TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 3, March 2018, Pages 2231–2262
http://dx.doi.org/10.1090/tran/7281
Article electronically published on November 1, 2017

# THE EXTENSION AND CONVERGENCE OF MEAN CURVATURE FLOW IN HIGHER CODIMENSION

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Abstract. In this paper, we investigate the convergence of the mean curvature flow of closed submanifolds in $\mathbb{R}^{n+q}$. We show that if the initial submanifold satisfies some suitable integral curvature conditions, then along the mean curvature flow it will shrink to a round point in finite time.

## 1. Introduction

Let $F_0 : M^n \to N^{n+q}$ be a smooth immersion from an $n$-dimensional Riemannian manifold without boundary to an $(n+q)$-dimensional Riemannian manifold. Consider a one-parameter family of smooth immersions $F : M \times [0,T) \to N$ satisfying

$$\begin{cases} \left(\frac{\partial}{\partial t}F(x,t)\right)^{\perp} = H(x,t), \\ F(x,0) = F_0(x), \end{cases}$$

where $\left(\frac{\partial}{\partial t}F(x,t)\right)^{\perp}$ is the normal component of $\frac{\partial}{\partial t}F(x,t)$, $H(x,t)$ is the mean curvature vector of $F_t(M)$ and $F_t(x) = F(x,t)$. We call $F : M \times [0,T) \to N$ the mean curvature flow with initial value $F_0 : M \to N$. This is the general form of the mean curvature flow, which is a nonlinear weakly parabolic system and is invariant under reparametrization of $M$. We can find a family of diffeomorphisms $\phi_t : M \to M$ for $t \in [0,T)$ such that $\bar{F}_t = F_t \circ \phi_t : M \to N$ satisfies $\frac{\partial}{\partial t}\bar{F}(x,t) = \bar{H}(x,t)$. We will study the (reparameterized) mean curvature flow

$$\begin{cases} \frac{\partial}{\partial t}F(x,t) = H(x,t), \\ F(x,0) = F_0(x). \end{cases} \tag{1.1}$$

The mean curvature flow was proposed by Mullins [14] to describe the formation of grain boundaries in annealing metals. In [2], Brakke introduced the motion of a submanifold by its mean curvature in arbitrary codimension and constructed a generalized varifold solution for all time. For the classical solution of the mean curvature flow, most work has been done on hypersurfaces. Huisken [9, 10] showed that if the initial hypersurface in a complete manifold with bounded geometry is compact and uniformly convex in some sense, then the mean curvature flow converges to a round point in finite time. Many other beautiful results have been obtained, and there are various approaches to studying the mean curvature flow of

Received by the editors September 23, 2016, and, in revised form, January 23, 2017.
2010 *Mathematics Subject Classification.* Primary 53C44, 53C40.
*Key words and phrases.* Mean curvature flow, submanifold, maximal existence time, convergence theorem, integral curvature.
This research was supported by the National Natural Science Foundation of China, Grant Nos. 11531012, 11371315, 11201416.

hypersurfaces (see [4,7], etc.). However, relatively little is known about the mean curvature flows of submanifolds in higher codimensions; see [18, 19, 22–24], etc., for example. Recently, Andrews-Baker [1] proved a convergence theorem for the mean curvature flow of closed submanifolds satisfying a suitable pointwise pinching condition in the Euclidean space.

For an $n$-dimensional submanifold $M$ in a Riemannian manifold, we denote by $g$ the induced metric on $M$. Let $A$ and $H$ be the second fundamental form and the mean curvature vector of $M$, respectively. The tracefree second fundamental form $\mathring{A}$ is defined by $\mathring{A} = A - \frac{1}{n} g \otimes H$. Denote by $||\cdot||_p$ the $L^p$-norm of a function or a tensor field. The volume of a compact manifold $\Sigma$ is denoted by $\mathrm{Vol}(\Sigma)$. In the present paper, we obtain the following convergence theorems for the mean curvature flow of closed submanifolds in the Euclidean space.

**Theorem 1.1.** *Let $F : M^n \to \mathbb{R}^{n+q}$ $(n \geq 3)$ be a smooth closed submanifold. Then for any fixed $p > 1$, there is a positive constant $C_1$ depending on $n, p, \mathrm{Vol}(M)$ and $||A||_{n+2}$, such that if*

$$||\mathring{A}||_p < C_1,$$

*then the mean curvature flow with $F$ as initial value has a unique solution $F : M \times [0, T) \to \mathbb{R}^{n+q}$ in a finite maximal time interval, and $F_t(M)$ converges uniformly to a point $x \in \mathbb{R}^{n+q}$ as $t \to T$. The rescaled immersions $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ converge in $C^\infty$-topology to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit $n$-sphere in some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+q}$.*

**Theorem 1.2.** *Let $F : M^n \to \mathbb{R}^{n+q}$ $(n \geq 3)$ be a smooth closed submanifold. Then for any fixed $p > n$, there is a positive constant $C_2$ depending on $n, p, \mathrm{Vol}(M)$ and $||H||_{n+2}$, such that if*

$$||\mathring{A}||_p < C_2,$$

*then the mean curvature flow with $F$ as initial value has a unique solution $F : M \times [0, T) \to \mathbb{R}^{n+q}$ in a finite maximal time interval, and $F_t(M)$ converges uniformly to a point $x \in \mathbb{R}^{n+q}$ as $t \to T$. The rescaled immersions $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ converge in $C^\infty$-topology to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit $n$-sphere in some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+q}$.*

Let $\mathbb{S}^n$ denote the unit $n$-sphere. Using the Chen-Willmore inequality on total mean curvature [3], we obtain the following convergence theorem for the mean curvature flow.

**Theorem 1.3.** *Let $F : M^n \to \mathbb{R}^{n+q}$ $(n \geq 3)$ be a smooth closed submanifold. Then there is a positive constant $C_3$ depending on $n, \mathrm{Vol}(M)$ and $||A||_{n+2}$, such that if*

$$||A||_n < n^{1/2}[\mathrm{Vol}(\mathbb{S}^n)]^{1/n} + C_3,$$

*then the mean curvature flow with $F$ as initial value has a unique solution $F : M \times [0, T) \to \mathbb{R}^{n+q}$ in a finite maximal time interval, and $F_t(M)$ converges uniformly to a point $x \in \mathbb{R}^{n+q}$ as $t \to T$. The rescaled immersions $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ converge in $C^\infty$-topology to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit $n$-sphere in some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+q}$.*

As immediate consequences of the convergence theorems, we obtain the following differentiable sphere theorems. First let $C_1$ be as in Theorem 1.1.

**Corollary 1.4.** *Let $F : M^n \to \mathbb{R}^{n+q}$ $(n \geq 3)$ be a smooth closed submanifold. If*

$$||\mathring{A}||_p < C_1$$

*for some $p > 1$, then $M$ is diffeomorphic to the unit $n$-sphere.*

Similarly, let $C_2$ be as in Theorem 1.2.

**Corollary 1.5.** *Let $F : M^n \to \mathbb{R}^{n+q}$ $(n \geq 3)$ be a smooth closed submanifold. If*

$$||\mathring{A}||_p < C_2$$

*for some $p > n$, then $M$ is diffeomorphic to the unit $n$-sphere.*

Also, let $C_3$ be as in Theorem 1.3. Then we have the following.

**Corollary 1.6.** *Let $F : M^n \to \mathbb{R}^{n+q}$ $(n \geq 3)$ be a smooth closed submanifold. If*

$$||A||_n < n^{1/2}[\mathrm{Vol}(\mathbb{S}^n)]^{1/n} + C_3,$$

*then $M$ is diffeomorphic to the unit $n$-sphere.*

We remark that in the above theorems and corollaries, we can replace the volume $\mathrm{Vol}(M)$ by a positive lower bound of $|H|$, in which case our method works without change.

The paper is organized as follows. In Section 2, we introduce some basic equations in submanifold theory and recall the evolution equations of the second fundamental form along the mean curvature flow. In Section 3, by using the Moser iteration and blow-up method for parabolic equations, we prove an extension theorem for the mean curvature flow. Theorems 1.1, 1.2 and 1.3 are proved in Section 4. In Section 5, we propose some unsolved problems on convergence of the mean curvature flow in higher codimension.

## 2. Preliminaries

Let $F : M^n \to N^{n+q}$ be a smooth immersion from an $n$-dimensional Riemannian manifold $M^n$ without boundary to an $(n+q)$-dimensional Riemannian manifold $N^{n+q}$. We shall make use of the following convention on the range of indices:

$$1 \leq i, j, k, \ldots \leq n, \quad 1 \leq A, B, C, \ldots \leq n+q, \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+q.$$

The Einstein sum convention is used to sum over the repeated indices.

Choose a local orthonormal frame field $\{e_A\}$ on $N$ such that the $e_i$'s are tangent to $M$. Let $\{\omega_A\}$ be the dual frame field of $\{e_A\}$. The metric $g$ and the volume form $d\mu$ of $M$ are $g = \sum \omega_i \otimes \omega_i$ and $d\mu = \omega_1 \wedge \cdots \wedge \omega_n$.

For any $x \in M$, denote by $N_xM$ the normal space of $M$ in $N$ at point $x$, which is the orthogonal complement of $T_xM$ in $F^*T_{F(x)}N$. Denote by $\bar{\nabla}$ the Levi-Civita connection on $N$. The Riemannian curvature tensor $\bar{R}$ of $N$ is defined by

$$\bar{R}(U, V)W = -\bar{\nabla}_U\bar{\nabla}_V W + \bar{\nabla}_V\bar{\nabla}_U W + \bar{\nabla}_{[U,V]}W$$

for vector fields $U, V$ and $W$ tangent to $N$. The induced connection $\nabla$ on $M$ is defined by

$$\nabla_X Y = (\bar{\nabla}_X Y)^\top$$

for $X, Y$ tangent to $M$, where $(\ )^\top$ denotes the tangential component. Let $R$ be the Riemannian curvature tensor of $M$.

Given a normal vector field $\xi$ along $M$, the induced connection $\nabla^\perp$ on the normal bundle is defined by

$$\nabla^\perp_X \xi = (\bar{\nabla}_X \xi)^\perp,$$

where $(\ )^\perp$ denotes the normal component. Let $R^\perp$ denote the normal curvature tensor.

The second fundamental form is defined to be

$$A(X, Y) = (\bar{\nabla}_X Y)^\perp$$

as a section of the tensor bundle $T^*M \otimes T^*M \otimes NM$, where $T^*M$ and $NM$ are the cotangential bundle and the normal bundle over $M$. The mean curvature vector $H$ is the trace of the second fundamental form.

The first covariant derivative of $A$ is defined as

$$(\widetilde{\nabla}_X A)(Y, Z) = \nabla^\perp_X A(Y, Z) - A(\nabla_X Y, Z) - A(Y, \nabla_X Z),$$

where $\widetilde{\nabla}$ is the connection on $T^*M \otimes T^*M \otimes NM$. Similarly, we can define the second covariant derivative of $A$. Under the local orthonormal frame field, the components of $A$ and its first and second covariant derivatives are

$$\begin{aligned} h^\alpha_{ij} &= \langle A(e_i, e_j), e_\alpha \rangle, \\ h^\alpha_{ijk} &= \langle (\widetilde{\nabla}_{e_k} A)(e_i, e_j), e_\alpha \rangle, \\ h^\alpha_{ijkl} &= \langle (\widetilde{\nabla}_{e_l} \widetilde{\nabla}_{e_k} A)(e_i, e_j), e_\alpha \rangle. \end{aligned}$$

The Laplacian of $A$ is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$.

We define the trace-free second fundamental form $\mathring{A}$ by $\mathring{A} = A - \frac{1}{n} g \otimes H$, whose components are $\mathring{A}^\alpha_{ij} = h^\alpha_{ij} - \frac{1}{n} h^\alpha_{kk} \delta_{ij}$. Obviously, we have $\mathring{A}^\alpha_{ii} = 0$.

Let

$$\begin{aligned} R_{ijkl} &= g(R(e_i, e_j)e_k, e_l), \\ \bar{R}_{ABCD} &= \langle \bar{R}(e_A, e_B)e_C, e_D \rangle, \\ R^\perp_{ij\alpha\beta} &= \langle R^\perp(e_i, e_j)e_\alpha, e_\beta \rangle. \end{aligned}$$

Then we have the following Gauss, Codazzi and Ricci equations:

$$\begin{aligned} R_{ijkl} &= \bar{R}_{ijkl} + h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}, \\ h^\alpha_{ijk} - h^\alpha_{ikj} &= -\bar{R}_{\alpha ijk}, \\ R^\perp_{ij\alpha\beta} &= \bar{R}_{ij\alpha\beta} + h^\alpha_{ik} h^\beta_{jk} - h^\alpha_{jk} h^\beta_{ik}. \end{aligned}$$

Suppose $F : M \times [0, T) \to N$ is the mean curvature flow with initial value $F_0 : M \to N$. We have the following evolution equations.

**Lemma 2.1** ([22]). *Along the mean curvature flow we have*

$$\frac{\partial}{\partial t} d\mu_t = -|H|^2 d\mu_t, \tag{2.1}$$

$$\begin{aligned}
\frac{\partial}{\partial t} h_{ij}^{\alpha} &= \Delta h_{ij}^{\alpha} + \bar{R}_{\alpha ijk,k} + \bar{R}_{\alpha kik,j} \\
&\quad -2\bar{R}_{lijk} h_{lk}^{\alpha} + 2\bar{R}_{\alpha\beta jk} h_{ik}^{\beta} + 2\bar{R}_{\alpha\beta ik} h_{jk}^{\beta} \\
&\quad -\bar{R}_{lkik} h_{lj}^{\alpha} - \bar{R}_{lkjk} h_{li}^{\alpha} + \bar{R}_{\alpha k\beta k} h_{ij}^{\beta} \\
&\quad -h_{im}^{\alpha}(h_{jm}^{\beta} h_{ll}^{\beta} - h_{km}^{\beta} h_{jk}^{\beta}) \\
&\quad -h_{km}^{\alpha}(h_{jm}^{\beta} h_{ik}^{\beta} - h_{km}^{\beta} h_{ij}^{\beta}) \\
&\quad -h_{ik}^{\beta}(h_{jl}^{\beta} h_{kl}^{\alpha} - h_{kl}^{\beta} h_{jl}^{\alpha}) \\
&\quad -h_{jk}^{\alpha} h_{ij}^{\beta} h_{ll}^{\beta} + h^{\beta}\langle e_{\alpha}, \bar{\nabla}_H e_{\beta}\rangle,
\end{aligned} \tag{2.2}$$

*where $\bar{R}_{ABCD,E}$ are the components of the first covariant derivative $\bar{\nabla}\bar{R}$ of $\bar{R}$.*

## 3. The extension of mean curvature flow

In this section, we prove the extension theorem for the mean curvature flow of submanifolds in arbitrary codimension. Throughout this section, we assume that the ambient space $N$ has bounded geometry. Recall that a Riemannian manifold is said to have bounded geometry if (i) the sectional curvature is bounded; (ii) the injective radius is bounded from below by a positive constant. We always assume that $N$ is a Riemannian manifold with bounded geometry satisfying $-K_1 \le K_N \le K_2$ for nonnegative constants $K_1$, $K_2$, and the injective radius of $N$ is bounded from below by a positive constant $i_N$.

In [9, 10], Huisken showed that if the second fundamental form is uniformly bounded, then the mean curvature flow can be extended over the time. Le-Šešum [12] and Xu-Ye-Zhao [27] obtained some integral conditions to extend the mean curvature flow of hypersurfaces in the Euclidean space independently. Later, Xu-Ye-Zhao [28] generalized these extension theorems to the case where the ambient space is a Riemannian manifold with bounded geometry. In the present paper, we generalize the extension theorems in [12, 27, 28] to the mean curvature flow of submanifolds in a Riemannian manifold with bounded geometry.

The following Sobolev inequality can be found in [8], which is a generalization of the remarkable work of Michael-Simon [13].

**Lemma 3.1** ([8]). *Let $M^n \subset N^{n+q}$ be an $n(\ge 2)$-dimensional closed submanifold in a Riemannian manifold $N^{n+q}$ with codimension $q \ge 1$. Denote by $i_N$ the positive lower bound of the injective radius of $N$ restricted on $M$. Assume the sectional curvature $K_N$ of $N$ satisfies $K_N \le b^2$. Let $h$ be a nonnegative $C^1$ function on $M$. Then*

$$\left(\int_M h^{\frac{n}{n-1}} d\mu\right)^{\frac{n-1}{n}} \le C(n,\alpha) \int_M \Big[|\nabla h| + h|H|\Big] d\mu,$$

*provided*

$$b^2(1-\alpha)^{-\frac{2}{n}}(\omega_n^{-1}\mathrm{Vol}(\mathrm{supp}\ h))^{\frac{2}{n}} \le 1 \ \textit{and}\ 2\rho_0 \le i_N,$$

*where*

$$\rho_0 = \begin{cases} b^{-1}\sin^{-1} b(1-\alpha)^{-\frac{1}{n}}(\omega_n^{-1}\mathrm{Vol}(\mathrm{supp}\ h))^{\frac{1}{n}} & \text{for } b \text{ real,} \\ (1-\alpha)^{-\frac{1}{n}}(\omega_n^{-1}\mathrm{Vol}(\mathrm{supp}\ h))^{\frac{1}{n}} & \text{for } b \text{ imaginary.} \end{cases}$$

*Here* $\omega_n$ *is the volume of the unit ball in* $\mathbb{R}^n$, $\alpha$ *is a free parameter,* $0 < \alpha < 1$, *and*

$$C(n,\alpha) = \frac{1}{2}\pi \cdot 2^n \alpha^{-1}(1-\alpha)^{-\frac{1}{n}}\frac{n}{n-1}\omega_n^{-\frac{1}{n}}.$$

*For* $b$ *imaginary, we may omit the factor* $\frac{1}{2}\pi$ *in the definition of* $C(n,\alpha)$.

**Lemma 3.2.** *Let* $M^n \subset N^{n+q}$ *be an* $n(\geq 3)$*-dimensional closed submanifold in a Riemannian manifold* $N^{n+q}$ *with codimension* $q \geq 1$. *Assume* $K_N \leq K_2$, *where* $K_2$ *is a nonnegative constant. Let* $f$ *be a nonnegative* $C^1$ *function on* $M$ *satisfying*

$$K_2(n+1)^{\frac{2}{n}}(\omega_n^{-1}\mathrm{Vol}(\mathrm{supp}\ f))^{\frac{2}{n}} \leq 1, \tag{3.1}$$

$$2K_2^{-\frac{1}{2}}\sin^{-1} K_2^{\frac{1}{2}}(n+1)^{\frac{1}{n}}(\omega_n^{-1}\mathrm{Vol}(\mathrm{supp}\ f))^{\frac{1}{n}} \leq i_N. \tag{3.2}$$

*Then*

$$||\nabla f||_2^2 \geq \frac{(n-2)^2}{4(n-1)^2(1+s)}\left[\frac{1}{C^2(n)}||f||^2_{\frac{2n}{n-2}} - H_0^2\left(1+\frac{1}{s}\right)||f||_2^2\right],$$

*where* $H_0 = \max_{x\in M}|H|$, $C(n) = C(n, \frac{n}{n+1})$ *and* $s > 0$ *is a free parameter.*

*Proof.* For all $g \in C^1(M)$, $g \geq 0$ satisfying (3.1) and (3.2), Lemma 3.1 implies

$$||g||_{\frac{n}{n-1}} \leq C(n)\int_M (|\nabla g| + g|H|)d\mu. \tag{3.3}$$

Substituting $g = f^{\frac{2(n-1)}{n-2}}$ into (3.3) gives

$$\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{n-1}{n}} \leq \frac{2(n-1)}{n-2}C(n)\int_M f^{\frac{n}{n-2}}|\nabla f|d\mu + C(n)\int_M |H|f^{\frac{2(n-1)}{n-2}}d\mu.$$

By Hölder's inequality, we get

$$\begin{aligned}\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{n-1}{n}} \leq\ & C(n)\Bigg[\frac{2(n-1)}{n-2}\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{1}{2}}\left(\int_M |\nabla f|^2 d\mu\right)^{\frac{1}{2}} \\ &+\left(\int_M H_0^2 f^2 d\mu\right)^{\frac{1}{2}}\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{1}{2}}\Bigg].\end{aligned}$$

Then

$$\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{n-2}{2n}} \leq C(n)\left[\frac{2(n-1)}{n-2}\left(\int_M |\nabla f|^2 d\mu\right)^{\frac{1}{2}} + \left(\int_M H_0^2 f^2 d\mu\right)^{\frac{1}{2}}\right].$$

This implies

$$||f||^2_{\frac{2n}{n-2}} \leq C^2(n)\left[\frac{4(n-1)^2(1+s)}{(n-2)^2}||\nabla f||_2^2 + H_0^2\left(1+\frac{1}{s}\right)||f||_2^2\right],$$

which is desired. □

Now we establish an inequality involving the maximal value of the squared norm of the mean curvature and its $L^{n+2}$-norm in the space-time.

**Lemma 3.3.** *Suppose that $F_t : M^n \to N^{n+q}$ $(n \geq 3)$ is the mean curvature flow solution for $t \in [0, T_0]$, where $N$ has bounded geometry. Then*

$$\max_{(x,t)\in M\times[\frac{T_0}{2},T_0]} |H|^2(x,t) \leq C\left(\int_0^{T_0}\int_{M_t}|H|^{n+2}d\mu_t dt\right)^{\frac{2}{n+2}},$$

*where $C$ is some constant depending only on $n$, $T_0$, $\sup_{(x,t)\in M\times[0,T_0]}|A|$, $K_1$, $K_2$ and $i_N$.*

*Proof.* In the following proof, we always denote by $C$ the constant depending on some quantities, including $n$, $T_0$, $\sup_{(x,t)\in M\times[0,T_0]}|A|$, $K_1$, $K_2$ and $i_N$, which may not be the same in different lines. We make use of Moser iteration for parabolic equations. Here we follow the computation in [6]. From the evolution equation of the second fundamental form in Lemma 2.1, we have the following differential inequality:

$$\frac{\partial}{\partial t}|H|^2 \leq \Delta|H|^2 + \beta|H|^2, \tag{3.4}$$

where $\beta$ is a positive constant depending only on $n$, $\sup_{(x,t)\in M\times[0,T_0]}|A|$, $K_1$ and $K_2$. For $0 < R < R' < \infty$ and $x_0 \in M$, we set

$$\eta = \begin{cases} 1, & x \in B_{g(0)}(x_0, R), \\ \eta \in [0,1] \text{ and } |\nabla\eta|_{g(0)} \leq \frac{1}{R'-R}, & x \in B_{g(0)}(x_0, R') \setminus B_{g(0)}(x_0, R), \\ 0, & x \in M \setminus B_{g(0)}(x_0, R'). \end{cases}$$

Since $\operatorname{supp}\eta \subseteq B_{g(0)}(x_0, R')$, we assume that $R'$ is sufficiently small such that $\eta$ satisfies (3.1) and (3.2) with respect to $g(0)$. On the other hand, the area of some fixed subset in $M$ is nonincreasing along the mean curvature flow; hence $\eta$ satisfies (3.1) and (3.2) with respect to each $g(t)$ for $t \in [0, T_0]$. Putting $f = |H|^2$ and $B(R') = B_{g(0)}(x_0, R')$, the inequality (3.4) implies that, for any $m \geq 2$,

$$\begin{aligned} &\frac{1}{m}\frac{\partial}{\partial t}\int_{B(R')} f^m\eta^2 d\mu_t \\ &\quad\leq \int_{B(R')}\left(\eta^2 f^{m-1}\Delta f d\mu_t + \beta f^m\eta^2\right)d\mu_t + \int_{B(R')}\frac{1}{m}f^m\eta^2\frac{\partial}{\partial t}d\mu_t \\ &\quad= \int_{B(R')}\left(\eta^2 f^{m-1}\Delta f d\mu_t + \beta f^m\eta^2\right)d\mu_t - \int_{B(R')}\frac{1}{m}f^{m+1}\eta^2 d\mu_t \\ &\quad\leq \int_{B(R')}\left(\eta^2 f^{m-1}\Delta f d\mu_t + \beta f^m\eta^2\right)d\mu_t. \end{aligned} \tag{3.5}$$

Here we have used the evolution equation of the volume form in Lemma 2.1. Integrating by parts we obtain

$$\begin{aligned} \int_{B(R')}\eta^2 f^{m-1}\Delta f d\mu_t &= -\frac{4(m-1)}{m^2}\int_{B(R')}|\nabla(f^{\frac{m}{2}}\eta)|^2 d\mu_t + \frac{4}{m^2}\int_{B(R')}|\nabla\eta|^2 f^m d\mu_t \\ &\quad+\frac{4(m-2)}{m^2}\int_{B(R')}\langle\nabla(f^{\frac{m}{2}}\eta), f^{\frac{m}{2}}\nabla\eta\rangle d\mu_t \\ &\leq -\frac{2}{m}\int_{B(R')}|\nabla(f^{\frac{m}{2}}\eta)|^2 d\mu_t + \frac{2}{m}\int_{B(R')}|\nabla\eta|^2 f^m d\mu_t. \end{aligned} \tag{3.6}$$

Thus by (3.5) and (3.6) we obtain

$$\begin{aligned}\frac{1}{m}\frac{\partial}{\partial t}\int_{B(R')} f^m\eta^2 d\mu_t \quad &\leq \quad -\frac{2}{m}\int_{B(R')} |\nabla(f^{\frac{m}{2}}\eta)|^2 d\mu_t \\ &\quad +\beta\int_{B(R')} f^m\eta^2 d\mu_t + \frac{2}{m}\int_{B(R')} |\nabla\eta|^2 f^m d\mu_t.\end{aligned}$$

This implies

$$\begin{aligned}\frac{\partial}{\partial t}\int_{B(R')} f^m\eta^2 d\mu_t \quad &+ \quad \int_{B(R')} |\nabla(f^{\frac{m}{2}}\eta)|^2 d\mu_t \\ &\leq \quad 2\int_{B(R')} |\nabla\eta|^2 f^m d\mu_t + \beta m\int_{B(R')} f^m\eta^2 d\mu_t. \qquad (3.7)\end{aligned}$$

For any $0 < \tau < \tau' < T_0$, define a function $\psi$ on $[0, T_0]$ by

$$\psi(t) = \begin{cases} 0 & 0 \leq t \leq \tau, \\ \frac{t-\tau}{\tau'-\tau} & \tau \leq t \leq \tau', \\ 1 & \tau' \leq t \leq T_0. \end{cases}$$

Then from (3.7) we get

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\psi\int_{B(R')} f^m\eta^2 d\mu_t\right) + \psi\int_{B(R')} |\nabla(f^{\frac{m}{2}}\eta)|^2 d\mu_t \\ &\qquad \leq 2\psi\int_{B(R')} |\nabla\eta|^2 f^m d\mu_t + (\beta m\psi + \psi')\int_{B(R')} f^m\eta^2 d\mu_t. \qquad (3.8)\end{aligned}$$

For any $t \in [\tau', T_0]$, integrating both sides of (3.8) on $[\tau, t]$ implies

$$\begin{aligned}\int_{B(R')} f^m\eta^2 d\mu_t &+ \int_{\tau'}^{t}\int_{B(R')} |\nabla(f^{\frac{m}{2}}\eta)|^2 d\mu_t dt \\ &\leq 2\int_{\tau}^{T_0}\int_{B(R')} |\nabla\eta|^2 f^m d\mu_t dt \\ &\quad + \left(\beta m + \frac{1}{\tau'-\tau}\right)\int_{\tau}^{T_0}\int_{B(R')} f^m\eta^2 d\mu_t dt. \qquad (3.9)\end{aligned}$$

By the Sobolev inequality in Lemma 3.2, we obtain

$$\begin{aligned}\left(\int_{B(R')} f^{\frac{mn}{n-2}}\eta^{\frac{2n}{n-2}} d\mu_t\right)^{\frac{n-2}{n}} \quad &= \quad ||f^{\frac{m}{2}}\eta||^2_{\frac{2n}{n-2}} \\ &\leq \quad \frac{4(n-1)^2(1+s)C^2(n)}{(n-2)^2}||\nabla(f^{\frac{m}{2}}\eta)||_2^2 \\ &\quad\quad +CC^2(n)\Big(1+\frac{1}{s}\Big)||f^{\frac{m}{2}}\eta||_2^2, \qquad (3.10)\end{aligned}$$

where $C$ depends on $n$ and $\sup_{(x,t)\in M\times[0,T_0]}|A|$. Combining (3.9) and (3.10) implies that

$$\begin{aligned}
&\int_{\tau'}^{T_0}\int_{B(R')} f^{m(1+\frac{2}{n})}\eta^{2(1+\frac{2}{n})}d\mu_t dt\\
\leq\;& \int_{\tau'}^{T_0}\left(\int_{B(R')} f^m\eta^2 d\mu_t\right)^{\frac{2}{n}}\left(\int_{B(R')} f^{\frac{nm}{n-2}}\eta^{\frac{2n}{n-2}}\mu_t\right)^{\frac{n-2}{n}} dt\\
\leq\;& \max_{t\in[\tau',T_0]}\left(\int_{B(R')} f^m\eta^2 d\mu_t\right)^{\frac{2}{n}}\times\int_{\tau}^{T_0}\Big[\frac{4(n-1)^2(1+s)C^2(n)}{(n-2)^2}||\nabla(f^{\frac{m}{2}}\eta)||_2^2\\
&+CC^2(n)\Big(1+\frac{1}{s}\Big)||f^{\frac{m}{2}}\eta||_2^2\Big]dt\\
\leq\;& C\max_{t\in[\tau',T_0]}\left(\int_{B(R')} f^m\eta^2 d\mu_t\right)^{\frac{2}{n}}\times\int_{\tau}^{T_0}\Big[||\nabla(f^{\frac{m}{2}}\eta)||_2^2+||f^{\frac{m}{2}}\eta||_2^2\Big]dt\\
\leq\;& C\Big[2\int_{\tau}^{T_0}\int_{B(R')}|\nabla\eta|^2 f^m d\mu_t dt\\
&+\Big(\beta m+\frac{1}{\tau'-\tau}\Big)\int_{\tau}^{T_0}\int_{B(R')} f^m\eta^2 d\mu_t dt\Big]^{1+\frac{2}{n}},
\end{aligned}\tag{3.11}$$

where $s=1$ and $C$ is a constant depending only on $n$ and $\sup_{(x,t)\in M\times[0,T_0]}|A|$.

Note that $|\nabla\eta|_{g(t)}\leq|\nabla\eta|^2_{g(0)}e^{lt}$, where $l=\max_{0\leq t\leq T_0}||\frac{\partial g}{\partial t}||_{g(t)}$. Thus

$$\begin{aligned}
\int_{\tau}^{T_0}\int_{B(R')}|\nabla\eta|^2 f^m d\mu_t dt &\leq \int_{\tau}^{T_0}\int_{B(R')}|\nabla\eta|^2_{g(0)}e^{lt}f^m d\mu_t dt\\
&\leq \frac{e^{CT_0}}{(R'-R)^2}\int_{\tau}^{T_0}\int_{B(R')} f^m d\mu_t dt
\end{aligned}$$

for some positive constant $C$ depending on $n$ and $\sup_{(x,t)\in M\times[0,T_0]}|A|$. This together with (3.11) implies that

$$\begin{aligned}
\int_{\tau}^{T_0}\int_{B(R)} f^{m(1+\frac{2}{n})}d\mu_t dt &\leq C\left(\beta m+\frac{1}{\tau'-\tau}+\frac{2e^{CT_0}}{(R'-R)^2}\right)^{1+\frac{2}{n}}\\
&\quad\times\left(\int_{\tau}^{T_0}\int_{B(R')} f^m d\mu_t dt\right)^{1+\frac{2}{n}},
\end{aligned}\tag{3.12}$$

where $C$ is a positive constant depending on $n$ and $\sup_{(x,t)\in M\times[0,T_0]}|A|$.

Putting $L(m,t,R)=\int_t^{T_0}\int_{B(R)} f^m d\mu_t dt$, we have from (3.12)

$$\begin{aligned}
L\Big(m\Big(1+\frac{2}{n}\Big),\tau',R\Big) &\leq C\left(\beta m+\frac{1}{\tau'-\tau}+\frac{2e^{CT_0}}{(R'-R)^2}\right)^{1+\frac{2}{n}}\\
&\quad\times L(m,\tau,R')^{1+\frac{2}{n}}.
\end{aligned}\tag{3.13}$$

We set

$$\mu=1+\frac{2}{n},\ m_k=\frac{n+2}{2}\mu^k,\ \tau_k=\Big(1-\frac{1}{\mu^{k+1}}\Big)t,\ R_k=\frac{R'}{2}\Big(1+\frac{1}{\mu^{k/2}}\Big).$$

Then it follows from (3.13) that

$$\begin{aligned} L(m_{k+1},\tau_{k+1},R_{k+1})^{\frac{1}{m_{k+1}}} &\\ \leq\ C^{\frac{1}{m_{k+1}}} &\left[\frac{(n+2)\beta}{2}+\frac{\mu^2}{\mu-1}\cdot\frac{1}{t}+\frac{4e^{CT_0}}{R'^2}\cdot\frac{\mu}{(\sqrt{\mu}-1)^2}\right]^{\frac{1}{m_k}} \\ &\times\mu^{\frac{k}{m_k}}L(m_k,\tau_k,R_k)^{\frac{1}{m_k}}. \end{aligned}$$

Hence

$$\begin{aligned} &L(m_{d+1},\tau_{d+1},R_{d+1})^{\frac{1}{m_{d+1}}} \\ &\leq C^{\sum_{k=0}^{d}\frac{1}{m_{k+1}}}\left[\frac{(n+2)\beta}{2}+\frac{\mu^2}{\mu-1}\cdot\frac{1}{t}+\frac{4e^{CT_0}}{R'^2}\cdot\frac{\mu}{(\sqrt{\mu}-1)^2}\right]^{\sum_{k=0}^{d}\frac{1}{m_k}} \\ &\quad\times\mu^{\sum_{k=0}^{d}\frac{k}{m_k}}L(m_0,\tau_0,R_0)^{\frac{1}{m_0}}. \end{aligned} \tag{3.14}$$

As $d\to+\infty$, we conclude from (3.14) that

$$\begin{aligned} f(x_0,t) \ \leq\ & C^{\frac{n}{n+2}}\Big(C+\frac{1}{t}+\frac{e^{CT_0}}{R'^2}\Big) \\ & \times\Big(1+\frac{2}{n}\Big)^{\frac{n}{2}}\left(\int_0^{T_0}\int_{M_t}f^{\frac{n+2}{2}}d\mu_t dt\right)^{\frac{2}{n+2}} \end{aligned} \tag{3.15}$$

for some positive constant $C$ depending on $n$, $\sup_{(x,t)\in M\times[0,T_0]}|A|$, $K_1$ and $K_2$.

Note that we choose $R'$ sufficiently small such that

$$K_2(n+1)^{\frac{2}{n}}(\omega_n^{-1}\mathrm{Vol}_{g(0)}(B(R')))^{\frac{2}{n}}\leq 1 \tag{3.16}$$

and

$$2K_2^{-\frac{1}{2}}\sin^{-1}K_2^{\frac{1}{2}}(n+1)^{\frac{1}{n}}(\omega_n^{-1}\mathrm{Vol}_{g(0)}(B(R')))^{\frac{1}{n}}\leq i_N. \tag{3.17}$$

For $g(0)$, there is a nonpositive constant $K$ depending on $n$, $\max_{x\in M_0}|A|$, $K_1$ and $K_2$ such that the sectional curvature of $M_0$ is bounded from below by $K$. By the Bishop-Gromov volume comparison theorem, we have

$$\mathrm{Vol}_{g(0)}(B(R'))\leq\mathrm{Vol}_K(B(R')),$$

where $\mathrm{Vol}_K(B(R'))$ is the volume of a ball with radius $R'$ in the $n$-dimensional complete simply connected space form with constant curvature $K$. Let $R'$ be the largest number such that

$$K_2(n+1)^{\frac{2}{n}}(\omega_n^{-1}\mathrm{Vol}_K(B(R')))^{\frac{2}{n}}\leq 1$$

and

$$2K_2^{-\frac{1}{2}}\sin^{-1}K_2^{\frac{1}{2}}(n+1)^{\frac{1}{n}}(\omega_n^{-1}\mathrm{Vol}_K(B(R')))^{\frac{1}{n}}\leq i_N.$$

Then $R'$ depends only on $n$, $K_1$, $K_2$, $i_N$ and $\sup_{(x,t)\in M\times[0,T_0]}|A|$, and $\mathrm{Vol}_{g(0)}(B(R'))$ satisfies (3.16) and (3.17). This together with (3.15) implies

$$\max_{(x,t)\in M\times[\frac{T_0}{2},T_0]}|H|^2(x,t)\leq C\left(\int_0^{T_0}\int_{M_t}|H|^{n+2}d\mu_t dt\right)^{\frac{2}{n+2}},$$

where $C$ is a constant depending on $n$, $T_0$ and $\sup_{(x,t)\in M\times[0,T_0]}|A|$, $K_1$, $K_2$ and $i_N$. □

Now we give a sufficient condition that assures the extension of the mean curvature flow of submanifolds in a Riemannian manifold.

**Theorem 3.4.** *Let $F_t : M^n \to N^{n+q}$ $(n \geq 3)$ be the mean curvature flow solution of closed submanifolds in a finite time interval $[0,T)$, where $N$ has bounded geometry. If*

(i) *there exist positive constants $a$ and $b$ such that $|A|^2 \leq a|H|^2 + b$ for $t \in [0,T)$ and*

(ii) $||H||_{\alpha, M\times[0,T)} = \left(\int_0^T \int_{M_t} |H|^\alpha d\mu_t dt\right)^{\frac{1}{\alpha}} < \infty$ *for some $\alpha \geq n+2$,*

*then this flow can be extended over time $T$.*

*Proof.* By Hölder's inequality, it is sufficient to prove the theorem for $\alpha = n+2$. We will argue by contradiction.

Suppose that the solution of the mean curvature flow can't be extended over $T$. Then the second fundamental form becomes unbounded as $t \to T$. From assumption (i), $|H|^2$ is unbounded also.

We choose a sequence of points $(x^{(i)}, t^{(i)}) \in M \times [0,T)$ such that $t^{(i)} \nearrow T$ as $i \to \infty$ and for each $i$, $x^{(i)}$ satisfies

$$|H|^2(x^{(i)}, t^{(i)}) = \max_{(x,t)\in M\times[0,t^{(i)}]} |H|^2(x,t).$$

Put

$$Q^{(i)} = |H|^2(x^{(i)}, t^{(i)}).$$

Then $\{Q^{(i)}\}_{i=1}^{\infty}$ is a nondecreasing sequence and $\lim_{i\to\infty} Q^{(i)} = \infty$. This together with $\lim_{i\to\infty} t^{(i)} = T > 0$ implies that there exists a positive integer $i_0$ such that $Q^{(i)}t^{(i)} \geq 1$ and $Q^{(i)} \geq 1$ for $i \geq i_0$. Let $h$ be the Riemannian metric on $N$. For $i \geq i_0$ and $t \in [0,1]$, we consider the rescaled mean curvature flows

$$F^{(i)}(t) = F\left(\frac{t-1}{Q^{(i)}} + t^{(i)}\right) : (M, g^{(i)}(t)) \to (N, Q^{(i)}h),$$

where $g^{(i)}(t) = F^{(i)}(t)^*(Q^{(i)}h)$. Let $H_{(i)}$ and $A^{(i)} = h^{(i)}_{jk}$ be the mean curvature vector and the second fundamental form of $F^{(i)}(t)$ respectively. Then we have

$$|H_{(i)}|^2(x,t) \leq 1 \quad \text{on} \quad M \times [0,1]. \tag{3.18}$$

From assumption (i) again, inequality (3.18) implies $|A^{(i)}| \leq C$, where $C$ is a constant independent of $i$. Since $(N,h)$ has bounded geometry and $Q^{(i)} \geq 1$ for $i \geq i_0$, $(N, Q^{(i)}h)$ also has bounded geometry for $i \geq i_0$ with the same bounding constants as $(N,h)$. It follows from Lemma 3.3 that for $i \geq i_0$,

$$\max_{(x,t)\in M^{(i)}\times[\frac{1}{2},1]} |H_{(i)}|^2(x,t) \leq C\left(\int_0^1 \int_{M_t} |H_{(i)}|^{n+2} d\mu_{g^{(i)}(t)} dt\right)^{\frac{2}{n+2}},$$

where $C$ is a constant independent of $i$.

By [5], there is a subsequence of pointed mean curvature flow solutions

$$F^{(i)}(t) : (M, g^{(i)}(t), x^{(i)}) \to (N, Q^{(i)}h),\ t \in [0,1]$$

that converges to a pointed mean curvature flow solution

$$\widetilde{F}(t) : (\widetilde{M}, \widetilde{g}(t), \widetilde{x}) \to \mathbb{R}^{n+q},\ t \in [0,1].$$

Denote by $\widetilde{H}$ the mean curvature vector of $\widetilde{F}$, $t \in [0,1]$. Then we have

$$\begin{aligned}\max_{(x,t)\in\widetilde{M}\times[\frac{1}{2},1]} |\widetilde{H}|^2(x,t) &\leq \lim_{i\to\infty} C\left(\int_0^1 \int_{M_t} |H_{(i)}|^{n+2} d\mu_{g^{(i)}(t)} dt\right)^{\frac{2}{n+2}} \\ &\leq \lim_{i\to\infty} C\left(\int_{t^{(i)}}^{t^{(i)}+(Q^{(i)})^{-1}} \int_{M_t} |H_{(i)}|^{n+2} d\mu dt\right)^{\frac{2}{n+2}} \\ &= 0. \end{aligned} \tag{3.19}$$

The equality in (3.19) holds because $\int_0^T \int_{M_t} |H|^{n+2} d\mu_t dt < +\infty$ and $(Q^{(i)})^{-1} \to 0$ as $i \to \infty$. On the other hand, according to the choice of the points, we have

$$|\widetilde{H}|^2(\widetilde{x}, 1) = \lim_{i\to\infty} |H_{(i)}|^2(x^{(i)}, 1) = 1.$$

This is a contradiction. $\square$

*Remark* 3.5. When $q = 1$, Theorem 3.4 generalizes the theorems in [12, 27, 28]. In fact, for $N^{n+1} = \mathbb{R}^{n+1}$, we have the following computations.

(i) If $h_{ij} \geq -C$ for $(x,t) \in M \times [0,T)$ with some $C \geq 0$, let $\lambda_i$, $i = 1, \ldots, n$, be the principal curvatures. Then $\lambda_i + C \geq 0$, which implies that

$$\sum_i (\lambda_i + C)^2 \leq n(\sum_i (\lambda_i + C))^2 \leq 2nH^2 + 2n^3C^2.$$

On the other hand,

$$\sum_i (\lambda_i + C)^2 = |A|^2 + 2CH + nC^2 \geq |A|^2 - H^2 + (n-1)C^2.$$

Hence $|A|^2 \leq (2n+1)H^2 + (2n^3 - n + 1)C^2$ for $t \in [0,T)$.

(ii) If $H > 0$ at $t = 0$, then there exists a positive constant $C$ such that $|A|^2 \leq CH^2$ at $t = 0$. By [9], we know that $H > 0$ for $t > 0$ and

$$\frac{\partial}{\partial t}\left(\frac{|A|^2}{H^2}\right) = \Delta\left(\frac{|A|^2}{H^2}\right) + \frac{2}{H}\left\langle \nabla H, \nabla\left(\frac{|A|^2}{H^2}\right)\right\rangle - \frac{2}{H^4}|H\nabla_i h_{jk} - \nabla_i H \cdot h_{jk}|^2.$$

By the maximum principle we obtain that $|A|^2/H^2$ is uniformly bounded from above by its initial data. Hence $|A|^2 \leq CH^2$ for $t \in [0,T)$.

For general $N^{n+1}$ with bounded geometry, we have similar computations. Hence our Theorem 3.4 is a generalization.

At the end of this section, we would like to propose the following.

**Open Problem 3.6.** Let $F_t : M \to N$ be the mean curvature flow solution of closed submanifolds in a finite time interval $[0,T)$. Suppose $||H||_{\alpha, M\times[0,T)} < \infty$ for some $\alpha \geq n+2$. Is there a positive constant $\omega$ such that the solution exists in $[0, T+\omega)$?

## 4. The convergence of mean curvature flow

In this section we obtain several convergence theorems for the mean curvature flow. The extension theorem proved in Section 3 will be used to give a positive lower bound on the existence time of the mean curvature flow.

We need the following Sobolev inequality for submanifolds in the Euclidean space, which is a consequence of the famous Sobolev inequality due to Michael-Simon [13].

**Lemma 4.1.** *Let $M$ be an $n(\geq 3)$-dimensional closed submanifold in $\mathbb{R}^{n+q}$. Then for all Lipschitz functions $v$ on $M$, we have*

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \leq C_n\left(\int_M |\nabla v|^2 d\mu + \int_M |H|^{n+2} d\mu \int_M v^2 d\mu\right),$$

*where $C_n$ is a positive constant depending only on $n$.*

*Proof.* The proof of the lemma for $q = 1$ was given in [12]. Using the same method we can prove the lemma for $q > 1$. $\square$

Now we begin to prove Theorem 1.1, which is restated as follows.

**Theorem 4.2.** *Let $F_0 : M^n \to \mathbb{R}^{n+q}$ $(n \geq 3)$ be a smooth closed submanifold. Then for any fixed $p > 1$, there is a positive constant $C_1$ depending on $n, p, \mathrm{Vol}(M_0)$ and $||A||_{n+2}$, such that if*

$$||\mathring{A}||_p < C_1,$$

*then the mean curvature flow with $F_0$ as initial value has a unique solution $F : M \times [0, T) \to \mathbb{R}^{n+q}$ in a finite maximal time interval, and $F_t(M)$ converges uniformly to a point $x \in \mathbb{R}^{n+q}$ as $t \to T$. The rescaled immersions $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ converge in $C^\infty$-topology to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit $n$-sphere in some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+q}$.*

*Proof.* We set $\Lambda = ||A||_{n+2}$. Denote by $T_{\max}$ the maximal existence time of the mean curvature flow with $F_0$ as initial value. It is easy to show that $T_{\max} < +\infty$ (see [24] for a proof).

We split the proof into several steps.

*Step* 1. For any fixed positive number $\varepsilon$, we first show that if

$$||\mathring{A}||_p < \varepsilon$$

for some $p > 1$, then $T_{\max}$ satisfies $T_{\max} > T_0$ for some positive constant $T_0$ depending on $n$, $p$, $\Lambda$ and independent of $\varepsilon$, and there hold $||A(t)||_{n+2} < 2\Lambda$, $||\mathring{A}(t)||_p < 2\varepsilon$ for $t \in [0, T_0]$.

Put

$$T = \sup\{t \in [0, T_{\max}) : ||A(t)||_{n+2} < 2\Lambda, ||\mathring{A}(t)||_p < 2\varepsilon\}.$$

We consider the mean curvature flow on the time interval $[0, T)$.

By the definition of $T$ we have $\int_{M_t} |A|^{n+2} d\mu_t \leq (2\Lambda)^{n+2}$ for $t \in [0, T)$. From Lemma 4.1 we have for a Lipschitz function $v$,

$$\left(\int_{M_t} v^{\frac{2n}{n-2}} d\mu_t\right)^{\frac{n-2}{n}} \leq C_n\left(\int_{M_t} |\nabla v|^2 d\mu_t + n^{\frac{n+2}{2}} (2\Lambda)^{n+2} \int_{M_t} v^2 d\mu_t\right). \tag{4.1}$$

From (2.2), we have

$$\frac{\partial}{\partial t}|A|^2 \leq \Delta|A|^2 + c_1|A|^4$$

for some positive constant $c_1$ depending only on $n$. Putting $u = |A|^2$, we have

$$\frac{\partial}{\partial t}u \leq \Delta u + c_1 u^2. \tag{4.2}$$

From (4.2) and (2.1) we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_{M_t} u^{\frac{n+2}{2}} d\mu_t &= \int_{M_t} \frac{n+2}{2} u^{\frac{n+2}{2}-1}\frac{\partial}{\partial t} u d\mu_t + \int_{M_t} u^{\frac{n+2}{2}}\frac{\partial}{\partial t} d\mu_t \\ &\leq \frac{n+2}{2}\int_{M_t} u^{\frac{n+2}{2}-1}(\Delta u + cu^2) d\mu_t - \int_{M_t} H^2 u^{\frac{n+2}{2}} d\mu_t \\ &\leq -\frac{4n}{n+2}\int_{M_t} |\nabla u^{\frac{n+2}{4}}|^2 d\mu_t + \frac{n+2}{2}c_1\int_{M_t} u^{\frac{n+2}{2}+1} d\mu_t. \end{aligned} \tag{4.3}$$

For the second term of the right hand side of (4.3), we have by Hölder's inequality

$$\begin{aligned}&\int_{M_t} u^{\frac{n+2}{2}+1} d\mu_t \leq \Big(\int_{M_t} u^{\frac{n+2}{2}} d\mu_t\Big)^{\frac{2}{n+2}} \cdot \Big(\int_{M_t} (u^{\frac{n+2}{2}})^{\frac{n+2}{n}} d\mu_t\Big)^{\frac{n}{n+2}} \\ &\leq \Big(\int_{M_t} u^{\frac{n+2}{2}} d\mu_t\Big)^{\frac{2}{n+2}} \cdot \Big(\int_{M_t} u^{\frac{n+2}{2}} d\mu_t\Big)^{\frac{2}{n+2}} \cdot \Big(\int_{M_t} (u^{\frac{n+2}{4}})^{\frac{2n}{n-2}} d\mu_t\Big)^{\frac{n-2}{n+2}} \\ &\leq \Big(\int_{M_t} u^{\frac{n+2}{2}} d\mu_t\Big)^{\frac{2}{n+2}} \cdot \Big(\int_{M_t} u^{\frac{n+2}{2}} d\mu_t\Big)^{\frac{2}{n+2}} \\ &\quad \times \Big[C_n\Big(\int_{M_t} |\nabla u^{\frac{n+2}{4}}|^2 d\mu_t + n^{\frac{n+2}{2}}(2\Lambda)^{n+2}\int_{M_t} u^{\frac{n+2}{2}} d\mu_t\Big)\Big]^{\frac{n}{n+2}} \\ &\leq \Big(\int_{M_t} u^{\frac{n+2}{2}} d\mu_t\Big)^{\frac{4}{n+2}} \cdot \Big[C_n^{\frac{n}{n+2}}\Big(\int_{M_t} |\nabla u^{\frac{n+2}{4}}|^2 d\mu_t\Big)^{\frac{n}{n+2}} \\ &\qquad + n^{\frac{n}{2}}(2\Lambda)^n C_n^{\frac{n}{n+2}}\Big(\int_{M_t} u^{\frac{n+2}{2}} d\mu_t\Big)^{\frac{2n}{n+2}}\Big] \\ &\leq n^{\frac{n}{2}}(2\Lambda)^n C_n^{\frac{n}{n+2}}\Big(\int_{M_t} u^{\frac{n+2}{2}} d\mu_t\Big)^2 + C_n^{\frac{n}{n+2}}\cdot\frac{2}{n+2}\epsilon^{\frac{n+2}{2}}\Big(\int_{M_t} u^{\frac{n+2}{2}} d\mu_t\Big)^2 \\ &\quad + C_n^{\frac{n}{n+2}}\cdot\frac{n}{n+2}\epsilon^{-\frac{n+2}{n}}\int_{M_t} |\nabla u^{\frac{n+2}{4}}|^2 d\mu_t \end{aligned} \tag{4.4}$$

for any $\epsilon > 0$. Combining (4.3) and (4.4), we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_{M_t} u^{\frac{n+2}{2}} d\mu_t \leq& \frac{n+2}{2}c_1\Big(n^{\frac{n}{2}}(2\Lambda)^n C_n^{\frac{n}{n+2}} + C_n^{\frac{n}{n+2}}\cdot\frac{2}{n+2}\epsilon^{\frac{n+2}{2}}\Big)\Big(\int_{M_t} u^{\frac{n+2}{2}} d\mu_t\Big)^2 \\ &+ \Big(\frac{n}{2}c_1 C_n^{\frac{n}{n+2}}\epsilon^{-\frac{n+2}{n}} - \frac{4n}{n+2}\Big)\int_{M_t} |\nabla u^{\frac{n+2}{4}}|^2 d\mu_t. \end{aligned} \tag{4.5}$$

Picking $\epsilon = \Big(\frac{(n+2)c_1 C_n^{\frac{n}{n+2}}}{8}\Big)^{\frac{n}{n+2}}$, inequality (4.5) reduces to

$$\frac{\partial}{\partial t}\int_{M_t} |A|^{n+2} d\mu_t \leq c_2\Big(\int_{M_t} |A|^{n+2} d\mu_t\Big)^2, \tag{4.6}$$

where $c_2 = \frac{n+2}{2}c_1\Big(n^{\frac{n}{2}}(2\Lambda)^n C_n^{\frac{n}{n+2}} + C_n^{\frac{n}{n+2}}\cdot\frac{2}{n+2}\Big(\frac{(n+2)c_1 C_n^{\frac{n}{n+2}}}{8}\Big)^{\frac{n}{2}}\Big)$.

From (4.6), we see by the maximum principle that, for $t \in [0, \min\{T, T_1\})$, where $T_1 = \frac{1-(\frac{2}{3})^{n+2}}{c_2\Lambda^{n+2}}$, there holds

$$||A(t)||_{n+2} < \frac{3}{2}\Lambda. \tag{4.7}$$

Now we consider the evolution equation of $|\mathring{A}|^2$. By a simple computation, we have

$$\frac{\partial}{\partial t}|\mathring{A}|^2 \le \Delta|\mathring{A}|^2 - 2|\nabla\mathring{A}|^2 + c_3|A|^2|\mathring{A}|^2, \tag{4.8}$$

where $c_3 \ge c_1$ is a positive constant depending only on $n$.

Define a tensor $\tilde{\mathring{A}}$ by $\tilde{\mathring{A}}_{ij}^{\alpha} = \mathring{A}_{ij}^{\alpha} + \sigma\eta^{\alpha}\delta_{ij}$, where $\eta^{\alpha} = 1$. Set $h_\sigma = |\tilde{\mathring{A}}| = (|\mathring{A}|^2 + nd\sigma^2)^{\frac{1}{2}}$. Then from (4.8) we have

$$\frac{\partial}{\partial t}h_\sigma \le \Delta h_\sigma + c_3|A|^2 h_\sigma. \tag{4.9}$$

For any $r \ge p > 1$, we have

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial t}\int_{M_t} h_\sigma^r d\mu_t &= \int_{M_t} h_\sigma^{r-1}\frac{\partial}{\partial t}h_\sigma d\mu_t + \frac{1}{r}\int_{M_t} h_\sigma^p \frac{\partial}{\partial t}d\mu_t \\
&\le -\frac{4(r-1)}{r^2}\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t + c_3\int_{M_t}|A|^2 h_\sigma^r d\mu_t.
\end{aligned} \tag{4.10}$$

For the second term of the right hand side of (4.10), we have the following estimate:

$$\begin{aligned}
\int_{M_t}|A|^2 h_\sigma^r d\mu_t &\le \Big(\int_{M_t}|A|^{n+2}d\mu_t\Big)^{\frac{2}{n+2}} \cdot \Big(\int_{M_t} h_\sigma^{r\cdot\frac{n+2}{n}} d\mu_t\Big)^{\frac{n}{n+2}} \\
&\le (2\Lambda)^2\Big(\int_{M_t} h_\sigma^r d\mu_t\Big)^{\frac{2}{n+2}} \cdot \Big(\int_{M_t}(h_\sigma^r)^{\frac{n}{n-2}}d\mu_t\Big)^{\frac{n-2}{n}\cdot\frac{n}{n+2}} \\
&\le (2\Lambda)^2\Big(\int_{M_t} h_\sigma^r d\mu_t\Big)^{\frac{2}{n+2}} \cdot \Big[C_n\Big(\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t \\
&\quad + n^{\frac{n+2}{2}}(2\Lambda)^{n+2}\int_{M_t} h_\sigma^r d\mu_t\Big)\Big]^{\frac{n}{n+2}} \\
&\le (2\Lambda)^2\Big(\int_{M_t} h_\sigma^r d\mu_t\Big)^{\frac{2}{n+2}} \cdot \Big[C_n^{\frac{n}{n+2}}\Big(\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t\Big)^{\frac{n}{n+2}} \\
&\quad + n^{\frac{n}{2}}(2\Lambda)^n C_n^{\frac{n}{n+2}}\Big(\int_{M_t} h_\sigma^r d\mu_t\Big)^{\frac{n}{n+2}}\Big] \\
&= n^{\frac{n}{2}}(2\Lambda)^{n+2}C_n^{\frac{n}{n+2}}\int_{M_t} h_\sigma^r d\mu_t \\
&\quad + (2\Lambda)^2 C_n^{\frac{n}{n+2}}\Big(\int_{M_t} h_\sigma^r d\mu_t\Big)^{\frac{2}{n+2}} \cdot \Big(\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t\Big)^{\frac{n}{n+2}} \\
&\le n^{\frac{n}{2}}(2\Lambda)^{n+2}C_n^{\frac{n}{n+2}}\int_{M_t} h_\sigma^r d\mu_t \\
&\quad + (2\Lambda)^2 C_n^{\frac{n}{n+2}} \cdot \frac{2}{n+2}\mu^{\frac{n+2}{2}}\int_{M_t} h_\sigma^r d\mu_t \\
&\quad + (2\Lambda)^2 C_n^{\frac{n}{n+2}} \cdot \frac{n}{n+2}\mu^{-\frac{n+2}{n}}\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t
\end{aligned} \tag{4.11}$$

for any $\mu > 0$. Therefore, combining (4.10) and (4.11) we have

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial t}\int_{M_t} h_\sigma^r d\mu_t \quad &\leq \quad \left(c_3(2\Lambda)^2 C_n^{\frac{n}{n+2}} \cdot \frac{n}{n+2}\mu^{-\frac{n+2}{n}} - \frac{4(r-1)}{r^2}\right)\int_{M_t} |\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t \\ &\quad +c_3\left(n^{\frac{n}{2}}(2\Lambda)^{n+2}C_n^{\frac{n}{n+2}} + (2\Lambda)^2 C_n^{\frac{n}{n+2}} \cdot \frac{2}{n+2}\mu^{\frac{n+2}{2}}\right)\int_M h_\sigma^r d\mu_t. \end{aligned} \tag{4.12}$$

Choose $\mu = \left(\frac{c_4 r^2 p}{3rp-4p+r}\right)^{\frac{n}{n+2}}$, where $c_4 = c_3(2\Lambda)^2 C_n^{\frac{n}{n+2}} \cdot \frac{n}{n+2}$. Then from (4.12), we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_{M_t} h_\sigma^r d\mu_t \quad &+ \quad \left(1-\frac{1}{p}\right)\int_{M_t} |\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t \\ &\leq \quad \left(c_5 + c_6\left(\frac{r^2 p}{3rp-4p+r}\right)^{\frac{n}{2}}\right)\cdot r \cdot \int_M h_\sigma^r d\mu_t, \end{aligned} \tag{4.13}$$

where $c_5 = c_3(2\Lambda)^2 C_n^{\frac{n}{n+2}}$ and $c_6 = c_3 n^{\frac{n}{2}}(2\Lambda)^{n+2}C_n^{\frac{n}{n+2}} \cdot \frac{2}{n+2} \cdot c_4^{\frac{n}{2}}$.

If we pick $r = p$, then (4.13) reduces to

$$\frac{\partial}{\partial t}\int_{M_t} h_\sigma^p d\mu_t \leq c_7 \int_M h_\sigma^p d\mu_t, \tag{4.14}$$

where $c_7 = \left(c_5 + c_6\left(\frac{p^2}{3p-3}\right)^{\frac{n}{2}}\right)\cdot p$. Letting $\sigma \to 0$, (4.14) becomes

$$\frac{\partial}{\partial t}\int_{M_t} |\mathring{A}|^p d\mu_t \leq c_7 \int_M |\mathring{A}|^p d\mu_t.$$

This implies by the maximum principle that, for $t \in [0, \min\{T, T_2\})$, where $T_2 = \frac{p \ln \frac{3}{2}}{c_7}$, there holds

$$||\mathring{A}(t)||_p < \frac{3}{2}\varepsilon. \tag{4.15}$$

Set $T_0 = \min\{T_1, T_2\}$. We claim that $T > T_0$. We prove this claim by contradiction. Suppose that $T \leq T_0$. Then (4.7) and (4.15) hold on $[0, T)$.

If $T < T_{\max}$, from the smoothness of the mean curvature flow we see that there exists a positive constant $\vartheta$ such that on $[0, T+\vartheta)$ we have

$$||A(t)||_{n+2} < \frac{5}{3}\Lambda, \quad ||\mathring{A}(t)||_p < \frac{5}{3}\varepsilon.$$

This contradicts the definition of $T$.

If $T = T_{\max}$, we will show that the mean curvature flow can be extended over time $T_{\max}$.

From (4.13), we have

$$\frac{\partial}{\partial t}\int_{M_t} h_\sigma^r d\mu_t + \left(1-\frac{1}{p}\right)\int_{M_t} |\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t \leq c_8 r^{n+1} \cdot \int_M h_\sigma^r d\mu_t, \tag{4.16}$$

where $c_8 = \max\{\frac{c_5}{p^n}, \frac{c_6}{(3p-3)^{\frac{n}{2}}}\}$.

As in the proof of Lemma 3.3, for any $\tau, \tau'$ such that $0 < \tau < \tau' < T_{\max} - \theta$, and for any $t \in [\tau', T_{\max} - \theta]$, where $\theta$ is a small positive constant, we have from (4.16) that

$$\begin{aligned}\int_{M_t} h_\sigma^r d\mu_t \quad &+ \quad \Big(1-\frac{1}{p}\Big)\int_{\tau'}^t\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t dt \\ &\le \quad \Big(c_8 r^{n+1}+\frac{1}{\tau'-\tau}\Big)\int_\tau^{T_{\max}-\theta}\int_{M_t} h_\sigma^r d\mu_t dt. \end{aligned}\tag{4.17}$$

As in (3.11), we have by (4.1)

$$\begin{aligned}&\int_{\tau'}^{T_{\max}-\theta}\int_{M_t} h_\sigma^{r(1+\frac{2}{n})} d\mu_t dt \\ \le\ & \int_{\tau'}^{T_{\max}-\theta}\Big(\int_{M_t} h_\sigma^r d\mu_t\Big)^{\frac{2}{n}}\cdot\Big(\int_{M_t} h_\sigma^{\frac{nr}{n-2}} d\mu_t\Big)^{\frac{n-2}{n}} dt \\ \le\ & \max_{t\in[\tau',T_{\max}-\theta]}\Big(\int_{M_t} h_\sigma^r d\mu_t\Big)^{\frac{2}{n}}\cdot\int_{\tau'}^{T_{\max}-\theta}\Big(\int_{M_t} h_\sigma^{\frac{nr}{n-2}} d\mu_t\Big)^{\frac{n-2}{n}} dt \\ \le\ & C_n\cdot\max_{t\in[\tau',T_{\max}-\theta]}\Big(\int_{M_t} h_\sigma^r d\mu_t\Big)^{\frac{2}{n}} \\ & \times\int_{\tau'}^{T_{\max}-\theta}\Big(\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t + n^{\frac{n+2}{2}}(2\Lambda)^{n+2}\int_{M_t} h_\sigma^r d\mu_t\Big) dt.\end{aligned}\tag{4.18}$$

From (4.17) and (4.18), we have

$$\begin{aligned}\int_{\tau'}^{T_{\max}-\theta}\int_{M_t} h_\sigma^{r(1+\frac{2}{n})} d\mu_t dt \quad &\le \quad c_9\Big(c_8 r^{n+1}+\frac{1}{\tau'-\tau}\Big)^{1+\frac{2}{n}} \\ &\quad \times\Big(\int_\tau^{T_{\max}-\theta}\int_{M_t} h_\sigma^r d\mu_t dt\Big)^{1+\frac{2}{n}},\end{aligned}\tag{4.19}$$

where $c_9 = C_n\cdot\max\{\frac{p}{p-1}, n^{\frac{n+2}{2}}(2\Lambda)^{n+2}T_0\}$.

We put

$$J(r,t)=\int_t^{T_{\max}-\theta}\int_{M_t} h_\sigma^r d\mu_t dt.$$

Then from (4.19) we have

$$J\Big(r\Big(1+\frac{2}{n}\Big),\tau'\Big)\le c_9\Big(c_8 r^{n+1}+\frac{1}{\tau'-\tau}\Big)^{1+\frac{2}{n}} J(r,\tau)^{1+\frac{2}{n}}.\tag{4.20}$$

We let

$$\mu = 1+\frac{2}{n},\quad r_k = p\mu^k,\quad \tau_k=\Big(1-\frac{1}{\mu^{k+1}}\Big)t.$$

Notice that $\mu>1$. From (4.20) we have

$$J(r_{k+1},\tau_{k+1})^{\frac{1}{r_{k+1}}}\le c_9^{\frac{1}{r_{k+1}}}\Big(c_8 p^{n+1}+\frac{\mu^2}{\mu-1}\cdot\frac{1}{t}\Big)^{\frac{1}{r_k}}\mu^{\frac{k}{r_k}\cdot(n+1)} J(r_k,\tau_k)^{\frac{1}{r_k}}.$$

Hence

$$
\begin{aligned}
J(r_{d+1},\tau_{d+1})^{\frac{1}{r_{d+1}}} \quad &\leq \quad c_9^{\sum_{k=0}^{d}\frac{1}{r_{k+1}}}\left(c_8 p^{n+1}+\frac{\mu^2}{\mu-1}\cdot\frac{1}{t}\right)^{\sum_{k=0}^{d}\frac{1}{r_k}} \\
&\qquad \times\mu^{(n+1)\cdot\sum_{k=0}^{d}\frac{k}{r_k}}J(p,t)^{\frac{1}{p}}.
\end{aligned}
$$

As $d\to+\infty$, we conclude that

$$
\begin{aligned}
h_\sigma(x,t) \quad &\leq \quad \left(1+\frac{2}{n}\right)^{\frac{n(n+1)(n+2)}{4p}} c_9^{\frac{n}{2p}}\left(c_8 p^{n+1}+\frac{(n+2)^2}{2nt}\right)^{\frac{n+2}{2p}} \\
&\qquad \times\left(\int_0^{T_{\max}-\theta}\int_{M_t} h_\sigma^p d\mu_t dt\right)^{\frac{1}{p}}.
\end{aligned} \tag{4.21}
$$

Now let $\sigma\to 0$ and $\theta\to 0$. Then we have for $t\in[\frac{T_{\max}}{2},T_{\max})$,

$$
|\mathring{A}|^2(x,t)\leq C(n,p,\Lambda,\varepsilon,T_{\max})<+\infty.
$$

This implies that

$$
|A|^2\leq a|H|^2+b
$$

on $[0,T_{\max})$ for some positive constants $a$ and $b$ independent of $t$. On the other hand, we also have

$$
\int_0^{T_{\max}}\int_{M_t}|H|^{n+2}d\mu_t dt<+\infty,
$$

since $T_{\max}<+\infty$. Now we apply Theorem 3.4 to conclude that the mean curvature flow can be extended over time $T_{\max}$. This is a contradiction. This completes the proof of the claim.

By the definition of $T$, for $t\in[0,T_0]$, we also have

$$
||A(t)||_{n+2}<2\Lambda,\quad ||\mathring{A}(t)||_p<2\varepsilon. \tag{4.22}
$$

This completes Step 1.

*Step* 2. We denote by $\mathrm{Vol}(\Sigma)$ the volume of a Riemannian manifold $\Sigma$ and set $V=\mathrm{Vol}(M_0)$. In this step we show that if we choose $\varepsilon$ sufficiently small, then at some time $T_3\in[\frac{T_0}{2},T_0]$, the mean curvature is bounded from below by a positive constant depending on $n$, $p$, $V$ and $\Lambda$.

Since the area of the submanifold is nonincreasing along the mean curvature flow, we see that for $t\in[0,T_{\max})$, there holds

$$
\mathrm{Vol}(M_t)\leq V. \tag{4.23}
$$

Since $M_t$ is a closed submanifold in the Euclidean space, by the Chen-Willmore inequality for the total mean curvature (for the proof see [3]), we have

$$
n^n\mathrm{Vol}(\mathbb{S}^n)\leq\int_{M_t}|H|^n d\mu_t\leq|H|^n_{\max}(t)\mathrm{Vol}(M_t)\leq|H|^n_{\max}(t)V.
$$

Here $|H|_{\max}(t)=\max_{M_t}|H|(\cdot,t)$. This implies that for $t\in[0,T_{\max})$, there holds

$$
|H|^2_{\max}(t)\geq n^n\mathrm{Vol}(\mathbb{S}^n)V^{-1}:=c_{10}. \tag{4.24}
$$

On the other hand, by [20] there is a positive constant $c_{11}$ depending only on $n$ such that for $t \in [0, T_{\max})$, we have

$$\operatorname{diam}(M_t) \le c_{11} \int_{M_t} |H|^{n-1} d\mu_t,$$

where $\operatorname{diam}(M_t)$ denotes the diameter of $M_t$. This together with the Hölder inequality, (4.22) and (4.23) implies for $t \in [0, T_{\max})$

$$\operatorname{diam}(M_t) \le c_{11} n^{\frac{n-1}{2}} (2\Lambda)^{n-1} V^{\frac{3}{n+2}} := c_{12}. \tag{4.25}$$

Since $T > T_0$, we consider the mean curvature flow on $[\frac{T_0}{2}, T_0]$.

As in (4.21), we have for $t \in [\frac{T_0}{2}, T_0]$

$$\begin{aligned} |\mathring{A}| &\le \left(1 + \frac{2}{n}\right)^{\frac{n(n+1)(n+2)}{4p}} c_9^{\frac{n}{2p}} \left(c_8 p^{n+1} + \frac{(n+2)^2}{nT_0}\right)^{\frac{n+2}{2p}} \cdot T_0^{\frac{1}{p}} \cdot 2\varepsilon \\ &:= c_{13}\varepsilon. \end{aligned} \tag{4.26}$$

Here $c_{13}$ depends on $n, p, V, \Lambda$ and is independent of $\varepsilon$.

For $u = |A|^2$, since $c_1 \le c_3$, we have by (4.2)

$$\frac{\partial}{\partial t} u \le \Delta u + c_3 |A|^2 u. \tag{4.27}$$

Then by a standard Moser iteration process as for $h_\sigma$ in Step 1, we have for $t \in [\frac{T_0}{2}, T_0]$

$$\begin{aligned} |A|^2 &\le \left(1 + \frac{2}{n}\right)^{\frac{n(n+1)}{2}} c_{15}^{\frac{n}{n+2}} \left(c_{14}\Big(\frac{n+2}{2}\Big)^{n+1} + \frac{(n+2)^2}{nT_0}\right) \cdot T_0^{\frac{2}{n+2}} \cdot 2\Lambda \\ &:= c_{16}. \end{aligned} \tag{4.28}$$

Here $c_{14} = \max\{\frac{c_5 2^n}{(n+2)^n}, \frac{c_6 2^{\frac{n}{2}}}{(3n)^{\frac{n}{2}}}\}$, and $c_{15} = C_n \cdot \max\{\frac{n+2}{n}, n^{\frac{n+2}{2}} (2\Lambda)^{n+2} T_0\}$.

Set

$$G = \Big(t - \frac{T_0}{2}\Big) |\nabla \mathring{A}|^2 + |\mathring{A}|^2.$$

We consider the evolution inequality of $G$ on $[\frac{T_0}{2}, T_0]$.

As in [1], we have

$$\nabla_t (\nabla \mathring{A}) = \nabla (\nabla_t \mathring{A}) + A * A * \nabla A.$$

Here $\nabla$ is the connection on the spatial vector bundle, which for each $t$ agrees with the Levi-Civita connection of $g(t)$. The evolution equation of $\mathring{A}$ is

$$\nabla_t \mathring{A} = \Delta \mathring{A} + A * A * A.$$

On the other hand, we have

$$\nabla (\Delta \mathring{A}) = \Delta (\nabla \mathring{A}) + A * A * \nabla A.$$

Hence

$$\nabla_t (\nabla \mathring{A}) = \Delta (\nabla \mathring{A}) + A * A * \nabla A.$$

This implies

$$\frac{\partial}{\partial t} |\nabla \mathring{A}|^2 \le \Delta |\nabla \mathring{A}|^2 + c_{17} |A|^2 |\nabla \mathring{A}|^2, \tag{4.29}$$

where $c_{17}$ is a positive constant depending only on $n$. Here we have used the inequality $|\nabla A|^2 \le \frac{3n}{2(n-1)}|\nabla \mathring{A}|^2$, which was proved in [1].

Combining (4.8) and (4.29) we have

$$\frac{\partial}{\partial t}G \le \Delta G + \left(\left(t - \frac{T_0}{2}\right)c_{17}|A|^2 - 1\right)|\nabla \mathring{A}|^2 + c_3|A|^2|\mathring{A}|^2. \tag{4.30}$$

From (4.26), (4.28) and (4.30), we have for $t \in [\frac{T_0}{2}, T_0]$

$$\frac{\partial}{\partial t}G \le \Delta G + \left(\left(t - \frac{T_0}{2}\right)c_{17}c_{16} - 1\right)|\nabla \mathring{A}|^2 + c_3c_{16}c_{13}^2\varepsilon^2. \tag{4.31}$$

Set $T_3 = \min\{T_0, \frac{T_0}{2} + \frac{1}{c_{17}c_{16}}\}$. Then $\frac{T_0}{2} \le T_3 \le T_0$. For $t \in [\frac{T_0}{2}, T_3]$, we have from (4.31)

$$\frac{\partial}{\partial t}G \le \Delta G + c_3c_{16}c_{13}^2\varepsilon^2.$$

By the maximum principle, this implies

$$G(t) - G\left(\frac{T_0}{2}\right) \le c_3c_{16}c_{13}^2\left(t - \frac{T_0}{2}\right)\varepsilon^2$$

for $t \in [\frac{T_0}{2}, T_3]$. Hence

$$\begin{aligned}\left(t - \frac{T_0}{2}\right)|\nabla \mathring{A}|^2 &\le |\mathring{A}|^2\left(\frac{T_0}{2}\right) + c_3c_{16}c_{13}^2\left(t - \frac{T_0}{2}\right)\varepsilon^2 \\ &\le c_{13}^2\varepsilon^2 + c_3c_{16}c_{13}^2\left(t - \frac{T_0}{2}\right)\varepsilon^2.\end{aligned}$$

Then for $t \in (\frac{T_0}{2}, T_3]$, there holds

$$|\nabla \mathring{A}|^2 \le \frac{c_{13}^2}{\left(t - \frac{T_0}{2}\right)}\varepsilon^2 + c_3c_{16}c_{13}^2\varepsilon^2. \tag{4.32}$$

On the other hand, from [1], we know that $|\nabla H|^2 \le \frac{3n^2}{2(n-1)}|\nabla \mathring{A}|^2$. Therefore, (4.32) implies that at $t = T_3$, we have

$$|\nabla H|^2 \le \frac{3n^2}{2(n-1)} \cdot \left(\frac{c_{13}^2}{\left(T_3 - \frac{T_0}{2}\right)} + c_3c_{16}c_{13}^2\right)\varepsilon^2 := c_{18}^2\varepsilon^2. \tag{4.33}$$

Now we consider the submanifold $M_{T_3}$ at time $T_3$. Let $x, y \in M_{T_3}$ be two points such that $|H|(x, T_3) = |H|_{\min}(T_3) := \min_{M_{T_3}} |H|(\cdot, T_3)$ and $|H|(y, T_3) = |H|_{\max}(T_3) := \max_{M_{T_3}} |H|(\cdot, T_3)$. Let $l : [0, L] \to M_{T_3}$ be the shortest geodesic such that $l(0) = x$ and $l(L) = y$. Define a function $\eta : [0, L] \to \mathbb{R}$ by $\eta(s) = |H|^2(l(s), T_3)$ for $s \in [0, L]$. Then $\eta(0) = |H|^2_{\min}(T_3)$ and $\eta(L) = |H|^2_{\max}(T_3)$. By the definition of $\eta$, we have

$$\left|\frac{d}{ds}\eta(s)\right| = \left|\frac{d}{ds}|H|^2(l(s), T_3)\right| \le \left|(\nabla|H|^2)(l(s), T_3)\right| \le \left|2(|H||\nabla H|)(l(s), T_3)\right|.$$

This together with (4.28) and (4.33) implies

$$\left|\frac{d}{ds}\eta(s)\right| \le 2n^{\frac{1}{2}}c_{16}c_{18}\varepsilon.$$

Then we have

$$\eta(L)-\eta(0)=\int_0^L \frac{d}{ds}\eta ds \leq \operatorname{diam}(M_{T_3})\cdot 2n^{\frac{1}{2}}c_{16}c_{18}\varepsilon. \tag{4.34}$$

Combining (4.24), (4.25) and (4.34), we obtain

$$|H|^2_{\min}(T_3)\geq c_{10}-c_{19}\varepsilon, \tag{4.35}$$

where $c_{19}=2n^{\frac{1}{2}}c_{16}c_{18}c_{12}$. We put

$$\varepsilon_1=\frac{c_{10}}{2c_{19}}.$$

Then if $\varepsilon\leq\varepsilon_1$, (4.35) implies that

$$|H|^2_{\min}(T_3)\geq\frac{c_{10}}{2}. \tag{4.36}$$

*Step* 3. In this step, we finish the proof of Theorem 4.2.

Consider the submanifold $M_{T_3}$. Set

$$\varepsilon_2=\frac{c_{10}^{\frac{1}{2}}}{[2n(n-1)]^{\frac{1}{2}}c_{13}}\quad\text{for}\quad n\geq 4,\quad\text{and}\quad \varepsilon_2=\frac{c_{10}^{\frac{1}{2}}}{3\sqrt{2}c_{13}}\quad\text{for}\quad n=3.$$

By (4.26) and (4.36), we see that if $\varepsilon\leq\min\{\varepsilon_1,\varepsilon_2\}$, then

$$|A|^2(T_3)\leq c_{13}^2\varepsilon_2^2+\frac{1}{n}|H|^2(T_3)\leq\frac{|H|^2(T_3)}{n-1}\quad\text{for}\quad n\geq 4,$$

and

$$|A|^2(T_3)\leq\frac{4}{9}|H|^2(T_3)\quad\text{for}\quad n=3.$$

We pick $C_1=\min\{\varepsilon_1,\varepsilon_2\}$, which depends only on $n$, $p$, $V$ and $\Lambda$. Then by the uniqueness of the mean curvature flow and the convergence theorem proved in [1], we conclude that the mean curvature flow with initial value $F_0$ converges to a round point in finite time. This completes the proof of Theorem 4.2. ☐

**Corollary 4.3.** *Let $F_0: M^n\to\mathbb{R}^{n+q}$ $(n\geq 3)$ be a smooth closed submanifold. Suppose that the mean curvature is nowhere vanishing. Then for any fixed $p>1$, there is a positive constant $C_1'$ depending on $n$, $p$, $\min_{M_0}|H|$ and $||A||_{n+2}$, such that if*

$$||\mathring{A}||_p<C_1',$$

*then the mean curvature flow with $F_0$ as initial value has a unique solution $F: M\times[0,T)\to\mathbb{R}^{n+q}$ in a finite maximal time interval, and $F_t(M)$ converges uniformly to a point $x\in\mathbb{R}^{n+q}$ as $t\to T$. The rescaled immersions $\widetilde{F}_t=\frac{F_t-x}{\sqrt{2n(T-t)}}$ converge in $C^\infty$-topology to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit $n$-sphere in some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+q}$.*

*Proof.* It is easy to see that we can choose $C_1$ in Theorem 4.2 such that $C_1=C_1(n,p,V,||A||_{n+2})$ depending on $n$, $p$, $||A||_{n+2}$ and the upper bound $V$ of the volume of $M_0$. Since

$$||A||_{n+2}\geq n^{\frac{1}{2}}||H||_{n+2}\geq n^{\frac{1}{2}}\operatorname{Vol}(M_0)^{\frac{1}{n+2}}\min_{M_0}|H|,$$

we have

$$\operatorname{Vol}(M)\leq n^{-\frac{n+2}{2}}(\min_{M_0}|H|)^{-(n+2)}||A||_{n+2}^{n+2}:=V'.$$

Then by Theorem 4.2, we can pick $C_1' = C_1(n, p, V', ||A||_{n+2})$, which depends on $n$, $p$, $\min_{M_0} |H|$ and $||A||_{n+2}$. $\square$

We restate Theorem 1.2 and give the proof as follows.

**Theorem 4.4.** *Let $F_0 : M^n \to \mathbb{R}^{n+q}$ $(n \geq 3)$ be a smooth closed submanifold. Then for any fixed $p > n$, there is a positive constant $C_2$ depending on $n, p, \mathrm{Vol}(M_0)$ and $||H||_{n+2}$, such that if*

$$||\mathring{A}||_p < C_2,$$

*then the mean curvature flow with $F_0$ as initial value has a unique solution $F : M \times [0, T) \to \mathbb{R}^{n+q}$ in a finite maximal time interval, and $F_t(M)$ converges uniformly to a point $x \in \mathbb{R}^{n+q}$ as $t \to T$. The rescaled immersions $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ converge in $C^\infty$-topology to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit $n$-sphere in some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+q}$.*

*Proof.* The idea to prove Theorem 4.4 is similar to the proof of Theorem 4.2. We set $\Lambda = ||H||_{n+2}$. Suppose

$$||\mathring{A}||_p < \varepsilon$$

for some fixed $p > n$ and assume $\varepsilon \in (0, 100]$. Set

$$T' = \sup\{t \in [0, T_{\max}) : ||H||_{n+2} < 2\Lambda, ||\mathring{A}||_p < 2\varepsilon\}.$$

As in the proof of Theorem 4.2, we consider the mean curvature flow on the time interval $[0, T')$.

For $|H|^2$, we have the inequality (see [1, 22] for the derivation)

$$\frac{\partial}{\partial t}|H|^2 \leq \Delta |H|^2 - 2|\nabla H|^2 + c_{20}|A|^2|H|^2$$

for some positive constant $c_{20}$ depending only on $n$. Set $w = |H|^2$. Then

$$\frac{\partial}{\partial t} w \leq \Delta w + c_{20}|\mathring{A}|^2 w + \frac{c_{20}}{n} w^2. \tag{4.37}$$

From (4.37) we have for $r > 1$

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial t}\int_{M_t} w^r d\mu_t \quad \leq \quad & -\frac{4(r-1)}{r^2}\int_{M_t} |\nabla w^{\frac{r}{2}}|^2 d\mu_t \\ & + c_{20}\int_{M_t} |\mathring{A}|^2 w^r d\mu_t + \frac{c_{20}}{n}\int_{M_t} w^{r+1} d\mu_t. \end{aligned} \tag{4.38}$$

Now we let $r = \frac{n+2}{2}$. As in (4.4), we have

$$\begin{aligned} &\int_{M_t} w^{\frac{n+2}{2}+1} d\mu_t \\ &\quad \leq C_n^{\frac{n}{n+2}} (2\Lambda)^n \Big(\int_{M_t} w^{\frac{n+2}{2}} d\mu_t\Big)^2 + C_n^{\frac{n}{n+2}} \cdot \frac{2}{n+2}\epsilon^{\frac{n+2}{2}} \Big(\int_{M_t} w^{\frac{n+2}{2}} d\mu_t\Big)^2 \\ &\qquad + C_n^{\frac{n}{n+2}} \cdot \frac{n}{n+2}\epsilon^{-\frac{n+2}{n}} \int_{M_t} |\nabla w^{\frac{n+2}{4}}|^2 d\mu_t \end{aligned} \tag{4.39}$$

for any $\epsilon > 0$.

As in (4.11), we have

$$
\begin{aligned}
&\int_{M_t} |\mathring{A}|^2 w^{\frac{n+2}{2}} d\mu_t \\
&\qquad \le \left( (200)^2 C_n^{\frac{n}{p}} (2\Lambda)^{\frac{n(n+2)}{p}} + (200)^2 C_n^{\frac{n}{p}} \frac{p-n}{p} \mu^{\frac{p}{p-n}} \right) \int_{M_t} w^{\frac{n+2}{2}} d\mu_t \\
&\qquad\quad + (200)^2 C_n^{\frac{n}{p}} \frac{n}{p} \mu^{-\frac{p}{n}} \int_{M_t} |\nabla w^{\frac{n+2}{4}}|^2 d\mu_t
\end{aligned} \tag{4.40}
$$

for any $\mu > 0$.

Therefore, combining (4.38), (4.39) and (4.40) we get

$$
\begin{aligned}
&\frac{2}{n+2} \cdot \frac{\partial}{\partial t} \int_{M_t} w^{\frac{n+2}{2}} d\mu_t \\
\le\ & \left( c_{20} (200)^2 C_n^{\frac{n}{p}} \cdot \frac{n}{p} \cdot \mu^{-\frac{p}{n}} + \frac{c_{20}}{n} \cdot C_n^{\frac{n}{n+2}} \cdot \frac{n}{n+2} \epsilon^{-\frac{n+2}{n}} \right. \\
&\left. - \frac{8n}{(n+2)^2} \right) \int_{M_t} |\nabla w^{\frac{r}{2}}|^2 d\mu_t \\
&+ c_{20} \left( (200)^2 C_n^{\frac{n}{p}} (2\Lambda)^{\frac{n(n+2)}{p}} + (200)^2 C_n^{\frac{n}{p}} \frac{p-n}{p} \mu^{\frac{p}{p-n}} \right) \int_{M_t} w^{\frac{n+2}{2}} d\mu_t \\
&+ \frac{c_{20}}{n} \left( C_n^{\frac{n}{n+2}} (2\Lambda)^n + C_n^{\frac{n}{n+2}} \cdot \frac{2}{n+2} \epsilon^{\frac{n+2}{2}} \right) \left( \int_{M_t} w^{\frac{n+2}{2}} d\mu_t \right)^2 .
\end{aligned} \tag{4.41}
$$

Now we pick

$$
\mu = \left( \frac{(200)^2 (n+2)^2}{4p} \cdot c_{20} C_n^{\frac{n}{p}} \right)^{\frac{n}{p}}, \quad \epsilon = \left( \frac{n+2}{4n} \cdot c_{20} C_n^{\frac{n}{n+2}} \right)^{\frac{n}{n+2}} .
$$

Then from (4.41), we have

$$
\frac{\partial}{\partial t} \int_{M_t} w^{\frac{n+2}{2}} d\mu_t \le c_{21} \int_{M_t} w^{\frac{n+2}{2}} d\mu_t + c_{22} \left( \int_{M_t} w^{\frac{n+2}{2}} d\mu_t \right)^2 , \tag{4.42}
$$

where

$$
c_{21} = (200)^2 c_{20} C_n^{\frac{n}{p}} \left[ (2\Lambda)^{\frac{n(n+2)}{p}} + \frac{p-n}{p} \left( \frac{(200)^2 (n+2)^2}{4p} \cdot c_{20} C_n^{\frac{n}{p}} \right)^{\frac{n}{p-n}} \right],
$$

$$
c_{22} = \frac{c_{20}}{n} C_n^{\frac{n}{n+2}} \left[ (2\Lambda)^n + \frac{2}{n+2} \left( \frac{n+2}{4n} \cdot c_{20} C_n^{\frac{n}{n+2}} \right)^{\frac{n}{2}} \right].
$$

Let $\rho(t)$ be the positive solution to the following Bernoulli equation:

$$
\begin{aligned}
\frac{d}{dt}\rho &= c_{21}\rho + c_{22}\rho^2, \\
\rho(0) &= \Lambda^{n+2}.
\end{aligned}
$$

Then

$$
\rho(t) = \frac{e^{c_{21} t}}{\frac{1}{\Lambda^{n+2}} + \frac{c_{22}}{c_{21}} - \frac{c_{22}}{c_{21}} e^{c_{21} t}}, \quad t \in \left[ 0, \frac{\ln \left( \frac{c_{21}}{c_{22} \Lambda^{n+2}} + 1 \right)}{c_{21}} \right).
$$

Set

$$T_1' = \frac{1}{c_{21}} \ln\left( \frac{1 + \frac{c_{22}}{c_{21}}\Lambda^{n+2}}{\left(\frac{2}{3}\right)^{n+2} + \frac{c_{22}}{c_{21}}\Lambda^{n+2}} \right);$$

then $\rho(t) \le \left(\frac{3}{2}\Lambda\right)^{n+2}$ for $t \in [0, T_1']$. By the maximum principle, we see that for $t \in [0, \min\{T', T_1'\})$, there holds

$$\int_{M_t} w^{\frac{n+2}{2}} d\mu_t < \left(\frac{3}{2}\Lambda\right)^{n+2}$$

or, equivalently,

$$||H(t)||_{n+2} < \frac{3}{2}\Lambda.$$

Next, from (4.9) we have

$$\frac{\partial}{\partial t} h_\sigma \le \Delta h_\sigma + c_3 |\mathring{A}|^2 h_\sigma + \frac{c_3}{n}|H|^2 h_\sigma. \tag{4.43}$$

From (4.43) we have for $r > 1$

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial t}\int_{M_t} h_\sigma^r d\mu_t \quad \le \quad & -\frac{4(r-1)}{r^2}\int_{M_t} |\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t \\ & + c_3 \int_{M_t} |\mathring{A}|^2 h_\sigma^r d\mu_t + \frac{c_3}{n}\int_{M_t} |H|^2 h_\sigma^r d\mu_t. \end{aligned} \tag{4.44}$$

As in (4.11), we have for $r \ge p > n$ and any $\nu,\ \varrho > 0$ that there hold

$$\begin{aligned} \int_{M_t} |\mathring{A}|^2 h_\sigma^r d\mu_t \quad \le \quad & \left(\int_{M_t} |\mathring{A}|^p d\mu_t\right)^{\frac{2}{p}} \left(\int_{M_t} (h_\sigma^r)^{\frac{p}{p-2}} d\mu_t\right)^{\frac{p-2}{p}} \\ \le \quad & (2\varepsilon)^2 \left(\int_{M_t} h_\sigma^r d\mu_t\right)^{1-\frac{n}{p}} \left(\int_{M_t} (h_\sigma^r)^{\frac{n}{n-2}} d\mu_t\right)^{\frac{n-2}{n}\cdot\frac{n}{p}} \\ \le \quad & (2\varepsilon)^2 \left(\int_{M_t} h_\sigma^r d\mu_t\right)^{1-\frac{n}{p}} \\ & \times \left[ C_n \left( \int_{M_t} |\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t + (2\Lambda)^{n+2}\int_{M_t} h_\sigma^r d\mu_t \right)\right]^{\frac{n}{p}} \\ \le \quad & (2\varepsilon)^2 C_n^{\frac{n}{p}} (2\Lambda)^{\frac{n(n+2)}{p}} \int_{M_t} h_\sigma^r d\mu_t \\ & + (2\varepsilon)^2 C_n^{\frac{n}{p}} \left[ \frac{p-n}{p} \nu^{\frac{p}{p-n}} \int_{M_t} h_\sigma^r d\mu_t + \frac{n}{p}\nu^{-\frac{p}{n}} \left( \int_{M_t} |\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t \right)\right] \\ \le \quad & \left( (200)^2 C_n^{\frac{n}{p}} (2\Lambda)^{\frac{n(n+2)}{p}} + (200)^2 C_n^{\frac{n}{p}} \frac{p-n}{p}\nu^{\frac{p}{p-n}} \right) \int_{M_t} h_\sigma^r d\mu_t \\ & + (200)^2 C_n^{\frac{n}{p}} \frac{n}{p} \nu^{-\frac{p}{n}} \left( \int_{M_t} |\nabla h_\sigma^{\frac{r}{2}}|^2 d\mu_t \right) \end{aligned} \tag{4.45}$$

and

$$\begin{aligned}
\int_{M_t}|H|^2h_\sigma^r d\mu_t &\le (2\Lambda)^2\Big(\int_{M_t}h_\sigma^r d\mu_t\Big)^{\frac{2}{n+2}}\Big[C_n^{\frac{n}{n+2}}\Big(\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2d\mu_t\Big)^{\frac{n}{n+2}}\\
&\quad +(2\Lambda)^nC_n^{\frac{n}{n+2}}\Big(\int_{M_t}h_\sigma^r d\mu_t\Big)^{\frac{n}{n+2}}\Big]\\
&\le (2\Lambda)^{n+2}C_n^{\frac{n}{n+2}}\int_{M_t}h_\sigma^r d\mu_t+(2\Lambda)^2C_n^{\frac{n}{n+2}}\frac{2}{n+2}\varrho^{\frac{n+2}{2}}\int_{M_t}h_\sigma^r d\mu_t\\
&\quad +(2\Lambda)^2C_n^{\frac{n}{n+2}}\frac{n}{n+2}\varrho^{-\frac{n+2}{n}}\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2d\mu_t. \qquad (4.46)
\end{aligned}$$

From (4.44), (4.45) and (4.46), we have

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial t}\int_{M_t}h_\sigma^r d\mu_t &\le \Big(c_3(200)^2C_n^{\frac{n}{p}}\frac{n}{p}\nu^{-\frac{p}{n}}+\frac{c_3}{n}(2\Lambda)^2C_n^{\frac{n}{n+2}}\frac{n}{n+2}\varrho^{-\frac{n+2}{n}}\\
&\quad -\frac{4(r-1)}{r^2}\Big)\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2d\mu_t\\
&\quad +\Big(c_3(200)^2C_n^{\frac{n}{p}}(2\Lambda)^{\frac{n(n+2)}{p}}+c_3(200)^2C_n^{\frac{n}{p}}\frac{p-n}{p}\nu^{\frac{p}{p-n}}\\
&\quad +\frac{c_3}{n}\Big((2\Lambda)^{n+2}C_n^{\frac{n}{n+2}}+(2\Lambda)^2C_n^{\frac{n}{n+2}}\frac{2}{n+2}\varrho^{\frac{n+2}{2}}\Big)\Big)\int_{M_t}h_\sigma^r d\mu_t. \qquad (4.47)
\end{aligned}$$

Pick

$$\nu^{\frac{p}{n+2}}=\varrho=\left(\frac{c_3(200)^2C_n^{\frac{n}{p}}\cdot\frac{n}{p}+\frac{c_3}{n}(2\Lambda)^2C_n^{\frac{n}{n+2}}\frac{n}{n+2}}{\frac{3r-4}{r^2}}\right)^{\frac{n}{n+2}}.$$

Since $r\ge p>n$, then

$$\nu^{\frac{p}{n+2}}=\varrho\le\left(\frac{c_3(200)^2C_n^{\frac{n}{p}}\cdot\frac{n}{p}+\frac{c_3}{n}(2\Lambda)^2C_n^{\frac{n}{n+2}}\frac{n}{n+2}}{3p-4}\right)^{\frac{n}{n+2}}\cdot r^{\frac{2n}{n+2}}:=c_{23}\cdot r^{\frac{2n}{n+2}}.$$

Then from (4.47), we have

$$\frac{\partial}{\partial t}\int_{M_t}h_\sigma^r d\mu_t+\int_{M_t}|\nabla h_\sigma^{\frac{r}{2}}|^2d\mu_t\le c_{24}r^{\frac{p+n}{p-n}+n}\int_{M_t}h_\sigma^r d\mu_t, \qquad (4.48)$$

where

$$\begin{aligned}
c_{24} &= c_3(200)^2C_n^{\frac{n}{p}}(2\Lambda)^{\frac{n(n+2)}{p}}+\frac{c_3}{n}(2\Lambda)^{n+2}C_n^{\frac{n}{n+2}}\\
&\quad +c_{23}^{\frac{n+2}{p-n}}\cdot c_3(200)^2C_n^{\frac{n}{p}}\frac{p-n}{p}+c_{23}^{\frac{n+2}{2}}\cdot\frac{c_3}{n}(2\Lambda)^2C_n^{\frac{n}{n+2}}\frac{2}{n+2}.
\end{aligned}$$

Letting $r=p$, we have from (4.48)

$$\frac{\partial}{\partial t}\int_{M_t}h_\sigma^p d\mu_t\le c_{24}p^{\frac{p+n}{p-n}+n}\int_{M_t}h_\sigma^p d\mu_t. \qquad (4.49)$$

Now we apply the maximum principle and let $\sigma\to 0$. Then for $t\in[0,\min\{T',T_2'\})$, where $T_2'=c_{24}^{-1}p^{-\frac{2n}{p-n}-n}\ln\frac{3}{2}$, there holds

$$||\mathring{A}(t)||_p<\frac{3}{2}\varepsilon.$$

Set $T_0' = \min\{T_1', T_2'\}$. As in Step 1 of the proof of Theorem 4.2, we can prove that $T' > T_0'$ by contradiction. In fact, from the smoothness of the mean curvature flow we exclude the case where $T' < T_{\max}$. For the case where $T' = T_{\max}$, since we have (4.48), which is similar in form to (4.16), we can apply the standard Moser process to obtain the following estimate for small $\theta > 0$:

$$\begin{aligned} h_\sigma(x,t) &\leq \left(1+\frac{2}{n}\right)^{\frac{n(n+2)}{4p}\left(\frac{p+n}{p-n}+n\right)} c_{25}^{\frac{n}{2p}}\left(c_{24}p^{\frac{p+n}{p-n}+n}+\frac{(n+2)^2}{2nt}\right)^{\frac{n+2}{2p}} \\ &\quad \times\left(\int_0^{T_{\max}-\theta}\int_{M_t} h_\sigma^p d\mu_t dt\right)^{\frac{1}{p}}. \end{aligned} \tag{4.50}$$

Here $c_{25} = C_n \cdot \max\{1, (2\Lambda)^{n+2}T_0'\}$.

Now we let $\sigma \to 0$ and $\theta \to 0$. Then we have for $t \in [\frac{T_{\max}}{2}, T_{\max})$,

$$|\mathring{A}|^2(x,t) \leq C'(n,p,\Lambda,\varepsilon,T_{\max}) < +\infty.$$

This implies that

$$|A|^2 \leq a'|H|^2 + b'$$

on $[0, T_{\max})$ for some positive constants $a'$ and $b'$ independent of $t$. On the other hand, we also have $\int_0^{T_{\max}}\int_{M_t}|H|^{n+2}d\mu_t dt < +\infty$. Applying Theorem 3.4 we conclude that the mean curvature flow can be extended over time $T_{\max}$. This is a contradiction.

We consider the mean curvature flow for $t \in [\frac{T_0'}{2}, T_0']$. As in (4.50), we have

$$\begin{aligned} |\mathring{A}|(x,t) &\leq \left(1+\frac{2}{n}\right)^{\frac{n(n+2)}{4p}\left(\frac{p+n}{p-n}+n\right)} c_{25}^{\frac{n}{2p}}\left(c_{24}p^{\frac{p+n}{p-n}+n}+\frac{(n+2)^2}{nT_0'}\right)^{\frac{n+2}{2p}} T_0'^{\frac{1}{p}} \cdot 2\varepsilon \\ &:= c_{26}\varepsilon. \end{aligned} \tag{4.51}$$

By (4.37), we have

$$\frac{\partial}{\partial t}w \leq \Delta w + c_{20}|\mathring{A}|^2 w + \frac{c_{20}}{n}|H|^2 w.$$

Then similarly as in (4.51), we get for $t \in [\frac{T_0'}{2}, T_0']$

$$\begin{aligned} |H|^2(x,t) &\leq \left(1+\frac{2}{n}\right)^{\frac{n(2n+1)}{2}} c_{25}^{\frac{n}{n+2}}\left(c_{27}(n+2)^{2n+1}+\frac{(n+2)^2}{nT_0'}\right)T_0'^{\frac{2}{n+2}} \cdot (2\Lambda)^2 \\ &:= c_{29}. \end{aligned} \tag{4.52}$$

Here

$$\begin{aligned} c_{27} &= c_{20}(200)^2 C_n^{\frac{n}{p}}(2\Lambda)^n + \frac{c_{20}}{n}(2\Lambda)^{n+2}C_n^{\frac{n}{n+2}} \\ &\quad + c_{28}^{\frac{n+2}{2}} \cdot c_3(200)^2 C_n^{\frac{n}{n+2}}\frac{2}{n+2} + c_{28}^{\frac{n+2}{2}} \cdot \frac{c_{20}}{n}(2\Lambda)^2 C_n^{\frac{n}{n+2}}\frac{2}{n+2}, \end{aligned}$$

and

$$c_{28} = \left(\frac{c_{20}(200)^2 C_n^{\frac{n}{n+2}} \cdot \frac{n}{n+2} + \frac{c_{20}}{n}(2\Lambda)^2 C_n^{\frac{n}{n+2}}\frac{n}{n+2}}{3n+2}\right)^{\frac{n}{n+2}}.$$

By (4.51) and (4.52), we have

$$|A|^2(x,t) \leq c_{26}^2 100^2 + \frac{c_{29}}{n} := c_{30}$$

for $t \in [\frac{T_0'}{2}, T_0']$. As in Step 2 of the proof of Theorem 4.2, we have for $t \in [0, T_{\max})$ that there hold

$$|H|^2_{\max}(t) \geq n^n \mathrm{Vol}(\mathbb{S}^n) V^{-1} := c_{31} \tag{4.53}$$

and

$$\mathrm{diam}(M_t) \leq c_{11}(2\Lambda)^{n-1} V^{\frac{3}{n+2}} := c_{32}, \tag{4.54}$$

where $V = \mathrm{Vol}(M_0)$.

Using a similar argument, for $t \in [\frac{T_0'}{2}, T_3']$, where $T_3' = \min\{T_0', \frac{T_0'}{2} + \frac{1}{c_{17}c_{30}}\}$, we have

$$|\nabla H|^2 \leq \frac{3n^2}{2(n-1)} \cdot \left( \frac{c_{26}^2}{\left(t - \frac{T_0'}{2}\right)} + c_3 c_{30} c_{26}^2 \right) \varepsilon^2 := c_{33}^2 \varepsilon^2. \tag{4.55}$$

Combining (4.53), (4.54) and (4.55), we obtain that, at time $T_3'$, there is

$$\varepsilon_1' = \frac{c_{31}}{2n^{\frac{1}{2}} c_{30} c_{32} c_{33}},$$

such that if $\varepsilon \leq \varepsilon_1'$, then

$$|H|^2_{\min}(T_3') \geq \frac{c_{31}}{2}. \tag{4.56}$$

Set

$$\varepsilon_2' = \frac{c_{31}^{\frac{1}{2}}}{[2n(n-1)]^{\frac{1}{2}} c_{27}} \quad \text{for} \quad n \geq 4, \quad \text{and} \quad \varepsilon_2' = \frac{c_{31}^{\frac{1}{2}}}{3\sqrt{2} c_{27}} \quad \text{for} \quad n = 3.$$

By (4.51) and (4.56), we see that if $\varepsilon \leq \min\{\varepsilon_1', \varepsilon_2', 100\}$, then

$$|A|^2(T_3') \leq c_{27}^2 \varepsilon_2^2 + \frac{1}{n}|H|^2(T_3') \leq \frac{|H|^2(T_3')}{n-1} \quad \text{for} \quad n \geq 4,$$

and

$$|A|^2(T_3') \leq \frac{4}{9}|H|^2(T_3') \quad \text{for} \quad n = 3.$$

Then we can pick $C_2 = \min\{\varepsilon_1', \varepsilon_2', 100\}$, which depends only on $n$, $p$, $V$ and $\Lambda$, and this completes the proof of Theorem 4.4. ∎

Using a similar argument as in the proof of Corollary 4.3, we have the following.

**Corollary 4.5.** *Let $F_0 : M^n \to \mathbb{R}^{n+q}$ $(n \geq 3)$ be a smooth closed submanifold. Suppose that the mean curvature is nowhere vanishing. Then for any fixed $p > n$, there is a positive constant $C_2'$ depending on $n$, $p$, $\min_{M_0} |H|$ and $||H||_{n+2}$, such that if*

$$||\mathring{A}||_p < C_2',$$

*then the mean curvature flow with $F_0$ as initial value has a unique solution $F : M \times [0, T) \to \mathbb{R}^{n+q}$ in a finite maximal time interval, and $F_t(M)$ converges uniformly to a point $x \in \mathbb{R}^{n+q}$ as $t \to T$. The rescaled immersions $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ converge in $C^\infty$-topology to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit $n$-sphere in some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+q}$.*

Now we give the proof of Theorem 1.3, which is restated as follows.

**Theorem 4.6.** *Let $F_0 : M^n \to \mathbb{R}^{n+q}$ $(n \geq 3)$ be a smooth closed submanifold. Then there is a positive constant $C_3$ depending on $n, \mathrm{Vol}(M)$ and $||A||_{n+2}$, such that if*

$$||A||_n < n^{1/2}[\mathrm{Vol}(\mathbb{S}^n)]^{1/n} + C_3,$$

*then the mean curvature flow with $F_0$ as initial value has a unique solution $F : M \times [0,T) \to \mathbb{R}^{n+q}$ in a finite maximal time interval, and $F_t(M)$ converges uniformly to a point $x \in \mathbb{R}^{n+q}$ as $t \to T$. The rescaled immersions $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ converge in $C^\infty$-topology to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit $n$-sphere in some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+q}$.*

*Proof.* The Chen-Willmore inequality says that for a closed submanifold $M^n$ in the Euclidean space, the total mean curvature satisfies

$$\int_M |H|^n d\mu \geq n^n \mathrm{Vol}(\mathbb{S}^n).$$

On the other hand, we have

$$\begin{aligned}\int_M |A|^n d\mu &= \int_M \left(|\mathring{A}|^2 + \frac{|H|^2}{n}\right)^{\frac{n}{2}} d\mu \\ &\geq \int_M |\mathring{A}|^n d\mu + n^{-\frac{n}{2}} \int_M |H|^n d\mu.\end{aligned}$$

Pick $C_3 = \left\{\left[C_1(n,n,V,||A||_{n+2})\right]^n + n^{\frac{n}{2}}\mathrm{Vol}(\mathbb{S}^n)\right\}^{\frac{1}{n}} - n^{\frac{1}{2}}[\mathrm{Vol}(\mathbb{S}^n)]^{\frac{1}{n}}$. Then if $||A||_n < n^{\frac{1}{2}}[\mathrm{Vol}(\mathbb{S}^n)]^{\frac{1}{n}} + C_3$, we have $||\mathring{A}||_n < C_1(n,n,V,||A||_{n+2})$. By Theorem 4.2, Theorem 4.6 follows. $\square$

Put $C_3' = \left\{\left[C_1'(n,n,\max_{M_0}|H|,||A||_{n+2})\right]^n + n^{\frac{n}{2}}\mathrm{Vol}(\mathbb{S}^n)\right\}^{\frac{1}{n}} - n^{\frac{1}{2}}[\mathrm{Vol}(\mathbb{S}^n)]^{\frac{1}{n}}$, where $C_1' = C_1'(n,p,\max_{M_0}|H|,||A||_{n+2})$ is as in Corollary 4.3. We also have the following corollary.

**Corollary 4.7.** *Let $F_0 : M^n \to \mathbb{R}^{n+q}$ $(n \geq 3)$ be a smooth closed submanifold. Suppose that the mean curvature is nowhere vanishing. Then there is a positive constant $C_3'$ depending on $n, \min_{M_0}|H|$ and $||A||_{n+2}$, such that if*

$$||A||_n < n^{1/2}[\mathrm{Vol}(\mathbb{S}^n)]^{1/n} + C_3',$$

*then the mean curvature flow with $F_0$ as initial value has a unique solution $F : M \times [0,T) \to \mathbb{R}^{n+q}$ in a finite maximal time interval, and $F_t(M)$ converges uniformly to a point $x \in \mathbb{R}^{n+q}$ as $t \to T$. The rescaled immersions $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ converge in $C^\infty$-topology to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit $n$-sphere in some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+q}$.*

## 5. Open problems

In this section, we propose several open problems for the convergence of the mean curvature flow of submanifolds. Denote by $\mathbb{F}^{n+q}(c)$ the $(n+q)$-dimensional complete simply connected space form of constant sectional curvature $c$. Let $M$ be an $n$-dimensional closed oriented submanifold in $\mathbb{F}^{n+q}(c)$ with $c \geq 0$. Shiohama-Xu [17] showed that if $|A|^2 < \alpha(n, |H|, c)$, then $M$ is homeomorphic to a sphere for $n \geq 4$ or diffeomorphic to a spherical space form for $n = 3$. Here

$$\alpha(n, |H|, c) = nc + \frac{n|H|^2}{2(n-1)} - \frac{n-2}{2(n-1)}\sqrt{|H|^4 + 4(n-1)c|H|^2}.$$

In [29], Xu-Zhao proved several differentiable sphere theorems for submanifolds satisfying suitable pinching conditions in a Riemannian manifold. Recently, Xu-Gu [26] strengthened Shiohama-Xu's topological sphere theorem for $c = 0$ to be a differentiable sphere theorem. Motivated by these sphere theorems and the convergence theorem for the mean curvature flow due to Andrews-Baker [1], we propose the following.

**Open Problem 5.1.** Let $M$ be an $n$-dimensional ($n \geq 3$) smooth closed submanifold in $\mathbb{F}^{n+q}(c)$ with $c > 0$. Let $M_t$ be the solution of the mean curvature flow with $M$ as initial submanifold. Suppose $M$ satisfies

$$|A|^2 < \alpha(n, |H|, c).$$

Then one of the following holds.

a) The mean curvature flow has a smooth solution $M_t$ on a finite time interval $0 \leq t < T$ and the $M_t$'s converge uniformly to a round point as $t \to T$.

b) The mean curvature flow has a smooth solution $M_t$ for all $0 \leq t < \infty$ and the $M_t$'s converge in the $C^\infty$-topology to a smooth totally geodesic submanifold $M_\infty$ in $\mathbb{F}^{n+q}(c)$.

In particular, $M$ is diffeomorphic to the standard $n$-sphere.

In [16], Shiohama-Xu obtained a topological sphere theorem for closed submanifolds satisfying $||\mathring{A}||_n < C(n)$ in $\mathbb{F}^{n+q}(c)$ with $c \geq 0$ for an explicit positive constant $C(n)$ depending only on $n$. The following problems arise out of this topological sphere theorem and our convergence theorems.

**Open Problem 5.2.** Let $M$ be an $n$-dimensional ($n \geq 2$) smooth closed submanifold in $\mathbb{R}^{n+q}$. Let $M_t$ be the solution of the mean curvature flow with $M$ as initial submanifold. Then there exists a positive constant $D(n)$ depending only on $n$, such that if $M$ satisfies

$$||\mathring{A}||_n < D(n),$$

then the mean curvature flow has a solution $M_t$ on a finite time interval $[0, T)$ and $M_t$ converges uniformly to a round point.

In particular, $M$ is diffeomorphic to the standard $n$-sphere.

For any 4-dimensional compact manifold $M$ which is homeomorphic to a sphere, we hope to show that there exists an isometric embedding of the 4-sphere into a Euclidean space such that $||\mathring{A}||_4$ is small enough in the sense of Theorems 1.2 or Open Problem 5.2. In fact, Shiohama-Xu [16] proved that for any 4-dimensional closed submanifold $M$ in a Euclidean space, we have $||\mathring{A}||_4 \geq C(\Sigma_{i=1}^3 \beta_i)^{1/4}$, where $C$ is a universal positive constant and $\beta_i$ is the $i$-th Betti number of $M$, $i = 1, 2, 3$. Therefore it's possible to isometrically embed a topological 4-sphere into a Euclidean space with small upper bound for $||\mathring{A}||_4$. If this can be done, then we can deduce that $M$ is diffeomorphic to a sphere. This may open a way to prove the smooth Poincaré conjecture in dimension 4, which is now one of the most challenging problems in geometry and topology.

In general, for a homotopy sphere $M$, we can try to find its embedding in Euclidean spaces with small integral norm $||\mathring{A}||_n$. Our results on mean curvature flow of arbitrary codimension reduce the problem of proving whether $M$ is diffeomorphic to a sphere to the problem of finding the optimal embeddings of $M$ into Euclidean spaces.

**Open Problem 5.3.** Let $M$ be an $n$-dimensional ($n \geq 2$) smooth closed submanifold in $\mathbb{F}^{n+q}(c)$ with $c > 0$. Let $M_t$ be the solution of the mean curvature flow with $M$ as initial submanifold. Then there exists a positive constant $E(n)$ depending only on $n$, such that if $M$ satisfies

$$||\mathring{A}||_n < E(n),$$

then one of the following holds.

a) The mean curvature flow has a smooth solution $M_t$ on a finite time interval $0 \leq t < T$ and the $M_t$'s converge uniformly to a round point as $t \to T$.

b) The mean curvature flow has a smooth solution $M_t$ for all $0 \leq t < \infty$, and the $M_t$'s converge in the $C^\infty$-topology to a smooth totally geodesic submanifold $M_\infty$ in $\mathbb{F}^{n+q}(c)$.

In particular, $M$ is diffeomorphic to the standard $n$-sphere.

## Acknowledgment

The authors would like to thank the referee for valuable comments and suggestions.

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