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CONVERGENCE ALMOST EVERYWHERE OF MULTIPLE FOURIER SERIES OVER CUBES

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ABSTRACT. We study convergence almost everywhere of multiple trigonometric Fourier series over cubes defined on the d-dimensional torus $\mathbb{T}^d$. We provide a new approach which allows us to prove the novel interpolation estimates for the Carleson maximal operators generated by the partial sums of the multiple Fourier series and all its conjugate series. Combining these estimates we show that these operators are bounded from a variant of the Arias-de-Reyna space QA^d to the weak L^1-space on $\mathbb{T}^d$. This implies that the multiple Fourier series of every function $f \in QA^d$ and all its conjugate series converge over cubes almost everywhere. By a close analysis of the space QA^d we prove that it contains a Lorentz space that strictly contains the Orlicz space $L(\log L)^d \log\log\log L(\mathbb{T}^d)$. This yields a significant improvement of a deep theorem proved by Antonov which was the best known result on the convergence of multiple Fourier series over cubes.

1. Introduction

The theory of Fourier series in the one-dimensional case has a rich history and is sufficiently well developed. There are many monographs related to this case; we mention here only the fundamental monograph of Zygmund [20]. The central problem in this theory is devoted to almost everywhere convergence of the Fourier series, and a large number of articles have been published. The general study of this problem was initiated in the pioneering work by Carleson [5] published in 1965, in which the answer to the famous Luzin's conjecture is given that the Fourier series of square summable functions converges almost everywhere. We refer to the book Arias-de-Reyna [3], which is an excellent source of information about the theory of Fourier series, in which the author gives the proof of the Carleson-Hunt theorem, following Carlson more than Hunt [10]. We point out that a new influential proof of Carleson's result was given by Fefferman in [9].

In the recent past the theory of multiple Fourier series was studied intensively. The aim of this paper is to prove novel theorems on almost everywhere convergence of multiple trigonometric Fourier series over cubes and all its conjugate series which cover the best known results. Before we formulate our results we begin with some definitions and notation.

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Throughout the paper, $\mathbb{T}$ will denote the torus. As usual we will identify $\mathbb{T}$ with $\mathbb{R}/2\pi\mathbb{Z}$ or more concretely with the interval $(-\pi,\pi]$, and then the functions defined on $\mathbb{T}$ can be interpreted as 2π-periodic functions defined on $\mathbb{R}$. But $\mathbb{T}$ can also be viewed as the unit circle $\mathbb{T}=\{z\in\mathbb{C};\,|z|=1\}=\{e^{it};\,t\in(-\pi,\pi]\}$. The Haar probability measure on $\mathbb{T}$ will be denoted by m; that is, $dm(e^{it})=dt/2\pi$. For every $f\in L^1(\mathbb{T})$, we denote the Fourier coefficients of f by

$$\widehat{f}(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\,e^{-itk}\,dt=\int_{\mathbb{T}}f(z)z^{-k}\,dm\,,\qquad k\in\mathbb{Z}.$$

Let $\mathbb{T}^d$ be the d-dimensional torus and m_d its Haar probability measure. Define for every $f\in L^1(\mathbb{T}^d)$ its Fourier coefficients by

$$\widehat{f}(\mathbf{k})=\int_{\mathbb{T}^d}f(\mathbf{x})\exp(-i\langle\mathbf{k},\mathbf{x}\rangle)\,dm_d\,,$$

where $\mathbf{k}=(k_1,k_2,\dots,k_d)\in\mathbb{Z}^d$, $\mathbf{x}=(x_1,x_2,\dots,x_d)\in\mathbb{T}^d$, $\langle\mathbf{k},\mathbf{x}\rangle=k_1x_1+k_2x_2+\cdots+k_dx_d$.

Given $f\in L^1(\mathbb{T}^d)$ and $n\in\mathbb{N}$, the multiple trigonometric Fourier series of f is given by

$$\sum_{\mathbf{k}\in\mathbb{Z}^d}\widehat{f}(\mathbf{k})\exp(i\langle\mathbf{k},\mathbf{x}\rangle),\tag{1}$$

and its Nth *cubic partial sum*, $S_Nf(\mathbf{x})$ at $\mathbf{x}\in\mathbb{T}^d$, is given by

$$S_Nf(\mathbf{x})=\sum_{\|\mathbf{k}\|_\infty\le N}\widehat{f}(\mathbf{k})\exp(i\langle\mathbf{k},\mathbf{x}\rangle),$$

where $\|\mathbf{k}\|_\infty=\max\{|k_1|,\dots,|k_d|\}$ for each $\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{Z}^d$.

Let $\widetilde{L^0}(\mathbb{T}^d)$ be the cone of m_d-measurable functions whose values lie in $[0,\infty]$. The *Carleson maximal operator* $\mathcal{S}\colon L^1(\mathbb{T}^d)\to\widetilde{L^0}(\mathbb{T}^d)$ is defined by

$$\mathcal{S}f(\mathbf{x})=\sup_{N\ge1}|S_Nf(\mathbf{x})|\,,\qquad f\in L^1(\mathbb{T}^d)\,,\quad\mathbf{x}\in\mathbb{T}^d\,.$$

Let $B=\{r_1,\dots,r_l\}$ be a non-empty subset of the set $\{1,\dots,d\}$. The multiple series of f given by

$$\sum_{\mathbf{k}\in\mathbb{Z}^d}\prod_{j=1}^{l}(-i\,\mathrm{sgn}\,k_{r_j})\widehat{f}(\mathbf{k})\exp(i\langle\mathbf{k},\mathbf{x}\rangle),\quad\mathbf{x}\in\mathbb{T}^d\,,\tag{2}$$

is called the *conjugate* of the series (1) with respect to the variables whose subscripts belong to B, or the B-*c*onjugate, and the Nth cubic partial sum $S_{N,B}\,f(\mathbf{x})$ of (2) is defined in analogy with the Nth cubic partial sum of (1).

We note that the series (1) has 2^d-1 conjugates and if B is empty, then (1) can be regarded as the B-conjugate ($\emptyset$-conjugate) of itself. In the case when $d=1$, the above definitions coincide with the usual definitions of a trigonometric series and its conjugate series of an integrable 2π-periodic function.

With the B-conjugate series of a function $f\in L^1(\mathbb{T}^d)$ we associate the B-conjugate function $\widetilde{f}_B$ defined as follows. For $\varepsilon(B)=(\varepsilon_{r_1},\dots,\varepsilon_{r_l})$, we let

$$T^{|B|}_{\varepsilon(B)}=\prod_{j=1}^{l}\big(\mathbb{T}\setminus[-\varepsilon_{r_j},\varepsilon_{r_j}]\big).$$

For $\mathbf{t} = (t_1, \dots, t_d)$ we let $\mathbf{t}(B) := (t_1(B), \dots, t_d(B))$, where $t_\nu(B) := t_{r_j}$ for $\nu = r_j$, $1 \le j \le l$ and $t_\nu(B) := 0$ for $\nu \ne r_j$. We define for every $\mathbf{x} \in \mathbb{T}^d$

$$(3) \quad \widetilde{f}_B(\mathbf{x}) = \Big(-\frac{1}{2\pi} \Big)^l \lim_{\varepsilon(B) \to 0+} \int_{T^{|B|}_{\varepsilon(B)}} f(\mathbf{x} + \mathbf{t}(B)) \Big(\prod_{j=1}^{l} \cot\Big(\frac{t_{r_j}}{2}\Big)\Big)\, dt_{r_1} \dots dt_{r_l} .$$

The limit in (3) is taken as the components of $\varepsilon(B)$ tend to 0 independently. If $B = \emptyset$, then we put $\widetilde{f}_B := f$.

Let D_N be the Dirichlet kernel and $\widetilde{D}_N$ the corresponding conjugate kernel,

$$D_N(t) = \frac{\sin\big(N + \frac{1}{2}\big)t}{2\sin\big(\frac{t}{2}\big)}, \qquad \widetilde{D}_N(t) = \frac{\cos\big(\frac{t}{2}\big) - \cos\big(N + \frac{1}{2}\big)t}{2\sin\big(\frac{t}{2}\big)} .$$

Note that the Nth cubic sum of the conjugate series (2) can be written

$$(4) \quad S_{N,B} f(\mathbf{x}) = \frac{1}{\pi^d} \int_{\mathbb{T}^d} \prod_{j \in B} \big(-\widetilde{D_N}(t_j)\big) \prod_{j \in \{1,\dots,d\} \setminus B} D_N(t_j) f(\mathbf{x} + \mathbf{t})\, d\mathbf{t}, \qquad \mathbf{x} \in \mathbb{T}^d ,$$

where the set B may be empty, with the convention that the empty product is equal to unity.

Throughout the paper for every $f \in L^1(\mathbb{T}^d)$ and each $B \subset \{1, \dots, d\}$ and $N \in \mathbb{N}$,

$$\widetilde{S_{N,B}}\, f(\mathbf{x}) := \max_{1 \le k \le N} |S_{k,B} f(\mathbf{x})|, \qquad \widetilde{\mathcal{S}_B} f(\mathbf{x}) := \max_{N \in \mathbb{N}} |\widetilde{S_{N,B}} f(\mathbf{x})|, \qquad \mathbf{x} \in \mathbb{T}^d .$$

Now we sketch the historical background for motivation of our study. In 1923 Kolmogorov [12] constructed a function $f \in L^1(\mathbb{T})$, whose Fourier series diverges almost everywhere, and in [20, 8.4] he showed that the Fourier series can diverge everywhere. The divergence of Fourier series on $\mathbb{T}$ has been investigated by many authors. The best known result in this direction is due to Konyagin [13]. It states that for an arbitrary non-decreasing function $\varphi\colon [0, \infty) \to [0, \infty)$ with $\varphi(t) = o\Big(t\sqrt{\frac{\log t}{\log\log t}}\Big)$ as $t \to \infty$, there is a function f integrable on $\mathbb{T}$ such that $\varphi(|f|) \in L^1(\mathbb{T})$ whose Fourier series is unboundedly divergent everywhere on $\mathbb{T}$.

In 1966 Carleson [5] justified Luzin's hypothesis and proved that the Fourier series of the functions in $L^2(\mathbb{T})$ converge almost everywhere; Hunt [10] proved that Carleson's result is also valid for functions in $L^p(\mathbb{T})$ for $1 < p < 2$. A more or less direct consequence of Carleson's and Hunt's results is the convergence a.e. of the cubic partial sums of the multiple Fourier series of functions in $L^p(\mathbb{T}^d)$ for any $d \ge 2$ and $p > 1$. Sjölin [17] extended this result to functions in the Orlicz space $L(\log L)^d \log\log L(\mathbb{T}^d)$. The next advance was made by Fefferman [8], who proved for functions in $L^p(\mathbb{T}^2)$, $p > 1$, the a.e. convergence of double Fourier series summed up over dilated of a fixed polygon in the plane. Tevzadze [19], proved the same if we consider partial sums over a nested increasing sequence of rectangles (with sides parallel to axes). On the one hand, Fefferman [7] showed that the result for the family of all those rectangles was not true even for continuous functions on $\mathbb{T}^2$.

The best known result on the divergence of multiple Fourier series over cubes on a set of positive measure was proved by Konyagin [13]; it states that for an arbitrary non-decreasing function $\varphi\colon [0, \infty) \to [0, \infty)$ with $\varphi(t) = o\big(t(\log t)^{d-1} \log\log t\big)$ as $t \to \infty$, there exists a function f integrable on $\mathbb{T}^d$ with $\varphi(|f|) \in L^1(\mathbb{T}^d)$ and with everywhere divergent Fourier series over cubes.

Antonov [2] proved a theorem extending his results from [1] on the convergence almost everywhere of ordinary Fourier series of functions belonging to certain classes to multiple Fourier series in the case of convergence over cubes. His deep result states that the multiple Fourier series of every function f in the Lorentz-Orlicz space $L(\log L)^d \log\log\log L(\mathbb{T}^d)$ and all its conjugates converge over cubes almost everywhere.

Our main result in this paper on almost everywhere convergence extends the best known results at present. Our methods hinge on a d-dimensional interpolation estimate of the Carleson maximal operators, which states that for each $B \subset \{1, \dots, d\}$ with $d \in \mathbb{N}$ there exists an absolute constant $C_d > 0$ such that the Carleson maximal operator $\widetilde{\mathcal{S}_B}$ generated by conjugate Fourier series satisfies the following interpolation estimates:

$$\|\widetilde{\mathcal{S}_B} f\|_{L_{1,\infty}(\mathbb{T}^d)} \le C\, \|f\|_{L^1(\mathbb{T}^d)} \left[\log\left(\frac{e\|f\|_{L^2(\mathbb{T}^d)}}{\|f\|_{L^1(\mathbb{T}^d)}}\right)\right]^d, \quad f \in L^2(\mathbb{T}^d), \tag{5}$$

where $L_{1,\infty}(\mathbb{T}^d)$ is the weak L^1-space on the d-dimensional torus $\mathbb{T}^d$. The mentioned estimates in higher dimensions allow us to prove that the Carleson maximal operator $\widetilde{\mathcal{S}_B}$ is bounded from a variant of the d-dimensional Arias-de-Reyna space QA^d to $L_{1,\infty}(\mathbb{T}^d)$; i.e., there exists a constant $C > 0$ such that for every $f \in QA^d$,

$$m_d\big(\{\mathbf{x} \in \mathbb{T}^d;\ \widetilde{\mathcal{S}_B} f(\mathbf{x}) > \lambda\}\big) \le \frac{C}{\lambda}\, \|f\|_{QA^d}, \quad \lambda > 0.$$

As a consequence we obtain that the multiple Fourier series of every function $f \in QA^d$ and all its conjugates converge over cubes almost everywhere. By a close analysis of the space QA^d defined in the paper, we prove that it contains a Lorentz space that strictly contains the Orlicz space $L(\log L)^d \log\log\log L(\mathbb{T}^d)$. This yields an improvement of the mentioned Antonov's result.

Throughout the paper we will use the following notation: given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural inclusion map of X in Y is continuous. If $X \hookrightarrow Y$ and $Y \hookrightarrow X$, we write $X = Y$. If f and g are real valued functions defined on a set A, then the symbol $f \asymp g$ means that $c_1\, g \le f \le c_2\, g$ on A for some positive constants c_1 and c_2.

2. Interpolation estimates for the Carleson maximal operators in one dimension

The main aim in this section is to prove the key results for the whole paper on interpolation type estimates for the Carleson maximal operators $\mathcal{S}$ and $\widetilde{\mathcal{S}}$. Before we state the main results we need some more definitions and notation.

The Hilbert transform $\mathcal{H}$ and the Riesz projection $\mathcal{R}$ are defined on $L^2(\mathbb{T})$, via the Fourier transform, by

$$\widehat{\mathcal{H}f}(m) = \begin{cases} -i\widehat{f}(m) & \text{if } m > 0, \\ 0 & \text{if } m = 0, \\ i\widehat{f}(m) & \text{if } m < 0, \end{cases} \qquad \widehat{\mathcal{R}f}(m) = \begin{cases} \widehat{f}(m) & \text{if } m \ge 0, \\ 0 & \text{if } m < 0. \end{cases}$$

$\mathcal{H}f$ is the conjugate function of f. Observe that we have

$$f + i\mathcal{H}f = 2\mathcal{R}f - \widehat{f}(0)\,. \tag{6}$$

We will need to consider some other partial sums and the maximal type operators. Let $f \in L^1(\mathbb{T})$ and $x \in \mathbb{T}$. Recall that

$$S_N f(x) = \sum_{k=-N}^{N} \widehat{f}(k) e^{ikx} \qquad \text{and} \qquad \mathcal{S} f(x) = \sup_{N \in \mathbb{N}} |S_N f(x)| .$$

The Fejér sums $\sigma_N f$ and the maximal Fejér operator are defined by

$$\sigma_N f = \frac{1}{N+1} \big(S_0 f + S_1 f + \cdots + S_N f \big), \qquad \Sigma f(x) = \sup_{N \in \mathbb{N}} |\sigma_N f(x)| .$$

We also define

$$S_N^+ f(x) = \mathcal{R}(S_N f)(x) = \sum_{k=0}^{N} \widehat{f}(k) e^{ikx} , \qquad \mathcal{S}^+ f(x) = \sup_{N \in \mathbb{N}} |S_N^+ f(x)| ,$$

and

$$\widetilde{S_N} f = \mathcal{H}(S_N f) , \qquad \widetilde{\mathcal{S}} f(x) = \sup_{N \in \mathbb{N}} |\widetilde{S_N} f(x)| .$$

Finally the Hardy-Littlewood maximal function $\mathcal{M} f$ is defined by

$$\mathcal{M} f(x) = \sup_{I \ni x} \frac{1}{m(I)} \int_I |f| \, dm ,$$

where the sup is taken over all intervals I containing the point x.

We will use some function spaces. We recall that a *quasi-Banach lattice* on a complete σ-finite measure space $(\Omega, \mu) := (\Omega, \Sigma, \mu)$ is defined to be a quasi-Banach space X which is a subspace of $L^0(\mu)$ (the topological linear space of all real measurable functions equipped with the topology of convergence in measure) such that there exists $u \in X$ with $u > 0$ a.e., and if $|f| \leq |g|$ a.e., where $g \in X$ and $f \in L^0(\mu)$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$. By a complex quasi-Banach lattice X we mean a complexification of X.

Given $f \in L^0(\mu)$, its distribution function is defined by $\mu_f(\lambda) = \mu(\{x \in \Omega;\ |f(\omega)| > \lambda\})$, and its decreasing rearrangement by $f^*(t) = f^*_\mu(t) = \inf\{\lambda \geq 0;\ \mu_f(\lambda) \ \leq t\}$ for $t \geq 0$. A (quasi-)Banach lattice $(X, \|\cdot\|_X)$ is called a *rearrangement invariant* (r.i. for short) (quasi-)Banach space provided $\mu_f = \mu_g$, $f \in X$ implies $g \in X$, and $\|f\|_X = \|g\|_X$.

In the paper we will consider r.i. spaces on a finite atomless measure space (Ω, μ). In this case the function $\varphi_X(t) := \|\chi_A\|_X$, where $\mu(A) = t$, $0 \leq t < \mu(\Omega)$, is called the *fundamental function* of X.

In the theory of pointwise convergence of Fourier series an important role is played by the weak L^1-space, Orlicz spaces and Lorentz spaces. We recall that if (Ω, μ) is a measure space and $0 < p < \infty$, then the weak L^p-space $L_{p,\infty}(\Omega)$ ($L_{p,\infty}$ for short) consists of all $f \in L^0(\mu)$ such that

$$\|f\|_{p,\infty} := \sup_{\lambda > 0} \lambda \, \mu_f(\lambda)^{1/p} < \infty .$$

It is well-known that $L_{p,\infty}$ is an r.i. quasi-Banach space when it is equipped with the quasi-norm $\|\cdot\|_{p,\infty}$.

Given an Orlicz function Φ (i.e., $\Phi\colon [0,\infty) \to [0,\infty)$ is an increasing, continuous function such that $\Phi(0) = 0$), we denote by $L_\Phi(\Omega)$ (L_Φ for short) the *Orlicz space* of all $f \in L^0(\mu)$ such that for some $\lambda > 0$,

$$\int_\Omega \Phi(\lambda |f|) \, d\mu < \infty.$$

It is easy to check that if there exists $C > 0$ such that $\Phi(t/C) \le \Phi(t)/2$ for all $t > 0$, then L_Φ is an r.i. quasi-Banach lattice with the quasi-norm $\|\cdot\|$ satisfying

$$\|f+g\|_\Phi \le C\big(\|f\|_\Phi + \|g\|_\Phi\big), \quad f,\, g \in L_\Phi,$$

where

$$\|f\|_\Phi := \inf\Big\{\lambda > 0;\ \int_\Omega \Phi\big(|f|/\lambda\big)\, d\mu \le 1\Big\}.$$

It is well-known that L_Φ is an r.i. Banach space provided Φ is a convex function.

An important example of r.i. spaces is Lorentz spaces. Let $\varphi\colon [0,\mu(\Omega)) \to [0,\infty)$ be a non-decreasing concave function, with $\varphi(0) = 0$. The *Lorentz space* $\Lambda_\varphi(\Omega)$ (Λ_φ for short) consists of all $f \in L^0(\mu)$ such that

$$\|f\|_{\Lambda_\varphi} := \int_0^{\mu(\Omega)} f^*(s)\, d\varphi(s) < \infty.$$

A special role is played by an Orlicz space denoted by $L\log L \log\log\log\ L(\Omega)$, which is generated by an Orlicz function $\Phi(t) = t\,(\overline{\log}\, t)\,(\overline{\log}\,\overline{\log}\,\overline{\log}\, t)$, where $\overline{\log}\, t := 1 + \log\, t$ for every $t \ge 1$, and $\overline{\log}\, t := t$, for $0 \le t \le 1$. Note that the case of an atomless probability measure space (Ω,μ) can be identified up to equivalence of norms with the Lorentz space (see, e.g., [4]) $\Lambda_\varphi(\Omega)$ with $\varphi\colon [0,1] \to [0,1]$ given by $\varphi(0) = 0$:

$$\varphi(t) \asymp t\,\overline{\log}\Big(\frac{1}{t}\Big)\,\overline{\log}\,\overline{\log}\,\overline{\log}\Big(\frac{1}{t}\Big), \quad t \in (0,1].$$

If ψ denotes the concave function $\psi\colon [0,1] \to [0,1]$ defined by $\psi(0) = 0$ and

$$\psi(t) = t\log\Big(\frac{e}{t}\Big), \quad 0 < t \le 1\,,$$

then the Lorentz space Λ_ψ on (Ω,μ) is denoted by $L\log L(\Omega)$ ($L\log\, L$ for short). Note that

$$\|f\|_{L\log\, L} = \int_0^1 f^*(t)\log\frac{1}{t}\, dt, \quad f \in L\log\, L,$$

and $L\log\, L(\mathbb{T})$ coincides up to equivalence of norms with the Orlicz space L_Φ generated by the function $\Phi(t) = t\log(1+t)$ for all $t \ge 0$.

We will denote by $QA(\mathbb{T})$ the quasi-Banach space introduced by Arias-de-Reyna in [4]; it is defined to be the space of all $f \in L^0(\mathbb{T})$ such that there exists a sequence $(f_n)_{n=1}^\infty$, $f_n \in L^\infty(\mathbb{T})$, such that

$$f = \sum_{n=1}^\infty f_n, \quad m\text{-a.e.}$$

and

$$\sum_{n=1}^\infty (1+\log\, n)\|f_n\|_1 \log\left(\frac{e\|f_n\|_\infty}{\|f_n\|_1}\right) < \infty\,.$$

QA is a quasi-Banach lattice on $\mathbb{T}$ equipped with the quasi-norm

$$\|f\|_{QA} = \inf \sum_{n=1}^\infty (1+\log\, n)\|f_n\|_1 \log\left(\frac{e\|f_n\|_\infty}{\|f_n\|_1}\right),$$

where the infimum is taken over all representations $f = \sum_{n=1}^\infty f_n$ of f as above.

We will use the following results. See [4, Theorem 9] for the first result and [6, Proposition 2.2] for the second one.

Theorem 2.1. *There exists a constant $C > 0$ such that, for every $f \in QA(\mathbb{T})$, we have*

$$\|\mathcal{S}f\|_{1,\infty} \leq C\|f\|_{QA}\,.$$

Proposition 2.2. *We have $QA(\mathbb{T}) \subset L\log L(\mathbb{T})$ with*

$$\|f\|_{L\log L} \leq \|f\|_{QA}, \quad f \in QA(\mathbb{T})\,.$$

Now we can state and prove the main results of this section. We point out that Theorem 2.3 and Theorem 2.4 below follow from a result in D. Lie's paper [15, Theorem 1.1 d)], however proved by a different method.

Theorem 2.3. *For every $p \in (1,\infty)$ there exists a constant $C_p > 0$ such that*

$$\|\mathcal{S}f\|_{1,\infty} \leq C_p\|f\|_1 \log\left(\frac{e\|f\|_p}{\|f\|_1}\right), \quad f \in L^p(\mathbb{T})\,.$$

Theorem 2.4. *For every $p \in (1,\infty)$ there exists a constant $C_p > 0$ such that*

$$\|\widetilde{\mathcal{S}}f\|_{1,\infty} \leq C_p\|f\|_1 \log\left(\frac{e\|f\|_p}{\|f\|_1}\right), \quad f \in L^p(\mathbb{T})\,.$$

Let us remark that usually the proofs of Carleson's theorem obtain bounds for a maximal operator that at the same time controls $\mathcal{S}$ and $\widetilde{\mathcal{S}}$. For instance in Arias-de-Reyna's book [3] this maximal operator is defined in page 45. The same operator appears in Fefferman [9] and in Lie [15]. So applying this and similar arguments to the case of $\mathcal{S}$, one can provide a proof of a result analogous to Theorem 2.1 for $\widetilde{\mathcal{S}}$. From this result Theorem 2.3 could be derived in the same way we are going to prove Theorem 2.4 from Theorem 2.1. However we are going to derive Theorem 2.4 from Theorem 2.3 by applying an estimate presented in Proposition 2.6, whose proof uses classical arguments. We find it interesting to include this different way that could be used in some cases where we only know the estimate for $\mathcal{S}$ and not for the bigger maximal operator.

In our study of multiple Fourier series we will use these results only for $p = 2$. It should be pointed out that the presence of the L^2-norm instead of the L^∞-norm which appears in Arias-de-Reyna's paper [4] is important. This difference allows us to perform an inductive passage to multiple dimensions, because Hilbert transform and Riesz projection are bounded on L^2 but not on L^∞.

Theorem 2.3 is a direct consequence of the following proposition and Theorem 2.1. As a byproduct the next proposition yields the proof of the following fact: if, in the definition of Arias-de-Reyna space $QA(\mathbb{T})$, we change the L^∞-norm by the L^p-norm, $p > 1$, we obtain the same space; we do not enlarge it as could be expected. See the Comment in [15, page 1242].

Proposition 2.5. *For every $p \in (1,\infty)$ there exists a constant $C_p > 0$ such that*

$$\|f\|_{QA} \leq C_p\|f\|_1 \log\left(\frac{e\|f\|_p}{\|f\|_1}\right), \quad f \in L^p(\mathbb{T})\,.$$

Proof. Take $f \in L^p(\mathbb{T})$. By homogeneity we can assume $\|f\|_1 = 1$. Let $b = \|f\|_p \geq 1$. We need to check $\|f\|_{QA} \leq C_p \log(eb)$. Let $\beta = eb$. Define the sequence (A_k) of pairwise disjoint measurable sets by

$$A_1 = \{|f| \leq \beta\}\,, \qquad A_k = \{\beta^{k-1} < |f| \leq \beta^k\}\,, \quad k \geq 2.$$

Define $f_k = f\chi_{A_k}$ and $a_k = \|f_k\|_1 = \int_{A_k} |f|\, dm$, for $k \in \mathbb{N}$. For $k \geq 2$ we have (by $\beta > 1$)

$$\beta^p \geq b^p \geq \int_{A_k} |f|^p\, dm \geq \int_{A_k} |f| \beta^{(p-1)(k-1)}\, dm = a_k \beta^{(p-1)(k-1)}$$

and whence $a_k \leq \beta^{2p-1-(p-1)k}$.

As we have $f = \sum_{k\geq 1} f_k$, we obtain the following bound for $\|f\|_{QA}$:

$$\|f\|_{QA} \leq \sum_{k\geq 1} (1+\log k)\|f_k\|_1 \log\left(\frac{e\|f_k\|_\infty}{\|f_k\|_1}\right) \leq \sum_{k\geq 1}(1+\log k) a_k \log\Big(\frac{e\beta^k}{a_k}\Big)\,.$$

Now observe that the function $x \mapsto x\log(ea/x)$ is increasing on the interval $(0,a]$ for every $a > 0$. Combining with $\beta \geq e$ and $a_k \leq \beta^{2p-1-(p-1)k} < \beta^k$ for each $k \geq 2$ yields

$$\begin{aligned} a_k \log\left(\frac{e\beta^k}{a_k}\right) &\leq \beta^{2p-1-(p-1)k} \log\left(\frac{e\beta^k}{\beta^{2p-1-(p-1)k}}\right) \leq \beta^{2p-1-(p-1)k} \log\left(\beta^{pk-2p+2}\right) \\ &= (pk-2p+2)\beta^{2p-1-(p-1)k}\log\beta, \quad k \geq 2. \end{aligned}$$

Let $A_p = (2p-1)/(p-1) \in (2,+\infty)$. If $k > A_p$, we have $2p-1-(p-1)k < 0$, and then

$$a_k \log\left(\frac{e\beta^k}{a_k}\right) \leq (pk-2p+2)e^{2p-1-(p-1)k}\log\beta\,, \qquad k > A_p\,.$$

For $k \leq A_p$, since $1 = \|f\|_1 \geq a_k$ and $\beta \geq e$,

$$a_k \log\left(\frac{e\beta^k}{a_k}\right) \leq \log(e\beta^k) \leq (k+1)\log\beta \leq (A_p+1)\log\beta\,, \qquad k \leq A_p\,.$$

In consequence we obtain

$$\begin{aligned} \|f\|_{QA} &\leq (1+\log A_p)(A_p+1)A_p\log\beta \\ &\qquad + \sum_{k>A_p} (1+\log k)(pk-2p+2)e^{2p-1-(p-1)k}\log\beta \\ &= C_p \log\beta, \end{aligned}$$

and this proves the result. $\square$

We will use the following proposition, inspired by [20, Theorem XIII.5.1], in the proof of Theorem 2.4. Before proving it we will provide first the proof of Theorem 2.4.

Proposition 2.6. *There exists a constant $C > 0$ such that, for every $f \in L^2(\mathbb{T})$ and every $x \in \mathbb{T}$, we have*

$$\widetilde{\mathcal{S}}f(x) \leq C\Big(\big[\mathcal{M}(|\mathcal{S}f|^{1/2})(x)\big]^2 + \Sigma(\mathcal{H}f)(x)\Big)\,.$$

Proof of Theorem 2.4. Applying Proposition 2.5 it is enough to prove

$$\|\widetilde{\mathcal{S}}f\|_{1,\infty} \leq C\|f\|_{QA}\,. \tag{7}$$

In order to prove (7) we use Proposition 2.6. We need to estimate $\|[\mathcal{M}(|\mathcal{S}f|^{1/2})]^2\|_{1,\infty}$ and $\|\Sigma(\mathcal{H}f)\|_{1,\infty}$.

By Zygmund's result (see, e.g., [20, Theorem VII.2.8]) the Hilbert transform $\mathcal{H}$ sends $L\log L(\mathbb{T})$ into $L^1(\mathbb{T})$. This fact and Proposition 2.2 provide the existence of a constant $C > 0$ such that

$$\|\mathcal{H}f\|_1 \le C\|f\|_{QA}, \qquad f \in QA(\mathbb{T}).$$

It is known (see, e.g., [3, pp. 23–24], that $\mathcal{M}g$ controls pointwise Σg; that is, $\Sigma g(x) \le C\mathcal{M}g(x)$, for every $x \in \mathbb{T}$ and every $g \in L^1(\mathbb{T})$. So, by Hardy-Littlewood's inequality,

$$\|\Sigma(\mathcal{H}f)\|_{1,\infty} \le C\|\mathcal{M}(\mathcal{H}f)\|_{1,\infty} \le C'\|\mathcal{H}f\|_1 \le C''\|f\|_{QA}\,. \tag{8}$$

By Proposition 2.1, $\mathcal{S}f \in L^{1,\infty}(\mathbb{T})$ for every $f \in QA(\mathbb{T})$. Therefore $|\mathcal{S}f|^{1/2}$ belongs to $L^{2,\infty}(\mathbb{T})$, for every $f \in QA(\mathbb{T})$ and

$$\||\mathcal{S}f|^{1/2}\|_{2,\infty} = \|\mathcal{S}f\|_{1,\infty}^{1/2} \le C\|f\|_{QA}^{1/2}\,. \tag{9}$$

The Hardy-Littlewood maximal operator $\mathcal{M}$ sends L^∞ into L^∞ and L^1 into $L^{1,\infty}$. By interpolation $\mathcal{M}$ sends $L^{2,\infty}$ into itself. Consequently, by (9), we have

$$\|[\mathcal{M}(|\mathcal{S}f|^{1/2})]^2\|_{1,\infty} = \|\mathcal{M}(|\mathcal{S}f|^{1/2})\|_{2,\infty}^2 \le C\||\mathcal{S}f|^{1/2}\|_{2,\infty}^2 \le C'\|f\|_{QA}\,. \tag{10}$$

Putting together the inequalities (8) and (10), using Proposition 2.6 and the fact that $\|\cdot\|_{1,\infty}$ is a quasi-norm, we deduce (7), and as we explained the theorem follows by Proposition 2.5. □

We need several lemmas for the proof of Proposition 2.6.

Lemma 2.7. *For $\alpha > 0$ and $k \in \mathbb{N}$ we denote $b_k(\alpha) = \sin(k\alpha) - \sin((k+1)\alpha)$. Let $f \in L^2(\mathbb{T})$, $n \in \mathbb{N}$ and $x \in \mathbb{T}$. We have*

$$2\sin(n\alpha)\widetilde{S_n}f(x) = S_nf(x-\alpha) - S_nf(x+\alpha) + 2\sum_{k=1}^{n-1}\widetilde{S_k}f(x)b_k(\alpha)\,. \tag{11}$$

Proof. Let $f \in L^2(\mathbb{T})$. We have, for $x \in \mathbb{T}$ and $\alpha > 0$,

$$\begin{aligned}\frac{1}{2}\big(S_nf(x-\alpha) - S_nf(x+\alpha)\big) &= \sum_{k=-n}^{n} i\widehat{f}(k)e^{ikx}\Big(\frac{e^{-ik\alpha}-e^{ik\alpha}}{2i}\Big)\\ &= \sum_{k=-n}^{n} -i\widehat{f}(k)e^{ikx}\sin(k\alpha)\\ &= \sum_{k=-n}^{n}\widehat{\mathcal{H}f}(k)e^{ikx}\sin(|k|\alpha) = \sum_{k=1}^{n}\big[\widetilde{S_k}f(x) - \widetilde{S_{k-1}}f(x)\big]\sin(k\alpha)\,,\end{aligned}$$

and by an Abel's summation, using that $\widetilde{S_0}f(x) = 0$,

$$= \sin(n\alpha)\widetilde{S_n}f(x) + \sum_{k=1}^{n-1}\widetilde{S_k}f(x)\big[\sin(k\alpha) - \sin((k+1)\alpha)\big]\,.$$

Now it is easy to deduce (11). □

Lemma 2.8. *Let $b_k(\alpha)$ be as in Lemma 2.7. Suppose $n \in \mathbb{N}$ and $\alpha \in [\pi/6n, 5\pi/6n]$. Then*

$$2n|b_{n-1}(\alpha)| + 2\sum_{k=1}^{n-2}(k+1)|b_k(\alpha) - b_{k+1}(\alpha)| \le 4\pi^2. \tag{12}$$

Proof. We have $b_k(\alpha) = -\int_{k\alpha}^{(k+1)\alpha} \cos t\, dt$. Therefore $|b_k(\alpha)| \le \alpha \le \pi/n$, and

$$|b_k(\alpha) - b_{k+1}(\alpha)| = \Big|\int_{k\alpha}^{(k+1)\alpha} [\cos t - \cos(t+\alpha)]\, dt\Big| \le \alpha^2 \le \frac{\pi^2}{n^2}\,.$$

The lemma follows easily since we have

$$2n|b_{n-1}(\alpha)| + 2\sum_{k=1}^{n-2}(k+1)|b_k(\alpha) - b_{k+1}(\alpha)| \le 2n\frac{\pi}{n} + 2\sum_{k=1}^{n-1} n\frac{\pi^2}{n^2} \le 4\pi^2.$$

□

Proof of Proposition 2.6. Let $n\in\mathbb{N}$, $x\in\mathbb{T}$ and $f\in L^2(\mathbb{T})$. Taking $\alpha\in[\pi/6n, 5\pi/6n]$ and applying Lemma 2.7, we get

$$2\sin(n\alpha)\widetilde{S_n}f(x) = S_nf(x-\alpha) - S_nf(x+\alpha) + 2\sum_{k=1}^{n-1}\widetilde{S_k}f(x)b_k(\alpha)\,. \tag{13}$$

Using that $\widetilde{S_k}f = S_k(\mathcal{H}f) = (k+1)\sigma_k(\mathcal{H}f) - k\sigma_{k-1}(\mathcal{H}f)$, for all $k \ge 1$, we have (by a new Abel's summation):

$$\begin{aligned}\sum_{k=1}^{n-1}\widetilde{S_k}f(x)b_k(\alpha) &= \sum_{k=1}^{n-1}[(k+1)\sigma_k(\mathcal{H}f)(x) - k\sigma_{k-1}(\mathcal{H}f)(x)]b_k(\alpha)\\ &= nb_{n-1}(\alpha)\sigma_{n-1}(\mathcal{H}f)(x) + \sum_{k=1}^{n-2}(k+1)\sigma_k(\mathcal{H}f)(x)[b_k(\alpha) - b_{k+1}(\alpha)]\,.\end{aligned}$$

By Lemma 2.8, since $|\sigma_k(\mathcal{H}f)(x)| \le \Sigma(\mathcal{H}f)(x)$, for every k, we deduce from (13) that

$$\begin{aligned}2|\sin(n\alpha)\widetilde{S_n}f(x)| &\le |S_nf(x-\alpha)| + |S_nf(x+\alpha)| + 4\pi^2\Sigma(\mathcal{H}f)(x)\\ &\le \mathcal{S}f(x-\alpha) + \mathcal{S}f(x+\alpha) + 4\pi^2\Sigma(\mathcal{H}f)(x)\,.\end{aligned}$$

Since $|\sin(n\alpha)| \ge 1/2$ for any $\alpha \in [\pi/6n, 5\pi/6n]$, we have

$$|\widetilde{S_n}f(x)| \le \Big(\inf_{\alpha\in[\pi/6n,5\pi/6n]} \mathcal{S}f(x-\alpha)^{1/2} + \mathcal{S}f(x+\alpha)^{1/2}\Big)^2 + 4\pi^2\Sigma(\mathcal{H}f)(x)\,. \tag{14}$$

The infimum can be bounded by the average over $[\pi/6n, 5\pi/6n]$:

$$\begin{aligned}&\inf_{\alpha\in[\pi/6n,5\pi/6n]} \mathcal{S}f(x-\alpha)^{1/2} + \mathcal{S}f(x+\alpha)^{1/2}\\ &\quad\le \frac{3n}{2\pi}\int_{\pi/6n}^{5\pi/6n}\big(\mathcal{S}f(x-\alpha)^{1/2} + \mathcal{S}f(x+\alpha)^{1/2}\big)\, d\alpha\\ &\quad\le \frac{3n}{2\pi}\int_{-5\pi/6n}^{5\pi/6n}\mathcal{S}f(x+t)^{1/2}\, dt \le \frac{5}{2}\mathcal{M}(|\mathcal{S}f|^{1/2})(x).\end{aligned}$$

Putting this estimate into (14), we get

$$|\widetilde{S_n}f(x)| \le \frac{25}{4}\big[\mathcal{M}(|\mathcal{S}f|^{1/2})(x)\big]^2 + 4\pi^2\Sigma(\mathcal{H}f)(x).$$

Taking the supremum in n the proposition follows. □

3. Interpolation estimates of the Carleson maximal operators in higher dimensions

The results of the previous section are one-dimensional, and it is natural to ask what happens in higher dimensions. Here we shall prove interpolation estimates for the Carleson maximal operators on the d-dimensional torus $\mathbb{T}^d$. In particular we prove the following result:

Theorem 3.1. *For each $d \in \mathbb{N}$ there exists $C_d > 0$ such that*

$$m_{\mathbb{T}^d}\big(\{\mathbf{x} \in \mathbb{T}^d;\ \mathcal{S}f(\mathbf{x}) > \lambda\}\big) \leq \frac{C_d}{\lambda}\, \|f\|_1 \left[\log\left(\frac{e\|f\|_2}{\|f\|_1}\right)\right]^d,$$

for every $f \in L^2(\mathbb{T}^d)$ and every $\lambda > 0$.

We provide two methods which are based on the one-dimensional case. The first one is based on some ideas from Antonov's paper [2]. The second one is also inspired in Antonov's paper, however with a different presentation using multipliers. This approach seems easier to handle and more flexible to be used in other contexts. We provide it with complete details.

3.1. **The standard approach.** In what follows, for a given $1 \leq \alpha < \infty$, we will denote by ψ_α the function defined on $[0,\infty)$ by $\psi_\alpha(0) = 0$, and

$$\psi_\alpha(t) = \begin{cases} \frac{1}{\alpha^\alpha}\, t \log^\alpha\left(\frac{e^\alpha}{t}\right) & \text{if } t \in (0,1], \\ 1 & \text{if } t \in (1,\infty). \end{cases}$$

In the case when $\alpha = 1$ we write ψ instead of ψ_1. It is easy to verify that ψ_α is a concave function on $[0,\infty)$ which is increasing on $[0,1]$.

We will need the following simple estimate which states that for every Lorentz space Λ_φ on an atomless probability space (Ω,μ) with $\varphi(0+) = 0$,

$$\|f\|_{\Lambda_\varphi} \leq \|f\|_\infty\, \varphi\Big(\frac{\|f\|_1}{\|f\|_\infty}\Big), \quad f \in L^\infty(\Omega). \tag{15}$$

To see this it is enough to use the formula (see [14, formula (5.4)])

$$\|f\|_{\Lambda_\varphi} = \int_0^\infty \varphi(\mu_f(\lambda))\, d\lambda, \quad f \in \Lambda_\varphi\,.$$

It is enough to show the estimate for $f \in L^\infty(\Omega)$ with $\|f\|_\infty \leq 1$.

Since $\mu_f(\lambda) = 0$ for all $\lambda \geq 1$, it follows by the above formula that

$$\|f\|_{\Lambda_\varphi} = \int_0^1 \varphi(\mu_f(\lambda))\, d\lambda\,.$$

Since φ is concave, the required estimate follows:

$$\|f\|_{\Lambda_\varphi} \leq \varphi\Big(\int_0^1 \mu_f(\lambda)\, d\lambda\Big) = \varphi(\|f\|_1)\,.$$

We now state and prove a lemma which we will need later. In the proof we will use the well-known fact that the Hilbert transform is bounded from $L\log L(\mathbb{T})$ to $L^1(\mathbb{T})$; i.e., there exists a constant $C > 0$ such that

$$\|\widetilde{f}\|_{L^1(\mathbb{T})} \leq C\, \|f\|_{L\log L(\mathbb{T})}, \quad f \in L\log L(\mathbb{T}).$$

Lemma 3.2. *Let g be a multiple trigonometric polynomial on $\mathbb{T}^d$ with $\|g\|_{L^\infty(\mathbb{T}^d)} \leq 1$, and let μ, $\nu \in \{-1, 0, 1\}$ with $\mu^2 + \nu^2 \neq 0$. Then there exists a constant $C > 0$ such that the function $h^{\mu,\nu}$ given by*

$$
\begin{aligned}
&h^{\mu,\nu}(x_1, \dots, x_d) \\
&\quad = -\frac{1}{2\pi} \int_{\mathbb{T}} g(x_1 + \mu t, x_2 + \nu t, x_3, \dots, x_d) \cot\left(\frac{t}{2}\right) dt, \qquad (x_1, \dots, x_d) \in \mathbb{T}^d,
\end{aligned}
$$

where the integral is understood in the sense of the principal value, is also a trigonometric polynomial and the inequality holds:

$$
\begin{aligned}
&\int_{\mathbb{T}^{d-1}} \left(\int_{\mathbb{T}} |h^{\mu,\nu}(x_1, \dots, x_d)|\, dx_1 \right) dx_2 \dots dx_d \\
&\qquad \leq C\, \psi\Big(\int_{\mathbb{T}^{d-1}} \left(\int_{\mathbb{T}} |g(x_1, x_2, \dots, x_d)|\, dx_1 \right) dx_2 \dots dx_d \Big).
\end{aligned}
$$

Proof. We assume without loss of generality that $\mu \neq 0$. Now observe that introducing the new variables $s_1 = x_1$, $s_2 = x_2 - (\nu/\mu)s_1$, $s_3 = x_3, \dots, s_d = x_d$, we obtain

$$
\begin{aligned}
h(x_1, \dots, x_d) &= -\frac{1}{2\pi} \int_{\mathbb{T}} g\Big(s_1 + \mu t, s_2 + \frac{\nu}{\mu} s_1 + \nu t, s_3, \dots, s_d\Big) \cot\left(\frac{t}{2}\right) dt \\
&= -\frac{1}{2\pi} \int_{\mathbb{T}} g\Big(s_1 + \mu t, s_2 + \frac{\nu}{\mu}(s_1 + \mu t), s_3, \dots, s_d\Big) \cot\left(\frac{t}{2}\right) dt\,.
\end{aligned}
$$

For fixed values $s_2, \dots, s_d$ the right-hand side of the last formula regarded as a function of s_1 is within a sign the conjugate function of $s_1 \mapsto g(s_1, s_2 + (\nu/\mu)s_1, s_3, \dots, s_d)$, and therefore combining the boundedness of the maximal conjugate function from $L \log L(\mathbb{T})$ to $L^1(\mathbb{T})$ with the estimate (15) for the Lorentz space $\Lambda_\psi = L \log L(\mathbb{T})$,

$$
\|f\|_{L \log L(\mathbb{T})} \leq \|f\|_{L^\infty(\mathbb{T})} \psi\left(\frac{\|f\|_{L^1(\mathbb{T})}}{\|f\|_{L^\infty(\mathbb{T})}} \right), \qquad f \in L^\infty(\mathbb{T}),
$$

we conclude that there exists a constant $C > 0$ such that

$$
\begin{aligned}
\int_{\mathbb{T}} \Big| \frac{1}{2\pi} \int_{\mathbb{T}} & g\Big(s_1 + \mu t, s_2 + \frac{\nu}{\mu}(s_1 + \mu t), s_3, \dots, s_d\Big) \cot\left(\frac{t}{2}\right) dt \Big| \Big)\, ds_1 \\
&\leq C \left\| g\Big(\cdot, s_2 + \frac{\nu}{\mu}(\cdot), s_3, \dots, s_d\Big) \right\|_{\Lambda_\psi(\mathbb{T})} \\
&\leq C\, \psi\Big(\left\| g\Big(\cdot, s_2 + \frac{\nu}{\mu}(\cdot), s_3, \dots, s_d\Big) \right\|_{L^1(\mathbb{T})} \Big).
\end{aligned}
$$

Combining this with the fact that the Jacobian of the transformation from $(s_1, \ldots, s_d)$ to $(x_1, \ldots, x_d)$ is equal to 1, we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^{d-1}} \left(\int_{\mathbb{T}} |h^{\mu,\nu}(x_1, \ldots, x_d)|\, dx_1 \right) dx_2 \ldots dx_d \\
&\quad \le C \int_{\mathbb{T}^{d-1}} \psi\Big(\Big\| g\Big(\cdot, s_2 + \frac{\nu}{\mu}(\cdot), s_3, \ldots, s_d\Big) \Big\|_{L^1(\mathbb{T})} \Big)\, ds_2 \ldots ds_d \\
&\quad \le C\, \psi\bigg(\int_{\mathbb{T}^{d-1}} \bigg(\int_{\mathbb{T}} \Big| g\Big(s_1, s_2 + \frac{\nu}{\mu} s_1, s_3, \ldots, s_d\Big) \Big|\, ds_1 \bigg) ds_2 \ldots ds_d \bigg) \\
&\quad = C\, \psi\bigg(\int_{\mathbb{T}^{d}} \Big| g\Big(s_1, s_2 + \frac{\nu}{\mu} s_1, s_3, \ldots, s_d\Big) \Big|\, ds_1 \ldots ds_d \bigg) \\
&\quad = C\, \psi\bigg(\int_{\mathbb{T}^{d}} |g(x_1, x_2, \ldots, x_d)|\, dx_1 \ldots dx_d \bigg),
\end{aligned}$$

and this completes the proof. □

We will need a variant of the Jensen inequality, which states that if (Ω, μ) is a probability space and $\phi\colon [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ is a concave function, then for every $f, g \in L^1(\Omega)$, we have

$$\int_{\Omega} \varphi(|f|, |g|)\, d\mu \le \varphi\Big(\int_{\Omega} |f|\, d\mu, \int_{\Omega} |g|\, d\mu \Big),$$

where $\varphi(s,t) := t\phi(s/t)$ for all $s \ge 0$, $t > 0$ and $\varphi(0,0) := 0$.

We are now ready to prove the theorem concerning interpolation estimates involving the maximal Carleson operator. The proof is based on some ideas from Antonov's paper [2].

Theorem 3.3. *Let $B \subset \{1, \ldots, d\}$ and $d \in \mathbb{N}$. There exists a constant $C_d > 0$ such that, for every $g \in L^2(\mathbb{T}^d)$, we have*

$$\big\| \widetilde{\mathcal{S}_B} g \big\|_{L_{1,\infty}(\mathbb{T}^d)} \le C_d\, \|g\|_{L^2(\mathbb{T}^d)} \psi_d \left(\frac{\|g\|_{L^1(\mathbb{T}^d)}}{\|g\|_{L^2(\mathbb{T}^d)}} \right). \tag{16}$$

Proof. It is enough to prove the theorem for any multiple trigonometric polynomial g on $\mathbb{T}^d$. Without loss of generality we may assume that $\|g\|_2 = 1$. We use induction with respect to d. We have already proved that the result is true with $d = 1$, $B = \emptyset$ and with $d = 1$, $B = \{1\}$.

Let $d \ge 2$ and assume that the inequality (16) holds for $d - 1$. Let g be an arbitrary polynomial. For a given $B \subset \mathbb{N}_d = \{1, \ldots, d\}$, we put $\overline{B} := \mathbb{N}_d \setminus B$. Following the proof of Theorem 2.1 in [2] we consider three cases:

Case (i). $1 \in \overline{B}$ and $2 \in \overline{B}$. Then

$$\widetilde{S_{N,B}} g(\mathbf{x}) = \frac{1}{\pi^d} \int_{\mathbb{T}^d} D_N(t_1) D_N(t_2) \Pi_N(t_3, \ldots, t_d) g(\mathbf{x} + \mathbf{t})\, d\mathbf{t}, \qquad \mathbf{x} \in \mathbb{T}^d, \tag{17}$$

where for any $(t_1, \ldots, t_d) \in \mathbb{T}^d$,

$$\Pi_N(t_3, \ldots, t_d) = \prod_{j \in B} \big(- \widetilde{D}_N(t_j) \big) \prod_{j \in \overline{B} \setminus \{1,2\}} D_N(t_j).$$

Combining trigonometric formulas

$$D_N(t_1) D_N(t_2) = D_N^1 + D_N^2 + \frac{1}{4},$$

where

$$D_N^1 := \frac{1}{2}\big[\widetilde{D}_N(t_2+t_1) - \widetilde{D}_N(t_2-t_1)\big]\frac{1}{2}\cot\Big(\frac{t_1}{2}\Big),$$
$$D_N^2 := \frac{1}{2}\big[\widetilde{D}_N(t_1+t_2) - \widetilde{D}_N(t_1-t_2)\big]\frac{1}{2}\cot\Big(\frac{t_2}{2}\Big),$$

yields (see [2, proof of Theorem 2.1])

$$\begin{aligned}
2\widetilde{\mathcal{S}_B}g(\mathbf{x}) \le \max_{N\in\mathbb{N}} \Big| &\frac{1}{\pi^{d-1}}\int_{\mathbb{T}^{d-1}} \widetilde{D}_N(u_2)\Pi_N(u_3,\dots,u_d) \\
&\times h^{1,-1}(x_1, x_2+u_2, x_3+u_3,\dots,x_d+u_d)\,du_2\dots du_d\Big| \\
+\max_{N\in\mathbb{N}}\Big| &\frac{1}{\pi^{d-1}}\int_{\mathbb{T}^{d-1}} \widetilde{D}_N(u_2)\Pi_N(u_3,\dots,u_d) \\
&\times h^{1,1}(x_1, x_2+u_2, x_3+u_3,\dots,x_d+u_d)\,du_2\dots du_d\Big| \\
+\max_{N\in\mathbb{N}}\Big| &\frac{1}{\pi^{d-1}}\int_{\mathbb{T}^{d-1}} \widetilde{D}_N(u_1)\Pi_N(u_3,\dots,u_d) \\
&\times h^{-1,1}(x_1+u_1, x_2, x_3+u_3,\dots,x_d+u_d)\,du_2\dots du_d\Big| \\
+\max_{N\in\mathbb{N}}\Big| &\frac{1}{\pi^{d-1}}\int_{\mathbb{T}^{d-1}} \widetilde{D}_N(u_2)\Pi_N(u_3,\dots,u_d) \\
&\times h^{1,1}(x_1, x_2+u_2, x_3+u_3,\dots,x_d+u_d)\,du_2\dots du_d\Big| \\
+\max_{N\in\mathbb{N}}\Big| &\frac{1}{2\pi^{d}}\int_{\mathbb{T}^{d}} \Pi_N(u_3,\dots,u_d) g(x_1+t_1,\dots,x_d+t_d)\,dt_1\dots dt_d\Big| \\
&= \widetilde{\mathcal{S}_B}^1 g(\mathbf{x}) + \widetilde{\mathcal{S}_B}^2 g(\mathbf{x}) + \widetilde{\mathcal{S}_B}^3 g(\mathbf{x}) + \widetilde{\mathcal{S}_B}^4 g(\mathbf{x}) + \widetilde{\mathcal{S}_B}^5 g(\mathbf{x}).
\end{aligned}$$

We claim that there exist universal constants C_j such that for each $1 \le j \le 5$,

$$\|\widetilde{\mathcal{S}_B}^j g\|_{L_{1,\infty}(\mathbb{T}^d)} \le C_j\,\psi_d\big(\|g\|_{L^1(\mathbb{T}^d)}\big). \tag{18}$$

For simplicity of presentation for each $d\in\mathbb{N}$, we put $\varphi_d(s,t) := t\psi_d(s/t)$ for all $s,t>0$ and $\varphi_d(0,0) := 0$.

Fix $x_1 \in \mathbb{T}$. By application of the induction hypothesis, it follows that for polynomial $h_{x_1}^{1,-1}$ given by $h_{x_1}^{1,-1}(x_2,\dots,x_d) = h^{1,-1}(x_1,x_2,\dots,x_d)$ for $(x_2,\dots,x_d)\in\mathbb{T}^{d-1}$ and $B' = B\setminus\{1\} \subset \{2,3,\dots,d\}$, we have

$$\begin{aligned}
\lambda\, m_{d-1}\Big\{(x_2,\dots,x_d)\in\mathbb{T}^{d-1};\ \max_{N\in\mathbb{N}}\Big|&\int_{\mathbb{T}^{d-1}}\widetilde{D}_N(u_2)\Pi_N(u_3,\dots,u_d) \\
&\times h^{1,-1}(x_1,x_2+u_2,\dots,x_d+u_d)\,du_2\dots du_d\Big| > \lambda\Big\} \\
&= m_{d-1}\Big\{(x_2,\dots,x_d)\in\mathbb{T}^{d-1};\ \max_{N\in\mathbb{N}}|S_{N,B'}h_{x_1}^{1,-1}(x_2,\dots,x_d)| > \lambda\Big\} \\
&\le C_{d-1}\varphi_{d-1}\big(\|h_{x_1}^{1,-1}\|_{L^1(\mathbb{T}^{d-1})}, \|h_{x_1}^{1,-1}\|_{L^2(\mathbb{T}^{d-1})}\big).
\end{aligned}$$

Thus, using the identity

$$\begin{aligned}
m_d\big\{&(x_1,\dots,x_d)\in\mathbb{T}^d;\ \widetilde{\mathcal{S}_B}^1 g(\mathbf{x}) > \lambda\big\} \\
&= \int_{\mathbb{T}} m_{d-1}\Big\{(x_2,\dots,x_d)\in\mathbb{T}^{d-1};\ \max_{N\in\mathbb{N}}\Big|\int_{\mathbb{T}^{d-1}}\widetilde{D}_N(u_2)\Pi_N(u_3,\dots,u_d) \\
&\times h^{1,-1}(x_1,x_2+u_2,\dots,x_d+u_d)\,du_2\dots du_d\Big| > \lambda\Big\}\,dx_1
\end{aligned}$$

yields

$$\lambda\, m_d\big\{(x_1,\dots,x_d)\in\mathbb{T}^d;\ \widetilde{S_B^1}g(\mathbf{x})>\lambda\big\} \\ \le C_{d-1}\int_{\mathbb{T}}\varphi_{d-1}\big(\|h_{x_1}^{1,-1}\|_{L^1(\mathbb{T}^{d-1})},\|h_{x_1}^{1,-1}\|_{L^2(\mathbb{T}^{d-1})}\big)\,dx_1\,.$$

Combining with Lemma 3.2 and the Jensen inequality, we get

$$\begin{aligned}
&\lambda\, m_d\big\{(x_1,\dots,x_d)\in\mathbb{T}^d;\ \widetilde{\mathcal{S}_B}^1 g(\mathbf{x})>\lambda\big\}\\
&\quad\le C_{d-1}\,\varphi_{d-1}\bigg(\int_{\mathbb{T}}\|h_{x_1}^{1,-1}\|_{L^1(\mathbb{T}^{d-1})}\,dx_1,\ \int_{\mathbb{T}}\|h_{x_1}^{1,-1}\|_{L^2(\mathbb{T}^{d-1})}\,dx_1\bigg)\\
&\quad\le C_{d-1}\,\varphi_{d-1}\bigg(\int_{\mathbb{T}^{d-1}}\bigg(\int_{\mathbb{T}}|h^{1,-1}(x_1,\dots,x_d)\,dx_1\bigg)\,dx_2\dots dx_d,\ \|h^{1,-1}\|_{L^2(\mathbb{T}^d)}\bigg)\\
&\quad\le \widetilde{C}_1\,\varphi_{d-1}\bigg(\int_{\mathbb{T}^{d-1}}\psi\bigg(\int_{\mathbb{T}}|g(x_1,\dots,x_d)|\,dx_1\bigg)\,dx_2\dots dx_d\,,\|g\|_{L^2(\mathbb{T}^d)}\bigg)\\
&\quad= \widetilde{C}_1\,\psi_{d-1}\bigg(\int_{\mathbb{T}^{d-1}}\psi\bigg(\int_{\mathbb{T}}|g(x_1,\dots,x_d)|\,dx_1\bigg)\,dx_2\dots dx_d\bigg)\\
&\quad\le \widetilde{C}_2\,\psi_{d-1}\big(\psi\big(\|g\|_{L^1(\mathbb{T}^d)}\big)\big).
\end{aligned}$$

Since $\psi_{d-1}(\psi(t))\le c_d\psi_d(t)$ with $c_d=d^d/(d-1)^{d-1}$ for every $0<t\le 1$ and $\|g\|_{L^1(\mathbb{T}^d)}\le 1$ (by $\|g\|_{L^2(\mathbb{T}^d)}=1$), we finally conclude that there exists a constant $C_1=C_1(d)$ such that

$$m_d\big\{(x_1,\dots,x_d)\in\mathbb{T}^d;\ \widetilde{\mathcal{S}_B}^1\, g(\mathbf{x})>\lambda\big\}\le\frac{C_1}{\lambda}\,\psi_d\big(\|g\|_{L^1(\mathbb{T}^d)}\big).$$

We can similarly verify that $\widetilde{S}^j g$ satisfies (18) with some constants C_j for each $1\le j\le 4$. It remains to show the estimate of $\widetilde{\mathcal{S}_B}^5 g$. In the case when $d=2$, we have $\Pi_N\equiv 1$, and so

$$\widetilde{\mathcal{S}_B}^5 g(\mathbf{x})\le\frac{1}{2\pi^d}\int_{\mathbb{T}^d}|g(\mathbf{t})|\,d\mathbf{t},\qquad \mathbf{x}\in\mathbb{T}^d,$$

yields

$$\big\|\widetilde{\mathcal{S}_B}^5 g\big\|_{L_{1,\infty}(\mathbb{T}^2)}\le 2^2\|g\|_{L^1(\mathbb{T}^2)}\le 4\|g\|_{L^2(\mathbb{T}^2)}\psi_2\left(\frac{\|g\|_{L^1(\mathbb{T}^2)}}{\|g\|_{L^2(\mathbb{T}^2)}}\right).$$

In the case when $d>2$ we have

$$\begin{aligned}
&m_d\big\{(x_1,\dots,x_d)\in\mathbb{T}^d;\ \widetilde{\mathcal{S}_B}^5 g(\mathbf{x})>\lambda\big\}\\
&\quad=\int_{\mathbb{T}^2}m_{d-2}\bigg\{(x_3,\dots,x_d)\in\mathbb{T}^{d-2};\ \max_{N\in\mathbb{N}}\bigg|\frac{1}{2\pi^d}\int_{\mathbb{T}^{d-2}}\Pi_N(u_3,\dots,u_d)\\
&\qquad\bigg(\int_{\mathbb{T}^2}g(x_1+t_1,x_2+t_2,\dots,x_d+t_d)\,dt_1\,dt_2\bigg)\,dt_3\dots dt_d\bigg|>\lambda\bigg\}\,dx_1\,dx_2\,.
\end{aligned}$$

We can easily verify that the required estimate (18) for $j=5$ follows by the inductive hypothesis applied for the polynomial

$$h_{x_1,x_2}(t_3,\dots,t_d)=\frac{1}{2\pi^2}\int_{\mathbb{T}^2}g(x_1+t_1,x_2+t_2,t_3,\dots,t_d)\,dt_1\,dt_2$$

in the case of dimension $d-2$.

Case (ii). Let $1 \in B$ and $2 \in \overline{B}$ (the case when $1 \in \overline{B}$ and $2 \in B$ can be treated the same way). In this case we have

$$\widetilde{S_{N,B}}g(\mathbf{x}) = -\frac{1}{\pi^d}\int_{\mathbb{T}^d} \widetilde{D}_N(t_1)D_N(t_2)\Pi_N(t_3,\dots,t_d)\,g(\mathbf{x}+\mathbf{t})\,d\mathbf{t}\,.$$

Again following the calculus shown in [2, proof of Theorem 2.1], we obtain

$$\begin{aligned}
2\widetilde{\mathcal{S}_B}g(\mathbf{x}) \le \max_{N\in\mathbb{N}} \Big| &\frac{1}{\pi^{d-1}}\int_{\mathbb{T}^{d-1}} \widetilde{D}_N(u_2)\Pi_N(u_3,\dots,u_d) \\
&\times h^{1,-1}(x_1, x_2+u_2, x_3+u_3,\dots,x_d+u_d)\,du_2\dots du_d\Big| \\
+\max_{N\in\mathbb{N}} \Big| &\frac{1}{\pi^{d-1}}\int_{\mathbb{T}^{d-1}} \widetilde{D}_N(u_2)\Pi_N(u_3,\dots,u_d) \\
&\times h^{1,1}(x_1, x_2+u_2, x_3+u_3,\dots,x_d+u_d)\,du_2\dots du_d\Big| \\
+\max_{N\in\mathbb{N}} \Big| &\frac{1}{\pi^{d-1}}\int_{\mathbb{T}^{d-1}} \widetilde{D}_N(u_1)\Pi_N(u_3,\dots,u_d) \\
&\times h^{1,0}(x_1, x_2+u_2, x_3+u_3,\dots,x_d+u_d)\,du_2\dots du_d\Big| \\
+\max_{N\in\mathbb{N}} \Big| &\frac{1}{\pi^{d-1}}\int_{\mathbb{T}^{d-1}} \widetilde{D}_N(u_1)\Pi_N(u_3,\dots,u_d) \\
&\times h^{-1,1}(x_1+u_1, x_2, x_3+u_3,\dots,x_d+u_d)\,du_2\dots du_d\Big| \\
+\max_{N\in\mathbb{N}} \Big| &\frac{1}{2\pi^{d}}\int_{\mathbb{T}^{d-1}} \widetilde{D}_N(u_1)\Pi_N(u_3,\dots,u_d) \\
&\times h^{1,1}h(x_1+u_1, x_2, x_3+u_3,\dots,x_d+u_d)\,du_1\dots du_d\Big|\,.
\end{aligned}$$

Case (iii). Let $1 \in B$ and $2 \in B$. Then via the trigonometric equalities

$$\widetilde{D}_N(t_1)\widetilde{D}_N(t_2) = D_N^5 + D_N^6,$$

where

$$\begin{aligned}
D_N^5 &:= \frac{1}{2}\big[\widetilde{D}_N(t_2+t_1) + \widetilde{D}_N(t_2-t_1) - 2\widetilde{D}_N(t_2)\big]\frac{1}{2}\cot\Big(\frac{t_1}{2}\Big), \\
D_N^6 &:= \frac{1}{2}\big[\widetilde{D}_N(t_1+t_2) + \widetilde{D}_N(t_2-t_1) - 2\widetilde{D}_N(t_2)\big]\frac{1}{2}\cot\Big(\frac{t_2}{2}\Big),
\end{aligned}$$

we can prove in a similar way as in the previous two cases that the required estimate holds. □

3.2. **Approach via multipliers.** Here we prove interpolation estimates for the Carleson maximal operators in higher dimensions by using multipliers. This approach seems to be more flexible and easy to handle. In fact, it can be used to prove, for a large class of functions, the almost everywhere convergence of multiple Fourier series summed up over the dilation of polytopes. To be more concrete, if P is a convex polytope in $\mathbb{R}^d$ with 0 in its interior and $f \in QA^d(\mathbb{T}^d)$ (see the definition of QA^d in Section 4), then we have

$$f(\mathbf{x}) = \lim_{\lambda\to+\infty} \sum_{\mathbf{k}\in(\lambda P)\cap\mathbb{Z}^d} \widehat{f}(\mathbf{k})\exp(i\langle\mathbf{k},\mathbf{x}\rangle), \qquad \text{for a.e. } \mathbf{x}\in\mathbb{T}^d.$$

This extends the main result of Fefferman in [8], where the above property was proved for P a polygon in the plane $\mathbb{R}^2$ and $f \in L^p(\mathbb{T}^2)$, $p > 1$. Details will appear in a forthcoming paper.

Let us describe the notation that we shall use. A *multiplier* on $L^2(\mathbb{T}^d)$ is a bounded operator $T\colon L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ such that there exists a bounded family $\{a_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{Z}^d}$ such that for every $f \in L^2(\mathbb{T}^d)$,

$$\widehat{Tf}(\mathbf{m}) = a_{\mathbf{m}}\widehat{f}(\mathbf{m}), \quad \mathbf{m} \in \mathbb{Z}^d\,.$$

We will denote then $a_{\mathbf{m}} = \widehat{T}(\mathbf{m})$, for all $\mathbf{m} \in \mathbb{Z}^d$. Every translation operator $f \mapsto f_{\mathbf{x}}$ is a multiplier, where $f_{\mathbf{x}}(\mathbf{y}) = f(\mathbf{x}+\mathbf{y})$ and $\mathbf{x}, \mathbf{y} \in \mathbb{T}^d$. In fact multipliers are the only bounded operators on $L^2(\mathbb{T}^d)$ commuting with all translation operators.

Notice that in the case when $d = 1$, the Hilbert transform $\mathcal{H}$ and the Riesz projection $\mathcal{R}$ are multipliers and

$$\widehat{\mathcal{H}}(m) = -i\,\mathrm{sgn}(m)\,, \qquad \widehat{\mathcal{R}}(m) = \chi_{[0,+\infty)}(m)\,, \qquad m \in \mathbb{Z}\,,$$

where, for $x \in \mathbb{R}$, we put $\mathrm{sgn}(x) = 1$ if $x > 0$, $\mathrm{sgn}(x) = -1$ if $x < 0$, and $\mathrm{sgn}(0) = 0$. The partial sums (over cubes) of the Fourier series S_N are also multipliers, and we have

$$\widehat{S_N}(\mathbf{m}) = \chi_{[-N,N]^d}(\mathbf{m}), \quad N \in \mathbb{N}, \quad \mathbf{m} \in \mathbb{Z}^d.$$

For every $B \subset \{1, 2, \ldots, d\}$, the operator sending f to its B-conjugate function is a multiplier too. Denoting this operator by $\mathcal{H}_B$, it is easy to see that

$$\widehat{\mathcal{H}_B}(\mathbf{m}) = \prod_{j\in B} \widehat{\mathcal{H}}(m_j) = \prod_{j\in B} (-i\ \mathrm{sgn}(m_j))\,, \qquad \mathbf{m} \in \mathbb{Z}^d.$$

As we have $\widetilde{S_{N,B}}f = S_N\big(\mathcal{H}_B f\big)$, the operator $\widetilde{S_{N,B}}$ is also a multiplier.

Given a sequence $(T_N)_{N\geq 1}$ of multipliers we will define its maximal operator $\mathcal{M}(T_N)$ by

$$\mathcal{M}(T_N)f(\mathbf{x}) = \sup_{N\geq 1} |T_N f(\mathbf{x})|\,, \qquad \mathbf{x} \in \mathbb{T}^d\,.$$

We will say that $(T_N)_{N\geq 1}$ is an *admissible sequence of multipliers in* $\mathbb{T}^d$ if there exists a constant $C > 0$ such that

$$\big\|\mathcal{M}(T_N)f\big\|_{1,\infty} \leq C\|f\|_1 \left[\log\left(\frac{e\|f\|_2}{\|f\|_1}\right)\right]^d, \qquad f \in L^2(\mathbb{T}^d). \tag{19}$$

Observe that for the sequence (S_N) of partial sums we have $\mathcal{S} = \mathcal{M}(S_N)$. Therefore in this language the statement of Theorem 3.1 is equivalent to saying that (S_N) is an admissible sequence of multipliers in $\mathbb{T}^d$. In the same way we have $\widetilde{\mathcal{S}_B} = \mathcal{M}(\widetilde{S_{N,B}})$, and the statement of Theorem 3.3 is equivalent to the fact that, for all $B \subset \{1, 2 \ldots, d\}$, the sequence $(\widetilde{S_{N,B}})$ is admissible.

In the following two lemmas we collect some properties of admissible sequences of multipliers. The proof of the first one is obvious and so we omit it.

Lemma 3.4. *For each integer $1 \leq j \leq J$, let $(T_N^j)_{N\geq 1}$ be an admissible sequence of multipliers in $\mathbb{T}^d$. If $\alpha_j \in \mathbb{C}$, for each $1 \leq j \leq J$, we define*

$$T_N = \sum_{j=1}^{J} \alpha_j T_N^j\,, \quad N \in \mathbb{N}\,.$$

Then $(T_N)_{N\geq 1}$ is an admissible sequence of multipliers in $\mathbb{T}^d$.

Lemma 3.5. *Let $(T_N)_{N\geq 1}$ be an admissible sequence of multipliers in $\mathbb{T}^d$, and let $\Phi\colon \mathbb{Z}^d \to \mathbb{Z}^d$ be an automorphism of the group $\mathbb{Z}^d$. Define, for each $N \geq 1$, the multiplier U_N by the formula*

$$\widehat{U_N}(\mathbf{m}) = \widehat{T_N}\big(\Phi(\mathbf{m})\big), \quad \mathbf{m} \in \mathbb{Z}^d .$$

Then $(U_N)_{N\geq 1}$ is also an admissible sequence of multipliers in $\mathbb{T}^d$.

Proof. There exists a continuous isomorphism $\varphi\colon \mathbb{T}^d \to \mathbb{T}^d$ such that

$$\exp(i\langle \mathbf{m}, \varphi(\mathbf{x})\rangle) = \exp(i\langle \Phi(\mathbf{m}), \mathbf{x}\rangle), \quad (\mathbf{m}, \mathbf{x}) \in \mathbb{Z}^d \times \mathbb{T}^d . \tag{20}$$

Indeed there exists a $d \times d$ matrix $(a_{j,k})$ with integer entries and determinant ± 1 such that

$$\Phi(\mathbf{m}) = \Big(\sum_{k=1}^{d} a_{1,k} m_k, \sum_{k=1}^{d} a_{2,k} m_k\, , \dots ,\, \sum_{k=1}^{d} a_{d,k} m_k\Big), \quad \mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{Z}^d.$$

To see this it is enough, considering equivalence classes in $\mathbb{R}/2\pi\mathbb{Z}$, to define φ by

$$\begin{aligned} &\varphi(x_1, x_2, \dots, x_m) \\ &\quad = \Big(\sum_{j=1}^{d} a_{j,1} x_j, \sum_{j=1}^{d} a_{j,2} x_j\, , \dots ,\, \sum_{j=1}^{d} a_{j,d} x_j\Big), \quad \mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{T}^d . \end{aligned}$$

This isomorphism φ preserves the measure m_d, and it follows by (20) that for every $f \in L^1(\mathbb{T}^d)$, $\mathbf{m} \in \mathbb{Z}^d$,

$$\widehat{f \circ \varphi}\big(\Phi(\mathbf{m})\big) = \widehat{f}(\mathbf{m})\,, \quad \mathbf{m} \in \mathbb{Z}^d . \tag{21}$$

This yields $U_N f = [T_N(f \circ \varphi)] \circ \varphi^{-1}$, for all $f \in L^2(\mathbb{T}^d)$ by

$$\widehat{U_N f}(\mathbf{m}) = \widehat{U_N}(\mathbf{m})\widehat{f}(\mathbf{m}) = \widehat{T_N}\big(\Phi(\mathbf{m})\big)\widehat{f \circ \varphi}\big(\Phi(\mathbf{m})\big) = \widehat{T_N(f \circ \varphi)}\big(\Phi(\mathbf{m})\big)\,,$$

and (21) implies $\widehat{g}\big(\Phi(\mathbf{m})\big) = \widehat{g \circ \varphi^{-1}}(\mathbf{m})$, for all $g \in L^1(\mathbb{T}^d)$.

Let $\mathcal{T} = \mathcal{M}(T_N)$ and $\mathcal{U} = \mathcal{M}(U_N)$ be the corresponding maximal operators. We have

$$\mathcal{U} f = [\mathcal{T}(f \circ \varphi)] \circ \varphi^{-1}\,, \quad f \in L^2(\mathbb{T}^d).$$

But, since φ and φ^{-1} preserve the measure $m_{\mathbb{T}^d}$ and (T_N) is admissible, we have

$$\begin{aligned} \|\mathcal{U} f\|_{1,\infty} = \|\mathcal{T}(f \circ \varphi)\|_{1,\infty} &\leq C\|f \circ \varphi\|_1 \Big[\log\Big(\frac{e\|f \circ \varphi\|_2}{\|f \circ \varphi\|_1}\Big)\Big]^d \\ &= C\|f\|_1 \Big[\log\Big(\frac{e\|f\|_2}{\|f\|_1}\Big)\Big]^d , \end{aligned}$$

for all $f \in L^2(\mathbb{T}^d)$, and the sequence (U_N) is admissible. $\square$

Define now, for every $\sigma \in \{0,1\}^d$ and every $N \geq 1$, the following subsets of $\mathbb{Z}^d$:

$$\Delta(\sigma, N) = \Big(\prod_{j=1}^{d} [\sigma_j, N]\Big) \cap \mathbb{Z}^d\,; \qquad C(\sigma, N) = \Big([\sigma_1, N] \times \prod_{j=2}^{d} [\sigma_j, +\infty)\Big) \cap \mathbb{Z}^d\,. \tag{22}$$

Define also the multipliers D_N^σ and P_N^σ by

$$\widehat{D_N^\sigma} = \chi_{\Delta(\sigma,N)} \qquad \text{and} \qquad \widehat{P_N^\sigma} = \chi_{C(\sigma,N)}\,.$$

As we have said, in order to prove Theorem 3.1 we just have to check that (S_N) is an admissible sequence in $\mathbb{T}^d$. Observing that $[-N,N]^d \cap \mathbb{Z}^d = ([-N,-1] \cup [0,N])^d \cap \mathbb{Z}^d$, it is easy to see that

$$\begin{aligned}\widehat{S_N}(\mathbf{m}) = \chi_{[-N,N]^d}(\mathbf{m}) &= \sum_{\sigma\in\{0,1\}^d} \chi_{\Delta(\sigma,N)}\big(E_\sigma(\mathbf{m})\big)\\ &= \sum_{\sigma\in\{0,1\}^d} \widehat{D_N^\sigma}\big(E_\sigma(\mathbf{m})\big), \quad \text{for all } \mathbf{m}\in\mathbb{Z}^d;\end{aligned}$$

where $E_\sigma(m_1,m_2,\dots,m_d) = \big((-1)^{\sigma_1}m_1,(-1)^{\sigma_2}m_2,\dots,(-1)^{\sigma_d}m_d\big)$. As every E_σ is an automorphism of $\mathbb{Z}^d$, applying Lemma 3.4 and Lemma 3.5, the proof of Theorem 3.1 will be done as soon as we prove the following.

Theorem 3.6. *For every $\sigma \in \{0,1\}^d$, the sequence $(D_N^\sigma)_{N\geq 1}$ is an admissible sequence of multipliers in $\mathbb{T}^d$.*

Theorem 3.6 is a direct consequence, via Lemma 3.4 and Lemma 3.5, of the following propositions to be proved in the following subsections.

Proposition 3.7. *For every $\sigma \in \{0,1\}^d$, there exist a finite sequence $(\Phi_j)_{j=1}^J$ of automorphism of $\mathbb{Z}^d$, a finite sequence $(\tau^j)_{j=1}^J$ in $\{0,1\}^d$, and a finite sequence of scalars $(\alpha_j)_{j=1}^J$ such that*

$$\widehat{D_N^\sigma}(\mathbf{m}) = \chi_{\Delta(\sigma,N)}(\mathbf{m}) = \sum_{j=1}^J \alpha_j \widehat{P_N^{\tau^j}}\left(\Phi_j(\mathbf{m})\right), \qquad \mathbf{m}\in\mathbb{Z}^d.$$

Proposition 3.8. *For every $\sigma \in \{0,1\}^d$, the sequence $(P_N^\sigma)_{N\geq 1}$ is an admissible sequence of multipliers in $\mathbb{T}^d$.*

3.3. **An inductive procedure of decomposition.** The aim of this section is to provide the proof of Proposition 3.7. The proof goes by induction on d.

Proof of Proposition 3.7. For $d=1$ the result is trivial because $D_N^\sigma = P_N^\sigma$ when $d=1$. Let us see the case $d=2$. We have four different $\sigma\in\{0,1\}^2$. We have to decompose the characteristic functions of the intersection with $\mathbb{Z}^2$ of the following sets:

$$[0,N]\times[0,N], \qquad [1,N]\times[1,N], \qquad [0,N]\times[1,N], \qquad \text{and} \qquad [1,N]\times[0,N].$$

We just need to consider the first three cases, because the fourth one is obtained from the third one with a permutation of coordinates which is an automorphism of $\mathbb{Z}^2$.

The reader can easily check that we have, for $\mathbf{m}=(m_1,m_2)\in\mathbb{Z}^2$:

$$\begin{aligned}\chi_{[0,N]\times[0,N]}(\mathbf{m}) &= \chi_{[0,N]}(m_1)\chi_{[0,+\infty)}(m_2) + \chi_{[0,N]}(m_2)\chi_{[0,+\infty)}(m_1)\\ &\quad - \chi_{[0,N]}(m_1)\chi_{[0,+\infty)}(m_2-m_1) - \chi_{[0,N]}(m_2)\chi_{[1,+\infty)}(m_1-m_2)\,.\end{aligned}$$

Therefore

$$\widehat{D_N^{(0,0)}}(\mathbf{m}) = \widehat{P_N^{(0,0)}}(\mathbf{m}) + \widehat{P_N^{(0,0)}}\big(\Phi_1(\mathbf{m})\big) - \widehat{P_N^{(0,0)}}\big(\Phi_2(\mathbf{m})\big) - \widehat{P_N^{(0,1)}}\big(\Phi_3(\mathbf{m})\big)\,,$$

where Φ_1, Φ_2 and Φ_3 are the automorphisms of $\mathbb{Z}^2$ defined by $\Phi_1(m_1,m_2) = (m_2,m_1)$, $\Phi_2(m_1,m_2) = (m_1,m_2-m_1)$ and $\Phi_3(m_1,m_2)=(m_2,m_1-m_2)$.

Analogously, we have, with the same meaning of Φ_1, Φ_2 and Φ_3:

$$\widehat{D_N^{(1,1)}}(\mathbf{m}) = \widehat{P_N^{(1,1)}}(\mathbf{m}) + \widehat{P_N^{(1,1)}}\big(\Phi_1(\mathbf{m})\big) - \widehat{P_N^{(1,0)}}\big(\Phi_2(\mathbf{m})\big) - \widehat{P_N^{(1,1)}}\big(\Phi_3(\mathbf{m})\big)$$

and

$$\widehat{D_N^{(0,1)}}(\mathbf{m}) = \widehat{P_N^{(0,1)}}(\mathbf{m}) + \widehat{P_N^{(1,0)}}\big(\Phi_1(\mathbf{m})\big) - \widehat{P_N^{(0,1)}}\big(\Phi_2(\mathbf{m})\big) - \widehat{P_N^{(1,0)}}\big(\Phi_3(\mathbf{m})\big).$$

The case $d = 2$ is complete.

Assume now that the statement of the theorem is true for $d-1 \geq 2$, and let us prove it for d. Let $\sigma \in \{0,1\}^d$ and write $\sigma' = (\sigma_2, \dots, \sigma_d) \in \{0,1\}^{d-1}$. Thus we can put $\sigma = (\sigma_1, \sigma')$. By the induction hypothesis, we can write

$$\chi_{\Delta(\sigma',N)}(m_2, \dots, m_d) = \sum_j \alpha_j \chi_{C(\tau^j,N)}\big(\Psi_j(m_2, \dots, m_d)\big)$$

where $\tau^j \in \{0,1\}^{d-1}$, and Ψ_j is an automorphism on $\mathbb{Z}^{d-1}$. Consequently

$$\begin{aligned}\widehat{D_N^{\sigma}}(\mathbf{m}) &= \chi_{[\sigma_1,N]}(m_1)\chi_{\Delta(\sigma',N)}(m_2, \dots, m_d) \\ &= \sum_j \alpha_j \chi_{[\sigma_1,N]}(m_1)\chi_{C(\tau^j,N)}\big(\Psi_j(m_2, \dots, m_d)\big).\end{aligned}$$

Therefore, we only need to see that we can decompose functions as

$$\chi_{[\sigma_1,N]}(m_1)\chi_{C(\tau,N)}\big(\Psi(m_2, \dots, m_d)\big), \tag{23}$$

where $\tau = (\tau_2, \dots, \tau_d) \in \{0,1\}^{d-1}$ and $\Psi \colon \mathbb{Z}^{d-1} \to \mathbb{Z}^{d-1}$ is an automorphism. We can say that

$$\Psi(m_2, \dots, m_d) = \big(\psi_2(m_2, \dots, m_d), \dots, \psi_d(m_2, \dots, m_d)\big)$$

where $\psi_j \colon \mathbb{Z}^{d-1} \to \mathbb{Z}$ are homomorphisms for $j = 2, \dots, d$. Then (23) can be written as

$$\chi_{[\sigma_1,N]}(m_1)\chi_{[\tau_2,N]}(\psi_2(m_2, \dots, m_d)) \prod_{j=3}^{d} \chi_{[\tau_j,+\infty)}(\psi_j(m_2, \dots, m_d)).$$

Now use the case $d = 2$ to decompose $\chi_{[\sigma_1,N]}(m_1)\chi_{[\tau_2,N]}(\psi_2)$ in order to obtain that (23) can be written as

$$\widehat{P_N^{\sigma^1}}(\Phi_1(\mathbf{m})) + \widehat{P_N^{\sigma^2}}(\Phi_2(\mathbf{m})) - \widehat{P_N^{\sigma^3}}(\Phi_3(\mathbf{m})) - \widehat{P_N^{\sigma^4}}(\Phi_4(\mathbf{m})), \tag{24}$$

where $\sigma^k \in \{0,1\}^d$, with $\sigma_j^k = \tau_j$ for $j = 3, \dots, d$ and $1 \leq k \leq 4$, and where $\Phi_k \colon \mathbb{Z}^d \to \mathbb{Z}^d$ are the automorphisms defined by

$$\begin{aligned}&\Phi_1(\mathbf{m}) = (m_1, \Psi(m_2, \dots, m_d)), \\ &\Phi_2(\mathbf{m}) = (\psi_2(m_2, \dots, m_d), m_1, \psi_3(m_2, \dots, m_d), \dots, \psi_d(m_2, \dots, m_d)), \\ &\Phi_3(\mathbf{m}) = (m_1, \psi_2(m_2, \dots, m_d) - m_1, \psi_3(m_2, \dots, m_d), \dots, \psi_d(m_2, \dots, m_d)), \quad \text{and} \\ &\Phi_4(\mathbf{m}) = (\psi_2(m_2, \dots, m_d), m_1 - \psi_2(m_2, \dots, m_d), \psi_3(m_2, \dots, m_d), \dots, \\ &\hspace{20em} \psi_d(m_2, \dots, m_d)).\end{aligned}$$

Then (24) yields the required decomposition for expressions like (23), and from the previous discussion $\widehat{D_N^{\sigma}}$ can be written as in the statement of Proposition 3.7 for every $\sigma \in \{0,1\}^d$. $\square$

3.4. **The basic admissible sequence of multipliers.** In this section we prove Proposition 3.8. First it will be convenient to introduce some more notation. For each integer $k \geq 0$ and every $f \in L^2(\mathbb{T}^d)$ we define

$$\Psi_k(f) = \|f\|_1 \left[\log\left(\frac{e\|f\|_2}{\|f\|_1}\right)\right]^k.$$

Observe that $\Psi_0(f) = \|f\|_1$. We will denote by P_N the multiplier P_N^σ for $\sigma = (0, 0, \dots, 0)$. That is, for $f \in L^2(\mathbb{T}^d)$,

$$\widehat{P_N f}(\mathbf{m}) = \begin{cases} \widehat{f}(\mathbf{m}) & \text{if } \mathbf{m} \in [0, N] \times [0, +\infty)^{d-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The proof of Proposition 3.8 will be finished if we prove that (P_N) is an admissible sequence of multipliers in $\mathbb{T}^d$. Indeed, if $\sigma \in \{0,1\}^d$ and we define $e_\sigma(\mathbf{x}) = \exp(i\langle\sigma, \mathbf{x}\rangle)$, $\mathbf{x} \in \mathbb{T}^d$, we have, for every $f \in L^2(\mathbb{T}^d)$,

$$P_N^\sigma f = e_\sigma \cdot P_{N-\sigma_1}(f \cdot \overline{e_\sigma}),$$

and therefore, for the maximal operators,

$$\mathcal{M}(P_N^\sigma)f = \mathcal{M}(P_N)(f\,\overline{e_\sigma}).$$

Since $\Psi_d(f) = \Psi_d(f\,\overline{e_\sigma})$, if (P_N) is admissible, then (P_N^σ) is admissible.

For each integer $N \geq 1$ and $1 \leq j \leq d$, we define $S_{1,N}^+$ and the jth Riesz projection $\mathcal{R}_j$ as the multipliers given by

$$\widehat{S_{1,N}^+}(\mathbf{m}) = \begin{cases} 1 & \text{if } 0 \leq m_1 \leq N, \\ 0 & \text{otherwise,} \end{cases} \qquad \widehat{\mathcal{R}_j}(\mathbf{m}) = \begin{cases} 1 & \text{if } 0 \leq m_j, \\ 0 & \text{if } 0 > m_j. \end{cases}$$

It is not difficult to check that, for every $f \in L^2(\mathbb{T}^d)$,

$$P_N f = S_{1,N}^+ \circ \mathcal{R}_2 \circ \mathcal{R}_3 \circ \cdots \circ \mathcal{R}_d f. \tag{25}$$

The proof of Proposition 3.8 will be a consequence of the two following lemmas and (25).

Lemma 3.9. *For each integer $1 \leq j \leq d$ and each integer $k \geq 0$, there exists a constant $C > 0$ such that*

$$\Psi_k(\mathcal{R}_j f) \leq C\Psi_{k+1}(f), \quad f \in L^2(\mathbb{T}^d).$$

Lemma 3.10. *There exists a constant $C > 0$ such that, for every $g \in L^2(\mathbb{T}^d)$,*

$$\left\|\mathcal{M}(S_{1,N}^+)g\right\|_{1,\infty} \leq C\Psi_1(g).$$

Proof of Proposition 3.8. Let $f \in L^2(\mathbb{T}^d)$ and define

$$g = \mathcal{R}_2 \circ \mathcal{R}_3 \circ \cdots \circ \mathcal{R}_d f.$$

Applying $d-1$ times the Lemma 3.9 we get

$$\Psi_1(g) \leq C\Psi_d(f). \tag{26}$$

Then, by (25), we have $P_N f = S_{1,N}^+ g$ and therefore

$$\mathcal{M}(P_N)f = \mathcal{M}(S_{1,N}^+)g.$$

Applying Lemma 3.10 and (26) we deduce that

$$\left\|\mathcal{M}(P_N)f\right\|_{1,\infty} \leq C\Psi_d(f).$$

We have proved that (P_N) is an admissible sequence of multipliers in $\mathbb{T}^d$ and, as explained before, we finish the proof of Proposition 3.8. $\square$

Proof of Lemma 3.9. There is no loss of generality in assuming that $j = 1$. We will prove first the case $k = 0$ and then we will use this case to prove the result for $k \geq 1$. For $k = 0$ and $d = 1$, we have $\mathcal{R}_1 = \mathcal{R}$, and by the relationship between the Hilbert transform and the Riesz projection (6), and Zygmund's result invoked in the proof of Theorem 2.4, there exists $C > 0$ such that

$$\|\mathcal{R}f\|_1 \leq C\|f\|_{L\log L}\,, \quad f \in L\log L(\mathbb{T}).$$

Applying then Proposition 2.2 and Proposition 2.5, there exists $C > 0$ so that

$$\|\mathcal{R}f\|_1 \leq C\Psi_1(f)\,, \quad f \in L^2(\mathbb{T}).$$

This finishes the proof for $k = 0$ and $d = 1$.

Assume now that $d \geq 2$. Given $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ we write $\mathbf{x}' = (x_2, \ldots, x_d)$, so that we have $\mathbf{x} = (x_1, \mathbf{x}')$. If $f\colon \mathbb{T}^d \to \mathbb{C}$, we denote by $f_{\mathbf{x}'}$ the function $f_{\mathbf{x}'}\colon \mathbb{T} \to \mathbb{C}$ defined by

$$f_{\mathbf{x}'}(t) = f(t, \mathbf{x}'), \quad t \in \mathbb{T}.$$

It is not difficult to check that, for every $f \in L^2(\mathbb{T}^d)$ and every $\mathbf{x} = (x_1, \mathbf{x}') \in \mathbb{T}^d$, we have

$$\mathcal{R}_1 f(\mathbf{x}) = \mathcal{R}_1 f(x_1, \mathbf{x}') = (\mathcal{R}f_{\mathbf{x}'})(x_1)\,.$$

Therefore, by the case $d = 1$, putting $d\mathbf{x}'$ by $dm_{d-1}(\mathbf{x}')$, we have

$$\|\mathcal{R}_1 f\|_1 = \int_{\mathbb{T}^{d-1}} \|\mathcal{R}f_{\mathbf{x}'}\|_1\, d\mathbf{x}' \leq C \int_{\mathbb{T}^{d-1}} \Psi_1(f_{\mathbf{x}'})\, d\mathbf{x}'\,.$$

The proof of the case $k = 0$ will finish if we prove

$$\int_{\mathbb{T}^{d-1}} \Psi_1(f_{\mathbf{x}'})\, d\mathbf{x}' \leq \Psi_1(f)\,. \tag{27}$$

In order to prove (27), by homogeneity, we can assume $\|f\|_1 = 1$. Then the measure $d\nu(\mathbf{x}') = \|f_{\mathbf{x}'}\|_1\, d\mathbf{x}'$ is a probability on $\mathbb{T}^{d-1}$ and we can apply Jensen's inequality to the concave function $t \mapsto \log t$ on $(0, \infty)$. We then have

$$\begin{aligned}\int_{\mathbb{T}^{d-1}} \Psi_1(f_{\mathbf{x}'})\, d\mathbf{x}' &= \int_{\mathbb{T}^{d-1}} \|f_{\mathbf{x}'}\|_1 \log\Big(\frac{e\|f_{\mathbf{x}'}\|_2}{\|f_{\mathbf{x}'}\|_1}\Big)\, d\mathbf{x}' \leq \log\Big(\int_{\mathbb{T}^{d-1}} \|f_{\mathbf{x}'}\|_1 \frac{e\|f_{\mathbf{x}'}\|_2}{\|f_{\mathbf{x}'}\|_1}\, d\mathbf{x}'\Big)\\ &= \log\Big(\int_{\mathbb{T}^{d-1}} e\|f_{\mathbf{x}'}\|_2\, d\mathbf{x}'\Big) \leq \log\Big(e\Big(\int_{\mathbb{T}^{d-1}} \|f_{\mathbf{x}'}\|_2^2\, d\mathbf{x}'\Big)^{1/2}\Big)\\ &= \log(e\|f\|_2) = \Psi_1(f)\,,\end{aligned}$$

since $\|f\|_1 = 1$. We have proved (27) and the case $k = 0$ is over.

By the case $k = 0$, there exists a constant $\kappa \geq 1$ such that

$$\|\mathcal{R}_1 f\|_1 \leq \kappa\|f\|_1 \log\Big(\frac{e\|f\|_2}{\|f\|_1}\Big)\,, \qquad f \in L^2(\mathbb{T}^d). \tag{28}$$

Now assume $k \geq 1$, and take $f \in L^2(\mathbb{T}^d)$. We consider two different cases:

$$\text{(a) } \kappa\|f\|_1 \log\Big(\frac{e\|f\|_2}{\|f\|_1}\Big) \leq e^{1-k}\|f\|_2\,, \qquad \text{and}$$

$$\text{(b) } \kappa\|f\|_1 \log\Big(\frac{e\|f\|_2}{\|f\|_1}\Big) > e^{1-k}\|f\|_2\,.$$

In case (b), first observe that there exists $B_k > 0$ such that $[\log(et)]^k \le B_k t$, for every $t \ge 1$, and this yields $\Psi_k(g) \le B_k \|g\|_2$, for every $g \in L^2(\mathbb{T}^d)$. Therefore, since $\mathcal{R}_1$ is a contraction on L^2 and $\log(e\|f\|_2/\|f\|_1) \ge 1$, we have

$$\begin{aligned}\Psi_k(\mathcal{R}_1 f) &\le B_k \|\mathcal{R}_1 f\|_2 \le B_k \|f\|_2 \le B_k \kappa e^{k-1} \|f\|_1 \log\Big(\frac{e\|f\|_2}{\|f\|_1}\Big)\\ &\le B_k \kappa e^{k-1} \|f\|_1 \Big[\log\Big(\frac{e\|f\|_2}{\|f\|_1}\Big)\Big]^{k+1} = B_k \kappa e^{k-1} \Psi_{k+1}(f)\,.\end{aligned}$$

In case (a), using (28), since the function $t \mapsto t\big[\log(e\|f\|_2/t)\big]^k$ is nondecreasing in the interval $(0, e^{1-k}\|f\|_2]$, we have (by $\kappa \log(e\|f\|_2/\|f\|_1) \ge 1$)

$$\begin{aligned}\Psi_k(\mathcal{R}_1 f) &= \|\mathcal{R}_1 f\|_1 \Big[\log\Big(\frac{e\|\mathcal{R}_1 f\|_2}{\|\mathcal{R}_1 f\|_1}\Big)\Big]^k \le \|\mathcal{R}_1 f\|_1 \Big[\log\Big(\frac{e\|f\|_2}{\|\mathcal{R}_1 f\|_1}\Big)\Big]^k\\ &\le \kappa \|f\|_1 \log\Big(\frac{e\|f\|_2}{\|f\|_1}\Big) \Big[\log\Big(\frac{e\|f\|_2}{\kappa\|f\|_1 \log(e\|f\|_2/\|f\|_1)}\Big)\Big]^k\\ &\le \kappa \|f\|_1 \log\Big(\frac{e\|f\|_2}{\|f\|_1}\Big) \Big[\log\Big(\frac{e\|f\|_2}{\|f\|_1}\Big)\Big]^k = \kappa \Psi_{k+1}(f)\,.\end{aligned}$$

□

Proof of Lemma 3.10. For $d = 1$, $S^+_{1,N}$ coincide with $S^+_N = \mathcal{R} \circ S_N$. By the relationship between the Hilbert transform and the Riesz projection (6), we deduce that

$$S^+_N f = \frac{1}{2}\big(S_N f + i\widetilde{S_N} f\big) + \frac{1}{2}\widehat{f}(0)\,, \quad f \in L^2(\mathbb{T}).$$

Taking maximal functions

$$\mathcal{M}(S^+_N) f = \mathcal{S}^+ f \le \mathcal{S} f + \widetilde{\mathcal{S}} f\,, \qquad f \in L^2(\mathbb{T}).$$

Applying Theorem 2.3 and Theorem 2.4 we obtain

(29)
$$\big\|\mathcal{S}^+ f\big\|_{1,\infty} = \big\|\mathcal{M}(S^+_N) f\big\|_{1,\infty} \le C\|f\|_1 \log\Big(\frac{e\|f\|_2}{\|f\|_1}\Big) = C\Psi_1(f)\,, \qquad f \in L^2(\mathbb{T}),$$

which is the case $d = 1$ of Lemma 3.10.

For $d \ge 2$, $f \in L^2(\mathbb{T}^d)$, and $\mathbf{x}' \in \mathbb{T}^{d-1}$ use, as before, the notation $f_{\mathbf{x}'}(t) = f(t, \mathbf{x}')$. As is easy to check, we have

$$S^+_{N,1} f(x_1, \mathbf{x}') = (S^+_N f_{\mathbf{x}'})(x_1)\,, \qquad (x_1, \mathbf{x}') \in \mathbb{T}^d\,,$$

and taking maximal functions,

$$\mathcal{M}(S^+_{N,1}) f(x_1, \mathbf{x}') = (\mathcal{S}^+ f_{\mathbf{x}'})(x_1)\,.$$

Therefore, for all $\lambda > 0$, using Fubini, (29) and (27), we have

$$\begin{aligned}&\lambda\, m_{\mathbb{T}^d}\big(\{(x_1, \mathbf{x}') : \mathcal{M}(S^+_{N,1}) f(x_1, \mathbf{x}') > \lambda\}\big)\\ &\qquad = \int_{\mathbb{T}^{d-1}} \lambda\, m\big(\{x_1 \in \mathbb{T}; (\mathcal{S}^+ f_{\mathbf{x}'})(x_1) > \lambda\}\big)\, d\mathbf{x}'\\ &\qquad \le \int_{\mathbb{T}^{d-1}} \big\|\mathcal{S}^+ f_{\mathbf{x}'}\big\|_{1,\infty}\, d\mathbf{x}' \le C \int_{\mathbb{T}^{d-1}} \Psi_1(f_{\mathbf{x}'})\, d\mathbf{x}' \le C\Psi_1(f)\,.\end{aligned}$$

Taking the supremum in $\lambda > 0$ we conclude that

$$\big\|\mathcal{M}(S^+_{1,N}) f\big\|_{1,\infty} \le C\Psi_1(f)\,, \qquad f \in L^2(\mathbb{T}).$$

□

We are now ready for the proof of Theorem 3.3 via use of multipliers; as we have said the statement of Theorem 3.3 is equivalent to the fact that, for all $B \subset \{1, 2 \ldots, d\}$, the sequence $(\widetilde{S_{N,B}})$ is an admissible sequence of multipliers in $\mathbb{T}^d$. Denoting $B^c = \{1, 2, \ldots, d\} \setminus B$, for every $\mathbf{m} \in \mathbb{Z}^d$, we have

$$\widehat{\widetilde{S_{N,B}}}(\mathbf{m}) = \prod_{j \in B} \left(i\chi_{[-N,-1]}(m_j) - i\chi_{[1,N]}(m_j) \right) \prod_{j \in B^c} \left(\chi_{[-N,-1]}(m_j) + \chi_{[0,N]}(m_j) \right).$$

If we develop this product as the sum of 2^d summands, we discover that the sequence $(\widetilde{S_{N,B}})$ is a linear combination of sequences of multipliers of the form $(F_N^{\sigma,\eta})$, σ, $\eta \in \{0,1\}^d$, where

$$\widehat{F_N^{\sigma,\eta}}(m_1, \ldots, m_d) = \chi_{\Delta(\sigma,N)}((-1)^{\eta_1} m_1, \ldots, (-1)^{\eta_d} m_d),$$

and $\Delta(\sigma, N) = \prod_{j=1}^d ([\sigma_j, N] \cap \mathbb{Z})$ as in (22).

Then by Theorem 3.6 and Lemma 3.5, the sequence $(F_N^{\sigma,\eta})$ is admissible for all σ and all $\eta \in \{0,1\}^d$. An application of Lemma 3.4 allows us to see that $(\widetilde{S_{N,B}})$ is an admissible sequence of multipliers in $\mathbb{T}^d$, and this concludes the proof of Theorem 3.3.

4. Arias-de-Reyna space on the d-dimensional torus

Based on some ideas from the paper [6], we give a self-contained treatment on a variant of Arias-de-Reyna's space $QA(\Omega)$ defined on a finite measure space (Ω, μ). In fact, Arias-de-Reyna [4] defined the space $QA(\mathbb{T})$ on the torus $\mathbb{T}$; here we are interested in the space defined on $\mathbb{T}^d$. The main aim is to show applications to study convergence almost everywhere of multiple Fourier series over cubes.

Recall that a non-negative φ defined on $[0, a)$ with $0 < a \leq \infty$ is called *quasi-concave* if it is non-decreasing on $[0, a)$ with $\varphi(0) = 0$ and $t \mapsto \varphi(t)/t$ is non-increasing on $(0, a)$. Notice that $\widetilde{\varphi}(t) := \inf_{s \in (0,a)} \left(1 + \frac{t}{s}\right)\varphi(s)$ for every $t \in [0, a)$ is a concave function $\widetilde{\varphi}$ satisfying

$$\varphi(t) \leq \widetilde{\varphi}(t) \leq 2\,\varphi(t), \quad t \in [0, a).$$

Let (Ω, μ) be a finite measure space and let $\varphi\colon [0, \mu(\Omega)] \to [0, \infty)$ be a quasi-concave function. A measurable function $f \in L^0(\mu)$ belongs to $QA_\varphi(\Omega)$ provided there exists a sequence $(f_n)_{n=1}^\infty$ with $f_n \in L^\infty(\Omega)$ such that

$$f = \sum_{n=1}^\infty f_n, \quad \mu\text{-a.e.}$$

and

$$\sum_{n=1}^\infty (1 + \log n) \|f_n\|_\infty \varphi\left(\frac{\|f_n\|_1}{\|f_n\|_\infty} \right) < \infty,$$

where $0/0 := 0$ by convention.

It is easy to check that $QA_\varphi(\Omega)$ is a quasi-Banach lattice on (Ω, μ) equipped with the quasi-norm

$$\|f\|_{QA_\varphi} = \inf \sum_{n=1}^\infty (1 + \log n) \|f_n\|_\infty \varphi\left(\frac{\|f_n\|_1}{\|f_n\|_\infty} \right),$$

where the infimum is taken over all representations $f = \sum_{n=1}^\infty f_n$ of f as shown above.

We will need some results concerning the space QA_φ. Since they can be proved using methods described in [6] in the case of the space $QA(\mathbb{T})$, we state them without proofs here.

Before we state the first result we recall that if f, g are functions in $L^1(\Omega)$, we write $g \preceq f$ provided that

$$\int_0^t g^*(s)ds \le \int_0^t f^*(s)\,ds, \qquad t > 0.$$

This defines the Hardy-Littlewood-Pólya ordering.

Lemma 4.1. *QA_φ is a quasi-Banach lattice on (Ω, μ) satisfying the following property: if $f \in QA_\varphi$ and $g \in L^1$ satisfy $g \preceq f$, then $g \in QA_\varphi$ and*

$$\|g\|_{QA_\varphi} \le \|f\|_{QA_\varphi}.$$

Consequently, QA_φ is a quasi-Banach r.i. space.

We state the following results. For complete proofs in the case of $QA(\mathbb{T})$, we refer to [6]. Since the proofs are similar we omit it.

Proposition 4.2. *Let (Ω, μ) be a probalilty measure space and let φ be a normalized concave function on $[0,1]$ (i.e., $\varphi(1) = 1$). Then the following continuous inclusion holds with norm less than or equal to 1:*

$$QA_\varphi \hookrightarrow \Lambda_\varphi\,.$$

To state the next result we recall that if $(X, \|\cdot\|)$ is a quasi-normed space whose dual separates the points, then the Mackey norm $\|\cdot\|^c$ on X is defined by

$$\|x\|^c = \inf\big\{\lambda > 0;\, x \in \lambda\,\mathrm{conv}(B_X)\big\}, \qquad x \in X,$$

where $\mathrm{conv}(B_X)$ is the convex hull of the unit ball $B_X = \{x \in X;\, \|x\| \le 1\}$. The completion of X equipped with the Mackey norm is denoted by $\widehat{X}$ and is usually called the Banach envelope of X.

It was shown in [6] that the Banach envelope of QA is isometrically isomorphic to $L\log L$. A similar proof gives the following.

Proposition 4.3. *Let (Ω, μ) be a probalilty measure space and let φ be a normalized concave function. Then the Banach envelope of QA_φ is isometrically isomorphic to Λ_φ.*

Given a concave function ϕ on $[0,1]$, following [6] we show how to construct on an atomless probability measure space (Ω, μ) a family of Lorentz spaces contained in QA_ϕ.

Given a sequence $s = (s_n) \in (0,1]^{\mathbb{N}}$, we define $\Lambda^{(s)} = \Lambda^{(s)}(\Omega)$ to be the space of all $f \in L^0(\Omega)$ such that there exists a sequence (f_n) with $f_n \in L^\infty(\Omega)$ satisfying $f = \sum_{n=1}^\infty f_n$ (convergence in $L^1(\Omega)$) with

$$\sum_{n=1}^\infty \max\big\{\|f_n\|_1, s_n\|f_n\|_\infty\big\}\frac{\phi(s_n)}{s_n}\,(1 + \log n) < \infty.$$

We equipped $\Lambda^{(s)}$ with the norm

$$\|f\|_{\Lambda^{(s)}} := \inf\Bigg\{\sum_{n=1}^\infty \max\big\{\|f_n\|_1, s_n\|f_n\|_\infty\big\}\frac{\phi(s_n)}{s_n}\,(1 + \log n);\; f = \sum_{n=1}^\infty f_n\Bigg\}.$$

Proposition 4.4. *For every positive sequence $s = (s_n) \in (0,1]^{\mathbb{N}}$, $\Lambda^{(s)}$ is an r.i. Banach space on (Ω, μ) such that*

$$\Lambda^{(s)} \hookrightarrow QA_\phi \tag{30}$$

with norm less than or equal to 1.

Proof. The proof that $\Lambda^{(s)}$ is a Banach space is standard. It is easy to check that $\Lambda^{(s)}$ is an exact interpolation space between $L^1(\Omega)$ and $L^\infty(\Omega)$; in particular this implies that $\Lambda^{(s)}$ is an r.i. space on Ω. To prove the required continuous inclusion, we observe that if $f \in L^\infty(\Omega)$ with $f \neq 0$, then the concavity of ϕ gives

$$\begin{aligned}\|f\|_\infty \phi\left(\frac{\|f\|_1}{\|f\|_\infty}\right) &\leq \|f\|_\infty \max\left\{1, \frac{\|f\|_1}{s_n\|f\|_\infty}\right\}\phi(s_n) \\ &= \max\left\{\|f\|_1, s_n\|f\|_\infty\right\}\frac{\phi(s_n)}{s_n}\,.\end{aligned}$$

Thus, if $f \in \Lambda^{(s)}$, for every $\varepsilon > 0$ there exists a sequence (f_n) with $f_n \in L^\infty(\Omega)$ such that $f = \sum_{n=1}^\infty f_n$ (convergence in $L^1(\Omega)$) and

$$\sum_{n=1}^\infty \max\left\{\|f_n\|_1, s_n\|f_n\|_\infty\right\}\frac{\phi(s_n)}{s_n}\,(1+\log\, n) \leq (1+\varepsilon)\|f\|_{\Lambda^{(s)}}.$$

Combining the above estimates, we conclude the result. $\square$

Theorem 4.5. *For every positive sequence $s \in (0,1]^{\mathbb{N}}$,*

$$\Lambda_{\widetilde{\varphi_s}} \hookrightarrow QA_\phi$$

with norm less than or equal to 1, *where φ_s is a quasi-concave function on $[0,1]$ defined by $\varphi_s(0) = 0$ and*

$$\varphi_s(t) = \inf_n \max\left\{1, \frac{t}{s_n}\right\}\phi(s_n)(1+\log n), \qquad 0 < t \leq 1.$$

Proof. Since $\varphi_s(t) \leq \widetilde{\varphi_s}(t)$ for all $t \in I$, it then follows from Proposition 4.4 that it is enough to prove that the fundamental function of the r.i. space $\Lambda^{(s)}$ is φ_s.

Fix $0 < t < 1$ and take any measurable set A with $\mu(A) = t$. Clearly, for each $n \geq 1$, we have

$$\begin{aligned}\|\chi_A\|_{\Lambda^{(s)}} &\leq \max\{t, s_n\}\frac{\phi(s_n)}{s_n}\,(1+\log\, n) \\ &= \max\left\{1, \frac{t}{s_n}\right\}\phi(s_n)\,(1+\log\, n), \qquad 0 < t \leq 1.\end{aligned}$$

Thus taking the infimum over all $n \geq 1$, we obtain that $\|\chi_A\|_{\Lambda^{(s)}} \leq \varphi_s(t)$.

To prove the converse, fix $\varepsilon > 0$ and let $\chi_A = \sum_n f_n$ (convergence in $L^1(\Omega)$) with $f_n \in L^\infty(\Omega)$, $f_n \geq 0$, and

$$\sum_{n=1}^\infty \max\left\{\|f_n\|_1, s_n\|f_n\|_\infty\right\}\frac{\phi(s_n)}{s_n}\,(1+\log\, n) \leq (1+\varepsilon)\|\chi_A\|_{\Lambda^{(s)}}.$$

Since $\|f_n\|_\infty \geq \frac{\|f_n\|_1}{t}$, we obtain

$$\begin{aligned}
(1+\varepsilon)\|\chi_A\|_{\Lambda^{(s)}} &\geq \sum_{n=1}^{\infty} \max\left\{\|f_n\|_1, s_n\|f_n\|_\infty\right\} \frac{\phi(s_n)}{s_n}(1+\log\, n) \\
&\geq \sum_{n=1}^{\infty} \max\left\{1, \frac{s_n}{t}\right\} \frac{\phi(s_n)}{s_n}(1+\log\, n)\,\|f_n\|_1 \\
&= \sum_{n=1}^{\infty} \max\left\{\frac{1}{s_n}, \frac{1}{t}\right\} \phi(s_n)(1+\log\, n)\,\|f_n\|_1 \\
&\geq \frac{1}{t}\inf_n \max\left\{1, \frac{t}{s_n}\right\} \phi(s_n)(1+\log\, n) \sum_{n=1}^{\infty}\|f_n\|_1 \\
&\geq \frac{1}{t}\varphi_s(t)\,\|\chi_A\|_1 = \varphi_s(t),
\end{aligned}$$

and the result follows letting ε tend to 0. □

Now we fix the probability measure space $(\mathbb{T}^d, m_d)$ and we consider the space QA_ϕ generated by $\phi := \psi_d$. In what follows we write QA^d for short instead of QA_{ψ_d} or $QA_{\psi_d}(\mathbb{T}^d)$.

Taking a special sequence (s_n), we obtain the following corollary (cf. [6]).

Corollary 4.6. *If $s = (s_n)$ with $s_n = e^{-e^n}$ for each $n \geq 1$, then*

$$\Lambda_{\widetilde{\varphi_s}}(\mathbb{T}^d) = L(\log\, L)^d \log\log\log L(\mathbb{T}^d) \hookrightarrow QA^d\,,$$

where $\varphi_s(t) = \inf_n \max\left\{s_n, t\right\} \frac{\psi_d(s_n)}{s_n}(1+\log n),\ 0 < t \leq 1$.

Proof. We observe that

$$\frac{1}{d^d}\, t\log^d\left(\frac{e^d}{t}\right) \leq t\log^d\left(\frac{e}{t}\right) \leq t\log^d\left(\frac{e^d}{t}\right), \quad 0 < t \leq 1\,,$$

yields $\psi_d \asymp \phi_d$ on $(0,1]$, where $\phi_d(t) := t\log^d\left(\frac{e^d}{t}\right)$ for every $0 < t \leq 1$. Thus for every $t \in [e^{-e^{k+1}}, e^{-e^k})$ with $k \geq d$, we have

$$\begin{aligned}
\varphi_s(t) &\asymp \inf_{n\geq 1} \max\left\{e^{-e^n}, t\right\} e^{dn}(1+\log\, n) \\
&= \min\left\{\inf_{1\leq n\leq k} \max\left\{e^{-e^n}, t\right\} e^{dn}(1+\log\, n),\ \inf_{n>k} \max\left\{e^{-e^n}, t\right\} e^{dn}(1+\log\, n)\right\} \\
&= \min\left\{\inf_{n\leq k} e^{-e^n} e^{dn}(1+\log\, n),\ t\inf_{n>k} e^{dn}(1+\log\, n)\right\} \\
&= \min\left\{e^{-e^k} e^{dk}(1+\log\, k),\ te^{d(k+1)}(1+\log\,(k+1))\right\} \\
&\asymp te^{dk}(1+\log\, k) \asymp \psi_d(t).
\end{aligned}$$

Since $\Lambda_{\psi_d}(\mathbb{T}^d) = L(\log\, L)^d \log\log\log L(\mathbb{T}^d)$, the proof is complete by Theorem 4.5. □

5. Applications to almost everywhere convergence

We apply our results to prove novel results on almost everywhere convergence of the multiple Fourier series over cubes. These results unify and extend the best known results at present.

Before presenting the results, we introduce some notation. Following Kalton [11], a quasi-Banach space X is said to be *logconvex* provided there exists a contant $K > 0$ such that

$$\|x_1 + \cdots + x_n\|_X \le K \sum_{k=1}^{n} (1 + \log\, k)\|x_k\|_X$$

for every finite set $\{x_1, \ldots, x_n\}$ in X.

It follows from the result due to Stein and Weiss [18] (see also Kalton [11]) that the weak L^1-space on any finite measure space is logconvex.

We can state and prove the main theorem of this section.

Theorem 5.1. *Let $d \in \mathbb{N}$ and $B \subset \{1, \ldots, d\}$. Then the maximal operator $\widetilde{\mathcal{S}_B}\colon QA^d \to L_{1,\infty}(\mathbb{T}^d)$ is bounded; i.e., there exists a constant $C > 0$ such that for every $f \in QA^d$,*

$$m_d\big(\{\mathbf{x} \in \mathbb{T}^d;\ \widetilde{\mathcal{S}_B} f(\mathbf{x}) > \lambda\}\big) \le \frac{C}{\lambda}\, \|f\|_{QA^d}, \quad \lambda > 0.$$

Therefore the multiple Fourier series of every function $f \in QA^d$ and all its conjugates converge over cubes almost everywhere.

Proof. Given $f \in QA^d$, there exists a sequence (f_n) in $L^\infty(\mathbb{T}^d)$ such $f = \sum_{n=1}^\infty f_n$ and

$$\sum_{n=1}^{\infty} \|f_n\|_\infty \psi_d\Big(\frac{\|f_n\|_1}{\|f_n\|_\infty}\Big) < \infty. \tag{31}$$

Clearly (31) implies that $\sum_{n=1}^\infty \|f_n\|_1 < \infty$. Hence $\sum_{n=1}^\infty |f_k|$ converges m_d-a.e. and so

$$\widehat{f}(\mathbf{k}) = \sum_{n=1}^{\infty} \widehat{f_n}(\mathbf{k}), \quad \mathbf{k} \in \mathbb{Z}^d.$$

This implies that for any $N \in \mathbb{N}$,

$$S_{N,B} f(\mathbf{x}) = \sum_{n=1}^{\infty} S_{N,B} f_n(\mathbf{x}), \quad \mathbf{x} \in \mathbb{T}^d.$$

In consequence

$$\widetilde{\mathcal{S}_B} f(\mathbf{x}) \le \sum_{n=1}^{\infty} \widetilde{\mathcal{S}_B} f_n(\mathbf{x}), \quad \mathbf{x} \in \mathbb{Z}^d.$$

Hence by Theorem 3.3 and the logconvexity of the space $L_{1,\infty}(\mathbb{T}^d)$ there exist constants $K > 0$ and $C_d > 0$ such that

$$\begin{aligned}
\|\widetilde{\mathcal{S}_B} f\|_{1,\infty} &\le C \sum_{n=1}^{\infty} (1 + \log\, n)\|\widetilde{\mathcal{S}_B} f_n\|_{1,\infty} \le KC_d \sum_{n=1} (1 + \log n)\|f_n\|_2 \psi_d\Big(\frac{\|f_n\|_1}{\|f_n\|_2}\Big) \\
&\le KC_d \sum_{n=1} (1 + \log n)\|f_n\|_\infty \psi_d\Big(\frac{\|f_n\|_1}{\|f_n\|_\infty}\Big).
\end{aligned}$$

Since $f \in QA^d$ was arbitrary, we conclude that there exists $C > 0$ such that

$$\|\widetilde{\mathcal{S}_B} f\|_{1,\infty} \le C\, \|f\|_{QA^d}, \quad f \in QA^d,$$

and this yields the required estimate:

$$m_d\big(\{\mathbf{x} \in \mathbb{T}^d;\ \widetilde{\mathcal{S}_B} f(\mathbf{x}) > \lambda\}\big) \le \frac{C}{\lambda}\, \|f\|_{QA^d}, \quad \lambda > 0.$$

The second part is standard; we define the sublinear operator $T_B\colon QA^d \to L_{1,\infty}(\mathbb{T}^d)$ by

$$T_B f(\mathbf{x}) = \limsup_{N,M\to\infty} \big|S_{N,B} f(\mathbf{x}) - S_{M,B} f(\mathbf{x})\big|, \quad f \in QA^d, \quad \mathbf{x} \in \mathbb{T}^d\,.$$

Since $T_B g \le 2\,\widetilde{\mathcal{S}_B} g$ for every $g \in QA^d$, it follows by (31) that

$$m_d\big(\big\{\mathbf{x} \in \mathbb{T}^d;\ T_B g(\mathbf{x}) > \lambda\big\}\big) \le \frac{2C}{\lambda}\,\|g\|_{QA^d}, \quad \lambda > 0\,.$$

Clearly for every trigonometric polynomial P on $\mathbb{T}^d$ we have $T_B P = 0$ and $T_B f = T_B(f-P)$. Applying the above estimate with $g = f - P$ yields

$$m_d\big(\big\{\mathbf{x} \in \mathbb{T}^d;\ T_B f(\mathbf{x}) > \lambda\big\}\big) \le \frac{2C}{\lambda}\,\|f-P\|_{QA^d}, \quad \lambda > 0\,.$$

Since the set of all trigonometric polynomials on $\mathbb{T}^d$ is a dense subspace in QA^d (by the density of $L^\infty(\mathbb{T}^d)$ in QA^d), it follows that

$$m_d\big(\big\{\mathbf{x} \in \mathbb{T}^d;\ T_B f(\mathbf{x}) > \lambda\big\}\big) = 0, \quad \lambda > 0\,.$$

This completes the proof. □

As an application of Theorem 5.1 and Corollary 4.6 we obtain the following important theorem of Antonov (see [2]).

Theorem 5.2. *Let $d \in \mathbb{N}$ and $B \subset \{1,\dots,d\}$. Then the maximal Carleson operator $\widetilde{\mathcal{S}_B}\colon L_\Phi(\mathbb{T}^d) \to L_{1,\infty}(\mathbb{T}^d)$ is bounded, where $L_\Phi(\mathbb{T}^d)$ is the Orlicz space $L(\log L)^d \log\log\log L$ on $(\mathbb{T}^d, m_d)$; i.e., there exists a constant $C>0$ such that for every $f \in L_\Phi(\mathbb{T}^d)$,*

$$m_d\big(\big\{\boldsymbol{x} \in \mathbb{T}^d;\ \widetilde{\mathcal{S}_B} f(\boldsymbol{x}) > \lambda\big\}\big) \le \frac{C}{\lambda}\,\|f\|_{L_\Phi(\mathbb{T}^d)}, \quad \lambda > 0.$$

Therefore the multiple Fourier series of every function $f \in L_\Phi(\mathbb{T}^d)$ and all its conjugates converge over cubes almost everywhere.

We show an analog of the result in the one-dimensional case showing that there exists a Lorentz space that strictly contains the Orlicz space $L(\log\, L)^d \log\log\log\, L$ which is contained in QA^d.

To state the next result we recall that a pair (A_0, A_1) of Banach spaces is called an interpolation couple provided A_0 and A_1 are continuously embedded in some Hausdorff topological vector space. For an interpolation couple, we define a Banach space $A_0 + A_1$ equipped with the norm

$$\|a\|_{A_0+A_1} = \inf\big\{\|a_0\|_{A_0} + \|a_1\|_{A_1};\ a = a_0 + a_1\big\}\,.$$

Since for every Banach lattice E on a measure space (Ω,μ) we have $E \hookrightarrow L^0(\mu)$, every pair (E_0, E_1) of Banach lattices is an interpolation pair. If E_0 and E_1 are r.i. spaces on an atomless measure space (Ω,μ), then $E_0 + E_1$ is also an r.i. space with the fundamental function $\varphi_{E_0+E_1} = \min\{\varphi_{E_0}, \varphi_{E_1}\}$.

We conclude with the d-dimensional analog of the one-dimensional result from [6, Theorem 2.3]. The proof is similar to the case $d=1$; however we include a proof for the reader's convenience.

Theorem 5.3. *There exists a Lorentz space $\Lambda_\varphi(\mathbb{T}^d)$ that strictly contains the Orlicz space $L(\log L)^d \log\log\log L(\mathbb{T}^d)$ and such that*

$$\Lambda_\varphi(\mathbb{T}^d) \hookrightarrow QA^d\,.$$

Proof. We recall that Orlicz space $L(\log L)^d \log\log\log(\mathbb{T}^d)$ coincides up to equivalence of norms with the Lorentz space $\Lambda_\phi(\mathbb{T}^d)$, where the normalized concave function ϕ satisfies

$$\phi(t) \asymp t \log^d\left(\frac{e}{t}\right) \log^+ \log^+ \log^+\left(\frac{1}{t}\right), \quad t \in (0,1]\,.$$

Let $s_n = e^{-e^{e^n}}$ for all $n \in \mathbb{N}$. Define a concave function φ on $[0,1]$ by $\varphi(0) = 0$ and

$$\varphi(t) = \min\left\{\phi(t), \widetilde{\varphi_s}(t)\right\}, \quad t \in (0,1]\,.$$

Clearly, we have $\Lambda_\varphi \hookrightarrow \Lambda_{\widetilde{\varphi_s}} + \Lambda_\phi$ and

$$\Lambda_\phi \hookrightarrow \Lambda_\varphi \hookrightarrow QA^d\,.$$

Since $\phi(s_n) \asymp e^{-e^{e^n}} e^{e^n} n$ and $\varphi_s(s_n) \leq e^{-e^{e^n}} e^{e^n}(1 + \log n)$ for each $n \geq 1$ and $\varphi_s \asymp \widetilde{\varphi_s}$, we have

$$\lim_{n\to\infty} \frac{\varphi(s_n)}{\phi(s_n)} = 0\,.$$

This implies that $\Lambda_\phi \neq \Lambda_\varphi$, and so the proof is complete. $\square$

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