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THE CLASSIFICATION PROBLEM FOR OPERATOR ALGEBRAIC VARIETIES AND THEIR MULTIPLIER ALGEBRAS

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Abstract. We study from the perspective of Borel complexity theory the classification problem for multiplier algebras associated with operator algebraic varieties. These algebras are precisely the multiplier algebras of irreducible complete Nevanlinna-Pick spaces. We prove that these algebras are not classifiable up to algebraic isomorphism using countable structures as invariants. In order to prove such a result, we develop the theory of turbulence for Polish groupoids, which generalizes Hjorth's turbulence theory for Polish group actions. We also prove that the classification problem for multiplier algebras associated with varieties in a finite-dimensional ball up to isometric isomorphism has maximum complexity among the essentially countable classification problems. In particular, this shows that Blaschke sequences are not smoothly classifiable up to conformal equivalence via automorphisms of the disc.

1. Introduction

Let $d \in \mathbb{N}$ be a natural number and let $\mathbb{B}_d$ denote the open unit ball in $\mathbb{C}^d$. The *Drury-Arveson space* H^2_d is the completion of the space of complex polynomials $\mathbb{C}[z_1, \ldots, z_d]$ in the variables $z_1, \ldots, z_d$ with respect to the inner product defined on monomials by

$$\left\langle z_1^{\alpha_1} \cdots z_d^{\alpha_d}, z_1^{\beta_1} \cdots z_d^{\beta_d} \right\rangle = \begin{cases} \frac{\alpha_1! \cdots \alpha_d!}{(\alpha_1 + \cdots + \alpha_d)!} & \text{if } \alpha_i = \beta_i \text{ for } 1 \leq i \leq d; \\ 0 & \text{otherwise.} \end{cases}$$

This space has been the focus of intensive study for over a decade. If $d = 1$, then the Drury-Arveson space is the classical Hardy space on the unit disc, which plays an important role in the theory of operators on Hilbert space. From the point of view of operator theory, H^2_d appears to be the correct generalization of the classical Hardy space to several variables. For a comprehensive treatment of the Drury-Arveson space, the reader is referred to [5] and to the survey article [29].

One can identify H^2_d with a Hilbert space of analytic functions on $\mathbb{B}_d$ (see, for example, [29, Section 3]). An *operator algebraic variety* is the set of common zeros in $\mathbb{B}_d$ of some subset of H^2_d. Observe that any algebraic variety is an operator

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algebraic variety. In turn, an operator algebraic variety is in particular an analytic variety.

A similar construction can be performed when $d = \infty$. In this case $\mathbb{C}^\infty$ should be regarded as the infinite-dimensional separable Hilbert space, and $\mathbb{B}_\infty$ as the norm open unit ball of $\mathbb{C}^\infty$. The Drury-Arveson space H^2_∞ is defined as above as the completion of the algebra of complex polynomials $\mathbb{C}[z]$ in the sequence of variables z_n for $n \in \mathbb{N}$. In this context a complex-valued function defined on an open subset of $\mathbb{B}_\infty$ is *analytic* if it admits a uniformly convergent power series representation.

The *multiplier algebra* of H^2_d is defined by

$$\mathrm{Mult}(H^2_d) = \{\varphi : \mathbb{B}_d \to \mathbb{C} : \varphi \cdot f \in H^2_d \text{ for all } f \in H^2_d\}.$$

Every element $\varphi \in \mathrm{Mult}(H^2_d)$ induces a bounded multiplication operator M_φ on H^2_d. The identification of φ with M_φ allows one to regard $\mathrm{Mult}(H^2_d)$ as an algebra of bounded linear operators on the Hilbert space H^2_d.

For an operator algebraic variety $V \subset \mathbb{B}_d$, we consider the algebra

$$\mathcal{M}_V = \{f|_V : f \in \mathrm{Mult}(H^2_d)\}.$$

Then $\mathcal{M}_V$ is an algebra of functions on V, and, in fact, can be regarded as the multiplier algebra of a Hilbert space of functions on V. Observe also that $\mathcal{M}_V$ can be identified with a quotient of $\mathrm{Mult}(H^2_d)$. These algebras are of particular interest because of a theorem of Agler and McCarthy [1], according to which every multiplier algebra of an irreducible complete Nevanlinna-Pick space can be identified with such an algebra.

The problem of classifying the multiplier algebras $\mathcal{M}_V$ has attracted considerable attention in the last few years [3, 4, 8–10, 14, 15, 20, 22]. In particular, we refer the reader to the survey article [28]. It is proved in [8]—see also [28, Section 4]—that for $d \in \mathbb{N}$ and operator algebraic varieties V, W in $\mathbb{B}_d$, the corresponding multiplier algebras $\mathcal{M}_V$ and $\mathcal{M}_W$ are (completely) isometrically isomorphic if and only if V and W are $\mathrm{Aut}(\mathbb{B}_d)$-*conformally equivalent*. This means that there exists an automorphism $F \in \mathrm{Aut}(\mathbb{B}_d)$ mapping V onto W. Here, $\mathrm{Aut}(\mathbb{B}_d)$ denotes the group of biholomorphic maps of $\mathbb{B}_d$ onto itself. Similar conclusions hold for $d = \infty$ as long as V and W have the same *affine codimension* [28, Subsection 4.1].

The situation for algebraic isomorphism is far less clear. (It should be noted that $\mathcal{M}_V$ and $\mathcal{M}_W$, being commutative and semisimple, are algebraically isomorphic if and only if they are isomorphic as Banach algebras.) Under some mild assumption on the varieties and for $d < \infty$, it was shown in [8] that if $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic, then V and W are biholomorphically equivalent. The converse fails in multiple ways; see [9]. On the other hand, the results of [10, 14] show that if V, W are homogeneous algebraic varieties in $\mathbb{B}_d$ for d finite, then $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic if and only if there exists an invertible linear map of $\mathbb{C}^d$ mapping V onto W. Recently, this result was extended in [15] by studying the algebras $\mathcal{M}_V$ from a different point of view.

In this paper, we investigate the classification problem for multiplier algebras $\mathcal{M}_V$ from the perspective of Borel complexity theory. Our main result is that the classification problem for multiplier algebras $\mathcal{M}_V$ up to algebraic isomorphism is intractable in the sense of Borel complexity theory.

Theorem 1.1. *The multiplier algebras $\mathcal{M}_V$, where V is an operator algebraic variety in $\mathbb{B}_\infty$, are not classifiable by countable structures up to algebraic isomorphism.*

This means that there is no explicit way to classify the multiplier algebras $\mathcal{M}_V$ using countable structures as complete invariants. In particular this rules out classification by K-theoretic data, modulo the routine check that the assignment of its K-theory to a Banach algebra is given by a Borel map; see [6, Chapters 5, 8, 9]. A more precise version of the statement of Theorem 1.1 will be given in Section 2. The proof of Theorem 1.1 is presented in Sections 3 and 5. In order to prove Theorem 1.1 we develop in Section 4 the theory of turbulence for Polish groupoids. This is a generalization of Hjorth's theory of turbulence for Polish group actions from [16]; see also [12, Chapter 10].

We also study the (completely) isometric classification problem for multiplier algebras $\mathcal{M}_V$ associated with operator algebraic varieties in $\mathbb{B}_d$ with d finite. In view of the results mentioned above, this amounts to classifying operator algebraic varieties in $\mathbb{B}_d$ up to $\mathrm{Aut}(\mathbb{B}_d)$-conformal equivalence. We are able to exactly determine the complexity of such a task.

Theorem 1.2. *For any $d \in \mathbb{N}$, the relation of $\mathrm{Aut}(\mathbb{B}_d)$-conformal equivalence of operator algebraic varieties in $\mathbb{B}_d$ is essentially countable and has maximum complexity among essentially countable equivalence relations.*

In particular, Theorem 1.2 shows that the $\mathrm{Aut}(\mathbb{B}_d)$-conformal equivalence classes of operator algebraic varieties in $\mathbb{B}_d$ cannot be explicitly parametrized by the points of a Polish space. In other words the relation of $\mathrm{Aut}(\mathbb{B}_d)$-conformal equivalence of operator algebraic varieties in $\mathbb{B}_d$ is not smooth. In fact, any class of complete invariants would have to be as complex as conceivable. We will explain in more detail the content of Theorem 1.2 in Section 2.

The rest of this article is organized as follows. In Section 2 we will recall the basic notions and results from Borel complexity theory that we will refer to. Section 3 gives a short introduction to reproducing kernel Hilbert spaces and their multiplier algebras. Here Theorem 1.1 is reduced to establishing a nonclassification result for sequences in $(0, 1]$ up to ℓ^∞-equivalence. This is established in Section 5 by means of turbulence theory for Polish groupoids, developed in Section 4. We conclude in Section 6 with the proof of Theorem 1.2.

2. Borel complexity theory

Borel complexity theory studies the relative complexity of classification problems in mathematics and offers tools to detect and prove obstructions to classification. In this framework, a classification problem is regarded as an *equivalence relation* on a *standard Borel space*. Perhaps after a suitable parametrization, this covers most of classification problems in mathematics. For example, operator algebraic varieties in $\mathbb{B}_d$ for $d \in \mathbb{N} \cup \{\infty\}$ are a collection $\mathcal{V}_d$ of nonempty closed subsets of $\mathbb{B}_d$. We will verify in the appendix that $\mathcal{V}_d$ is a Borel subset of the space of nonempty closed subsets of $\mathbb{B}_d$ endowed with the Effros Borel structure [19, Section 12.C]. This shows that operator algebraic varieties form a standard Borel space when endowed with the induced Borel structure [19, Proposition 12.1]. The relation of $\mathrm{Aut}(\mathbb{B}_d)$-conformal equivalence of varieties in $\mathbb{B}_d$ can then be regarded as an equivalence relation on this standard Borel space. Similarly, the multiplier algebras $\mathcal{M}_V$ are naturally parametrized by the varieties themselves, and one can regard algebraic isomorphisms of the algebras $\mathcal{M}_V$ as an equivalence relation on the standard Borel space of varieties described above.

Borel complexity theory aims to compare the complexity of different classification problems. The fundamental notion of comparison is Borel reducibility. If E and F are equivalence relations on standard Borel spaces X and Y respectively, then a *Borel reduction* from E to F is a Borel function $f : X \to Y$ with the property that

$$f(x)Ff(x') \quad \text{if and only if} \quad xEx'$$

for every $x, x' \in X$. The relation E is *Borel reducible* to F—in formulas $E \leq_B F$—if there exists a Borel reduction from E to F. This amounts to saying that one can assign to the elements of X complete invariants up to E that are F-equivalence classes, and moreover such an assignment is *constructive* in the sense that it is given by a Borel map at the level of the spaces. We say that E and F are Borel bireducible and write $E \sim_B F$ if $E \leq_B F$ and $F \leq_B E$. The notion of Borel reducibility was first introduced in [11, Definition 2]. A complete survey on Borel complexity theory can be found in [12].

Some distinguished equivalence relations are used as benchmarks of complexity to draw a hierarchy of classification problems is mathematics. The first natural benchmark is provided by the relation $=_\mathbb{R}$ of equality of real numbers. An equivalence relation is *smooth* if it is Borel reducible to $=_\mathbb{R}$. (One can replace $\mathbb{R}$ with any other standard Borel space [19, Theorem 15.6].) For example, the relation of isomorphism of locally finite *rooted* trees is smooth [12, Theorem 13.2.3].

Smooth equivalence relations represent the lowest level complexity. A more ample class is given by considering Borel equivalence relations that are countable or essentially countable. An equivalence relation E on a standard Borel space X is *Borel* if it is a Borel subset of the product $X \times X$. A Borel equivalence relation E is *countable* if its classes are countable and *essentially countable* if it is Borel reducible to a countable one. Clearly, a smooth equivalence relation is, in particular, essentially countable. The relation E_0 of tail equivalence of binary sequences is countable but not smooth [12, Subsection 6.1]. More generally the orbit equivalence relation of a Borel action of a countable group on a standard Borel space is countable. There exists a countable Borel equivalence relation E_∞ that has maximum complexity among (essentially) countable Borel equivalence relations. One can describe E_∞ as the relation of isomorphism of locally finite trees or graphs [12, Theorem 13.2.4]. In the proof of Theorem 1.2, we will use the following equivalent description of E_∞. Let F_2 be the free group on two generators and $\{0,1\}^{F_2}$ the space of subsets of F_2 endowed with the product topology. The group F_2 naturally acts on $\{0,1\}^{F_2}$ by translation. The corresponding orbit equivalence relation $E(F_2, 2)$ is Borel bireducible with E_∞ [12, Theorem 7.3.8].

A more generous notion of classifiability for equivalence relations is being *classifiable by countable structures*. An equivalence relation is classifiable by countable structures if it is Borel reducible to the relation of isomorphism within some Borel class of structures in some first order language. Equivalently an equivalence relation is classifiable by countable structures if it is Borel reducible to the orbit equivalence relation of a continuous action of S_∞ on a Polish space [12, Section 3.6]. The Polish group S_∞ is the group of permutations of $\mathbb{N}$ with the topology of pointwise convergence [12, Section 2.4]. Any (essentially) countable equivalence relation is in particular classifiable by countable structures [17, Lemma 2.4, Lemma 2.5]. Again, there exists an equivalence relation of maximum complexity among those that are classifiable by countable structures. Such an equivalence relation can be described, for instance, as the relation of isomorphism of countable trees or graphs [11, Theorem 1].

3. Kernels and multiplier algebras

A *reproducing kernel Hilbert space* is a Hilbert space $\mathcal{H}$ which consists of functions on a set X such that for every $w \in X$, the functional

$$f \mapsto f(w)$$

on $\mathcal{H}$ is bounded. Thus, there exists $K_w \in \mathcal{H}$ such that

$$f(w) = \langle f, K_w \rangle_{\mathcal{H}}$$

for all $f \in \mathcal{H}$. The function $K : X \times X \to \mathbb{C}$ defined by $K(z, w) = K_w(z)$ is called the reproducing kernel of $\mathcal{H}$, and it is easy to check that K is positive semi-definite in the sense that for any $n \in \mathbb{N}$ and any collection of points $x_1, \dots, x_n \in X$, the $n \times n$-matrix

$$\Big(K(x_i, x_j) \Big)_{i,j=1}^{n}$$

is positive semi-definite. Conversely, if K is a *kernel* on X, that is, $K : X \times X \to \mathbb{C}$ is positive semi-definite, then there exists a unique reproducing kernel Hilbert space $\mathcal{H}(K)$ of functions on X such that K is the reproducing kernel of $\mathcal{H}(K)$; see for example [2, Section 2.2]. If K is a kernel on X and $Y \subset X$, then we denote by $K|_Y$ the kernel on Y given by the restriction of K to $Y \times Y$. A kernel is *irreducible* if for any $z, w \in X$ one has that $K(z, w) \neq 0$, and furthermore if $z \neq w$, then K_z and K_w are linearly independent.

The *multiplier algebra* of the reproducing kernel Hilbert space $\mathcal{H}$ is defined by

$$\mathrm{Mult}(\mathcal{H}) = \{\varphi : X \to \mathbb{C} : \varphi \cdot f \in \mathcal{H} \text{ for all } f \in \mathcal{H}\}.$$

It is a standard consequence of the closed graph theorem that for $\varphi \in \mathrm{Mult}(\mathcal{H})$, the operator M_φ on $\mathcal{H}$ defined by $M_\varphi f = \varphi \cdot f$ is bounded. We will always assume that $1 \in \mathcal{H}$, so that we may identify an element φ of $\mathrm{Mult}(\mathcal{H})$ with its multiplication operator M_φ. This identification endows $\mathrm{Mult}(\mathcal{H})$ with the structure of a non-selfadjoint algebra of operators on $\mathcal{H}$. A good reference for reproducing kernel Hilbert spaces and their multiplier algebras is the book [2].

We will be interested in kernels with the *complete Nevanlinna-Pick* (NP) *property*. A kernel K on a set X is said to have the Nevanlinna-Pick property if, given points $z_1, \dots, z_n \in X$ and complex numbers $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, the existence of a multiplier φ on $\mathcal{H}$ with $||M_\varphi|| \leq 1$ and

$$\varphi(z_i) = \lambda_i \quad (i = 1, \dots, n)$$

is equivalent to positive semi-definiteness of the matrix

$$\Big(K(z_i, z_j)(1 - \lambda_i \overline{\lambda_j}) \Big)_{i,j=1}^{n}.$$

The kernel is said to have the complete Nevanlinna-Pick property if the same result holds for matrix-valued interpolation of arbitrary matrix size. More information on the complete Nevanlinna-Pick property can be found in [2, Section 5]. If K satisfies the complete Nevanlinna-Pick property, then $\mathcal{H}(K)$ is said to be a *complete Nevanlinna-Pick space*. We say that $\mathcal{H}(K)$ is irreducible if K is irreducible.

Perhaps the most important example of a complete Nevanlinna-Pick space is the *Drury-Arveson space*. For $d \in \mathbb{N} \cup \{\infty\}$, we denote the Drury-Arveson kernel on $\mathbb{B}_d$ by

$$A_d\left(z, w\right) = \frac{1}{1 - \langle z, w\rangle}.$$

This kernel has the complete NP property, and H^2_d is the reproducing kernel Hilbert space associated with the space $\mathcal{H}(A_d)$ (see [2, Section 8.1]). If $V \subset \mathbb{B}_d$, then the restriction $A_d|_V$ of the Drury-Arveson kernel is a kernel on V. The associated complete NP space $H^2_d|_V$ is the closed linear span of functions on V of the form $z \mapsto \frac{1}{1-\langle z,w\rangle}$ for $w \in V$. It is a consequence of the Nevanlinna-Pick property that every multiplier of $H^2_d|_V$ is the restriction of a multiplier of H^2_d to V; thus the multiplier algebra $\mathrm{Mult}(H^2_d|_V)$ coincides with the algebra $\mathcal{M}_V$ defined in the introduction. In particular, this endows $\mathcal{M}_V$ with the structure of a non-selfadjoint operator algebra. According to a theorem of Agler and McCarthy, H^2_∞ is universal in the sense that every separable irreducible complete Nevanlinna-Pick space can be identified with $H^2_\infty|_V$ for some operator algebraic variety $V \subset \mathbb{B}_\infty$; see [2, Section 8]. The article [29] offers a comprehensive survey on the Drury-Arveson space and its properties.

We will observe in the appendix that the collection $\mathcal{V}_\infty$ of operator algebraic varieties $V \subset \mathbb{B}_\infty$ is a Borel subset of the space of nonempty closed subspaces of $\mathbb{B}_\infty$ endowed with the Effros Borel structure [19, Section 12.C]. Therefore we can state more precisely Theorem 1.1 as follows.

Theorem 3.1. *The equivalence relation on the space $\mathcal{V}_\infty$ of operator algebraic varieties in $\mathbb{B}_\infty$ defined by $V \sim W$ if and only if $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic is not classifiable by countable structures.*

An important class of irreducible complete NP kernels are the unitarily invariant complete NP kernels on $\mathbb{B}_d$. By definition, these kernels admit a power series representation

$$K^{(d)}_{\boldsymbol{a}}(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n$$

for $z, w \in \mathbb{B}_d$, where $\boldsymbol{a} = (a_n)$ is a sequence of positive numbers such that $a_0 = 1$, the power series $\sum_{n=0}^\infty a_n t^n$ has radius of convergence 1, $\sum_{n=0}^\infty a_n = \infty$, and there exists a sequence $\boldsymbol{b} = (b_n)$ of nonnegative numbers such that

$$\sum_{n=0}^{\infty} a_n t^n = \frac{1}{1 - \sum_{n=1}^\infty b_n t^n}; \tag{1}$$

see [15, Section 7]. We let $\mathcal{A} \subset (0, \infty)^{\mathbb{N}}$ denote the set of such sequences. It is not difficult to see that the set $\mathcal{B}$ of pairs $(\boldsymbol{a}, \boldsymbol{b})$ of sequences satisfying (1) is a Borel subset of $(0, \infty)^{\mathbb{N}} \times [0, \infty)^{\mathbb{N}}$. From this and the fact that every section of $\mathcal{B}$ has cardinality at most 1, it follows that $\mathcal{A}$ is the range of an injective Borel map and hence Borel by [19, Corollary 15.2].

It follows from the universality result of Agler and McCarthy mentioned above that for every $\boldsymbol{a} \in \mathcal{A}$, there exists a variety $V_{\boldsymbol{a}} \subset \mathbb{B}_\infty$ such that $\mathcal{M}_{V_{\boldsymbol{a}}}$ is completely isometrically isomorphic to $\mathrm{Mult}(\mathcal{H}(K^{(d)}_{\boldsymbol{a}}))$. Furthermore the variety $V_{\boldsymbol{a}}$ can be taken as the image of $\mathbb{B}_d$ under an embedding $j_{\boldsymbol{a}} : \mathbb{B}_d \to \mathbb{B}_\infty$. The discussion preceding Proposition 11.8 in [15] shows that the embedding $j_{\boldsymbol{a}}$ can be explicitly defined in terms of $\boldsymbol{a}$. One can use this to show that there exists a Borel map $\boldsymbol{a} \mapsto V_{\boldsymbol{a}}$ from $\mathcal{A}$ to $\mathcal{V}_\infty$ such that $\mathcal{M}_{V_{\boldsymbol{a}}}$ is completely isometrically isomorphic to $\mathrm{Mult}(H(K_{\boldsymbol{a}}^{(d)}))$. Therefore, in order to establish Theorem 3.1, it is enough to prove the following result.

Theorem 3.2. *Let $d \in \mathbb{N}$. The relation $\sim_d$ on the space $\mathcal{A}$ defined by $\boldsymbol{a} \sim_d \boldsymbol{a}'$ if and only if* $\mathrm{Mult}(H(K_{\boldsymbol{a}}^{(d)}))$ *and* $\mathrm{Mult}(H(K_{\boldsymbol{a}'}^{(d)}))$ *are algebraically isomorphic is not classifiable by countable structures.*

To prove this result, we will consider a special class of unitarily invariant complete NP kernels. We say that a (necessarily nonincreasing) sequence $\boldsymbol{a} = (a_n)$ in $(0,1]^{\mathbb{N}}$ is *admissible log-convex* if $a_0 = 1$, $(\frac{a_n}{a_{n+1}})_n$ is nonincreasing and converges to 1, and $\sum_{n=0}^{\infty} a_n = \infty$. Let $\mathcal{A}_0 \subset (0,1]^{\mathbb{N}}$ be the Borel set of admissible log-convex sequences. Log-convexity of $\boldsymbol{a}$ implies that there exists a sequence (b_n) of nonnegative numbers as in equation (1); see [2, Lemma 7.38]. Therefore, $\mathcal{A}_0 \subset \mathcal{A}$.

We consider on $\mathcal{A}_0$ the relation $E_{\mathcal{A}_0}$ defined by $\boldsymbol{a} E_{\mathcal{A}_0} \boldsymbol{a}'$ if and only if $\boldsymbol{a}$ and $\boldsymbol{a}'$ have the same growth or, using Landau's notation, $\boldsymbol{a} = \Theta(\boldsymbol{a}')$. This means that there are constants $c, C > 0$ such that $c \leq a_n'/a_n \leq C$ for every $n \in \mathbb{N}$. The equivalence of (ii) and (iii) in [15, Corollary 11.7] shows that the relations $\sim_d$ and $E_{\mathcal{A}_0}$ coincide on $\mathcal{A}_0$. Therefore, it only remains to show that the relation $E_{\mathcal{A}_0}$ is not classifiable by countable structures. This will be proved in Section 5.

We mention here that the same proof also shows that the algebras $A(K)$ for K a unitarily invariant complete NP kernel on $\mathbb{B}_d$ are not classifiable by countable structures up to algebraic isomorphism. Here $A(K)$ denotes the closure of the polynomials in $\mathrm{Mult}(\mathcal{H}(K))$; see [15, Section 6]. One can also observe that, for $d \in \mathbb{N}$, the collection $\mathcal{K}$ of unitarily invariant complete NP kernels is Borel. It follows from Theorem 3.2 that the relation on $\mathcal{K}$ defined by $K \sim K'$ if and only if $\mathrm{Mult}(\mathcal{H}(K))$ and $\mathrm{Mult}(\mathcal{H}(K'))$ are algebraically isomorphic is not classifiable by countable structures.

4. Turbulence for Polish groupoids

The main goal of this section is to introduce the notion of *turbulence* for Polish groupoids and to generalize to this setting Hjorth's turbulence theorem. A *groupoid* can be seen as a tuple $(G, G^0, s, r, \cdot, i, (\ \ )^{-1})$ where G^0 and G are sets, s, r are functions $G \to G^0$, i is a function $G^0 \to G$, $\cdot$ is a function from the set $G^2 := \{(\gamma, \rho) \in G \times G : s(\gamma) = r(\rho)\}$ to G, and $\gamma \mapsto \gamma^{-1}$ is a function from G to G. These functions are assumed to satisfy the following relations:

- $s(i(x)) = r(i(x)) = x$ for every $x \in G^0$;
- $\gamma \cdot i(s(\gamma)) = \gamma = i(r(\gamma)) \cdot \gamma$ for every $\gamma \in G$;
- $s(\gamma \cdot \rho) = s(\rho)$ and $r(\gamma \cdot \rho) = r(\gamma)$ for every $\gamma, \rho \in G$;
- $(\gamma \cdot \rho) \cdot \tau = \gamma \cdot (\rho \cdot \tau)$ for every $\gamma, \rho, \tau \in G$ such that $(\gamma, \rho) \in G^2$ and $(\rho, \tau) \in G^2$;
- $\gamma^{-1} \cdot \gamma = s(\gamma)$ and $\gamma \cdot \gamma^{-1} = r(\gamma)$ for every $\gamma \in G$.

The elements of G^0 are called the *objects* of the groupoid, while the elements of G are called the *arrows*. Given an arrow γ the objects $s(\gamma)$ and $r(\gamma)$ are called the source and range of γ, respectively, while γ^{-1} is called the inverse arrow of γ. The arrow $i(x)$ associated with x is called the *identity arrow* of X. The maps $s, r : G \to G^0$ are called source and range maps, respectively, while the partially defined binary operation $\cdot$ is called composition of arrows. As is customary, in the following we identify every object with the corresponding identity arrow, and we denote the composition of arrows $\gamma \cdot \rho$ simply by $\gamma\rho$. A groupoid can be equivalently defined as a small category where every morphism is an isomorphism. In this case, the arrows of the groupoid are just the morphisms of the category.

A *Polish groupoid* is a groupoid endowed with a topology that

(1) has a countable basis of Polish open sets,
(2) makes composition and inversion of arrows continuous and open,
(3) makes for every $x \in G^0$ the set Gx of arrows of G with source x a Polish subspace of G, and
(4) makes the set of objects G^0 a Polish subspace of G.

Polish groupoids have been introduced and studied in [24, 25]. In [21] several fundamental results about Polish group actions are generalized to Polish groupoids. We assume in the following that G is a Polish groupoid. The *orbit equivalence relation* of G is the equivalence relation E_G on G^0 defined by xE_Gy if and only if there exists $\gamma \in G$ such that $s(\gamma) = x$ and $r(\gamma) = y$. If $A, B \subset G$ we let AB be the set of all compositions $\gamma\rho$ for $\gamma \in A$ and $\rho \in B$ such that $r(\rho) = s(\gamma)$. We write $A\gamma$ for $A\{\gamma\}$ when $A \subset G$ and $\gamma \in G$. In particular if $x \in G^0$, then Ax is the set of elements of A with source x. If X is a G_δ subset of G^0, denote by $G|_X$ the Polish groupoid $XGX = \{\gamma \in G : s(\gamma), r(\gamma) \in X\}$ endowed with the subspace topology. This is called the *restriction* of G to X. If x is an object of G and V is a neighborhood of x in G, then the *local orbit* $\mathcal{O}(x, V)$ is the set of all points that can be reached from x by applying elements of V. In formulas

$$\mathcal{O}(x, V) = \bigcup_{n \in \mathbb{N}} r[V^n x].$$

Definition 4.1. An object x of G is *turbulent* if for every neighborhood V of x the local orbit $\mathcal{O}(x, V)$ is somewhere dense. The groupoid G is *generically preturbulent* if the set of turbulent objects with dense orbit is a comeager subset of G^0. If moreover every orbit is meager, then G is *generically turbulent*.

In the rest of this section we will often tacitly use the following version of the classical Kuratowski–Ulam theorem; see [21, Lemma 2.9.1].

Fact 4.2. Suppose that X is a second countable topological space, Y is a Polish space, and $f : X \to Y$ is open and continuous. If $A \subset X$ is analytic, then A is comeager if and only if $f^{-1}\{y\} \cap A$ is comeager in $f^{-1}\{y\}$ for comeager many $y \in Y$.

For example, it follows from Fact 4.2 that if X is a dense G_δ subspace of G^0 and G is generically (pre)turbulent, then $G|_X$ is generically (pre)turbulent.

Suppose that H is a Polish group and Y is a Polish H-space, i.e. a Polish space endowed with a continuous action of H. Let G be the Polish action groupoid associated with the Polish H-space Y as in [21, Subsection 2.7]. Observe that the orbit equivalence relation E_G coincides with the orbit equivalence relation E_H^Y. Furthermore it is not difficult to verify that G is a generically (pre)turbulent groupoid as in Definition 4.1 if and only if Y is a generically (pre)turbulent H-space in the sense of [12, Definition 10.3.3].

Recall the following terminology from Borel complexity of equivalence relations. If E and F are equivalence relations on standard Borel spaces X and Y, then an (E,F)-homomorphism is a function $f : X \to Y$ that maps E-classes into F-classes. A generic (E,F)-homomorphism is a function $f : X \to Y$ that is an (E,F)-homomorphism when restricted to some comeager subset of X. An equivalence relation E on a standard Borel space X is *generically S_∞-ergodic* if for every Polish S_∞-space Y and every Baire-measurable generic $(E, E^Y_{S_\infty})$-homomorphism, there exists a comeager subset of X that is mapped by f into a single S_∞-orbit. It is well known that an equivalence relation is classifiable by countable structures if and only if it is Borel reducible to the orbit equivalence relation of a Polish S_∞-space; see [12, Theorem 11.3.8].

The following is the main consequence of turbulence for Polish groupoids.

Theorem 4.3. *Suppose that G is a generically preturbulent Polish groupoid. Then the associated orbit equivalence relation E_G is generically S_∞-ergodic.*

Corollary 4.4. *If G is a generically turbulent Polish groupoid, then the orbit equivalence relation E_G is not classifiable by countable structures.*

Theorem 4.3 generalizes the original result of Hjorth [16, Section 3] from Polish group actions to Polish groupoids. Polish groupoids provide a natural setting to present the proof of Hjorth's turbulence theorem even in the case of Polish group actions. Indeed in the course of the proof one looks at the action "restricted" to a (not necessarily invariant) G_δ subspace; see for example [12, Theorem 10.4.2]. Such a restriction is not a Polish group action in general, even when one starts with a Polish group action. It is nonetheless a Polish groupoid.

The following lemma is the groupoid analog of [16, Lemma 3.17]. In the following, if V is an open subset of G, we write $\forall^* \gamma \in V$ to mean "for a comeager set of $\gamma \in V$".

Lemma 4.5. *Suppose that G is a Polish groupoid, H is a Polish group, and Y is a Polish H-space. If $f : G^0 \to Y$ is a Baire-measurable generic $\left(E_G, E^Y_H\right)$-homomorphism, then there exists a comeager subset C of G^0 such that for every $x \in C$ and every open neighborhood W of 1_H in H there exists a neighborhood V of x such that for every $x' \in s[V] \cap C$ and for a comeager set of $\gamma \in Vx'$,*

$$f(r(\gamma)) \in W f(x').$$

Proof. After replacing G with the restriction of G to a dense G_δ subset of G^0, we can assume that f is a continuous $\left(E_G, E^Y_H\right)$-homomorphism [12, Exercise 2.3.2]. Furthermore it is enough to prove that for every open neighborhood W of 1_H there is a comeager subset C of X such that for every $x \in C$ there exists a neighborhood V of x in G such that $\forall x' \in s[V] \cap C$, $\forall^* \gamma \in Vx'$, $f(r(\gamma)) \in W f(x')$. Fix an open neighborhood W of 1_H and an open neighborhood W_0 of 1_H such that $W_0^{-1} = W_0$ and $W_0^2 \subset W$. Fix a sequence (h_n) in H such that

$$\bigcup_{n \in \mathbb{N}} W_0 h_n = H.$$

For every $n \in \mathbb{N}$, the set

$$B_n = \{(z,y) \in Y \times Y \mid z \in W_0 h_n y\}$$

is analytic. Therefore the set

$$A_n = \{\gamma \in G : f(r(\gamma)) \in W_0 h_n f(s(\gamma))\}$$

is analytic by [19, Proposition 22.1]. By [19, Proposition 8.22] there exists an open subset O_n of G such that $O_n \bigtriangleup A_n$ is meager. Set $D_n = A_n \cap O_n$, and observe that $D_n D_n^{-1}$ is a comeager subset of $O_n O_n^{-1}$. Since G is the union of A_n for $n \in \mathbb{N}$, the union O of O_n for $n \in \mathbb{N}$ is an open dense subset of G. In particular $r[O]$ is an open subset of G^0. Define now, for $n \in \mathbb{N}$, $\widetilde{O}_n$ to be the set of $\gamma \in O_n$ such that $r(\gamma)$ does not belong to the closure of the union of $r[O_i]$ for $i < n$. Let $\widetilde{O}$ be the union of $\widetilde{O}_n$ for $n \in \mathbb{N}$, and observe that $r[\widetilde{O}]$ is an open dense subset of G^0. For every $n \in \mathbb{N}$ set $\widetilde{D}_n = D_n \cap \widetilde{O}_n$ and observe that $\widetilde{D}_n$ is a comeager subset of $\widetilde{O}_n$. Therefore there exists a comeager subset C_n of $r[\widetilde{O}_n] = s[\widetilde{O}_n \widetilde{O}_n^{-1}]$ such that for every $x \in C_n$, $\widetilde{D}_n \widetilde{D}_n^{-1} x$ is a comeager subset of $\widetilde{O}_n \widetilde{O}_n^{-1} x$. Define C to be the union of C_n for $n \in \mathbb{N}$, and observe that C is a comeager subset of G^0. We claim that C satisfies the desired conclusions. Fix $x \in C$ and $n \in \mathbb{N}$ such that $x \in C_n$. We have that $\widetilde{O}_n \widetilde{O}_n^{-1}$ is an open neighborhood of x. Furthermore for every $x' \in C_n = C \cap s[\widetilde{O}_n \widetilde{O}_n^{-1}]$, $\widetilde{D}_n \widetilde{D}_n^{-1} x'$ is comeager in $\widetilde{O}_n \widetilde{O}_n^{-1} x'$. If $\rho, \gamma \in \widetilde{D}_n$, then

$$f(r(\gamma)) \in W_0 h_n f(s(\gamma)) \quad \text{and} \quad f(r(\rho)) \in W_0 h_n f(s(\rho)).$$

Therefore

$$f(r(\rho\gamma^{-1})) = f(r(\rho)) \in W_0 h_n f(s(\rho)) \subset W_0 W_0^{-1} f(s(\rho\gamma^{-1})) \subset W f(s(\rho\gamma^{-1})).$$

This concludes the proof. □

We now explain how one can deduce Theorem 4.3 from Lemma 4.5.

Proof of Theorem 4.3. Fix an enumeration $(V_k)_{k\in\mathbb{N}}$ of a basis of Polish open subsets of G and a compatible complete metric d_Y on Y bounded by 1. Suppose that d is the metric in S_∞ defined by

$$\log_2 d(\sigma, \rho) = -\min\{n \in \mathbb{N} : \sigma(n) \neq \rho(n)\}$$

for $\sigma, \rho \in S_\infty$. We also consider the complete metric

$$D(\sigma, \rho) = d(\sigma, \rho) + d(\sigma^{-1}, \rho^{-1})$$

on S_∞. Define e to be the identity of S_∞, and

$$N_k = \left\{\sigma \in S_\infty : d(\sigma, e) < 2^{-k}\right\}$$

for $k \in \mathbb{N}$. As in the proof of Hjorth's turbulence theorem for Polish group actions [12, Theorem 10.4.2], one can deduce from Lemma 4.5 that there exists a dense G_δ subset C_0 of G^0 with the following properties:

- $f|_{C_0}$ is a continuous $(E_G, E^Y_{S_\infty})$-homomorphism;
- every element of C_0 has dense orbit;
- for every $m \in \mathbb{N}$ and $x \in V_m \cap C_0$ the local orbit $\mathcal{O}(x, V_m)$ is somewhere dense;
- for every $x \in C_0$ and $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $x \in V_m$ and $\forall x' \in s[V_m]$, $\forall^* \gamma \in V_m x'$, $f(r(\gamma)) \in N_k f(x')$.

Let C be the set of $x \in C_0$ such that $\forall^* \gamma \in Gx$, $r(\gamma) \in C_0$, and observe that C is a dense G_δ subset of G^0 [21, Lemma 2.10.6]. After replacing G with the restriction $G|_C$ of G to C and V_k with $V_k \cap G|_C$, we can assume that $C = G^0$.

Fix $x_0, y_0 \in G^0$. We claim that $f(x) E^Y_{S_\infty} f(y)$. We will define by recursion on $i \geq 0$ elements x_i, y_i of G^0, g_i, h_i of S_∞, and $n_x(i), n_y(i)$ of $\mathbb{N}$, such that the following conditions hold:

- $g_0 = h_0 = e$;
- $x_0 \in V_{n_x(0)}$ and $y_0 \in V_{n_y(0)}$;
- $g_i f(x) = f(x_i)$ and $h_i f(y) = f(y_i)$;

- $x_{i+1} \in V_{n_x(i)} \cap \mathcal{O}\left(x_i, V_{n_x(i)}\right)$ and $y_{i+1} \in V_{n_y(i)} \cap \mathcal{O}\left(y_i, V_{n_y(i)}\right)$;
- the d_Y-diameter of $f\left[G^0 \cap V_{n_x(i)}\right]$ is at most 2^{-i};
- $\mathcal{O}\left(x_i, V_{n_x(i)}\right)$ is dense in $V_{n_y(i)} \cap G^0$, and $\mathcal{O}\left(y_i, V_{n_y(i)}\right)$ is dense in $V_{n_x(i+1)} \cap G^0$;
- $d(g_i, g_{i+1}) \leq 2^{-i}$ and $d(h_i, h_{i+1}) \leq 2^{-i}$;
- if $i > 0$ and $k_x(i) = \max\left\{g_i(\lambda), g_i^{-1}(\lambda) \mid \lambda \leq i\right\}$, then $\forall z \in s\left[V_{n_x(i)}\right]$, $\forall^* \gamma \in V_{n_x(i)} z$, $f(r(\gamma)) \in N_{k_x(i)} f(z)$;
- if $i \geq 0$ and $k_y(i) = \max\left\{h_i(\lambda), h_i^{-1}(\lambda) \mid \lambda \leq i\right\}$, then $\forall z \in s\left[V_{n_y(i)}\right]$, $\forall^* \gamma \in V_{n_y(i)} z$, $f(r(\gamma)) \in N_{k_y(i)} f(z)$.

Granted the construction, the sequences $(g_i), (h_i)$ in S_∞ are D-Cauchy and hence converge to elements $g, h \in S_\infty$. Furthermore $d_Y\left(g_i f(x), h_i f(y)\right) \to 0$ and hence $gf(x) = hf(y)$. This concludes the proof that $f(x) E_{S_\infty}^Y f(y)$.

We assume recursively that we have defined $x_i, y_i, g_i, h_i, n_x(i), n_y(i)$ and explain how to define $x_{i+1}, g_{i+1}, n_x(i+1)$. The definition of $y_{i+1}, h_{i+1}, n_y(i+1)$ is similar. We have that the local orbit $\mathcal{O}\left(y_i, V_{n_y(i)}\right)$ is somewhere dense. Pick a nonempty open subset W of $V_{n_y(i)}$ that is contained in the closure of $\mathcal{O}\left(y_i, V_{n_y(i)}\right)$. By recursive hypothesis we have that $\mathcal{O}\left(x_i, V_{n_x(i)}\right)$ is dense in W. Let $\gamma_0, \ldots, \gamma_{\ell-1} \in V_{n_x(i)}$ such that, setting $z_j = s\left(\gamma_j\right)$ for $j < \ell$ and $z_\ell = r\left(\gamma_{\ell-1}\right)$, one has that $z_0 = x_i$, $z_\ell \in W$, and $z_{j+1} = r\left(\gamma_j\right)$ for $j < \ell$. Since by inductive assumption we have that $\forall z \in s\left[V_{n_x(i)}\right]$, $\forall^* \gamma \in V_{n_x(i)} z$, $f(r(\gamma)) \in N_{k_x(i)} f(z)$, after modifying the sequence $(\gamma_0, \ldots, \gamma_{\ell-1})$ we can assume that, for every $j < \ell$, $f\left(z_{j+1}\right) = p_j f\left(z_j\right)$ for some $p_j \in N_{k_x(i)}$. Therefore $f(z_\ell) = pf(z)$ where $p = p_{\ell-1} p_{\ell-2} \cdots p_0 \in N_{k_x(i)}$. We may then let $x_{i+1} = z_\ell$, $g_{i+1} = pg_i$, $k_x(i+1) = \max\left\{g_{i+1}(\lambda), g_{i+1}^{-1}(\lambda) : \lambda \leq i+1\right\}$, and $n_x(i+1) \in \mathbb{N}$ such that $x_{i+1} \in V_{n_x(i+1)}$ and $\forall x' \in s\left[V_{n_x(i+1)}\right]$, $\forall^* \gamma \in V_{n_x(i+1)} x'$, $f(r(\gamma)) \in N_{k_x(i+1)} f(x')$. This concludes the definition of $x_{i+1}, g_{i+1}, n_x(i+1)$. $\square$

5. Admissible log-convex sequences

Recall from Section 3 that a sequence $\boldsymbol{a}$ in $(0,1]^{\mathbb{N}}$ is *admissible log-convex* if $a_0 = 1$, $(\frac{a_n}{a_{n+1}})_n$ is nonincreasing and converges to 1, and $\sum_n a_n = \infty$. The set $\mathcal{A}_0 \subset (0,1]^{\mathbb{N}}$ of admissible log-convex sequences is Borel. We say that two admissible log-convex sequences $\boldsymbol{a}$ and $\boldsymbol{a}'$ are *growth equivalent* if there exist $c, C > 0$ such that $c \leq \frac{a_n'}{a_n} \leq C'$ for every $n \in \mathbb{N}$. We denote by $E_{\mathcal{A}_0}$ the corresponding equivalence relation on $\mathcal{A}_0$. The main goal of this section is to prove the following result:

Proposition 5.1. *Admissible log-convex sequences considered up to growth equivalence are not classifiable by countable structures.*

However, it is not difficult to verify that admissible log-convex sequences are classifiable by the orbits of a Polish group action up to growth equivalence. This means that there exists a continuous Polish group action $G \curvearrowright X$ such that $E_{\mathcal{A}_0}$ is Borel reducible to the orbit equivalence relation E_G^X. The crucial point is that if

$$B = \{(-\log(a_n)) : (a_n) \in A\} \subset (0, \infty)^{\mathbb{N}},$$

then

$$H = \{z \in \ell^\infty : \text{ there exist } x, y \in B \text{ with } x - y = z\}$$

is a subgroup of ℓ^∞ which is separable in the ℓ^∞-metric, and two sequences $(a_n), (a_n')$ in A have the same growth if and only if $(-\log(a_n))$ and $(-\log(a_n'))$ belong to the same H-orbit under translation.

The rest of this section is dedicated to the proof of Proposition 5.1. Consider the equivalence relation F on $(0,1)^{\mathbb{N}}$ defined by

$$\boldsymbol{s} F \boldsymbol{s}' \text{ if and only if } \sup_n \left| \sum_{k<n} \left(\prod_{i\leq k} s_i - \prod_{i\leq k} s_i' \right) \right| < \infty.$$

Define furthermore the Borel function

$$\begin{aligned} (0,1)^{\mathbb{N}} &\to (0,1]^{\mathbb{N}} \\ \boldsymbol{s} &\mapsto f(\boldsymbol{s}) = \exp\left(-\sum_{k<n} \prod_{i\leq k} s_i \right)_{n\in\mathbb{N}} \end{aligned}$$

where the empty sum is 0. Observe that for $s \in (0,1)^{\mathbb{N}}$, we have that $f(\boldsymbol{s})_0 = 1$, $f(\boldsymbol{s})$ is log-convex and $f(\boldsymbol{s})_n / f(\boldsymbol{s})_{n+1} \geq 1$ for all $n \in \mathbb{N}$. Let $X \subset (0,1)^{\mathbb{N}}$ be the set of $\boldsymbol{s} \in (0,1)^{\mathbb{N}}$ such that $f(\boldsymbol{s}) \in \mathcal{A}_0$. Using the fact that $f(\boldsymbol{s}) \in \mathcal{A}_0$ if and only if $f(\boldsymbol{s})$ is not summable, it is not difficult to verify that X is a dense G_δ subset of $(0,1)^{\mathbb{N}}$. The restriction $f|_X$ of f to X is a Borel reduction from $F|_X$ to $E_{\mathcal{A}_0}$. It is thus enough to show that $F|_X$ is not classifiable by countable structures.

Lemma 5.2. *F has meager classes.*

Proof. Fix $\boldsymbol{s} \in (0,1)$. We want to show that the F-class of $\boldsymbol{s}$ is meager. We can assume without loss of generality that $\prod_{i\leq k} s_i \to 0$ for $k \to \infty$, as the set of such $\boldsymbol{s}$ is a comeager subset of $(0,1)^{\mathbb{N}}$. Fix $m \in \mathbb{N}$ and let K_m be the (closed) set of $\boldsymbol{t} \in (0,1)^{\mathbb{N}}$ such that, for every $n \in \mathbb{N}$,

$$\left| \sum_{k<n} \left(\prod_{i\leq k} s_i - \prod_{i\leq k} t_i \right) \right| \leq m.$$

Observe that if $\boldsymbol{t}^0 \in K$ and $n_0 \in \mathbb{N}$, then the element $\boldsymbol{t}$ of $(0,1]^{\mathbb{N}}$ defined by

$$t_i = \begin{cases} t_i^0 & \text{for } i \leq n_0, \\ 1 - 2^{-i} & \text{otherwise} \end{cases}$$

does not belong to K_m. Therefore K_m is nowhere dense. Finally observe that the F-class of $\boldsymbol{s}$ is $\bigcup_m K_m$. ☐

Now let Γ be the subgroup of $\mathbb{R}_+^{\mathbb{N}}$ containing those sequences $\boldsymbol{g}$ such that

$$\sum_n \left| \prod_{k\leq n} g_k - 1 \right| < \infty.$$

Observe that Γ is indeed a subgroup of $\mathbb{R}_+^{\mathbb{N}}$. In fact suppose that $\boldsymbol{g}, \boldsymbol{h} \in \Gamma$. Fix $n_0 \in \mathbb{N}$ such that

$$\left| \prod_{k\leq n} g_k - 1 \right| \leq \frac{1}{2}$$

for every $n \geq n_0$. Then

$$\sum_{n\in\mathbb{N}} \left| \prod_{k\leq n} g_k^{-1} - 1 \right| \leq \sum_{n<n_0} \left| \prod_{k\leq n} g_k^{-1} - 1 \right| + 2 \sum_{n\in\mathbb{N}} \left| \prod_{k\leq n} g_k - 1 \right| < \infty$$

and hence $\boldsymbol{g}^{-1} \in \Gamma$. Furthermore

$$\sum_{n\in\mathbb{N}}\left|\prod_{k\leq n} g_k h_k - 1\right| \leq \sum_{n<n_0}\left|\prod_{k\leq n} g_k h_k - 1\right| + \frac{3}{2}\sum_{n\in\mathbb{N}}\left|\prod_{k\leq n} h_k - 1\right| + \sum_{n}\left|\prod_{k\leq n} g_k - 1\right| < \infty$$

and hence $\boldsymbol{gh} \in \Gamma$. Consider the coordinate-wise multiplication action $\Gamma \curvearrowright \mathbb{R}_+^{\mathbb{N}}$. Let E be the restriction to $(0,1)^{\mathbb{N}}$ of the orbit equivalence relation of the action $\Gamma \curvearrowright \mathbb{R}_+^{\mathbb{N}}$. Since F has meager classes, $X \subset (0,1)^{\mathbb{N}}$ is comeager, and $E \subset F$, in order to prove that $F|_X$ is not classifiable by countable structures it is enough to show that E is generically S_∞-ergodic.

Define the bi-invariant metric d_Γ on Γ by setting

$$d_\Gamma(\boldsymbol{g}, \boldsymbol{h}) = \sum_{n\in\mathbb{N}}\left|\prod_{k\leq n} g_k - \prod_{k\leq n} h_k\right|.$$

We claim that d_Γ induces a Polish topology on Γ. To this end, consider the injective map $\Phi : \mathbb{R}_+^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ defined by

$$\boldsymbol{a} \mapsto \left(\left(\prod_{k\leq n} a_k\right) - 1\right)_n.$$

Observe that the restriction of Φ to Γ is an isometry from (Γ, d_Γ) to ℓ^1 endowed with the ℓ^1-metric. Furthermore the image of Γ under Φ is a G_δ subset of ℓ^1, since $\boldsymbol{b} \in \Phi[\Gamma]$ if and only if $b_n > -1$ for every $n \in \mathbb{N}$. Since a G_δ subspace of a Polish space is Polish [19, Theorem 3.11], this concludes the proof that d_Γ induces a Polish topology on Γ.

If $\boldsymbol{g} \in \Gamma$ and $\boldsymbol{s} \in (0,1)^{\mathbb{N}}$, define $\boldsymbol{gs} \in \mathbb{R}_+^{\mathbb{N}}$ by

$$(\boldsymbol{gs})_n = g_n s_n.$$

Consider now the groupoid

$$G = \left\{(\boldsymbol{g}, \boldsymbol{s}) \in \Gamma \times (0,1)^{\mathbb{N}} : \boldsymbol{gs} \in (0,1)^{\mathbb{N}}\right\}.$$

Composition and inversion of arrows in G are defined by

$$(\boldsymbol{g}, \boldsymbol{s})(\boldsymbol{h}, \boldsymbol{t}) = (\boldsymbol{gh}, \boldsymbol{t})$$

whenever $\boldsymbol{ht} = \boldsymbol{s}$, and

$$(\boldsymbol{g}, \boldsymbol{s})^{-1} = \left(\boldsymbol{g}^{-1}, \boldsymbol{gs}\right).$$

Being a closed subset of $\Gamma \times (0,1)^{\mathbb{N}}$, G is Polish with the induced topology. Clearly composition and inversion of arrows are continuous. Furthermore the map $(\mathbf{1}, \boldsymbol{s}) \mapsto \boldsymbol{s}$ allows one to identify the set of objects of G with $(0,1)^{\mathbb{N}}$. It remains to show that composition of arrows is open. To this purpose it is enough to show that the source map

$$\begin{array}{ccc} G & \to & (0,1)^{\mathbb{N}} \\ (\boldsymbol{g}, \boldsymbol{s}) & \mapsto & \boldsymbol{s} \end{array}$$

is open; see [26, Exercise I.1.8]. Suppose that $(\boldsymbol{g}, \boldsymbol{s}) \in G$ and U is an open neighborhood of $(\boldsymbol{g}, \boldsymbol{s})$. Thus there exist $\varepsilon > 0$ and $N \in \mathbb{N}$ such that U contains all the pairs $(\boldsymbol{h}, \boldsymbol{t}) \in G$ such that $d_\Gamma(\boldsymbol{g}, \boldsymbol{h}) < \varepsilon$ and $|s_n - t_n| < \varepsilon$ for $n \leq N$. Suppose that $\varepsilon > \eta > 0$ is such that $g_n(s_n + \eta) < 1$ for every $n \leq N$. Consider the neighborhood

W of $\boldsymbol{s}$ consisting of those $\boldsymbol{t} \in (0,1)^{\mathbb{N}}$ such that $|s_n - t_n| < \eta$ for every $n \leq N$. We claim that $s[U] \supset W$. In fact if $\boldsymbol{t} \in W$ we have that for $n \leq N$,

$$g_n t_n \leq g_n (s_n + \eta) < 1$$

and therefore $(\boldsymbol{g}, \boldsymbol{t}) \in U$.

In the following lemma we establish that G is a turbulent Polish groupoid. Together with Theorem 4.3, this implies that its associated orbit equivalence relation E is generically S_∞-ergodic, concluding the proof of Proposition 5.1.

Lemma 5.3. *Any element $\boldsymbol{s}$ of $(0,1)^{\mathbb{N}}$ is a turbulent object with dense orbit for the Polish groupoid G.*

Proof. It is easy to see that the orbit of $\boldsymbol{s}$ is dense. It remains to show that for any neighborhood V of $(\boldsymbol{1}, \boldsymbol{s})$ in G the local orbit $\mathcal{O}(\boldsymbol{s}, V)$ is somewhere dense. Without loss of generality we can assume that there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that if

$$U = \left\{ \boldsymbol{t} \in (0,1)^{\mathbb{N}} : \forall n \leq n_0, \left| \frac{t_n}{s_n} - 1 \right| < \varepsilon \right\}$$

and

$$W = \{ \boldsymbol{g} \in \Gamma : d_\Gamma (\boldsymbol{g}, \boldsymbol{1}) < \varepsilon \},$$

then $V = (W \times U) \cap G$. We claim that the local orbit $\mathcal{O}(\boldsymbol{s}, V)$ is dense in U. Fix $\boldsymbol{t} \in U$ and $n_1 \geq n_0$. Let $N \in \mathbb{N}$, to be determined later. Set

$$g_k = \begin{cases} \sqrt[N]{t_k/s_k} & \text{for } k \leq n_1, \\ \prod_{j \leq n_1} \sqrt[N]{s_j/t_j} & \text{for } k = n_1 + 1, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that, for N large enough, we have that $\boldsymbol{g} \in \Gamma$, $d(\boldsymbol{g}, \boldsymbol{1}) < \varepsilon$, and $\boldsymbol{g}^i \boldsymbol{s} \in U$ for every $i \leq N$. Finally observe that $g_k^N s_k = t_k$ for $k \leq n_1$. This concludes the proof that the local orbit $\mathcal{O}(\boldsymbol{s}, V)$ is dense in U. Since this is true for every neighborhood V of $\boldsymbol{s}$ in G, $\boldsymbol{s}$ is a turbulent point for G. □

6. Conformal equivalence of operator algebraic varieties

Fix $d \in \mathbb{N}$ and let $\mathcal{V}_d$ be the space of operator algebraic varieties in $\mathbb{B}_d$. Denote by $\mathrm{Aut}(\mathbb{B}_d)$ the group of conformal automorphisms of $\mathbb{B}_d$. The *pseudo-hyperbolic distance* ρ on $\mathbb{B}_d$ is defined by

$$\rho(a, b) = \|\varphi_a(b)\|,$$

where $\|\cdot\|$ is the usual Euclidean norm and φ_a is the conformal automorphism of $\mathbb{B}_d$ which interchanges 0 and a defined in [27, Subsection 2.2.1]. Recall that ρ is a proper metric (since its closed balls coincide with Euclidean closed balls) that induces the usual topology in $\mathbb{B}_d$. Furthermore, $\mathrm{Aut}(\mathbb{B}_d)$ is a closed subgroup of the group of isometries of $(\mathbb{B}_d, \rho)$, and hence a locally compact Polish group when endowed with the compact-open topology. More information about conformal automorphisms of $\mathbb{B}_d$ can be found in [27, Chapter 2]. Consider the Borel action of $\mathrm{Aut}(\mathbb{B}_d)$ on $\mathcal{V}_d$ defined by $(\alpha, V) \mapsto \alpha[V]$. Observe that the relation $E_{\mathrm{Aut}(\mathbb{B}_d)}^{\mathcal{V}_d}$ of $\mathrm{Aut}(\mathbb{B}_d)$-conformal equivalence of operator algebraic varieties in $\mathbb{B}_d$ is the orbit equivalence relation associated with this action. Therefore, it follows from [18, Theorem 1.1] that $E_{\mathrm{Aut}(\mathbb{B}_d)}^{\mathcal{V}_d}$ is essentially countable.

The rest of this section is devoted to proving Theorem 1.2, asserting that $E_{\mathrm{Aut}(\mathbb{B}_d)}^{\mathcal{V}_d}$ has in fact maximum complexity among essentially countable equivalence relations.

As explained in the introduction, the same conclusion will then apply to the relation of (completely) isometric isomorphisms of multiplier algebras $\mathrm{Mult}(H^2_\infty|_V)$ for $V \in \mathcal{V}_d$.

Observe that the canonical inclusion of $\mathbb{B}_d$ into $\mathbb{B}_{d+1}$ induces an inclusion of $\mathcal{V}_d$ into $\mathcal{V}_{d+1}$. According to the following proposition, this inclusion is a Borel reduction from the relation of $\mathrm{Aut}(\mathbb{B}_d)$-conformal equivalence on $\mathcal{V}_d$ to the relation of $\mathrm{Aut}(\mathbb{B}_{d+1})$-conformal equivalence on $\mathcal{V}_{d+1}$. We mention that this result also follows from [8, Theorem 4.4].

Proposition 6.1. *Let $X, Y \subset \mathbb{B}_d$ be subsets. Then X and Y are conformally equivalent via an element of* $\mathrm{Aut}(\mathbb{B}_d)$ *if and only if they are conformally equivalent via an element of* $\mathrm{Aut}(\mathbb{B}_{d+1})$.

Proof. By [27, Section 2.2.8], every conformal automorphism of $\mathbb{B}_d$ extends to a conformal automorphism of $\mathbb{B}_{d+1}$. This establishes one direction.

Conversely, suppose that $F \in \mathrm{Aut}(\mathbb{B}_{d+1})$ maps X onto Y, and let $G \subset \mathrm{Aut}(\mathbb{B}_{d+1})$ denote the subgroup of all automorphisms which fix $\mathbb{B}_d$. We wish to show that X and Y are G-equivalent. Since $\mathrm{Aut}(\mathbb{B}_d)$ acts transitively on $\mathbb{B}_d$ [27, Theorem 2.2.3], and since every element of $\mathrm{Aut}(\mathbb{B}_d)$ extends to an element of G, the subgroup G acts transitively on $\mathbb{B}_d$. We may therefore assume that $0 \in X$ and $0 \in Y$. By Proposition 2.4.2 in [27] and the discussion preceding it, F maps the affine span of X onto the affine span of Y. Hence, F maps $\mathrm{span}(X) \cap \mathbb{B}_d$ onto $\mathrm{span}(Y) \cap \mathbb{B}_d$, where span denotes the linear span. Since $\mathrm{span}(X) \cap \mathbb{B}_d$ and $\mathrm{span}(Y) \cap \mathbb{B}_d$ are themselves unitarily equivalent to complex balls of dimension $e \leq d$, and since automorphisms of $\mathbb{B}_e$ extend to automorphisms of higher-dimensional balls, we conclude that there exists a map $\widetilde{F} \in G$ such that $F|_{\mathrm{span}(X)\cap\mathbb{B}_d} = \widetilde{F}|_{\mathrm{span}(X)\cap\mathbb{B}_d}$. This completes the proof. □

Therefore to establish the desired lower bound on the complexity of $E^{\mathcal{V}_d}_{\mathrm{Aut}(\mathbb{B}_d)}$ it suffices to consider the case $d = 1$. The 1-dimensional complex Euclidean ball is usually called the *disc* and denoted by $\mathbb{D}$. The elements of $\mathcal{V}_1 \setminus \{\mathbb{D}\}$ are called *Blaschke sequences.* For background material on Blaschke sequences, the reader is referred to [13, Chapter II, Section 2]. Recall from Section 2 that the orbit equivalence relation $E(F_2, 2)$ associated with the left translation action of the free group F_2 on its subsets has maximum complexity among essentially countable equivalence relation. We will now show that $E(F_2, 2)$ is Borel reducible to the relation $E^{\mathcal{V}_d}_{\mathrm{Aut}(\mathbb{B}_d)}$ of $\mathrm{Aut}(\mathbb{D})$-conformal equivalence of Blaschke sequences. To this end, we will adapt the proof of [17, Theorem 4.1].

The lower bound in [17, Theorem 4.1] is achieved by encoding the action of F_2 on $\{0,1\}^{F_2}$ by translation. The crucial point in this proof is that $\mathrm{Aut}(\mathbb{D})$ contains a copy of F_2 such that the orbit of every point in $\mathbb{D}$ is discrete. We require something stronger, namely that the orbit of every point is a Blaschke sequence.

Proposition 6.2. *There exists a discrete group* $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ *which is isomorphic to F_2 such that*

$$\sum_{g\in\Gamma} (1 - |g(z)|) < \infty$$

for every $z \in \mathbb{D}$.

Proof. Let g_1 and g_2 be two conformal automorphisms of $\mathbb{D}$ which generate a Schottky group (see Chapter II, Section 1 in [7]), and let Γ be the group generated by

g_1 and g_2. Then Γ is isomorphic to F_2 by [7, Chapter II, Proposition 1.6]. By the same proposition, the closure of the Dirichlet domain $\mathcal{D}_0(\Gamma)$ of Γ contains non-trivial arcs in $\partial\mathbb{D}$ (see [7, Chapter I, Section 2.3] for the definition of the Dirichlet domain). In particular, the Lebesgue measure of $\overline{\mathcal{D}_0(\Gamma)} \cap \partial\mathbb{D}$ is strictly positive. In this situation, [30, Theorem XI.4] applies to show that

$$\sum_{g\in\Gamma}(1-|g(0)|) < \infty.$$

Finally, the argument preceding Theorem XI.3 in [30] shows that this sum is finite if 0 is replaced with an arbitrary point $z \in \mathbb{D}$. □

It seems worthwhile to give a concrete example of two conformal automorphisms of $\mathbb{D}$ which generate a group Γ as in the statement of the proposition. Let $\mathbb{H}$ denote the upper half-plane in $\mathbb{C}$. Recall that $\mathbb{D}$ and $\mathbb{H}$ are conformally equivalent via the Cayley map

$$\begin{array}{ccc} \mathbb{H} & \to & \mathbb{D} \\ z & \mapsto & \dfrac{z-i}{z+i}. \end{array}$$

This map induces an isomorphism of topological groups between $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{H})$. Moreover, $\mathrm{Aut}(\mathbb{H})$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ via the map that assigns to the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R})$ the corresponding *Möbius transformation*

$$z \mapsto \frac{az+b}{cz+d}.$$

Let $\Phi : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{Aut}(\mathbb{D})$ denote the isomorphism obtained by composing the two isomorphisms above. The group Λ considered in the proof of [17, Theorem 4.1] is generated by the images of

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

under Φ. The group Λ is isomorphic to F_2, but the orbit of 0 under Λ is not a Blaschke sequence. This follows from the following facts:

- Λ has finite index in $\mathrm{PSL}(2,\mathbb{Z})$, and
- the orbit of 0 under $\Phi\left[\mathrm{PSL}_2(\mathbb{Z})\right]$ is not a Blaschke sequence, as its conical limit set on $\partial\mathbb{D}$ has positive Lebesgue measure; see [7, Chapter II, Section 3.1].

Moreover, Λ is not a Schottky group, but just a generalized Schottky group in the sense of [7, Chapter II, Section 1.1]. However, if we let $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ denote the group generated by the images of

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix},$$

then it is not hard to see that Γ is indeed a Schottky group and thus satisfies the conclusion of the proposition.

In the proof of the next theorem, we require the following elementary observation.

Lemma 6.3. *Let (X,d) be a metric space and let $x^{(0)}, x^{(1)}, x^{(2)}$ and $y^{(0)}, y^{(1)}, y^{(2)}$, $y^{(3)}$ by points in X such that*

$$d(x^{(i)}, x^{(j)}) = d(y^{(i)}, y^{(j)})$$

for $0 \le i, j \le 2$ *and such that the distances* $d(y^{(i)}, y^{(j)})$ *are all distinct for* $0 \le i < j \le 3$. *If* $\theta : X \to X$ *is an isometry such that*

$$\theta(\{x^{(0)}, x^{(1)}, x^{(2)}\}) \subset \{y^{(0)}, y^{(1)}, y^{(2)}, y^{(3)}\},$$

then $\theta(x^{(i)}) = y^{(i)}$ *for* $0 \le i \le 2$.

Proof. The assumptions on the distances and the fact that θ is an isometry imply that

$$\begin{aligned}\theta(\{x^{(0)}, x^{(1)}\}) &= \{y^{(0)}, y^{(1)}\},\\ \theta(\{x^{(0)}, x^{(2)}\}) &= \{y^{(0)}, y^{(2)}\}, \text{ and}\\ \theta(\{x^{(1)}, x^{(2)}\}) &= \{y^{(1)}, y^{(2)}\}.\end{aligned}$$

This is only possible if $\theta(x^{(i)}) = y^{(i)}$ for $0 \le i \le 2$. □

We are now ready to prove the main result of this section.

Theorem 6.4. *The relation* $E(F_2, 2)$ *is Borel reducible to the relation of* $\mathrm{Aut}(\mathbb{D})$*-conformal equivalence of Blaschke sequences.*

Proof. The proof is an adaptation of the proof of the lower bound in [17, Theorem 4.1]. The details are as follows.

Let Γ be a group as in Proposition 6.2. We will identify F_2 with Γ. Moreover, let ρ be the pseudohyperbolic metric on $\mathbb{D}$, and for $z \in \mathbb{D}$ and $\varepsilon > 0$, let

$$D_\varepsilon(z) = \{y \in \mathbb{D} : \rho(y, z) < \varepsilon\}.$$

We will explicitly construct four Blaschke sequences

$$B_i = \{x_g^{(i)} : g \in F_2\}$$

for $0 \le i \le 3$ and find $\varepsilon > 0$ with the following properties:

(1) $gx_h^{(i)} = x_{gh}^{(i)}$ for $g, h \in F_2$ and $0 \le i \le 3$;
(2) $x_g^{(i)} \in D_{\varepsilon/5}(x_g^{(0)})$ for $g \in F_2$ and $0 \le i \le 3$;
(3) $D_{\varepsilon/2}(x_g^{(0)}) \cap (B_0 \cup B_1 \cup B_2 \cup B_3) = \{x_g^{(i)} : 0 \le i \le 3\}$;
(4) the distances $\rho(x_g^{(i)}, x_g^{(j)})$ do not depend on $g \in F_2$ and are all distinct and positive for $0 \le i < j \le 3$.

The construction proceeds as follows. Let $x_1^{(0)} \in \mathbb{D}$ be arbitrary and set $x_g^{(0)} = g(x_1^{(0)})$ for $g \in F_2$. Let $B_0 = \{x_g^{(0)} : g \in F_2\}$. Then B_0 is a Blaschke sequence. In particular, there exists $\varepsilon > 0$ such that

$$D_\varepsilon(x_1^{(0)}) \cap B_0 = \{x_1^{(0)}\}.$$

Choose distinct points $x_1^{(i)} \in D_{\varepsilon/5}(x_1^{(0)}) \setminus \{x_1^{(0)}\}$ for $i \in \{1, 2, 3\}$ such that the pseudohyperbolic distances $\rho(x_1^{(i)}, x_1^{(j)})$ for $i < j$ are all different from each other, and define $x_g^{(i)} = g(x_1^{(i)})$ for $i \in \{1, 2, 3\}$ and $g \in F_2$. Moreover, set $B_i = \{x_g^{(i)} : g \in F_2\}$. Using the fact that every $g \in F_2$ is an isometry with respect to ρ, properties (1)–(4) are now easy to verify.

Given $A \subset F_2$, let

$$V_A = B_0 \cup B_1 \cup B_2 \cup \{x_g^{(3)} : g \in A\}.$$

We will show that $A = gB$ for some $g \in F_2$ if and only if V_A and V_B are $\mathrm{Aut}(\mathbb{D})$-conformally equivalent. Clearly, if $g \in F_2$ such that $gA = B$, then $g[V_A] = V_B$;

hence V_A and V_B are $\mathrm{Aut}(\mathbb{D})$-conformally equivalent. Conversely, assume that there exists $\theta \in \mathrm{Aut}(\mathbb{D})$ with $\theta[V_A] = V_B$. We will show that there exists $g \in F_2$ such that $\theta = g$. Since $x_1^{(0)} \in V_A$, there exists $g \in F_2$ and $i \in \{0, 1, 2, 3\}$ such that $\theta(x_1^{(0)}) = x_g^{(i)}$. Observe that for $k \in \{1, 2\}$, we have

$$\rho(\theta(x_1^{(k)}), x_g^{(i)}) = \rho(\theta(x_1^{(k)}), \theta(x_1^{(0)})) = \rho(x_1^{(k)}, x_1^{(0)}) < \varepsilon/5$$

by condition (2). By the same condition, $\rho(x_g^{(i)}, x_g^{(0)}) < \varepsilon/5$, hence

$$\theta(x_1^{(k)}) \in D_{\varepsilon/2}(x_g^{(0)}).$$

Therefore, condition (3) implies that

$$\theta(\{x_1^{(0)}, x_1^{(1)}, x_1^{(2)}\}) \subset \{x_g^{(i)} : 0 \leq i \leq 3\}.$$

In light of condition (4), an application of Lemma 6.3 shows that $\theta(x_1^{(i)}) = x_g^{(i)}$ for $0 \leq i \leq 2$. This means that θ and g are two Möbius transformations which agree on three points. Consequently, $\theta = g$; see for example [23, Theorem 10.10]. We finish the proof by showing that $gA = B$. Note that if $h \in A$, then $x_h^{(3)} \in V_A$. Therefore, $x_{gh}^{(3)} = g(x_h^{(3)}) = \theta(x_h^{(3)}) \in V_B$, so $gh \in B$. This shows that $gA \subset B$. Similarly, $g^{-1}B \subset A$, so $gA = B$, as desired. $\square$

Appendix

Recall that if X is a Polish space, then the space $F(X)$ of nonempty closed subsets of X is a standard Borel space when endowed with the Effros Borel structure [19, Section 12.C]. This is the Borel structure generated by the sets

$$\{K \in F(X) : K \cap U \neq \varnothing\}$$

where U ranges over the open subsets of X. The Kuratowski–Ryll-Nardzewski theorem asserts that there exists a sequence (σ_n) of Borel maps from $F(X)$ to X such that $(\sigma_n(A))$ enumerates a dense subset of A for every nonempty closed subset A of X [19, Theorem 12.13].

Fix $d \in \mathbb{N} \cup \{\infty\}$, and let $\mathcal{V}_d \subset F(\mathbb{B}_d)$ be the set of operator algebraic varieties in $\mathbb{B}_d$. For $d \leq d'$ the canonical inclusion $\mathbb{B}_d \subset \mathbb{B}_{d'}$ induces a Borel injection from $\mathcal{V}_d$ into $\mathcal{V}_{d'}$.

Proposition. *The set $\mathcal{V}_d$ of operator algebraic varieties in $\mathbb{B}_d$ is a Borel subset of $F(\mathbb{B}_d)$.*

Proof. Let H_d^2 be the Drury-Arveson space. Observe that $\mathcal{V}_d$ is the image of $F(H_d^2)$ under the Borel map that assigns to a closed subset S of H_d^2 the operator algebraic variety V_S of common zeros of elements of S. Therefore $\mathcal{V}_d$ is analytic. By [19, Theorem 14.7] it remains to show that $\mathcal{V}_d$ is coanalytic. Fix a dense subset $\mathcal{D}$ in the unit ball of H_d^2 and a sequence (σ_n) of Borel functions from $F(\mathbb{B}_d)$ to $\mathbb{B}_d$ such that $(\sigma_n(S))$ enumerates a dense subset of S for every nonempty closed subset S of $\mathbb{B}_d$. We claim that a closed subset V of $\mathbb{B}_d$ is an operator algebraic variety if and only if for every $x \in \mathbb{B}_d$ either $x \in V$ or there exists a strictly positive rational number ε such that for every $n \in \mathbb{N}$ there exists $f \in \mathcal{D}$ such that $|f(x)| \geq \varepsilon$ and $|f(\sigma_i(V))| \leq 2^{-n}$ for every $i \leq n$.

One implication is obvious. For the other implication it is enough to observe that if $x \in \mathbb{B}_d$ and (f_n) is a sequence in the unit ball of H_d^2, then (f_n) has an accumulation point f in the unit ball of H_d^2 with respect to the weak topology. If furthermore $|f_n(x)| \geq \varepsilon$ and $|f(\sigma_i(V))| \leq 2^{-n}$ for every $i \leq n$, then $f(x) \neq 0$ and f vanishes on V. This shows that V is the set of common zeroes of all the elements of H^2_d that vanish on V, and hence it is an operator algebraic variety. $\square$

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