_{l_1'}, i_{l_1'}} \cdots u_{q_{l_2'}, i_{l_2'}} = u_{q_{l_1'}, i_{l_1'}}.$$

By Lemma 5.12, if we fix the indices of $\mathcal{Q}\setminus(q_{l_1'},\dots,q_{l_2'})$, then $q_{l_1'},\dots,q_{l_2'}$ can be any integers such that $q_{l_1'}=\cdots=q_{l_2'}$ and $\max\{q_{l_1'-1},q_{l_2'+1})\}<q_{l_1'}\le n$. Therefore, we have

$$\begin{aligned}
&\sum_{\max\{q_{l_1'-1},q_{l_2'+1})\}<q_{l_1'}\le n} u_{q_{l_1'-1},i_{l_1'-1}}u_{q_{l_1'},i_{l_1'}}u_{q_{l_2'+1},i_{l_2'+1}}\\
=&\sum_{1\le q_{l_1'}\le n} u_{q_{l_1'-1},i_{l_1'-1}}u_{q_{l_1'},i_{l_1'}}u_{q_{l_2'+1},i_{l_2'+1}}\\
=&\ u_{q_{l_1'-1},i_{l_1'-1}}u_{q_{l_2'+1},i_{l_2'+1}}.
\end{aligned}$$

The first equality holds because the extra terms are 0. The second equality uses the monotone universal condition of $M_i(n,k)$. Let $L''=L-l_2'+l_1'+1\le L'$; then $\mathcal{J}\setminus(j_{l_1'},\dots,j_{l_2'})\in[n]^{L''}$. By Lemma 5.10, $\mathcal{Q}\setminus(q_{l_1'},\dots,q_{l_2'})\sim_m\mathcal{J}\setminus(j_{l_1'},\dots,j_{l_2'})$. If we denote by $(i_1',\dots,i_{L''}')$ the sequence $\mathcal{I}\setminus(i_{l_1'},\dots,i_{l_2'})$, then we have

$$\begin{aligned}
&\sum_{(q_1,\dots,q_L)\sim_m\mathcal{J}} u_{q_1,i_1}\cdots u_{q_L,i_L}\mathbf{P}\\
=&\sum_{(q_1',\dots,q_{L''}')\sim_m\mathcal{J}\setminus(j_{l_1'},\dots,j_{l_2'})} u_{q_1',i_1'}\cdots u_{q_{L''}',i_{L''}'}\mathbf{P}\\
=&\begin{cases}\mathbf{P} & \text{if } \mathcal{J}\setminus(j_{l_1'},\dots,j_{l_2'})\sim_m\mathcal{I}\setminus(i_{l_1'},\dots,i_{l_2'}),\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

The last equality comes from the assumption of our induction. By Lemma 5.10 and Lemma 5.11, $\mathcal{J}\setminus(j_{l_1'},\dots,j_{l_2'})\sim_m\mathcal{I}\setminus(i_{l_1'},\dots,i_{l_2'})$ iff $\mathcal{J}\sim_m\mathcal{I}$. The proof is complete. □

Now, we show that operator valued monotone finite sequences of random variables are monotonically spreadable.

Definition 5.14. Let $\mathcal{I}=(i_1,\dots,i_L)$ be a sequence of ordered indices and $a=\min\{i_1,\dots,i_L\}$. We call the set $\S(\mathcal{I})=\{l|i_l=a\}$ the positions of the smallest elements of $\mathcal{I}$. An interval of $(i_{l_1'},\dots,i_{l_2'})$ is called a hill of $\mathcal{I}$ if $i_{l_1'-1}=i_{l_2'+1}=a$ and $i_{l_3}'\neq a$ for all $l_1'\le l_3'\le l_2'$. Here, we assume that $i_0=i_{L+1}=a$ for convenience.

Example. $(1,2,3,4,1,2,1)$ has two hills $(2,3,4)$ and (2). $(1,2,1,3,4)$ has two hills (2) and $(3,4)$. $(1,1,1,1,1)$ has no hill.

Lemma 5.15. *Given two sequences $\mathcal{I}=\{i_1,\dots,i_L\},\mathcal{J}=\{j_1,\dots,j_L\}\in[n]^L$ such that $\mathcal{I}\sim_m\mathcal{J}$, then $\S(\mathcal{I})=\S(\mathcal{J})$. Let $(i_{l_1'},\dots,i_{l_2'})$ be a hill of $\mathcal{I}$; then*

$$(i_{l_1'},\dots,i_{l_2'})\sim_m(j_{l_1'},\dots,j_{l_2'}).$$

Proof. We just need to check the elements of $\mathcal{J}$ one by one. Suppose

$$\S(\mathcal{I})=\{l_1''<\cdots<l_{k'}''\},$$

where k' is the cardinality of $\S(\mathcal{I})$. Let $b=\min\{j_1,\dots,j_L\}$. We need to show that $j_{l_1''}=\cdots=j_{l_{k'}''}=b$ and $j_l>b$ for all $l\notin\S(\mathcal{I})$.

Given an integer $1\le p<k'$, we have

$$i_l>a=i_{l_p''}=i_{l_{p+1}''}$$

for all $l_p''<l<l_{p+1}''$. According to the definition of $\sim_m$ and Lemma 5.5, we have

$$j_{l_p''}=j_{l_{p+1}''}$$

and

$$j_l > \max\{j_{l''_p}, j_{l''_{p+1}}\}$$

for all $l''_p < l < l''_{p+1}$. What's left is to check the elements j_l with $l < l''_1$ or $l > l''_{k'}$. If there exists $l < l''_1$ such that $j_l \leq j_{l''_1}$, we choose the greatest such l. Then, we have

$$j_{l'} > \max\{j_l, j_{l''_1}\}$$

for all $l < l' < l''_1$. Therefore, we have

$$i_l \leq i_{l''_1},$$

which is a contradiction. This implies that

$$j_l > j_{l''_1}$$

for all $l < l''_1$. Similarly we have

$$j_l > j_{l''_1}$$

for all $l > l''_k$. Therefore, $j_{l''_1} = \cdots = j_{l''_{k'}} = \min\{j_1, \ldots, j_L\}$. The last statement of this lemma is obvious from the definition of $\sim_m$. □

Given $\mathcal{I} = \{i_1, \ldots, i_L\} \in \mathbb{Z}^L$, we will denote $x_{\mathcal{I}} = x_{i_1} x_{i_2} \cdots x_{j_L}$ for short.

Proposition 5.16. *Let $(\mathcal{A}, \mathcal{B}, E)$ be an operator valued probability space and $(x_i)_{i=1,\ldots,n}$ be a sequence of random variables in $\mathcal{A}$. If $(x_i)_{i=1,\ldots,n}$ are identically distributed and monotonically independent, then for indices sequences $\mathcal{I} = \{i_1, \ldots, i_L\}, \mathcal{J} = \{j_1, \ldots, j_L\} \in [n]^L$ such that $\mathcal{I} \sim_m \mathcal{J}$, $L \in \mathbb{N}$, we have*

$$E[x_{\mathcal{I}}] = E[x_{\mathcal{J}}].$$

Proof. When $L = 1$, the statement is true since the sequence is identically distributed.

Suppose the statement is true for all $L \leq L' \in \mathbb{N}$. Let us consider the case $L = L' + 1$:

If $\mathcal{I}$ has no hill, then $i_1 = \cdots = i_L$, which implies that $j_1 = \cdots = j_L$. The statement is true, since the sequence is identically distributed.

Suppose $\mathcal{I}$ has hills $\mathcal{I}_1, \ldots, \mathcal{I}_l$ and $a = \min\{i_1, \ldots, i_L\}$. Then, $x_{\mathcal{I}}$ can be written as

$$x_a^{n_1} x_{\mathcal{I}_1} x_a^{n_2} x_{\mathcal{I}_2} \cdots x_a^{n_l} x_{\mathcal{I}_l} x_a^{n_{l+1}},$$

where $n_2, \ldots, n_l \in \mathbb{N}$ and $n_1, n_{l+1} \in \mathbb{N} \cup \{0\}$. Since the x_i's are monotonically independent, we have

$$E[x_{\mathcal{I}}] = E[x_a^{n_1} E[x_{\mathcal{I}_1}] x_a^{n_2} E[x_{\mathcal{I}_2}] \cdots x_a^{n_l} E[x_{\mathcal{I}_l}] x_a^{n_{l+1}}].$$

Let $b = \min\{j_1, \ldots, j_L\}$. By Lemma 5.15, $\mathcal{J}$ has hills $\mathcal{J}_1, \ldots, \mathcal{J}_l$ whose positions of elements correspond to the positions of elements of $\mathcal{I}_1, \ldots, \mathcal{I}_l$ and $\mathcal{J}_{l'} \sim_m \mathcal{J}_{l'}$ for all $1 \leq l' \leq k'$. Therefore, we have

$$\begin{aligned} E[x_{\mathcal{J}}] &= E[x_b^{n_1} E[x_{\mathcal{J}_1}] x_b^{n_2} E[x_{\mathcal{J}_2}] \cdots x_b^{n_l} E[x_{\mathcal{J}_l}] x_b^{n_{l+1}}] \\ &= E[x_b^{n_1} E[x_{\mathcal{I}_1}] x_b^{n_2} E[x_{\mathcal{I}_2}] \cdots x_b^{n_l} E[x_{\mathcal{I}_l}] x_b^{n_{l+1}}] \\ &= E[x_a^{n_1} E[x_{\mathcal{I}_1}] x_a^{n_2} E[x_{\mathcal{I}_2}] \cdots x_a^{n_l} E[x_{\mathcal{I}_l}] x_a^{n_{l+1}}] \\ &= E[x_{\mathcal{I}}], \end{aligned}$$

where the second equality follows the induction and the third equality holds because x_a and x_b are identically distributed. The proof is complete. □

Proposition 5.17. *Let $(\mathcal{A}, \mathcal{B}, E)$ be an operator valued probability space, and $(x_i)_{i=1,\ldots,n}$ be a sequence of random variables in $\mathcal{A}$ which are identically distributed and monotonically independent with respect to E. Let ϕ be a state on $\mathcal{A}$ such that $\phi(\cdot) = \phi(E[\cdot])$. Then, $(x_i)_{i=1,\ldots,n}$ is monotonically spreadable with respect to ϕ.*

Proof. For fixed natural numbers $n, k \in \mathbb{N}$, let $(u_{i,j})_{i=1,\ldots,n;j=1,\ldots,k}$ be standard generators of $M_i(n,k)$. Let $\mathcal{J} = (j_1, \ldots, j_L) \in [k]^L$ and denote $x_{j_1} \cdots x_{j_L}$ by $x_{\mathcal{J}}$. We denote the equivalent class of $[n]^L$ associated with $\sim_m$ by $\overline{[n^L]}$. For each $\mathcal{I} \in [n]^L$, we denote $u_{i_1,j_1} \cdots u_{i_L,j_L}$ by $u_{\mathcal{I},\mathcal{J}}$. Then, by Proposition 5.13, we have

$$\begin{aligned}
& \sum_{\mathcal{I}\in[n]^L} \phi(x_{\mathcal{I}})\mathbf{P}u_{\mathcal{I},\mathcal{J}}\mathbf{P} \\
= & \sum_{\mathcal{I}\in[n]^L} \phi(E[x_{\mathcal{I}}])\mathbf{P}u_{\mathcal{I},\mathcal{J}}\mathbf{P} \\
= & \sum_{\bar{Q}\in\overline{[n]^L}} \sum_{\mathcal{I}\in\bar{Q}} \phi(E[x_{\mathcal{I}}])\mathbf{P}u_{\mathcal{I},\mathcal{J}}\mathbf{P} \\
= & \sum_{\mathcal{J}\notin\bar{Q}\in\overline{[n]^L}} \sum_{\mathcal{I}\in\bar{Q}} \phi(E[x_{\mathcal{I}}])\mathbf{P}u_{\mathcal{I},\mathcal{J}}\mathbf{P} + \sum_{\mathcal{J}\in\bar{Q}\in\overline{[n]^L}} \sum_{\mathcal{I}\in\bar{Q}} \phi(E[x_{\mathcal{I}}])\mathbf{P}u_{\mathcal{I},\mathcal{J}}\mathbf{P} \\
= & \sum_{\mathcal{J}\notin\bar{Q}\in\overline{[n]^L}} \sum_{\mathcal{I}\in\bar{Q}} \phi(E[x_{\mathcal{Q}}])\mathbf{P}u_{\mathcal{I},\mathcal{J}}\mathbf{P} + \sum_{\mathcal{I}\sim_m\mathcal{J}} \phi(E[x_{\mathcal{J}}])\mathbf{P}u_{\mathcal{I},\mathcal{J}}\mathbf{P} \\
= & \; 0 + \phi(E[x_{\mathcal{J}}])\mathbf{P} \\
= & \; \phi(x_{\mathcal{J}})\mathbf{P}.
\end{aligned}$$

Since n, k are arbitrary, the proof is complete. ☐

6. Tail algebras

In the previous work on distributional symmetries, infinite sequences of objects are indexed by natural numbers. For these kinds of infinite sequences, the conditional expectations in de Finetti type theorems are defined via the limit of unilateral shifts. It was shown in [15] that a unilateral shift is an isometry from $\mathcal{A}$ into itself if $(\mathcal{A}, \phi)$ is a W^*-probability space generated by a spreadable sequence of random variables and ϕ is faithful. Therefore, a normal conditional expectation defined via the limit of unilateral shifts exists under a very weak condition; i.e. the sequence of random variables just needs to be spreadable. However, our work is in a more general situation where the state ϕ is not necessarily faithful. In our framework, we will provide an example in which the sequence is monotonically spreadable, but the unilateral shift is not an isometry. Therefore, we cannot get an extended de Finetti type theorem for monotone independence in the usual way. Therefore, we will consider bilateral sequences of random variables. Here, we begin with an interesting example.

6.1. **Unbounded spreadable sequences.**

Example. Let $\mathcal{H}$ be the standard 2-dimensional Hilbert space with orthonormal basis

$$\{v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}.$$

Let $p, A, x \in B(\mathcal{H})$ be operators on $\mathcal{H}$ with the following matrix forms:

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\mathscr{H} = \bigotimes_{n=1}^{\infty} \mathcal{H}$ be the infinite tensor product of $\mathcal{H}$. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of selfadjoint operators in $B(\mathscr{H})$ defined as follows:

$$x_i = \bigotimes_{n=1}^{i-1} A \otimes x \otimes \bigotimes_{m=1}^{\infty} p.$$

Let ϕ be the vector state $\langle \cdot v, v\rangle$ on $\mathcal{H}$ and let $\Phi = \bigotimes_{n=1}^{\infty} \phi$ be a state on $B(\mathscr{H})$. It is obvious that $\Phi(x_i^n) = \phi(x^n)$ for i. Therefore, the sequence $(x_i)_{i\in\mathbb{N}}$ is identically distributed. For any $y, z \in B(\mathcal{H})$, an elementary computation shows that

$$\phi(ypz) = \phi(y)\phi(z).$$

For convenience, we denote $A^{\otimes i-1} = \bigotimes_{n=1}^{i-1} A$ and $P^{\otimes\infty} = \bigotimes_{n=1}^{\infty} P$. Also, we denote $x_{i_1}\cdots x_{i_L} = x_{\mathcal{I}}$ for $\mathcal{I} = (i_1, \ldots, i_L) \in \mathbb{N}^L$. We will show that the sequence $\{x_i\}_{i\in\mathbb{N}}$ is $M_i(n,k)$-spreadable with respect to Φ.

Lemma 6.1. *For indices sequences $\mathcal{I} = (i_1, \ldots, i_L), \mathcal{J} = (j_1, \ldots, j_L) \in [n]^L$ such that $\mathcal{I} \sim_m \mathcal{J}$ and $L \in \mathbb{N}$, we have*

$$\Phi(x_{\mathcal{I}}) = \Phi(x_{\mathcal{J}}).$$

Proof. When $L = 1$, the statement is true since the sequence is identically distributed.

Suppose the statement is true for all $L \le L'$. Let us consider the case $L = L'+1$. If $\mathcal{I}$ has no hill, then $i_1 = \cdots = i_L$, which implies that $j_1 = \cdots = j_L$. The statement is true for this case, because the sequence is identically distributed. Also, we denote by $x_i^{(n)}$ the n-th component of x_i. Then,

$$x_i^{(n)} = \begin{cases} a & \text{if } n < i, \\ x & \text{if } n = i, \\ p & \text{if } n > i, \end{cases}$$

and $x_{\mathcal{I}}^{(n)} = x_{i_1}^{(n)} x_{i_2}^{(n)} \cdots x_{i_L}^{(n)}$.

According to the definition of Φ, we have that

$$\Phi(x_{i_1}x_{i_2}\cdots x_{j_L}) = \prod_{n=1}^{\infty} \phi(\prod_{l=1}^{L} x_i^{(n)}).$$

Suppose that $\mathcal{I}$ has hills $\mathcal{I}_1, \ldots, \mathcal{I}_l$ and $a = \min\{i_1, \ldots, i_L\}$. Then $x_{\mathcal{I}}$ can be written as

$$x_a^{n_1} x_{\mathcal{I}_1} x_a^{n_2} x_{\mathcal{I}_2} \cdots x_a^{n_l} x_{\mathcal{I}_l} x_a^{n_{l+1}}$$

and

$$\phi(\prod_{l=1}^{L} x_i^{(n)}) = \begin{cases} 1 & \text{if } n < a, \\ \phi(x^{n_1} A^{|\mathcal{I}_1|} x^{n_2} A^{|\mathcal{I}_2|} \cdots x^{n_l} A^{|\mathcal{I}_l|} x^{n_{l+1}}) & \text{if } n = a, \\ \phi(p x_{\mathcal{I}_1}^{(n)} p x_{\mathcal{I}_2}^{(n)} p \cdots p x_{\mathcal{I}_l}^{(n)} p) & \text{if } n > a. \end{cases}$$

It follows that

$$\phi(\prod_{l=1}^{L} x_i^{(n)}) = \prod_{n\ge\min\{\mathcal{I}\}}^{\infty} \phi(\prod_{l=1}^{L} x_i^{(n)}).$$

Because

$$\phi(p x_{\mathcal{I}_1}^{(n)} p x_{\mathcal{I}_2}^{(n)} p \cdots p x_{\mathcal{I}_l}^{(n)} p) = \phi(x_{\mathcal{I}_1}^{(n)})\phi(x_{\mathcal{I}_2}^{(n)}) \cdots \phi(x_{\mathcal{I}_l}^{(n)}),$$

we have

$$\begin{aligned}
& \Phi(x_{i_1}x_{i_2}\cdots x_{j_L}) \\
= \; & \phi(x^{n_1}A^{|\mathcal{I}_1|}x^{n_2}A^{|\mathcal{I}_2|}\cdots x^{n_l}A^{|\mathcal{I}_l|}x^{n_{l+1}})\prod_{n>a}^{\infty}\phi(px_{\mathcal{I}_1}^{(n)}px_{\mathcal{I}_2}^{(n)}p\cdots px_{\mathcal{I}_l}^{(n)}p) \\
= \; & \phi(x^{n_1}A^{|\mathcal{I}_1|}x^{n_2}A^{|\mathcal{I}_2|}\cdots x^{n_l}A^{|\mathcal{I}_l|}x^{n_{l+1}})\prod_{n>a}^{\infty}\phi(x_{\mathcal{I}_1}^{(n)})\phi(x_{\mathcal{I}_2}^{(n)})\cdots\phi(x_{\mathcal{I}_l}^{(n)}) \\
= \; & \phi(x^{n_1}A^{|\mathcal{I}_1|}x^{n_2}A^{|\mathcal{I}_2|}\cdots x^{n_l}A^{|\mathcal{I}_l|}x^{n_{l+1}})\Phi(x_{\mathcal{I}_1})\Phi(x_{\mathcal{I}_2})\cdots\Phi(x_{\mathcal{I}_l}).
\end{aligned}$$

Let $b = \min\{j_1,\ldots,j_L\}$. By Lemma 5.15, $\mathcal{J}$ has hills $\mathcal{J}_1,\ldots,\mathcal{J}_l$ whose positions of elements correspond to the positions of elements of $\mathcal{I}_1,\ldots,\mathcal{I}_l$ and $\mathcal{J}_{l'} \sim_m \mathcal{J}_{l'}$ for all $1 \le l' \le k'$. Therefore, we have

$$\begin{aligned}
\Phi(x_{\mathcal{J}}) &= \Phi(x_{i_1}x_{i_2}\cdots x_{i_L}) \\
&= \phi(x^{n_1}A^{|\mathcal{J}_1|}x^{n_2}A^{|\mathcal{J}_2|}\cdots x^{n_l}A^{|\mathcal{J}_l|}x^{n_{l+1}})\Phi(x_{\mathcal{J}_1})\Phi(x_{\mathcal{J}_2})\cdots\Phi(x_{\mathcal{J}_l}) \\
&= \phi(x^{n_1}A^{|\mathcal{I}_1|}x^{n_2}A^{|\mathcal{I}_2|}\cdots x^{n_l}A^{|\mathcal{I}_l|}x^{n_{l+1}})\Phi(x_{\mathcal{I}_1})\Phi(x_{\mathcal{I}_2})\cdots\Phi(x_{\mathcal{I}_l}) \\
&= \Phi(x_{\mathcal{I}}),
\end{aligned}$$

where the second equality follows from induction, the fact that $\mathcal{J}_k \sim_m \mathcal{I}_k$ and $|\mathcal{J}_k| = |\mathcal{I}_k|$ for all $1 \le k \le l$. ☐

Proposition 6.2. *The joint distribution of* $(x_i)_{i\in\mathbb{N}}$ *with respect to* Φ *is monotonically spreadable.*

Proof. Fix $n > k \in \mathbb{N}$, and let $\{u_{i,j}\}_{i=1,\ldots,n;j=1,\ldots,k}$ be the set of standard generators of $M_i(n,k)$. For all $\mathcal{I} = (i_1,\ldots,i_L) \in [k]^L$, we denote by $\overline{[n]^L}$ the $\sim_m$ equivalence classes of $[n]^L$. Then we have

$$\begin{aligned}
& \mathbf{P}(\mu_{x_1,\ldots,x_n}\otimes id_{M_i(n,k)})(\alpha_{n,k}^{(m)}(X_{\mathcal{I}}))\mathbf{P} \\
= \; & \sum_{\mathcal{J}\in[n]^L}\mu_{x_1,\ldots,x_n}(X_{\mathcal{J}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} \\
= \; & \sum_{\bar{\mathcal{Q}}\in\overline{[n]^L}}\sum_{\mathcal{J}\in\bar{\mathcal{Q}}}\mu_{x_1,\ldots,x_n}(X_{\mathcal{J}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} \\
= \; & \sum_{\mathcal{Q}\in\overline{[n]^L}\setminus\{\bar{\mathcal{I}}\}}\sum_{\mathcal{J}\in\bar{\mathcal{Q}}}\mu_{x_1,\ldots,x_n}(X_{\mathcal{J}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} + \sum_{\mathcal{J}\sim_m\mathcal{I}}\mu_{x_1,\ldots,x_n}(X_{\mathcal{J}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} \\
= \; & \sum_{\mathcal{Q}\in\overline{[n]^L}\setminus\{\bar{\mathcal{I}}\}}\sum_{\mathcal{J}\in\bar{\mathcal{Q}}}\mu_{x_1,\ldots,x_n}(X_{\mathcal{Q}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} + \sum_{\mathcal{J}\sim_m\mathcal{I}}\mu_{x_1,\ldots,x_n}(X_{\mathcal{I}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} \\
= \; & \sum_{\mathcal{Q}\in\overline{[n]^L}\setminus\{\bar{\mathcal{I}}\}}\mu_{x_1,\ldots,x_n}(X_{\mathcal{Q}})\sum_{\mathcal{J}\in\bar{\mathcal{Q}}}\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} + \sum_{\mathcal{J}\sim_m\mathcal{I}}\mu_{x_1,\ldots,x_n}(X_{\mathcal{I}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} \\
= \; & \sum_{\mathcal{Q}\in\overline{[n]^L}\setminus\{\bar{\mathcal{I}}\}}\mu_{x_1,\ldots,x_n}(X_{\mathcal{Q}})\cdot 0 + \sum_{\mathcal{J}\sim_m\mathcal{I}}\mu_{x_1,\ldots,x_n}(X_{\mathcal{I}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} \\
= \; & \sum_{\mathcal{J}\sim_m\mathcal{I}}\mu_{x_1,\ldots,x_n}(X_{\mathcal{I}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} \\
= \; & \mu_{x_1,\ldots,x_n}(X_{\mathcal{I}})\mathbf{P}.
\end{aligned}$$

The proof is complete. ☐

By direct computations, we have

$$(\prod_{i=1}^{n}x_{n+1-i})v^{\otimes\infty} = w^{\otimes n}\otimes v^{\otimes\infty}$$

and

$$x_{n+1}(w^{\otimes n} \otimes v^{\otimes \infty}) = 2^n w^{\otimes n+1} \otimes v^{\otimes \infty}. \tag{3}$$

Let $(\mathcal{H}', \pi', \xi')$ be the GNS representation of the von Neumann algebra generated by $(x_i)_{i=1,\ldots,\infty}$ associated with Φ. We have

$$\|\pi'(x_{n+1})\| \leq \|x_{n+1}\| = 2^n,$$

but equation (3) shows that $\|\pi'(x_{n+1})\| \geq 2^n$. Therefore, $\|\pi'(x_{n+1})\| = 2^n$.

Let $\mathcal{A}$ be the von Neumann algebra generated by $\pi(x_i)$'s. Then, there is no bounded endomorphism α on $\pi(\mathcal{A})$ such that $\alpha(\pi(x_i)) = \pi(x_{i+1})$.

6.2. **Tail algebras of bilateral sequences of random variables.** We have shown that in a W^*-probability space with a nondegenerated normal state, the unilateral shift of a spreadable unilateral sequence of random variables may not be extended to a bounded endomorphism. Therefore, in general, we cannot define a normal condition expectation by taking the limit of unilateral shifts of variables. In $(\mathcal{A}, \phi)$, a W^*-probability space with a faithful state, the norm of a selfadjoint random variable $x \in \mathcal{A}$ is controlled by the moments of X, i.e.

$$\|x\| = \lim_{n\to\infty} \phi(|x|^n)^{\frac{1}{n}}.$$

But, in nondegenerated W^*-probability spaces, the norm of a random variable depends on all mixed moments which involve it. As a kind of partial distributional symmetry, spreadability cannot provide relations between all mixed moments, which means a spreadable sequence can be unbounded. To create a well defined conditional expectation, we consider spreadable sequences of random variables indexed by $\mathbb{Z}$ but not $\mathbb{N}$. As a consequence, we will have two choices to take limits on defining normal conditional expectations and tail algebras. Before studying tail algebras of bilateral sequences, we introduce some necessary notation and assumptions here.

Let $(\mathcal{A}, \phi)$ be a W^*-probability space generated by a spreadable bilateral sequence of bounded random variables $(x_i)_{i\in\mathbb{Z}}$ and let ϕ be a nondegenerated normal state. We assume that the unit of $\mathcal{A}$ is contained in the WOT-closure of the nonunital algebra generated by $(x_i)_{i\in\mathbb{Z}}$. Let $(\mathcal{H}, \pi, \xi)$ be the GNS representation of $\mathcal{A}$ associated with ϕ. Then, $\{\pi(P(x_i|i \in \mathbb{Z}))\xi | P \in \mathbb{C}\langle X_i | i \in \mathbb{Z}\rangle\}$ is dense in $\mathcal{H}$. For convenience, we will denote $\pi(y)\xi$ by $\hat{y}$ for all $y \in \mathcal{A}$. When there is no confusion, we will write y for $\pi(y)$. We denote by A_{k+} the nonunital algebra generated by $(x_i)_{i\geq k}$ and A_{k-} the nonunital algebra generated by $(x_i)_{i\leq k}$. Let $\mathcal{A}_k^+$ and $\mathcal{A}_k^-$ be the WOT-closure of A_{k+} and A_{k-}, respectively.

Definition 6.3. Let $(\mathcal{A}, \phi)$ be a nondegenerated noncommutative W^*-probability space, and let $(x_i)_{i\in\mathbb{Z}}$ be a bilateral sequence of bounded random variables in $\mathcal{A}$ such that $\mathcal{A}$ is the WOT closure of the nonunital algebra generated by $(x_i)_{i\in\mathbb{Z}}$. The positive tail algebra $\mathcal{A}_{tail}^+$ of $(x_i)_{i\in\mathbb{Z}}$ is defined as follows:

$$\mathcal{A}_{tail}^+ = \bigcap_{k>0} \mathcal{A}_k^+.$$

In the opposite direction, we define the negative tail algebra $\mathcal{A}_{tail}^-$ of $(x_i)_{i\in\mathbb{Z}}$ as follows:

$$\mathcal{A}_{tail}^- = \bigcap_{k<0} \mathcal{A}_k^-.$$

Remark 6.4. In general, the positive tail algebra and the negative tail algebra are different.

Even though our framework looks quit different from the framework in [15], we can show that there exists a normal bounded shift of the sequence in a similar way. For completeness, we provide the details here.

Lemma 6.5. *There exists a unitary map $U:\mathcal{H}\to\mathcal{H}$ such that $U(P(x_i|i\in\mathbb{Z}))\xi = P(x_{i+1}|i\in\mathbb{Z})\xi$.*

Proof. Since $(x_i)_{i\in\mathbb{Z}}$ is spreadable, we have

$$\phi((P(x_i|i\in\mathbb{Z}))^*P(x_i|i\in\mathbb{Z})) = \phi((P(x_{i+1}|i\in\mathbb{Z}))^*P(x_{i+1}|i\in\mathbb{Z})).$$

This implies that

$$U(P(x_i|i\in\mathbb{Z})\xi) = P(x_{i+1}|i\in\mathbb{Z})\xi$$

is a well defined isometry on $\{\pi(P(x_i|i\in\mathbb{Z}))\xi|P\in\mathbb{C}\langle X_i|i\in\mathbb{Z}\rangle\}$. Since $\{\pi(P(x_i|i\in\mathbb{Z}))\xi|P\in\mathbb{C}\langle X_i|i\in\mathbb{Z}\rangle\}$ is dense in $\mathcal{H}$, U can be extended to the whole space $\mathcal{H}$. It is obvious that $\{\pi(P(x_i|i\in\mathbb{Z}))\xi|P\in\mathbb{C}\langle X_i|i\in\mathbb{Z}\rangle\}$ is contained in the range of U. Therefore, the extension of U is a unitary map on $\mathcal{H}$. □

Now, we can define an automorphism α on $\mathcal{A}$ by the following formula:

$$\alpha(y) = UyU^{-1}.$$

Lemma 6.6. *α is the bilateral shift of $(x_i)_{i\in\mathbb{Z}}$, i.e.*

$$\alpha(x_k) = x_{k+1}$$

for all $k\in\mathbb{Z}$.

Proof. For all $y = P(x_i|i\in\mathbb{Z})\xi$, we have

$$\alpha(x_k)y = Ux_kU^{-1}P(x_i|i\in\mathbb{Z})\xi = Ux_kP(x_{i-1}|i\in\mathbb{Z})\xi = x_{k+1}P(x_i|i\in\mathbb{Z})\xi.$$

By the density of $\{\pi(P(x_i|i\in\mathbb{Z}))\xi|P\in\mathbb{C}\langle X_i|i\in\mathbb{Z}\rangle\}$, we have $\alpha(x_k) = x_{k+1}$. The proof is complete. □

Since α is a normal automorphism of $\mathcal{A}$, we have

Corollary 6.7. *For all $k\in\mathbb{Z}$, we have $\alpha(\mathcal{A}_k^+) = \mathcal{A}_{k+1}^+$.*

Lemma 6.8. *Fix $n\in\mathbb{Z}$. Let $y_1, y_2\in A_{n-}$. Then, we have*

$$\langle\alpha^l(a)\hat{y}_1,\hat{y}_2\rangle = \langle a\hat{y}_1,\hat{y}_2\rangle,$$

where $l\in\mathbb{N}$ and $a\in\mathcal{A}_{n+1}^+$.

Proof. It is sufficient to prove the statement under the assumption that $l=1$. Since $a\in\mathcal{A}_{n+1}^+$, by Kaplansky's theorem, there exists a sequence $(a_m)_{m\in\mathbb{N}}\subset A_{(n+1)+}$ such that $\|a_m\|\le\|a\|$ for all m and a_m converges to a in WOT. Then, by the spreadability of $(x_i)_{i\in\mathbb{Z}}$, we have

$$\langle\alpha(a)\hat{y}_1,\hat{y}_2\rangle = \lim_{m\to\infty}\langle\alpha(a_m)\hat{y}_1,\hat{y}_2\rangle = \lim_{m\to\infty}\phi(y_2^*a_m\hat{y}_1) = \langle a\hat{y}_1,\hat{y}_2\rangle.$$

□

In the following context, we fix $k\in\mathbb{Z}$.

Lemma 6.9. *For all $a \in \mathcal{A}_k^+$, we have that*

$$E^+[a] = WOT - \lim_{l\to\infty} \alpha^l(a)$$

exists. Moreover, $E^+[a] \in \mathcal{A}_{tail}^+$.

Proof. For all $y_1, y_2 \in \{\pi(P(x_i|i \in \mathbb{Z}))\xi | P \in \mathbb{C}\langle X_i|i \in \mathbb{Z}\rangle\}$, there exists $n \in \mathbb{Z}$ such that $y_1, y_2 \in \mathcal{A}_{n-}$. For all $l > n - k$, we have $\alpha^l(a) \in \mathcal{A}_{(n+1)+}$. By Lemma 6.8, we have

$$\langle \alpha^{n+1-k}(a)y_1, y_2\rangle = \langle \alpha^{n+2-k}(a)y_1, y_2\rangle = \cdots .$$

Therefore,

$$\lim_{l\to\infty} \langle \alpha^l(a)y_1, y_2\rangle = \langle \alpha^{n+1-k}(a)y_1, y_2\rangle.$$

$\alpha^l(a)$ converges pointwisely to an element $E^+[a]$. Since for all $n > 0$, we have $\alpha^l(a) \in \mathcal{A}_n^+$ for all $l > n - k + 1$ it follows that $WOT - \lim_{l\to\infty} \alpha^l(a) \in \mathcal{A}_n^+$ for all n. Hence, $E^+[a] \in \mathcal{A}_{tail}^+$. □

Proposition 6.10. *E^+ is normal on $\mathcal{A}_k^+$ for all $k \in \mathbb{Z}$.*

Proof. Let $(a_m)_{m\in\mathbb{N}} \subset \mathcal{A}_k^+$ be a bounded sequence which converges to 0 in WOT. For all $y_1, y_2 \in \{\pi(P(x_i|i \in \mathbb{Z}))\xi | P \in \mathbb{C}\langle X_i|i \in \mathbb{Z}\rangle\}$, there exists $n \in \mathbb{Z}$ such that $y_1, y_2 \in \mathcal{A}_{n-}$. Then, we have

$$\lim_{m\to\infty} \langle E^+[a_m]y_1, y_2\rangle = \lim_{m\to\infty} \langle \alpha^{n+1-k}(a_m)y_1, y_2\rangle = 0.$$

The last equality holds because α^l is normal for all $l \in \mathbb{N}$. The proof is complete. □

Remark 6.11. E^+ is defined on $\bigcup_{k\in\mathbb{Z}} \mathcal{A}_k^+$ but not on $\mathcal{A}$. In general, we cannot extend E^+ to the whole algebra $\mathcal{A}$.

Lemma 6.12. *We have that $E^+[a] = a$ for all $a \in \mathcal{A}_{tail}^+$.*

Proof. For all $\hat{y}_1, \hat{y}_2 \in \{\pi(P(x_i|i \in \mathbb{Z}))\xi | P \in \mathbb{C}\langle X_i|i \in \mathbb{Z}\rangle\}$, there exists $n \in \mathbb{Z}$ such that $y_1, y_2 \in A_{n-}$. Since $a \in \mathcal{A}_{tail}^+ \subset \mathcal{A}_{n+1}^+$, by Kaplansky's theorem, there exists a sequence $(a_m)_{m\in\mathbb{N}} \subset A_{(n+1)+}$ such that $a_m \to a$ in WOT and $\|a_m\| \le \|a\|$ for all m. Then, we have

$$\langle a\hat{y}_1, \hat{y}_2\rangle = \lim_{m\to\infty} \langle a_m\hat{y}_1, \hat{y}_2\rangle = \lim_{m\to\infty} \langle \alpha(a_m)\hat{y}_1, \hat{y}_2\rangle = \langle \alpha(a)\hat{y}_1, \hat{y}_2\rangle.$$

Since y_1, y_2 are arbitrary, we have $a = \alpha(a)$. □

Remark 6.13. One should be careful that $\mathcal{A}_{tail}^+$ could be a proper subset of the fixed points set of α.

Lemma 6.14. *We have*

$$E^+[a_1ba_2] = a_1E^+[b]a_2$$

for all $b \in \mathcal{A}_k^+$, $a_1, a_2 \in \mathcal{A}_{tail}^+$.

Proof. By Lemma 6.12, we have

$$E^+[a_1ba_2] = \lim_{l\to\infty} \alpha^l(a_1ba_2) = \lim_{l\to\infty} \alpha^l(a_1)\alpha^l(b)\alpha^l(a_2) = \lim_{l\to\infty} a_1\alpha^l(b)a_2 = a_1E^+[b]a_2.$$

□

7. Conditional expectations of bilateral monotonically spreadable sequence

In this section, we assume that the joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable.

Lemma 7.1. *Fix $n > k \in \mathbb{N}$, and let $(u_{i,j})_{i=1,\dots,n;\, j=1,\dots,k}$ be the standard generators of $M_i(n,k)$. Then, we have*

$$\phi(a_1x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_2\cdots b_{m-1}x_{i_m}^{l_m}a_2)\mathbf{P} = \sum_{j_1,\dots,j_m=1}^{n}\phi(a_1x_{j_1}^{l_1}b_1x_{j_2}^{l_2}b_2\cdots b_{m-1}x_{j_m}^{l_m}a_2)\mathbf{P}u_{j_1,i_1}\cdots u_{j_m,i_m}\mathbf{P},$$

where $1\le i_1,\dots,i_m\le k$, $b_1,\dots,b_{m-1}\in A_{(n+1)+}$ and $a_1,a_2\in A_{0-}$.

Proof. Without loss of generality, we assume that there exist $n_1,n_2\in\mathbb{N}$ such that

$$a_1,a_2\in A_{[-n_1+1,0]}$$

and

$$b_1,\dots,b_{m-1}\in A_{[n+1,n_2+k]}.$$

Since the map is linear, we just need to consider the case that a_1,a_2 and $b_1,\dots,b_{m-1}$ are products of $(x_i)_{i\in\mathbb{Z}}$. Let

$$a_1 = x_{s_{1,1}}\cdots x_{s_{1,t_1}}$$

and

$$a_2 = x_{s_{2,1}}\cdots x_{s_{2,t_2}}$$

for some $t_1,t_2\in\mathbb{N}$ and $-n_1+1\le s_{c,d}\le 0$. Let

$$b_i = x_{r_{i,1}}\cdots x_{r_{i,t'_i}}$$

for $t'_1,\dots,t'_{m-1}\in\mathbb{N}\cup\{0\}$ and $n+1\le r_{c,d}\le k+n_2$. Then, $(x_{-n_1+1},\dots,x_{n+n_2})$ is a sequence of length $n+n_1+n_2$; we denote it by $(y_1,\dots,y_{n+n_1+n_2})$. Let $n' = n+n_1+n_2$ and $k' = k+n_1+n_2$. By our assumption, $a_1x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_2\cdots b_{m-1}x_{i_m}^{l_m}a_2$ is in the algebra generated by $(y_1,\dots,y_{k'})$. Let $(u'_{i,j})_{i=1,\dots,n';\, j=1,\dots,k'}$ be the standard generators of $M_i(n',k')$ and $\mathbf{P}'$ be the invariant projection. Let π be the C^*-homomorphism in Lemma 3.13 and id be the identity on $\mathbb{C}\langle X_1,\dots,,X_{n'}\rangle$. Since $1\le s_{c,d}+n_1\le n_1$, we have

$$(id\otimes\pi)(\alpha_{n',k'}^{(m)}(X_{s_{i,1}+n_1}\cdots X_{s_{i,t_1}}+n_1)) = X_{s_{i,1}+n_1}\cdots X_{s_{i,t_1}+n_1}\otimes\mathbf{P}.$$

Since $n_1+n+1\le r_{c,d}+n_1\le n_1+n_2+k$, we have

$$(id\otimes\pi)(\alpha_{n',k'}^{(m)}(X_{r_{i,1}+n_1}\cdots X_{r_{1,t'_1}}+n_1)) = X_{r_{i,1}+n_1+n-k}\cdots X_{r_{i,t'_i}+n_1+n-k}\otimes I,$$

where I is the identity of $M_i(n,k)$. According to our assumption, we have $1\le i_t\le k$ for $t=1,\dots,m$. Then

$$(id\otimes\pi)(\alpha_{n',k'}^{(m)}(X_{i_t+n_1}^{l_t}) = \sum_{j_t=1}^{n}X_{j_t+n_1}^{l_t}\otimes u_{j_t,i_t}.$$

According to the monotone spreadability of $(y_1, \dots, y_{n'})$ and Lemma 3.13, we have

$$\begin{aligned}
&\phi(a_1 x_{i_1}^{l_1} b_1 x_{i_2}^{l_2} b_2 \cdots b_{m-1} x_{i_m}^{l_m} a_2)\mathbf{P} \\
= \;& \mu_{y_1,\dots,y_{k'}}(X_{s_{1,1}+n_1} \cdots X_{s_{1,t_1}+n_1} X_{i_1+n_1}^{l_1} \cdots X_{i_m+n_1}^{l_m} X_{s_{1,1}+n_1} \cdots X_{s_{2,t_2}+n_1})\pi(\mathbf{P}') \\
= \;& \mathbf{P}(\mu_{y_1,\dots,y_{n'}} \otimes \pi)(\alpha_{n',k'}^{(m)}(X_{s_{1,1}+n_1} \cdots X_{s_{1,t_1}+n_1} X_{i_1+n_1}^{l_1} \\
&\qquad \cdots X_{i_m+n_1}^{l_m} X_{s_{1,1}+n_1} \cdots X_{s_{2,t_2}+n_1}))\mathbf{P} \\
= \;& \sum_{j_1,\dots,j_m=1}^{n} \mu_{y_1,\dots,y_{n'}}(X_{s_{1,1}+n_1} \cdots X_{s_{1,t_1}+n_1} X_{j_1+n_1}^{l_1} X_{r_{1,1}+n_1+n-k} \cdots \\
& X_{r_{m-1,t'_{m-1}+n_1}+n-k} X_{j_m}^{l_m+n_1} X_{s_{1,1}+n_1} \cdots X_{s_{2,t_2}})\mathbf{P} u_{j_1,i_1} \cdots u_{j_m,i_m}\mathbf{P}.
\end{aligned}$$

Notice that $(y_1, \dots, y_{n'})$ is spreadable and $n+1 \le r$. The above equation becomes

$$\begin{aligned}
&\phi(a_1 x_{i_1}^{l_1} b_1 x_{i_2}^{l_2} b_2 \cdots b_{m-1} x_{i_m}^{l_m} a_2)\mathbf{P} \\
= \;& \sum_{j_1,\dots,j_m=1}^{n} \mu_{y_1,\dots,y_{n'}}(X_{s_{1,1}+n_1} \cdots X_{s_{1,t_1}+n_1} X_{j_1+n_1}^{l_1} X_{r_{1,1}+n_1} \cdots \\
& X_{r_{m-1,t'_{m-1}+n_1}} X_{j_m+n_1}^{l_m} X_{s_{1,1}+n_1} \cdots X_{s_{2,t_2}})\mathbf{P} u_{j_1,i_1} \cdots u_{j_m,i_m}\mathbf{P} \\
= \;& \sum_{j_1,\dots,j_m=1}^{n} \phi(x_{s_{1,1}} \cdots x_{s_{1,t_1}} x_{j_1}^{l_1} x_{r_{1,1}} \cdots x_{r_{m-1,t'_{m-1}}} x_{j_m}^{l_m} x_{s_{1,1}} \cdots x_{s_{2,t_2}}) \\
& \times \mathbf{P} u_{j_1,i_1} \cdots u_{j_m,i_m}\mathbf{P} \\
= \;& \sum_{j_1,\dots,j_m=1}^{n} \phi(a_1 x_{j_1}^{l_1} b_1 x_{j_2}^{l_2} b_2 \cdots b_{m-1} x_{j_m}^{l_m} a_2)\mathbf{P} u_{j_1,i_1} \cdots u_{j_m,i_m}\mathbf{P}.
\end{aligned}$$

The proof is complete. □

Lemma 7.2. *Fix $n > k \in \mathbb{N}$, let $(u_{i,j})_{i=1,\dots,n;\, j=1,\dots,k}$ be the standard generators of $M_i(n,k)$. Then, we have*

$$\begin{aligned}
&E^+[x_{i_1}^{l_1} b_1 x_{i_2}^{l_2} b_2 \cdots b_{m-1} x_{i_m}^{l_m}] \otimes \mathbf{P} \\
&\quad = \sum_{j_1,\dots,j_m=1}^{n} E^+[x_{j_1}^{l_1} b_1 x_{j_2}^{l_2} b_2 \cdots b_{m-1} x_{j_m}^{l_m}] \otimes \mathbf{P} u_{j_1,i_1} \cdots u_{j_m,i_m}\mathbf{P},
\end{aligned}$$

where $1 \le i_1, \dots, i_m \le k$, $b_1, \dots, b_{m-1} \in A_{(n+1)+}$.

Proof. It is necessary to check that the two sides of the equation are equal to each other pointwisely, i.e.

$$\begin{aligned}
(4) \quad &\phi(a_1 E^+[x_{i_1}^{l_1} b_1 x_{i_2}^{l_2} b_2 \cdots b_{m-1} x_{i_m}^{l_m}] a_2)\mathbf{P} \\
&\quad = \sum_{j_1,\dots,j_m=1}^{n} \phi(a_1 E^+[x_{j_1}^{l_1} b_1 x_{j_2}^{l_2} b_2 \cdots b_{m-1} x_{j_m}^{l_m}] a_2)\mathbf{P} u_{j_1,i_1} \cdots u_{j_m,i_m}\mathbf{P}
\end{aligned}$$

for all $a_1, a_2 \in A_{[-\infty,\infty]}$. Given $a_1, a_2 \in A_{[-\infty,\infty]}$, there exists $M \in \mathbb{N}$ such that $a_1, a_2 \in A_{M-}$. Then,

$$\alpha^{-m}(a_1), \alpha^{-m}(a_2) \in A_{0-}$$

for all $m > M$. By Lemma 7.1, we have

$$\begin{aligned}
&\phi(\alpha^{-m}(a_1) x_{i_1}^{l_1} b_1 x_{i_2}^{l_2} b_2 \cdots b_{m-1} x_{i_m}^{l_m} \alpha^{-m}(a_2))\mathbf{P} \\
= \;& \sum_{j_1,\dots,j_m=1}^{n} \phi(\alpha^{-m}(a_1) x_{j_1}^{l_1} b_1 x_{j_2}^{l_2} b_2 \cdots b_{m-1} x_{j_m}^{l_m} \alpha^{-m}(a_2))\mathbf{P} u_{j_1,i_1} \cdots u_{j_m,i_m}\mathbf{P}.
\end{aligned}$$

Therefore, for all $m > M$,we have

$$\begin{aligned}&\phi(a_1\alpha^m(x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_2\cdots b_{m-1}x_{i_m}^{l_m})a_2)\mathbf{P}\\ =&\sum_{j_1,\dots,j_m=1}^{n}\phi(a_1\alpha^m(x_{j_1}^{l_1}b_1x_{j_2}^{l_2}b_2\cdots b_{m-1}x_{j_m}^{l_m})a_2)\mathbf{P}u_{j_1,i_1}\cdots u_{j_m,i_m}\mathbf{P}.\end{aligned}$$

Letting m go to $+\infty$, we get equation (4).

The proof is complete since a_1, a_2 are arbitrary. □

Proposition 7.3. *Let $(\mathcal{A},\phi)$ be a W^*-probability space, $(x_i)_{i\in\mathbb{Z}}$ a sequence of self-adjoint random variables in $\mathcal{A}$, and E^+ be the conditional expectation onto the positive tail algebra $\mathcal{A}_{tail}^+$. Assume that the joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable. Then the same is true for the joint distribution with respect to E^+, i.e. for fixed $n > k \in \mathbb{N}$ and $(u_{i,j})_{i=1,\dots,n;\,j=1,\dots,k}$ the standard generators of $M_i(n,k)$, we have that*

$$\begin{aligned}&E^+[x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_2\cdots b_{m-1}x_{i_m}^{l_m}]\otimes\mathbf{P}\\ &\quad=\sum_{j_1,\dots,j_m=1}^{n}E^+[x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_2\cdots b_{m-1}x_{i_m}^{l_m}]\otimes\mathbf{P}u_{j_1,i_1}\cdots u_{j_m,i_m}\mathbf{P},\end{aligned}$$

$1\le i_1,\dots,i_m\le k$, $l_1,\dots,l_m\in\mathbb{N}$ and $b_1,\dots,b_n\in\mathcal{A}_{tail}^+$.

Proof. Since $b_1,\dots,b_{m-1}\in\mathcal{A}_{tail}^+\in\mathcal{A}_n^+$, by Kaplansky's theorem, there exist sequences

$$\{b_{s,t}\}_{s=1,\dots m-1;t\in\mathbb{N}}\subset A_{n+}$$

such that $\|b_{s,t}\|\le\|b_s\|$ and $\lim_{n\to\infty}b_{s,t}=b_s$ in SOT for each $s=1,\dots,m-1$. Therefore,

$$SOT-\lim_{t_1\to\infty}x_{i_1}^{l_1}b_{1,t_1}x_{i_2}^{l_2}b_{2,t_2}\cdots b_{m-1,t_m}x_{i_m}^{l_m}=x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_{2,t_2}\cdots b_{m-1,t_m}x_{i_m}^{l_m}.$$

By Lemma 7.2, we have

$$\begin{aligned}&E^+[x_{i_1}^{l_1}b_{1,t_1}x_{i_2}^{l_2}b_{2,t_2}\cdots b_{m-1,t_m}x_{i_m}^{l_m}]\otimes\mathbf{P}\\ &\quad=\sum_{j_1,\dots,j_m=1}^{n}E^+[x_{j_1}^{l_1}b_{1,t_1}x_{j_2}^{l_2}b_{2,t_2}\cdots b_{m-1,t_{m-1}}x_{j_m}^{l_m}]\otimes\mathbf{P}u_{j_1,i_1}\cdots u_{j_m,i_m}\mathbf{P}.\end{aligned}$$

Letting t_1 go to $+\infty$, by normality of E^+, we have

$$\begin{aligned}&E^+[x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_{2,t_2}\cdots b_{m-1,t_m}x_{i_m}^{l_m}]\otimes\mathbf{P}\\ &\quad=\sum_{j_1,\dots,j_m=1}^{n}E^+[x_{j_1}^{l_1}b_1x_{j_2}^{l_2}b_{2,t_2}\cdots b_{m-1,t_{m-1}}x_{j_m}^{l_m}]\otimes\mathbf{P}u_{j_1,i_1}\cdots u_{j_m,i_m}\mathbf{P}.\end{aligned}$$

Again, taking $t_2,\dots,t_{m-1}$ to $+\infty$, we have

$$\begin{aligned}(5)\qquad&E^+[x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_2\cdots b_{m-1}x_{i_m}^{l_m}]\otimes\mathbf{P}\\ &\quad=\sum_{j_1,\dots,j_m=1}^{n}E^+[x_{j_1}b_1x_{j_2}b_2\cdots b_{m-1}x_{j_m}]\otimes\mathbf{P}u_{j_1,i_1}\cdots u_{j_m,i_m}\mathbf{P}.\end{aligned}$$

□

If $i_s = i_{s+1}$ for some s, according to the universal conditions of $M_i(n,k)$, the terms on the right hand side do not vanish only if $j_s = j_{s+1}$. Therefore, we can shorten the product on the right hand side of (5) if $i_s = i_{s+1}$ for some s. We have

Proposition 7.4. *Let $(\mathcal{A},\phi)$ be a W^*-probability space, let $(x_i)_{i\in\mathbb{Z}}$ be a sequence of selfadjoint random variables in $\mathcal{A}$, and let E^+ be the conditional expectation onto the positive tail algebra $\mathcal{A}_{tail}^+$. Assume that the joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable. For fixed $n>k\in\mathbb{N}$ and $(u_{i,j})_{i=1,\dots,n;\,j=1,\dots,k}$ the standard generators of $M_i(n,k)$, we have that*

$$E^+[p_1(x_{i_1})\cdots p_m(x_{i_m})]\otimes\mathbf{P}=\sum_{j_1,\dots,j_m=1}^{n}E^+[p_1(x_{j_1})\cdots p_m(x_{j_m})]\otimes\mathbf{P}u_{j_1,i_1}\cdots u_{j_m,i_m}\mathbf{P},$$

whenever $1\le i_1,\dots,i_m\le k$, $i_1\ne\cdots\ne i_m$ and $p_1,\dots,p_m\in\mathcal{A}_{tail}^+\langle X\rangle_0$.

Lemma 7.5. *Let $(\mathcal{A},\phi)$ be a W^*-probability space, $(x_i)_{i\in\mathbb{Z}}$ a sequence of selfadjoint random variables in $\mathcal{A}$, and E^+ the conditional expectation onto the positive tail algebra $\mathcal{A}_{tail}^+$. Assume that the joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable. Then*

$$E^+[p_1(x_{i_1})\cdots p_s(x_{i_s})\cdots p_m(x_{i_m})]=E^+[p_1(x_{i_1})\cdots E^+[p_s(x_{i_s})]\cdots p_m(x_{i_m})]$$

whenever $i_s>i_t$ for all $t\ne s$, $i_1\ne\cdots\ne i_m$ and $p_1,\dots,p_m\in\mathcal{A}_{tail}^+\langle X\rangle_0$.

Proof. Since $(x_i)_{i\in\mathbb{Z}}$ is spreadable, by Lemma 6.9, we have that

$$\alpha(p_t(x_{i_t}))=p_t(\alpha(x_{i_t}))$$

and

$$E^+[\alpha^{k'}(a)]=E^+[a]$$

for all $a\in\bigcup_{n'\in\mathbb{Z}}\mathcal{A}_{n'}^+$ and $k'\in\mathbb{Z}$.

Therefore, it is sufficient to prove the statement under the assumption that $i_1,\dots,i_m>0$. Let $i_s=k$, $(u_{i,j})_{i=1,\dots,n+1;\,j=1,\dots,k}$ the standard generators of $M_i(n+k,k)$. By Proposition 7.4, we have

$$E^+[p_1(x_{i_1})\cdots p_m(x_{i_m})]\otimes\mathbf{P}=\sum_{j_1,\dots,j_m=1}^{n+k}E^+[p_1(x_{j_1})\cdots p_m(x_{j_m})]\otimes\mathbf{P}u_{j_1,i_1}\cdots u_{j_m,i_m}\mathbf{P}.$$

Let $l_1=\cdots=l_{k-1}=1$ and $l_k=n+1$. By Proposition 3.12, we have

$$\begin{aligned}&E^+[p_1(x_{i_1})\cdots p_s(x_{i_s})\cdots p_m(x_{i_m})]\otimes\mathbf{P}\\&\quad=\frac{1}{n+1}\sum_{j_s=k}^{n+k}E^+[p_1(x_{i_1})\cdots p_s(x_{j_s})\cdots p_m(x_{i_m})]\otimes\mathbf{P}.\end{aligned}$$

Since n is arbitrary and E^+ is normal on $\mathcal{A}_0^+$, we have

$$\begin{aligned}&E^+[p_1(x_{i_1})\cdots p_s(x_{i_s})\cdots p_m(x_{i_m})]\\=\;&\tfrac{1}{n+1}\sum_{j_s=k}^{n+k}E^+[p_1(x_{i_1})\cdots p_s(x_{j_s})\cdots p_m(x_{i_m})]\\=\;&\mathrm{WOT}-\lim_{n\to\infty}E^+[p_1(x_{i_1})\cdots(\tfrac{1}{n+1}\sum_{j_s=k}^{n+k}p_s(x_{j_s}))\cdots p_m(x_{i_m})]\\=\;&\mathrm{WOT}-\lim_{n\to\infty}E^+[p_1(x_{i_1})\cdots(\tfrac{1}{n+1}\sum_{t=0}^{n}\alpha^t(p_s(x_{i_s})))\cdots p_m(x_{i_m})]\\=\;&\mathrm{WOT}-\lim_{n\to\infty}E^+[p_1(x_{i_1})\cdots E^+[p_s(x_{i_s})]\cdots p_m(x_{i_m})].\end{aligned}$$

The proof is complete. ∎

Now, we consider the case that the maximal index is not unique.

Proposition 7.6. *Let $(\mathcal{A},\phi)$ be a W^*-probability space, $(x_i)_{i\in\mathbb{Z}}$ a sequence of self-adjoint random variables in $\mathcal{A}$, and E^+ the conditional expectation onto the positive tail algebra $\mathcal{A}^+_{tail}$. Assume that the joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable. Then*

$$E^+[p_1(x_{i_1})\cdots p_s(x_{i_s})\cdots p_m(x_{i_m})] = E^+[p_1(x_{i_1})\cdots E^+[p_s(x_{i_s})]\cdots p_m(x_{i_m})]$$

whenever $i_s = \max\{i_1,\dots,i_n\}$ for all $t\neq s$, $i_1\neq\cdots\neq i_m$ and $p_1,\dots,p_m\in\mathcal{A}^+_{tail}\langle X\rangle_0$.

Proof. Again, we can assume that $i_1,\dots,i_t>0$ and $\max\{i_1,\dots,i_m\}=k$. Suppose the number k appears t times in the sequence, which are $\{i_{l_j}\}_j=1,\dots,t$ such that $i_{l_j}=k$ and $l_1<l_2<\cdots<l_t$. Fix n,k and consider $M_i(n+k,k)$. By Proposition 7.4 and Proposition 3.12, we have

$$\begin{aligned}
& E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{i_{l_1}})\cdots p_{l_2}(x_{i_{l_2}})\cdots p_m(x_{i_m})]\otimes P\\
= & \sum_{j_{l_1},j_{l_2},\dots j_{l_t}=k}^{k+n} E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]\\
& \qquad \otimes PP_{j_{l_1},k}PP_{j_{l_2},k}P\cdots u_{j_{l_t},k}P\\
= & \tfrac{1}{(n+1)^t}\sum_{j_{l_1},j_{l_2},\dots j_{l_t}=k}^{k+n} E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]\otimes P\\
= & \tfrac{1}{(n+1)^t}\Big(\sum_{j_{l_s}\neq j_{l_r}\text{ if } s\neq r}^{N} E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]\otimes P\\
& + \sum_{j_{l_s}=j_{l_t}\text{ for some } s\neq t}^{N} E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]\otimes P\Big).
\end{aligned}$$

In the first part of the sum, apply Lemma 7.5 on indices $j_{l_1},\dots j_{l_{l_t}}$ recursively. It follows that

$$\begin{aligned}
& E^+[p_1(x_{i_1})\cdots p_s(x_{j_{l_1}})\cdots p_s(x_{j_{l_2}})\cdots p_m(x_{i_m})]\\
& \quad = E^+[p_1(x_{i_1})\cdots E^+[p_{l_1}(x_{j_{l_1}})]\cdots E^+[p_{l_2}(x_{j_{l_2}})]\cdots p_m(x_{i_m})].
\end{aligned}$$

Since $E^+[p_s(x_{j_{l_1}})] = E^+[p_s(x_k)]$ for all $j_{l_1},\dots,j_{l_t}$,

$$\begin{aligned}
& E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]\\
& \quad = E^+[p_1(x_{i_1})\cdots E^+[p_{l_1}(x_k)]\cdots E^+[p_{l_2}(x_k)]\cdots p_m(x_{i_m})].
\end{aligned}$$

Then, we have

$$\begin{aligned}
& \tfrac{1}{(n+1)^t}\Big(\sum_{j_{l_s}\neq j_{l_r}\text{ if } s\neq r}^{N} E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]\otimes P\\
= & \frac{\prod_{s=0}^{t-1}(n+1-s)}{n+1^t} E^+[p_1(x_{i_1})\cdots E^+[p_{l_1}(x_k)]\cdots E^+[p_{l_2}(x_k)]\cdots p_m(x_{i_m})]\otimes P,
\end{aligned}$$

which converges to $E^+[p_1(x_k)]\cdots E^+[p_s(x_k)]\cdots p_m(x_{i_m})]\otimes P$ in norm as n goes to $+\infty$.

In the second part of the sum, we have

$$\begin{aligned}
& \|E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]\| \\
\leq\ & \|p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})\| \\
\leq\ & \|p_1(x_{i_1})\|\cdots\|p_{l_1}(x_{j_{l_1}})\|\cdots\|p_{l_2}(x_{j_{l_2}})\|\cdots\|p_m(x_{i_m})\| \\
\leq\ & \|p_1(x_1)\|\cdots\|p_{l_1}(x_1)\|\cdots\|p_{l_2}(x_1)\|\cdots\|p_m(x_1)\|,
\end{aligned}$$

which is finite. Therefore,

$$\begin{aligned}
& \|\sum_{j_{l_s}=j_{l_t}\text{ for some } s\neq t}^{N} E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]\| \\
\leq\ & (1-\tfrac{\prod_{s=0}^{t-1}(n+1-s)}{(n+1)^t})\|p_1(x_1)\|\cdots\|p_{l_1}(x_1)\|\cdots\|p_{l_2}(x_1)\|\cdots\|p_m(x_1)\|
\end{aligned}$$

goes to 0 as n goes to $+\infty$.

Therefore, we have

$$\begin{aligned}
& E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{i_{l_1}})\cdots p_{l_2}(x_{i_{l_2}})\cdots p_m(x_{i_m})] \\
&\quad = E^+[p_1(x_{i_1})\cdots E^+[p_{l_1}(x_k)]\cdots E^+[p_{l_2}(x_k)]\cdots p_m(x_{i_m})].
\end{aligned}$$

Similarly we can show that

$$\begin{aligned}
& E^+[p_1(x_{i_1})\cdots p_{l_1}(x_k)\cdots E^+[p_s(x_{i_s})]\cdots p_{l_2}(x_k)\cdots p_m(x_{i_m})] \\
&\quad = E^+[p_1(x_{i_1})\cdots E^+[p_{l_1}(x_k)]\cdots E^+[p_{l_2}(x_k)]\cdots p_m(x_{i_m})],
\end{aligned}$$

which implies that

$$E^+[p_1(x_{i_1})\cdots p_s(x_{i_s})\cdots p_m(x_{i_m})] = E^+[p_1(x_{i_1})\cdots E^+[p_s(x_{i_s})]\cdots p_m(x_{i_m})].$$

□

8. De Finetti type theorem for monotone spreadability

8.1. **Proof of Main Theorem 1.1.** Now, we prove our main theorem for monotone independence.

Theorem 8.1. *Let $(\mathcal{A},\phi)$ be a nondegenerated W^*-probability space and $(x_i)_{i\in\mathbb{Z}}$ be a bilateral infinite sequence of selfadjoint random variables which generate $\mathcal{A}$. Let $\mathcal{A}_k^+$ be the WOT closure of the nonunital algebra generated by $\{x_i|i\geq k\}$. Then the following are equivalent:*

(a) *The joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable.*
(b) *For all $k\in\mathbb{Z}$, there exists a ϕ-preserving conditional expectation $E_k:\mathcal{A}_k^+\to\mathcal{A}_{tail}^+$ such that the sequence $(x_i)_{i\geq k}$ is identically distributed and monotone with respect to E_k. Moreover, $E_k|_{\mathcal{A}_{k'}}=E_{k'}$ when $k\geq k'$.*

Proof. "(b) ⇒ (a)" follows from Proposition 5.17.

We will prove "(a) ⇒ (b)" by induction. Since the sequence is spreadable, it suffices to prove (a) ⇒ (b) for $k=1$:

By the results in the previous two sections, there exists a conditional expectation $E_k:\mathcal{A}_k^+\to\mathcal{A}_{tail}^+$ such that the sequence $(x_i)_{i\geq k}$ is identically distributed with respect to E_k and $E_k|_{\mathcal{A}_{k'}}=E_{k'}$ when $k\geq k'$. Actually, E_k is the restriction of E^+ on $\mathcal{A}_k^+$. Since the sequence is spreadable, we just need to show that the sequence $(x_i)_{i\in\mathbb{N}}$ is monotonically independent with respect to E_1, i.e.

$$(6)\quad E^+[p_1(x_{i_1})\cdots p_s(x_{i_s})\cdots p_m(x_{i_m})] = E^+[p_1(x_{i_1})\cdots E^+[p_s(x_{i_s})]\cdots p_m(x_{i_m})],$$

$i_{s-1}<i_s>i_{s+1}$, $i_1\neq\cdots\neq i_m$, $i_1,\ldots,i_m\in\mathbb{N}$ and $p_1,\ldots,p_m\in\mathcal{A}_{tail}^+\langle X\rangle$.

Now, we prove this equality by induction on the maximal index of $\{i_1,\dots,i_m\}$: When $\max\{i_1,\dots,i_m\}=1$, then equality is true because $i_s=1$ and the length of the sequence $(i_1,\dots,i_m)$ can only be 1.

Suppose the equality holds for $\max\{i_1,\dots,i_m\}=n$. When $\max\{i_1,\dots,i_m\}=n+1$, we have two cases.

Case 1 ($i_s=n+1$). In this case the equality follows Proposition 7.6.

Case 2 ($i_s\le n$). Suppose the number $n+1$ appears t times in the sequence, which is $\{i_{l_j}\}_j=1,\dots,t$ such that $i_{l_j}=k$ and $l_1<l_2<\cdots<l_t$. Since $i_{s-1}<i_s>i_{s+1}$, $i_{s-1},i_s,i_{s+1}\ne n+1$. By Proposition 7.6, we have

$$\begin{aligned}&E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{i_{l_1}})\cdots p_{s-1}(x_{i_{s-1}})p_s(x_{i_s})p_{s+1}(x_{i_{s+1}})\cdots p_{l_t}(x_{i_{l_t}})\cdots p_m(x_{i_m})]\\&\quad=E^+[p_1(x_{i_1})\cdots E^+[p_{l_1}(x_{i_{l_1}})]\cdots p_{s-1}(x_{i_{s-1}})p_s(x_{i_s})p_{s+1}(x_{i_{s+1}})\\&\qquad\cdots E^+[p_{l_t}(x_{i_{l_t}})]\cdots p_m(x_{i_m})].\end{aligned}$$

Notice that

$$\begin{aligned}&p_1(x_{i_1})\cdots E^+[p_{l_1}(x_{i_{l_1}})]\cdots p_{s-1}(x_{i_{s-1}})p_s(x_{i_s})p_{s+1}(x_{i_{s+1}})\cdots E^+[p_{l_t}(x_{i_{l_t}})]\\&\qquad\cdots p_m(x_{i_m})\in\mathcal{A}^+_{tail}\langle X_1,\dots,X_n\rangle,\end{aligned}$$

so by induction, we have

$$\begin{aligned}&E^+[p_1(x_{i_1})\cdots E^+[p_{l_1}(x_{i_{l_1}})]\cdots p_{s-1}(x_{i_{s-1}})p_s(x_{i_s})p_{s+1}(x_{i_{s+1}})\\&\qquad\cdots E^+[p_{l_t}(x_{i_{l_t}})\cdots p_m(x_{i_m})]\\&=E^+[p_1(x_{i_1})\cdots E^+[p_{l_1}(x_{i_{l_1}})]\cdots p_{s-1}(x_{i_{s-1}})E^+[p_s(x_{i_s})]p_{s+1}(x_{i_{s+1}})\\&\qquad\cdots E^+[p_{l_t}(x_{i_{l_t}})]\cdots p_m(x_{i_m})]\\&=E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{i_{l_1}})\cdots p_{s-1}(x_{i_{s-1}})E^+[p_s(x_{i_s})]p_{s+1}(x_{i_{s+1}})\cdots p_{l_t}(x_{i_{l_t}})\\&\qquad\cdots p_m(x_{i_m})].\end{aligned}$$

The last equality follows Proposition 7.6. This is our desired conclusion. □

8.2. **Conditional expectation E^-.** We do not know whether we can extend E^+ to the whole space $\mathcal{A}$. But, the conditional expectation E^- can be extended to the whole algebra $\mathcal{A}$ if the bilateral sequence $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable. Given $a,b,c\in A_{[-\infty,\infty]}$, there exists $L\in\mathbb{N}$ such that $a,b,c\in A_{[-L,L]}$. Therefore, $\alpha^{-3L}(b)\in\mathcal{A}_{[-4L,-3L]}$. Since $(x_{-4L},x_{-4L+1},\dots)$ is monotonically independent with respect to E^+, we have

$$\begin{aligned}&\phi(aE^-[b]c)\\=&\lim_{n\to\infty}\phi(a\alpha^{-n}(b)c)\\=&\lim_{n\to\infty,n>4L}\phi(a\alpha^{-n}(b)c)\\=&\lim_{n\to\infty,n>4L}\phi(E^+[a\alpha^{-n}(b)c])\\=&\lim_{n\to\infty,n>4L}\phi(E^+[E^+[a]\alpha^{-n}(b)E^+[c]])\\=&\lim_{n\to\infty}\phi(E^+[a]\alpha^{-n}(b)E^+[c])\\=&\lim_{n\to\infty}\phi(E^+[a]E^-[b]E^+[c]).\end{aligned}$$

Since $\mathcal{A}$ is generated by countably many operators, by Kaplansky's density theorem, for all $y\in\mathcal{A}$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}\subset A_{[-\infty,\infty]}$ such that $\|y_n\|\le\|y\|$

for all n and y_n converges to y in WOT. Then, for all $a, c \in A_{[-\infty,\infty]}$ we have

$$\lim_{n\to\infty} \phi(aE^-[y_n]c) = \lim_{n\to\infty} \phi(E^+[a]y_n E^+[c]) = \phi(E^+[a]yE^+[c]).$$

Therefore, $E^-[y_n]$ converges to an element y' pointwisely. Moreover, y' depends only on y. If we define $E^-[y] = y'$, then we have

Proposition 8.2. *Let $(\mathcal{A}, \phi)$ be a nondegenerated W^*-probability space and $(x_i)_{i\in\mathbb{Z}}$ be a bilateral infinite sequence of selfadjoint random variables which generate $\mathcal{A}$. If $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable, then the negative conditional expectation E^- can be extended to the whole algebra $\mathcal{A}$ such that*

$$\phi(aE^-[y]b) = \phi(E^+[a]yE^+[c])$$

for all $y \in \mathcal{A}$ and $a, c \in A_{[-\infty,\infty]}$. Moreover, the extension is normal.

9. De Finetti type theorem for boolean spreadability

In this section, we assume that $(\mathcal{A}, \phi)$ is a W^*-probability space with a nondegenerated normal state and $\mathcal{A}$ is generated by a bilateral sequence of random variables $(x_i)_{i\in\mathbb{Z}}$ which is boolean spreadable with respect to ϕ.

Lemma 9.1. *Let $y_i = x_{-i}$ for all $i \in \mathbb{Z}$. Then $(y_i)_{i\in\mathbb{Z}}$ is also boolean spreadable.*

Proof. Since we have assumed that $(x_i)_{i\in\mathbb{Z}}$, $(x_i)_{i=-n,\dots,n}$, $(x_i)_{i=1,\dots,2n+1}$ and $(x_i)_{i=-1,\dots,-2n-1}$ have the same spreadability (we can shift indices). By Proposition 3.22, it suffices to show that $(y_i)_{i=1,\dots,n}$ is boolean spreadable for all $n \in \mathbb{N}$. Given a natural number $k < n$, assume the standard generators of $B_i(n, k)$ are $\{u_{i,j}\}_{i=1,\dots,n;j=1,\dots,k}$ and assume invariant projection $\mathbf{P}$.

Consider the matrix $\{u'_{i,j}\}_{i=1,\dots,n;j=1,\dots,k}$ such that $u'_{i,j} = u_{n+1-i,k+1-j}$. It is obvious that the entries of the matrix are orthogonal projections and

$$\sum_{i=1}^{n} u'_{i,j}\mathbf{P} = \sum_{i=1}^{n} u_{i,k+1-j}\mathbf{P} = \mathbf{P}.$$

Given $j, j', i, i' \in \mathbb{N}$ such that $1 \le j < j' \le k$ and $1 \le i \le i' \le n$, we have $n + 1 - i \le n + 1 - i'$ and $k + 1 - j < k + 1 - j'$. Therefore,

$$u'_{i,j}u'_{i',j'} = u_{n+1-i,k+1-j}u_{n+1-i',k+1-j'} = 0.$$

This implies that $\{u'_{i,j}\}_{i=1,\dots,n;j=1,\dots,k}$ and $\mathbf{P}$ satisfy the universal conditions of $B_i(n, k)$. It follows that there exists a unital C^*-homomorphism $\Phi : B_i(n, k) \to B_i(n, k)$ such that

$$\Phi(u_{i,j}) = u'_{i,j} \text{ and } \Phi(\mathbf{P}) = \mathbf{P}.$$

Let $z_i = x_{i-n-1}$ for $i = 1, \dots, n$. Since $(x_i)_{i\in\mathbb{Z}}$ are boolean spreadable, $(z_i)_{i=1,\dots,n}$ is boolean spreadable. Therefore, for $i_1, \dots, i_L \in [k]$, we have

$$\begin{aligned}
& \phi(y_{i_1}\cdots y_{i_L})\mathbf{P} \\
= \ & \phi(y_{n-k+i_1}\cdots y_{n-k+i_L})\mathbf{P} \\
= \ & \phi(x_{-n+k-i_1}\cdots x_{-n+k-i_L})\mathbf{P} \\
= \ & \Phi(\phi(z_{k+1-i_1}\cdots z_{k+1-i_L})\mathbf{P}) \\
= \ & \Phi(\sum_{j_1,\dots,j_L=1}^{n} \phi(z_{j_1}\cdots z_{j_L})\mathbf{P}u_{j_1,k+1-i_1}\cdots u_{j_L,k+1-i_L}\mathbf{P}) \\
= \ & \sum_{j_1,\dots,j_L=1}^{n} \phi(z_{j_1}\cdots z_{j_L})\mathbf{P}u_{n+1-j_1,i_1}\cdots u_{n+1-j_L,i_L}\mathbf{P} \\
= \ & \sum_{j_1,\dots,j_L=1}^{n} \phi(x_{j_1-n-1}\cdots x_{j_L-n-1})\mathbf{P}u_{n+1-j_1,i_1}\cdots u_{n+1-j_L,i_L}\mathbf{P} \\
= \ & \sum_{j_1,\dots,j_L=1}^{n} \phi(y_{n+1-j_1}\cdots y_{n+1-j_L})\mathbf{P}u_{n+1-j_1,i_1}\cdots u_{n+1-j_L,i_L}\mathbf{P} \\
= \ & \sum_{j_1,\dots,j_L=1}^{n} \phi(y_{j_1}\cdots y_{j_L})\mathbf{P}u_{j_1,i_1}\cdots u_{j_L,i_L}\mathbf{P},
\end{aligned}$$

which completes the proof. □

Proposition 9.2. *$(\mathcal{A}, \phi)$ is a W^*-probability space with a nondegenerated normal state, and $\mathcal{A}$ is generated by a bilateral sequence of random variables $(x_i)_{i\in\mathbb{Z}}$ and $(x_i)_{i\in\mathbb{Z}}$ which are boolean spreadable. Then, E^- and E^+ can be extended to the whole algebra $\mathcal{A}$. Moreover, $E^- = E^+$.*

Proof. Since $(x_i)_{i\in\mathbb{Z}}$ is boolean spreadable, $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable. By Proposition 8.2, E^- can be extended to the whole algebra. By Lemma 9.1, $(x_{-i})_{i\in\mathbb{Z}}$ is also boolean spreadable and its negative-conditional expectation is exactly the positive conditional expectation of $(x_i)_{i\in\mathbb{Z}}$. Therefore, E^+ can also be extended to the whole algebra $\mathcal{A}$ normally. Given $a, b, c \in A_{[-\infty,\infty]}$, by Proposition 8.2, we have

$$\begin{aligned}
\phi(aE^-[b]c) &= \phi(E^+[a]bE^+[c]) \\
&= \phi(E^+[E^+[a]bE^+[c]]) \\
&= \phi(E^+[a]E^+[b]E^+[c]) \\
&= \lim_{n\to\infty} \phi(\alpha^n(a)E^+[b]E^+[c]) \\
&= \lim_{n\to\infty}\lim_{m\to\infty} \phi(\alpha^n(a)E^+[b]\alpha^m(c)).
\end{aligned}$$

Notice that, for fixed n, m,

$$\phi(\alpha^n(a)E^+[b]\alpha^m(c)) = \phi(\alpha^n(a)\alpha^L(b)\alpha^m(c))$$

for $L \in \mathbb{N}$ which is large enough. Since $(x_{-i})_{i\in\mathbb{Z}}$ is monotonically spreadable, by Theorem 1.1, $(x_{-i})_{i\in\mathbb{Z}}$ is monotonically independent with respect to E^-. Therefore, we have

$$\begin{aligned} & \phi(\alpha^n(a)E^+[b]\alpha^m(c)) \\ = & \phi(\alpha^n(a)\alpha^L(b)\alpha^m(c)) \\ = & \phi(E^-[\alpha^n(a)\alpha^L(b)\alpha^m(c)]) \\ = & \phi(E^-[\alpha^n(a)]E^-[\alpha^L(b)]E^-[\alpha^m(c)]) \\ = & \phi(E^-[a]E^-[b]E^-[c]) \\ = & \phi(E^-[E^-[a]bE^-[c]]) \\ = & \phi(E^-[a]bE^-[c]) \\ = & \phi(aE^+[b]c) \end{aligned}$$

and

$$\begin{aligned} \phi(aE^-[b]c) &= \phi(E^+[a]bE^+[c]) \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\phi(\alpha^n(a)E^+[b]\alpha^m(c)) \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\phi(aE^+[b]c) \\ &= \phi(aE^+[b]c). \end{aligned}$$

This implies that $E^+[b] = E^-[b]$ for all $b \in A_{[-\infty,\infty]}$. Since $\mathcal{A}$ is the WOT closure of $A_{[-\infty,\infty]}$, the proof is complete. □

Corollary 9.3. *$(\mathcal{A}, \phi)$ is a W^*-probability space with a nondegenerated normal state, and $\mathcal{A}$ is generated by a bilateral sequence of random variables $(x_i)_{i\in\mathbb{Z}}$ and $(x_i)_{i\in\mathbb{Z}}$ which are boolean spreadable. Then, the positive tail algebra and the negative tail algebra of $(x_i)_{i\in\mathbb{Z}}$ are the same.*

Now, we are ready to prove Theorem 1.3.

Theorem 9.4. *Let $(\mathcal{A}, \phi)$ be a nondegenerated W^*-probability space and $(x_i)_{i\in\mathbb{Z}}$ be a bilateral infinite sequence of selfadjoint random variables which generate $\mathcal{A}$ as a von Neumann algebra. Then the following are equivalent:*

(a) *The joint distribution of $(x_i)_{i\in\mathbb{N}}$ is boolean spreadable.*

(b) *The sequence $(x_i)_{i\in\mathbb{Z}}$ is identically distributed and boolean independent with respect to the ϕ-preserving conditional expectation E^+ onto the nonunital positive tail algebra of the $(x_i)_{i\in\mathbb{Z}}$.*

Proof. "(b) ⇒ (a)". If the sequence $(x_i)_{i\in\mathbb{Z}}$ is identically distributed and boolean independent with respect to a ϕ-preserving conditional expectation E, then the sequence $(x_i)_{i\in\mathbb{Z}}$ is boolean exchangeable by Theorem 7.1 in [17]. According to the diagram in section 4, $(x_i)_{i\in\mathbb{Z}}$ is boolean spreadable.

"(a) ⇒ (b)". By Proposition 9.2, $(x_i)_{i\in\mathbb{Z}}$ is monotone with respect to E^+, $(x_{-i})_{i\in\mathbb{Z}}$ is monotone with respect to E^- and $E^+ = E^-$. Therefore,

$$\begin{aligned} & E^+[p_1(x_{i_1})\cdots p_m(x_{i_m})] = E^+[p_1(x_{i_1})]E^+[p_2(x_{i_2})\cdots p_m(x_{i_m})] = \cdots \\ = & E^+[p_1(x_{i_1})]E^+[p_2(x_{i_2})]\cdots E^+[p_m(x_{i_m})] \end{aligned}$$

whenever $i_1 \neq \cdots \neq i_m$ and $p_1, \ldots, p_m \in \mathcal{A}^+_{tail}\langle X\rangle$. The proof is complete. □

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