KNOTS AND LINKS OF COMPLEX TANGENTS

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ABSTRACT. It is shown that every knot or link is the set of complex tangents of a 3-sphere smoothly embedded in the 3-dimensional complex space. We show in fact that a 1-dimensional submanifold of a closed orientable 3-manifold can be realised as the set of complex tangents of a smooth embedding of the 3-manifold into the 3-dimensional complex space if and only if it represents the trivial integral homology class in the 3-manifold. The proof involves a new application of singularity theory of differentiable maps.

1. INTRODUCTION

An immersion f of a C^∞-smooth manifold M into the complex space $\mathbb{C}^n$ is said to be *totally real* if $df_x(T_xM) \cap J(df_x(T_xM)) = \{0\}$ for each point $x \in M$ and the complex structure J. If, on the contrary, $df_x(T_xM)$ contains a complex line, such a point x is said to be *a complex tangent*. Totally real immersions and embeddings have long been important topics in differential geometry (see e.g. [1, 3, 12, 14]). The behaviour of complex tangents is also apparently interesting and has been extensively studied (see e.g. [4, 6, 7, 9, 13, 17, 29]).

In this paper we show that a 1-dimensional submanifold L of a closed orientable 3-manifold M^3 can be realised as the set of complex tangents of a C^∞-smooth embedding M^3 into $\mathbb{C}^3$ if and only if the homology class $[L]$ vanishes in $H_1(M^3; \mathbb{Z})$. Ali M. Elgindi has obtained, in his pioneering paper [7], a similar result mainly for a knot in the 3-sphere S^3, namely, in the case where L is a single circle and $M^3 = S^3$, in which, however, the embedding of S^3 into $\mathbb{C}^3$ ought to have a degenerate point and cannot be taken to be C^∞-smooth (see also [9]). In his argument, Akbulut and King's result [2] has played a crucial role to relate the two seemingly unrelated objects — geometry of complex tangents and topology of knots. Our approach is quite different; we employ instead Saeki's theorem on singularities of stable maps in the spirit of differential topology. This enables us to avoid dealing with the degeneracy and to study knots and links in a general orientable 3-manifold.

A stable map between manifolds, which we will define later in terms of the conditions of local forms, is a C^∞-smooth map which differs from neighbouring maps in the mapping space only by diffeomorphisms of the source and target manifolds. The notion of a stable map can be naturally regarded as a high-dimensional variant of a Morse function and has attracted attention as a tool to analyse the topology

of a manifold. We especially focus on *liftable* stable maps from 3-manifolds to the plane, that is, those stable maps which can factor through immersions into $\mathbb{R}^4$, and reveal that stable maps are useful to study the geometry of a real submanifold in a complex space.

Our proof is not complicated; it contains two main ideas: a refinement of Saeki's theorem claiming that any integrally null-homologous link is the singular set of a liftable stable map to the plane (Theorem 7.1) and a gimmick to lift the stable map into an immersion in $\mathbb{C}^3$ whose complex tangents form the given link (Theorem 6.3). At the final step, with the aid of a totally real version of Whitney's trick (Gromov [14] and Forstnerič [12]), we eliminate double points of the immersion so as to obtain the main theorem (Theorem 8.1).

Our argument focusing on the liftability goes back to an interesting corollary on stable maps (Corollary 6.5). Namely, we show that the singular set of a liftable stable map from a closed orientable 3-manifold M^3 to the plane represents the trivial integral homology class in $H_1(M^3;\mathbb{Z})$. This can be regarded as a refined version of the well-known Thom polynomial [28] stating that the singular set of a stable map from a closed orientable 3-manifold to the plane represents the trivial $\mathbb{Z}/2\mathbb{Z}$-coefficient homology class.

In what follows, the term "C^∞-smooth" will be referred to simply as "smooth". All manifolds and maps between manifolds shall be supposed to be smooth, unless otherwise stated.

2. COMPLEX TANGENTS

Let $f\colon M^k \to \mathbb{C}^n$ be a smooth immersion. As mentioned in §1, a point $x \in M^k$ is said to be *a complex tangent* if $df_x(T_x M^k)$ contains a complex line, that is,

$$df_x(T_x M^k) \cap J(df_x(T_x M^k)) \neq \{0\}$$

holds. If f has no complex tangents it is said to be *totally real*.

We deal mainly with embeddings of closed orientable 3-manifolds into $\mathbb{C}^3$. According to Lai [17, Theorem 2.3] (see also [29, Proposition (2.1)] and [7, Proposition 3]), for a smooth generic immersion of M^3 into $\mathbb{C}^3$, the set of complex tangents is empty or forms a codimension two submanifold of M^3.

On the other hand, based on the h-principle due to Gromov [14], it has been shown [3, 12] that any compact orientable 3-manifold admits a totally real embedding in $\mathbb{C}^3$. More precisely, any immersion $f\colon M^3 \to \mathbb{C}^3$ of a compact orientable 3-manifold M^3 is regularly homotopic to a totally real immersion, and moreover, if f is regularly homotopic to an embedding, then it is regularly homotopic to a totally real embedding. This implies, in a sense, that the existence of complex tangents is not an obstruction to total reality. Therefore, our interest is rather in their global behaviours.

Elgindi has initiated the study of topology of complex tangents of embeddings of the 3-sphere in a series of papers [7–9]. In addition to the result mentioned in §1, he has shown in [9] that for any given knot K, an embedded circle in S^3, there exists a smooth embedding of S^3 into $\mathbb{C}^3$ with complex tangents forming a knot isotopic to K *or* a 2-component link isotopic to two unlinked copies of K. The latter case, however, cannot be excluded, and after all it seems that the argument is facing a difficult trade-off. The problems of resolving this and dealing with knots and

links in a general 3-manifold are posed at the end of [9]. We will offer satisfactory solutions to these problems in §8.

Regarding the set of complex tangents, our first observation is the following.

Theorem 2.1. *For a smooth generic immersion of a closed orientable 3-manifold M^3 into $\mathbb{C}^3$, the integral homology class represented by the set of complex tangents vanishes in $H_1(M^3; \mathbb{Z})$.*

Proof. Denote by $G_{6,3}$ the Grassmann manifold of 3-planes in $\mathbb{R}^6 = \mathbb{C}^3$. Let $G_{6,3}^{\mathrm{TR}}$ be the open subset of $G_{6,3}$ consisting of totally real 3-planes and put $W := G_{6,3} \smallsetminus G_{6,3}^{\mathrm{TR}}$, which turns out to be a codimension two closed orientable submanifold of $G_{6,3}$ (see [8, §2 and Theorem 3]).

To a smooth immersion $f \colon M^3 \to \mathbb{C}^3$, we associate the Gauss map $\Gamma_f \colon M^3 \to G_{6,3}$ defined by $\Gamma_f(p) = df(T_p M^3)$. For a generic f, the Gauss map Γ_f becomes a continuous map transverse to W, and the set of complex tangents of f is just the codimension two closed orientable submanifold $\Gamma_f^{-1}(W)$ of M^3.

As mentioned above, f is regularly homotopic to a totally real immersion [3,12]. This implies that there exists a homotopy from Γ_f to a map $M^3 \to G_{6,3}$ with image inside $G_{6,3}^{\mathrm{TR}}$. Such a homotopy determines the map

$$\widetilde{\Gamma}_f \colon M^3 \times [0,1] \to G_{6,3}$$

such that $\widetilde{\Gamma}_f$ restricted to $M^3 \times \{0\}$ coincides with Γ_f and $\widetilde{\Gamma}_f(M^3 \times \{1\}) \cap W = \emptyset$. By a small perturbation (fixed on $M^3 \times \{0\}$) if necessary, we can make $\widetilde{\Gamma}_f$ transverse to W. Then, the inverse image $\widetilde{\Gamma}_f^{-1}(W)$ gives an orientable submanifold bounded by $\Gamma_f^{-1}(W)$ in M^3, which implies that the set of complex tangents of f is null-homologous in M^3. $\square$

Remark 2.2. A similar statement in the case of homology with coefficients in $\mathbb{R}$ has been proved in [6,17].

3. Knots and links in 3-manifolds

The coherent band surgery for links in $\mathbb{R}^3$, which is equivalent to the move called *nullification* or a sort of *rational tangle surgery*, has been extensively studied particularly in relation with the study of enzyme actions on DNA (see [11] for example). We consider here the notion of coherent band surgery for links in a general 3-manifold M^3, that is, copies of circles embedded in M^3.

Definition 3.1. Let L be a 1-dimensional submanifold of n-manifold M and $b \colon I \times I \to M^n$ an embedding such that $b(I \times I) \cap L = b(I \times \partial I)$, where $n \geq 3$ and $I = [0,1]$. Then,

$$L' = (L \smallsetminus b(I \times \partial I)) \cup b(\partial I \times I)$$

is said to be the link L obtained by *the band surgery* along the band b. If L is an oriented link and L' has the orientation compatible with $L \smallsetminus b(I \times \partial I)$, the link L' is said to be obtained by *the coherent band surgery* (see Figure 1).

The following proposition has been implicitly proved in [23, Lemma 3.9].

Proposition 3.2. *Let L and L' be closed oriented 1-dimensional submanifolds of a closed n-dimensional manifold M where $n \geq 3$. Then, the homology class $[L]$ is equal to $[L']$ in $H_1(M; \mathbb{Z})$ if and only if L' is isotopic to a 1-dimensional oriented submanifold obtained from L by a finite iteration of coherent band surgeries.*
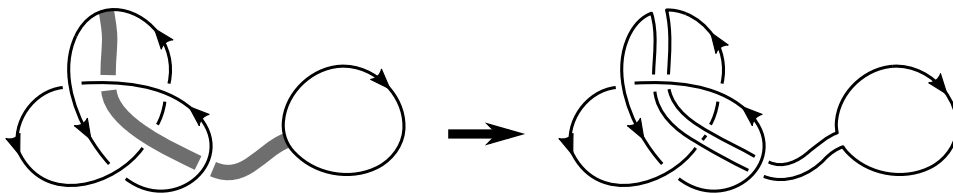
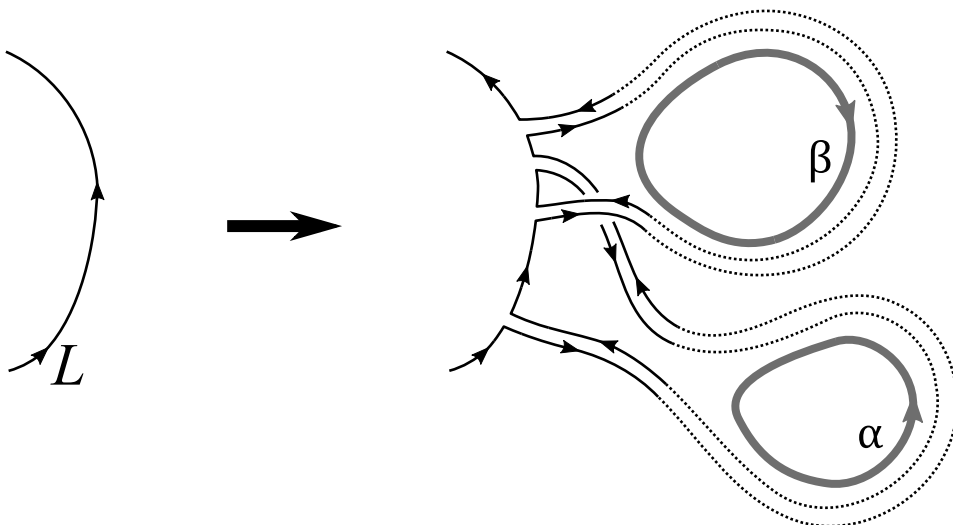FIGURE 1. The coherent band surgery

FIGURE 2. The commutator through coherent band surgeries

Proof. The proof is done essentially as in the proof of [23, Lemma 3.9].

Since the coherent band surgery clearly does not change the homology class represented by the submanifold, the necessity is obvious.

Now suppose that L and L' are integrally homologous in M. By suitable coherent band surgeries, we may assume they are both connected. Then the proof requires just the third and fourth paragraphs of the proof of [23, Lemma 3.9]. Namely, we first need to show that the action of a commutator $\alpha\beta\alpha^{-1}\beta^{-1}$, for $\alpha, \beta \in \pi_1(M^3, x)$ and a point $x \in L$, can be realised by an iteration of coherent band surgeries (see Figure 2). Thus, in view of the fact that $H_1(M; \mathbb{Z})$ can be identified with the abelianisation of $\pi_1(M^3, x)$, we may assume that L and L' represent the same class in $\pi_1(M^3, x)$; hence we then need to show that the unknotting operation (the "crossing change" up to isotopy) can be realised by an iteration of coherent band surgeries (see Figure 3). See [23, pages 1150–1151] for details. $\square$

4. STABLE MAPS FROM 3-MANIFOLDS TO THE PLANE

We introduce stable maps from 3-manifolds to the plane. Although the notion of a stable map can be defined for more general source and target manifolds, those from 3-manifolds to the plane, in particular, have been extensively studied (see
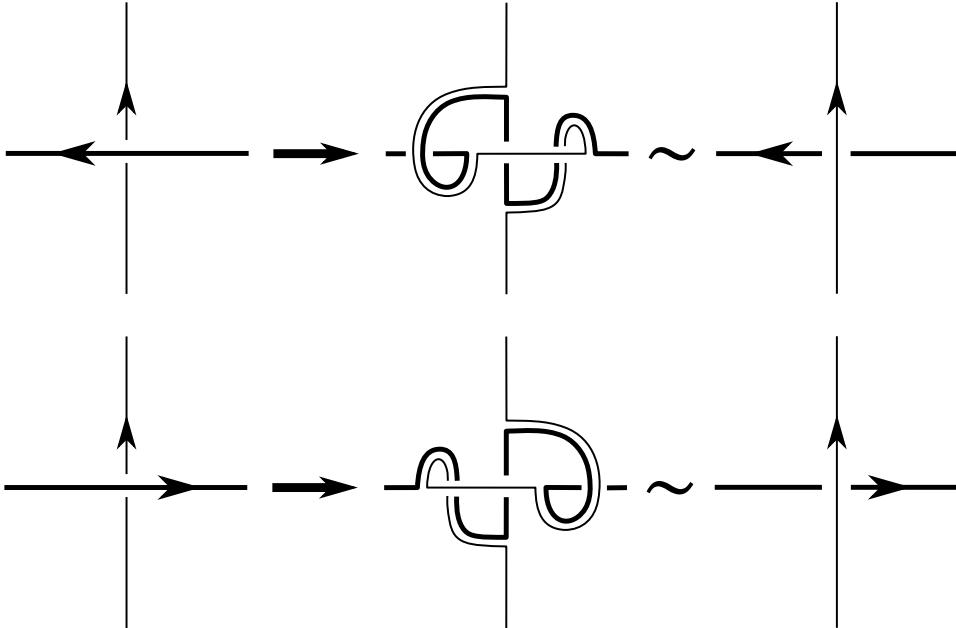
FIGURE 3. The unknotting operation through coherent band surgeries

[16, 18, 24] and [31] for example). Thus we adopt here a common definition in the dimensions (see e.g. [16] or [18, p. 6]).

For a smooth map $f\colon M \to N$ between smooth manifolds, we denote by $S(f)$ the set of singular points; that is, we put

$$S(f) = \{p \in M \,|\, \operatorname{rank} df_p \leq \min(\dim M, \dim N)\}$$

and call it *the singular set* of f.

Definition 4.1. A smooth map $f\colon M^3 \to \mathbb{R}^2$ is called *a stable map* if it satisfies the following conditions.

    I. Local conditions:

        For each point $p \in M^3$, there exist local coordinates (x, y, z) centred at p and (X, Y) centred at $f(p)$ with which f has one of the following forms:

    (L1) $(X \circ f, Y \circ f) = (x, y)$ (p is called *a regular point*),

    (L2) $(X \circ f, Y \circ f) = (x, y^2 + z^2)$ (p is called *a definite fold point*),

    (L3) $(X \circ f, Y \circ f) = (x, y^2 - z^2)$ (p is called *an indefinite fold point*),

    (L4) $(X \circ f, Y \circ f) = (x, xy + y^3 + z^2)$ (p is called *a cusp point*).

    II. Global conditions:

    (G1) For each cusp point $p \in M^3$, we have $f^{-1}(f(p)) \cap S(f) = \{p\}$.

    (G2) f restricted to $(S(f) \smallsetminus \{\text{cusp points}\})$ is an immersion with normal crossings.

Remark 4.2. The set of all stable maps $M^3 \to \mathbb{R}^2$ from a compact 3-manifold M^3 to the plane is open and dense in the mapping space $C^\infty(M^3, \mathbb{R}^2)$, the set of all smooth maps $M^3 \to \mathbb{R}^2$ endowed with the Whitney C^∞-topology [20]. Hence, every smooth map $M^3 \to \mathbb{R}^2$ can be approximated by a stable map.
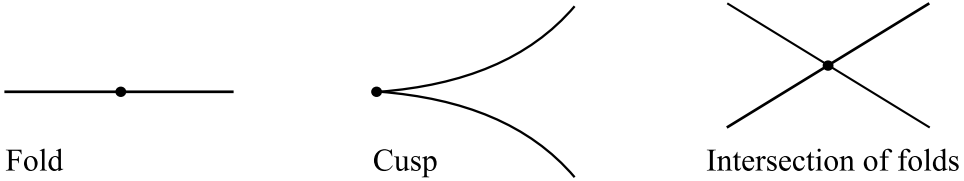
Fold   Cusp   Intersection of folds

FIGURE 4. Local images of singular points

Remark 4.3. For a stable map $f\colon M^3 \to \mathbb{R}^2$ from a compact 3-manifold M^3 to the plane, its singular set $S(f)$ forms a compact smooth 1-dimensional submanifold of M^3. It consists of smooth arcs of definite folds or indefinite folds and isolated cusp points, where definite and indefinite fold arcs meet. Figure 4 depicts the local image of the singular points.

The 2-dimensional regions divided by the lines consist of the images of regular points. The regular fibre, the inverse image of a regular point, consists of several copies of circles. Therefore, if we describe such regular fibres and how they degenerate in crossing the singular lines, we can recover the given map f locally. This is successfully done in the fundamental paper [31] due to Minoru Yamamoto, who has studied in detail stable maps from 3-manifolds to the plane and their deformations with lots of clear figures. We will often refer to the figures of his paper [31].

A generic homotopy $f_t\colon M^3 \to \mathbb{R}^2$ ($t \in [-1,1]$) between two stable maps f_0 and f_1 has been studied in [5, 26]. The germ of such a generic homotopy f_t, in suitable local coordinates (x, y, z) of M^3 and (X, Y) of $\mathbb{R}^2$, is given by one of the following:

(i) $(X \circ f_t, Y \circ f_t) = (x, y^3 + yx^2 + z^2 + yt)$ (*Lips*),
(ii) $(X \circ f_t, Y \circ f_t) = (x, y^3 - yx^2 + z^2 + yt)$ (*Beaks*),
(iii) $(X \circ f_t, Y \circ f_t) = (x, y^4 + yx \pm z^2 + y^2t)$ (*Swallowtail*);

in addition,

(iv) an *intersection of a fold and a cusp*,
(v) a *non-transversal intersection of two folds*,
(vi) an *intersection of three folds*

may occur as codimension one multigerms (see Figure 5). Each of the above homotopies passes through a non-stable map at $t = 0$, *a bifurcation point*. Note that only in (i), (ii), and (iii) do the types of singularities change through the bifurcation point. Note further that the case (iii) does not change *the set* of singular points.

5. SAEKI'S THEOREM

A striking result that every knot or link in S^3 is the singular set of a stable map from S^3 to the plane is a consequence of the following theorem due to Saeki [23, Theorem 2.2].

Theorem 5.1 (Saeki [23, Corollary 6.3]). *Let M^3 be a closed orientable 3-manifold and let L be a closed 1-dimensional submanifold of M^3. Then there exists a stable map $f\colon M^3 \to \mathbb{R}^2$ with $S(f) = L$ if and only if the $\mathbb{Z}/2\mathbb{Z}$-coefficient homology class $[L]_2$ vanishes in $H_1(M^3; \mathbb{Z}/2\mathbb{Z})$.*
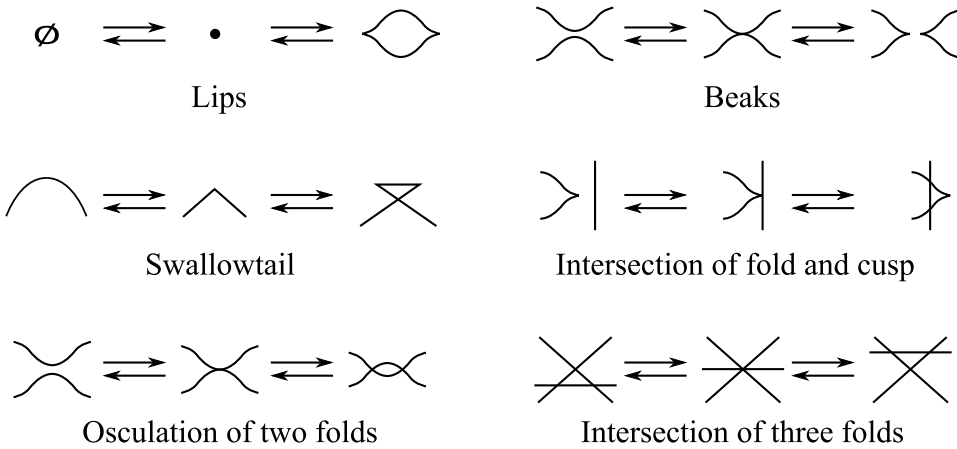
We review the outline of Saeki's proof.

FIGURE 5. Generic homotopies

His first observation is that any two 1-dimensional submanifolds (links) L_1 and L_2 in M^3 can be connected by a finite iteration of (not necessarily coherent) band surgeries if and only if $[L_1]_2 = [L_2]_2$ in $H_1(M^3; \mathbb{Z}/2\mathbb{Z})$ ([23, Lemma 3.9]; compare it with Proposition 3.2).

The second point is that the Beaks, one of the seven generic homotopies introduced in §4, affect the singular set just as the band surgery. More precisely, we can deform a given stable map $f \colon M^3 \to \mathbb{R}^2$, so that an arbitrary band surgery is performed on the singular set $S(f)$ by suitably iterating the following four types of generic homotopies (see [23, Remark 4.2]):

(H0) *Lips from left to right* in Figure 5,

(H1) *Beaks from right to left* in Figure 5,

(H2) *Swallowtail from left to right* in Figure 5, and

(H3) *Intersection of a fold and a cusp from left to right* in Figure 5.

Specifically, these homotopies are used as follows. First, if the stable map has the empty singular set, by using (H0) we can make it non-empty (note that we do not need this since a stable map to the plane necessarily has the non-empty singular set). Then, if we need to perform a band surgery in some place on the singular set, we use (H2) to yield cusps in the correct locations, which serve as "footholds" for the necessary "band", and then extend the band from one foot toward the other by a series of (H3). The Beaks (H1) in the final stretch complete the desired band surgery on the singular set.

Thus, recalling that among the above four generic homotopies only the Lips and the Beaks may possibly change the isotopy class (link type) of $S(f) \subset M^3$ (and the Lips are not used in our case), the proof is outlined as follows. *The "only if" part is nothing but the Thom polynomial: for any stable map $f \colon M^3 \to \mathbb{R}^2$ the homology class $[S(f)]_2$ represents the dual of the second Stiefel–Whitney class $w_2(M^3)$ and hence always vanishes in $H_1(M^3; \mathbb{Z}/2\mathbb{Z})$ (Thom [28]). For the "if" part, beginning with an arbitrary stable map $f_{init.} \colon M^3 \to \mathbb{R}^2$ we can modify it into $f' \colon M^3 \to \mathbb{R}^2$ through a finite iteration of (H1), (H2) and (H3) so that the singular set $S(f')$ is isotopic to any given L, as long as $[L]_2 = 0 \in H_1(M^3; \mathbb{Z}/2\mathbb{Z})$. Finally,*

by composing a suitable homeomorphism (derived from an ambient isotopy) of M^3 with f' we obtain the desired map $f\colon M^3 \to \mathbb{R}^2$ with $S(f) = L$ (see [23, p. 1155]).

Remark 5.2. From now on, we will call the above stable map $f_{init.}\colon M^3 \to \mathbb{R}^2$ *the initial stable map* in Saeki's construction.

6. LIFTABLE STABLE MAPS AND COMPLEX TANGENTS

Given a stable map the problem whether it can be lifted to an immersion (or an embedding) has been studied by many authors (the references in [25] might be convenient), including Haefliger [15] and Harold Levine [18]. Haefliger [15] has studied the lifting problem of a stable map from a surface to $\mathbb{R}^2$ and obtained a necessary and sufficient condition for being lifted to an immersion into $\mathbb{R}^3$. H. Levine [19] has studied the analogous problem of the existence (and classification [18]) of immersions into $\mathbb{R}^4$ over stable maps from 3-manifolds to $\mathbb{R}^2$.

Definition 6.1. Let $f\colon M^3 \to \mathbb{R}^2$ be a stable map from an orientable 3-manifold to $\mathbb{R}^2$. We shall say that f is *liftable* or has *an immersion lift* $\widetilde{f}$ in $\mathbb{R}^4$ if there exists an immersion $\widetilde{f}\colon M^3 \to \mathbb{R}^4$ such that $\pi \circ \widetilde{f} = f$ for the projection $\pi\colon \mathbb{R}^4 \to \mathbb{R}^2$.

Remark 6.2. Note that any closed orientable 3-manifold M^3 admits a liftable stable map to $\mathbb{R}^2$. To obtain such a stable map, we only need to compose any generic immersion $M^3 \to \mathbb{R}^4$ with a generic projection $\mathbb{R}^4 \to \mathbb{R}^2$ [21].

The following seems the first indication of a relation between liftable stable maps and complex tangents.

Theorem 6.3. *Let $g_1 = (f_1, f_2)\colon M^3 \to \mathbb{R}^2$ be a stable map of a closed orientable 3-manifold which has an immersion lift $\widetilde{g_1} = (f_1, f_2, f_3, f_4)$. Then, the map*

$$G = (f_1, f_2, f_3, f_4, f_1, -f_2)\colon M^3 \longrightarrow \mathbb{R}^6 = \mathbb{C}^3$$

defines a smooth immersion of M^3 into $\mathbb{C}^3$ the set of whose complex tangents coincides with the singular set $S(g_1)$ of g_1.

Proof. Put $g_2 := (f_3, f_4)$ and $g_3 := (f_1, -f_2)$, so that we have

$$G = (g_1, g_2, g_3)\colon M^3 \longrightarrow \mathbb{C}^3.$$

First, we show that the immersion G is totally real on $M^3 \smallsetminus S(g_1)$. Take any regular point p of g_1 and consider the differential map

$$dG_p : T_p M^3 \longrightarrow T_{G(p)}\mathbb{C}^3 = \mathbb{C}^3 = \mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \mathbb{C}_3$$

at p. Then, the image $dG_p(T_p M^3)$ contains no complex line for the following reason.

Suppose that $dG_p(T_p M^3)$ contains a complex line l_p. Then, via the projections $T_{G(p)}\mathbb{C}^3 \to \mathbb{C}_i$ for $i = 1, 2$, and 3, the line l_p should be mapped holomorphically onto or zero to each $\mathbb{C}_i$. Since $\dim_{\mathbb{R}} \ker (dg_1)_p = 1$, we see that l_p should be mapped holomorphically onto both $\mathbb{C}_1$ and $\mathbb{C}_3$. But this is impossible since, by the definition of G, if l_p is mapped holomorphically onto one, then it should be mapped anti-holomorphically onto the other. Thus, G is totally real on $M^3 \smallsetminus S(g_1)$.

Next, we show that for any singular point $p \in S(g_1)$, the image $dG_p(T_p M^3)$ contains a complex line. Since g_1 is a stable map, along the smooth link $L := S(g_1)$, $\ker dg_1|_L$ is a real 2-dimensional vector bundle. Furthermore, since $\widetilde{g_1} = (g_1, g_2)$ is an immersion, $(dg_2)_p$, for $p \in L$, gives a linear isomorphism between $\ker (dg_1)_p$

and $\mathbb{C}_2$. Since $\ker{(dg_1)}_p = \ker{(dg_3)}_p$, the image $dG_p(\ker{(dg_1)}_p)$ is equal to $(dg_2)_p(\ker{(dg_1)}_p)$, which is nothing but the complex line $T_{g_2(p)}\mathbb{C} = \mathbb{C}_2$ in $T_{G(p)}\mathbb{C}^3$.

We have thus shown that the set of complex tangents of the immersion G coincides with the singular set $S(g_1)$ of the stable map g_1. $\qed$

Remark 6.4. In Theorem 6.3, since the singular set $L = S(g_1)$ is a 1-dimensional submanifold of M^3 and the map $g_1 = (f_1, f_2)$ restricted to L, except at isolated cusp points, is a self-transverse immersion to the plane, we may assume that the immersion $\widetilde{g_1} = (f_1, f_2, f_3, f_4)$ restricted to L is an embedding, by slightly perturbing f_3 and f_4 if necessary. Then, there is a tubular neighbourhood N of L such that $\widetilde{g_1}|_N$ is an embedding. Thus we can choose the immersion G so that its restriction to N is an embedding.

The following is an easy consequence of Theorems 2.1 and 6.3. As mentioned in §1, it is intriguing to compare Corollary 6.5 with the usual Thom polynomial [28].

Corollary 6.5. *Let $f\colon M^3 \to \mathbb{R}^2$ be a liftable stable map from a closed orientable 3-manifold M^3. Then, $[S(f)] = 0 \in H_1(M;\mathbb{Z})$.*

7. An orientation of the singular set of liftable stable maps and an extension of Saeki's theorem

As mentioned in §6, H. Levine [19] has studied the lifting problem of stable maps from 3-manifolds to the plane and given an example of a non-liftable stable map from an orientable 3-manifold ([19, Example 2, p. 288]). His example is based on a certain necessary condition for the existence of an immersion lift in terms of the rotation numbers of regular fibres and an appropriate orientation on the singular set (see also [18, Theorem, p. 55 and Proposition, p. 59]). We recall it briefly here.

Let $f\colon M^3 \to \mathbb{R}^2$ be a stable map from an oriented 3-manifold with an immersion lift $\widetilde{f} = (f, h)\colon M^3 \to \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, and let $\pi\colon \mathbb{R}^4 \to \mathbb{R}^2$ be the projection onto the first factor. Then, each component C of the regular fibre of f over a point $x \in \mathbb{R}^2$, which can be compatibly oriented with respect to the orientations of M^3 and $\mathbb{R}^2$, is immersed by h into $\mathbb{R}^2 = \pi^{-1}(x)$, so that we can consider the rotation number $r(C)$ of this immersion. Then, according to [19, p. 288], for such a liftable stable map f we can take an orientation of the singular set $S(f)$ such that the following equations hold in Figure 6, which depicts the image of definite and indefinite fold points up to regular circle components:

- $r(C) = 1$ (resp. $= -1$) if the arc of definite folds is oriented upward (resp. downward), and
- $r(C_2) + r(C_3) - r(C_1) = 1$ (resp. $= -1$) if the arc of indefinite folds is oriented upward (resp. downward).

We shall call it *a good orientation* of $S(f)$ in what follows. Note that the choice of a good orientation is not unique.

The rest of this section is devoted to extending Saeki's theorem slightly. Namely, we show that in Theorem 5.1 the stable map f can be chosen to be liftable if the integral homology class $[L]$ represented by the given link L vanishes.

Theorem 7.1. *Let M^3 be a closed orientable 3-manifold and let L be a closed 1-dimensional submanifold of M^3. Then, there exists a liftable stable map $f\colon M^3 \to \mathbb{R}^2$ with $S(f) = L$ if and only if the $\mathbb{Z}$-coefficient homology class $[L]$ vanishes in $H_1(M^3;\mathbb{Z})$.*

definite folds   indefinite fold
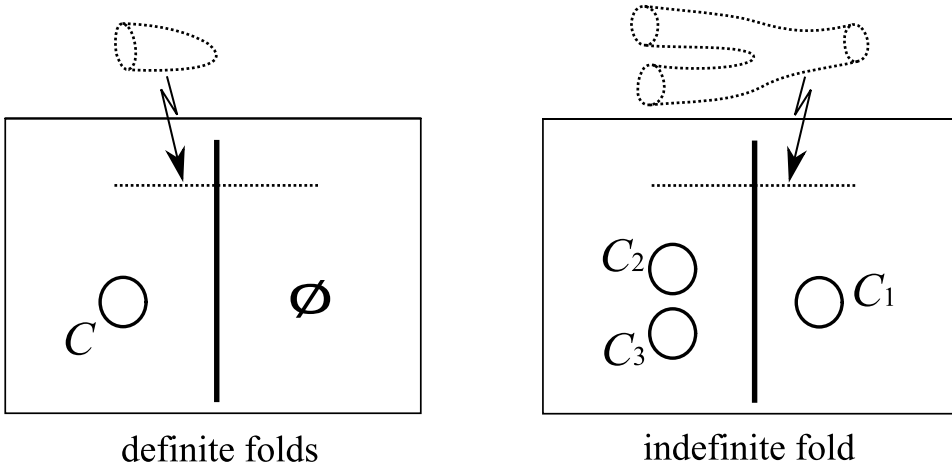
FIGURE 6. A good orientation of the singular set

Proof. The "only if" part is nothing but Corollary 6.5.

Suppose that $[L] = 0 \in H_1(M^3; \mathbb{Z})$ and consider any orientation on L. In Saeki's construction, we can choose the initial stable map $f_{init} \colon M^3 \to \mathbb{R}^2$ (Remark 5.2) so that it has an immersion lift in $\mathbb{R}^4$ (Remark 6.2). Take a good orientation on the singular set $S(f_{init})$ (see the beginning of this section). Then, $[S(f_{init})] = 0 \in H_1(M^3; \mathbb{Z})$ by Corollary 6.5, and hence $S(f_{init})$ can be related to L through a finite iteration of coherent band surgeries by Proposition 3.2. Therefore, we only have to check that the coherent cases of (H1) in addition to (H2) and (H3) in §5 keep the "liftability".

To see this, we first list all the necessary local homotopies, including the information on how regular fibres degenerate in crossing the image of the singular points and how the singular fibres are deformed during the homotopies (up to regular circle components). This is carried out in Figures 7–16, based on the classification given in [31, Theorem 4.7]. Our Figures 7–16 just correspond to the eleven figures appearing in [31, Figure 8(a)] (the case $\mathrm{III}^a(l)$ corresponding to Lips is excluded).

In Figures 7–16, we use the solid lines for definite folds and the dotted lines for indefinite folds. Bigger circles represent the regular fibre over each point of the region, and smaller circles drawn near the lines indicate how they are deformed and degenerate there. An explanation about the circles with shade is given below.

What we want to show is that in each homotopy in Figures 7–16 if we have an immersion lift in $\mathbb{R}^4$ at the beginning (the left), then we can construct an immersion lift in $\mathbb{R}^4$ at the end (the right) consistently. Actually we will prove more; that is, each generic homotopy above can be covered by a regular homotopy in $\mathbb{R}^4$ if it has an immersion lift in $\mathbb{R}^4$ at the beginning.

Now suppose that we have an immersion lift in $\mathbb{R}^4$, in the left column of each figure. Then, the regular fibre over each point of the 2-dimensional regions — the disjoint union of copies of circles — is immersed into $\mathbb{R}^2$ (that is, a fibre of the projection $\mathbb{R}^4 \to \mathbb{R}^2$). At this point we see that some fibre components, vanishing as they travel towards the definite folds, should be immersed into $\mathbb{R}^2$ with rotation number ± 1. In Figures 7–16, we shade such circles. Particularly in Figure 7, the goodness of the orientation on $S(f)$ determines whether $+1$ or -1 should be chosen;
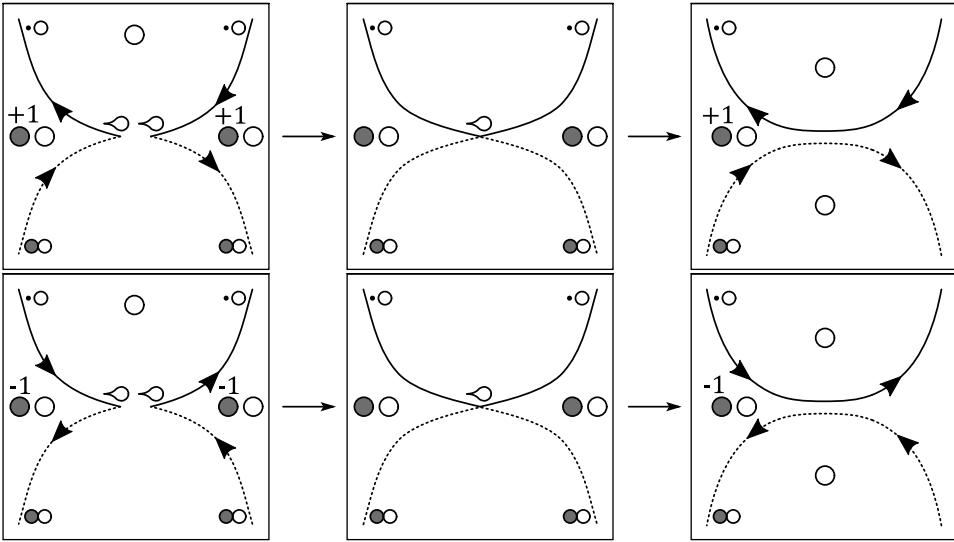
FIGURE 7. Beaks with coherent orientations (see [31, Figure 6(a) (2) and Figure 8(a) $III^a(b)$])
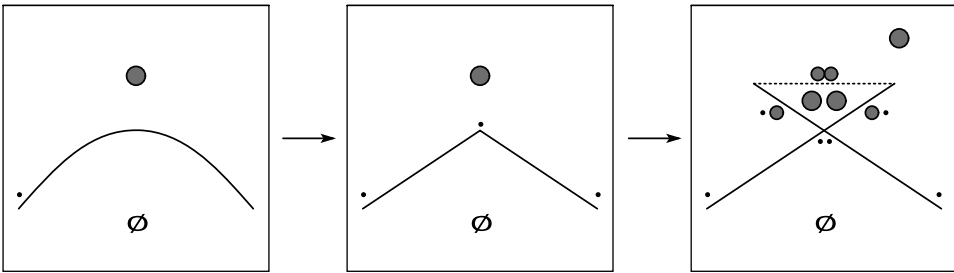

FIGURE 8. D-swallowtail (see [31, Figure 6(a) (3) and Figure 8(a) III^b])
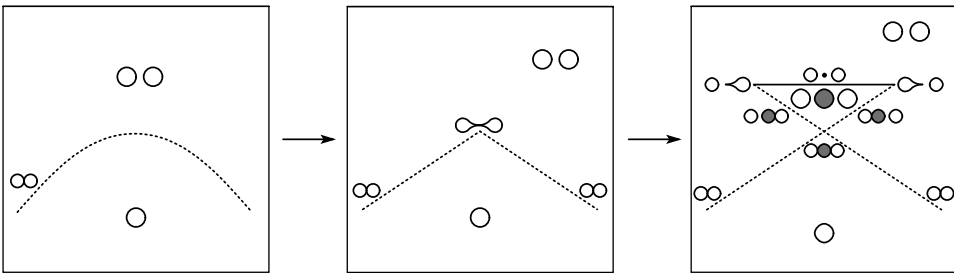

FIGURE 9. I-swallowtail (see [31, Figure 6(a) (4) and Figure 8(a) III^c])

the numbers in Figure 7 imply it (but what is important here is that the two shaded circles of the left column have the same number in each of the two cases of Figure 7). Note that we do not know how other circles (with no shade) are immersed into $\mathbb{R}^2$.
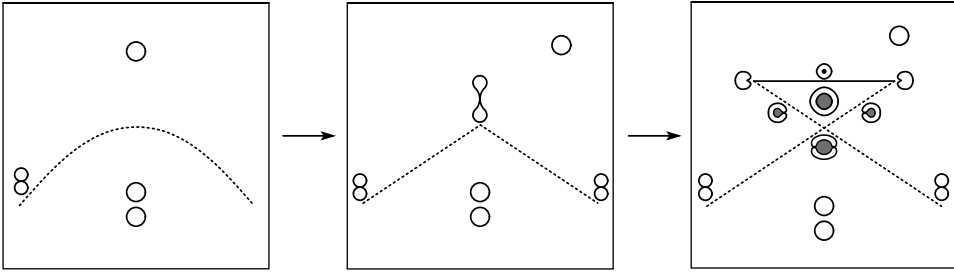
FIGURE 10. I-swallowtail (see [31, Figure 6(a) (4) and Figure 8(a) III^d])
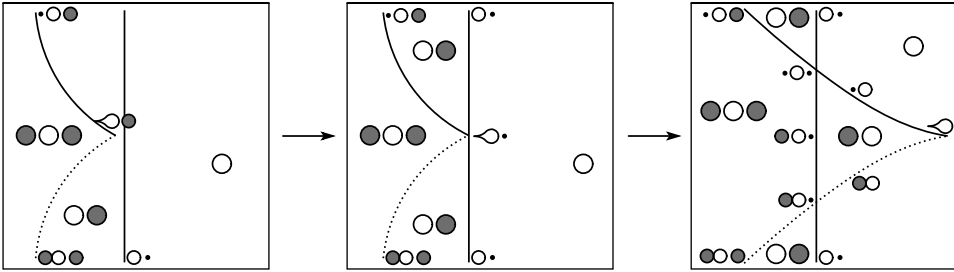

FIGURE 11. cusp-plus-D fold (type 1) (see [31, Figure 6(b) (5) and Figure 8(a) $III_1^{0,a}$])
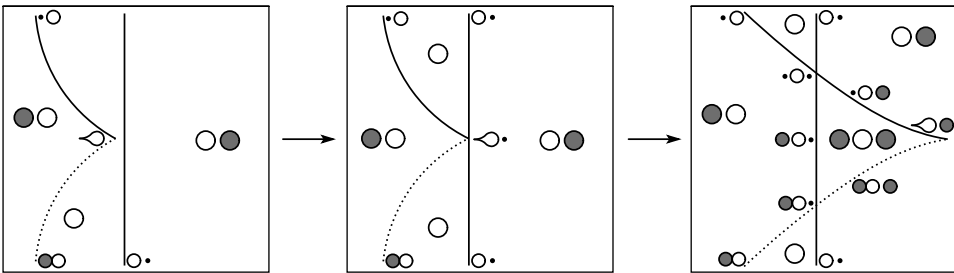

FIGURE 12. cusp-plus-D fold (type 2) (see [31, Figure 6(b) (6) and Figure 8(a) $III_2^{0,a}$])

Thus, by focusing on those shaded circles, it turns out that we can construct an immersion lift for the right column, so that circles with shade are immersed trivially (with rotation number ± 1) and the other circles are immsersed in the inherited ways from the left column.

For example, in the right column of Figure 11, we need to determine how the two circles of the (newly generated) central region are immersed into $\mathbb{R}^2$: the figure shows that we can do this by immersing the shaded circle trivially and the other circle similarly to the circle of the rightmost region. The other figures can be likewise understood. However, Figure 10 seems a little complicated. In the right column of Figure 10, it is obvious that the two circles in the bottom region should be immersed into $\mathbb{R}^2$ in a similar way as those of the left column. The shade of the singular fibre at the intersection of the lines indicates that these immersed circles,
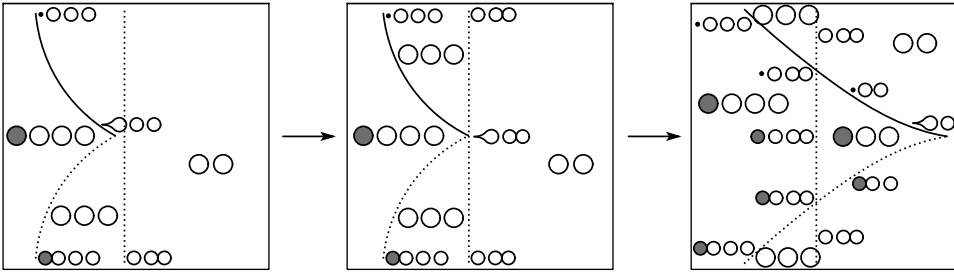
FIGURE 13. cusp-plus-I fold (see [31, Figure 6(b) (7) and Figure 8(a) $\mathrm{III}_1^{1,a}$])
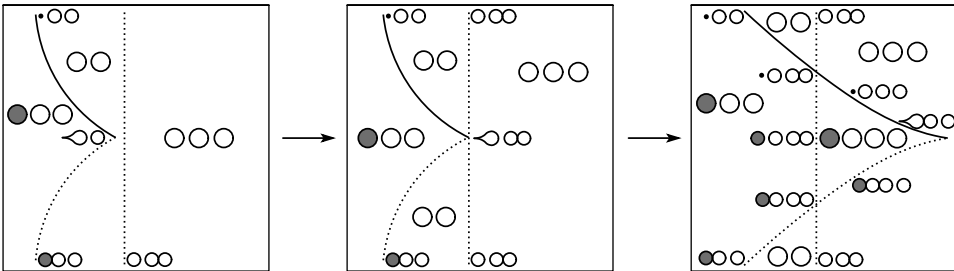

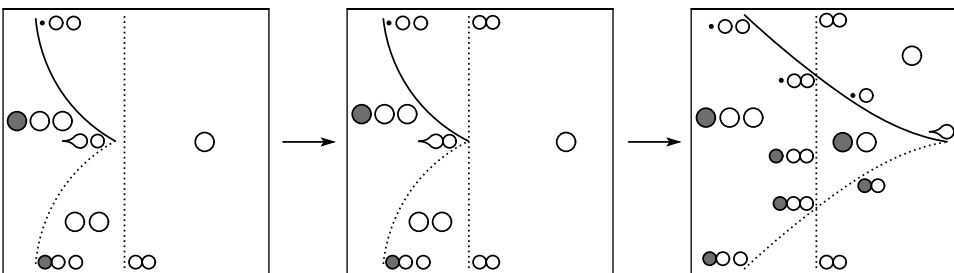FIGURE 14. cusp-plus-I fold (see [31, Figure 6(b) (7) and Figure 8(a) $\mathrm{III}_2^{1,a}$]

FIGURE 15. cusp-plus-I fold (see [31, Figure 6(b) (7) and Figure 8(a) III_1^e]

nearing the intersection of the singular lines, osculate at two points in such a way that we could span an immersed disk at the shaded portion. For the two circles of the central region, one should be immersed similarly as the circle of the top region and the other (with shade) should be immersed trivially. We can thus determine a consistent immersion lift for the right column. (In Figure 10, all the shaded portions in the right column shrink to a point at the bifurcation point, that is, in the center column).

Finally we see that the situations of the degenerations of fibres in the left and the right columns can be continuously connected via the center column (the bifurcation point). Therefore, we could obtain a covering regular homotopy in $\mathbb{R}^4$ of each generic homotopy. $\qquad\square$
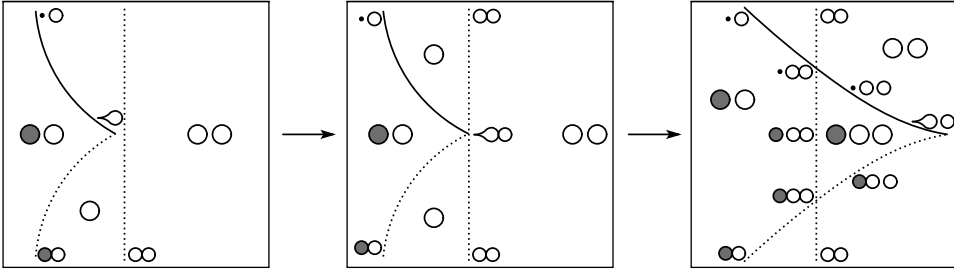
FIGURE 16. cusp-plus-I fold (see [31, Figure 6(b) (7) and Figure 8(a) III$_2^e$]

Remark 7.2. For an immersion of M^3 in $\mathbb{R}^4$ the precomposition by a homeomorphism of M^3 does not change the regular homotopy class of the immersion (see [30] and [27, Remark 3.4]). Therefore, in view of the last paragraph of the proof of Theorem 7.1, the immersion lift of the resultant stable map and that of the initial stable map belong to the same regular homotopy class.

8. KNOTS AND LINKS OF COMPLEX TANGENTS

We are now ready to state the main theorem.

Theorem 8.1. *Let M^3 be a closed orientable 3-manifold and let L be a closed 1-dimensional submanifold of M^3. Then, there exists a smooth embedding $F\colon M^3 \to \mathbb{C}^3$ the set of whose complex tangents coincides with L if and only if $[L] = 0 \in H_1(M^3; \mathbb{Z})$.*

Proof. The "only if" part follows from Theorem 2.1. Suppose that $[L] = 0 \in H_1(M^3; \mathbb{Z})$.

By Whitney's theorem, we can embed M^3 into $\mathbb{R}^6$ such that it has the trivial normal bundle. Furthermore, by the Compression Theorem [22], such an embedding $G'\colon M^3 \to \mathbb{R}^6$ can be chosen so that its composition with the projection to $\mathbb{R}^4$ becomes an immersion, which we denote by $\widetilde{f'}\colon M^3 \to \mathbb{R}^4$. By further composing $\widetilde{f'}$ with a generic projection $\mathbb{R}^4 \to \mathbb{R}^2$, we obtain a stable map, denoted by $f' = (f_1', f_2')\colon M^3 \to \mathbb{R}^2$.

We have thus obtained the liftable stable map $f' = (f_1', f_2')\colon M^3 \to \mathbb{R}^2$ with the immersion lift $\widetilde{f'} = (f_1', f_2', f_3', f_4')\colon M^3 \to \mathbb{R}^4$, whose composition $j \circ \widetilde{f'}$ with the inclusion $j\colon \mathbb{R}^4 \to \mathbb{R}^6$ is regularly homotopic to the embedding $G'\colon M^3 \to \mathbb{R}^6$.

Since $[L] = 0$, by using this stable map f' as the initial stable map (see Remark 5.2) in the proof of Theorem 7.1, we obtain a stable map f which satisfies $S(f) = L$ and has an immersion lift $\widetilde{f} = (f_1, f_2, f_3, f_4)\colon M^3 \to \mathbb{R}^4$. By Remark 7.2, furthermore, we see that $\widetilde{f}$ is regularly homotopic to $\widetilde{f'}$.

By Theorem 6.3, we obtain the immersion $G'' = (f_1, f_2, f_3, f_4, f_1, -f_2)$ whose complex tangents form $S(f) = L$. Then, it is clear that G'' is regularly homotopic to the embedding $G'\colon M^3 \to \mathbb{R}^6$. By Remark 6.4, we may assume that the immersion G'' is already an embedding on a tubular neighbourhood N of L. Furthermore, since the condition of total reality is an open condition, by slightly perturbing G'' on $M^3 \smallsetminus N$, we obtain a new immersion G which has only transverse double points away from N and is still totally real on $M^3 \smallsetminus N$. Thus, the self-transverse immersion

G has only isolated double points lying in $M^3 \smallsetminus N$, and the set of complex tangents of G coincides with L.

Finally we apply to $G|_{M^3 \smallsetminus N}$ the relative h-principle for totally real embeddings (see Gromov [14], Eliashberg and Mishachev [10], and Forstnerič [12] for example). Since the immersion G is regularly homotopic to an embedding, is totally real on $M^3 \smallsetminus L$, and is already an embedding on N, we can find a smooth embedding $F \colon M^3 \to \mathbb{C}^3$ such that $F|_{M^3 \smallsetminus L}$ is totally real and $F|_N = G|_N$ by using the relative h-principle. Thus $F \colon M^3 \to \mathbb{C}^3$ is the desired embedding with complex tangents forming L. $\qquad\square$

Corollary 8.2. *Every knot or link in S^3 can be realised as the set of complex tangents of a smooth embedding $F \colon S^3 \to \mathbb{C}^3$.*

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References

[1] Patrick Ahern and Walter Rudin, *Totally real embeddings of S^3 in $\mathbf{C}^3$*, Proc. Amer. Math. Soc. **94** (1985), no. 3, 460–462, DOI 10.2307/2045235. MR787894

[2] S. Akbulut and H. King, *All knots are algebraic*, Comment. Math. Helv. **56** (1981), no. 3, 339–351, DOI 10.1007/BF02566217. MR639356

[3] Michèle Audin, *Fibrés normaux d'immersions en dimension double, points doubles d'immersions lagragiennes et plongements totalement réels* (French), Comment. Math. Helv. **63** (1988), no. 4, 593–623, DOI 10.1007/BF02566781. MR966952

[4] Errett Bishop, *Differentiable manifolds in complex Euclidean space*, Duke Math. J. **32** (1965), 1–21. MR0200476

[5] Eduardo Chincaro, *Bifurcations of Whitney maps*, Ph.D. thesis, Tese de doutorado, IMPA (1978).

[6] A. V. Domrin, *A description of characteristic classes of real submanifolds in complex manifolds in terms of RC-singularities* (Russian, with Russian summary), Izv. Ross. Akad. Nauk Ser. Mat. **59** (1995), no. 5, 19–40, DOI 10.1070/IM1995v059n05ABEH000039; English transl., Izv. Math. **59** (1995), no. 5, 899–918. MR1360632

[7] Ali M. Elgindi, *On the topological structure of complex tangencies to embeddings of S^3 into $\mathbb{C}^3$*, New York J. Math. **18** (2012), 295–313. MR2928578

[8] Ali M. Elgindi, *A topological obstruction to the removal of a degenerate complex tangent and some related homotopy and homology groups*, Internat. J. Math. **26** (2015), no. 5, 1550025, 16. MR3345506

[9] Ali M. Elgindi, *Totally real perturbations and nondegenerate embeddings of S^3*, New York J. Math. **21** (2015), 1283–1293. MR3441643

[10] Y. Eliashberg and N. Mishachev, *Introduction to the h-principle*, Graduate Studies in Mathematics, vol. 48, American Mathematical Society, Providence, RI, 2002. MR1909245

[11] C. Ernst and D. W. Sumners, *A calculus for rational tangles: applications to DNA recombination*, Math. Proc. Cambridge Philos. Soc. **108** (1990), no. 3, 489–515, DOI 10.1017/S0305004100069383. MR1068451

[12] Franc Forstnerič, *On totally real embeddings into $\mathbf{C}^n$*, Exposition. Math. **4** (1986), no. 3, 243–255. MR880125

[13] Franc Forstnerič, *Complex tangents of real surfaces in complex surfaces*, Duke Math. J. **67** (1992), no. 2, 353–376, DOI 10.1215/S0012-7094-92-06713-5. MR1177310

[14] Mikhael Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 9, Springer-Verlag, Berlin, 1986. MR864505

[15] André Haefliger, *Quelques remarques sur les applications différentiables d'une surface dans le plan* (French), Ann. Inst. Fourier. Grenoble **10** (1960), 47–60. MR0116357

[16] León Kushner, Harold Levine, and Paulo Porto, *Mapping three-manifolds into the plane. I*, Bol. Soc. Mat. Mexicana (2) **29** (1984), no. 1, 11–33. MR790729

[17] Hon Fei Lai, *Characteristic classes of real manifolds immersed in complex manifolds*, Trans. Amer. Math. Soc. **172** (1972), 1–33, DOI 10.2307/1996329. MR0314066

[18] Harold Levine, *Classifying immersions into $\mathbf{R}^4$ over stable maps of 3-manifolds into $\mathbf{R}^2$*, Lecture Notes in Mathematics, vol. 1157, Springer-Verlag, Berlin, 1985. MR814689

[19] Harold Levine, *Stable mappings of 3-manifolds into the plane*, Singularities (Warsaw, 1985), Banach Center Publ., vol. 20, PWN, Warsaw, 1988, pp. 279–289. MR1101845

[20] J. N. Mather, *Stability of C^∞ mappings. VI: The nice dimensions*, Proceedings of Liverpool Singularities-Symposium, I (1969/70), Lecture Notes in Math., Vol. 192, Springer, Berlin, 1971, pp. 207–253. MR0293670

[21] John N. Mather, *Generic projections*, Ann. of Math. (2) **98** (1973), 226–245, DOI 10.2307/1970783. MR0362393

[22] Colin Rourke and Brian Sanderson, *The compression theorem. I*, Geom. Topol. **5** (2001), 399–429, DOI 10.2140/gt.2001.5.399. MR1833749

[23] Osamu Saeki, *Constructing generic smooth maps of a manifold into a surface with prescribed singular loci* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **45** (1995), no. 4, 1135–1162. MR1359844

[24] Osamu Saeki, *Simple stable maps of 3-manifolds into surfaces*, Topology **35** (1996), no. 3, 671–698, DOI 10.1016/0040-9383(95)00034-8. MR1396772

[25] Osamu Saeki and Masamichi Takase, *Desingularizing special generic maps*, J. Gökova Geom. Topol. GGT **7** (2013), 1–24. MR3153918

[26] J. Sotomayor, *Bifurcation of whitney maps $\mathbf{R}^n \to \mathbf{R}^2$ and critical pareto sets*, Proceedings of a Symposium titled "Applications of Topology and Dynamical Systems" held at the University of Warwick, Coventry, 1973/1974.

[27] Masamichi Takase, *An Ekholm-Szűcs-type formula for codimension one immersions of 3-manifolds up to bordism*, Bull. Lond. Math. Soc. **39** (2007), no. 1, 39–45, DOI 10.1112/blms/bdl009. MR2303517

[28] R. Thom, *Les singularités des applications différentiables* (French), Ann. Inst. Fourier, Grenoble **6** (1955), 43–87. MR0087149

[29] S. M. Webster, *The Euler and Pontrjagin numbers of an n-manifold in $\mathbf{C}^n$*, Comment. Math. Helv. **60** (1985), no. 2, 193–216, DOI 10.1007/BF02567410. MR800003

[30] Wu Wen-tsün, *On the immersion of C^∞-3-manifolds in a Euclidean space*, Sci. Sinica **13** (1964), 335–336. MR0172302

[31] Minoru Yamamoto, *First order semi-local invariants of stable maps of 3-manifolds into the plane*, Proc. London Math. Soc. (3) **92** (2006), no. 2, 471–504, DOI 10.1112/S0024611505015534. MR2205725

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