ASYMPTOTIC SYZYGIES OF STANLEY-REISNER RINGS OF ITERATED SUBDIVISIONS

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ABSTRACT. Inspired by recent results of Ein, Lazarsfeld, Erman and Zhou on the non-vanishing of Betti numbers of high Veronese subrings, we describe the behavior of the Betti numbers of Stanley-Reisner rings associated with iterated barycentric or edgewise subdivisions of a given simplicial complex. Our results show that for a simplicial complex Δ of dimension $d-1$ and for $1 \leq j \leq d-1$ the number of 0's in the jth linear strand of the minimal free resolution of the rth barycentric or edgewise subdivision is bounded above only in terms of d and j (and independently of r).

1. INTRODUCTION

In recent work, Ein, Lazarsfeld and Erman [17, 18] and Zhou [28] study the asymptotic behavior of syzygies of algebraic varieties under high Veronese embeddings. In particular, they treat the case of the syzygies of rth Veronese embeddings $v_r(\mathbb{P}^n)$ of projective space $\mathbb{P}^n$. Roughly speaking, they prove that for large r the syzygies of $v_r(\mathbb{P}^n)$ are non-zero for most of the homological positions and internal degrees that are allowed by the restrictions imposed by the projective dimension and by the Castelnuovo-Mumford regularity. Similar results, but with less precise bounds, are obtained for arithmetically Cohen-Macaulay varieties and are conjectured in general.

The goal of this paper is to prove that a similar behavior occurs also for the syzygies of Stanley-Reisner rings of iterated barycentric subdivisions and edgewise subdivisions. These two combinatorial operations on simplicial complexes have some formal similarity (but also some important dissimilarity) to the formation of Veronese subalgebras. Moreover, as we explain later on, the edgewise subdivision of a simplicial complex is closely related, via Gröbner deformations, with the formation of Veronese subalgebras of the associated Stanley-Reisner ring.

Let $\mathbb{K}$ be a field and let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring. Let I be a homogeneous ideal of S such that $I \subset (x_1, \dots, x_n)^2$. Let $A = S/I = \bigoplus_{i \geq 0} A_i$ be a standard graded $\mathbb{K}$-algebra. Denote by $\beta_{i,j}(A)$ the graded Betti number of A, i.e.,

$$\beta_{i,j}(A) = \dim_{\mathbb{K}} \operatorname{Tor}_i^S(A, \mathbb{K})_j = \dim_{\mathbb{K}} H_i(m_A, A)_j.$$

Here $H_i(m_A, A)$ denotes the ith Koszul homology of the maximal homogeneous ideal $m_A = \bigoplus_{i \geq 1} A_i$ of A, and the index j on the right always denotes the selection of the jth homogeneous component.

The rth Veronese algebra of A is by definition

$$A^{(r)} = \bigoplus_{i \geq 0} A_{ir}.$$

One of the main results of Ein and Lazarsfeld [17, Theorem B] asserts that for every integer j in the interval $[1, n-1]$ the graded Betti numbers $\beta_{i,i+j}(S^{(r)})$ of the rth Veronese subalgebra of the polynomial ring S are asymptotically (i.e., for large r) non-zero for every integer i in an interval $[a_j, b_j]$ with specified endpoints a_j, b_j that depend on j. Note that the Castelnuovo-Mumford regularity of $S^{(r)}$ is always $\leq n-1$ with equality for large r. Hence it is clear that $\beta_{i,i+j}(S^{(r)}) = 0$ for j outside the interval $[1, n-1]$ with the exception of $\beta_{0,0}(S^{(r)}) = 1$. Comparing the size of the intervals $[a_j, b_j]$ with the projective dimension of $S^{(r)}$ (i.e., the length of the minimal free resolution) one deduces that for large r the Betti number $\beta_{i,i+j}(S^{(r)})$ is non-zero for most of the values of i, j, which are allowed by the restrictions imposed by the value of the projective dimension and the Castelnuovo-Mumford regularity.

In [18] a similar result, but with a less precise description of the intervals, is proved for the Betti numbers $\beta_{i,i+j}(A^{(r)})$ of the Veronese subalgebras of an arbitrary Cohen-Macaulay algebra A. In particular, it follows that for every $j = 1, \ldots, \dim A - 1$ one has

$$(1.1) \qquad \lim_{r \to \infty} \frac{\#\{i \ : \ \beta_{i,i+j}(A^{(r)}) \neq 0\}}{\operatorname{pdim}(A^{(r)})} = 1.$$

Furthermore, the authors conjecture that the same behavior holds for an arbitrary standard graded $\mathbb{K}$-algebra. Note also that, at least when the variety associated to A is smooth, for $j = \dim(A)$ the limit in (1.1) is 0 since the number of non-zero syzygies in that strand is bounded independently of d; see [17, Cor. 5.2].

Let Δ be a simplicial complex and denote by $\mathbb{K}[\Delta]$ the corresponding Stanley-Reisner ring. We consider two combinatorial operations on simplicial complexes: the iterated barycentric subdivision and edgewise subdivisions. We will denote by $\operatorname{sd}^r(\Delta)$ the rth iterated barycentric subdivision of Δ and by $\Delta^{\langle r \rangle}$ the rth edgewise subdivision of Δ, whose definition is recalled in Section 1.1.

We study the asymptotic behavior of $\beta_{i,i+j}(\mathbb{K}[\operatorname{sd}^r(\Delta)])$ and $\beta_{i,i+j}(\mathbb{K}[\Delta^{\langle r \rangle}])$. The main results we prove are Theorem 3.14, Proposition 3.19, Theorem 4.10 and Proposition 4.14, whose central content is the following:

Theorem 1.1. *Let Δ be an arbitrary simplicial complex of dimension $d-1 > 0$. Let $\Delta(r)$ be either the iterated barycentric subdivision $\operatorname{sd}^r(\Delta)$ or the edgewise subdivision $\Delta^{\langle r \rangle}$ of Δ. Then for large r the Castelnuovo-Mumford regularity of $\mathbb{K}[\Delta(r)]$ is given by*

$$\operatorname{reg}(\mathbb{K}[\Delta(r)]) = \begin{cases} d-1, & \text{if } \widetilde{H}_{d-1}(\Delta; \mathbb{K}) = 0, \\ d, & \text{if } \widetilde{H}_{d-1}(\Delta; \mathbb{K}) \neq 0. \end{cases}$$

Furthermore:

(1) *For every $j = 1, \ldots, d-1$ one has that $\#\{i \ : \ \beta_{i,i+j}(\mathbb{K}[\Delta(r)]) = 0\}$ is bounded above in terms of d, j (and independently of r). In particular,*

$$\lim_{r \to \infty} \frac{\#\{i \ : \ \beta_{i,i+j}(\mathbb{K}[\Delta(r)]) \neq 0\}}{\operatorname{pdim}(\mathbb{K}[\Delta(r)])} = 1.$$

(2) *If* $\widetilde{H}_{d-1}(\Delta; \mathbb{K}) \neq 0$, *then*

$$\lim_{r \to \infty} \frac{\#\{i \ : \ \beta_{i,i+d}(\mathbb{K}[\Delta(r)]) \neq 0\}}{\mathrm{pdim}(\mathbb{K}[\Delta(r)])}$$

is a rational number in the interval $[0,1)$ *that can be described in terms of the minimal* $(d-1)$-*cycles of* Δ.

The limit in (2) does not depend on whether one takes iterated barycentric subdivision or edgewise subdivision. Moreover, for any rational number in the interval $[0,1)$ and any d, we construct a $(d-1)$-dimensional Cohen-Macaulay complex that realizes the specified limit. This limit measures the proportion of the number of facets contained in a minimal $(d-1)$-cycle of Δ compared to the total number of facets of Δ. In particular, the limit equals 0 if Δ is a manifold without boundary.

Brun and Römer prove in [9, Corollary 6.5] that for every simplicial complex Δ there is a term order $\prec$ such that the Veronese subalgebra $\mathbb{K}[\Delta]^{(r)}$ of $\mathbb{K}[\Delta]$ has a Gröbner deformation to $\mathbb{K}[\Delta^{\langle r \rangle}]$. Betti numbers can only increase under a Gröbner deformation, and hence one has

$$(1.2) \qquad\qquad \beta_{i,i+j}(\mathbb{K}[\Delta]^{(r)}) \quad \leq \quad \beta_{i,i+j}(\mathbb{K}[\Delta^{\langle r \rangle}])$$

for every r, i, j. In particular, the assertion about the limit in Theorem 1.1(1) for $\Delta^{\langle r \rangle}$ when Δ is Cohen-Macaulay can be deduced from the results of Ein, Erman and Lazarsfeld in [18].

Note, however, the conclusion from Theorem 1.1(1) that the cardinality of $\{i : \beta_{i,i+j}(\mathbb{K}[\Delta(r)]) = 0\}$ is bounded only in terms of j, d is simply not true for iterated Veronese. The latter assertion can be deduced from the results in [16, Theorem 1]. Another important difference between our results and the ones from [17, 18] lies in the limit of the relative size of the dth strand. In our context the limit behavior yields a number in $[0, 1)$ and any number is possible. In the Veronese case, at least for the coordinate rings of smooth varieties in characteristic 0, the length on the dth strand is constant. Hence the corresponding limit is 0; see [17, Eq. (1.3), p. 607].

One may ask if replacing the Stanley-Reisner ideal of the edgewise subdivision $\Delta^{\langle r \rangle}$ on the right-hand side of (1.2) by a different initial ideal of the ideal from the left-hand side can lead to equality in (1.2). Already for $\Delta = \Delta_{d-1}$ and $\mathbb{K}[\Delta] = S$, the polynomial ring on $n = d$ variables, this is not, in general, the case. The ideal defining $S^{(r)}$ is well known to be generated by quadrics for all r. Hence any initial ideal achieving equality in (1.2) must be generated by quadrics as well. For monomial ideals generated by quadrics Theorem 4.1 from [13] provides an upper bound on the regularity which depends on the number of steps the resolution is linear. The number of steps for which the resolution of $S^{(r)}$ is linear has been studied intensively under the name N_p property; see for example [8]. In particular it is known that, in arbitrary characteristic, the resolution of $S^{(r)}$ is linear for r steps. Evaluating the inequality from [13, Theorem 4.1] at the known bounds shows no initial ideal whose resolution is linear for the same number of steps that can achieve regularity $d - 1$ at least for certain combinations of values of d and r as, for example, those given in [1, 7.5].

Our methods are mostly geometric and combinatorial and are based on Hochster's formula expressing the Betti numbers in terms of homologies of induced subcomplexes. The case when $\Delta = \Delta_{d-1}$ is the full $(d-1)$-simplex turns out to be crucial. A key observation for this analysis is (see Proposition 4.2) that links of

faces of $\Delta_{d-1}^{\langle r \rangle}$, whose interior lies in the interior of Δ_{d-1}, are barycentric subdivisions of boundaries of full simplices. This essentially reduces the analysis of the Betti numbers of $\mathbb{K}[\Delta^{\langle r \rangle}]$ and of $\mathbb{K}[\mathrm{sd}^r(\Delta)]$ to the analysis of the Betti numbers of $\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]$. One of the referees pointed out that this argument is similar, in spirit, to the one used by Ein, Erman and Lazarsfeld in [18]. This analysis is performed in Section 3 and appears to be of independent interest.

We remark that many of the arguments that we present go through for arbitrary subdivision operations that satisfy mild assumptions. The actual formulation of the assumption is technical and does not give too much insight, but the requirement is twofold. First, starting with a subdivision of the $(d-1)$-simplex (see [26] for the definition) this subdivision has to induce a well-defined subdivision operation on any pure $(d-1)$-dimensional simplicial complex. In particular, the subdivision can be applied iteratively. The second technical requirement roughly says that when iteratively subdividing a simplex "sufficiently many" new vertices are created in the interior of the simplex. The conclusion will be that for a simplicial complex Δ of dimension $d-1$ and a suitable subdivision operation Sub, if for some $j < d$, $1 \leq k$ and $r \geq 0$ we have $\beta_{k,k+j}(\mathbb{K}[\mathrm{Sub}^r(\Delta)]) \neq 0$, then

$$\lim_{r \to \infty} \frac{\#\{i \ : \ \beta_{i,i+j}(\mathbb{K}[\mathrm{Sub}^r(\Delta)]) \neq 0\}}{\mathrm{pdim}(\mathbb{K}[\mathrm{Sub}^r(\Delta)])} = 1.$$

Examples of suitable subdivision operations can be found in various articles; see for example [12, p. 414] and [27, Section 4]. Note that there are suitable subdivision operations that fail to produce non-zero $\beta_{i,i+j}(\mathrm{Sub}(\mathbb{K}[\Delta]))$ for every $1 \leq j \leq d-1$ and for some i. Consider for example the subdivision of $(d-1)$-dimensional simplicial complexes where we subdivide by coning the boundary of each $(d-1)$-simplex over a point in the interior of the simplex. Already for the 3-simplex, none of the induced subcomplexes obtained by iterating this construction will have non-trivial homology in dimension 1.

1.1. **Notation and background.** Let Δ be a simplicial complex on ground set Ω. We call an element $F \in \Delta$ a *face* of Δ with *dimension* $\dim F := \#F - 1$. The *dimension* $\dim \Delta$ of Δ is then the maximal dimension of one of its faces. If $\dim \Delta = d-1$, then we write $f^\Delta = (f_{-1}^\Delta, \ldots, f_{d-1}^\Delta)$ for the *f-vector* of Δ, where f_i^Δ counts the number of i-dimensional faces in Δ.

Sometimes we are interested in subsets of Δ that are not necessarily simplicial complexes themselves. Let $\Gamma \subseteq \Delta$ be such a subset. Then we consider Γ as a partially ordered set ordered by inclusion and write $\Delta(\Gamma)$ for its *order complex* that is the set of all subsets of Γ that are linearly ordered. If $\Gamma = \Delta \setminus \{\emptyset\}$, then $\mathrm{sd}(\Delta) := \Delta(\Gamma)$ is the *barycentric subdivision* of Δ. The other subdivision operation that is important for this paper is the edgewise subdivision. Assume Δ is $(d-1)$-dimensional with vertex set $[n] := \{1, \ldots, n\}$ and let $r \geq 1$ be a positive integer. Set $\Omega_{r,n} := \{(i_1, \ldots, i_n) \in \mathbb{N}^n \ : \ i_1 + \cdots + i_n = r\}$. Denote by $\mathfrak{e}_i$ the ith unit vector of $\mathbb{R}^n$. By the obvious identification, we can consider Δ as a simplicial complex over the vertex set $\Omega_{1,n} = \{\mathfrak{e}_1, \ldots, \mathfrak{e}_n\}$. For $i \in [n]$ set $\mathfrak{u}_i := \mathfrak{e}_i + \cdots + \mathfrak{e}_n$ and for $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ let $\mathfrak{i}(a) := \sum_{l=1}^n a_l \cdot \mathfrak{u}_l$. The rth *edgewise subdivision* of Δ is the simplicial complex $\Delta^{\langle r \rangle}$ on ground set $\Omega_{r,n}$ such that $F \subseteq \Omega_{r,n}$ is a simplex in $\Delta^{\langle r \rangle}$ if and only if

   (i) $\bigcup_{a \in F} \mathrm{supp}(a) \in \Delta$, and
   (ii) for all $a, \tilde{a} \in F$ either $\mathfrak{i}(a - \tilde{a}) \in \{0,1\}^n$ or $\mathfrak{i}(\tilde{a} - a) \in \{0,1\}^n$.

If we denote by $|\Delta|$ the geometric realization of Δ, then we can choose realizations such that $|\Delta| = |\mathrm{sd}(\Delta)| = |\Delta^{\langle r \rangle}|$. If F is a face of Δ, we sometimes write $|F|$ to denote the geometric realization of the subcomplex 2^F of Δ. By $\partial|\Delta|$ we denote the boundary of $|\Delta|$.

For a simplicial complex Δ on ground set Ω its *Stanley-Reisner ideal* I_Δ is the ideal in $S = \mathbb{K}[x_\omega \; : \; \omega \in \Omega]$ generated by monomials $\prod_{\omega \in N} x_\omega$ for $N \notin \Delta$. $\mathbb{K}[\Delta] := S/I_\Delta$ is called the *Stanley-Reisner ring* of Δ. For background on the algebraic invariants of $\mathbb{K}[\Delta]$ studied in this paper we refer to [10, 19].

2. Basic algebraic invariants under Veronese, barycentric and edgewise subdivision

In this section we recall how some basic invariants of standard graded $\mathbb{K}$-algebras and Stanley-Reisner rings behave under the Veronese operation on the algebra and barycentric/edgewise subdivision on the simplicial complex.

In the following $A = \bigoplus_{i=0}^{\infty} A_i$ will denote a standard graded $\mathbb{K}$-algebra of Krull dimension $d > 0$ and $H^i_{m_A}(A)$ will denote the ith local cohomology module with respect to the maximal homogeneous ideal $m_A = \bigoplus_{i=1}^{\infty} A_i$ of A. Furthermore, Δ will denote a simplicial complex of dimension $d - 1 \geq 0$. We denote by $t_1(A)$ the largest degree of a minimal generator of the defining ideal of A as a quotient of the polynomial ring in $\dim A_1$ variables. Hence $t_1(\mathbb{K}[\Delta])$ is the largest cardinality of a minimal non-face of Δ and $t_1(\mathbb{K}[\Delta]) = 2$ if and only if Δ is a flag complex. Set $a_d(A) = \max\{j \; : \; H^d_{m_A}(A)_j \neq 0\}$ and

$$w(\Delta, \mathbb{K}) = \begin{cases} d - 1, & \text{if } \widetilde{H}_{d-1}(\Delta; \mathbb{K}) = 0, \\ d, & \text{if } \widetilde{H}_{d-1}(\Delta; \mathbb{K}) \neq 0. \end{cases}$$

We have:

TABLE 1

	(1)	(2)	(3)	(4)
	$A^{(r)}$	$\mathbb{K}[\Delta]^{(r)}$	$\mathbb{K}[\Delta^{\langle r \rangle}]$	$\mathbb{K}[\mathrm{sd}^r(\Delta)]$
dim	$\dim A$	$\dim \mathbb{K}[\Delta]$	$\dim \mathbb{K}[\Delta]$	$\dim \mathbb{K}[\Delta]$
depth	$\geq \mathrm{depth}\, A$	$\mathrm{depth}\, \mathbb{K}[\Delta]$	$\mathrm{depth}\, \mathbb{K}[\Delta]$	$\mathrm{depth}\, \mathbb{K}[\Delta]$
t_1	2 *	2 *	Lemma 4.1	2
reg	$d - 1,$ if $a_d(A) < 0$ * $\quad$ $d,$ if $a_d(A) \geq 0$	$w(\Delta, \mathbb{K})$ *	$w(\Delta, \mathbb{K})$ *	$w(\Delta, \mathbb{K})$

An asterik * in Table 1 signals that the formula holds for large values of r. For these formulas estimates holding for all r may also exist; for example, $t_1(A^{(r)}) \leq \max\{2, \lceil t_1(A)/r \rceil\}$ holds for every r.

The data in column (1) are obtained by applying the formula that relates the local cohomology before and after applying the Veronese functor (see [21, Thm. 3.1.1]) and the assertion that $H^d_{m_A}(A)_j \neq 0$ for every $j \leq a_d(A)$ (see [11, Prop. 2.2]). It might happen that $\mathrm{depth}\, A^{(r)} > \mathrm{depth}\, A$ for every $r > 1$. Take, for example, $A = \mathbb{K}[x, y]/(x^2, xy)$ or, if one wants a domain, $A = \mathbb{K}[x^4, x^3y, xy^3, y^4]$.

The data in column (2) are obtained using [21, Thm. 3.1.1] and Hochster's formula for local cohomology modules of Stanley-Reisner rings [10, Thm. 5.3.8].

Next we turn to the data in column (3). The row on dimension is clear by construction. The invariance of depth under edgewise subdivision follows from Munkres' result [25, Thm. 3.1] since $|\Delta| = |\Delta^{\langle r \rangle}|$. In Lemma 4.1 we prove that if Δ is not a simplex, one has either $t_1(\mathbb{K}[\Delta^{\langle r \rangle}]) = t_1(\mathbb{K}[\Delta]) - 1$ or $t_1(\mathbb{K}[\Delta^{\langle r \rangle}]) = t_1(\mathbb{K}[\Delta])$ depending on certain features of Δ that we describe. If Δ is a simplex, then one has $t_1(\mathbb{K}[\Delta^{\langle r \rangle}]) = 2$ for every $r > 1$.

The formula for the regularity will be proved in Corollary 4.5.

Finally, for the data in column (4), the dimension and the value of t_1 are clear by construction. The formula for regularity follows from [22, Prop. 2.6]. Again Munkres' result [25, Thm. 3.1] and $|\Delta| = |\mathrm{sd}(\Delta)|$ show the invariance of depth. This also follows through a more direct argument from [22, Cor. 2.5].

3. Barycentric subdivision

In this section we provide the analysis of the Betti numbers $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta)])$ of the Stanley-Reisner ring of the barycentric subdivision of a simplicial complex Δ. Besides its combinatorial appeal it is crucial for the proof of our main result Theorem 4.9.

We have recorded already in Table 1 how basic algebraic invariants behave under barycentric subdivision. As the depth is preserved, it follows from the Auslander-Buchsbaum formula that the projective dimension of the barycentric subdivision is given by

$$\mathrm{pdim}(\mathbb{K}[\mathrm{sd}(\Delta)]) = \mathrm{pdim}(\mathbb{K}[\Delta]) + \sum_{i \geq 1} f_i^{\Delta}.$$

Recall Hochster's formula for the graded Betti numbers of Stanley-Reisner rings [10, Thm. 5.5.1]:

$$(3.1) \qquad \beta_{i,i+j}(\mathbb{K}[\Delta]) = \sum_{\substack{W \subseteq [n] \\ \#W = i+j}} \dim_{\mathbb{K}} \widetilde{H}_{j-1}(\Delta_W; \mathbb{K}),$$

where $\Delta_W = \{F \in \Delta \ : \ F \subseteq W\}$. In particular, if Δ is a simplicial complex on vertex set Ω,

$$\beta_{i,i+j}(\mathbb{K}[\Delta]) \neq 0 \qquad \Leftrightarrow \qquad \exists W \subseteq \Omega, \ \#W = i+j \text{ such that } \widetilde{H}_{j-1}(\Delta_W; \mathbb{K}) \neq 0.$$

We are interested in the range of the different strands in the minimal resolution of $\mathbb{K}[\mathrm{sd}(\Delta)]$. More precisely, for all $1 \leq j \leq \mathrm{reg}(\mathbb{K}[\mathrm{sd}(\Delta)])$, we would like to identify the set of the i's such that $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta)]) \neq 0$ in terms of invariants of Δ.

3.1. **Betti numbers for barycentric subdivisions of simplices.** We start our analysis with the study of the Betti numbers of the barycentric subdivision of the $(d-1)$-dimensional simplex Δ_{d-1}. Note that in this case one has

$$\mathrm{pdim}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) = 2^d - d - 1.$$

We start with a definition.

Definition 3.1. Given integers d and j such that $d \geq 1$ and $1 \leq j \leq d - 1$ we set

$$m_j(d) = \begin{cases} j, & \text{if } j \leq d/2, \\ 2^{a+2}(c+d-j) - 2d + j, & \text{if } j \geq d/2 \text{ where } (2j-d) = a(d-j) + c \\ & \text{with } a, c \in \mathbb{N} \text{ and } 0 \leq c < d - j. \end{cases}$$

When d is clear from the context, we will suppress it from the notation and simply use m_j for $m_j(d)$.

These numbers play an important role in the following results, and, as we will see in the proofs, they arise by considerations related to the search of $(j-1)$-spheres as subcomplexes of $\mathrm{sd}(\Delta_{d-1})$ induced by as few vertices as possible.

Theorem 3.2. *Let $d \geq 1$. Let Δ_{d-1} be the $(d-1)$-simplex and let $\mathrm{sd}(\Delta_{d-1})$ be its barycentric subdivision. Then:*

(i) *If $1 \leq j \leq \frac{d}{2}$, then*

$$\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \begin{cases} = 0 \text{ for } 0 \leq i \leq j-1, \\ \neq 0 \text{ for } j \leq i \leq 2^d - d - 1 - m_{d-j-1}, \\ = 0 \text{ for } 2^d - 2d + j < i \leq 2^d - d - 1. \end{cases}$$

(ii) *If $\frac{d}{2} < j \leq d-2$, then*

$$\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \begin{cases} = 0 \text{ for } 0 \leq i \leq j-1, \\ \neq 0 \text{ for } m_j \leq i \leq 2^d - 2d + j, \\ = 0 \text{ for } 2^d - 2d + j < i \leq 2^d - d - 1. \end{cases}$$

(iii) $\beta_{i,i+d-1}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0$ *if and only if $i = 2^d - d - 1$.*

The theorem identifies whether $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})])$ is zero or not, except for the cases

- $1 \leq j \leq \frac{d}{2}$ and $2^d - d - m_{d-j-1} \leq i \leq 2^d - 2d + j$, and
- $\frac{d}{2} < j \leq d-2$ and $j \leq i \leq m_j - 1$.

For these values of j and i we do not know whether the corresponding Betti numbers vanish.

We now formulate some crucial lemmas and propositions, which will lead to a proof of Theorem 3.2. First, we introduce some notation that will be frequently used throughout this section. For $1 \leq j \leq d-1$ we set

$$l_j(d-1) = \min\{i \ : \ \beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0\}$$

and

$$u_j(d-1) = \max\{i \ : \ \beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0\}.$$

We determine bounds for $l_j(d-1)$ and $u_j(d-1)$ by constructing $(j-1)$-spheres as subcomplexes of $\mathrm{sd}(\Delta_{d-1})$ induced by as few vertices as possible.

In general we can construct induced $(j-1)$-spheres of $\mathrm{sd}(\Delta_{d-1})$ in the following way. Let $(i_1, \dots, i_r) \in \mathbb{N}^r$ such that

$$(3.2) \qquad i_1 + \cdots + i_r + (r-1) \quad = \quad j-1,$$

$$(3.3) \qquad i_1 + \cdots + i_r + 2r \quad \leq \quad d.$$

Let $\mathbb{S}^n$ denote the n-dimensional sphere. Note that (3.2) implies

$$\mathbb{S}^{j-1} \cong \mathbb{S}^{i_1} * \cdots * \mathbb{S}^{i_r},$$

where $*$ denotes the join operator. We set $W_{i_1} := \{A \ : \ \emptyset \neq A \subsetneq [i_1 + 2]\}$ and for $2 \leq \ell \leq r$ we let

$$(3.4) \quad W_{i_\ell} := \{A \cup [i_1 + \cdots + i_{\ell-1} + 2(\ell-1)] \ :$$
$$\emptyset \neq A \subsetneq [i_1 + \cdots + i_\ell + 2\ell] \setminus [i_1 + \cdots + i_{\ell-1} + 2(\ell-1)]\}.$$

As an abstract simplicial complex, the restriction $\mathrm{sd}(\Delta_{d-1})_{W_{i_\ell}}$ is isomorphic to the barycentric subdivision of the boundary of an $(i_\ell+1)$-simplex. Hence, geometrically, $\mathrm{sd}(\Delta_{d-1})_{W_{i_\ell}}$ is an i_ℓ-sphere. Moreover,

$$
\mathrm{sd}(\Delta_{d-1})_{\bigcup_{\ell=1}^r W_{i_\ell}} = \mathop{\Huge *}\limits_{\ell=1}^r \mathrm{sd}(\Delta_{d-1})_{W_{i_\ell}}
$$

(3.5)
$$
\cong \mathop{\Huge *}\limits_{\ell=1}^r \mathbb{S}^{i_\ell} = \mathbb{S}^{j-1}.
$$

Note that

$$
\#\left(\bigcup_{\ell=1}^r W_{i_\ell}\right) = \sum_{\ell=1}^r (2^{i_\ell+2} - 2).
$$

We observe the following.

Lemma 3.3. *The numbers m_j defined in* Definition 3.1 *satisfy the following equality:*

(3.6)
$$
m_j + j = \min\left\{\sum_{\ell=1}^r (2^{i_\ell+2} - 2) \; : \; \begin{array}{l} (i_1,\ldots,i_r) \in \mathbb{N}^r, \\ i_1 + \cdots + i_r + (r-1) = j - 1, \\ i_1 + \cdots + i_r + 2r \le d \end{array} \right\}.
$$

The proof of Lemma 3.3 will be given in Section 5. Hence we obtain:

Corollary 3.4. *The number $m_j + j$ is an upper bound for the minimal cardinality of a subset W of the set of vertices of $\mathrm{sd}(\Delta_{d-1})$ such that $\widetilde{H}_{j-1}(\mathrm{sd}(\Delta_{d-1})_W; \mathbb{K}) \ne 0$. In particular, m_j is an upper bound for $l_j(d-1)$ for every $j = 1,\ldots,d-1$.*

Our next aim is to improve the bounds for $l_j(d-1)$ and to provide bounds for $u_j(d-1)$. To this end we need two lemmas. As we have already observed in Table 1 the barycentric subdivision of any simplicial complex is a flag complex; i.e., all its minimal non-faces are of size 2. In particular:

Lemma 3.5. *Let Δ be a $(d-1)$-dimensional simplicial complex on vertex set $[n]$ and let $W \subseteq \Delta \setminus \{\emptyset\}$ be a subset of the vertex set of $\mathrm{sd}(\Delta)$. Then $\mathrm{sd}(\Delta)_W$ is a flag complex or a full simplex.*

We prove the next lemma by adapting arguments used for the proof of [20, Lem. 2.1.14].

Lemma 3.6. *Let z be a j-cycle in a $(d-1)$-dimensional flag simplicial complex Δ, which is not a boundary. Then the simplices in the support of z contain at least $2(j+1)$ vertices.*

Proof. We proceed by induction on j. If z is 0-cycle, then there are at least 2 simplices in its support. If $j = 1$, then there exist at least 3 vertices in the simplices in the support of z. If z contains exactly 3 vertices in its support, Δ cannot be flag. Let $j \ge 2$. Assume that z is a j-cycle which is not a boundary, and assume that the set of vertices in the simplices in the support of z is minimal. Then in the simplices in the support of z, there exist two vertices v and w that are not connected by an edge in Δ. Otherwise, the vertices in the simplices from the support of z form a simplex, and hence z is a boundary. Let $z = \sum_{\sigma \in \Delta} a_\sigma \sigma$. We write $z = vz_1 + z_2$, where

$$
z_1 = \sum_{\substack{v \in \sigma \in \Delta \\ a_\sigma \ne 0}} a_\sigma(\sigma \setminus \{v\}) \text{ and } z_2 = \sum_{\substack{v \notin \sigma \in \Delta \\ a_\sigma \ne 0}} a_\sigma \sigma.
$$

It follows that z_1 has to be a $(j-1)$-cycle since otherwise $\partial z \neq 0$. Moreover, z_1 cannot be a boundary since otherwise the set of vertices in the simplices in the support of z is not minimal. Indeed, if $z_1 = \partial z_3$, then $z_3 + z_2$ is a j-cycle, which is not a boundary. In particular, this contradicts the minimality of z. We conclude by induction that the simplices in the support of z_1 contain at least $2j$ vertices. Since the simplices in the support of z contain the two additional vertices v and w, there are at least $2(j+1)$ vertices in the simplices in the support of z. $\square$

Now using Hochster's formula (3.1), Corollary 3.4, Lemma 3.5 and Lemma 3.6, we obtain the following bounds on $l_j(d-1)$ and $u_j(d-1)$.

Proposition 3.7.

(i) *For* $1 \leq j \leq \frac{d}{2}$ *one has* $l_j(d-1) = j$.

(ii) *For* $\frac{d}{2} < j \leq d-2$ *one has* $j \leq l_j(d-1) \leq m_j$.

(iii) *For* $1 \leq j \leq \frac{d}{2}$ *one has* $2^d - d - 1 - m_{d-j-1} \leq u_j(d-1) \leq 2^d - 2d + j$.

(iv) *For* $\frac{d}{2} < j \leq d-2$ *one has* $u_j(d-1) = 2^d - 2d + j$.

(v) $l_{d-1}(d-1) = u_{d-1}(d-1) = 2^d - d - 1$.

Proof. Since $\mathrm{sd}(\Delta_{d-1})$ is a cone over the barycentric subdivision of the boundary of Δ_{d-1}, which is a triangulation of a sphere, it follows that $\mathrm{sd}(\Delta_{d-1})$ is Gorenstein. From graded Poincaré duality on the Koszul homology of Gorenstein rings (see [10, Thm. 3.4.5]) we deduce that

$$(3.7) \qquad \beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) = \beta_{2^d-d-1-i,\, 2^d-2-i-j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]).$$

Hence assertions (i) and (iv) are equivalent, and similarly assertions (ii) and (iii) are equivalent. We will now show (i) and (ii).

By Corollary 3.4 we know that $l_j(d-1) \leq m_j$ and hence in particular $l_j(d-1) \leq j$ if $1 \leq j \leq \frac{d}{2}$. To show equality in (i) note that by Lemma 3.5 all induced subcomplexes of $\mathrm{sd}(\Delta_{d-1})$ are flag. Hence we can infer from Lemma 3.6 that a $(j-1)$-cycle of $\mathrm{sd}(\Delta_{d-1})$ is supported on at least $2j$ vertices, which implies by Hochster's formula (3.1) that $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) = 0$ for $1 \leq i \leq j-1$ and equality in (i) follows. The same argument also shows the lower bound for $l_j(d-1)$ in (ii).

Finally (v) follows immediately from (3.7) since $\beta_{i,i}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) = 0$ for every $i > 0$ and $\beta_{0,0}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) = 1$. $\square$

After establishing the bounds on $l_j(d-1)$ and $u_j(d-1)$ we next turn to a sequence of lemmas showing that there are no internal 0s in the intervals we have identified.

Lemma 3.8. *Let* $1 \leq j \leq d-2$. *Then we have* $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0$ *for*

$$2^{j+1} - 2 - j \leq i \leq 2^d - 2^{d-j} - 1 - j.$$

Proof. Let G be a j-dimensional face of Δ_{d-1} and consider the set $V_{<G}$ of vertices of $\mathrm{sd}(\Delta_{d-1})$ that correspond to faces of Δ_{d-1} properly contained in G. Then $\mathrm{sd}(\Delta_{d-1})_{V_{<G}}$ is the barycentric subdivision of the boundary of G and hence triangulates a $(j-1)$-sphere and $\widetilde{H}_{j-1}(\mathrm{sd}(\Delta_{d-1})_{V_{<G}}; \mathbb{K}) \neq 0$. Let F be an arbitrary face of G of dimension $j-1$. Then F is contained in the support of the homology $(j-1)$-cycle z of the boundary of G. Let V_F be the set of vertices in $\mathrm{sd}(\Delta_{d-1})$ that correspond to faces of Δ_{d-1} that are neither subsets nor supersets of F. For any subset W of V_F the restriction of $\mathrm{sd}(\Delta_{d-1})$ to $V_{<G} \cup W$ has non-vanishing $(j-1)^{\mathrm{st}}$

reduced homology group. For this consider the image $\mathrm{sd}(z)$ of z in the chain complex of $\mathrm{sd}(\Delta_{d-1})_{V_{<G} \cup W}$. Then $\mathrm{sd}(z)$ contains any $(j-1)$-simplex of $\mathrm{sd}(2^F)$ in its support. Each of these simplices contains the vertex F. But, if $\mathrm{sd}(z)$ were a boundary, then it could only be the boundary of a chain that contained a j-simplex with F as a vertex. Since F is $(j-1)$-dimensional, this simplex must also contain a vertex corresponding to a proper superset of F, but such a simplex does not exist in $\mathrm{sd}(\Delta_{d-1})_{V_{<G} \cup W}$.

This and elementary counting shows that $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0$ for

$$2^{j+1} - 2 - j = \#V_{<G} - j \leq i \leq \#(V_{<G} \cup V_F) - j = 2^d - 2^{d-j} - 1 - j.$$

$\square$

Proposition 3.9. *Let* $1 \leq j \leq \frac{d}{2}$. *Then we have* $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0$ *for*

$$j \leq i \leq 2^{2j-3} + j.$$

Proposition 3.9 follows from Lemma 3.10 applied to the sequence $(i_1, \ldots, i_j) = (0, \ldots, 0)$ for $1 \leq j \leq \frac{d}{2}$.

Lemma 3.10. *Let* $r \geq 2$ *and* $1 \leq j \leq d - 2$. *Let* $(i_1, \ldots, i_r) \in \mathbb{N}^r$ *such that*

(i) $i_1 + \cdots + i_r + (r - 1) = j - 1$.
(ii) $i_1 + \cdots + i_r + 2r \leq d$.

Then $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0$ *for all*

$$\sum_{\ell=1}^{r}(2^{i_\ell+2} - 2) - j \leq i \leq \sum_{\ell=1}^{r}(2^{i_\ell+2} - 2) + (2^{i_r+2} - 2)2^{i_2+\cdots+i_{r-1}+2r-4} - j.$$

Proof. We recall the definition of the sets W_{i_ℓ} from (3.4). We set $W_{i_1} := \{A \,:\, \emptyset \neq A \subsetneq [i_1 + 2]\}$ and for $2 \leq \ell \leq r$ we let

$$W_{i_\ell} := \{A \cup [i_1 + \cdots + i_{\ell-1} + 2(\ell - 1)] \,:$$
$$\emptyset \neq A \subsetneq [i_1 + \cdots + i_\ell + 2\ell] \setminus [i_1 + \cdots + i_{\ell-1} + 2(\ell - 1)]\}.$$

We further define

$$C(i_1, \ldots, i_r) = \left\{A \cup B \,:\, \begin{smallmatrix} \emptyset \neq A \subsetneq [i_1 + \cdots + i_r + 2r] \setminus [i_1 + \cdots + i_{r-1} + 2(r-1)] \\ B \subseteq [i_1 + \cdots + i_{r-1} + 2(r-1)] \setminus [i_1 + 2] \end{smallmatrix}\right\}.$$

First observe, that for $1 \leq j \leq r - 1$, $A \in W_{i_j}$ and $B \in C(i_1, \ldots, i_r)$ the set $\{A, B\}$ is a non-face of $\mathrm{sd}(\Delta_{d-1})$. Indeed, since $A \cap ([i_1 + \cdots + i_r + 2r] \setminus [i_1 + \cdots + i_{r-1} + 2(r-1)]) = \emptyset$ but $B \cap ([i_1 + \cdots + i_r + 2r] \setminus [i_1 + \cdots + i_{r-1} + 2(r-1)]) \neq \emptyset$, we could only have $A \subseteq B$. In this case, we arrive at a contradiction, since we know that $A \cap [i_1 + 2] \neq \emptyset$, but $B \cap [i_1 + 2] = \emptyset$. Hence, it follows that for any $D \subseteq C(i_1, \ldots, i_r)$ it holds that

$$\mathrm{sd}(\Delta_{d-1})_{\bigcup_{\ell=1}^r W_{i_\ell} \cup D} = \mathrm{sd}(\Delta_{d-1})_{\bigcup_{\ell=1}^r W_{i_\ell}} \cup \mathrm{sd}(\Delta_{d-1})_{W_{i_r} \cup D}.$$

Moreover,

$$\mathrm{sd}(\Delta_{d-1})_{\bigcup_{\ell=1}^r W_{i_\ell}} \cap \mathrm{sd}(\Delta_{d-1})_{W_{i_r} \cup D} = \mathrm{sd}(\Delta_{d-1})_{W_{i_r}}.$$

We claim that

(3.8) $$\widetilde{H}_\ell(\mathrm{sd}(\Delta_{d-1})_{W_{i_r} \cup D}; \mathbb{K}) = \widetilde{H}_\ell(\mathrm{sd}(\Delta_{d-1})_{W_{i_r}}; \mathbb{K})$$

for any $\ell \in \mathbb{N}$. If we have shown (3.8), we can deduce from (3.5) and the Mayer-Vietoris sequence in reduced homology for the above decomposition that

$$\widetilde{H}_\ell(\mathrm{sd}(\Delta_{d-1})_{\bigcup_{\ell=1}^r W_{i_\ell} \cup D}; \mathbb{K}) = \begin{cases} \mathbb{K}, & \text{if } \ell = i_1 + \cdots + i_r + (r-1), \\ 0, & \text{otherwise,} \end{cases}$$

for all $D \subseteq C(i_1, \ldots, i_r)$. This then in particular shows the assertion of the proposition since $\#C(i_1, \ldots, i_r) = (2^{i_r+2} - 2)2^{i_2 + \cdots + i_{r-1} + 2r - 4}$ and $\#\left(\bigcup_{\ell=1}^r W_{i_\ell}\right) = \sum_{\ell=1}^r (2^{i_\ell + 2} - 2)$.

We now show claim (3.8). The key idea is to interpret $\mathrm{sd}(\Delta_{d-1})_{W_{i_r} \cup D}$ as the order complex of a poset. We distinguish two cases.

Case 1: $D = C(i_1, \ldots, i_r)$. First, note that $W_{i_r} \cup C(i_1, \ldots, i_r)$ can be written in the following form:

$$W_{i_r} \cup C(i_1, \ldots, i_r) = \left\{ A \cup B : \begin{smallmatrix} \emptyset \neq A \subsetneq [i_1 + \cdots + i_r + 2r] \setminus [i_1 + \cdots + i_{r-1} + 2(r-1)] \\ B \subseteq [i_1 + \cdots + i_{r-1} + 2(r-1)] \setminus [i_1 + 2] \text{ or } B = [i_1 + \cdots + i_{r-1} + 2(r-1)] \end{smallmatrix} \right\}.$$

To simplify notation, we set

$$P_1 = 2^{[i_1 + \cdots + i_r + 2r] \setminus [i_1 + \cdots + i_{r-1} + 2(r-1)]}$$
$$- \{\emptyset, [i_1 + \cdots + i_r + 2r] \setminus [i_1 + \cdots + i_{r-1} + 2(r-1)]\}$$

and

$$P_2 = 2^{[i_1 + \cdots + i_{r-1} + 2(r-1)] \setminus [i_1 + 2]} \cup \{[i_1 + \cdots + i_{r-1} + 2(r-1)]\}.$$

Since $([i_1 + \cdots + i_r + 2r] \setminus [i_1 + \cdots + i_{r-1} + 2(r-1)]) \cap [i_1 + \cdots + i_{r-1} + 2(r-1)] = \emptyset$ any set $E \in W_{i_r} \cup C(i_1, \ldots, i_r)$ has a unique decomposition $E = E_1 \cup E_2$, where $E_1 \in P_1$ and $E_2 \in P_2$. In the following we consider $W_{i_r} \cup C(i_1, \ldots, i_r)$, P_1 and P_2 as partially ordered sets with order relation given by set inclusion. By the above arguments the map

$$\Phi : \begin{cases} W_{i_r} \cup C(i_1, \ldots, i_r) & \to & P_1 \times P_2 \\ A \cup B & \mapsto & (A, B) \end{cases}$$

is well-defined. Here, for the two posets P_1 and P_2, we write $P_1 \times P_2$ for the partially ordered set on the Cartesian product with $(p_1, p_2) \leq (p_1', p_2')$ if and only if $p_s \leq p_s'$ in P_s for $s \in \{1, 2\}$. It is now straightforward to verify that Φ defines an isomorphism of partially ordered sets. Moreover, the poset $P_1 \times P_2$ is easily seen to be isomorphic to $P_1' \times P_2'$, where $P_1' = 2^{[i_r+2]} - \{\emptyset, [i_r+2]\}$, $P_2' = 2^{[i_2 + \cdots + i_{r-1} + 2r - 4]} \cup \{\hat{1}\}$ with order relation being inclusion and $\hat{1}$ being an artificial maximal element of P_2'. It then follows that the order complexes of $W_{i_r} \cup C(i_1, \ldots, i_r)$ and $P_1' \times P_2'$ are isomorphic. Note that the order complex of $W_{i_r} \cup C(i_1, \ldots, i_r)$ is $\mathrm{sd}(\Delta_{d-1})_{W_{i_r} \cup C(i_1, \ldots, i_r)}$. We now look at the following map:

$$f : \begin{cases} P_1' \times P_2' & \to & P_1' \times P_2' \\ (A, B) & \mapsto & (A, \hat{1}). \end{cases}$$

Then f is a poset map and satisfies $f^2((A, B)) = f((A, B)) \geq (A, B)$. Hence f is a closure operator. Thus by [3, Cor. 10.12] the order complexes of $P_1' \times P_2'$ and $f(P_1' \times P_2')$ are homotopy equivalent. Since the projection on the first coordinate is an isomorphism of $f(P_1' \times P_2')$ and P_1' it follows that the order complexes of $P_1' \times P_2'$ and P_1' are homotopy equivalent.

In particular, their homology groups are equal. Finally, since the order complex of P_1' is the barycentric subdivision of the boundary complex of an $(i_r + 1)$-simplex, we obtain

$$\widetilde{H}_\ell(\mathrm{sd}(\Delta_{d-1})_{W_{i_r} \cup C(i_1,\ldots,i_r)}; \mathbb{K}) = \begin{cases} \mathbb{K}, & \text{if } \ell = i_r, \\ 0, & \text{otherwise.} \end{cases}$$

Since by the discussion preceding (3.5) we have $\mathrm{sd}(\Delta_{d-1})_{W_{i_r}} \cong \mathbb{S}^{i_r}$, this concludes Case 1.

Case 2: $\emptyset \neq D \subsetneq C(i_1,\ldots,i_r)$. Using the map Φ defined in Case 1, and by an analogous argument as in this case, one sees that $W_{i_r} \cup D$ is isomorphic to a subposet P_D of $P_1 \times P_2$ that contains $P_1 \times \{[i_1 + \cdots + i_{r-1} + 2(r-1)]\}$. For this note that $B \cup [i_1 + \cdots + i_{r-1} + 2(r-1)] \in W_{i_r}$ for all $\emptyset \neq B \neq [i_1 + \cdots + i_r + 2r] \setminus [i_1 + \cdots + i_{r-1} + 2(r-1)]$. The identification of $P_1 \times P_2$ and $P_1' \times P_2'$ provides a copy P_D' of P_D inside $P_1' \times P_2'$ such that $P_1' \times \{\hat{1}\} \subseteq P_D'$. Now the restriction $f|_{P_D'}$ of f to P_D' is a closure operator on P_D'. By $P_1' \times \{\hat{1}\} \subseteq P_D'$ the projection on the first coordinate gives an isomorphism of the image of f_{P_D}' and P_1'. As in Case 1 we conclude that

$$\widetilde{H}_\ell(\mathrm{sd}(\Delta)_{W_{i_r} \cup D}; \mathbb{K}) = \begin{cases} \mathbb{K}, & \text{if } \ell = i_r, \\ 0, & \text{otherwise.} \end{cases}$$

This finishes the proof. $\qquad\square$

The following technical lemmas show that the preceding constructions are sufficient to deduce $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0$ for every i in the intervals we have identified.

Lemma 3.11. *Let* $d \geq 3$, $\frac{d}{2} < j \leq d-1$ *be integers and let* $0 \leq i_1 \leq i_2 \leq \cdots \leq i_r$ *be a sequence of integers such that*

(i) $i_1 = 0$,
(ii) $i_1 + \cdots + i_r + (r-1) = j-1$,
(iii) $i_1 + \cdots + i_r + 2r \leq d$.

Then

$$(2^{i_r+2} - 2)2^{i_2 + \cdots + i_{r-1} + 2r - 4} + \sum_{\ell=1}^{r}(2^{i_\ell+2} - 2) \geq 2^{j+1} - 2.$$

Lemma 3.12. *Let* $d \geq 3$, $\frac{d}{2} < j \leq d-1$, $r \geq 2$ *be integers and let* $0 \leq i_1 \leq i_2 \leq \cdots \leq i_r$ *be a sequence of integers such that*

(i) $i_1 \geq 1$,
(ii) $i_1 + \cdots + i_r + (r-1) = j-1$,
(iii) $i_1 + \cdots + i_r + 2r \leq d$.

Set $j_1 = i_1 - 1$, $j_2 = i_2 + 1$ *and* $j_\ell = i_\ell$ *for* $3 \leq \ell \leq r$. *Then*

$$(2^{i_r+2} - 2)2^{i_2 + \cdots + i_{r-1} + 2r - 4} + \sum_{\ell=1}^{r}(2^{i_\ell+2} - 2) \geq \sum_{\ell=1}^{r}(2^{j_\ell+2} - 2).$$

The proofs of Lemma 3.11 and Lemma 3.12 are given in Section 5.

If we combine Lemma 3.10, Lemma 3.11 and Lemma 3.12 we obtain the following.

Lemma 3.13. *Let $d \geq 1$ and let $\frac{d}{2} < j \leq d - 2$. Then $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0$ for all*

$$m_j \leq i \leq 2^{j+1} - 2 - j.$$

Proof. Let $0 \leq i_1 \leq i_2 \leq \cdots \leq i_r$ be a sequence of integers such that

(i) $i_1 + \cdots + i_r + (r - 1) = j - 1$,
(ii) $i_1 + \cdots + i_r + 2r \leq d$, and
(iii) $m_j = \sum_{\ell=1}^{r} (2^{i_\ell+2} - 2) - j$.

If $r = 1$, then we have $m_j = 2^{j+1} - 2 - j$, and it suffices to show that

$$\beta_{m_j, m_j+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0.$$

But this is true by Proposition 3.7(ii).

Assume $r \geq 2$. It follows from Lemma 3.10 that $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0$ for all

$$(3.9) \qquad m_j \leq i \leq \sum_{\ell=1}^{r} (2^{i_\ell+2} - 2) + (2^{i_r+2} - 2)2^{i_2 + \cdots + i_{r-1} + 2r - 4} - j.$$

If $i_1 = 0$, then Lemma 3.11 directly yields the claim. If $i_1 \geq 1$, then we infer from (3.9) and Lemma 3.12 that $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0$ for all

$$m_j \leq i \leq \sum_{\ell=1}^{r} (2^{j_\ell+2} - 2) - j,$$

where $j_1 = i_1 - 1$, $j_2 = i_2 + 1$ and $j_\ell = i_\ell$ for $3 \leq \ell \leq r$. If we apply Lemma 3.10 to the ordered sequence given by $\{j_1, \ldots, j_r\}$, then we can conclude that $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0$ for all

$$m_j \leq i \leq \sum_{\ell=1}^{r} (2^{j_\ell+2} - 2) + (2^{j_r+2} - 2)2^{j_2 + \cdots + j_{r-1} + 2r - 4} - j.$$

If $j_1 = 0$, the claim follows again from Lemma 3.11. If $j_1 > 0$, then we go on perturbing the sequence; i.e., we subtract 1 from its minimum element and add 1 to its second smallest element until the smallest element equals 0. Applying Lemma 3.10 and Lemma 3.12 in each step of this process and using Lemma 3.11 in the last step, the claim follows. □

We finally provide the proof of Theorem 3.2.

Proof of Theorem 3.2. By the same arguments as in the proof of Proposition 3.7 assertions (i) and (ii) are equivalent. We provide the proof of (i).

Using

$$\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) = \beta_{2^d-d-1-i, 2^d-i-j-2}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})])$$

(see (3.7)) it follows from Lemma 3.13 that $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0$ for

$$2^d - d - 1 - (2^{d-1-j+1} - 2 - (d - 1 - j)) \leq i \leq 2^d - d - 1 - m_{d-1-j},$$

i.e., for

$$(3.10) \qquad 2^d - 2^{d-j} - j \leq i \leq 2^d - d - 1 - m_{d-1-j}.$$

In addition, we know from Proposition 3.9 and Lemma 3.8 that $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0$ for $j \leq i \leq 2^{2j-3} + j$ and for $2^{j+1} - 2 - j \leq i \leq 2^d - 2^{d-j} - 1 - j$. Since $2^d - 2^{d-j} - 1 - j \geq 2^{2j-3} + j \geq 2^{j+1} - 2 - j \geq j$ for $d \geq 3$ and $j \geq 1$, we obtain $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0$ for $j \leq i \leq 2^d - 2^{d-j} - 1 - j$. Combining this with (3.10) shows the first part of the claim. The second part, concerning the vanishing

of certain Betti numbers, follows from Proposition 3.7(i) and (ii) combined with (3.7).

(iii) follows from Proposition 3.7(v). □

3.2. **Asymptotic behavior of Betti numbers for barycentric subdivisions.**
In this section, we do not restrict our attention to barycentric subdivisions of simplices anymore but consider iterated barycentric subdivisions of arbitrary simplicial complexes. More precisely, let Δ be a $(d-1)$-simplicial complex and, for $r \in \mathbb{N}$, let $\mathrm{sd}^r(\Delta)$ be its rth iterated barycentric subdivision, defined by $\mathrm{sd}^0(\Delta) := \Delta$, $\mathrm{sd}^1(\Delta) := \mathrm{sd}(\Delta)$ and $\mathrm{sd}^r(\Delta) := \mathrm{sd}(\mathrm{sd}^{r-1}(\Delta))$. Given a non-negative integer $0 \leq j \leq \mathrm{reg}(\mathbb{K}[\mathrm{sd}(\Delta)]) = \mathrm{reg}(\mathbb{K}[\mathrm{sd}^r(\Delta)])$ we are interested in the relative proportion of non-zero Betti numbers $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}^r(\Delta)])$ compared to the projective dimension if r tends to infinity. That is, given j, we want to study the quantity

$$\frac{\#\{i \ : \ \beta_{i,i+j}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) \neq 0\}}{\mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)])}$$

if r goes to infinity. Our main result in this section is the following.

Theorem 3.14. *Let $d-1 \geq 1$ and let Δ be a $(d-1)$-dimensional simplicial complex. Let $N(d)$ be the number of vertices of $\mathrm{sd}^3(\Delta_{d-1}) \setminus \partial(\mathrm{sd}^3(\Delta_{d-1}))$. Then, for $r \geq 3$ we have $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) \neq 0$ in the following cases:*

(i) $1 \leq j \leq \frac{d}{2}$ *and* $j \leq i \leq \mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) + \mathrm{depth}(\mathbb{K}[\Delta]) - N(d) + 2^d - d - 1 - m_{d-j-1}$,

(ii) $\frac{d}{2} < j \leq d-2$ *and* $m_j \leq i \leq \mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) + \mathrm{depth}(\mathbb{K}[\Delta]) - N(d) + 2^d - 2d + j$,

(iii) $j = d-1$ *and* $2^d - d - 1 \leq j \leq \mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) + \mathrm{depth}(\mathbb{K}[\Delta]) - N(d) + 2^d - d - 1$.

Proof. Since Δ is a $(d-1)$-dimensional simplicial complex, there exists a $(d-1)$-dimensional face $H \in \Delta$. Let $F \in \mathrm{sd}^{r-3}(2^H)$ be a $(d-1)$-dimensional face of $\mathrm{sd}^{r-3}(2^H)$. Choose a $(d-1)$-dimensional face G of $\mathrm{sd}^2(2^F) \subseteq \mathrm{sd}^{r-1}(\Delta)$ that lies completely in the interior of $\mathrm{sd}^2(2^F)$. After an additional subdivision it is guaranteed that none of the vertices of $\mathrm{sd}(2^G)$ are connected by an edge to any of the vertices on the boundary of $\mathrm{sd}^3(2^F)$. Moreover, by construction, $\mathrm{sd}(2^G) = \mathrm{sd}^r(\Delta)_{\{A \ : \ \emptyset \neq A \subseteq G\}}$; i.e., $\mathrm{sd}(2^G)$ is an induced subcomplex of $\mathrm{sd}^r(\Delta)$. For simplicity, we use V_G to denote the vertex set of $\mathrm{sd}(2^G)$ and V_F to denote the vertices in $\mathrm{sd}^3(2^F) \setminus \partial(\mathrm{sd}^3(2^F))$. Observe that we have $V_G \subsetneq V_F$. Moreover, let V_Δ be the vertex set of $\mathrm{sd}^r(\Delta)$ and set $V := V_\Delta \setminus V_F$. Since in $\mathrm{sd}^3(\Delta)$ there is no edge connecting $\mathrm{sd}(2^G)$ and $\partial(\mathrm{sd}^3(2^F))$, there cannot exist an edge passing from V_G to V. Hence, for any $A \subseteq V$ and $B \subseteq V_G$, it follows that $\mathrm{sd}^r(\Delta)_{A \cup B} = \mathrm{sd}^r(\Delta)_A \cup \mathrm{sd}^r(\Delta)_B$, and, using that $V \cap V_G = \emptyset$, we infer that $\mathrm{sd}^r(\Delta)_{A \cup B}$ is disconnected with connected components $\mathrm{sd}^r(\Delta)_A$ and $\mathrm{sd}^r(\Delta)_B$. This, in particular, implies that

$$\widetilde{H}_j(\mathrm{sd}^r(\Delta)_{A \cup B}; \mathbb{K}) = \widetilde{H}_j(\mathrm{sd}^r(\Delta)_A; \mathbb{K}) \oplus \widetilde{H}_j(\mathrm{sd}^r(\Delta)_B; \mathbb{K}).$$

Using that $\widetilde{H}_j(\mathrm{sd}^r(\Delta)_B; \mathbb{K}) = \widetilde{H}_j(\mathrm{sd}(2^G)_B; \mathbb{K})$ we conclude that if $B \subseteq V_G$ is such that $\widetilde{H}_{j-1}(\mathrm{sd}(2^G)_B; \mathbb{K}) \neq 0$, then $\widetilde{H}_{j-1}(\mathrm{sd}^r(\Delta)_{A \cup B}; \mathbb{K}) \neq 0$ for any $A \subseteq V$. Now Hochster's formula (3.1) implies that $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) \neq 0$ for $l_G(j) \leq i \leq u_G(j) + \#V$, where $l_G(j)$ and $u_G(j)$ denote the beginning and the end of the jth strand in

the resolution of $\mathbb{K}[\mathrm{sd}(\langle G \rangle)]$. If $1 \leq j \leq \frac{d}{2}$, it now follows from Theorem 3.2(i) that $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) \neq 0$ for

$$
\begin{aligned}
j \leq i &\leq 2^d - d - 1 - m_{d-j-1} + \#V \\
&= \#V_\Delta - \#V_F + 2^d - d - 1 - m_{d-j-1} \\
&= \mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) + \mathrm{depth}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) - \#V_F + 2^d - d - 1 - m_{d-j-1}.
\end{aligned}
$$

Since the depth is invariant under taking barycentric subdivisions [22, Cor. 2.5] and since $\#V_F = N(d)$, this shows (i).

The claims in (ii) and (iii) can be seen by the same arguments using parts (ii) and (iii) of Theorem 3.2, respectively. □

As an immediate consequence of Theorem 3.14 and the fact that

$$
\lim_{r \to \infty} \mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) = \infty
$$

we obtain:

Corollary 3.15. *Let $d - 1 \geq 1$ and let Δ be a $(d-1)$-dimensional simplicial complex. For $1 \leq j \leq d-1$ one has that $\#\{i \ : \ \beta_{i,i+j}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) = 0\}$ is bounded above in terms of j and d (and independent of r). In particular,*

$$
\lim_{r \to \infty} \frac{\#\{i \ : \ \beta_{i,i+j}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) \neq 0\}}{\mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)])} = 1
$$

for every $j = 1, \ldots, d-1$.

From the properties of barycentric subdivisions listed in Table 1 we know that for $r \geq 1$ one has

$$
\mathrm{reg}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) = \begin{cases} d - 1, & \text{if } \widetilde{H}_{d-1}(\Delta; \mathbb{K}) = 0, \\ d, & \text{if } \widetilde{H}_{d-1}(\Delta; \mathbb{K}) \neq 0. \end{cases}
$$

In case $\widetilde{H}_{d-1}(\Delta; \mathbb{K}) = 0$, Theorem 3.14 covers all strands of the minimal free resolution of $\mathbb{K}[\mathrm{sd}^r(\Delta)]$. However, in the second case, Theorem 3.14 does not provide a statement for the last strand of the resolution. Indeed, we will see that in this case the situation becomes more involved and the behavior depends on the geometry of the original simplicial complex Δ. Before we can state the precise result, we need to introduce some notation and recall some work from [14]. For a $(d-1)$-dimensional simplicial complex with f-vector $(f_{-1}^\Delta, \ldots, f_{d-1}^\Delta)$ the polynomial

$$
f^\Delta(t) = \sum_{j=0}^{d} f_{j-1}^\Delta t^{d-j}
$$

is called the *f-polynomial* of Δ. In [6], Brenti and Welker study the behavior of the f-polynomial of $\mathrm{sd}^r(\Delta)$ and show that, as $r \to \infty$, all but one root of $f^{\mathrm{sd}^r(\Delta)}(t)$ converge to negative real numbers, depending only on the dimension of Δ, and the last root goes to infinity (see [6, Thm. 3]. This statement was then made more explicit in [14], where "limit polynomials" for the normalized f-polynomials were provided. We recall the construction of those polynomials. Let $\Lambda_d := (\lambda_{i,j})_{-1 \leq i,j \leq d-1}$ be the $(d+1) \times (d+1)$ matrix, where $\lambda_{-1,-1} = 1$ and $\lambda_{i,-1} = 0$ if $i \neq -1$, and $\lambda_{i,j}$ counts the number of j-dimensional faces in the interior of the first barycentric subdivision of an i-dimensional simplex otherwise.

It is shown in [14, Lem. 3.4] that Λ_d has eigenvalues $0!, 1!, 2!, 3!, \ldots, d!$. Let D_d be the diagonal matrix of these eigenvalues (in the stated order) and let P_d be the corresponding matrix of eigenvectors that diagonalizes Λ_d, i.e., $D_d = P_d^{-1} \Lambda_d P_d$. We define $M_{d,d}$ as the $(d+1) \times (d+1)$ matrix, whose only non-zero entry is a 1 in the lower right corner. Finally, let $\underline{t} := (t^d, t^{d-1}, \ldots, t^1, t^0)^T$. It is proven in [14, Sect. 3] that for a $(d-1)$-dimensional simplicial complex Δ the sequence of normalized f-polynomials $\left(\frac{1}{(d!)^r} f^{\mathrm{sd}^r(\Delta)}(t) \right)_{r \geq 1}$ converges coefficientwise to the polynomial

$$p_\infty^\Delta(t) := (f^\Delta P_d) M_{d,d} (P_d)^{-1} \underline{t}.$$

By definition of P_d its last column is the eigenvector of the matrix Λ_d to the eigenvalue $d!$. Since Λ_d is a lower triangular matrix with $\lambda_{d-1,d-1} = d!$, we have $\Lambda_d \mathfrak{e}_{d+1} = d! \mathfrak{e}_{d+1}$ and we can hence choose $\mathfrak{e}_{d+1}$ as the last column of P_d. Let $p_{d-1,2}^{-1}$ denote the entry of P_d^{-1} in the last row and second column. Having set up this additional notation, the above discussion directly yields the following.

Corollary 3.16. *Let Δ be a $(d-1)$-dimensional simplicial complex. Then*

$$\lim_{r \to \infty} \frac{1}{(d!)^r} f_0^{\mathrm{sd}^r(\Delta)} = p_{d-1,2}^{-1} f_{d-1}^\Delta.$$

The following simple lemma is needed for the main result of this section and follows immediately from the fact that in a $(d-1)$-dimensional simplicial complex there are no boundaries in dimension $d-1$.

Lemma 3.17. *Let Δ, Δ' be $(d-1)$-dimensional simplicial complexes such that there are geometric realizations for which every $(d-1)$-simplex of Δ is the union of some $(d-1)$-simplices of Δ'. Let $\sigma_1, \ldots, \sigma_\ell$ be a basis of the cycle space of Δ in dimension $d-1$ and $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_\ell$ their images in the cycle space of Δ'. Then every $(d-1)$-cycle of Δ' is a unique linear combination of $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_\ell$.*

The following is a simple consequence of the transformation of f-vectors under barycentric subdivision (see, e.g., [6, Lem. 1]).

Lemma 3.18. *Let Δ, Δ' be two $(d-1)$-dimensional simplicial complexes such that for some $0 \leq i \leq d-1$ we have $f_i^\Delta > f_i^{\Delta'}$ and $f_j^\Delta = f_j^{\Delta'}$ for $i < j \leq d-1$. Then there exists R such that for $r \geq R$ we have*

$$f_j^{\mathrm{sd}^r(\Delta)} > f_j^{\mathrm{sd}^r(\Delta')} \text{ for } 0 \leq j \leq i \text{ and}$$
$$f_j^{\mathrm{sd}^r(\Delta)} = f_j^{\mathrm{sd}^r(\Delta')} \text{ for } i < j \leq d-1.$$

The preceding lemma motivates the following minimality concept for ℓ-cycles of a simplicial complex Δ. We say that the ℓ-cycle $\sigma \neq 0$ is *minimal* among the ℓ-cycles of Δ if there is no ℓ-cycle $\sigma' \neq 0$ such that for the simplicial complexes $\tilde{\sigma}$ and $\widetilde{\sigma'}$ induced by the support of the cycles there is an index i satisfying $f_i^{\tilde{\sigma}} > f_i^{\widetilde{\sigma'}}$ and $f_j^{\tilde{\sigma}} = f_j^{\widetilde{\sigma'}}$ for $i < j \leq d-1$. We can now formulate our result concerning the last strand of the resolution of $\mathbb{K}[\mathrm{sd}^r(\Delta)]$, assuming that Δ has non-trivial homology in top-dimension.

Proposition 3.19. *Let $d - 1 \geq 1$ and let Δ be a $(d-1)$-dimensional simplicial complex such that $\widetilde{H}_{d-1}(\Delta; \mathbb{K}) \neq 0$. Further let σ be a minimal homology $(d-1)$-cycle of Δ and let*

$$\widetilde{\sigma} = \{F \in \Delta \ : \ F \subseteq G \text{ for some } G \text{ in the support of } \sigma\}$$

be the corresponding induced subcomplex of Δ. Then

(i) *for $r \geq 1$, $\beta_{i,i+d}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) \neq 0$ for $\#V_r^\sigma - d \leq i \leq \mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)])$, where V_r^σ denotes the vertex set of $\widetilde{\sigma}^{\langle r \rangle}$. If r is large, then in addition $\beta_{i,i+d}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) = 0$ for $0 \leq i < \#V_r^\sigma - d$.*

(ii)

$$\lim_{r \to \infty} \frac{\#\{i \ : \ \beta_{i,i+d}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) \neq 0\}}{\mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)])} = 1 - \frac{f_{d-1}^{\widetilde{\sigma}}}{f_{d-1}^{\Delta}}.$$

Proof. One observes that $\mathrm{sd}^r(\Delta)_{V_r^\sigma} = \mathrm{sd}^r(\widetilde{\sigma})$. Since σ is a homology $(d-1)$-cycle of Δ, the rth barycentric subdivision $\mathrm{sd}^r(\widetilde{\sigma})$ of its induced complex $\widetilde{\sigma}$ gives rise to a homology $(d-1)$-cycle σ_r of $\mathrm{sd}^r(\Delta)$, and we conclude that $\widetilde{H}_{d-1}(\mathrm{sd}^r(\Delta)_{V_r^\sigma}; \mathbb{K}) \neq 0$. It follows from Hochster's formula (3.1) that for $i = \#V_r^\sigma - d$ we have $\beta_{i,i+d}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) \neq 0$. Moreover, if we consider induced subcomplexes $\mathrm{sd}^r(\Delta)_A$ of $\mathrm{sd}^r(\Delta)$ with $V_r^\sigma \subseteq A$, then σ_r will remain a homology $(d-1)$-cycle in $\mathrm{sd}^r(\Delta)_A$. This in particular implies that $\beta_{i,i+d}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) \neq 0$ for $\#V_r^\sigma - d \leq i \leq \mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)])$, and hence the part of (i) concerning the non-vanishing Betti numbers follows. For the vanishing part, let τ be a $(d-1)$-cycle of $\mathrm{sd}^r(\Delta)$. By Lemma 3.17 it follows that $\tilde{\tau}$ is the union of some $\mathrm{sd}^r(\tilde{\sigma}')$ for $(d-1)$-cycles σ' of Δ. Thus for r large enough by Lemma 3.18 it follows that the vertex set of $\tilde{\tau}$ is of larger cardinality than the vertex set of $\mathrm{sd}^r(\tilde{\sigma})$ for the minimal $(d-1)$-cycle σ of Δ. Now Hochster's formula (3.1) shows the vanishing.

It remains to show (ii). Let V_r^Δ be the vertex set of $\mathrm{sd}^r(\Delta)$. We know from (i) that

$$\frac{1}{\mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)])} \#\{i \ : \ \beta_{i,i+d}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) \neq 0\}$$

$$= \frac{1}{\mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)])} \left(\mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) - (\#V_r^\sigma - d - 1)\right)$$

$$= 1 - \frac{\#V_r^\sigma - d - 1}{\#V_r^\Delta - \mathrm{depth}(\mathbb{K}[\mathrm{sd}^r(\Delta)])}$$

$$= 1 - \frac{\frac{1}{(d!)^r}(\#V_r^\sigma - d - 1)}{\frac{1}{(d!)^r}(\#V_r^\Delta - \mathrm{depth}(\mathbb{K}[\Delta]))}.$$

We infer from Corollary 3.16 that numerator and denominator of this fraction converge to $p_{d-1,2}^{-1} f_{d-1}^{\widetilde{\sigma}}$ and $p_{d-1,2}^{-1} f_{d-1}^{\Delta}$, respectively. Thus, we can finally conclude that

$$\lim_{r \to \infty} \frac{1}{\mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)]}) \#\{i \ : \ \beta_{i,i+d}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) \neq 0\}$$

$$= 1 - \frac{p_{d-1,2}^{-1} f_{d-1}^{\widetilde{\sigma}}}{p_{d-1,2}^{-1} f_{d-1}^{\Delta} p_{d+1}} = 1 - \frac{f_{d-1}^{\widetilde{\sigma}}}{f_{d-1}^{\Delta}}.$$

$\square$

We now provide an example that shows that for any d indeed any rational number in the half-open interval $[0, 1)$ can occur as a limit in (ii) of the above proposition.

Example 3.20. Let $\frac{p}{q} \in [0, 1) \cap \mathbb{Q}$.

Case 1: $\frac{p}{q} = 0$. Let Δ be the boundary of a $(d-1)$-simplex. In this case, the only minimal $(d-1)$-homology cycle of Δ is Δ itself, and it follows from Proposition 3.19 that

$$\lim_{r \to \infty} \frac{\#\{i \ : \ \beta_{i,i+d}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) \neq 0\}}{\mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)])} = 0.$$

Indeed, for any r, we have $\beta_{i,i+d}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) \neq 0$ if and only if $i = \mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)])$.

Case 2: $p > 0$. Our construction relies on a result of Lee [23, Cor. 4.4.15] and Björner and Linusson [5, Thm. 1, Thm. 7]. They showed that for any d, there exists $N(d) \in \mathbb{N}$ such that for all even numbers $n \geq N(d)$ there exists a simple d-polytope with n vertices or, by taking the dual, a simplicial d-polytope with n facets, i.e., a simplicial polytopal $(d-1)$-sphere with n facets.

By this result there exists a simplicial $(d-1)$-sphere Δ_1 with $2N(d)(q-p)$ facets. Let Δ be the Cohen-Macaulay complex obtained from Δ_1 by stacking $2N(d)p$ copies of the $(d-1)$-simplex over a specified $(d-2)$-face of Δ_1. In this case Δ_1 is the only minimal $(d-1)$-homology cycle of Δ and Proposition 3.19 yields

$$\lim_{r \to \infty} \frac{\#\{i \ : \ \beta_{i,i+d}(\mathbb{K}[\mathrm{sd}^r(\Delta)]) \neq 0\}}{\mathrm{pdim}(\mathbb{K}[\mathrm{sd}^r(\Delta)])} = 1 - \frac{2N(d)(q-p)}{2N(d)q} = 1 - \frac{q-p}{q} = \frac{p}{q}.$$

4. EDGEWISE SUBDIVISIONS

4.1. **Algebraic invariants.** We have listed in Table 1 how the basic invariants behave under the rth edgewise subdivision of a $(d-1)$-dimensional simplicial complex Δ. It remains to provide a precise statement concerning the largest degree $t_1(\Delta^{\langle r \rangle})$ of a minimal generator of $I_{\Delta^{\langle r \rangle}}$.

Lemma 4.1. *Let Δ be a $(d-1)$-dimensional simplicial complex. Let $N(\Delta)$ be the set of minimal non-faces of Δ of cardinality $t_1(\mathbb{K}[\Delta])$. Then:*

(i) *If Δ is flag or $\Delta = \Delta_{d-1}$, then $t_1(\Delta^{\langle r \rangle}) = 2$.*

(ii) *If Δ is neither flag nor a $(d-1)$-simplex, the following two cases can occur. If there exists $F \in N(\Delta)$ and a vertex $v \in \Delta$ such that $\partial(F) * \{v\} \subseteq \Delta$, then, for any $r \geq 2$, $t_1(\mathbb{K}[\Delta^{\langle r \rangle}]) = t_1(\mathbb{K}[\Delta])$. Otherwise, $t_1(\mathbb{K}[\Delta^{\langle r \rangle}]) = t_1(\mathbb{K}[\Delta]) - 1$.*

Proof. Since flag-ness is preserved under edgewise subdivisions, we have $t_1(\Delta^{\langle r \rangle}) = 2$ if Δ is flag. If Δ is a $(d-1)$-simplex, then $\Delta^{\langle r \rangle}$ is flag by the definition of edgewise subdivision, which shows $t_1(\Delta^{\langle r \rangle}) = 2$.

To show (ii), first we prove that $t_1(\Delta^{\langle r \rangle}) \leq t_1(\Delta)$. For this aim, let $G = \{v_1, \ldots, v_m\}$ be a minimal non-face of $\Delta^{\langle r \rangle}$. Then at least one of the two conditions in the definition of edgewise subdivisions fails. If the second one fails, we have $|G| = 2$, and hence G gives rise to a minimal generator of $I_{\Delta^{\langle r \rangle}}$ of degree $2 < t_1(\Delta)$. So assume only the first condition fails, i.e., $H := \bigcup_{a \in G} \mathrm{supp}(a) \notin \Delta$. There exists $F \in N(\Delta)$ such that $F \subseteq H$. Since G is minimal, for any $1 \leq j \leq m$, there exists $i_j \in F$ such that $i_j \in \mathrm{supp}(v_j)$ and $i_j \notin \mathrm{supp}(v_\ell)$ for $1 \leq \ell \leq m$ and $\ell \neq j$. This implies that $\#G = m \leq s$, which finally shows $t_1(\Delta^{\langle r \rangle}) \leq t_1(\Delta)$.

Next, we show $t_1(\Delta^{\langle r \rangle}) \geq t_1(\Delta) - 1$. Let $F \in N(\Delta)$. Without loss of generality let $F = [s]$, where $s = t_1(\Delta)$. For $1 \leq i \leq s - 1$ set $v_i = \mathfrak{e}_i + (r-1)\mathfrak{e}_s$ and let $G = \{v_1, \ldots, v_{s-1}\}$. Then, $H \in \Delta^{\langle r \rangle}$ for any $H \subsetneq G$ but $G \notin \Delta^{\langle r \rangle}$; i.e., G is a minimal non-face of $\Delta^{\langle r \rangle}$. This implies the claimed inequality.

Now assume that there exists $F \in N(\Delta)$ and a vertex $v \in \Delta$ such that $\partial(F) * \{v\} \subseteq \Delta$. Without loss of generality assume $F = [s]$ and $v = s + 1$. We set $w_i = \mathfrak{e}_i + (r-1)\mathfrak{e}_{s+1}$ for $1 \leq i \leq s$ and $G = \{w_1, \ldots, w_s\}$. By minimality of F we have $H \in \Delta^{\langle r \rangle}$ for any $H \subsetneq G$. Since F is a non-face, we further know that $G \notin \Delta^{\langle r \rangle}$. Hence, G is a minimal non-face of $\Delta^{\langle r \rangle}$ of cardinality $t_1(\Delta)$, which implies that $t_1(\Delta^{\langle r \rangle}) \geq t_1(\Delta)$ and hence $t_1(\Delta^{\langle r \rangle}) = t_1(\Delta)$ in this case.

Finally assume that for all $F \in N(\Delta)$ and all vertices $v \in \Delta$ we have $\partial(F) * \{v\} \not\subseteq \Delta$. Let $G = \{v_1, \ldots, v_m\}$ be a minimal non-face of $\Delta^{\langle r \rangle}$ such that $F \bigcup_{i=1}^m \mathrm{supp}(v_i)$, where F is a minimal non-face of Δ. Without loss of generality assume $F = [s]$. It follows from the second paragraph of this proof that $\#G \leq \#F$. If $\#G < \#F$, then G corresponds to a minimal generator of $I_{\Delta^{\langle r \rangle}}$ of degree $< t_1(\Delta)$. So assume $\#G = \#F$. From the second paragraph of this proof it follows that for any $1 \leq j \leq s$ there exists a unique $1 \leq i_j \leq s = m$, such that $j \in \mathrm{supp}(v_{i_j})$. Since $\{v_\ell, v_k\} \in \Delta^{\langle r \rangle}$ for any $1 \leq \ell < k \leq s$, it must hold that $(v_{i_j})_j = 1$ for $1 \leq j \leq s$. Hence, any vertex $v_j \in G$ is of the form $v_j = \mathfrak{e}_{i_j} + z_j$, where $z_j \in \Delta^{\langle r-1 \rangle}$ and $\mathrm{supp}(z_j) \cap [s] = \emptyset$. Since G is a minimal non-face of $\Delta^{\langle r \rangle}$, the boundary of G is a subcomplex of $\Delta^{\langle r \rangle}$, and hence $\partial(F) * \bigcup_{i=1}^s \mathrm{supp}(z_i)$ is a subcomplex of Δ. Since $\bigcup_{i=1}^s \mathrm{supp}(z_i) \neq \emptyset$ $(r \geq 2)$, we arrive at a contradiction and the claim follows. $\qquad\square$

In the following we will analyze the Betti numbers of the rth edgewise subdivision of the $(d-1)$-dimensional simplex Δ_{d-1}. We start by showing that, locally, $\Delta_{d-1}^{\langle r \rangle}$ behaves as a barycentric subdivision of the boundary of a $(d-1)$-simplex.

Proposition 4.2. *Let $r \geq d$ be positive integers. Then for any face F of $\Delta_{d-1}^{\langle r \rangle}$ satisfying*

$$(4.1) \qquad \partial |F| \;\; = \;\; |F| \cap \partial |\Delta_{d-1}^{\langle r \rangle}|,$$

the link $\mathrm{lk}_{\Delta_{d-1}^{\langle r \rangle}}(F)$ is abstractly isomorphic to the barycentric subdivision of the boundary of a $(d - \#F)$-simplex.

Proof. To prove the statement we will first list two facts that allow us to simplify the situation.

- If F and G are two faces of equal dimension in $\Delta_{d-1}^{\langle r \rangle}$ satisfying (4.1), then it is straightforward to show that their links are isomorphic as simplicial complexes.
- Let F_1 be a face of $\Delta_{d-1}^{\langle r \rangle}$ and F_2 a face of $\Delta_{d-1}^{\langle s \rangle}$ both satisfying (4.1) in their respective complexes. If $\#F_1 = \#F_2 = t$ and $r, s \geq d - t + 1$, then it is easy to show that $\mathrm{lk}_{\Delta_{d-1}^{\langle r \rangle}}(F_1)$ and $\mathrm{lk}_{\Delta_{d-1}^{\langle s \rangle}}(F_2)$ are also isomorphic as simplicial complexes.

Combining those two reductions, we will now show the claim for a specific face F in the $(d - \#F + 1)^{\mathrm{st}}$ edgewise subdivision of Δ_{d-1}. More precisely, let s be a

fixed positive integer. Let

$$w_i = (\underbrace{1,\ldots,1}_{d-s},\underbrace{0,\ldots,0}_{i-1},\underbrace{1}_{\text{position } d+i-s},\underbrace{0,\ldots,0}_{s-i})$$

$$= \sum_{j=1}^{d-s} \mathfrak{e}_j + \mathfrak{e}_{d-s+i},$$

for $1 \le i \le s$. Let F be the abstract simplex whose geometric realization has vertex set $\{w_1,\ldots,w_s\}$. Then all vertices of F lie on the boundary and so does the convex hull of any $s-1$ subset of the vertices. But F itself does not lie on the boundary of the simplex. Hence F satisfies (4.1). It now suffices to show that $\mathrm{lk}_{\Delta_{d-1}^{\langle d-s+1\rangle}}(F)$ is isomorphic to the barycentric subdivision of the boundary of a $(d-s)$-simplex. Let V be the vertex set of $\mathrm{lk}_{\Delta_{d-1}^{\langle d-s+1\rangle}}(F)$. We define a map $\Phi : V \to \{A \,:\, \emptyset \ne A \subsetneq [d-s] \cup \{d\}\}$ by

$$v \mapsto \begin{cases} \mathrm{supp}(\mathfrak{i}(v-w_1)) \cap [d-s], & \text{if } \mathfrak{i}(v-w_1) \in \{0,1\}^d, \\[2mm] [d-s] \setminus \mathrm{supp}(\mathfrak{i}(v-w_1)) \cup \{d\}, & \text{if } \mathfrak{i}(v-w_1) \in \{-1,0\}^d. \end{cases}$$

We claim that Φ is a bijection and moreover that it induces an isomorphism between $\mathrm{lk}_{\Delta_{d-1}^{\langle d-s+1\rangle}}(F)$ and $\mathrm{sd}(\partial\Delta_{d-s})$ as simplicial complexes, where for the purpose of this proof Δ_{d-s} denotes the $(d-s)$-simplex on vertex set $[d-s] \cup \{d\}$.

Injectivity: Let $u, v \in \mathrm{lk}_{\Delta_{d-1}^{\langle d-s+1\rangle}}(F)$ and $u \ne v$. If $\mathfrak{i}(u-w_1) \in \{0,1\}^d$ and $\mathfrak{i}(v-w_1) \in \{-1,0\}^d$ (or vice versa), then $d \in \Phi(v)$ but $d \notin \Phi(u)$. Hence, $\Phi(u) \ne \Phi(v)$ in this case. Now, let $\mathfrak{i}(u-w_1) \in \{0,1\}^d$ and $\mathfrak{i}(v-w_1) \in \{0,1\}^d$. Assume, by contradiction, that $\Phi(u) = \Phi(v)$, i.e.,

$$\mathrm{supp}(\mathfrak{i}(u-w_1)) \cap [d-s] = \mathrm{supp}(\mathfrak{i}(v-w_1)) \cap [d-s]$$

and hence the first $d-s$ components of $\mathfrak{i}(u-w_1)$ and $\mathfrak{i}(v-w_1)$ are equal. This implies that the first $d-s$ components of u and v coincide. If $\mathfrak{i}(u-w_1)_{d-s} = \mathfrak{i}(v-w_1)_{d-s} = 1$, it must hold that $u_{d-s+1} = \cdots = u_d = 0$ and $v_{d-s+1} = \cdots = v_d = 0$ since $\sum_{j=1}^d u_j = \sum_{j=1}^d v_j = d-s+1$. This in particular implies that $u = v$. If $\mathfrak{i}(u-w_1)_{d-s} = \mathfrak{i}(v-w_1)_{d-s} = 0$, we can infer from $(w_1)_{d-s+1} = 1$ and $\mathfrak{i}(u-w_1) \in \{0,1\}^d$ that $u_{d-s+1} = v_{d-s+1} = 1$. As in the previous case, we conclude that $u_{d-s+2} = \cdots = u_d = 0$ and $v_{d-s+2} = \cdots = v_d = 0$. Hence, again, $u = v$. Finally let $\mathfrak{i}(u-w_1) \in \{-1,0\}^d$ and $\mathfrak{i}(v-w_1) \in \{-1,0\}^d$. Similar arguments as in the previous case show that we must have $\Phi(u) \ne \Phi(v)$.

Surjectivity: Let $\emptyset \ne G \subsetneq [d-s] \cup \{d\}$. First assume that $d \notin G$. We define a vector $v \in \mathbb{Z}^d$ by setting $v_1 = 1$ if $1 \notin G$ and $v_1 = 2$ if $1 \in G$ and successively, for $2 \le j \le d-s$,

$$v_j = \begin{cases} -\sum_{\ell=1}^{j-1} v_\ell + j, & \text{if } j \notin G, \\[2mm] -\sum_{\ell=1}^{j-1} v_\ell + j + 1, & \text{if } j \in G. \end{cases}$$

Moreover, $v_{d-s+1} = 0$ if $d-s \in G$ and $v_{d-s+1} = 1$ if $d-s \notin G$. For $d-s+2 \le j \le d$, we set $v_j = 0$. It is straightforward to verify that $v \in \mathrm{lk}_{\Delta_{d-1}^{\langle d-s+1\rangle}}(F)$ and $\Phi(v) = G$. Now, suppose $d \in G$. In this case, we define a vector $v \in \mathbb{Z}^d$ by setting

$v_1 = 1$ if $1 \in G$ and $v_1 = 0$ if $1 \notin G$. For $2 \le j \le d - s$ we successively set

$$v_j = \begin{cases} -\sum_{\ell=1}^{j-1} v_\ell + j, & \text{if } j \notin G, \\ -\sum_{\ell=1}^{j-1} v_\ell + j - 1, & \text{if } j \in G. \end{cases}$$

For $d - s + 2 \le j \le d - 1$ we set $v_j = 0$. In addition, we let $v_{d-s+1} = 1$ and $v_d = 0$ if $d - s \notin G$ and $v_{d-s+1} = 0$ and $v_d = 1$ if $d - s \in G$. One can easily check that indeed $v \in \mathrm{lk}_{\Delta_{d-1}^{\langle d-s+1 \rangle}}(F)$ and $\Phi(v) = G$, which completes the proof of surjectivity of Φ.

We can extend the map Φ to $\mathrm{lk}_{\Delta_{d-1}^{\langle d-s+1 \rangle}}(F)$ by mapping a face $A = \{a_1, \ldots, a_t\}$ of $\mathrm{lk}_{\Delta_{d-1}^{\langle d-s+1 \rangle}}(F)$ to $\{\Phi(a_1), \ldots, \Phi(a_t)\}$. We need to show that $\mathrm{lk}_{\Delta_{d-1}^{\langle d-s+1 \rangle}}(F)$ and $\mathrm{sd}(\partial \Delta_{d-s})$ are isomorphic as simplicial complexes. Since both complexes are flag, it suffices to show that Φ induces an isomorphism between their 1-skeleta. Let $\{v, w\} \in \mathrm{lk}_{\Delta_{d-1}^{\langle d-s+1 \rangle}}(F)$. Then, we have either $\mathrm{i}(v - w) \in \{0, 1\}^d$ or $\mathrm{i}(v - w) \in \{-1, 0\}^d$. Without loss of generality we can assume that $\mathrm{i}(w - v) \in \{0, 1\}^d$. Moreover, since $v \in \mathrm{lk}_{\Delta_{d-1}^{\langle d-s+1 \rangle}}(F)$, we know that $\mathrm{i}(v - w_1) \in \{0, 1\}^d$ or $\mathrm{i}(v - w_1) \in \{-1, 0\}^d$.

Case 1: $\mathrm{i}(v - w_1) \in \{0, 1\}^d$. Let $1 \le \ell \le d - s$ such that $\mathrm{i}(v - w_1)_\ell = 1$. It follows that

$$\mathrm{i}(w - w_1)_\ell = \mathrm{i}(w - v)_\ell + \mathrm{i}(v - w_1)_\ell$$
$$\ge \mathrm{i}(v - w_1)_\ell = 1,$$

since $\mathrm{i}(w - v) \in \{0, 1\}^d$. Since $w \in \mathrm{lk}_{\Delta_{d-1}^{\langle d-s+1 \rangle}}(F)$, we conclude that $\mathrm{i}(w - w_1)_\ell = 1$ and $\mathrm{i}(w - w_1) \in \{0, 1\}^d$. We can deduce that $\mathrm{supp}(\mathrm{i}(v - w_1)) \subsetneq \mathrm{supp}(\mathrm{i}(w - w_1))$, and hence $\Phi(v) \subsetneq \Phi(w)$, i.e., $\{\Phi(v), \Phi(w)\} \in \mathrm{sd}(\partial \Delta_{d-s})$.

Case 2: $\mathrm{i}(v - w_1) \in \{-1, 0\}^d$. We consider the two subcases, that $\mathrm{i}(w - w_1) \in \{-1, 0\}^d$ and $\mathrm{i}(w - w_1) \in \{0, 1\}^d$. Suppose that we are in the first case. If we have $\mathrm{i}(w - w_1)_\ell = -1$ for some $1 \le \ell \le d - s$, then it follows from $\mathrm{i}(v - w) \in \{-1, 0\}^d$ that

$$\mathrm{i}(v - w_1)_\ell = \mathrm{i}(v - w)_\ell + \mathrm{i}(w - w_1)_\ell$$
$$\le \mathrm{i}(w - w_1)_\ell = -1.$$

This implies that $\mathrm{supp}(\mathrm{i}(w - w_1)) \subsetneq \mathrm{supp}(\mathrm{i}(v - w_1))$ and hence

$$\Phi(v) = [d - s] \setminus \mathrm{supp}(\mathrm{i}(v - w_1)) \cup \{d\} \subsetneq [d - s] \setminus \mathrm{supp}(\mathrm{i}(w - w_1)) \cup \{d\} = \Phi(w).$$

In particular, $\{\Phi(v), \Phi(w)\}$ is an edge of $\mathrm{sd}(\partial(\Delta_{d-s}))$. It remains to consider the case $\mathrm{i}(w - w_1) \in \{0, 1\}^d$. Let $1 \le \ell \le d - s$ with $\mathrm{i}(w - w_1)_\ell = 1$. We have

$$\mathrm{i}(v - w_1)_\ell = \mathrm{i}(v - w)_\ell + \mathrm{i}(w - w_1)_\ell$$
$$= \mathrm{i}(v - w)_\ell + 1.$$

Since $\mathrm{i}(v - w) \in \{-1, 0\}^d$ and $\mathrm{i}(v - w_1) \in \{-1, 0\}^d$, it must hold that $\mathrm{i}(v - w)_\ell = -1$ and $\mathrm{i}(v - w_1)_\ell = 0$. Using that $d \in \Phi(v)$ but $d \notin \Phi(w)$, we can finally conclude that

$$\Phi(w) = \mathrm{supp}(\mathrm{i}(w - w_1)) \cap [d - s] \subsetneq [d - s] \setminus \mathrm{supp}(\mathrm{i}(v - w_1)) \cup \{d\} = \Phi(v)$$

and thus $\{\Phi(v), \Phi(w)\} \in \mathrm{sd}(\partial \Delta_{d-s})$. This finishes the proof of containment of the 1-skeleton of $\mathrm{lk}_{\Delta_{d-1}^{\langle d-s+1 \rangle}}(F)$ in the 1-skeleton of $\mathrm{sd}(\partial \Delta_{d-s})$. We omit the proof of the other inclusion since it follows from a similar reasoning. $\square$

The following corollary is a special case of Proposition 4.2.

Corollary 4.3. *Let $r \geq d$ be positive integers. Then for any vertex $v = (i_1, \ldots, i_n)$ in the interior of $\Delta_{d-1}^{\langle r \rangle}$ the link $\operatorname{lk}_{\Delta_{d-1}^{\langle r \rangle}}(v)$ is abstractly isomorphic to the barycentric subdivision of the boundary of a $(d-1)$-simplex.*

Note that necessarily $r \geq d$ if there is an interior vertex of $\Delta_{d-1}^{\langle r \rangle}$. This will also be crucial for Corollary 4.5. Using Corollary 4.3 and Theorem 3.2 we get the following bounds for the non-vanishing of Betti numbers in the edgewise subdivision. We use the constant m_j defined by (3.6).

Corollary 4.4. *Let $r \geq d$. Then*

  (i) *For $1 \leq j \leq \frac{d}{2}$ we have $\beta_{i,i+j}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}]) \neq 0$ for $j \leq i \leq 2^d - d - 1 - m_{d-1-j}$.*

  (ii) *For $\frac{d}{2} < j \leq d-2$ we have $\beta_{i,i+j}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}]) \neq 0$ for $m_j \leq i \leq 2^d - 2d + j$.*

  (iii) *For $j = d-1$, we have $\beta_{2^d-1-d, 2^d-1-d+j}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}]) \neq 0$.*

The next corollary shows that Corollary 4.4 covers all but possibly the last strand in the minimal free resolution of $\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}]$.

Corollary 4.5. *Let Δ be a $(d-1)$-dimensional simplicial complex. Then*

$$\operatorname{reg}(\mathbb{K}[\Delta^{\langle r \rangle}]) = \begin{cases} d-1, & \text{if } \widetilde{H}_{d-1}(\Delta; \mathbb{K}) = 0 \text{ and } r \geq d, \\ d, & \text{if } \widetilde{H}_{d-1}(\Delta; \mathbb{K}) \neq 0. \end{cases}$$

Moreover, for $1 \leq r \leq d-1$ one has $\operatorname{reg}(\mathbb{K}[\Delta^{\langle r \rangle}]) \geq \max(\operatorname{reg}(\mathbb{K}[\Delta]), r-1)$.

Proof. Let $r \geq d$ and let v be an interior vertex of the rth edgewise subdivision of a $(d-1)$-simplex $F \in \Delta$. Then $\operatorname{lk}_{\Delta^{\langle r \rangle}}(v) = \operatorname{lk}_{(2^F)^{\langle r \rangle}}(v)$, and it follows from Corollary 4.3 that $\operatorname{lk}_{\Delta^{\langle r \rangle}}(v)$ is abstractly isomorphic to the barycentric subdivision of the boundary of a $(d-1)$-simplex. Let V be the vertex set of $\operatorname{lk}_{\Delta^{\langle r \rangle}}(v)$. Then $\operatorname{lk}_{\Delta^{\langle r \rangle}}(v) = (\Delta^{\langle r \rangle})_V$. Using Hochster's formula (3.1) we conclude that $\operatorname{reg}(\mathbb{K}[\Delta^{\langle r \rangle}]) \geq d-1$. If $\widetilde{H}_{d-1}(\Delta; \mathbb{K}) = 0$, then we also have $\widetilde{H}_{d-1}(\Delta^{\langle r \rangle}; \mathbb{K}) = 0$ and, moreover, $\widetilde{H}_{d-1}((\Delta^{\langle r \rangle})_W; \mathbb{K}) = 0$ for any subset W of the vertices of $\Delta^{\langle r \rangle}$. Hence, Hochster's formula (3.1) implies that $\operatorname{reg}(\mathbb{K}[\Delta^{\langle r \rangle}]) \leq d-1$, which shows the first part.

If $\widetilde{H}_{d-1}(\Delta; \mathbb{K}) \neq 0$, then we also have $\widetilde{H}_{d-1}(\Delta^{\langle r \rangle}; \mathbb{K}) \neq 0$ and by an application of Hochster's formula (3.1) we infer that $\operatorname{reg}(\mathbb{K}[\Delta^{\langle r \rangle}]) \geq d$. On the other hand, the regularity of $\mathbb{K}[\Gamma]$ of any $(d-1)$-dimensional simplicial complex Γ cannot exceed d and the claim follows.

For the last part, we first show $\operatorname{reg}(\mathbb{K}[\Delta^{\langle r \rangle}]) \geq \operatorname{reg}(\mathbb{K}[\Delta])$. Let $\operatorname{reg}(\mathbb{K}[\Delta]) = s$. By Hochster's formula (3.1) there exists a subset W of the vertex set of Δ such that $\widetilde{H}_{s-1}(\Delta_W; \mathbb{K}) \neq 0$. This implies that $\widetilde{H}_{s-1}((\Delta_W)^{\langle r \rangle}; \mathbb{K}) \neq 0$. Let W_r be the vertex set of $(\Delta_W)^{\langle r \rangle}$. Since $(\Delta^{\langle r \rangle})_{W_r}$ is a deformation retract of $(\Delta_W)^{\langle r \rangle}$, it follows that $\widetilde{H}_{s-1}((\Delta^{\langle r \rangle})_{W_r}; \mathbb{K}) \neq 0$, and by Hochster's formula (3.1) we conclude that $\operatorname{reg}(\mathbb{K}[\Delta^{\langle r \rangle}]) \geq \operatorname{reg}(\mathbb{K}[\Delta])$. To see the other inequality, consider an $(r-1)$-dimensional face F of Δ. Let V_F be the vertex set of $(\partial(2^F))^{\langle r \rangle}$. Since $(2^F)^{\langle r \rangle}$ has an interior vertex, we infer $(\Delta^{\langle r \rangle})_{V_F} = (\partial(F))^{\langle r \rangle}$ and hence $\widetilde{H}_{r-2}((\Delta^{\langle r \rangle})_{V_F}; \mathbb{K}) \neq 0$. The claim follows from Hochster's formula (3.1). $\qquad \square$

The above corollary in particular shows that the regularity can only increase under arbitrary edgewise subdivision.

The upper bounds for the non-vanishing of the Betti numbers in Corollary 4.4 can be improved further. Indeed, it can be shown that the strands in the Betti diagram of the rth edgewise subdivision of $\Delta_{d-1}^{\langle r \rangle}$ run up to the projective dimension. To provide these results we need to better understand the topology of edgewise subdivisions.

Lemma 4.6. *Let Δ be a simplicial complex on ground set Ω such that $|\Delta|$ is a regular triangulation of a $(d-1)$-ball and let $F \in \Delta$. If $\partial|F| = |F| \cap \partial|\Delta|$, then there are deformation retractions from $|\Delta| \setminus |F|$ to $|\Delta_{\Omega \setminus F}|$ to $|\text{lk}_\Delta(F)|$.*

Proof. By [4, Lem. 4.7.27] or [24, Lem. 70.1] $|\Delta_{\Omega \setminus F}|$ is a deformation retract of $|\Delta| \setminus |F|$. Since $|\Delta|$ is a regular triangulation, we know that $|\text{star}_\Delta(F)|$ is convex. Let p be an interior point of $|F|$. Consider the map that sends a point $q \in |\Delta_{\Omega \setminus F}|$ to the intersection of $\partial|\text{star}_\Delta(F)|$ and the line segment through p and q. Note that this intersection is well-defined since $|\text{star}_\Delta(F)|$ is convex. The image of this map is $|\text{star}_\Delta(F)| \setminus |F|$. Thus, the map defines a deformation retract between $|\Delta_{\Omega \setminus F}|$ and $\partial|\text{star}_\Delta(F)| \setminus |F|$. Let Γ be the simplicial complex whose geometric realization is $\partial|\text{star}_\Delta(F)|$. Another application of [4, Lem. 4.7.27] or [24, Lem. 70.1] shows that $|\Gamma_{\Omega \setminus F}| = |\text{lk}_\Delta(F)|$ is a deformation retract of $|\Gamma| \setminus |F|$. □

Note that by definition $\partial|F| = \emptyset$ if F is a 0-dimensional face.

Lemma 4.7. *Let Δ be a simplicial complex on ground set Ω such that $|\Delta|$ is a regular triangulation of a $(d-1)$-ball and let $F \in \Delta$ such that $\partial|F| \subseteq \partial|\Delta|$. Let $B \subseteq \Omega \setminus F$ with $\text{lk}_\Delta(F) \subseteq 2^B$. Then $\widetilde{H}_{d-1-\#F}(|\Delta_B|; \mathbb{K}) \neq 0$.*

Proof. Let A be the vertex set of $\text{lk}_\Delta(F)$. Since Δ is a regular triangulation of a ball, $|\text{star}_\Delta(F)|$ is convex and hence the points from A are in convex position. For a face G of Δ_A we have $G \in \text{star}_\Delta(F) = 2^F * \text{lk}_\Delta(F)$. Hence $G \in \text{lk}_\Delta(F)$ and $\Delta_A = \text{lk}_\Delta(F)$. Thus in this case the assertion follows from the fact that the assumptions imply that $\widetilde{H}_{d-1-\#F}(\text{lk}_\Delta(F); \mathbb{K}) = \mathbb{K}$. We know from Lemma 4.6 that $|\Delta_A| = |\text{lk}_\Delta(F)|$ is a deformation retract of $|\Delta_{\Omega \setminus F}|$. Thus the inclusion $\text{lk}_\Delta(F) \hookrightarrow \Delta_{\Omega \setminus F}$ induces a map in homology that sends the generator of $\widetilde{H}_{d-1-\#F}(\text{lk}_\Delta(F); \mathbb{K}) = \mathbb{K}$ identically to the generator of $\widetilde{H}_{d-1-\#F}(\Delta_{\Omega \setminus F}; \mathbb{K}) = \mathbb{K}$. In particular, if we choose a $(d-1-\#F)$-cycle σ representing the homology class $\widetilde{H}_{d-1-\#F}(\text{lk}_\Delta(F); \mathbb{K}) = \mathbb{K}$, then σ also represents the homology class $\widetilde{H}_{d-1-\#F}(\Delta_{\Omega \setminus F}; \mathbb{K}) = \mathbb{K}$. In particular, σ is not a boundary in $\Delta_{\Omega \setminus F}$. Let $A \subseteq B \subseteq \Omega \setminus F$. Then B supports σ and by $\Delta_B \subseteq \Delta_{\Omega \setminus F}$ it cannot be a boundary in Δ_B. In particular, we have $\widetilde{H}_{d-1-\#F}(|\Delta_B|; \mathbb{K}) \neq 0$. □

We can now use the previous two lemmas to derive the main result of this section. In order to apply these lemmas we need to construct faces $F \in \Delta_{d-1}^{\langle r \rangle}$ satisfying the property that $\partial|F| = |F| \cap \partial|\Delta|$.

Lemma 4.8. *Let $r \geq d$. For $0 \leq j \leq d-2$, let $v^{(j)} \in \Omega_{r-1, d-j-1}$ such that all coordinates are greater than 0. Let*

$$F_j := \{(\underbrace{1, 0, \ldots, 0}_{j-1}, v^{(j)}), (\underbrace{0, 1, 0, \ldots, 0}_{j-1}, v^{(j)}), \ldots, (\underbrace{0, \ldots, 0, 1}_{j-1}, v^{(j)})\}.$$

Then $F_j \in \Delta_{d-1}^{\langle r \rangle}$ and $\partial|F_j| = |F_j| \cap \partial|\Delta_{d-1}^{\langle r \rangle}|$.

Proof. Since $v^{(j)} \in \Omega_{r-1,d-j-1}$ it follows that $F_j \in \Delta_{d-1}^{\langle r \rangle}$. Moreover, for any j-tuple of vertices there exists $1 \le k \le d-j-1$ such that their k^{th} coordinates all equal 0. Hence they lie on the hyperplane $x_k = 0$ and therefore on a facet of $\Delta_{d-1}^{\langle r \rangle}$. This shows $\partial |F_j| \subseteq \partial |\Delta|$. Moreover any point in the interior of $|F_j|$ lies in the interior of $|\Delta_{d-1}^{(r)}|$ since $v^{(j)}$ has only non-zero coordinates. $\square$

Now we state the main result of the section, which improves the bounds provided in Corollary 4.4.

Theorem 4.9. *Let $r \ge d$. Then:*

  (i) *If $1 \le j \le \frac{d}{2}$, then*

$$\beta_{i,i+j}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}]) \begin{cases} = 0 \text{ for } 0 \le i \le j-1, \\ \ne 0 \text{ for } j \le i \le \text{pdim}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}]). \end{cases}$$

  (ii) *If $\frac{d}{2} < j \le d-2$, then*

$$\beta_{i,i+j}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}]) \begin{cases} = 0 \text{ for } 0 \le i \le j-1, \\ \ne 0 \text{ for } m_j \le j \le \text{pdim}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}]). \end{cases}$$

  (iii) *For $j = d-1$ we have*

$$\beta_{i,i+j}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}]) \ne 0 \text{ for } 2^d - 1 - d \le i \le \text{pdim}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}]).$$

Proof. The statements concerning the vanishing of Betti numbers in (i) and (ii) follow from the same arguments as the corresponding statements in the proof of Proposition 3.7 since $\Delta_{d-1}^{\langle r \rangle}$ is a flag complex.

(i) By Corollary 4.4 it suffices to show that $\beta_{i,i+j}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}]) \ne 0$ for $2^d - d - 1 - m_{d-1-j} < i \le \text{pdim}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}]) = \#\Omega_{r,d} - d$. Let F_j be as in Lemma 4.8 and let A_j be the vertex set of $\text{lk}_{\Delta_{d-1}(r)}(F_j)$. It follows from Lemma 4.7 that

$$\widetilde{H}_{d-1-j-1}(\Delta_{d-1}^{\langle r \rangle})_{B_j}; \mathbb{K}) \ne 0$$

for $A_j \subseteq B_j \subseteq \Omega_{r,d} \setminus F_j$. By Hochster's formula (3.1) we conclude that

$$\beta_{i,i+d-j-1}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}]) \ne 0$$

for $\#A_j - (d-j-1) \le i \le \#\Omega_{r,d} - (j+1) - (d-j-1) = \text{pdim}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}])$. By Proposition 4.2 we know that $\text{lk}_{\Delta_{d-1}(r)}(F_j)$ is abstractly isomorphic to the boundary of the barycentric subdivision of a $(d-1-j)$-simplex. Hence $\#A_j = 2^{d-j} - 2$. It remains to show that $2^{d-j} + j \le 2^d - m_{d-1-j}$ or equivalently $m_{d-1-j} \le 2^d - 2^{d-j} - j$. From (3.10) we can infer that $m_{d-1-j} \le 2^{d-j} + j - d - 1$ if $1 \le j \le \frac{d}{2}$. Since it can be easily verified that $2^{d-j} + j - d - 1 \le 2^d - 2^{d-j} - j$ for $1 \le j \le \frac{d}{2}$, the claim follows. The proofs of (ii) and (iii) use similar reasoning. $\square$

4.2. Asymptotic behavior of Betti numbers for edgewise subdivisions.

The focus in this section lies on the study of Betti numbers of edgewise subdivisions of arbitrary simplicial complexes; i.e., we do not restrict our attention to simplices anymore. Similarly as we did for iterated barycentric subdivisions, as r grows, we want to determine the relative amount of non-zero Betti numbers $\beta_{i,i+j}(\mathbb{K}[\Delta^{\langle r \rangle}])$,

for j fixed, compared to the projective dimension of $\mathbb{K}[\Delta^{\langle r \rangle}]$. More precisely, given $1 \leq j \leq d-1$, we study the quantity

$$\frac{\#\{i \ : \ \beta_{i,i+j}(\mathbb{K}[\Delta^{\langle r \rangle}]) \neq 0\}}{\mathrm{pdim}(\mathbb{K}[\Delta^{\langle r \rangle}])}.$$

Our main result in this section, paralleling Theorem 3.14, is the following:

Theorem 4.10. *Let $d - 1 \geq 1$ and let Δ be a $(d-1)$-dimensional simplicial complex and let $r \geq 2d$ be a positive integer. Then, $\beta_{i,i+j}(\mathbb{K}[\Delta^{\langle r \rangle}]) \neq 0$ in the following cases:*

   (i) $1 \leq j \leq \frac{d}{2}$ *and* $j \leq i \leq \binom{2d-1}{d-1} - d + \mathrm{pdim}(\mathbb{K}[\Delta^{\langle r \rangle}]) + \mathrm{depth}(\mathbb{K}[\Delta]) - \binom{3d-1}{d-1}$.

   (ii) $\frac{d}{2} \leq j \leq d - 2$ *and* $m_j \leq i \leq \binom{2d-1}{d-1} - d + \mathrm{pdim}(\mathbb{K}[\Delta^{\langle r \rangle}]) + \mathrm{depth}(\mathbb{K}[\Delta]) - \binom{3d-1}{d-1}$.

   (iii) $j = d - 1$ *and* $2^d - d - 1 \leq i \leq \binom{2d-1}{d-1} - d + \mathrm{pdim}(\mathbb{K}[\Delta^{\langle r \rangle}]) + \mathrm{depth}(\mathbb{K}[\Delta]) - \binom{3d-1}{d-1}$.

The proof of this theorem follows a similar strategy as the proof of Theorem 3.14. The main idea is to isolate a subcomplex Γ that is isomorphic to the dth edgewise subdivision of a $(d-1)$-simplex from the rest of the complex $\Delta^{\langle r \rangle}$. Then we apply Theorem 4.9 to Γ. Eventually, one uses that any homologically non-trivial induced subcomplex of $\Gamma^{\langle r \rangle}$ gives rise to homologically non-trivial induced subcomplexes of $\Delta^{\langle r \rangle}$ when adding vertices not connected to $\Gamma^{\langle r \rangle}$. We make this more precise in the following proof.

Proof. Since Δ is of dimension $d - 1$, we can choose a $(d-1)$-dimensional face F of Δ. If $r \geq 2d$, there exist subcomplexes of $(2^F)^{\langle r \rangle}$ that are isomorphic to $\Delta_{d-1}^{\langle 2d \rangle}$. Let $\widetilde{\Delta}$ be one of those subcomplexes. Note that there exists more than one such subcomplex as soon as $r > 2d$. In $\widetilde{\Delta}$ there is another subcomplex Γ that is isomorphic to $\Delta_{d-1}^{\langle d \rangle}$ and that completely lies in the interior of $\widetilde{\Delta}$. Just peel off the "outer" layers of $\widetilde{\Delta}$. In particular, there does not exist any edge in $\Delta^{\langle r \rangle}$ that connects some vertex in Γ to some vertex in $\Delta^{\langle r \rangle} \setminus \widetilde{\Delta}$. Let V_Γ, $V_{\widetilde{\Delta}}$ and V_Δ denote the vertex sets of Γ, $\widetilde{\Delta}$ and $\Delta^{\langle r \rangle}$, respectively. By construction, we have $\Gamma = (\Delta^{\langle r \rangle})_{V_\Gamma}$ and thus Γ is an induced subcomplex of $\Delta^{\langle r \rangle}$. Arguing as in the proof of Theorem 3.14 one shows that

$$\widetilde{H}_j(\Delta_{A \cup B}^{\langle r \rangle}; \mathbb{K}) \neq 0$$

for any $A \subseteq V_\Gamma$ with $\widetilde{H}_j(\Gamma_A; \mathbb{K}) \neq 0$ and any $B \subseteq V_\Delta \setminus V_{\widetilde{\Delta}}$. Let $l_\Gamma(j)$ and $u_\Gamma(j)$ denote the beginning and the end of the jth strand of the resolution of $\mathbb{K}[\Gamma]$. Using Hochster's formula (3.1) we conclude that $\beta_{i,i+j}(\mathbb{K}[\Delta^{\langle r \rangle}]) \neq 0$ for $l_\Gamma(j) \leq i \leq u_\Gamma(j) + \#V_\Delta - \#V_{\widetilde{\Delta}}$. Let $1 \leq j \leq \frac{d}{2}$. In this case, Theorem 3.14(i) implies that $\beta_{i,i+j}(\mathbb{K}[\Delta^{\langle r \rangle}]) \neq 0$ if

$$j \leq i \leq \mathrm{pdim}(\mathbb{K}[\Delta_{d-1}^{\langle d \rangle}]) + \mathrm{pdim}(\mathbb{K}[\Delta^{\langle r \rangle}]) + \mathrm{depth}(\mathbb{K}[\Delta^{\langle r \rangle}]) - \#V_{\widetilde{\Delta}}$$

$$= \binom{2d-1}{d-1} - d + \mathrm{pdim}(\mathbb{K}[\Delta^{\langle r \rangle}]) + \mathrm{depth}(\mathbb{K}[\Delta]) - \binom{3d-1}{d-1},$$

where we use that $\Delta_{d-1}^{\langle r \rangle}$ has $\binom{d+r-1}{d-1}$ many vertices and that the depth is invariant under taking edgewise subdivisions (see Table 1). This shows (i). We omit the

proofs of (ii) and (iii) since they follow from similar reasoning, using parts (ii) and
(iii) of Corollary 4.4.   □

As an immediate consequence of Theorem 4.10 and of the fact that
$\lim_{r \to \infty} \mathrm{pdim}(\mathbb{K}[\Delta^{\langle r \rangle}]) = \infty$ we obtain:

Corollary 4.11. *Let $d - 1 \geq 1$ and let Δ be a $(d-1)$-dimensional simplicial
complex. For $1 \leq j \leq d - 1$ one has that $\#\{i : \beta_{i,i+j}(\mathbb{K}[\Delta^{\langle r \rangle}]) = 0\}$ is bounded
above in terms of j and d. In particular*

$$\lim_{r \to \infty} \frac{\#\{i \ : \ \beta_{i,i+j}(\mathbb{K}[\Delta^{\langle r \rangle}]) \neq 0\}}{\mathrm{pdim}(\mathbb{K}[\Delta^{\langle r \rangle}])} = 1$$

for every $j = 1, \ldots, d - 1$.

As for iterated barycentric subdivision, the above theorem does not cover the
last strand of the resolution of the edgewise subdivision in case the regularity is d or
equivalently when the simplicial complex has homology in top-dimension. It turns
out that in this setting the asymptotic behavior of the last strand of the resolution
depends on the simplicial complex. We now recall a special case of Theorem 5.1
from [15].

Proposition 4.12. *Let Δ be a $(d-1)$-dimensional simplicial complex. Then, as
$r \to \infty$,*

$$\frac{f_0^{\Delta^{\langle r \rangle}}}{r^{d-1}} \quad \to \quad \frac{f_{d-1}^{\Delta}}{(d-1)!}.$$

A more simple fact paralleling Lemma 3.18 can be easily deduced from the f-
vector transformation in edgewise subdivisions (see [7, Prop. 4.2]).

Lemma 4.13. *Let Δ, Δ' be two $(d-1)$-dimensional simplicial complexes such that
for some $0 \leq i \leq d - 1$ we have $f_i^{\Delta} > f_i^{\Delta'}$ and $f_j^{\Delta} = f_j^{\Delta'}$ for $i < j \leq d - 1$. Then
there exists R such that for $r \geq R$ we have*

$$f_j^{\Delta^{\langle r \rangle}} > f_j^{\Delta'^{\langle r \rangle}} \ \text{for } 0 \leq j \leq i,$$
$$f_j^{\Delta^{\langle r \rangle}} = f_j^{\Delta'^{\langle r \rangle}} \ \text{for } i < j \leq d - 1.$$

These results are crucial for the next proposition, which treats the asymptotics
of the last strand of the resolution of $\mathbb{K}[\Delta^{\langle r \rangle}]$ if the $(d-1)$-dimensional simplicial
complex Δ has homology in top-dimension, i.e., if $\mathrm{reg}(\mathbb{K}[\Delta^{\langle r \rangle}]) = d$.

Proposition 4.14. *Let $d - 1 \geq 1$ and let Δ be a $(d-1)$-dimensional simplicial
complex such that $\widetilde{H}_{d-1}(\Delta; \mathbb{K}) \neq 0$. Further let σ be a minimal homology $(d-1)$-
cycle of Δ and let*

$$\widetilde{\sigma} = \{F \in \Delta \ : \ F \subseteq G \text{ for some } G \text{ in the support of } \sigma\}$$

be the corresponding induced subcomplex of Δ, whose vertex set is V_r^{σ}. Then:

   (i) *For $r \geq 1$, $\beta_{i,i+d}(\mathbb{K}[\Delta^{\langle r \rangle}]) \neq 0$ for $\#V_r^{\sigma} - d \leq i \leq \mathrm{pdim}(\mathbb{K}[\Delta^{\langle r \rangle}])$. If r is
       large, then in addition $\beta_{i,i+d}(\mathbb{K}[\Delta^{\langle r \rangle}]) = 0$ for $0 \leq i < \#V_r^{\sigma} - d$.*
   (ii)

$$\lim_{r \to \infty} \frac{\#\{i \ : \ \beta_{i,i+d}(\mathbb{K}[\Delta^{\langle r \rangle}]) \neq 0\}}{\mathrm{pdim}\mathbb{K}[\Delta^{\langle r \rangle}]} = 1 - \frac{f_{d-1}^{\widetilde{\sigma}}}{f_{d-1}^{\Delta}}.$$

Proof. Since $(\Delta^{\langle r \rangle})_{V_r^\sigma} = \widetilde{\sigma}^{\langle r \rangle}$, and since σ is a homology $(d-1)$-cycle, it holds that $\widetilde{H}_{d-1}((\Delta^{\langle r \rangle})_{V_r^\sigma}; \mathbb{K}) \neq 0$. Thus, using Hochster's formula (3.1), we infer that $\beta_{i,i+d}(\mathbb{K}[\Delta^{\langle r \rangle}]) \neq 0$ for $i = \#V_r^\sigma - d$. By the same reasoning as in the proof of Proposition 3.19 we can further conclude that $\beta_{i,i+d}(\mathbb{K}[\Delta^{\langle r \rangle}]) \neq 0$ for $\#V_r^\sigma - d \leq i \leq \mathrm{pdim}(\mathbb{K}[\Delta^{\langle r \rangle}])$, which shows the non-vanishing in (i). For the vanishing we use the minimality of σ and Lemma 4.13 in the same way as in the proof of Proposition 3.19.

We now prove (ii). Let V_r^Δ denote the vertex set of $\Delta^{\langle r \rangle}$. It follows from (i) that

$$\frac{1}{\mathrm{pdim}(\mathbb{K}[\Delta^{\langle r \rangle}])} \#\{i \ : \ \beta_{i,i+d}(\mathbb{K}[\Delta^{\langle r \rangle}]) \neq 0\}$$

$$= \frac{1}{\mathrm{pdim}(\mathbb{K}[\Delta^{\langle r \rangle}])}(\mathrm{pdim}(\mathbb{K}[\Delta^{\langle r \rangle}]) - (\#V_r^\sigma - d - 1))$$

$$= 1 - \frac{\#V_r^\sigma - d - 1}{\#V_r^\Delta - \mathrm{depth}(\mathbb{K}[\Delta^{\langle r \rangle}])}$$

$$= 1 - \frac{\frac{1}{r^{d-1}}(\#V_r^\sigma - d - 1)}{\frac{1}{r^{d-1}}(\#V_r^\Delta - \mathrm{depth}(\mathbb{K}[\Delta]))}$$

$$= 1 - \frac{\frac{1}{r^{d-1}}(f_0^{\widetilde{\sigma}^{\langle r \rangle}} - d - 1)}{\frac{1}{r^{d-1}}(f_0^{\Delta^{\langle r \rangle}} - \mathrm{depth}(\mathbb{K}[\Delta]))}.$$

As r goes to infinity, Proposition 4.12 implies that $\frac{\#\{i \ : \ \beta_{i,i+d}(\mathbb{K}[\Delta^{\langle r \rangle}]) \neq 0\}}{\mathrm{pdim}\mathbb{K}[\Delta^{\langle r \rangle}]}$ approaches $\frac{f_{d-1}^{\widetilde{\sigma}}}{f_{d-1}^\Delta}$. $\square$

Since the limits in Proposition 3.19 and Proposition 4.14 coincide, Example 3.20 shows that for any d any rational number in the half-open interval $[0, 1)$ can occur as a limit in Proposition 4.14.

5. APPENDIX

In this section we provide the proofs of the statements that require only manipulations of simple numerical expressions.

Proof of Lemma 3.3. We set

$$(5.1) \qquad n_j = \min\left\{\sum_{\ell=1}^r (2^{i_\ell+2} - 2) : \begin{array}{l} (i_1, \ldots, i_r) \in \mathbb{N}^r, \\ i_1 + \cdots + i_r + (r-1) = j - 1, \\ i_1 + \cdots + i_r + 2r \leq d \end{array}\right\} - j.$$

Our goal is to prove that $n_j = m_j$ where the m_j's are defined in Definition 3.1. We may write the second condition in (5.1) as

$$(5.2) \qquad\qquad\qquad i_1 + \cdots + i_r = j - r$$

and the third as

$$j + r \leq d,$$

or, equivalently,

$$(5.3) \qquad\qquad\qquad r \leq d - j.$$

Note also that (5.2) implies that $r \leq j$ since $(i_1, \ldots, i_r) \in \mathbb{N}^r$. Therefore $r \leq \min\{j, d-j\}$. Summing up, we can rewrite the definition of n_j as

$$(5.4) \quad n_j = \min_{r=1,\ldots,\min\{j,d-j\}} \min\left\{\left(\sum_{\ell=1}^{r} 2^{i_\ell+2}\right) - 2r \; : \; \begin{matrix} (i_1, \ldots, i_r) \in \mathbb{N}^r, \\ i_1 + \cdots + i_r = j - r \end{matrix}\right\} - j.$$

Observe the following:

Claim 1. Assume that $r < \min\{j, d-j\}$, $i = (i_1, \ldots, i_r) \in \mathbb{N}^r$ and $i_1 + \cdots + i_r = j - r$. Note that $j - r > 0$ and hence one of the i_k is positive, say $i_1 > 0$. Then we set $u = (u_1, \ldots, u_{r+1})$ where $u_1 = i_1 - 1$, $u_k = i_k$ for $k = 2, \ldots, r$ and $u_{r+1} = 0$. By construction,

$$u_1 + \cdots + u_{r+1} = i_1 + \cdots + i_r - 1 = j - (r+1),$$

and hence u is a "valid" vector. We want to show that the "contribution" of u is strictly smaller than that of the vector i. Hence we consider the difference of the "contributions" of the vector i and the vector u:

$$\left(\sum_{l=1}^{r} 2^{i_l+2}\right) - 2r - \left(\left(\sum_{l=1}^{r+1} 2^{u_l+2}\right) - 2(r+1)\right),$$

which is

$$2^{i_1+2} - 2r - 2^{u_1+2} - 2^{u_{r+1}+2} + 2(r+1),$$

that is,

$$2^{i_1+2} - 2^{i_1+1} - 2^2 + 2,$$

that is,

$$2^{i_1+2} - 2^{i_1+1} - 2,$$

which is clearly positive (since $i_1 > 0$).

Claim 2. Now suppose that $i_1 + \cdots + i_r = j - r$ and that there are two of the i_k whose difference is > 1, say $i_2 - i_1 > 1$. Now we define $u = (u_1, \ldots, u_r)$ by $u_1 = i_1 + 1, u_2 = i_2 - 1$ and $u_k = i_k$ for $k > 2$. We want to show that the contribution of u is smaller than the one of i. Hence we consider the difference of the contributions and we have

$$2^{i_1+2} + 2^{i_2+2} - 2^{i_1+3} - 2^{i_2+1}.$$

We want to show that it is positive. We may factor out 2^{i_1+2} and we have to show that

$$1 + 2^{i_2-i_1} - 2 - 2^{i_2-i_1-1}$$

is positive, that is,

$$2^{i_2-i_1} - 1 - 2^{i_2-i_1-1}$$

is positive. This is clearly true.

Taking Claim 1 and Claim 2 into consideration we have that the minimum in the expression of (5.4) for n_j is obtained when $r = \min\{j, d-j\}$ and for the vector $(i_1, \ldots, i_r)$ such that $i_1 \geq i_2 \geq \cdots \geq i_r$ and $i_1 - i_r \leq 1$. Now if $j \leq d - j$, i.e., $j \leq d/2$, then $r = j$ and the corresponding vector is $(i_1, \ldots, i_j) = (0, \ldots, 0)$. It follows then that $n_j = j$.

If instead $d - j \leq j$, i.e., $j \geq d/2$, then $r = d - j$ and the corresponding vector $(i_1, \ldots, i_{d-j})$ is obtained as follows. Since

$$i_1 + \cdots + i_{d-j} = j - (d - j) = 2j - d$$

and $i_1 \geq i_2 \geq \cdots \geq i_{d-j}$ and $i_1 - i_{d-j} \leq 1$ it must hold that

$$i_k = \begin{cases} a+1 & \text{for } k = 1, \ldots, c, \\ a & \text{for } k = c+1, \ldots, d-j, \end{cases}$$

where a and c are non-negative integers such that

$$(2j - d) = a(d - j) + c$$

and $0 \leq c < d - j$. Hence we have

$$n_j = \left(\sum_{\ell=1}^{d-j} 2^{i_\ell+2} \right) - 2(d-j) - j = c2^{a+3} + (d-j-c)2^{a+2} - 2d + j$$

$$= 2^{a+2}(2c + d - j - c) - 2d + j = 2^{a+2}(c + d - j) - 2d + j.$$

Hence we have shown that n_j is equal to m_j. $\qquad \square$

Proof of Lemma 3.11. We have the following chain of inequalities:

$$(2^{i_r+2} - 2)2^{i_2+\cdots+i_{r-1}+2r-4} = 2^{i_r+2} \left(1 - \frac{1}{2^{i_r+1}} \right) 2^{i_2+\cdots+i_{r-1}+2r-4}$$

$$\geq 2^{i_r+2} \left(1 - \frac{1}{2} \right) 2^{i_2+\cdots+i_{r-1}+2r-4}$$

$$= 2^{i_r+1+i_2+\cdots+i_{r-1}+2r-4}.$$

Since $i_1 = 0$ by (i), it follows from (ii) that $i_2 + \cdots + i_{r-1} + i_r + (r - 1) = j - 1$ and hence

$$(2^{i_r+2} - 2)2^{i_2+\cdots+i_{r-1}+2r-4} \geq 2^{j+r-3}.$$

For $r \geq 4$ the claim follows. We need to treat the cases $r \in \{1, 2, 3\}$ separately.

Case 1: $r = 1$. By (ii) it then follows that $j = 1$ and hence $d < 2$, which is a contradiction to the assumptions.

Case 2: $r = 2$. It follows from (ii) that $i_2 = j - 2$ and hence

$$(2^{i_r+2} - 2)2^{i_2+\cdots+i_{r-1}+2r-4} + \sum_{l=1}^{r}(2^{i_l+2} - 2)$$

$$= (2^j - 2) + (2^{0+2} - 2) + (2^j - 2)$$

$$= 2 \cdot 2^j - 2 = 2^{j+1} - 2,$$

which shows the claim.

Case 3: $r = 3$. By (i) and (ii) it holds that $i_2 + i_3 = j - 3$. From this, we obtain

$$(2^{i_r+2} - 2)2^{i_2+\cdots+i_{r-1}+2r-4} + \sum_{l=1}^{r}(2^{i_l+2} - 2)$$

$$= (2^{i_3+2} - 2)2^{i_2+2} + (2^{0+2} - 2) + (2^{i_2+2} - 2) + (2^{i_3+2} - 2)$$

$$= 2^{i_2+i_3+4} - 2^{i_2+3} + 2^{i_2+2} + 2^{i_3+2} - 2$$

$$= 2^{j+1} - 2^{i_2+2} + 2^{i_3+2} - 2$$

$$\geq 2^{j+1} - 2,$$

where for the last inequality we use $i_2 \leq i_3$. $\qquad \square$

Proof of Lemma 3.12. Since

$$\sum_{\ell=1}^{r}(2^{i_\ell+2}-2) - \sum_{\ell=1}^{r}(2^{j_\ell+2}-2) = 2^{i_1+2} + 2^{i_2+2} - 2^{j_1+2} - 2^{j_2+2}$$
$$= 2^{i_1+2} + 2^{i_2+2} - 2^{i_1+1} - 2^{i_2+3} = 2^{i_1+1} - 2^{i_2+2},$$

we need to show that

(5.5) $$(2^{i_r+2}-2)2^{i_2+\cdots+i_{r-1}+2r-4} + 2^{i_1+1} - 2^{i_2+2} \geq 0.$$

We have

$$(2^{i_r+2}-2)2^{i_2+\cdots+i_{r-1}+2r-4} = 2^{i_r+2}\left(1 - \frac{1}{2^{i_r+1}}\right)2^{i_2+\cdots+i_{r-1}+2r-4}$$
$$\geq 2^{i_r+2}\left(1 - \frac{1}{2}\right)2^{i_2+\cdots+i_{r-1}+2r-4}$$
$$= 2^{i_r+1}2^{i_2+\cdots+i_{r-1}+2r-4} = 2^{i_2+\cdots+i_{r-1}+i_r+2r-3}.$$

For $r \geq 3$ the last expression is $\geq 2^{i_2+i_r+6-3} \geq 2^{i_2+2}$, from which follows (5.5) in this case. If $r = 2$, we obtain

$$(2^{i_r+2}-2)2^{i_2+\cdots+i_{r-1}+2r-4} = 2^{i_2+2} - 2.$$

Hence, (5.5) also holds in this case.   □

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