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CONNECTIONS BETWEEN UNIT-REGULARITY, REGULARITY, CLEANNESS, AND STRONG CLEANNESS OF ELEMENTS AND RINGS

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Abstract. We construct an example of a unit-regular ring which is not strongly clean, answering an open question of Nicholson. We also characterize clean matrices with a zero column, and this allows us to describe an interesting connection between unit-regular elements and clean elements. Next we study in arbitrary rings those elements whose powers are regular, and provide a method for constructing inner inverses which satisfy many additional strong relations. As a corollary we show that if each of the powers $a, a^2, \dots, a^n$ is a regular element in some ring R (for some $n \geq 1$), then there exists $w \in R$ such that $a^k w^k a^k = a^k$ and $w^k a^k w^k = w^k$ for $1 \leq k \leq n$. Similar statements are also obtained for unit-regular elements. The paper ends with a large number of examples elucidating further connections (and disconnections) between cleanness, regularity, and unit-regularity.

1. Introduction

Regular rings were defined by von Neumann in his study of continuous geometries and have become a staple of noncommutative ring theory due to their simple definition and connection to decomposition theory. An element a in a ring R is said to be (von Neumann) *regular* if there exists some $r \in R$ with $ara = a$. We denote the set of all regular elements of R by $\operatorname{reg}(R)$, and if $\operatorname{reg}(R) = R$ one says that R is a *regular ring*. The element r is called an *inner inverse* for a, and it need not be unique.

As defined by Ehrlich [12], an element $a \in R$ is *unit-regular* if it has an inner inverse which is a unit of R. The set of unit-regular elements is denoted by $\operatorname{ureg}(R)$, and if $\operatorname{ureg}(R) = R$ we say R is a *unit-regular ring*. Examples include all semisimple rings and all commutative regular rings. However, there are regular rings which are not unit-regular. Additional information on regular rings can be found in the textbook [14]. The utility of these definitions is most easily seen by the following classical result.

Lemma 1.1. *Let M_k be a right k-module for some ring k, and let $a \in R := \operatorname{End}(M)$. We have $a \in \operatorname{reg}(R)$ if and only if $\ker(a)$ and $\operatorname{im}(a)$ are direct summands of M. Moreover, $a \in \operatorname{ureg}(R)$ if and only if $a \in \operatorname{reg}(R)$ and $\ker(a) \cong \operatorname{coker}(a)$.*

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Proof. See Exercises 4.14A$_1$ and 4.14C from [18], and the comments following them. The second exercise appears as part of [13, Theorem 1]. Both contain the additional hypothesis that the entire ring R is regular, but the proofs can be modified to avoid this assumption. □

An important generalization of regular rings arose in the work of Warfield [24], based on earlier work of Crawley and Jónsson on direct sum decomposition theory [10]. Nicholson [19] showed that Warfield's rings are exactly those for which idempotents lift modulo every one-sided ideal, and they are called the *exchange rings*.

Many exchange rings satisfy additional properties, which Nicholson described in [19] and [20]. Following that work, we say that an element $a \in R$ is *clean* if $a = e+u$ for some idempotent e and some unit u of R. The additive decomposition $a = e + u$ is sometimes called a *clean decomposition.* If a has some clean decomposition $a = e + u$ where additionally e and u commute, then we say a is *strongly clean.* When every element of R is (strongly) clean, then one says that R is a (*strongly*) *clean ring.* Surprisingly, most natural examples of exchange rings are clean rings, including all endomorphism rings of continuous modules [5], all commutative (and more generally abelian) exchange rings [19, Proposition 1.8], and all strongly π-regular rings [20, Theorem 1]. (The last two classes are also strongly clean.)

Bergman has constructed a regular (hence exchange) ring which is not clean; see [9, p. 4746] and [22, Example 3.1]. All other examples of nonclean exchange rings seem to be based on this single example. Remarkably, unit-regularity of *rings* does imply cleanness by [9, Theorem 5] and [8, Theorem 1].

This raises the question of whether every unit-regular ring is strongly clean, which appears as the fourth of five questions in [20]. In Section 2 we answer this question in the negative by constructing a unit-regular ring which is not strongly clean. (Thus, of the five questions from [20], only Questions 1 and 2 remain open.) It is also well-known that unit-regular *elements* are not always clean. We clarify this relationship in Section 3 by showing that there is a natural element-wise extension of unit-regularity which is equivalent to an enhanced form of cleanness, proven in Theorem 3.14. One of the conditions of that theorem leads us, in Section 4, to consider powers of regular elements. We prove that if a is an element of a ring R such that $a, a^2, \dots, a^n$ are all regular in R, then there exists some $w \in R$ such that $a^i w^j a^k = w^{j-i} a^j w^{j-k}$ for all $0 \le i, k \le j \le n$. In particular, w satisfies the "power inner inverse" conditions $a^k w^k a^k = a^k$ and $w^k a^k w^k = w^k$ for all $1 \le k \le n$. Similar statements hold when a is unit-regular. We finish by proving some natural limitations on these theorems and providing many examples.

Rings in this paper are associative and have 1, but are not necessarily commutative. Modules are unital, and endomorphisms will be written on the side opposite the scalars. The set of units in a ring R will be denoted by $U(R)$, while the set of idempotents is $\mathrm{idem}(R)$.

2. Unit-regular rings are not always strongly clean

This section is devoted to constructing a unit-regular ring which is not strongly clean. The ring we will create is isomorphic to a complicated example of Bergman found in [14, Example 5.12], given there as an instance of a unit-regular ring with a regular subring which is not unit-regular. Instead of following the steps taken there to define the ring as a set of certain endomorphisms, we find it easier to view the ring as a set of infinite $\mathbb{Z} \times \mathbb{Z}$ matrices, so we will form the ring along independent lines. (As shown to us after writing this paper, O'Meara in [21] has recast Bergman's example in similar terms.)

Let F be a field. Let $F(\!(t)\!)$ denote the field of formal Laurent series over F. We define R to be the set of all $\mathbb{Z} \times \mathbb{Z}$ matrices $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ over F such that there exist $m, n \in \mathbb{Z}$ and $f(t) = \sum_{k=k_0}^{\infty} a_k t^k \in F(\!(t)\!)$ with the following properties:

- (R1) if $i \geq m$ or $j < n$, then $a_{i,j} = a_{j-i}$ (where we set $a_k = 0$ if $k < k_0$);
- (R2) the submatrix $A_0 = (a_{i,j})_{i<m,j\geq n}$ has finite rank (i.e., it has only finitely many linearly independent columns or, equivalently, finitely many linearly independent rows).

Thus, the matrix $A \in R$ in the above definition is of the following form:

$$
\begin{array}{cc}
 & \hphantom{\ddots\quad a_{n-m-1}\quad a_{n-m-1}}\; n \\
m & \left(\begin{array}{ccc|ccc}
\ddots & \vdots & & & & \\
\ddots & a_{n-m} & \vdots & & A_0 & \\
 & a_{n-m-1} & a_{n-m} & & & \\
\hline
\iddots & & a_{n-m-1} & a_{n-m} & \cdots & \\
 & \iddots & & a_{n-m-1} & a_{n-m} & \cdots \\
 & & \iddots & & \ddots & \ddots
\end{array}\right)
\end{array}
$$

The indices of the entries $a_{i,j}$ in this matrix increase when we move down or right (respectively). The lower left corner of A_0 is at position $(m-1, n)$.

The small letter m on the left of the horizontal line indicates that the row directly below this line is the mth row of A (i.e. the row $(a_{m,j})_{j\in\mathbb{Z}}$). Similarly, the small letter n above the vertical line indicates that the column directly to the right of the vertical line is the nth column of A (i.e. $(a_{i,n})_{i\in\mathbb{Z}}$). In subsequent computations, we will often use these small letters since they will simplify notation.

Note that in the definition of R given above, we may always increase m or decrease n by any finite number, because these changes do not affect the finite rank condition on A_0. Moreover, the condition (R2) in the above definition may be made independent of m and n by simply saying that $(a_{i,j})_{i<0,j\geq 0}$ has finite rank or alternatively that $(a_{i,j})_{i<p,j\geq q}$ has finite rank for *some* (arbitrary) $p, q \in \mathbb{Z}$.

There is one more way of characterizing the finite rank condition (R2). Note that if the matrix $A_0 = (a_{i,j})_{i<m,j\geq n}$ has finite rank, then there exists $c \geq 1$ such that each row in A_0 is a linear combination of the bottom c rows of A_0. Writing a decomposition

$$A_0 = \left(\begin{array}{c} A_2 \\ \hline A_1 \end{array}\right),$$

with A_1 composed of the last c rows of A_0, this means that we can factor $A_2 = YA_1$ for a suitable infinite matrix Y (with c columns). Moreover, since A_1 has only finitely many independent columns, we can find $d \geq 1$ such that A_1 decomposes as

$$A_1 = \left(\begin{array}{c|c} X_0 & X_0 Z \end{array}\right)$$

where X_0 contains the first d columns of A_1 and Z is an infinite matrix with d rows. Putting this together, we get a decomposition of A_0:

$$A_0 = \begin{pmatrix} YX_0 & YX_0Z \\ X_0 & X_0Z \end{pmatrix}. \tag{2.1}$$

It is clear that such a decomposition is also sufficient for A_0 to have finite rank.

Let us now proceed with our construction. We endow the set R with the usual matrix addition and multiplication. It is clear that addition on R is well-defined. Let us briefly sketch the proof of why multiplication is also well-defined. First, it is clear that if $A, B \in R$, then AB is well-defined as a $\mathbb{Z} \times \mathbb{Z}$ matrix over F, because rows of A are bounded from the left and columns of B are bounded from below. Next, if the Laurent series $f(t), g(t) \in F((t))$ correspond to A and B respectively, then a straightforward computation shows that $f(t)g(t) \in F((t))$ corresponds to AB in the sense that condition (R1) holds for that Laurent series (for sufficiently large m and sufficiently small n). And finally, to see that AB satisfies the finite rank condition, decompose

$$A = \begin{array}{cc} & \begin{array}{cc} \phantom{A_{11}} & \!\!\!\!0 \end{array} \\ 0 & \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \end{array} \qquad \text{and} \qquad B = \begin{array}{cc} & \begin{array}{cc} \phantom{B_{11}} & \!\!\!\!0 \end{array} \\ 0 & \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right) \end{array}.$$

Since A_{12} and B_{12} have finite ranks, $A_{11}B_{12}$ and $A_{12}B_{22}$ also have finite ranks, and hence the upper right corner of AB, which is $A_{11}B_{12} + A_{12}B_{22}$, also has finite rank.

By checking the ring axioms directly (associativity of multiplication being the only step that isn't completely trivial), this proves R is a ring. For every $A \in R$, we denote by $\psi(A) \in F((t))$ the Laurent series that corresponds to A. Note that $\psi : R \to F((t))$ is a surjective ring homomorphism.

Proposition 2.2. *The ring R constructed above is unit-regular.*

Proof. Let $A = (a_{i,j}) \in R$. Fix $m, n \in \mathbb{Z}$ and $f(t) = \sum_{k \geq k_0} a_k t^k$ such that $a_{i,j} = a_{j-i}$ if $i \geq m$ or $j < n$. First, suppose that $f(t) \neq 0$. We may assume that $a_{k_0} \neq 0$ and, after increasing m, that $m + k_0 > n$. Decompose A as

$$A = \begin{array}{cc} & \begin{array}{ccc} & n & m+k_0 \end{array} \\ \begin{array}{c} \\ n-k_0 \\ m \\ \end{array} & \left(\begin{array}{c|c|c} S & Y & W \\ \hline 0 & X_0 & Z \\ \hline 0 & 0 & T \end{array}\right) \end{array}.$$

Observe that the matrices S and T are upper triangular, constant on diagonals, and nonzero on the main diagonal. Therefore, they are invertible. In fact, the inverses S^{-1} and T^{-1} are also upper triangular and constant on diagonals, with entries that are precisely the coefficients of $f(t)^{-1}$ (with the leading term on the main diagonal). Moreover, since any finite matrix ring over a field is unit-regular, there exists an invertible matrix $U_0 \in \mathbb{M}_{m+k_0-n}(F)$ such that $X_0U_0X_0 = X_0$. Now define

$$U := \begin{array}{cc} & \begin{array}{ccc} \phantom{S^{-1}} & n-k_0 & m \end{array} \\ \begin{array}{c} \\ n \\ m+k_0 \\ \end{array} & \left(\begin{array}{c|c|c} S^{-1} & -S^{-1}YU_0 & S^{-1}(YU_0Z - W)T^{-1} \\ \hline 0 & U_0 & -U_0ZT^{-1} \\ \hline 0 & 0 & T^{-1} \end{array}\right) \end{array}.$$

Since U_0 and W have finite ranks, $S^{-1}(YU_0Z - W)T^{-1}$ has finite rank, hence $U \in R$. Note that U is invertible, with the inverse

$$U^{-1} = \begin{array}{cc} & \begin{array}{ccc} & n & m+k_0 \end{array} \\ \begin{array}{c} \\ n-k_0 \\ m \\ \end{array} & \left(\begin{array}{c|c|c} S & Y & W \\ \hline 0 & U_0^{-1} & Z \\ \hline 0 & 0 & T \end{array}\right) \end{array}.$$

One also easily verifies that $AUA = A$, which completes the proof in this case.

Now suppose that $f(t) = 0$. Since $A_0 = (a_{i,j})_{i<m, j\geq n}$ has finite rank, we can apply decomposition (2.1), so that A becomes

$$A = \begin{array}{cc} & \begin{array}{ccc} & n & n+d \end{array} \\ \begin{array}{c} \\ m-c \\ m \\ \end{array} & \left(\begin{array}{c|c|c} 0 & YX_0 & YX_0Z \\ \hline 0 & X_0 & X_0Z \\ \hline 0 & 0 & 0 \end{array}\right) \end{array}. \tag{2.3}$$

We may assume that $c = d$ by increasing one or the other as needed. Taking an invertible $U_0 \in \mathbb{M}_c(F)$ with $X_0U_0X_0 = X_0$, one now easily verifies that

$$U := \begin{array}{cc} & \begin{array}{ccc} & m-c & m \end{array} \\ \begin{array}{c} \\ n \\ n+c \\ \end{array} & \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & U_0 & 0 \\ \hline 0 & 0 & I \end{array}\right) \end{array}$$

is invertible in R, with inverse

$$U^{-1} = \begin{array}{cc} & \begin{array}{ccc} & n & n+c \end{array} \\ \begin{array}{c} \\ m-c \\ m \\ \end{array} & \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & U_0^{-1} & 0 \\ \hline 0 & 0 & I \end{array}\right) \end{array},$$

and that $AUA = A$. This completes the proof. $\square$

Theorem 2.4. *The ring R constructed above is not strongly clean.*

Proof. Let

$$A := \begin{array}{cc} & \begin{array}{cccc} & -2 & -1 & 0 \end{array} \\ \begin{array}{c} \\ -1 \\ 0 \\ 1 \\ \end{array} & \left(\begin{array}{c|c|c|c} I & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right) \end{array} \in R.$$

That is, $A = (a_{i,j})$ where $a_{i,i-1} = 1$ for all $i \in \mathbb{Z} \setminus \{0\}$, $a_{-1,-1} = 1$, and $a_{i,j} = 0$ for all other pairs (i, j). We will prove that A is not strongly clean in R.

Let $E = (e_{i,j}) \in R$ be any idempotent such that $AE = EA$. We have two possible cases, $\psi(E) = 0$ or $\psi(E) = 1$. First, suppose that $\psi(E) = 0$. We claim that in this case $e_{i,j} = 0$ whenever $i \geq 0$ and $j \in \mathbb{Z}$. Assume by way of contradiction that there are some $i \geq 0$ and $j \in \mathbb{Z}$ such that $e_{i,j} \neq 0$. Suppose that i is the largest integer with this property, so that $e_{i',j'} = 0$ whenever $i' > i$ and $j' \in \mathbb{Z}$. In this case, the $(i + 1, j)$ entry of EA is 0, while the $(i + 1, j)$ entry of AE is $e_{i,j} \neq 0$, contradicting the fact that $AE = EA$. So E has the stated property. But this means that the 0th row of $A - E$ is all zeros, and hence $A - E$ is not a unit.

Now suppose that $\psi(E) = 1$. Then $E' = (e'_{i,j}) = 1 - E$ satisfies $\psi(E') = 0$ and $AE' = E'A$. Similarly as above, we can show that this implies $e'_{i,j} = 0$ whenever $i \in \mathbb{Z}$ and $j < 0$. Accordingly, $A - E = A - 1 + E'$ has the (-1)th column all zeros and hence it cannot be a unit. This proves that A is not strongly clean. $\square$

Remark 2.5. (1) In the proof of this theorem, the only fact we needed about idempotents $E \in \operatorname{idem}(R)$ is that $\psi(E) \in \{0, 1\}$.

(2) In the above proof, one could prove that $A - E$ is not a unit by merely assuming that $AE = EAE$, rather than $AE = EA$. In fact, if $AE = EAE$ and $\psi(E) = 0$, then the same argument as above shows that $E = (e_{i,j})$ must satisfy $e_{i,j} = 0$ for all $i \geq 0$ and $j \in \mathbb{Z}$. Hence $A - E$ cannot be a unit as it has a zero 0th row. Similarly, if $AE = EAE$ and $\psi(E) = 1$, then $E' = (e'_{i,j}) = 1 - E$ satisfies $\psi(E') = 0$ and $E'A = E'AE'$, from which we obtain $e'_{i,j} = 0$ for all $i \in \mathbb{Z}$ and $j < 0$. Hence $A - E = A - 1 + E'$ cannot be a unit as it has a zero (-1)th column. This shows that R is not even *capably clean* in the terminology of [6]. In that paper it is shown that one-sided continuous, Dedekind-finite, regular rings are capably clean. In agreement with this fact, one can check directly that the ring R we have defined is not left or right continuous.

It may be interesting to note that we can view R as a subring of the endomorphism ring of $F(\!(t)\!)$. Given an element $f(t) = \sum_{k \geq k_0} a_k t^k \in F(\!(t)\!)$, we can identify the Laurent series as an infinite row vector $(a_k)_{k \in \mathbb{Z}}$ which has zero entries for sufficiently negative indices. The ring R acts on such vectors by right multiplication. In this way one can identify the ring R we have constructed with the ring Q of endomorphisms given in Example 5.12 from [14]. However, we will not give the formal details of this identification, since that example is quite involved, and we don't need this fact.

3. Element-wise connections between unit-regularity and clean decompositions

As shown in [9] unit-regular rings are clean rings, and as later clarified in [8] unit-regularity for rings is equivalent to a strengthened form of cleanness. The hypotheses of these statements cannot be significantly weakened, since there are regular rings which are not clean. Nor can the conclusions of these statements be significantly strengthened, because the example from the previous section demonstrates that unit-regular rings need not be strongly clean rings.

The purpose of this section is to clarify the connection between unit-regular elements and clean elements by showing that a "doubly unit-regular" condition is equivalent to a strengthened form of cleanness. This result should come as something of a surprise, since it was shown in [17] that unit-regular elements are not necessarily clean. Before we state our main result, we first generalize some results in the literature by characterizing when matrices with a column of zeros are clean.

To begin we recall an easy alternate characterization of when $a \in R$ is clean.

Lemma 3.1 ([25, Proposition 2])**.** *Let a be an element of a ring R. The following are equivalent:*

(1) *There exist $e \in \operatorname{idem}(R)$ and $u \in U(R)$ such that $a = e + u$.*
(2) *There exist $g \in \operatorname{idem}(R)$ and $v \in U(R)$ such that $g = gva$ and $1 - g = -(1 - g)v(1 - a)$.*

Moreover, there is a natural bijection between the two conditions. Given u, e, we can take $v = u^{-1}$ and $g = 1 - u^{-1}eu$; conversely, given v, g, we can take $u = v^{-1}$ and $e = 1 - v^{-1}gv$.

Let R be a ring with an idempotent e and $a \in Re$. In the Peirce decomposition corresponding to e and $f := 1 - e$, we can write a as

$$a = \begin{pmatrix} \alpha & 0 \\ \tau & 0 \end{pmatrix}, \tag{3.2}$$

where $\alpha := ea \in eRe$ and $\tau := fa \in fRe$. The following lemma characterizes when such an element a is clean in R, in terms of the properties of the "corner" elements α and τ. The lemma generalizes [22, Proposition 2.2], which deals with matrices of the form (3.2) with the additional property $\tau = 0$, and also [17, Theorem 3.2], which deals with matrices in $\mathbb{M}_2(K)$ with a zero row (or column) in the case when the base ring K is *commutative.*

Lemma 3.3. *Let R be a ring, $e \in \mathrm{idem}(R)$, $f := 1 - e$, and $a \in Re$. Put $\alpha := ea$ and $\tau := fa$ as above. The following are equivalent:*

(1) *The element a is clean in R.*

(2) *There exist $\varepsilon \in \mathrm{idem}(eRe)$, $\mu \in U(eRe)$, $\beta \in eRf$, and $\gamma \in fRe$ such that*
$$\varepsilon = \varepsilon\mu\alpha + \varepsilon\beta(\tau + \gamma\alpha) \qquad \text{and} \qquad e - \varepsilon = -(e - \varepsilon)\mu(e - \alpha).$$

(3) *There exist $\varepsilon \in \mathrm{idem}(eRe)$, $\mu \in U(eRe)$, $\beta \in eRf$, and $\gamma \in fRe$ such that*
$$\alpha = \varepsilon + \mu + (e - \varepsilon)\beta(\tau + \gamma\alpha).$$

Proof. (1) $\Rightarrow$ (2): We imitate the proof of [22, Proposition 2.2]. Assume that a is clean. Write $g = gva$ and $1 - g = -(1 - g)v(1 - a)$ where $g \in \mathrm{idem}(R)$ and $v \in U(R)$. Since $g \in Ra \subseteq Re$, it must be the case that g has a zero second column (in the e-f-Peirce decomposition), so that

$$g = \begin{pmatrix} \varepsilon & 0 \\ \chi & 0 \end{pmatrix}$$

for some $\varepsilon \in \mathrm{idem}(eRe)$ and $\chi \in fR\varepsilon$. Write $\zeta := e - \varepsilon$ and set

$$v = \begin{pmatrix} \pi & \beta \\ \sigma & \delta \end{pmatrix}.$$

By matrix expansion of the equation $g = gva$ we get the following two equalities:

$$\varepsilon = \varepsilon\pi\alpha + \varepsilon\beta\tau, \tag{3.4}$$
$$\chi = \chi\pi\alpha + \chi\beta\tau. \tag{3.5}$$

(Note that (3.5) also follows from (3.4) since $\chi = \chi\varepsilon$.) Furthermore, from $1 - g = -(1 - g)v(1 - a)$ we get the following four equations:

$$\zeta = -\zeta\pi(e - \alpha) + \zeta\beta\tau, \tag{3.6}$$
$$0 = \zeta\beta, \tag{3.7}$$
$$-\chi = -(\sigma - \chi\pi)(e - \alpha) + (\delta - \chi\beta)\tau, \tag{3.8}$$
$$f = -\delta + \chi\beta. \tag{3.9}$$

The invertibility of v, in conjunction with (3.9), yields that

$$\begin{pmatrix} e & \beta \\ 0 & f \end{pmatrix} \begin{pmatrix} e & 0 \\ -\chi & f \end{pmatrix} v = \begin{pmatrix} \pi + \beta(\sigma - \chi\pi) & 0 \\ \sigma - \chi\pi & -f \end{pmatrix}$$

is invertible. As the lower-right corner is a unit (in the corner ring fRf) and the entire matrix is a lower-triangular invertible matrix, we see that $\mu := \pi + \beta(\sigma - \chi\pi)$ is invertible in eRe.

From (3.7) we have $\zeta\pi = \zeta\mu$, so that (3.6) becomes $\zeta = -\zeta\mu(e-\alpha)$, which is the second of the two equations we need. Moreover, (3.4) yields

$$\varepsilon = \varepsilon(\mu - \beta(\sigma - \chi\pi))\alpha + \varepsilon\beta\tau = \varepsilon\mu\alpha + \varepsilon\beta(\tau - (\sigma - \chi\pi)\alpha).$$

Taking $\gamma := -\sigma + \chi\pi$ gives the other needed equation. (It may be interesting to note that we never needed to use (3.8).)

(2) $\Rightarrow$ (1): Suppose that $\varepsilon = \varepsilon\mu\alpha + \varepsilon\beta(\tau + \gamma\alpha)$ and $\zeta = -\zeta\mu(e-\alpha)$ for some $\varepsilon = e - \zeta \in \mathrm{idem}(eRe)$, $\mu \in U(eRe)$, $\beta \in eRf$ and $\gamma \in fRe$. Write $\tau' := \tau + \gamma\alpha - \gamma \in fRe$, and let

$$g := \begin{pmatrix} \varepsilon & 0 \\ \tau'\varepsilon & 0 \end{pmatrix}$$

and

$$v := \begin{pmatrix} e & 0 \\ \tau'\varepsilon & f \end{pmatrix} \begin{pmatrix} e & -\varepsilon\beta \\ 0 & f \end{pmatrix} \begin{pmatrix} (e + \varepsilon\beta\tau'\zeta)\mu & 0 \\ \tau'\zeta\mu - \gamma & -f \end{pmatrix}.$$

Clearly, $g \in \mathrm{idem}(R)$ and $v \in U(R)$, and a straightforward verification shows that $g = gva$ and $1 - g = -(1-g)v(1-a)$. Thus a is clean in R by Lemma 3.1.

(2) $\Rightarrow$ (3): The proof of this implication and its converse are similar to the proof of Lemma 3.1, but we include the details for completeness. Given the two equations of (2), adding them together we get

$$e = \varepsilon\mu\alpha + \varepsilon\beta(\tau + \gamma\alpha) - (e - \varepsilon)\mu(e - \alpha) = \mu\alpha + \varepsilon\beta(\tau + \gamma\alpha) - (e - \varepsilon)\mu,$$

so that

$$\alpha = \mu^{-1}(e - \varepsilon)\mu + \mu^{-1} - \mu^{-1}\varepsilon\beta(\tau + \gamma\alpha),$$

which gives the desired equation taking $\varepsilon' := \mu^{-1}(e-\varepsilon)\mu$, $\mu' := \mu^{-1}$, $\beta' := -\mu^{-1}\beta$, and $\gamma' := \gamma$.

(3) $\Rightarrow$ (2): Starting with the equation given in (3), multiplying on the left by $\mu^{-1}\varepsilon$ gives $\mu^{-1}\varepsilon\alpha = \mu^{-1}\varepsilon + \mu^{-1}\varepsilon\mu$. Hence

$$\mu^{-1}\varepsilon\mu = -(\mu^{-1}\varepsilon\mu)\mu^{-1}(e - \alpha). \tag{3.10}$$

Taking $\varepsilon' := \mu^{-1}(e - \varepsilon)\mu$ and $\mu' := \mu^{-1}$, then (3.10) is exactly the second equation of (2). Similarly, multiplying (3) on the left by $\mu^{-1}(e - \varepsilon)$ gives the first equation of (2). □

Remark 3.11. (1) If $\tau = 0$, then Lemma 3.3 says that $a = \alpha \in eRe$ is clean in R if and only if

$$\alpha = \varepsilon + \mu + (e - \varepsilon)\beta\gamma\alpha \tag{3.12}$$

for some $\varepsilon \in \mathrm{idem}(eRe)$, $\mu \in U(eRe)$, $\beta \in eRf$, and $\gamma \in fRe$. Thus a is *weakly clean* in eRe, following the terminology of [22, Definition 2.3].

Conversely, if $a \in eRe$ is weakly clean in eRe and

$$e = exfye \text{ for some } x, y \in R, \tag{3.13}$$

then a satisfies (3.12) and hence is clean in R. However, without assuming any extra condition such as (3.13), weakly clean elements of corner rings need not be clean in the entire ring. This is easy to see by taking $e = 1$ and R to be any weakly clean ring which is not clean, such as in [22, Example 3.1].

(2) Lemma 3.3 provides an easy way to see that regular elements in rings with stable range one are always clean, which was originally proved in [23, Theorem 3.3].

To see this, let R have stable range one and write $a = ara \in R$ with $r \in R$. Setting $e := ra$ we have $a \in Re$, so that a decomposes as in (3.2). Since $e = era = ere\alpha + erf\tau$ and eRe has stable range one, we have $e = \mu\alpha + \omega erf\tau$ for some $\mu \in U(eRe)$ and $\omega \in eRe$. Thus, $\varepsilon := e$, μ, $\beta := \omega erf$, and $\gamma := 0$ satisfy Lemma 3.3(2), and hence a is clean in R.

We are now prepared to present the main theorem of this section, which demonstrates that certain unit-regular elements, which we call "doubly unit-regular", possess an extended version of the clean property.

As mentioned previously, not all unit-regular elements are clean. To get around this problem, we assume two instances of unit-regularity; once for the original element, and once for a corner of the element. However, as it turns out, unit-regularity is only needed for the corner and mere regularity for the original element.

Theorem 3.14. *Let a be an element of a ring R. The following are equivalent:*

(1) *There exists $u \in U(R)$ with $aua = a$ such that writing $e := ua \in \mathrm{idem}(R)$, the element eae is unit-regular in eRe.*
(2) *There exists $r \in R$ with $ara = a$ such that writing $e := ra \in \mathrm{idem}(R)$, the element eae is unit-regular in eRe.*
(3) *There exist $g \in \mathrm{idem}(R)$ and $v \in U(R)$ such that putting $h := 1 - g$ we have $g = gva$, $h = -hv(1-a)$, $hvh = -h$, and $gvhvgvh = -gvh$.*
(4) *There exist $e \in \mathrm{idem}(R)$ and $u \in U(R)$ such that $a = e + u$, $aR \cap eR = (0)$, and $a^2R \cap aeR = (0)$.*
(5) *There exist $e \in \mathrm{idem}(R)$ and $u \in U(R)$ such that $a = e + u$, $aR \cap eR = (0)$, and $a^2R \cap aeaR = (0)$.*
(6) *There exist $e \in \mathrm{idem}(R)$ and $u \in U(R)$ such that $a = e + u$ with $au^{-1}a = a$ and $a^2u^{-2}a^2 = a^2$.*

Proof. (1) $\Rightarrow$ (2) is a tautology.

(2) $\Rightarrow$ (3): Put $f := 1-e$. Since $a \in Re$, we can write $a = \left(\begin{smallmatrix} \alpha & 0 \\ \tau & 0 \end{smallmatrix}\right)$ in the e-f-Peirce decomposition. As α is unit-regular by hypothesis, we can find $\mu \in U(eRe)$ such that $\alpha\mu\alpha = \alpha$. Now

$$e = e^3 = erae = ereae + erfae = ere\alpha + erf\tau$$

gives $erf\tau(e - \mu\alpha) = (e - ere\alpha)(e - \mu\alpha) = e - \mu\alpha$. Multiplying on the left by the idempotent $e - \mu\alpha$ yields

$$(e - \mu\alpha)erf\tau(e - \mu\alpha) = e - \mu\alpha$$

so that

$$e = \mu\alpha + (e - \mu\alpha)erf\tau(e - \mu\alpha) = \mu\alpha + (e - \mu\alpha)erf(\tau - \tau\mu\alpha). \tag{3.15}$$

Hence $\varepsilon := e$, μ, $\beta := (e - \mu\alpha)erf$, and $\gamma := -\tau\mu$ satisfy the conditions of Lemma 3.3(2). Accordingly, a is clean in R, i.e. $g = gva$ and $h = -hv(1-a)$, with $g = 1 - h \in \mathrm{idem}(R)$ and $v \in U(R)$. Moreover, the proof of the implication (2) $\Rightarrow$ (1) in Lemma 3.3 tells us that we can take

$$g := \begin{pmatrix} e & 0 \\ \tau' & 0 \end{pmatrix} \quad \text{and} \quad v := \begin{pmatrix} e & 0 \\ \tau' & f \end{pmatrix}\begin{pmatrix} e & -\beta \\ 0 & f \end{pmatrix}\begin{pmatrix} \mu & 0 \\ -\gamma & -f \end{pmatrix}$$

where $\tau' := \tau + \gamma\alpha - \gamma$ (since $\varepsilon = e$ and hence $\zeta = 0$). It is a matter of a routine verification that g, h, and v satisfy $hvh = -h$, so it only remains to see that $gvhvgvh = -gvh$. By direct computation we get

$$gvh = \begin{pmatrix} -\beta\tau' & \beta \\ -\tau'\beta\tau' & \tau'\beta \end{pmatrix} \quad \text{and} \quad gvhvgvh = \begin{pmatrix} \beta(\gamma+\tau')\beta\tau' & -\beta(\gamma+\tau')\beta \\ \tau'\beta(\gamma+\tau')\beta\tau' & -\tau'\beta(\gamma+\tau')\beta \end{pmatrix},$$

so we only need to see that $\beta(\gamma+\tau')\beta = \beta$. This follows from $\beta = (e-\mu\alpha)\beta$ and

$$\beta(\gamma+\tau') = \beta(\tau+\gamma\alpha) = (e-\mu\alpha)erf(\tau-\tau\mu\alpha) = e-\mu\alpha,$$

where the last equality follows from (3.15).

(3) $\Rightarrow$ (4): Assuming (3), Lemma 3.1 gives us an idempotent $e := v^{-1}hv$ and a unit $u := v^{-1}$ such that $a = e + u$. From $hvh = -h$ we have $-ve = -hv \in \operatorname{idem}(R)$. The orthogonal complement of this idempotent is precisely $1 + ve = 1 + v(a - v^{-1}) = va$. As orthogonal idempotents have disjoint images, it follows that $vaR \cap (-ve)R = (0)$, which is nothing but $aR \cap eR = (0)$.

To prove the remaining equality $a^2R \cap aeR = (0)$, first apply $hv = hvg - h$ and $vh = gvh - h$ to get

$$\begin{aligned} v^2ae &= v(va)e = v(1+hv)e = (1+vh)ve = (1+vh)hv \\ &= (gvh+g)(hvg-h) = gvhvg - gvh \end{aligned}$$

and

$$\begin{aligned} v^2a^2 &= v(va)a = v(1+hv)a = (1+vh)va = (1+vh)(1+hv) \\ &= (gvh+g)(hvg+g) = gvhvg + g. \end{aligned}$$

Using $gvhvgvh = -gvh$, it is now a routine verification to see that $-v^2ae$ and v^2a^2 are orthogonal idempotents. Therefore $v^2a^2R \cap (-v^2ae)R = (0)$, which readily gives the needed equality.

(4) $\Rightarrow$ (5) is trivial since $aeaR \subseteq aeR$.

(5) $\Rightarrow$ (6): Assume that (5) holds. Denote $f := 1 - e$; then

$$\begin{aligned} f(au^{-1}a - a) &= fa(u^{-1}a - 1) = f(e+u)(u^{-1}a-1) \\ &= fu(u^{-1}a-1) = f(a-u) = fe = 0. \end{aligned}$$

Hence $au^{-1}a - a \in aR \cap eR = (0)$ and therefore $au^{-1}a = a$. Moreover, from

$$eu^{-1}a = (a-u)u^{-1}a = au^{-1}a - a = 0$$

we have $afau^{-2}a = af(e+u)u^{-2}a = afu^{-1}a = au^{-1}a = a$, which gives

$$a^2u^{-2}a^2 - a^2 = aeau^{-2}a^2 + afau^{-2}a^2 - a^2 = aeau^{-2}a^2.$$

Thus, $a^2u^{-2}a^2 - a^2 \in a^2R \cap aeaR = (0)$, which yields $a^2u^{-2}a^2 = a^2$.

(6) $\Rightarrow$ (1): Write $e' := u^{-1}a$. We will prove that $e'ae' = e'a$ is unit-regular in $e'Re'$. Set $f := 1 - e$. From

$$fue' = fuu^{-1}a = fa = f(e+u) = fu$$

and

$$e'fu = u^{-1}afu = u^{-1}(e+u)fu = fu$$

we have $fu \in e'Re'$. This, together with $ee' = eu^{-1}a = (a-u)u^{-1}a = 0$ and $au^{-1}e = au^{-1}(a-u) = 0$, gives

$$fue'u^{-1}e' = fuu^{-1}e' = fe' = e'$$

and

$$e'u^{-1}e'fu = e'u^{-1}fu = u^{-1}au^{-1}fu = u^{-1}au^{-1}u = u^{-1}a = e'.$$

Hence $e'u^{-1}e'$ is a unit in $e'Re'$, with inverse fu. We also have

$$e'ae'u^{-1}e'a = e'au^{-1}e'a = u^{-1}a^2u^{-2}a^2 = u^{-1}a^2 = e'a,$$

which proves that indeed $e'a \in \mathrm{ureg}(e'Re')$. This completes the proof of the theorem. □

Condition (6) of Theorem 3.14 is visually left-right symmetric, and so we could replace any of the other conditions by their left-right analogue. Further, assume $a \in R$ is unit-regular. Let r, r' be any two inner inverses, so $ara = ar'a = a$. Putting $e = ra$ and $e' = r'a$ we claim that eae is unit-regular in eRe if and only if $e'ae'$ is unit-regular in $e'Re'$, and hence conditions (1) and (2) do not depend on which (unit) inner inverse one chooses. This can be shown directly, but is also a consequence of the following:

Proposition 3.16. *If R is the endomorphism ring of some left k-module ${}_kM$, then the conditions of* Theorem 3.14 *are also equivalent to:*

(7) *The element $a \in \mathrm{End}(M)$ is regular in* $\mathrm{End}(M)$, *and $a|_{Ma} \in \mathrm{End}(Ma)$ is unit-regular in* $\mathrm{End}(Ma)$.

Proof. (2) ⇔ (7): Assume $a \in R$ is regular, and let $r \in R$ be any inner inverse. Putting $e := ra \in \mathrm{idem}(R)$, we then have $Ma = Me$. There is a natural identification $\mathrm{End}(Ma) = \mathrm{End}(Me) \cong eRe$. Under this identification, $a|_{Ma}$ corresponds to eae. Thus eae is unit-regular if and only if $a|_{Ma}$ is as well. □

Remark 3.17. (1) As Theorem 3.14 is an element-wise statement, in principle one should be able to give precise formulas for some clean decomposition of a, using the following two conditions: (A) $ara = a$ for some $r \in R$ and (B) $w \in U(raRra)$ is an inner inverse for the element $(ra)a(ra)$ in the corner ring $raRra$. Indeed, put $t = w^{-1} \in U(raRra)$. With some work we obtain the relations

$$\begin{array}{llll} (1)\ ara = a, & (2)\ a^2wa = a^2, & (3)\ wawa = wa, & (4)\ tawa = ta, \\ (5)\ wra = w, & (6)\ raw = w, & (7)\ tra = t, & (8)\ rat = t, \\ (9)\ wt = ra, & (10)\ tw = ra. & & \end{array}$$

Putting

$$\begin{aligned} e :=& 1 - ra + tr + ar^2a - artr - awaw - ra^2r + ra^2w \\ &+ ar^2a^2r - ar^2a^2w - awar^2a + awar^2a^2w \end{aligned}$$

then a direct computation (which we performed using a computer and only the ten relations above) shows $e^2 = e$ and $u := a - e$ is a unit satisfying condition (6) of Theorem 3.14, whose inverse v has 53 monomials in its support when written in the letters a, r, t, w! (It is possible that a different choice for e may lead to a slightly simpler expression for v. However, adjoining an inverse for r does not simplify any of the formulas given here.)

(2) As one may expect, the statements (4) and (5) of Theorem 3.14 are *not* equivalent in the sense that a fixed idempotent e and a unit u would satisfy the conditions of (4) if and only if they would satisfy the conditions of (5). For example, taking $a = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix} -1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ in the 2×2 matrix ring over a field, one easily checks that a, e, u satisfy (5) but $a^2R \cap aeR \neq (0)$.

A well-known result of Ara says that strongly π-regular rings have stable range 1. A key step in the proof is the observation that any regular, nilpotent element of an exchange ring is a unit-regular element (by [1, Theorem 2], but also see [16]). One might ask: Is the assumption that R is an exchange ring necessary? If $a \in \mathrm{reg}(R)$ and $a^2 = 0$, then Proposition 3.16 tells us a is doubly unit-regular, and thus *both* unit-regular and clean.

However, to end this section we construct an example of a ring S and a regular element $a \in S$ such that $a^3 = 0$, but a is not unit-regular in S. Our construction is based on the ring $F\langle x, y \ : \ x^2 = 0\rangle$. The following lemma lists some properties of this ring.

Lemma 3.18. *Let F be a field and $R := F\langle x, y \ : \ x^2 = 0\rangle$. The following hold:*

(1) *If $ab = 0$ for some nonzero $a, b \in R$, then $a \in Rx$ and $b \in xR$. In particular, the set of nilpotent elements of R is precisely $Rx \cap xR = Fx + xRx$.*
(2) *The idempotents of R are trivial, so $\mathrm{idem}(R) = \{0, 1\}$.*
(3) *The ring R is Dedekind-finite, meaning $ab = 1$ implies $ba = 1$ for all $a, b \in R$.*
(4) *Units in R are exactly the elements of the form $\mu + a$ where $\mu \in F \setminus \{0\}$ and $a \in Rx \cap xR$. In particular, $U(R) + Fx \subseteq U(R)$, and $1 - yx \notin U(R)$.*

Proof. (1) is found in [7, Example 9.3].

(2) If $e \neq 1$ is an idempotent in R, then e is both a left and a right zero divisor, so that $e \in Rx \cap xR$ by (1). Hence e is a nilpotent and thus $e = 0$. (Alternatively, this follows by an easy minimal degree argument.)

(3) follows from (2) since $ab = 1$ always implies that ba is a nonzero idempotent (when $1 \neq 0$).

(4) Clearly $(F \setminus \{0\}) + (Rx \cap xR) \subseteq U(R)$, so it suffices to prove the other inclusion. Let $u \in U(R)$. We may write $u = \mu + u_1 + u_2 + u_3 + u_4$ with $\mu \in F$, $u_1 \in Rx \cap xR$, $u_2 \in Rx \cap yR$, $u_3 \in Ry \cap xR$ and $u_4 \in Ry \cap yR$. Clearly, $\mu \neq 0$. We need to prove that $u_2 = u_3 = u_4 = 0$.

Let $v := u^{-1} = \nu + v_1 + v_2 + v_3 + v_4$, with $\nu \in F$, $v_1 \in Rx \cap xR$, $v_2 \in Rx \cap yR$, $v_3 \in Ry \cap xR$, and $v_4 \in Ry \cap yR$. We have $xu \cdot vx = 0$ and $xu, vx \neq 0$, so that (1) yields $xu \in Rx$. This gives $xu_4 \in Rx$, so that $u_4 = 0$. Similarly, $v_4 = 0$.

Suppose that $u_2 \neq 0$ and $v_2 \neq 0$. Taking any monomial p in u_2 of the largest degree and any monomial q in v_2 of the largest degree, we see that the monomial pq cannot cancel with any other monomial in the product uv, so that $uv \neq 1$, which is a contradiction. Thus $u_2 \neq 0$ forces $v_2 = 0$. Similarly, $u_3 \neq 0$ forces $v_3 = 0$. Therefore, if both $u_2, u_3 \neq 0$, then $v = \nu + v_1$, which gives $u = v^{-1} = \nu^{-1} - \nu^{-2}v_1$, a contradiction. Hence $u_2 = 0$ or $u_3 = 0$; we may assume by symmetry that $u_3 = 0$.

Finally, suppose that $u_2 \neq 0$, so that $v_2 = 0$. Then

$$x = xvu = x(\nu + v_1 + v_3)(\mu + u_1 + u_2) = \nu x(\mu + u_1 + u_2) = \mu\nu x + \nu x u_2.$$

Hence $xu_2 \in Fx$, which is again a contradiction. Thus $u_2 = 0$, which completes the proof. $\square$

Example 3.19. There exists a ring S and an element $a \in \mathrm{reg}(S)$ with $a^3 = 0$, but $a \notin \mathrm{ureg}(S)$.

Proof. Let F be a field, and set $R := F\langle x, y \,:\, x^2 = 0\rangle$. Let $I := R(1 - yx)$, which is a left ideal in R. We let S be the unital subring of $\mathbb{M}_2(R)$ given by

$$S := \begin{pmatrix} R & I \\ R & F + I \end{pmatrix}.$$

The element $A := \left(\begin{smallmatrix} x & 0 \\ 1 & 0 \end{smallmatrix}\right)$ is regular with inner inverse $\left(\begin{smallmatrix} y & 1-yx \\ 0 & 0 \end{smallmatrix}\right)$. Further $A^3 = 0$.

Assume, by way of contradiction, that A is unit-regular in S, so $A = AUA$ for some unit U of S. As $E := UA$ is an idempotent with a second column of zeros, it is of the form

$$E = \begin{pmatrix} e & 0 \\ te & 0 \end{pmatrix}$$

for some $t \in R$, and some nonzero idempotent $e^2 = e \in R$. The ring R has only trivial idempotents so $e = 1$. Now $V := U^{-1}\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$ is an invertible matrix in S satisfying

$$V\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = U^{-1}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = U^{-1}E = A;$$

hence the first columns of V and A coincide, so we may write $V = \left(\begin{smallmatrix} x & a \\ 1 & b \end{smallmatrix}\right)$ for some $a \in I$ and $b \in F + I$.

Now, since V is invertible in the larger ring $\mathbb{M}_2(R)$, we see that

$$V\begin{pmatrix} -b & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a - xb & x \\ 0 & 1 \end{pmatrix}$$

is also invertible, and hence $c := a - xb \in U(R)$. On the other hand $xb \in I + Fx$ so that $c \in I + Fx$. Writing $c = \lambda x + c'(1 - yx)$ with $\lambda \in F$ and $c' \in R$, by Lemma 3.18(4) we have

$$c'(1 - yx) = c - \lambda x \in U(R)$$

so that $1 - yx \in U(R)$, yielding the needed contradiction. $\square$

Remark 3.20. Ara and O'Meara have also proven the existence of a regular, nilpotent element which is not unit-regular, independently and simultaneously with this work. They work directly with the generic ring $F\langle a, x \,:\, axa = a, xax = x, a^3 = 0\rangle$, but after seeing our example they have subsequently shown that in fact their example is isomorphic to the one given here (see [2])! As their methods are different from ours, we continue to include our example here.

It may be interesting to note that there is a special case of the "nilpotent regular element problem" which has a positive solution. (O'Meara has shown us in personal communication that he also independently discovered this result.)

Proposition 3.21. *If $a, b \in R$ are nilpotents in a ring R and $aba = a$, then $a \in \mathrm{ureg}(R)$.*

Proof. Let $u := b + (1 + a)^{-1}(1 - ab)$. Clearly, $ua = ba$ and so $aua = aba = a$. We also compute

$$\begin{aligned} u &= (1 + a)^{-1}(1 + a)b + (1 + a)^{-1}(1 - ab) = (1 + a)^{-1}(b + ab + 1 - ab) \\ &= (1 + a)^{-1}(1 + b) \in U(R). \end{aligned}$$ $\square$

This proposition leaves open the possibility that a nilpotent element with a strongly π-regular inner inverse is unit-regular, but we were unable to solve that problem.

4. Elements with regular powers

Condition (6) of Theorem 3.14 is interesting not only because it tells us that any doubly unit-regular element a is clean but also because it shows us that in a unit-regular ring, every element a has a unit inner inverse u such that u^2 is an inner inverse of a^2. This unexpected result raises the question if, for example, one could also find u such that additionally, u^3 would be an inner inverse of a^3. In this section we show that in fact such a generalization can be obtained, and much more. First, we can find v such that v^k is an inner inverse of a^k, for all k up to some arbitrary (but fixed) upper bound n. Second, we can replace the "unit-regularity" assumption just by regularity. (Of course, in this case v is no longer a unit.) Third, we can reduce the strength of our assumptions to merely saying that the powers $a, a^2, \dots, a^n$ are regular rather than positing regularity of the whole ring. Hence, only extremely minimal conditions on the element a are required to find the desired element v.

Our first result gives this generalization. Since its statement will be further strengthened shortly, we just call it a "lemma" here. In subsequent proofs, x^0 will always mean 1 for any element x of a ring.

Lemma 4.1. *Let R be a ring. If $a \in R$ is an element such that $a, a^2, \dots, a^n \in \operatorname{reg}(R)$ for some $n \geq 1$, then there exists $r \in R$ such that*

$$a^i r^j a^j = r^{j-i} a^j \qquad \text{and} \qquad a^j r^j a^i = a^j r^{j-i} \qquad \text{for all } 0 \leq i \leq j \leq n. \tag{4.2}$$

In particular, $a^j r^j a^j = a^j$ for all $1 \leq j \leq n$. Moreover, if $a \in \operatorname{ureg}(R)$, then one can take $r \in U(R)$.

Proof. We prove the statement by induction on n. If $n = 1$ there is nothing to prove, so let $n \geq 2$ and suppose that the lemma holds for integers less than n. Take $a \in R$ such that $a, a^2, \dots, a^n \in \operatorname{reg}(R)$. By the inductive hypothesis we can find $r_0 \in R$ satisfying (4.2) with $n-1$ in place of n. Moreover, if $a \in \operatorname{ureg}(R)$, then we can also assume that $r_0 \in U(R)$.

Define the idempotents $e := r_0 a$, $f := 1 - e$, $e' := a r_0$, and $f' := 1 - e'$, so that $af = 0$ and $f'a = 0$. Further take $x \in R$ with $a^n x a^n = a^n$. Let us prove that

$$r := (1 + f a^{n-1} x a) r_0 (1 + a x a^{n-1} f')$$

satisfies all the desired properties. First observe that if $r_0 \in U(R)$, then r is a unit since it is a product of three units (note that $f a^{n-1} x a$ and $a x a^{n-1} f'$ are nilpotents). It remains to see that (4.2) holds.

We prove (4.2) by way of induction on j. If $j = 0$ there is nothing to prove, so let $j \geq 1$. If $j \leq n-1$, then the inductive assumption on r_0 gives $r_0 a^j = a^{j-1} r_0^j a^j \in a^{j-1}R$ and consequently $f a^{n-1} = a^{n-1} - r_0 a^n = a^{j-1} a^{n-j} - (r_0 a^j) a^{n-j} \in a^{j-1} R$, so that

$$r a^j = (1 + f a^{n-1} x a) r_0 (1 + a x a^{n-1} f') a^j = (1 + f a^{n-1} x a) r_0 a^j \in a^{j-1} R. \tag{4.3}$$

Moreover, $r a^n \in a^{n-1} R$ also holds (i.e. the containment $r a^j \in a^{j-1} R$ of (4.3) when $j = n$), since

$$\begin{aligned} r a^n &= (1 + f a^{n-1} x a) r_0 a^n = (1 + a^{n-1} x a - r_0 a^n x a) r_0 a^n \\ &= r_0 a^n + a^{n-1} x a^n - r_0 a^n x a^n = a^{n-1} x a^n. \end{aligned}$$

Hence, in any case we may write $ra^j = a^{j-1}t_j$ for some $t_j \in R$. Now, to prove (4.2), first consider the case $i = j$. Applying the inductive hypothesis on j we have

$$\begin{aligned} a^j r^j a^j &= a^j r^{j-1} r a^j = a^j r^{j-1} a^{j-1} t_j = a(a^{j-1} r^{j-1} a^{j-1}) t_j \\ &= a a^{j-1} t_j = a r a^j = a r_0 a^j = a^j, \end{aligned}$$

as desired. In the remaining case $i \leq j-1$, apply the inductive hypothesis on j again to get

$$a^i r^j a^j = a^i r^{j-1} r a^j = a^i r^{j-1} a^{j-1} t_j = r^{j-1-i} a^{j-1} t_j = r^{j-1-i} r a^j = r^{j-i} a^j.$$

This proves the first half of the equations in (4.2). For the remaining equations $a^j r^j a^i = a^j r^{j-i}$, just note that r is defined in a left-right symmetric way, so that the equations hold by symmetry. ☐

With this lemma at hand, we can now prove an even stronger statement. We would like to thank George Bergman for pointing out to us that the set of relations (4.2) we had discovered could be further enlarged, as was done in an early draft of [4] but under slightly different assumptions. We also want to note that many of the later results in this section were inspired by subsequent discussions with Bergman, and questions raised in [4].

Theorem 4.4. *Let R be a ring. If $a \in R$ is an element such that $a, a^2, \ldots, a^n \in$ $\operatorname{reg}(R)$ for some $n \geq 1$, then there exists $v \in R$ such that*

$$a^i v^j a^k = v^{j-i} a^j v^{j-k} \qquad \text{for all } 0 \leq i, j, k \leq n \text{ with } i, k \leq j \text{ and } i + k \leq j. \tag{4.5}$$

In particular, v satisfies (4.2). Moreover, if $a \in \operatorname{ureg}(R)$, then one can take $v \in U(R)$.

Proof. Let $n \geq 1$ and $a \in R$ be as in the theorem. By Lemma 4.1 we can find $r \in R$ such that (4.2) holds and such that $r \in U(R)$ if $a \in \operatorname{ureg}(R)$. For each $m = 2, \ldots, n$, set

$$x_m := r a^m r^m - a^{m-1} r^m a^{m-1} r^{m-1}$$

and define

$$v := r + \sum_{m=2}^{n} x_m.$$

We will prove that v satisfies the properties stated in the theorem.

First observe that from $ara = a$ and $a^m r^m a^{m-1} = a^m r$ we have $a x_m = a^m r^m - a^m r r^{m-1} = 0$ for each $m = 2, \ldots, n$, so that $av = ar$. From this we can prove that

$$a^j v^j = a^j r^j \qquad \text{for each } j = 0, \ldots, n. \tag{4.6}$$

Indeed, in the case $j = 0$ there is nothing to prove. Proceeding inductively, let $j \geq 1$, and then use the inductive hypothesis and $a^j r^j a = a^j r^{j-1}$ to get

$$\begin{aligned} a^j v^j &= a(a^{j-1} v^{j-1}) v = a(a^{j-1} r^{j-1}) v \\ &= a^j r^{j-1} v = a^j r^j a v = a^j r^j a r = a^j r^{j-1} r = a^j r^j, \end{aligned}$$

as desired.

Next, let us prove that

$$v a^k r^k = a^{k-1} r^{k-1} v a r \qquad \text{for each } k = 1, \ldots, n. \tag{4.7}$$

If $k=1$ there is nothing to prove, so fix $k \geq 2$. For each $2 \leq m \leq k$ we have $a^m r^m a^k = a^k$ and $a^{m-1}r^{m-1}a^k = a^k$, which, together with $a^{m-1}r^m a^m = ra^m$, yield

$$x_m a^k = ra^k - a^{m-1}r^m a^k = ra^k - a^{m-1}r^m a^m a^{k-m} = ra^k - ra^m a^{k-m} = 0,$$

so that $x_m a^k r^k = 0$. Moreover, for each $k+1 \leq m \leq n$ we have $a^m r^m a^k = a^m r^{m-k}$ and $a^{m-1}r^{m-1}a^k = a^{m-1}r^{m-1-k}$, which yield

$$\begin{aligned} x_m a^k r^k &= ra^m r^{m-k} r^k - a^{m-1}r^m a^{m-1} r^{m-1-k} r^k \\ &= ra^m r^m - a^{m-1} r^m a^{m-1} r^{m-1} = x_m. \end{aligned}$$

These two observations together yield

$$va^k r^k = ra^k r^k + \sum_{m=2}^{n} x_m a^k r^k = ra^k r^k + \sum_{m=k+1}^{n} x_m.$$

Now let us compute $a^{k-1}r^{k-1}var$. First observe, similarly as above, that $a^{k-1}r^{k-1}x_m ar = x_m$ whenever $k+1 \leq m \leq n$, and that $a^{k-1}r^{k-1}x_m = a^{k-1}r^k(a^m r^m - a^{m-1}r^{m-1})$ whenever $2 \leq m \leq k$. This yields

$$a^{k-1}r^{k-1}var = a^{k-1}r^k ar + \sum_{m=2}^{k} a^{k-1}r^k(a^m r^m - a^{m-1}r^{m-1}) + \sum_{m=k+1}^{n} x_m.$$

In the first of the two sums on the right-hand side all terms but two cancel, so that the sum is equal to $a^{k-1}r^k a^k r^k - a^{k-1}r^k ar = ra^k r^k - a^{k-1}r^k ar$. It follows that

$$a^{k-1}r^{k-1}var = a^{k-1}r^k ar + ra^k r^k - a^{k-1}r^k ar + \sum_{m=k+1}^{n} x_m = ra^k r^k + \sum_{m=k+1}^{n} x_m.$$

Hence both sides of (4.7) are indeed equal for every k.

Now we are ready to prove that v satisfies (4.5). We prove the statement by induction on k. If $k=0$, then $i=j$, so that the equations in (4.5) are tautologies. Now let $k \geq 1$ and assume that the equations in (4.5) hold for all integers $0 \leq i', k' \leq j' \leq n$ with $i'+k' \geq j'$ and $k' < k$. We may also assume that $i \neq j$ since otherwise (4.6) gives $a^j v^j a^k = a^j r^j a^k = a^j r^{j-k} = a^j v^{j-k}$, as needed. Thus $0 \leq i, k-1 \leq j-1$ with $i+(k-1) \geq j-1$, so that the inductive hypothesis yields $a^i v^{j-1} a^{k-1} = v^{j-1-i} a^{j-1} v^{j-k}$. This, together with (4.7), gives

$$a^i v^j a^k = a^i v^{j-1}(va^k r^k)a^k = a^i v^{j-1}(a^{k-1}r^{k-1}var)a^k = v^{j-1-i}a^{j-1}v^{j-k}r^{k-1}vara^k.$$

Now, applying (4.6) and (4.7) again, we see that this is further equal to

$$\begin{aligned} v^{j-1-i}a^{j-1}r^{j-k}r^{k-1}vara^k &= v^{j-1-i}(a^{j-1}r^{j-1}var)a^k = v^{j-1-i}va^j r^j a^k \\ &= v^{j-i}a^j r^{j-k} = v^{j-i}a^j v^{j-k}. \end{aligned}$$

This completes the inductive step.

To conclude the proof, we need to see that $v \in U(R)$ whenever $r \in U(R)$. But this is obvious because $x_m = x_m ar$ and $ax_m = 0$ for each m, so that v can be written as $v = (1 + \sum_{m=2}^{n} x_m a)r$ where $\sum_{m=2}^{n} x_m a$ is a square-zero nilpotent. □

The element v in the above theorem satisfies, besides the "power inner inverse" condition $a^j v^j a^j = a^j$, many other additional equations. For example, (4.5) implies that the idempotents $a^i v^i$ and $a^j v^j$ (resp. $v^i a^i$ and $v^j a^j$) commute for all $i, j \leq n$, and $a^i v^i$ and $v^j a^j$ commute whenever $i + j \leq n$.

It is well-known that if a is a regular element in an arbitrary ring, then there exists w such that $awa = a$ and $waw = w$. Therefore, one might wonder if one could obtain w such that, additionally to (4.5), equations like $w^j a^j w^j = w^j$ would hold. In the following theorem we show that such a w can indeed be obtained. Of course, in this case we lose the invertibility of w (even if a is unit-regular), since $waw = w$ with w a unit forces $a = w^{-1}$ to be a unit.

Theorem 4.8. *Let R be a ring. If $a \in R$ is an element such that $a, a^2, \dots, a^n \in \operatorname{reg}(R)$ for some $n \geq 1$, then there exists $w \in R$ such that*

$$a^i w^j a^k = w^{j-i} a^j w^{j-k} \qquad \text{for all } 0 \leq i, j, k \leq n \text{ with } i, k \leq j. \tag{4.9}$$

In particular, w satisfies $a^j w^j a^j = a^j$ and $w^j a^j w^j = w^j$ when $j \leq n$.

Proof. Let v be any element satisfying the statement of Theorem 4.4, and put $w := vav$. An easy induction shows that $w^j = v^j a^j v^j$ for each $j = 1, \dots, n$. Hence $w^{j-i} a^j = v^{j-i} a^{j-i} v^{j-i} a^j = v^{j-i} a^j$ and similarly $a^j w^{j-k} = a^j v^{j-k}$ for all $0 \leq i, k \leq j \leq n$, which in turn gives

$$a^i w^j a^k = a^i (v^j a^j v^j) a^k = v^{j-i} a^j v^{j-k} = w^{j-i} a^j w^{j-k},$$

as desired. □

Remark 4.10. If we additionally assume $a \in \operatorname{ureg}(R)$ in Theorem 4.8, then we can take $w = vav$ with v a unit satisfying (4.5).

One might ask if there are any other relations that can be forced on the element w. The inner inverse constructed in Theorem 4.8 was chosen to satisfy symmetric relations, but there are indeed many other inner inverses which satisfy *nonsymmetric* relations. Thus, (4.9) does not represent the only type of inner inverse possible. Bergman has constructed an interesting example along these lines in [4].

In the remainder of this section we will discuss other limitations and consequences of Theorems 4.4 and 4.8.

Remark 4.11. (1) Given that

$$ax_1a = a \quad \text{and} \quad a^2x_2a^2 = a^2 \tag{4.12}$$

hold in some ring R, since the results above are completely constructive, it is straightforward to describe $z \in R$ satisfying $aza = a$ and $a^2z^2a^2 = a^2$. The element r constructed in Lemma 4.1 when $n = 2$ is

$$\begin{aligned} r \quad := \quad & x_1 + ax_2ax_1 + x_1ax_2a - x_1a^2x_2ax_1 - x_1ax_2a^2x_1 \\ & + ax_2ax_2a - ax_2ax_2a^2x_1 - x_1a^2x_2ax_2a + x_1a^2x_2ax_2a^2x_1 \end{aligned}$$

and has nine monomials in its support, whereas the element v constructed in Theorem 4.4 and the element w of Theorem 4.8 each have twenty-four monomials in their support. On the other hand, the following solution also works, with only three monomials:

$$z := x_1 + ax_2ax_1 - x_1a^2x_2ax_1.$$

(2) As one might expect, if a is an element of a unit-regular ring R and r is some (invertible) element with $ara = a$, then $a^kr^ka^k = a^k$ can easily fail for every $k \geq 2$. For example, let $R := \mathbb{M}_2(F(x))$, with F a field and $F(x)$ the field of rational

functions over F. Put $a := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $r := \left(\begin{smallmatrix} 1 & 1 \\ x & x^2 \end{smallmatrix}\right)$. An easy inductive argument shows that

$$r^k = \begin{pmatrix} s_k(x) & t_k(x) \\ u_k(x) & v_k(x) \end{pmatrix}$$

with $\deg(s_k) = 2k-3$, $\deg(t_k) = 2k-2$, $\deg(u_k) = 2k-1$, and $\deg(v_k) = 2k$, when $k \geq 2$. Thus, $a^k r^k a^k \neq a^k$ when $k \geq 2$. A quick check shows, however, that $ara = a$.

The next corollary, which is immediate from Theorem 4.4, is also a consequence of [15, Lemma 3.2].

Corollary 4.13. *If a is a unit-regular element in a regular ring R, then a^k is unit-regular for all $k \geq 1$.*

If a is a nilpotent element, a very strong classification for when all powers of a are unit-regular is given by Theorem 3.6 in [3].

To finish this section, we discuss examples showing that the statement of Theorem 4.8 cannot be significantly improved. First, we prove that each of the n different regularity conditions given as hypotheses in Theorem 4.8 is independent of the others.

Example 4.14. Let $I \subseteq \mathbb{Z}_{>0}$ be an arbitrary subset of positive integers, and let F be any field. There exist an F-algebra A and an element $a \in A$ such that $a^k \in \operatorname{ureg}(A)$ whenever $k \in I$, and $a^k \notin \operatorname{reg}(A)$ whenever $k \in \mathbb{Z}_{>0} \setminus I$.

Proof. We first construct, for each $n \geq 1$, an algebra A_n and an element $a_n \in A_n$ such that $a_n^k \in \operatorname{ureg}(A_n)$ for all $k \neq n$, and $a_n^n \notin \operatorname{reg}(A_n)$. If $n = 1$, take $A_1 := F[t]/(t^2)$ (the ring of polynomials modulo the ideal generated by t^2) and $a_1 := t$. Then clearly $a_1 \notin \operatorname{reg}(A_1)$ and $a_1^k = 0 \in \operatorname{ureg}(A_1)$ if $k \geq 2$. If $n \geq 2$, take $A_n := M_n(F[t]/(t^2))$ and define

$$a_n := \begin{pmatrix} 0 & & & & \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & t \end{pmatrix} \in A_n \quad \text{and} \quad u_n := \begin{pmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & 0 \end{pmatrix} \in U(A_n).$$

One then easily verifies that $a_n^k u_n^k a_n^k = a_n^k$ for every $k = 1, \ldots, n-1$ and $a_n^{n+1} = 0$. Hence $a_n^k \in \operatorname{ureg}(A_n)$ whenever $k \neq n$. It is also easy to see that a_n^n lies in the Jacobson radical of A_n and hence $a_n^n \notin \operatorname{reg}(A_n)$, as desired.

With the above algebras A_n and elements $a_n \in A_n$ at hand, now take

$$A := \prod_{n \in J} A_n \quad \text{and} \quad a := (a_n)_{n \in J} \in A,$$

where $J := \mathbb{Z}_{>0} \setminus I$. Since $a_n^k \in \operatorname{ureg}(A_n)$ for every $n \neq k$, we have $a^k \in \operatorname{ureg}(A)$ whenever $k \in \mathbb{Z}_{>0} \setminus J = I$. Also, since $a_n^n \notin \operatorname{reg}(A_n)$ for every n, we have $a^k \notin \operatorname{reg}(A)$ whenever $k \in J$. Hence $a \in A$ indeed satisfies the desired properties. ☐

In view of Theorem 4.8, a natural and important question arises: Assuming that a^k is regular for *each* k, can one obtain w such that (4.9) holds for *unbounded* j? In particular, this would imply that w^k is an inner inverse of a^k, for *every* k. The following example, however, demonstrates that in a general ring such an "unbounded" w does not always exist.

Example 4.15. For every field F, there exist an F-algebra A and an element $a \in A$ all of whose powers are unit-regular in A, but there is no element $w \in A$ satisfying $a^k w^k a^k = a^k$ for every $k \geq 1$.

Proof. Let A_n and $a_n \in A_n$ be as in the proof of Example 4.14. Since the algebras A_n are finite dimensional, we can embed each A_n into a unit-regular algebra B_n (for example, we can take the vector space endomorphism ring $B_n := \mathrm{End}_F(A_n)$). Let $B := \prod_{n \in \mathbb{Z}_{>0}} B_n$, and let A be the subalgebra of all sequences $(x_n)_{n \geq 1} \in B$ such that $x_n \in A_n$ for all but finitely many n. Let us prove that

$$a := (a_n)_{n \geq 1} \in A$$

satisfies the desired properties. First note that for each $k \geq 1$ we have $a^k \in \mathrm{ureg}(A)$, because $a_n^k \in \mathrm{ureg}(B_n)$ for $n \leq k$ and $a_n^k \in \mathrm{ureg}(A_n)$ for $n \geq k+1$. Assume, to the contrary, that there exists $w := (w_n)_{n \geq 1} \in A$ such that $a^k w^k a^k = a^k$ for each $k \geq 1$. Taking any n such that $w_n \in A_n$, this yields $a_n^k w_n^k a_n^k = a_n^k$ for each $k \geq 1$. In particular, $a_n^n w_n^n a_n^n = a_n^n$, which means that a_n^n is regular in A_n, a contradiction. $\square$

Example 4.15 still leaves open the possibility that the unbounded "power inverse condition" could hold in some special classes of rings, such as regular (or at least unit-regular) rings. This was left as an open problem in a previous version of this paper, but Bergman has shown in [4] the existence of a *unit-regular* ring R and an element $a \in R$ such that there is no $w \in R$ satisfying $a^k w^k a^k = a^k$ for all $k \geq 1$! Even though these examples show that the "power inner inverse" condition may not always hold for all powers simultaneously, there are some situations where this does happen. For instance, by general universal algebra techniques, an ultrapower R' of a regular ring R based on a nonprincipal ultrafilter on the natural numbers has this property; indeed every element $a \in R'$ has an inner inverse satisfying (4.9) for every $n \geq 0$ simultaneously.

Another special case occurs when right annihilators of powers of elements stabilize.

Corollary 4.16. *Let R be any ring and assume $a, a^2, \ldots, a^n \in \mathrm{reg}(R)$ for some $n \geq 1$. If $\mathrm{ann}_r(a^{n-1}) = \mathrm{ann}_r(a^n)$, then there exists $v \in R$ such that $a^k v^k a^k = a^k$ for all $k \geq 1$ (and, in particular, all powers of $a \in R$ are regular). If in addition $a \in \mathrm{ureg}(R)$, then we can take $v \in U(R)$.*

Proof. Construct v as in Theorem 4.4, and follow the notation given there. Since $a^n(1 - v^n a^n) = 0$, we have $a^{n-1}(1 - v^n a^n) = 0$ and therefore

$$a^{n-1} = a^{n-1} v^n a^n = v a^n$$

by (4.5) with $i = n-1$ and $j = k = n$. From this, an easy induction shows that $v^k a^k = v^{n-1} a^{n-1}$ for all $k \geq n-1$. Hence, for any $k \geq n-1$,

$$a^k v^k a^k = a^k v^{n-1} a^{n-1} = a^{k-n+1}(a^{n-1} v^{n-1} a^{n-1}) = a^{k-n+1} a^{n-1} = a^k. \qquad \square$$

As a special case of this corollary we have:

Corollary 4.17. *If R is a regular ring of bounded index of nilpotence (so, in particular, R is unit-regular), then for every element $a \in R$ there exists $v \in U(R)$ such that $a^k v^k a^k = a^k$ for all $k \geq 1$.*

There is a further natural situation where elements satisfy this strong form of the power inner inverse condition, which was pointed out to us by T.Y. Lam in personal communication, and has the added benefit of yielding additional information about the powered inner inverses. We thank him for allowing us to include it here.

Proposition 4.18. *Let R be any ring and suppose $a \in \operatorname{ureg}(R)$. Writing $a = eu$ with $e \in \operatorname{idem}(R)$ and $u \in U(R)$, suppose further that $eue = ue$. Then for every $k \geq 1$, the element u^{-k} is an inner inverse for a^k.*

Proof. Clearly $au^{-1}a = euu^{-1}eu = e^2u = eu = a$, establishing the $k = 1$ case. Now fixing $k \geq 2$, we recursively compute

$$a^k = (eu)^k = (eueu)(eu)^{k-2} = ueu(eu)^{k-2} = u(eu)^{k-1} = \cdots = u^{k-1}(eu).$$

Thus $a^k u^{-k} a^k = u^{k-1}(eu)u^{-k}u^{k-1}(eu) = u^{k-1}(eu) = a^k$, as needed. □

Remark 4.19. Note that the elements of Proposition 4.18 also satisfy the conditions of Corollary 4.16, since it happens that $\operatorname{ann}_r(a^k) = (1 - u^{-k}a^k)R = (1 - u^{-k}u^{k-1}eu)R = (1 - u^{-1}eu)R$ holds for all $k \geq 1$.

Another important situation when the unbounded power inner inverse condition holds is when the ring is regular and (left or right) self-injective. An example of such a ring is the endomorphism ring of a vector space. Thus the following result, in particular, shows that the unbounded power inner inverse condition holds for endomorphisms of a vector space.

Proposition 4.20. *Let R be a regular left or right self-injective ring. For every $a \in R$ there exists $w \in R$ such that $a^n w^n a^n = a^n$ for all $n \geq 1$.*

Proof. Without loss of generality we assume that R is right self-injective. Let $a \in R$, and for every $n \geq 1$ pick $r_n \in R$ such that $a^n r_n a^n = a^n$. We construct a sequence $(a_n)_{n\geq 1}$ of elements in R and a sequence $(w_n)_{n\geq 1}$ of homomorphisms of right R-modules $w_n : I_n = a_1R + \cdots + a_nR \to R$ inductively as follows. First set $a_1 := a$ and $w_1 := r_1 : aR \to R$ (that is, $w_1(x) = r_1x$ for all $x \in aR$). Now suppose that $n \geq 2$ and that $a_{n'}, w_{n'}$ have already been defined for all $n' < n$. Set

$$a_n := w_{n-1}(a_{n-1})a.$$

In order to define w_n, first note that since finitely generated submodules of projective modules over a regular ring are direct summands [14, Theorem 1.11] we can write a direct sum decomposition $I_n = I_{n-1} \oplus J_n$ for some submodule J_n. Set, with respect to this decomposition,

$$w_n|_{I_{n-1}} := w_{n-1} \quad \text{and} \quad w_n|_{J_n} := r_n a^{n-1} : J_n \to R.$$

This defines a_n and w_n for every n.

We claim that

$$a^n w_n(a_n) = a^n \tag{4.21}$$

for every n. For $n = 1$ this is clear. Proceeding inductively, let $n \geq 2$ and suppose that the statement holds for all $k < n$. Write $a_n = x + y$ where $x = a_1x_1 + \cdots + a_{n-1}x_{n-1} \in I_{n-1}$ and $y \in J_n$. By the inductive hypothesis we have $a^n w_n(a_k) = a^n w_k(a_k) = a^{n-k}a^k w_k(a_k) = a^{n-k}a^k = a^n$ for all $k < n$, hence

$$a^n w_n(x) = a^n w_n(a_1x_1 + \cdots + a_{n-1}x_{n-1}) = a^n(x_1 + \cdots + x_{n-1}).$$

Moreover, using the inductive hypothesis we easily see that $a^{n-1}a_k = a^n$ for each $k = 1, \ldots, n$, so that

$$w_n(y) = r_n a^{n-1} y = r_n a^{n-1}(a_n - a_1x_1 - \cdots - a_{n-1}x_{n-1}) = r_n a^n - r_n a^n(x_1 + \cdots + x_{n-1}).$$

These two equalities, together with $a^n r_n a^n = a^n$, now give

$$\begin{aligned} a^n w_n(a_n) &= a^n w_n(x) + a^n w_n(y) \\ &= a^n(x_1 + \cdots + x_{n-1}) + a^n(r_n a^n - r_n a^n(x_1 + \cdots + x_{n-1})) = a^n, \end{aligned}$$

proving the claim.

Now define $w : \sum_{n\geq 1} I_n \to R$ by setting $w(x) = w_n(x)$, where n is chosen subject only to $x \in I_n$. Since R is right self-injective, we can extend w to a homomorphism $w : R \to R$ and consider it as an element of R. In order to see that w fulfills the statement of the proposition, it suffices to see that $w^n a^n = w_n(a_n)$ for each $n \geq 1$. (Indeed, the equality $a^n w^n a^n = a^n$ then follows from (4.21).) The case $n = 1$ is trivial, so let $n \geq 2$ and proceed inductively. Applying the inductive hypothesis gives $w^n a^n = w(w^{n-1}a^{n-1})a = ww_{n-1}(a_{n-1})a = wa_n = w_n(a_n)$, which completes the proof. □

We finish this section with two examples showing that the power inner inverse condition for unit-regular elements does *not* imply cleanness, even if we add the extremely restrictive condition that we are working in a regular ring! So, in particular, unit-regular elements of exchange rings need not be clean.

Example 4.22. There exists a regular ring S, an element $a \in S$, and a unit $w \in U(S)$, such that $a^k w^k a^k = a^k$ for all $k \geq 1$, but a is not clean in S.

Proof. We use an example of Bergman which can be found in [11, Theorem 6.11]. Let F be a field and put

$$\begin{aligned} R := \Big\{ A = (a_{i,j})\ (i,j \in \mathbb{Z}_{>0}) \ :\ & \text{there exist } n \geq 0 \text{ and } \psi(A) = \textstyle\sum_{k\geq k_0} a_k t^k \in F(\!(t)\!) \\ & \text{such that } a_{i,j} = a_{j-i} \text{ whenever } i > n \text{ and } j \geq 1 \Big\}. \end{aligned}$$

These are the $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ column-finite matrices over F which have constant diagonals outside a finite set of rows, and thus this is just the ring R from [22, Example 3.1] defined in another way. As in [22], note that the set map ψ yields a homomorphism $\psi : R \to F(\!(t)\!)$. Put $K := \ker(\psi)$, so that

$$\begin{aligned} K = \{A = (a_{i,j}) \in R \ :\ & \text{there exists } n \geq 0 \text{ such that } a_{i,j} = 0 \\ & \text{whenever } i > n \text{ and } j \geq 1\} \end{aligned}$$

is the set of matrices which are zero outside a finite set of rows. From [22] we know that R is a regular ring which is not clean (in fact, the matrices $A \in R$ with $\psi(A) \notin F[\![t]\!]$ are not clean in R).

Let S be the subring of $\mathbb{M}_2(R)$ defined as

$$S := \begin{pmatrix} R & K \\ K & R \end{pmatrix}.$$

Note that S is regular by [14, Lemma 1.3] since it has an ideal $J := \mathbb{M}_2(K)$ such that J and $S/J \cong F(\!(t)\!) \times F(\!(t)\!)$ are both regular. Let $\alpha \in R$ denote the right shift operator, i.e. $\alpha = (a_{i,j})$ where $a_{i,j} = 1$ if $i - j = 1$ and $a_{i,j} = 0$ otherwise. Set

$$a := \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \in S.$$

The element a is not clean in S. Indeed, suppose to the contrary that $a = e + u$ for some $e \in \mathrm{idem}(S)$ and $u \in U(S)$. The homomorphism $\psi : R \to F(\!(t)\!)$ naturally induces a homomorphism $\psi : S \to F(\!(t)\!) \times F(\!(t)\!)$. Write $\psi(e) = (\varepsilon_1, \varepsilon_2)$ and

$\psi(u) = (\mu_1, \mu_2)$. Note that $\varepsilon_1, \varepsilon_2 \in \{0,1\}$ as e is an idempotent, and also note that $\mu_1, \mu_2 \neq 0$ as u is a unit. From $a = e + u$ we have

$$(t^{-1}, 0) = \psi(a) = \psi(e) + \psi(u) = (\varepsilon_1, \varepsilon_2) + (\mu_1, \mu_2),$$

so that $\mu_1 = t^{-1} - \varepsilon_1$ and $\mu_2 = -\varepsilon_2$. Hence $\varepsilon_2 = 1$, so that either $\psi(u) = (t^{-1}, -1)$ or $\psi(u) = (t^{-1}-1, -1)$. Thinking of S as consisting of infinite matrices over F, this means that u looks like $\operatorname{diag}(\alpha, -I)$ or $\operatorname{diag}(\alpha - I, -I)$ outside of finitely many rows. A quick check shows that in none of these cases can u be a surjective endomorphism of $(\bigoplus_{i\geq 1} F)^2$, and hence cannot be a unit in S. This proves that a is not clean in S.

In order to construct the inner inverse $w \in U(S)$ of a, denote by $\alpha' \in R$ the left shift operator, and let $\sigma = (s_{i,j}) \in K$ be the matrix with $s_{i,j} = 1$ if $i = j = 1$ and $s_{i,j} = 0$ otherwise. Set

$$w := \begin{pmatrix} \alpha' & 0 \\ \sigma & \alpha \end{pmatrix} \in S \quad \text{and} \quad w' := \begin{pmatrix} \alpha & \sigma \\ 0 & \alpha' \end{pmatrix} \in S.$$

Considering $\alpha'\alpha = 1$ and $\alpha\alpha' = 1 - \sigma$, we easily check that $ww' = w'w = 1$, so that w is a unit in S. Moreover, an easy verification shows that $w^k a^k = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for all $k \geq 1$, which also gives $a^k w^k a^k = a^k$. $\square$

This example still leaves open the possibility that by forcing certain algebraic expressions to be unit-regular, the element might become clean. For instance, if $a, 1-a \in \operatorname{ureg}(R)$, is a clean in R? The answer to this question is no in general. We can prove this by generalizing the previous example.

Example 4.23. For any field F, there exists a regular F-algebra S and an element $a \in S$ such that for every $p(x) \in F[x]$ it happens that $p(a) \in \operatorname{ureg}(S)$, but a is not clean in S.

Proof. We assume F is algebraically closed by replacing it with its algebraic closure if necessary. Define the ring R and the ideal K just as in the proof of Example 4.22. Similarly, let $\alpha \in R$ denote the right shift operator, $\alpha' \in R$ the left shift operator, and $\sigma \in R$ the matrix with 1 in the upper-left corner and zeros elsewhere (just as in the previous example). Let $\Lambda := F \cup \{\diamond\}$ and let S be the set of all $\Lambda \times \Lambda$ matrices $A = (a_{i,j})_{i,j\in\Lambda}$ over R, with only finitely many nonzero off-diagonal entries, each of which belong to K.

The set S is a ring under the usual matrix addition and multiplication. Further S is regular because it has the ideal

$$J := \{A = (a_{i,j}) \in S \ : \ a_{i,j} \in K \ \text{ for all } \ i,j \in \Lambda\}$$

such that J and $S/J \cong \prod_{\lambda\in\Lambda} F(\!(t)\!)$ are both regular. (Elements of J look like finite matrices over K, and since $\mathbb{M}_n(K)$ is regular, so are the elements of J.)

Let $a \in S$ be the diagonal matrix with α in the $(\diamond, \diamond)$ coordinate and λI in the (λ, λ) coordinate for each $\lambda \in F$. Our first order of business will be to prove that a is not clean in S. Suppose to the contrary that $a = e + u$ for some $e \in \operatorname{idem}(S)$ and $u \in U(S)$. As e, u and u^{-1} all have only finitely many nonzero off-diagonal entries, this means

$$a_0 = \operatorname{diag}(\alpha, \lambda_2 I, \lambda_3 I, \ldots, \lambda_n I) = e_0 + u_0$$

for some $n \geq 2$ and some distinct elements $\lambda_2, \ldots, \lambda_n \in F$, with $e_0 \in \mathrm{idem}(S_n)$ and $u_0 \in U(S_n)$, where S_n denotes the subring of $\mathbb{M}_n(R)$ with all off-diagonal entries lying in K.

The homomorphism $\psi : R \to F(\!(t)\!)$ naturally induces a homomorphism $\psi : S_n \to F(\!(t)\!)^n$. Write $\psi(e_0) = (\varepsilon_1, \ldots, \varepsilon_n)$ and $\psi(u_0) = (\mu_1, \ldots, \mu_n)$. Clearly each $\varepsilon_i \in \{0, 1\}$ as e_0 is an idempotent, and also each $\mu_i \neq 0$ as u_0 is a unit. Similarly as in the previous example, $a_0 = e_0 + u_0$ yields

$$(t^{-1}, \lambda_2, \ldots, \lambda_n) = \psi(a_0) = \psi(e_0) + \psi(u_0) = (\varepsilon_1, \ldots, \varepsilon_n) + (\mu_1, \ldots, \mu_n),$$

so that $\mu_1 = t^{-1} - \varepsilon_1$ and $\mu_i = \lambda_i - \varepsilon_i \in F$ if $i \geq 2$. Hence we have either $\psi(u_0) = (t^{-1}, \mu_2, \ldots, \mu_n)$ or $\psi(u_0) = (t^{-1} - 1, \mu_2, \ldots, \mu_n)$ where $\mu_i \in F$ if $i \geq 2$, which means that u_0 looks like $\mathrm{diag}(\alpha, \mu_2 I, \ldots, \mu_n I)$ or $\mathrm{diag}(\alpha - I, \mu_2 I, \ldots, \mu_n I)$ outside of finitely many rows. As before, we see that in neither of these two cases can u_0 be surjective as an endomorphism of $(\bigoplus_{i \geq 1} F)^n$ and hence cannot be a unit in S_n, a contradiction. Therefore a is not clean in S.

Let us prove now that $p(a)$ is unit-regular in S for every $p(x) \in F[x]$. We may assume that $p(x)$ is nonconstant. As F is algebraically closed, there exists $\lambda \in F$ with $p(\lambda) = 0$, so that $p(a)$ is a diagonal matrix with at least one diagonal entry zero. From this we see that it suffices to prove that $z := \left(\begin{smallmatrix} p(\alpha) & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is unit-regular in S_2. Write $\beta := p(\alpha)$ and let β' be any left inverse of β in R. Define $v := \left(\begin{smallmatrix} \beta' & 0 \\ 1-\beta\beta' & \beta \end{smallmatrix}\right)$. Clearly, $v \in S_2$, and v is invertible with $v^{-1} := \left(\begin{smallmatrix} \beta & 1-\beta\beta' \\ 0 & \beta' \end{smallmatrix}\right)$, and a quick verification shows that $zvz = z$. Thus $z \in \mathrm{ureg}(S_2)$ and hence $p(a) \in \mathrm{ureg}(S)$. $\square$

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