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HARDY-HODGE DECOMPOSITION OF VECTOR FIELDS IN $\mathbb{R}^n$

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ABSTRACT. We prove that an $\mathbb{R}^{n+1}$-valued vector field on $\mathbb{R}^n$ is the sum of the traces of two harmonic gradients, one in each component of $\mathbb{R}^{n+1} \setminus \mathbb{R}^n$, and of an $\mathbb{R}^n$-valued divergence free vector field. We apply this to the description of vanishing potentials in divergence form. The results are stated in terms of Clifford Hardy spaces, the structure of which is important for our study.

1. INTRODUCTION

Decomposing a complex function on the line as the sum of the traces of two holomorphic functions, one in each half plane cut out by the line, is a classical topic from complex analysis that lies at the root of many developments in harmonic analysis. Indeed, such a decomposition features the Hilbert transformation, which is the prototype of a Calderòn-Zygmund operator, whose $C^{1,\alpha}$ and L^p boundedness was historically the starting point of elliptic regularity theory [6,19,20,22]. This decomposition is also a cornerstone of solutions to Riemann-Hilbert problems, which are especially meaningful in spectral theory [5] and have provided in recent years striking advances in the theory of orthogonal polynomials [4]. Moreover, it is instrumental for defining and studying Hankel and Toeplitz operators, which play a fundamental role in complex approximation and were successfully applied to issues of basic importance in control and signal analysis [3,13,14,16]. Besides, in a Hilbertian framework, the decomposition was used to obtain sparse representations of analytic signals of scalar-valued signals in various classical contexts ([18] and subsequent papers by these authors).

Specifically, given a complex-valued function $f \in L^p(\mathbb{R}),\ 1 < p < \infty$, one has

$$f = f^+ + f^-, \tag{1.1}$$

where

$$f^\pm(x) = \lim_{y\to 0\pm} \frac{\pm 1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-(x+iy)}dt,$$

where $f^\pm$ are, respectively, nontangential boundary limit functions of holomorphic functions of one complex variable in, respectively, the Hardy spaces $H^p(\mathbb{C}^\pm)$ of the

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upper and lower half planes. The Hardy space functions are given by

$$f^{\pm}(z) = \frac{\pm 1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}dt, \quad z = x+iy,\ \pm y > 0;$$

see [6].

Now, under the standard identification $\mathbb{C} \sim \mathbb{R}^2$, a holomorphic function may be regarded as the gradient of a harmonic function, and this way (1.1) says that an $\mathbb{R}^2$-valued vector field on $\mathbb{R}$ of L^p class may be decomposed as the sum of the traces of two harmonic gradients, arising from harmonic functions in the upper and lower half plane respectively. The question that we raise in this paper is whether such a decomposition is possible in higher dimension, namely whether a vector field in $L^p(\mathbb{R}^n, \mathbb{R}^{n+1})$ is the sum of the traces of two gradients of functions harmonic in the two half-spaces cut out by $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$. The answer is no in general, but the next best thing is that a decomposition becomes possible if a third summand is allowed, which takes the form of a divergence free vector field tangent to $\mathbb{R}^n$. This fact was observed in [1] when $n = 2$ and used to characterize silent magnetization distributions on a plane. We presently carry this decomposition over to every n. When projected onto $\mathbb{R}^n$, it yields back the classical Hodge decomposition of an L^p tangent vector field on $\mathbb{R}^n$ as the sum of a gradient and of a divergence free component. This is why we call our decomposition of $L^p(\mathbb{R}^n, \mathbb{R}^{n+1})$ vector fields the Hardy-Hodge decomposition.

Formally the decomposition can be surmised from Hodge theory for 1-currents supported on a hypersurface in ambient space [7, Sec. 2.8], but the estimates needed to control L^p-norms of the objects involved pertain to the Calderòn-Zygmund theory. In this connection, it would be pedantic to introduce currents to speak of vector fields on linear submanifolds, but it is convenient to use the formalism of Clifford analysis, which provides us with a substitute for complex variables and is well adapted to handle higher dimensional singular integrals. In fact, Clifford analysis is also suited to extend the result to vector fields on more general submanifolds, although such a generalization lies beyond the scope of the present paper.

The latter is organized as follows. In Section 2 we recall some basic facts from Clifford analysis and Clifford Hardy spaces, most of which can be found in [8], and we study the structure of boundary functions in detail, along with density properties of rational-like functions. In Sections 3 and 4, we prove the Hardy-Hodge decomposition and some variants thereof. Finally, in Section 5, we discuss an application to nonuniqueness for inverse potential problems in divergence form.

2. Preliminaries

Let $n \geq 3$ be an integer and Φ be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Hereafter, we put $L^p(\mathbb{R}^n, E)$ for the familiar Lebesgue space of functions on $\mathbb{R}^n$ with values in a Banach space E (typically $E = \Phi^m$) whose norm to the p-th power is integrable, and we often write $L^p(\mathbb{R}^n)$ for simplicity if E is understood from the context.

We adopt standard notation in Clifford analysis; see [8]. In particular, we denote by $Cl(n, \Phi)$ the Clifford algebra generated over Φ by $\mathbf{e}_1, \cdots, \mathbf{e}_n$ with $\mathbf{e}_0 = 1$ and $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij},\ i, j = 1, \cdots, n$. We indicate with $\mathcal{S}$ the collection of subsets of $\{1, \cdots, n\}$. Then, the elements of the canonical basis of $Cl(n, \Phi)$, viewed as a vector space over Φ, are denoted as $\mathbf{e}_S,\ S \in \mathcal{S}$, where $e_S = \mathbf{e}_{j_1}\cdots\mathbf{e}_{j_k}$ if

$S = \{1 \le j_1 < \cdots < j_k \le n\}$. A generic member of $Cl(n,\Phi)$ can thus be written as $x = \sum_{S\in\mathcal{S}} x_S \mathbf{e}_S$ with $x_S \in \Phi$. When S is empty, we write $\mathbf{e}_\emptyset = \mathbf{e}_0 = 1$. The conjugate of x, denoted as $\overline{x}$, is defined to be $\sum_{S\in\mathcal{S}} (-1)^{|S|} x_S \mathbf{e}_S$, where $|S|$ indicates the cardinality of S. By convention, a 0-form is a scalar. A k-form is a sum $\sum_{S\in\mathcal{S}_k} x_S\mathbf{e}_S$ where $\mathcal{S}_k$ indicates those members of $\mathcal{S}$ with cardinality k. Clearly, $Cl(n,\Phi)$ is a 2^n-dimensional linear space over Φ. A 1-form is also called a *vector*, denoted with an underscore: $\underline{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$. Clifford vectors are identified with Euclidean vectors in $\mathbb{R}^n$. The sum of a 0-form and a 1-form is called a *para-vector*, and if x is a para-vector we let $\underline{x}$ be its vector part: $x = x_0\mathbf{e}_0 + \underline{x}$. This is consistent with our previous notation for vectors. The norm of $x \in Cl(n,\Phi)$ is defined to be $|x| = (\sum_{S\in\mathcal{S}} |x_S|^2)^{1/2}$, which is derived from the inner product $\langle x, y\rangle = \sum_{S\in\mathcal{S}} x_S\overline{y}_S$. If both x, y are para-vectors, then their Clifford product $xy = -\langle x, y\rangle + x \wedge y$, where the exterior product $x\wedge y$ is a 2-form similar to the exterior product of differential forms from geometry:

$$x \wedge y = \sum_{j<k} (x_j y_k - x_k y_j)\mathbf{e}_j\mathbf{e}_k.$$

Let $\mathbb{R}^n_1 = \{x = x_0 + \underline{x} \mid x_0 \in \mathbb{R},\ \underline{x} \in \mathbb{R}^n\}$, $\mathbb{R}^n_{1,\pm} = \{x = x_0 + \underline{x} \mid \pm x_0 > 0,\ \underline{x} \in \mathbb{R}^n\}$, and $\mathbb{R}^{n+1}_\pm = \{x = \underline{x} + x_{n+1}\mathbf{e}_{n+1} \mid \pm x_{n+1} > 0\}$, noting that $\mathbb{R}^n_1, \mathbb{R}^n_{1,\pm}$ are included in $Cl(n,\mathbb{R})$ while $\mathbb{R}^{n+1}_\pm$ is contained in $\mathbb{R}^{n+1} \subset Cl(n+1,\mathbb{R})$ (via the identification of Clifford vectors with Euclidean vectors). For $k = 0, \cdots, n+1$, we introduce the partial derivatives $\partial_k = \partial/\partial_{x_k}$ and subsequently we define

$$\begin{aligned} D_0 &= \partial_0, & D_n &= \mathbf{e}_1\partial_1 + \cdots + \mathbf{e}_n\partial_n, \\ D &= D_0 + D_n, & D_{n+1} &= \mathbf{e}_1\partial_1 + \cdots + \mathbf{e}_n\partial_n + \mathbf{e}_{n+1}\partial_{n+1}. \end{aligned} \tag{2.2}$$

A $Cl(n,\Phi)$ or $Cl(n+1,\Phi)$-valued function f such that $Df = 0$ ($fD = 0$) or $D_{n+1}f = 0$ ($fD_{n+1} = 0$) on an open set of $\Omega \subset \mathbb{R}^n_1$ or $\Omega \subset \mathbb{R}^{n+1}$ is called *left-monogenic* (*right-monogenic*) on Ω. By convention, coordinates in the case of D are denoted by $x_0, \cdots, x_n$, whereas in the case of D_{n+1} they are written $x_1, \cdots, x_{n+1}$. If a function is both left- and right-monogenic, we call it *two-sided-monogenic.* Let us stress that when applying the differential operators (2.2), the partials ∂_j commute with the e_k but the e_j do not, so that it generally matters whether the operator gets applied from the left or the right.

Note that $(D_0 - D_n)D = \Delta$ (resp. $D^2_{n+1} = -\Delta$) where $\Delta = \sum_{j=0}^n \partial^2_{x_j}$ (resp. $\Delta = \sum_{j=1}^n \partial^2_{x_j}$) is the ordinary Laplacian. Therefore left- or right-monogenic functions have harmonic components; in particular they are real analytic on Ω and there is no difference being monogenic in the distributional or in the strong sense. When f is para-vector-valued in $Cl(n,\Phi)$ and we write $f = f_0 + f_1\mathbf{e}_1 + \cdots + f_n\mathbf{e}_n$, it is readily checked that $Df = 0$ if and only if

$$\partial_0 f_0 = \sum_{j=1}^n \partial_j f_j \quad \text{with} \quad \partial_j f_k = \partial_k f_j \quad \text{and} \quad \partial_0 f_j = -\partial_j f_0 \quad \text{for} \quad 1 \le j < k \le n, \tag{2.3}$$

and similarly when $fD = 0$. In particular, a vector or para-vector-valued function which is left-monogenic must also be right-monogenic. In the same manner, when f is vector valued in $Cl(n+1,\Phi)$ with $f = f_1\mathbf{e}_1 + \cdots + f_{n+1}\mathbf{e}_{n+1}$, we have that

$D_{n+1}f = 0$ if and only if

$$\sum_{j=1}^{n+1} \partial_j f_j = 0 \quad \text{and} \quad \partial_j f_k - \partial_k f_j = 0 \quad \text{for} \quad 1 \le j < k \le n+1, \tag{2.4}$$

and the same if $fD_{n+1} = 0$. Thus, vector-valued left-monogenic functions are right-monogenic.

By (2.3), a para-vector-valued function $f = f_0 + f_1\mathbf{e}_1 + \cdots + f_n\mathbf{e}_n$, where the f_j are real valued, is monogenic if and only if $(-f_0, f_1, \cdots, f_n)$ is a harmonic gradient, meaning that it is the gradient of a harmonic function. The components of a harmonic gradient are sometimes referred to as a conjugate harmonic system, or a Riesz system of functions; *cf.* [21]. When $\mathbb{R}_1^n$ gets identified with $\mathbb{R}^{n+1}$, the fact that $(-f_0, f_1, \cdots, f_n)$ is a harmonic gradient amounts to saying that $\overline{f} = f_0 - f_1\mathbf{e}_1 - \cdots - f_n\mathbf{e}_n$ is a harmonic gradient. Likewise, it follows from (2.4) that a vector-valued function in $Cl(n+1, \mathbb{R})$, say $f = f_1\mathbf{e}_1 + \cdots + f_{n+1}\mathbf{e}_{n+1}$, is monogenic if and only if $(f_1, \cdots, f_{n+1})$ is a harmonic gradient. Identifying vectors in $Cl(n+1, \mathbb{R})$ with $\mathbb{R}^{n+1}$, we simply say in this case that f is a harmonic gradient.

Let $g : \mathbb{R}_{1,\pm}^n \to \mathbb{R}_1^n$. For $1 < p < \infty$, we say g belongs to the Hardy space $H^p(\mathbb{R}_{1,\pm}^n, \mathbb{R}_1^n)$ if $Dg = 0$ in $\mathbb{R}_{1,\pm}^n$ and

$$\|g\|_{H_\pm^p}^p \triangleq \sup_{\pm x_0 > 0} \int_{\mathbb{R}^n} |g(x_0 + \underline{x})|^p d\underline{x} < \infty. \tag{2.5}$$

We refer to the above Hardy spaces as being of para-vector type, or also of *inhomogeneous* type. Thus, $g \in H^p(\mathbb{R}_{1,\pm}^n, \mathbb{R}_1^n)$ if and only if $\overline{g}$ is a harmonic gradient which moreover satisfies the p-norm boundedness (2.5) in the relevant half-space. Equivalently, since each of the functions composing a conjugate harmonic system is harmonic, it follows from (2.5) and standard estimates on harmonic functions (see *e.g.* [21, Ch. II, Thm. 3.7 and eqn. (3.18)]) that a para-vector-valued monogenic function in $\mathbb{R}_{1,\pm}^n$ lies in $H^p(\mathbb{R}_{1,\pm}^n, \mathbb{R}_1^n)$ if and only if the nontangential maximal function given by

$$\mathcal{M}_\alpha g(\underline{x}) \triangleq \sup_{x_0 + \underline{x} \in \Gamma_\alpha(\underline{x})} |g(x_0 + \underline{x})| \tag{2.6}$$

lies in $L^p(\mathbb{R}^n, \mathbb{R})$ with equivalence of norms: $\|g\|_{H_\pm^p} \le \|\mathcal{M}_\alpha g\|_{L^p(\mathbb{R}^n)} \le C_\alpha \|g\|_{H_\pm^p}$. Here, to each $\alpha > 0$ and $\underline{x} \in \mathbb{R}^n$, the notation $\Gamma_\alpha(\underline{x})$ stands for the cone

$$\Gamma_\alpha(\underline{x}) = \{y_0 + \underline{y} \in \mathbb{R}_{1,\pm}^n, \ |\underline{y} - \underline{x}| < \alpha |y_0|\},$$

and the precise value of α is irrelevant except that the constants will depend on it.

Likewise, for $g : \mathbb{R}_\pm^{n+1} \to \mathbb{R}^{n+1} \subset Cl(n+1, \mathbb{R})$ and $1 < p < \infty$, we say that g belongs to the Hardy space $H^p(\mathbb{R}_\pm^{n+1}, \mathbb{R}^{n+1})$ if $D_{n+1}g = 0$ in the half-space $\mathbb{R}_\pm^{n+1}$ and

$$\|g\|_{H_{\pm,h}^p} = \sup_{\pm x_{n+1} > 0} \int_{\mathbb{R}^n} |g(\underline{x} + x_{n+1}\mathbf{e}_{n+1})|^p d\underline{x} < \infty. \tag{2.7}$$

Note that in the previous equation $\underline{x}$ refers to a vector in $Cl(n, \mathbb{R})$ viewed as a vector in $Cl(n+1, \mathbb{R})$ whose $(n+1)$-th component is zero. We refer to the latter Hardy spaces as being of vector type or homogeneous type, which is the reason for the subscript "h" in the notation for the norm. Thus, we have that $f = f_1\mathbf{e}_1 + \cdots + f_{n+1}\mathbf{e}_{n+1}$ lies in $H^p(\mathbb{R}_\pm^{n+1}, \mathbb{R}^{n+1})$ if and only if it is a harmonic gradient in $\mathbb{R}^{n+1}_{\pm}$ satisfying the p-boundedness condition (2.7). The latter is again equivalent to the L^p boundedness on $\mathbb{R}^n$ of the nontangential maximal function

$$\mathcal{M}_{\alpha,h}g(\underline{x}) \triangleq \sup_{\underline{x}+x_{n+1}\mathbf{e}_{n+1}\in\Gamma_{\alpha,h}(\underline{x})} |g(\underline{x}+x_{n+1}\mathbf{e}_{n+1})|, \tag{2.8}$$

where this time

$$\Gamma_{\alpha,h}(\underline{x}) = \{\underline{y}+y_{n+1}\mathbf{e}_{n+1} \in \mathbb{R}^{n+1}_{\pm},\ |\underline{y}-\underline{x}| < \alpha|y_{n+1}|\}.$$

In fact the passage from nonhomogeneous to homogeneous Hardy spaces is rather mechanical, trading x_0 and 1 for x_{n+1} and $\mathbf{e}_{n+1}$ while changing g_0 into $-g_{n+1}$.

Next, recall the local Fatou theorem asserting that a harmonic function in $\mathbb{R}^{n+1}_{+}$ which is nontangentially bounded at almost every point of a set $G \subset \mathbb{R}^n$ has a nontangential limit at almost every point of G. Here, nontangential refers to the fact that bounds and limits are sought in cones $\Gamma_{\alpha,h}(\underline{x})$ for arbitrary but fixed $\alpha > 0$; see [21, Thm. 3.19]. In view of (2.5) and (2.7), it follows from the local Fatou theorem that each component of a Hardy function (resp. homogeneous Hardy function) has a nontangential limit at almost every point of $\mathbb{R}^n \subset Cl(n,\mathbb{R})$ (resp. $\mathbb{R}^n\times\{0\} \subset \mathbb{R}^{n+1} \subset Cl(n+1,\mathbb{R})$). This defines boundary values for such functions. Now, it is an important and peculiar property of left- or right-monogenic functions that they can be recovered as Cauchy integrals of their boundary values; see [8, Cor. 3.20] and [11, 12]. Specifically, let us discuss the case of left-monogenic $Cl(n,\mathbb{R})$-valued functions on $\mathbb{R}^n_{1,+}$ as a prototypical example. If g is such a function and if $\mathcal{M}_\alpha g \in L^p(\mathbb{R}^n)$ for some $p \in (1,\infty)$, then g has nontangential limit a.e. on $\mathbb{R}^n$ since its components are harmonic functions with L^p nontangential maximal function, and this nontangential limit clearly lies in $L^p(\mathbb{R}^n)$ because it is dominated by the nontangential maximal function. Then, denoting the nontangential limit by g again and letting ω_n designate the volume of the unit sphere of dimension n, we get that $g = C^+g$, where

$$\begin{aligned} C^+g(x) &= \frac{1}{\omega_n}\int_{\mathbb{R}^n} \frac{\overline{\underline{y}-x}}{|\underline{y}-x|^{n+1}}(-e_0)g(\underline{y})d\underline{y} \\ &= \frac{1}{\omega_n}\int_{\mathbb{R}^n} \frac{x_0}{|x-\underline{y}|^{n+1}}g(\underline{y})d\underline{y} + \frac{1}{\omega_n}\int_{\mathbb{R}^n} \frac{x_1-y_1}{|x-\underline{y}|^{n+1}}(-\mathbf{e}_1)g(\underline{y})d\underline{y} \\ &\quad + \cdots + \frac{1}{\omega_n}\int_{\mathbb{R}^n} \frac{x_n-y_n}{|x-\underline{y}|^{n+1}}(-\mathbf{e}_n)g(\underline{y})d\underline{y}, \end{aligned} \tag{2.9}$$

with $x = x_0\mathbf{e}_0+x_1\mathbf{e}_1+\cdots+x_n\mathbf{e}_n \in \mathbb{R}^n_{1,+}$. Here, the presence of $-e_0$ in the definition of C^+ is because $-e_0$ is the exterior unit normal to $\mathbb{R}^n_{1,+}$; see [8, 12].

Conversely, if g is any $Cl(n,\mathbb{R})$-valued function in $L^p(\mathbb{R}^n)$, the formula for C^+g makes good sense and defines a left-monogenic function in $\mathbb{R}^n_{1,+}$ since the kernel is left-monogenic (as is easily verified). Invoking the Plemelj formula [12], we get at almost all points $\underline{x} \in \mathbb{R}^n$ that there exist nontangential limits of C^+g, denoted with a curly $\mathcal{C}^+g$, which are given by

$$\mathcal{C}^+g(\underline{x}) = \frac{1}{2}[g(\underline{x}) + Hg(\underline{x})] = \frac{1}{2}(I+H)g(\underline{x}), \tag{2.10}$$

where

$$\begin{aligned} Hg(\underline{x}) &= \frac{2}{\omega_n}\lim_{\varepsilon\to 0^+}\int_{|\underline{x}-\underline{y}|>\varepsilon}\frac{x_1-y_1}{|\underline{x}-\underline{y}|^{n+1}}(-\mathbf{e}_1)g(\underline{y})d\underline{y} \\ &\quad+\cdots+\frac{2}{\omega_n}\lim_{\varepsilon\to 0^+}\int_{|\underline{x}-\underline{y}|>\varepsilon}\frac{x_n-y_n}{|\underline{x}-\underline{y}|^{n+1}}(-\mathbf{e}_n)g(\underline{y})d\underline{y} \\ &\triangleq \sum_{k=1}^{n}(-\mathbf{e}_k)R_k(g)(\underline{x}), \end{aligned}$$

where

$$R_k(g)(\underline{x}) = \frac{2}{\omega_n}\lim_{\varepsilon\to 0^+}\int_{|\underline{x}-\underline{y}|>\varepsilon}\frac{x_k-y_k}{|\underline{x}-\underline{y}|^{n+1}}g(\underline{y})d\underline{y}$$

is the k-th Riesz transformation of g, $k = 1,\cdots,n$. As an operator, R_k has multiplier $-i\kappa_k/|\underline{\kappa}|$ in the Fourier domain ($\underline{\kappa} = \kappa_1\mathbf{e}_1 + \cdots + \kappa_n\mathbf{e}_n$ denoting the Fourier variable), and it maps $L^p(\mathbb{R}^n,\mathbb{R})$ into itself for $1 < p < \infty$; see [20, Ch. III, Secs. 4.3-4.4]. The operator H defines the Hilbert transformation in the present context, and it satisfies the relation $H^2 = I$, where I is the identity.

We note that our definition of H is consistent with the classical definition of the Hilbert transform in dimension 1, given *e.g.* in [2]. If we designate the latter with H_1, we get when $n = 1$, with $\mathbf{e}_1 = -i$, that

$$Hg = -\mathbf{e}_1\frac{1}{\pi}\left(\text{p.v.}\frac{1}{(\cdot)}*\right)g = i\frac{1}{\pi}\left(\text{p.v.}\frac{1}{(\cdot)}*\right)g \triangleq iH_1g,$$

where "p.v." indicates the principal value. Thus, in view of the well known relation $H_1^2 = -I$, we have that $H^2 = (iH_1)^2 = I$, which is consistent with the n-dimensional case. Similar considerations apply to functions in $H^p(\mathbb{R}^n_{1,-},\mathbb{R}^n_1)$, only trading C^+ for its opposite (because the outer normal to $\mathbb{R}^n_{1,-}$ is $\mathbf{e}_0$ and not $-\mathbf{e}_0$) and letting x now range over $\mathbb{R}^n_{1,-}$. This time letting $\mathcal{C}^-g$ stand for the nontangential limit on $\mathbb{R}^n$, this results in the Plemelj formula

$$\mathcal{C}^-g(\underline{x}) = \frac{1}{2}(I-H)g(\underline{x}). \tag{2.11}$$

In the sequel we denote by $\mathrm{Sc}\{x\}$ the scalar part of a Clifford number $x \in Cl(n,\Phi)$, which is the 0-form of x, and by $\mathrm{Nsc}\{x\}$ the nonscalar part of x, which is the sum of all the k-forms of x, $k = 1,\cdots,2^n$.

We pointed out already that each function in $H^p(\mathbb{R}^n_{1,\pm},\mathbb{R}^n_1)$ is naturally associated to the conjugate of a harmonic gradient. In the next lemma, we identify the latter as being the gradient of a Newton potential, and we describe the boundary values of Hardy functions.

Lemma 2.1. *If $g = g_0\mathbf{e}_0+\cdots+g_n\mathbf{e}_n \in H^p(\mathbb{R}^n_{1,\pm},\mathbb{R}^n_1)$, where each g_k is real valued, then its nontangential limit on $\mathbb{R}^n$, still denoted as g, satisfies $g = (I\pm H)g_0$, that is, $g_k = \mp R_kg_0$ for $k = 1,\cdots,n$. Conversely, each function on $\mathbb{R}^n$ of the form $(I\pm H)\varphi$, with $\varphi \in L^p(\mathbb{R}^n,\mathbb{R})$, is the nontangential limit of a function in $H^p(\mathbb{R}^n_{1,\pm},\mathbb{R}^n_1)$, and the Hardy norm is equivalent to $\|\varphi\|_{L^p(\mathbb{R}^n)}$. Moreover, the conjugate of each $g \in H^p(\mathbb{R}^n_{1,\pm},\mathbb{R}^n_1)$ is the harmonic gradient of the Newton potential of $\mp 2/(n-1)$ times its scalar part g_0, namely:*

$$g(x) = \partial_0\mathcal{N}_{g_0}(x) - \sum_{k=1}^{n}\partial_k\mathcal{N}_{g_0}(x)\mathbf{e}_k, \tag{2.12}$$

with

$$\mathcal{N}_{g_0}(x) \triangleq \mp\frac{2}{(n-1)\omega_n}\int_{\mathbb{R}^n}\frac{g_0(\underline{y})}{|x-\underline{y}|^{n-1}}d\underline{y}, \qquad x\in\mathbb{R}^n_{1,\pm}.$$

Proof. By the Cauchy-Clifford formula, the function g in the upper half-space may be expressed as the Cauchy integral over $\mathbb{R}^n$ of its boundary limit function. Therefore the Plemelj theorem implies that $(1/2)(I\pm H)g = g$, hence $\pm Hg = g$. By comparing the terms of degree 1 on the left- and the right-hand sides of the above identity we get $\mp\mathbf{e}_k R_k g_0 = \mathbf{e}_k g_k$ for $k=1,\cdots,n$, ensuring that $g=(I\pm H)g_0$. Invoking the Plemelj theorem again, this implies that the Cauchy integral of $2g_0$ is a left-monogenic function whose nontangential boundary value is g. Hence, by the Cauchy formula, this function must coincide with g on $\mathbb{R}^n_{1,\pm}$. Now, when identifying $\mathbb{R}^n_1$ with $\mathbb{R}^{n+1}$, the conjugate of the Cauchy kernel is $-1/(n-1)$ times the gradient of the Newton kernel $1/|x-\cdot|^{n-1}$ and taking gradient commutes with the integration, as the integrand does not have singularity, so we conclude that the conjugate of g is the gradient of the Newton potential of $\mp 2g_0/(n-1)$ (the $\mp$ arises because of the presence of $\mp\mathbf{e}_0$ in the Cauchy integral). Conversely, by (2.10), a function on $\mathbb{R}^n$ of the form $(I\pm H)\varphi$, where φ is scalar valued in $L^p(\mathbb{R}^n)$, is the nontangential limit of the Cauchy integral $C^{\pm}(2\varphi)$ which is indeed para-vector-valued and monogenic. To see that it lies in $H^p(\mathbb{R}^n_{1,\pm},\mathbb{R}^n_1)$, observe from what precedes that its conjugate is the gradient of the Newton potential $\mathcal{N}_\varphi$ of $\mp 2\varphi/(n-1)$. In particular, by inspection of formula (2.12) (where g_0 is set to φ), we find that $\mathrm{Sc}\{(C^{\pm}\varphi)(x_0,\underline{x})\} = (P_{x_0}*\varphi)(\underline{x})$ at every $(x_0,\underline{x})\in\mathbb{R}^n_{1,\pm}$, where the symbol "$*$" indicates convolution and P_{x_0} is the Poisson kernel at level x_0:

$$P_{x_0}(\underline{x}) = \frac{2}{\omega_n}\frac{|x_0|}{(x_0^2+|\underline{x}|^2)^{(n+1)/2}}, \qquad \underline{x}\in\mathbb{R}^n. \tag{2.13}$$

Since P_{x_0} has unit norm in $L^1(\mathbb{R}^n)$ for all x_0, it follows that

$$\left\|\mathrm{Sc}\left\{(C^{\pm}\varphi)(x_0,\cdot)\right\}\right\|_{L^p(\mathbb{R}^n)} \le \|\varphi\|_{L^p(\mathbb{R}^n)}, \quad \forall x_0, \tag{2.14}$$

implying that the scalar part of $C^{\pm}\varphi$ meets the p-boundedness condition (2.5). To show that the vector part also satisfies this condition, let us work on $\mathbb{R}^n_{1,+}$, as the argument for $\mathbb{R}^n_{1,-}$ is similar. Fix $z_0>0$ and consider the para-vector-valued function on $\mathbb{R}^n_{1,+}$ given by $F(x_0,\underline{x}) = (C^+\varphi)(z_0+x_0,\underline{x})$. Clearly it is monogenic, and we get upon applying Hölder's inequality to (2.9) that $\|F\|_{H^p_+}\le c$ for some constant $c=c(z_0)$; see definition (2.5). Hence $F\in H^p(\mathbb{R}^n_{1,\pm},\mathbb{R}^n_1)$, and it is obvious that its nontangential limit on $\mathbb{R}^n$ is $C^+\varphi(z_0,\cdot)$. Thus, by the previous part of the proof, it holds that

$$(C^+\varphi)(x_0,\cdot) = (I+H)\mathrm{Sc}\{F(0,\cdot)\} = (I+H)\mathrm{Sc}\left\{(C^+\varphi)(x_0,\cdot)\right\}.$$

In view of the definition of H (*cf.* the equation following (2.10)) and the L^p-boundedness of Riesz transforms, we now deduce from (2.14) that $C^+\varphi$ satisfies (2.5), as desired. We also proved that $\|C^+\varphi\|_{H^p_+}\le c\|\varphi\|_{L^p(\mathbb{R}^n)}$ for some constant c independent of φ, and since $\|\varphi\|_{L^p(\mathbb{R}^n)}$ is obviously less than the L^p norm of the nontangential maximal function, which itself is equivalent to the Hardy norm, as pointed out after (2.6), the proof is complete. □

When dealing with functions in $H^p(\mathbb{R}^{n+1}_\pm,\mathbb{R}^{n+1})$, the Cauchy formula is the same except that $\mp\mathbf{e}_0$ gets replaced by $\mp\mathbf{e}_n$ (the outer normal to $\mathbb{R}^{n+1}_\pm$), and in the

Plemelj formula H is changed into $H\mathbf{e}_{n+1} = \sum_{k=1}^n(-\mathbf{e}_k\mathbf{e}_{n+1})R_k$; see [8, Ch. 2, Sec. 5]. An argument analogous to the previous one shows that $f = f_1\mathbf{e}_1+\cdots+f_{n+1}\mathbf{e}_{n+1}$ lies in $H^p(\mathbb{R}^{n+1}_\pm, \mathbb{R}^{n+1})$ if and only if $f_k = \pm R_k f_{n+1}$ for $k = 1, \cdots, n$ and that f is $D_{n+1}\mathcal{N}$, where $\mathcal{N}$ is the Newton potential of $\mp 2g_{n+1}/(n-1)$.

Observe that Lemma 2.1 and its analog for homogeneous spaces entail that $H^p(\mathbb{R}^n_{1,\pm}, \mathbb{R}^n_1)$ (resp. $H^p(\mathbb{R}^{n+1}_\pm, \mathbb{R}^{n+1})$) is a Banach space isomorphic to $L^p(\mathbb{R}^n, \mathbb{R})$, with norm equivalent to the L^p-norm of the trace of the scalar part (resp. $(n+1)$-th component). Observe also from this, since $L^q(\mathbb{R}^n, \mathbb{R}) \cap L^p(\mathbb{R}^n, \mathbb{R})$ is dense in $L^p(\mathbb{R}^n, \mathbb{R})$ for any $q \in (1, \infty)$, that $H^p(\mathbb{R}^{n+1}_\pm, \mathbb{R}^{n+1}) \cap H^q(\mathbb{R}^{n+1}_\pm, \mathbb{R}^{n+1})$ is dense in $H^p(\mathbb{R}^{n+1}_\pm, \mathbb{R}^{n+1})$.

Lemma 2.1 easily implies a result which is of interest in its own right and parallels the density of rational functions in holomorphic Hardy spaces of index $p \in (1, \infty)$ on the half-plane [6]. Note that rational functions with simple poles are conjugate of gradients of logarithmic potentials of discrete measures with finite support. In the present context, analogs of rational functions with simple poles are conjugates of gradients of Newton potentials of discrete measures with finite support. Specifically, if we let

$$\begin{aligned} R_x(y) &= \overline{\nabla_y\left(\frac{1}{\omega_n|x-y|^{n-1}}\right)} \\ &= \frac{n-1}{\omega_n}\left(\frac{x_0-y_0}{|x-y|^{n+1}} - \sum_{j=1}^n \frac{x_j-y_j}{|x-y|^{n+1}}\mathbf{e}_j\right), \quad y \in \mathbb{R}^n_{1,+},\ x \in \mathbb{R}^n_{1,-}, \end{aligned}$$

then $R_x \in H^p(\mathbb{R}^n_{1,+}, \mathbb{R}^n_1)$ as a function of y for fixed x, and we have the following result.

Corollary 2.2. *The span of* $\{R_x\}_{x\in\mathbb{R}^n_{1,-}}$ *is dense in* $H^p(\mathbb{R}^n_{1,+}, \mathbb{R}^n_1)$ *for* $1 < p < n$.

Proof. It follows from Lemma 2.1 that $H^p(\mathbb{R}^n_{1,+}, \mathbb{R}^n_1)$ is isomorphic to $L^p(\mathbb{R}^n, \mathbb{R})$ with equivalence of norms, the isomorphism being

$$L^p(\mathbb{R}^n, \mathbb{R}) \ni h \ \mapsto\ C^+\Big(h - \sum_{k=1}^n (R_k h)\mathbf{e}_k\Big) \in H^p(\mathbb{R}^n_{1,+}, \mathbb{R}^n_1). \tag{2.15}$$

The inverse image of R_x under this isomorphism is $(1-n)/2$ times the Poisson kernel $P_{x_0}(\underline{x} - \underline{y})$ defined in (2.13). Thus, by the Hahn-Banach theorem, the asserted density is equivalent to the fact that no nonzero function in $L^{p'}(\mathbb{R}^n, \mathbb{R})$, with $1/p + 1/p' = 1$, can have vanishing Poisson integral. This, however, drops out immediately from the property that the Poisson kernel is an approximate identity, thereby achieving the proof. □

When saying that a vector-valued function $f = f_1\mathbf{e}_1 + \cdots + f_n\mathbf{e}_n$ on $\mathbb{R}^n$ is divergence free, we mean that $\operatorname{div} f = \sum_{k=1}^n \partial_k f_k = 0$. This is to be understood in the generalized function sense that amounts to the relation $\sum_{k=1}^n R_k f_k = 0$ or, equivalently, $\sum_{k=1}^n \xi_k \hat{f}_k(\xi) = 0$, through the inverse Fourier transformation, again to be understood in the generalized function sense if $p > 2$ so that the Fourier transform is really a distribution. The space of vector-valued divergence free maps in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ is a closed subset thereof and thus a Banach space in its own right that we denote by $D^p(\mathbb{R}^n)$. Though initially defined on $\mathbb{R}^n$ only, a divergence free vector field extends naturally to $\mathbb{R}^n_{1,\pm}$ (resp. $\mathbb{R}^{n+1}_\pm$) into an $\mathbb{R}^n$-valued map F which is independent of x_0 (resp. x_{n+1}). This function need not be monogenic, but it satisfies $\mathrm{Sc}\{DF\} = 0$ (resp. $\mathrm{Sc}\{D_{n+1}F\} = 0$).

3. Hardy-Hodge decomposition of para-vector-valued functions in $L^p(\mathbb{R}^n)$

Theorem 3.1. *Let f be a para-vector-valued function in $L^p(\mathbb{R}^n, Cl(n,\Phi))$, $1 < p < \infty$. Then f is uniquely decomposed as $f = f^+ + f^- + f^0$, all in $L^p(\mathbb{R}^n)$, such that $f^\pm$ are para-vector-valued, being the nontangential boundary limits of some two functions in, respectively, $H^p(\mathbb{R}^n_{1,\pm}, \mathbb{R}^n_1)$, and f^0 is vector valued and divergence free. Moreover, for all p in the indicated range the decomposition is unique and, for $p = 2$,*

$$\|f\|^2 = \|f^+\|^2 + \|f^-\|^2 + \|f^0\|^2. \tag{3.16}$$

In fact, this decomposition induces a topological direct sum:

$$L^p(\mathbb{R}^n, \mathbb{R}^n_1) = H^p(\mathbb{R}^n_{1,+}, \mathbb{R}^n_1) \oplus H^p(\mathbb{R}^n_{1,-}, \mathbb{R}^n_1) \oplus D^p(\mathbb{R}^n). \tag{3.17}$$

Proof. Let $f(\underline{x}) = \sum_{k=0}^n f_k(\underline{x})\mathbf{e}_k$ be in $L^p(\mathbb{R}^n)$, where $f_0(\underline{x}), f_1(\underline{x}), \cdots, f_n(\underline{x})$ are scalar valued and $\underline{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n \in \mathbb{R}^n$.

By (2.10) and (2.11), since $H^2 = I$, we have that $\mathcal{C}^\pm$ is a projection:

$$\begin{aligned} \frac{1}{2}(I+H)f(\underline{x}) &= [\frac{1}{2}(I+H)]^2 f(\underline{x}), \\ \frac{1}{2}(I-H)f(\underline{x}) &= [\frac{1}{2}(I-H)]^2 f(\underline{x}). \end{aligned}$$

Then we have

$$\begin{aligned} f(\underline{x}) &= \frac{1}{2}(I+H)f(\underline{x}) + \frac{1}{2}(I-H)f(\underline{x}) \\ &= [\frac{1}{2}(I+H)]^2 f(\underline{x}) + [\frac{1}{2}(I-H)]^2 f(\underline{x}) \\ &= \frac{1}{2}(I+H)\left[\mathrm{Sc}\{\frac{1}{2}(I+H)f(\underline{x})\} + \mathrm{Nsc}\{\frac{1}{2}(I+H)f(\underline{x})\}\right] \\ &+ \frac{1}{2}(I-H)\left[\mathrm{Sc}\{\frac{1}{2}(I-H)f(\underline{x})\} + \mathrm{Nsc}\{\frac{1}{2}(I-H)f(\underline{x})\}\right] \\ &= \frac{1}{2}(I+H)[\mathrm{Sc}\{\frac{1}{2}(I+H)f(\underline{x})\}] + \frac{1}{2}(I-H)[\mathrm{Sc}\{\frac{1}{2}(I-H)f(\underline{x})\}] \\ &+ \frac{1}{2}(I+H)[\mathrm{Nsc}\{\frac{1}{2}(I+H)f(\underline{x})\}] + \frac{1}{2}(I-H)[\mathrm{Nsc}\{\frac{1}{2}(I-H)f(\underline{x})\}]. \end{aligned} \tag{3.18}$$

Consider the function given by the last line of the above chain of equalities, viz.

$$\frac{1}{2}(I+H)[\mathrm{Nsc}\{\frac{1}{2}(I+H)f(\underline{x})\}] + \frac{1}{2}(I-H)[\mathrm{Nsc}\{\frac{1}{2}(I-H)f(\underline{x})\}]. \tag{3.19}$$

It can be computed directly through

$$\begin{aligned} &\frac{1}{2}(I+H)[\mathrm{Nsc}\{\frac{1}{2}(I+H)f(\underline{x})\}] \\ = {} & \frac{1}{4}(I+H)[\sum_{k=1}^n f_k\mathbf{e}_k + \mathrm{Nsc}\{Hf\}] \\ = {} & \frac{1}{4}\left\{\sum_{k=1}^n f_k\mathbf{e}_k + \mathrm{Nsc}\{Hf\} + \sum_{k=1}^n H[f_k\mathbf{e}_k] + H[\mathrm{Nsc}\{Hf\}]\right\} \end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}(I-H)[\mathrm{Nsc}\{\frac{1}{2}(I-H)f(\underline{x})\}] \\
= \quad &\frac{1}{4}(I-H)[\sum_{k=1}^{n} f_k\mathbf{e}_k - \mathrm{Nsc}\{Hf\}] \\
= \quad &\frac{1}{4}\left\{\sum_{k=1}^{n} f_k\mathbf{e}_k - \mathrm{Nsc}\{Hf\} - \sum_{k=1}^{n} H[f_k\mathbf{e}_k] + H[\mathrm{Nsc}\{Hf\}]\right\}.
\end{aligned}$$

By adding these relations together, we have that

$$\begin{aligned}
&\frac{1}{2}(I+H)[\mathrm{Nsc}\{\frac{1}{2}(I+H)f(\underline{x})\}] + \frac{1}{2}(I-H)[\mathrm{Nsc}\{\frac{1}{2}(I-H)f(\underline{x})\}] \\
= \quad &\frac{1}{2}\left\{\sum_{k=1}^{n} f_k\mathbf{e}_k + H[\mathrm{Nsc}\{Hf\}]\right\}. \qquad (3.20)
\end{aligned}$$

As f is para-vector valued, relation (3.18) and the fact that H maps scalar-valued functions to vector-valued functions together imply that the quantity in (3.20) is a para-vector; therefore $H[\mathrm{Nsc}\{Hf\}]$ is a para-vector. Now we work out its expression.

Since

$$Hf = \sum_{k=1}^{n} R_k(f_k) - \sum_{k=1}^{n} R_k(f_0)\mathbf{e}_k + H \wedge \underline{f},$$

where $\underline{f} = \sum_{k=1}^{n} f_k\mathbf{e}_k$, we have that

$$\mathrm{Nsc}\{Hf\} = -\sum_{k=1}^{n} R_k(f_0)\mathbf{e}_k + H \wedge \underline{f}.$$

Consequently, since we need to collect terms only of the forms 0 and 1, and because Riesz transforms commute, we obtain

$$\begin{aligned}
H[\mathrm{Nsc}\{Hf\}] \quad &= \quad (-\sum_{k=1}^{n} R_k^2)f_0 + \sum_{k=1}^{n}\left[(\sum_{l\neq k} -R_l^2)f_k + R_k\sum_{l\neq k} R_l f_l\right]\mathbf{e}_k \\
&= \quad f_0 + \sum_{k=1}^{n}\left[(\sum_{l\neq k} -R_l^2)f_k + R_k\sum_{l\neq k} R_l f_l\right]\mathbf{e}_k.
\end{aligned}$$

Substituting back into (3.20), we get that

$$\begin{aligned}
&\frac{1}{2}(I+H)[\mathrm{Nsc}\{\frac{1}{2}(I+H)f(\underline{x})\}] + \frac{1}{2}(I-H)[\mathrm{Nsc}\{\frac{1}{2}(I-H)f(\underline{x})\}] \\
= \quad &\frac{1}{2}\left\{f(\underline{x}) + \sum_{k=1}^{n}\left[(\sum_{l\neq k} -R_l^2)f_k + R_k\sum_{l\neq k} R_l f_l\right]\mathbf{e}_k\right\}.
\end{aligned}$$

Therefore, by (3.18),

$$\begin{aligned} f(\underline{x}) &= \frac{1}{2}(I+H)\left[\mathrm{Sc}\{\frac{1}{2}(I+H)f(\underline{x})\}\right]+\frac{1}{2}(I-H)\left[\mathrm{Sc}\{\frac{1}{2}(I-H)f(\underline{x})\}\right] \\ &\quad +\frac{1}{2}\left\{f(\underline{x})+\sum_{k=1}^{n}\left[(\sum_{l\neq k}-R_l^2)f_k+R_k\sum_{l\neq k}R_lf_l\right]\mathbf{e}_k\right\}. \end{aligned}$$

Finally,

$$\begin{aligned} f(\underline{x}) &= (I+H)\left[\mathrm{Sc}\{\frac{1}{2}(I+H)f(\underline{x})\}\right]+(I-H)\left[\mathrm{Sc}\{\frac{1}{2}(I-H)f(\underline{x})\}\right] \\ &\quad +\sum_{k=1}^{n}\left[(\sum_{l\neq k}-R_l^2)f_k+R_k\sum_{l\neq k}R_lf_l\right]\mathbf{e}_k. \end{aligned} \tag{3.21}$$

It is apparent that $(I+H)[\mathrm{Sc}\{\frac{1}{2}(I+H)f(\underline{x})\}]$ and $(I-H)[\mathrm{Sc}\{\frac{1}{2}(I-H)f(\underline{x})\}]$ are para-vector valued, and it follows from Lemma 2.1 that they are boundary values of functions in $H^p(\mathbb{R}^n_{1,+},\mathbb{R}^n_1)$ and $H^p(\mathbb{R}^n_{1,-},\mathbb{R}^n_1)$ respectively. Now we show that

$$\sum_{k=1}^{n}\left[(\sum_{l\neq k}-R_l^2)f_k+R_k\sum_{l\neq k}R_lf_l\right]\mathbf{e}_k$$

is divergence free. For this, by the last remark of the last section, it suffices to show that

$$\sum_{k=1}^{n}R_k\left[(\sum_{l\neq k}-R_l^2)f_k+R_k\sum_{l\neq k}R_lf_l\right]=0. \tag{3.22}$$

The above, however, is obvious since Riesz transformations commute, and thus we obtain the desired decomposition $f(\underline{x})=f^++f^-+f^0$, where

$$f^+=(I+H)[\mathrm{Sc}\{\frac{1}{2}(I+H)f(\underline{x})\}]\in H^p(\mathbb{R}^n_{1,+},\mathbb{R}^n_1), \tag{3.23}$$

$$f^-=(I-H)[\mathrm{Sc}\{\frac{1}{2}(I-H)f(\underline{x})\}]\in H^p(\mathbb{R}^n_{1,-},\mathbb{R}^n_1), \tag{3.24}$$

and

$$f^0=\sum_{k=1}^{n}\left[(\sum_{l\neq k}-R_l^2)f_k+R_k\sum_{l\neq k}R_lf_l\right]\mathbf{e}_k \tag{3.25}$$

is divergence free.

Next we prove uniqueness. This is equivalent to showing that if we have a decomposition of the zero function $0=f^++f^-+f^0$, then it must be that $f^+=f^-=f^0=0$. Indeed, in that case we may write

$$0=(I+H)(f^++f^-+f^0)=2f^++(I+H)f^0=2f^++f^0+Hf^0,$$

where we used that $f^+=Hf^+$ by (2.10) since $\mathcal{C}^+f^+=f^+$, and also that $Hf^-=-f^-$ by (2.11) since $\mathcal{C}^-f^-=f^-$. Note that since f^0 is divergence free, the scalar part of Hf^0 is zero, and thus only the 2-form part of Hf^0 is possibly nonzero. However, the last equality shows that the 2-form part also has to be zero, because all the other terms are para-vectors. We thus conclude that $2f^++f^0=0$. The same reasoning gives us $2f^- + f^0 = 0$. These together yield $f^+ = f^-$. By applying $I + H$ to both sides we get $f^+ = 0$, hence also $f^- = 0$, and consequently $f^0 = 0$. This establishes uniqueness and shows that (3.17) holds as a direct sum. In addition, since $f^\pm$ and f^0 are continuous functions of f in $L^p(\mathbb{R}^n, Cl(n,\Phi))$ by (3.23), (3.24), (3.25) and the L^p continuity of the Riesz transformations, we see that the projections in (3.17) are continuous; hence the sum is topological by the open mapping theorem. Finally, when $p = 2$, we show the Pythagoras type relation

$$\|f\|^2 = \|f^+\|^2 + \|f^-\|^2 + \|f^0\|^2. \tag{3.26}$$

First, since f is para-vector valued, we obviously have that

$$\|f\|^2 = \int_{\mathbb{R}^2} f\overline{f} d\underline{x}.$$

Hence, to prove (3.26), it suffices to establish the following orthogonality relations:

$$\int_{\mathbb{R}^n} f^+\overline{f^-} d\underline{x} = \int_{\mathbb{R}^n} f^-\overline{f^+} d\underline{x} = 0 \tag{3.27}$$

and

$$\int_{\mathbb{R}^n} (f^+\overline{f^0} + f^0\overline{f^+}) d\underline{x} = \int_{\mathbf{R}^n} (f^-\overline{f^0} + f^0\overline{f^-}) d\underline{x} = 0. \tag{3.28}$$

Let us show (3.27). Recall from Lemma 2.1 that if $g \in H^2(\mathbb{R}^n_{1,\pm}, \mathbb{R}^n_1)$, then $g = (I \pm H)g_0$, and consequently, taking Fourier transforms (the Fourier transform of a vector-valued function is computed componentwise), we get that $\hat{g} = 2\chi_\pm \hat{g_0}$, where $\chi_\pm$ are multipliers for the Hardy space projections $\chi_\pm(\underline{\xi}) = \frac{1}{2}(1 \pm i\frac{\underline{\xi}}{|\underline{\xi}|})$, that satisfy $\chi_\pm^2 = \chi_\pm$ and $\chi_+ + \chi_- = 1$ as well as $\chi_+\chi_- = \chi_-\chi_+ = 0$. Here we used the expression for the multiplier of R_k in the Fourier domain; see [12, 20]. Applying these remarks to $g = f^\pm$ and using Parseval's Theorem, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^2} f^+\overline{f^-} d\underline{x} \\
=\ &\int_{\mathbb{R}^2} (f^+)^\wedge(\underline{\xi})\overline{(f^-)^\wedge(\underline{\xi})} d\underline{\xi} \\
=\ &4\int_{\mathbb{R}^2} \chi_+(\underline{\xi})\chi_-(\underline{\xi})\hat{f_0^+}\overline{\hat{f_0^-}}\, d\underline{\xi} \\
=\ &0,
\end{aligned}$$

where we used the relation $\chi_+(\underline{\xi})\chi_-(\underline{\xi}) = 0$ for all ξ. The proof of the second equality relation in (3.27) is similar.

Now, let us show that equality (3.28) holds. Indeed,

$$\begin{aligned}
\int_{\mathbb{R}^n} (f^+\overline{f^0} + f^0\overline{f^+}) d\underline{x} &= 2\mathrm{Sc}\left\{\int_{\mathbb{R}^n} f^+\overline{f^0} d\underline{x}\right\} \\
&= \mathrm{Sc}\left\{\int_{\mathbb{R}^n} \left(1 + i\frac{\underline{\xi}}{|\underline{\xi}|}\right) \widehat{f_0}(\xi)\overline{\widehat{f^0}}(\underline{\xi}) d\underline{\xi}\right\} \\
&= \mathrm{Sc}\left\{\int_{\mathbb{R}^n} \widehat{f_0}(\xi)\overline{\frac{i\underline{\xi}}{|\underline{\xi}|}\widehat{f^0}(\underline{\xi})} d\underline{\xi}\right\} \\
&= 0,
\end{aligned}$$

where the last equality used the relation $(i\underline{\xi}/|\underline{\xi}|)\widehat{f^0}(\underline{\xi}) = 0$, the latter being a consequence of the fact that f^0 is divergence free. The proof is complete. $\square$

Remark. If, alternatively, we use the scalar product

$$\langle f, g\rangle = \mathrm{Sc} \int_{\mathbb{R}^n} f\overline{g}d\underline{x},$$

then indeed the decomposition $f = f^+ + f^- + f^0$ is orthogonal. Moreover, since we observed after the proof of Lemma 2.1 that $H^2(\mathbb{R}^n_{1,\pm}, \mathbb{R}^n_1) \cap H^p(\mathbb{R}^n_{1,\pm}, \mathbb{R}^n_1)$ is dense in $H^p(\mathbb{R}^n_{1,\pm}, \mathbb{R}^n_1)$, we deduce that (3.27) holds as soon as $f^+ \in H^p(\mathbb{R}^n_{1,+}, \mathbb{R}^n_1)$ and $f^- \in H^{p'}(\mathbb{R}^n_{1,-}, \mathbb{R}^n_1)$ with $1/p + 1/p' = 1$:

(3.29)
$$\int_{\mathbb{R}^n} f^+\overline{f^-}d\underline{x} = \int_{\mathbb{R}^n} f^-\overline{f^+}d\underline{x} = 0, \qquad f^+ \in H^p(\mathbb{R}^n_{1,+}, \mathbb{R}^n_1), \quad f^- \in H^{p'}(\mathbb{R}^n_{1,-}, \mathbb{R}^n_1).$$

Likewise, (3.28) generalizes to

$$\begin{aligned}\int_{\mathbb{R}^n} (f^+\overline{f^0} + f^0\overline{f^+})d\underline{x} &= \int_{\mathbf{R}^n} (f^-\overline{f^0} + f^0\overline{f^-})d\underline{x} \\ &= 0, \quad f^\pm \in H^p(\mathbb{R}^n_{1,\pm}, \mathbb{R}^n_1), \quad f^0 \in D^{p'}(\mathbb{R}^n).\end{aligned} \tag{3.30}$$

A few comments are in order:

- If (3.17) gets projected onto the last n components, and since $(\pm R_1 h, \cdots, \pm R_n h)$ is a gradient vector field on $\mathbb{R}^n$ (*i.e.* the gradient of the trace of a solution to the Neumann problem on $\mathbb{R}^{n+1}_\pm$ with inner normal derivative h a.e. on $\mathbb{R}^n$), we recover the classical Helmoltz-Hodge decomposition of vector fields from $L^p(\mathbb{R}^n, \mathbb{R}^n)$ into the sum of a rotational free and divergence free vector field [9].
- Decomposition (3.17) generalizes to higher dimensions the standard decomposition of a complex-valued function in $L^p(\mathbb{R})$ into the sum of a function belonging to the holomorphic Hardy space $H^p(\mathbb{R}^2_+)$ and a function belonging to the holomorphic Hardy space $H^p(\mathbb{R}^2_-)$. The difference in dimension bigger than 1 is that a divergence free term must be added, for in this case not every vector field is a gradient. Note, since R_k and the divergence operator preserve realness, that Theorem 3.1 carries over to Clifford-valued maps and Clifford Hardy spaces with complex coefficients.

4. Variations

Next we consider the homogeneous case on $\mathbb{R}^n$. We regard $\mathbb{R}^n$ as being the subspace $\mathbb{R}^n \times \{0\}$ of $\mathbb{R}^{n+1}$. When considering Clifford 1-forms as Euclidean vectors, we mean that 1-forms in $Cl(n, \mathbb{R})$ get identified with 1-forms in $Cl(n+1, \mathbb{R})$ whose coefficient of $\mathbf{e}_{n+1}$ is zero. Note that $Cl(n, \mathbb{R})$ can be viewed as the subalgebra of $Cl(n+1, \mathbb{R})$ generated by $\mathbf{e}_1, \cdots, \mathbf{e}_n$.

Theorem 4.1. *Let $f \in L^p(\mathbb{R}^n, Cl(n+1, \mathbb{R}))$ be an $(n+1)$-vector-valued function, $1 < p < \infty$. Then f is uniquely decomposed as $f = f^+ + f^- + f^0$, all in $L^p(\mathbb{R}^n)$, such that $f^\pm$ are the nontangential boundary limits of some two functions in, respectively, $H^p(\mathbb{R}^{n+1}_\pm, \mathbb{R}^{n+1})$, while f^0 is vector valued in $Cl(n, \mathbb{R})$ and divergence free. For all p in the indicated range, the decomposition is unique and topological. Moreover, for $p = 2$,*

$$\|f\|^2 = \|f^+\|^2 + \|f^-\|^2 + \|f^0\|^2.$$

By factorizing out $\mathbf{e}_{n+1}$ one can, in particular, reduce the proof of Theorem 4.1 to that of Theorem 3.1. More precisely, noting that $\mathbf{e}_{n+1}^{-1} = -\mathbf{e}_{n+1}$, we use the relation

$$\sum_{k=1}^{n+1} f_k\mathbf{e}_k = [\sum_{k=1}^{n+1} f_k\mathbf{e}_k\mathbf{e}_{n+1}^{-1}]\mathbf{e}_{n+1}.$$

For the Cauchy kernels in the two settings, one has

$$\frac{\overline{\sum_{k=1}^n y_k\mathbf{e}_k - (\sum_{k=1}^{n+1} x_k\mathbf{e}_k)}}{|\sum_{k=1}^n y_k\mathbf{e}_k - (\sum_{k=1}^{n+1} x_k\mathbf{e}_k)|^{n+1}} = -\mathbf{e}_{n+1}\frac{\overline{\sum_{k=1}^n y_k\mathbf{e}_k\mathbf{e}_{n+1}^{-1} - (\sum_{k=1}^n x_k\mathbf{e}_k\mathbf{e}_{n+1}^{-1} + x_{n+1})}}{|\sum_{k=1}^n y_k\mathbf{e}_k\mathbf{e}_{n+1}^{-1} - (\sum_{k=1}^n x_k\mathbf{e}_k\mathbf{e}_{n+1}^{-1} + x_{n+1})|^{n+1}}.$$

This corresponds to the relation between the two Dirac operators:

$$\sum_{k=1}^{n+1} \partial_k\mathbf{e}_k = (\sum_{k=1}^{n+1} \partial_k\mathbf{e}_k\mathbf{e}_{n+1}^{-1})\mathbf{e}_{n+1}.$$

Letting $\tilde{\mathbf{e}}_k = \mathbf{e}_k\mathbf{e}_{n+1}^{-1},\ k = 1, \cdots, n$, one reduces the proof of Theorem 4.1 to that of Theorem 3.1 (compare [17]).

Theorem 4.1 can alternatively be rewritten without mentioning Clifford analysis as

Theorem 4.2. *Every vector field $f \in L^p(\mathbb{R}^n, \mathbb{R}^{n+1})$, $1 < p < \infty$, may be uniquely decomposed as $f = f^+ + f^- + f^0$, where $f^\pm \in L^p(\mathbb{R}^n, \mathbb{R}^{n+1})$ are, respectively, the nontangential boundary limits of some harmonic gradients on $\mathbb{R}^{n+1}_\pm$, which satisfy* (2.7), *while $f^0 \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ is divergence free. The decomposition is topological, and for $p = 2$ there holds*

$$\|f\|^2 = \|f^+\|^2 + \|f^-\|^2 + \|f^0\|^2.$$

From the analog of Lemma 2.1 for homogeneous Hardy spaces (see the discussion after the proof of that lemma), we know that boundary limits of harmonic gradients on $\mathbb{R}^{n+1}_\pm$ satisfying (2.7) are those members of $L^p(\mathbb{R}^n, \mathbb{R}^{n+1})$ of the type $(\pm R_1h, \cdots, \pm R_nh, h)$ with $h \in L^p(\mathbb{R}^n, \mathbb{R})$, where all the "+" signs and, respectively, all the "−" signs are taken. In such form, Theorem 4.2 was proven in [1] when $n = 2$, also for more general function spaces.

It is worth contrasting Theorem 3.1 and Theorem 4.1 with their quaternionic counterparts. The space $\mathbb{H}$ of real quaternions consists of numbers $q = q_0 + \underline{q}$, with $\underline{q} = q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3$, where $q_j \in \mathbb{R}$ and $\mathbf{e}_1, \mathbf{e}_2$ are as before, but, additionally, $\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2$. We identify the linear space consisting of all $\underline{q}$ with the space $\mathbb{R}^3$, and we put $\mathbb{H}_\pm$ for those quaternions with, respectively, $\pm q_0 > 0$. We say that a quaternionic-valued function f is left-quaternionic if $(D_0 + D_3)f = 0$ (see (2.2)), but this time the relation $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3$ is taken into account. The definitions of quaternionic Hardy spaces $H^p(\mathbb{H}_\pm, \mathbb{H})$ as spaces of left quaternionic functions in $\mathbb{H}_\pm$ meeting the analogs of (2.5) now run parallel to those for inhomogeneous Hardy spaces.

Theorem 4.3. *Let $f \in L^p(\mathbb{R}^3, \mathbb{H})$, $1 < p < \infty$. Then f is uniquely decomposed as $f = f^+ + f^-$ such that $f^\pm \in L^p(\mathbb{R}^3, \mathbb{H})$ are nontangential boundary limit functions* *of some two functions in, respectively, $H^p(\mathbb{H}_\pm, \mathbb{H})$. Moreover, for $p = 2$, f^+ and f^- are orthogonal:*

$$\|f\|^2 = \|f^+\|^2 + \|f^-\|^2.$$

Proof. This is an immediate consequence of the corresponding Plemelj formula, for functions obtained from the Cauchy formula are all quaternionic-valued.

Remark. The reason why the divergence free term f^0 can be omitted in Theorem 4.3 is the closedness of multiplication in the quaternionic field. At the same time, the interpretation of $f^\pm$ as traces of harmonic gradients is lost. To see the difference in the analysis of $H^p(\mathbb{R}^3_{1,\pm}, \mathbb{R}^3_1)$ and $H^p(\mathbb{H}_\pm, \mathbb{H})$, recall that if $f(x_0 + \underline{x}) = f_0(x_0+\underline{x})\mathbf{e}_0 + f_1(x_0+\underline{x})\mathbf{e}_1 + f_2(x_0+\underline{x})\mathbf{e}_2 + f_3(x_0+\underline{x})\mathbf{e}_3$ lies in $H^p(\mathbb{R}^3_{1,+}, \mathbb{R}^3_1)$, then

$$\begin{cases} \frac{\partial f_0}{\partial x_0} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}, \\ \frac{\partial f_0}{\partial x_i} = -\frac{\partial f_i}{\partial x_0}, \quad i = 1, 2, 3, \\ \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \quad i \neq 0, j \neq 0, i \neq j, \end{cases} \tag{4.31}$$

while $f(q_0+\underline{q}) = f_0(q_0+\underline{q})\mathbf{e}_0 + f_1(q_0+\underline{q})\mathbf{e}_1 + f_2(q_0+\underline{q})\mathbf{e}_2 + f_3(q_0+\underline{q})\mathbf{e}_3 \in H^p(\mathbb{H}_+, \mathbb{H})$ will imply

$$\begin{cases} \frac{\partial f_0}{\partial x_0} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}, \\ \frac{\partial f_1}{\partial x_2} = \frac{\partial f_3}{\partial x_0} + \frac{\partial f_0}{\partial x_3} + \frac{\partial f_2}{\partial x_1}, \\ \frac{\partial f_2}{\partial x_3} = \frac{\partial f_1}{\partial x_0} + \frac{\partial f_0}{\partial x_1} + \frac{\partial f_3}{\partial x_2}, \\ \frac{\partial f_3}{\partial x_1} = \frac{\partial f_2}{\partial x_0} + \frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3}. \end{cases} \tag{4.32}$$

The system of equations (4.31) implies the system of equations (4.32), but not conversely. □

5. Uniqueness issues for potentials in divergence form

In [1], the Hardy-Hodge decomposition was introduced when $n = 2$ to characterize silent magnetizations with support in $\mathbb{R}^2$. This issue can be recast as that of describing vanishing potentials in divergence form. Recall that the Newton potential of a distribution φ on $\mathbb{R}^{n+1}$ is the convolution of φ with $1/(\omega_n|x|^{n-1})$, wherever it exists. The potential is said to be in divergence form if φ can be taken to be the divergence of some $\mathbb{R}^{n+1}$-valued distribution ψ:

$$P_{\operatorname{div}\psi}(x) = \frac{1}{\omega_n}\int \frac{1}{|x-y|^{n-1}} \operatorname{div}\psi(y) = -\frac{1}{\omega_n}\int \nabla_y\left(\frac{1}{|x-y|^{n-1}}\right)\cdot\psi(y), \tag{5.33}$$

where ∇_y indicates the gradient with respect to the variable y, where the dot indicates Euclidean scalar product. When $n = 2$, in the quasi-static approximation to Maxwell's equations, (5.33) formally expresses the magnetic potential of the magnetization ψ (cf. [10, Sec. 5.9.C]). Those x (if any) for which this expression makes good sense depend of course on ψ. We shall be concerned with the case where ψ is supported on a hyperplane $\mathcal{P}$ and has L^p density there. Specifically, if we write

$$\mathcal{P} = \{x \in \mathbb{R}^{n+1},\ x\cdot u = a\} \tag{5.34}$$

for some $u \in \mathbb{R}^{n+1}$ and $a \in \mathbb{R}$, it means that $\psi = f \otimes \delta_0(x\cdot u - a)$ where $f = (f_0, \cdots, f_n) \in L^p(\mathcal{P}, \mathbb{R}^{n+1})$ and δ_0 indicates the Dirac mass at 0. Then, (5.33) becomes

$$P_{\operatorname{div}\psi}(x) = \frac{n-1}{\omega_n}\int_{\mathcal{P}} \frac{f(y)\cdot(x-y)}{|x-y|^{n+1}}\,dy, \tag{5.35}$$

which is well defined for all $x \notin \mathcal{P}$, more generally for all x not in the support of f.

Let

$$\mathcal{H}_\pm = \{x \in \mathbb{R}^{n+1},\ \pm(x\cdot u - a) > 0\}$$

denote the two half-spaces whose union is $\mathbb{R}^{n+1}\setminus\mathcal{P}$, the complement of $\mathcal{P}$. The question that we raise is:

for which f does it happen that $P_{\operatorname{div}\psi}(x) = 0$ for all $x \in \mathcal{H}_\pm$?

From the physical viewpoint, when $n = 1$ it amounts to describing those magnetizations with L^p density supported on a plane which are silent from one side of that plane, meaning that they generate no magnetic field in the corresponnding half-space. These cannot be detected by measuring devices and account a good deal for the ill-posedness of inverse magnetization problems [15]. The result below gives an answer to the question in terms of the Hardy-Hodge decomposition, thereby generalizing to higher dimension results from [1] for $n = 2$.

It will be convenient to define the Clifford Hardy spaces $H^p(\mathcal{H}_\pm, \mathbb{R}^n_1)$ consisting of para-vector-valued monogenic functions g on $\mathcal{H}_\pm$ meeting the condition

$$\sup_{\pm b>0}\int_{\mathcal{P}_b} |g|^p\,dm < \infty, \qquad \mathcal{P}_b \triangleq \{x \in \mathbb{R}^{n+1},\ \pm(x\cdot u - a) = b\}, \tag{5.36}$$

where dm indicates the differential of Lebesgue measure. Just as in the case of inhomogeneous Hardy spaces $H^p(\mathbb{R}^n_{1,\pm}, \mathbb{R}^n_1)$, condition (5.36) may be replaced by the $L^p(\mathcal{P}, \mathbb{R})$-boundedness of the nontangential maximal function, computed this time over cones with vertex on $\mathcal{P}$ and axis parallel to u. Functions in $H^p(\mathcal{H}_\pm, \mathbb{R}^n_1)$ have nontangential limits in $L^p(\mathcal{P}, \mathbb{R}^n_1)$, of which they are the Cauchy-Clifford integral, and they can be identified with their nontangential limit. In fact, if $\mathfrak{R}$ is any orientation preserving affine isometry mapping $\mathcal{P}$ to $\{0\}\times\mathbb{R}^n$, we have that f belongs to $H^p(\mathcal{H}_\pm, \mathbb{R}^n_1)$ if and only if $f\circ\mathfrak{R}$ belongs to $H^p(\mathbb{R}^n_{1,\pm}, \mathbb{R}^n_1)$. As an extra piece of notation, we use $u^\perp$ for the vector space orthogonal to u in $\mathbb{R}^{n+1}$, which is the linear hyperplane parallel to $\mathcal{P}$.

Theorem 5.1. *Let $\mathcal{P}\subset\mathbb{R}^{n+1}$ be a hyperplane defined by (5.34) and $\psi\in L^p(\mathcal{P}, \mathbb{R}^{n+1})$ with $1 < p < \infty$. Then the potential $P_{\operatorname{div}\psi}$ vanishes on $\mathcal{H}_\pm$ if and only if ψ is the sum of a member of $H^p(\mathcal{H}_\pm, \mathbb{R}^n_1)$ and of a divergence free vector field tangent to $\mathcal{P}$ in $L^p(\mathcal{P}, u^\perp)$. The potential $P_{\operatorname{div}\psi}$ vanishes on $\mathbb{R}^{n+1}\setminus\mathcal{P}$ (that is, on both $\mathcal{H}_+$ and $\mathcal{H}_-$) if and only if it is a divergence free function in $L^p(\mathcal{P}, u^\perp)$.*

Proof. Because the statement is invariant under orientation preserving affine isometries, we may assume that $\mathcal{P} = \{0\}\times\mathbb{R}^n$ so that $\mathcal{H}_\pm = \mathbb{R}^n_{1,\pm}$. Let us single out the components of ψ as $\psi_0, \cdots, \psi_n$ and identify ψ with the para-vector-valued function $\psi = \psi_0\mathbf{e}_0 + \psi_1\mathbf{e}_1 + \cdots + \psi_n\mathbf{e}_n$. Set $\psi = \psi^+ + \psi^- + \psi^0$ for the Hardy-Hodge decomposition from Theorem 3.1. For $x \in \mathbb{R}^n_{1,\pm}$, it is easily checked that $y \mapsto x - y/|x-y|^{n+1}$ lies in $H^q(\mathbb{R}^n_{1,\pm}, \mathbb{R}^n_1)$. Thus, it follows from (5.35), (3.29) and (3.30) that $P_{\operatorname{div}(\psi^\mp+\psi_0)} \equiv 0$ on $\mathbb{R}^n_{1,\pm}$. Therefore, the assumption that $P_{\operatorname{div}\psi} \equiv 0$ on $\mathbb{R}^n_{1,\pm}$ reduces to $P_{\operatorname{div}\psi^\mp} \equiv 0$ on $\mathbb{R}^n_{1,\pm}$. Now, comparing (5.35) and (2.9), we find this is equivalent to

$$\operatorname{Sc}\left\{C^\pm\psi^\mp\right\}(\bar{x}) = 0, \qquad x \in \mathbb{R}^n_{1,\pm},$$

which amounts to

$$\mathrm{Sc}\left\{C^{\mp}\psi^{\mp}\right\}(\xi) = 0, \qquad \xi \in \mathbb{R}^n_{1,\mp}. \tag{5.37}$$

Since $\psi^{\mp} \in H^p(\mathbb{R}^n_{1,\mp}, \mathbb{R}^n_1)$, we have by the Cauchy-Clifford formula that $C^{\mp}\psi^{\mp}(\xi) = \psi^{\mp}(\xi)$; therefore (5.35) means that $\mathrm{Sc}\{\psi^{\mp}\}$ vanishes on $\mathbb{R}^n_{1,\mp}$ and so does its nontangential limit on $\mathbb{R}^n$. But we know from Lemma 2.1 that the $L^p(\mathbb{R}^n)$-norm of the nontangential limit of the scalar part is an equivalent norm on $H^p(\mathbb{R}^n_{1,\mp}, \mathbb{R}^n_1)$, hence $\psi^{\mp} = 0$. This proves the first assertion of the theorem. To establish the second assertion, observe from what precedes that if $P_{\mathrm{div}\,\psi} = 0$ both in $\mathbb{R}^n_{1,+}$ and $\mathbb{R}^n_{1,-}$, then $\psi^{\pm} = 0$, and thus $\psi = \psi^0$ is vector valued and divergence free. The proof is complete. $\square$

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