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VERTEX ALGEBRAIC INTERTWINING OPERATORS AMONG GENERALIZED VERMA MODULES FOR $\widehat{\mathfrak{sl}(2,\mathbb{C})}$

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Abstract. We construct vertex algebraic intertwining operators among certain generalized Verma modules for $\widehat{\mathfrak{sl}(2,\mathbb{C})}$ and calculate the corresponding fusion rules. Additionally, we show that under some conditions these intertwining operators descend to intertwining operators among one generalized Verma module and two (generally non-standard) irreducible modules. Our construction relies on the irreducibility of the maximal proper submodules of generalized Verma modules appearing in the Garland-Lepowsky resolutions of standard $\widehat{\mathfrak{sl}(2,\mathbb{C})}$-modules. We prove this irreducibility using the composition factor multiplicities of irreducible modules in Verma modules for symmetrizable Kac-Moody Lie algebras of rank 2, given by Rocha-Caridi and Wallach.

1. Introduction

Intertwining operators are fundamental objects in the representation theory of vertex operator algebras and in the construction of conformal field theories. Intertwining operators among a triple of modules for a vertex operator algebra V correspond to V-module homomorphisms from the tensor product of two modules (if it exists) into the third ([HLZ1], [HL]). Since tensor products of V-modules cannot be obtained from the tensor products of the underlying vector spaces, intertwining operators are essential to understanding the representation theory of V. Indeed, the existence of tensor products depends on the finiteness of the fusion rules, that is, the dimensions of spaces of intertwining operators. Moreover, associativity of intertwining operators ([H1], [HLZ2]) and modular invariance for traces of products of intertwining operators ([H2]) are crucial for showing that modules for certain vertex operator algebras form modular tensor categories ([H4]; see also the review article [HL]).

In [FZ], for vertex operator algebras for which a suitable category of weak modules is semisimple, Frenkel and Zhu identified the spaces of intertwining operators among V-modules with suitable spaces constructed from left modules and bimodules for the Zhu's algebra $A(V)$. In [Li3], Li gave a generalization and a proof of this result. In particular, this approach can be used to describe the intertwining operators among standard (that is, integrable highest weight) modules for the affine Lie algebra $\widehat{\mathfrak{g}}$ (where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra) at a fixed level $\ell \in \mathbb{N}$. These $\widehat{\mathfrak{g}}$-modules are modules for the generalized Verma module vertex operator algebra $V^{M(\ell\Lambda_0)}$ constructed using the fundamental weight Λ_0 of $\widehat{\mathfrak{g}}$ corresponding to the affine simple root α_0 that is not a simple root of $\mathfrak{g}$. (See Section 2 for more on

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generalized Verma modules and the notation we use in this paper, and see Section 4 for a review of the vertex operator algebra structure on $V^{M(\ell\Lambda_0)}$.)

In the present paper, we construct intertwining operators among generalized Verma modules for $V^{M(\ell\Lambda_0)}$ and their (generally non-standard) irreducible quotients in the case that $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. Since generalized Verma modules are typically reducible but indecomposable, the category of $V^{M(\ell\Lambda_0)}$-modules is not semisimple, and we cannot use the method in [FZ] for constructing intertwining operators among generalized Verma modules for $V^{M(\ell\Lambda_0)}$ or for calculating fusion rules. Moreover, we cannot use Li's generalization of [FZ] because [Li3] constructs intertwining operators among triples of modules where two modules are generalized Verma modules but the third is the contragredient of a generalized Verma module. Instead, we develop new methods for constructing intertwining operators, which use the structure of the generalized Verma modules under consideration.

We use two results from the representation theory of Kac-Moody Lie algebras to determine the structure of generalized Verma modules for $\widehat{\mathfrak{sl}(2,\mathbb{C})}$. The first result, which more generally illustrates the importance of generalized Verma modules in representation theory, is the construction by Garland and Lepowsky in [GL] of resolutions of standard modules for symmetrizable Kac-Moody Lie algebras. These resolutions enabled Garland and Lepowsky to prove the Macdonald-Kac formulas for symmetrizable Kac-Moody Lie algebras. For $\widehat{\mathfrak{sl}(2,\mathbb{C})}$, we focus on resolutions by generalized Verma modules: If Λ is a dominant integral weight of $\widehat{\mathfrak{sl}(2,\mathbb{C})}$, there is an exact sequence

$$\cdots \to V^{M(w_j(\Lambda+\rho)-\rho)} \xrightarrow{d_j} V^{M(w_{j-1}(\Lambda+\rho)-\rho)} \xrightarrow{d_{j-1}} \cdots \xrightarrow{d_2} V^{M(w_1(\Lambda+\rho)-\rho)} \xrightarrow{d_1} V^{M(\Lambda)} \xrightarrow{\Pi_\Lambda} L(\Lambda) \to 0,$$

where the d_j are $\widehat{\mathfrak{sl}(2,\mathbb{C})}$-module homomorphisms, Π_Λ denotes the projection from $V^{M(\Lambda)}$ to its irreducible quotient $L(\Lambda)$, and the w_j are certain elements of length j in the Weyl group.

Secondly, we use a multiplicity result of Rocha-Caridi and Wallach [RW2] for symmetrizable Kac-Moody Lie algebras of rank 2, that composition factor multiplicities of irreducible modules in Verma modules equal 1 or 0. This is a special case of the Kazhdan-Lusztig multiplicity formula, conjectured in [DGK] and proved by Kashiwara-Tanisaki in [Ka], [KaT] and Casian in [C], that composition factor multiplicities are given by values of Kazhdan-Lusztig polynomials. Combining the result of Rocha-Caridi and Wallach with the Garland-Lepowsky resolutions, we prove our main result on the structure of generalized Verma modules for $\widehat{\mathfrak{sl}(2,\mathbb{C})}$: for a generalized Verma $\widehat{\mathfrak{sl}(2,\mathbb{C})}$-module appearing in the Garland-Lepowsky resolution of a standard $\widehat{\mathfrak{sl}(2,\mathbb{C})}$-module, the maximal proper submodule is irreducible.

Our main result states that under certain conditions, there is a natural vector space isomorphism between the space of intertwining operators of type $\binom{V^{M(r)}}{V^{M(p)}\ V^{M(q)}}$ and

$$\mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(M(p)\otimes M(q), M(r)),$$

where $V^{M(p)}$, for instance, represents the generalized Verma module of level $\ell \in \mathbb{N}$ induced from the $(p+1)$-dimensional irreducible $\mathfrak{sl}(2,\mathbb{C})$-module $M(p)$ for $p \in \mathbb{N}$.

The main difficulty in constructing an intertwining operator $\mathcal{Y}$ from an $\mathfrak{sl}(2,\mathbb{C})$-homomorphism $f : M(p) \otimes M(q) \to M(r)$ is to first construct

$$\mathcal{Y} : M(p) \otimes M(q) \to V^{M(r)}\{x\}$$

satisfying appropriate properties, where $V^{M(r)}\{x\}$ denotes the space of formal series involving general complex powers of x with coefficients in $V^{M(r)}$. It is then comparatively routine to extend $\mathcal{Y}$ to an intertwining operator of type $\binom{V^{M(r)}}{V^{M(p)}\ V^{M(q)}}$.

Given a homomorphism f from $M(p) \otimes M(q)$ into $M(r)$, there is a very natural way to extend this to a map

$$\bar{\mathcal{Y}} : M(p) \otimes M(q) \to (V^{M(r)})'\{x\},$$

where $(V^{M(r)})'$ is the contragredient of $V^{M(r)}$. In fact, $\bar{\mathcal{Y}}$ extends to an intertwining operator of type $\binom{(V^{M(r)})'}{V^{M(p)}\ V^{M(q)}}$ by Li's results in [Li3]. However, $(V^{M(r)})' \not\cong V^{M(r)}$ since the contragredient of a reducible generalized Verma module is not a generalized Verma module. Nonetheless, there is a (degenerate) bilinear form on $V^{M(r)}$ whose radical is the maximal proper submodule $J(r)$. We use our theorem on the irreducibility of $J(r)$ to show that the image of $\bar{\mathcal{Y}}$ in $(V^{M(r)})'$ annihilates $J(r)$, so that we can use the bilinear form on $V^{M(r)}$ to transport $\bar{\mathcal{Y}}$ to a map

$$\mathcal{Y}^K : M(p) \otimes M(q) \to K(r)\{x\},$$

where $K(r)$ is an $\mathfrak{sl}(2,\mathbb{C})$-module complement of $J(r)$ in $V^{M(r)}$. (Note that $K(r)$ is not an $\widehat{\mathfrak{sl}(2,\mathbb{C})}$-module since $V^{M(r)}$ is indecomposable.)

Using an analogous procedure, again using the irreducibility of $J(r)$, we can next construct a map

$$\mathcal{Y}^J : M(p) \otimes M(q) \to J(r)\{x\}$$

such that the operator

$$\mathcal{Y} = \mathcal{Y}^K + \mathcal{Y}^J : M(p) \otimes M(q) \to V^{M(r)}\{x\}$$

satisfies a commutator formula that then allows its extension to an intertwining operator of type $\binom{V^{M(r)}}{V^{M(p)}\ V^{M(q)}}$. In this way, we completely determine the space of intertwining operators of type $\binom{V^{M(r)}}{V^{M(p)}\ V^{M(q)}}$. Additionally, we show that intertwining operators of type $\binom{V^{M(r)}}{V^{M(p)}\ V^{M(q)}}$ induce intertwining operators of type $\binom{L(r)}{V^{M(p)}\ L(q)}$ under certain easily-checked conditions.

Recalling the connection between vertex algebraic intertwining operators and tensor products of modules for a vertex operator algebra, we expect that our results here will contribute to solving the problem of constructing tensor categories of $V^{M(\ell\Lambda_0)}$-modules. In particular, our determination of the fusion rules can be interpreted as calculations of the multiplicities of the modules $V^{M(r)}$ in tensor products of $V^{M(p)}$ and $V^{M(q)}$. It would be interesting to see how our results here can be generalized to larger classes of $\widehat{\mathfrak{sl}(2,\mathbb{C})}$-modules, or to generalized Verma modules for higher-rank affine Kac-Moody Lie algebras. The primary restriction on our methods here is that they require knowing the structure of generalized Verma modules. For instance, we cannot expect maximal proper submodules of generalized Verma modules for higher-rank Kac-Moody Lie algebras to be irreducible. Additional research is currently under way to develop methods to bypass this problem.

This paper is structured as follows. In Section 2 we recall constructions and results that we shall need from the theory of affine Kac-Moody Lie algebras. In

Section 3, we recall the Kazhdan-Lusztig multiplicity formula and the Garland-Lepowsky resolutions that we shall then use to prove the irreducibility of maximal proper submodules for generalized Verma modules for $\widehat{\mathfrak{sl}(2,\mathbb{C})}$. In Section 4, we recall the vertex operator algebra and module structures on generalized Verma modules for an affine Lie algebra as well as the definition of intertwining operator among modules for a vertex operator algebra. In Section 5, we review results on bilinear pairings between generalized Verma modules that we shall need in Section 6, where we prove our main theorem on intertwining operators among generalized Verma modules for $\widehat{\mathfrak{sl}(2,\mathbb{C})}$. In Section 7, we prove conditions under which an intertwining operator of type $\binom{V^{M(r)}}{V^{M(p)}\, V^{M(q)}}$ descends to an intertwining operator of type $\binom{L(r)}{V^{M(p)}\, L(q)}$, and in Section 8 we construct examples of intertwining operators among $\widehat{\mathfrak{sl}(2,\mathbb{C})}$-modules using our theorems. We also show by counterexample that the conditions on our theorems are necessary. Finally, in the Appendix, we give a proof that a map

$$\mathcal{Y} : M(p) \otimes M(q) \to V^{M(r)}\{x\}$$

satisfying appropriate conditions extends to an intertwining operator

$$\mathcal{Y} : V^{M(p)} \otimes V^{M(q)} \to V^{M(r)}\{x\}.$$

Note that although our main concern in this paper is with $\widehat{\mathfrak{sl}(2,\mathbb{C})}$, some results are proved for general untwisted affine Kac-Moody algebras.

2. Affine Kac-Moody Lie algebras

In this section we shall recall material that we shall need on Kac-Moody Lie algebras. See references such as [Hu], [Le1], [K], and [MP] for more details.

We take $\mathfrak{g}$ to be a finite-type Kac-Moody algebra with indecomposable Cartan matrix, that is, $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\mathfrak{g}$ has the triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+,$$

where $\mathfrak{g}_\pm$ are the sums of the positive and negative root spaces of $\mathfrak{g}$, respectively. We use $\langle \cdot, \cdot \rangle$ to denote the symmetric non-degenerate invariant bilinear form on $\mathfrak{g}$ such that long roots have square length 2.

Using $\mathfrak{g}$ and the bilinear form $\langle \cdot, \cdot \rangle$, we can construct the affine Lie algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$$

with $\mathbf{k}$ central and all other bracket relations defined by

$$[g \otimes t^m, h \otimes t^n] = [g, h] \otimes t^{m+n} + m\langle g, h\rangle \delta_{m+n,0}\mathbf{k}$$

for $g, h \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. We have the decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_- \oplus \widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_+,$$

where $\widehat{\mathfrak{g}}_\pm = \coprod_{n \in \pm\mathbb{Z}_+} \mathfrak{g} \otimes t^n$ and $\widehat{\mathfrak{g}}_0 = \mathfrak{g} \oplus \mathbb{C}\mathbf{k}$.

We now construct generalized Verma modules for $\widehat{\mathfrak{g}}$, in the sense of [GL] and [Le2]. For a dominant integral weight λ of $\mathfrak{g}$ and $\ell \in \mathbb{C}$, we take $M(\lambda, \ell)$ to be the $\widehat{\mathfrak{g}}_0$-module which is the irreducible $\mathfrak{g}$-module with highest weight λ on which $\mathbf{k}$ acts as ℓ. We then have the generalized Verma module

$$V^{M(\lambda,\ell)} = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_+)} M(\lambda, \ell) \cong U(\widehat{\mathfrak{g}}_-) \otimes_{\mathbb{C}} M(\lambda, \ell),$$

where the linear isomorphism follows from the Poincaré-Birkhoff-Witt theorem. The scalar ℓ is called the *level* of $V^{M(\lambda,\ell)}$. For $g \in \mathfrak{g}$ and $n \in \mathbb{Z}$, we use the notation $g(n)$ to denote the action of $g \otimes t^n$ on a $\widehat{\mathfrak{g}}$-module. Then the generalized Verma module $V^{M(\lambda,\ell)}$ is linearly spanned by vectors of the form

$$g_1(-n_1)\cdots g_k(-n_k)u,$$

where $g_i \in \mathfrak{g}$, $n_i > 0$, and $u \in M(\lambda, \ell)$.

We will also need the semidirect product Lie algebra

$$\widetilde{\mathfrak{g}} = \widehat{\mathfrak{g}} \rtimes \mathbb{C}\mathbf{d},$$

where $[\mathbf{d}, \mathbf{k}] = 0$ and

$$[\mathbf{d}, g \otimes t^n] = n(g \otimes t^n)$$

for $g \in \mathfrak{g}$, $n \in \mathbb{Z}$. The Lie algebra $\widetilde{\mathfrak{g}}$ is the *Kac-Moody Lie algebra* associated with a certain generalized Cartan matrix formed by adding one extra row and column to the Cartan matrix of $\mathfrak{g}$. The Kac-Moody algebra $\widetilde{\mathfrak{g}}$ has the triangular decomposition

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_- \oplus \mathfrak{H} \oplus \widetilde{\mathfrak{g}}_+,$$

where $\widetilde{\mathfrak{g}}_\pm = \mathfrak{g}_\pm \oplus \widehat{\mathfrak{g}}_\pm$ and

$$\mathfrak{H} = \mathfrak{h} \oplus \mathbb{C}\mathbf{k} \oplus \mathbb{C}\mathbf{d}$$

is a Cartan subalgebra of $\widetilde{\mathfrak{g}}$. If $h_1, \dots, h_l \in \mathfrak{h}$ are the coroots of $\mathfrak{g}$, then $\mathfrak{H}$ has a basis $h_0, h_1, \dots, h_l, \mathbf{d}$ where $h_0 = -h_\theta + \mathbf{k}$ and h_θ denotes the coroot associated to the longest root θ of $\mathfrak{g}$.

Note that

$$\mathfrak{H}^* = \mathfrak{h}^* \oplus \mathbb{C}\mathbf{k}' \oplus \mathbb{C}\mathbf{d}',$$

where $\mathbf{k}'$ and $\mathbf{d}'$ are dual to $\mathbf{k}$ and $\mathbf{d}$, respectively. The simple roots of $\widetilde{\mathfrak{g}}$ are given by $\alpha_0, \alpha_1, \dots, \alpha_l$, where $\alpha_1, \dots, \alpha_l \in \mathfrak{h}^*$ are the simple roots of $\mathfrak{g}$ and $\alpha_0 = -\theta + \mathbf{d}'$. We recall the action of the Weyl group $\mathcal{W}$ of $\widetilde{\mathfrak{g}}$ on $\mathfrak{H}^*$: $\mathcal{W}$ is the group generated by the simple reflections $r_0, r_1, \dots, r_l$ where

$$r_i(\Lambda) = \Lambda - \Lambda(h_i)\alpha_i$$

for $\Lambda \in \mathfrak{H}^*$ and $0 \le i \le l$.

Recall the set of integral weights of $\widetilde{\mathfrak{g}}$

$$P = \{\Lambda \in \mathfrak{H}^* \,|\, \Lambda(h_i) \in \mathbb{Z} \ \text{ for all } 0 \le i \le l\}$$

and the set of dominant integral weights

$$P^+ = \{\Lambda \in \mathfrak{H}^* \mid \Lambda(h_i) \in \mathbb{N} \ \text{ for all } 0 \le i \le l\}.$$

We will in particular fix $\rho \in P^+$ such that $\rho(h_i) = 1$ for $0 \le i \le l$. We will also need the set

$$P_1 = \{\Lambda \in \mathfrak{H}^* \,|\, \Lambda(h_i) \in \mathbb{N} \ \text{ for all } 1 \le i \le l\}.$$

Remark 2.1. Since the coroots of $\widetilde{\mathfrak{g}}$ are contained in $\mathfrak{h} \oplus \mathbb{C}\mathbf{k}$, if $\Lambda \in P^+$, so is $\Lambda + h\mathbf{d}'$ for any $h \in \mathbb{C}$. Thus we can define the dominant integral weights of $\widehat{\mathfrak{g}}$ to be the elements $\Lambda \in \mathfrak{h}^* \oplus \mathbb{C}\mathbf{k}'$ such that $\Lambda(h_i) \in \mathbb{N}$ for $0 \le i \le l$. Then

$$\Lambda = \lambda + \ell\mathbf{k}'$$

is a dominant integral weight of $\widehat{\mathfrak{g}}$ if and only if $\ell \in \mathbb{N}$ and λ is a dominant integral weight of $\mathfrak{g}$ satisfying $\langle \lambda, \theta \rangle \le \ell$.

For $\Lambda \in \mathfrak{H}^*$, let $\mathbb{C}_\Lambda$ denote the one-dimensional $\mathfrak{H} \oplus \widetilde{\mathfrak{g}}_+$-module on which $\widetilde{\mathfrak{g}}_+$ acts trivially and $\mathfrak{H}$ acts according to Λ. We define the *Verma module*

$$V^\Lambda = U(\widetilde{\mathfrak{g}}) \otimes_{U(\mathfrak{H} \oplus \widetilde{\mathfrak{g}}_+)} \mathbb{C}_\Lambda \cong U(\widetilde{\mathfrak{g}}_-) \otimes_{\mathbb{C}} \mathbb{C}_\Lambda.$$

We can also give generalized Verma modules for $\widehat{\mathfrak{g}}$ a $\widetilde{\mathfrak{g}}$-module structure. Note that for any $\Lambda \in P_1$ we can write

$$\Lambda = \lambda + \ell \mathbf{k}' + h\mathbf{d}', \tag{2.1}$$

where λ is a dominant integral weight of $\mathfrak{g}$ and $\ell, h \in \mathbb{C}$. Then $M(\lambda, \ell)$ becomes a $\widehat{\mathfrak{g}}_0 \oplus \mathbb{C}\mathbf{d}$-module on which $\mathbf{d}$ acts as h, and $V^{M(\lambda,\ell)}$ becomes a $\widetilde{\mathfrak{g}}$-module via

$$\mathbf{d} \cdot g_1(-n_1) \cdots g_k(-n_k)u = (h - n_1 - \ldots - n_k) g_1(-n_1) \cdots g_k(-n_k)u$$

for $g_i \in \mathfrak{g}$, $n_i > 0$, and $u \in M(\lambda, \ell)$. Note that as a $\widetilde{\mathfrak{g}}$-module, $V^{M(\lambda,\ell)}$ is generated by a highest weight vector of weight Λ.

Remark 2.2. We will typically use the notation $V^{M(\Lambda)} = V^{M(\lambda,\ell)}$ for Λ as in (2.1) when we need to consider a generalized Verma module as a $\widetilde{\mathfrak{g}}$-module.

For $\Lambda \in \mathfrak{H}^*$, the Verma module V^Λ is the universal highest weight $\widetilde{\mathfrak{g}}$-module with highest weight Λ; thus for $\Lambda \in P_1$, there is a surjection $\eta : V^\Lambda \to V^{M(\Lambda)}$ taking a basis vector v_0 of $\mathbb{C}_\Lambda$ to a highest weight vector generating $V^{M(\Lambda)}$. Similarly, for $\Lambda \in P_1$, $V^{M(\Lambda)}$ is the universal highest weight $\widetilde{\mathfrak{g}}$-module such that for any $\widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_+ \oplus \mathbb{C}\mathbf{d}$-module homomorphism from $M(\Lambda)$ into a $\widetilde{\mathfrak{g}}$-module W, there is a unique extension to a $\widetilde{\mathfrak{g}}$-module homomorphism from $V^{M(\Lambda)}$ into W. Both V^Λ and $V^{M(\Lambda)}$ have unique maximal proper submodules, the sum of all submodules that do not intersect the highest weight space. Since $V^{M(\Lambda)}$ is a quotient of V^Λ, both have the same unique irreducible quotient $L(\Lambda)$. We will use the notation $J(\Lambda)$ for the maximal proper submodule of $V^{M(\Lambda)}$.

Remark 2.3. For $\Lambda \in P_1$ as in (2.1), we will also use the notation $J(\lambda)$ for $J(\Lambda)$ and $L(\lambda)$ for $L(\Lambda)$, especially if we are considering these modules as $\widehat{\mathfrak{g}}$-modules and if ℓ is understood.

Recall that a $\widetilde{\mathfrak{g}}$-module W is called $\mathfrak{H}$-*semisimple* if

$$W = \coprod_{\Lambda \in \mathfrak{H}^*} W_\Lambda,$$

where

$$W_\Lambda = \{w \in W \mid h \cdot w = \Lambda(h)w \text{ for } h \in \mathfrak{H}\}.$$

The subspace W_Λ is called a *weight space*, and $\Lambda \in \mathfrak{H}^*$ is called a *weight* if $W_\Lambda \neq 0$. We use Q^+ to denote the subset of $\mathfrak{H}^*$ consisting of non-negative integral combinations of the simple roots $\alpha_0, \ldots, \alpha_l$. Recall that $\mathfrak{H}^*$ is partially ordered via $\alpha \leq \beta$ if and only if $\beta - \alpha \in Q^+$. For $\beta \in \mathfrak{H}^*$, set $D(\beta) = \{\alpha \in \mathfrak{H}^* \mid \alpha \leq \beta\}$.

We recall the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ whose objects are $\widetilde{\mathfrak{g}}$-modules W satisfying:

(i) W is $\mathfrak{H}$-semisimple with finite-dimensional weight spaces.
(ii) There exist finitely many elements $\beta_1, \ldots, \beta_k \in \mathfrak{H}^*$ such that any weight Λ of M belongs to some $D(\beta_i)$.

The category $\mathcal{O}$ is stable under the operations of taking submodules, quotients, and finite direct sums. All highest weight $\widetilde{\mathfrak{g}}$-modules, including Verma modules and generalized Verma modules, are in category $\mathcal{O}$. We shall need the following proposition.

Proposition 2.4. *Suppose W is a module in category $\mathcal{O}$ and $\Lambda \in \mathfrak{H}^*$. Then there exists an increasing filtration $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_t = W$ of submodules of W and a subset J of $\{1, \dots, t\}$ such that*

(i) *for $j \in J$, $W_j/W_{j-1} \cong L(\Lambda_j)$ for some $\Lambda_j \geq \Lambda$ and*
(ii) *for $j \in \{1, \dots, t\} - J$ and any $\mu \geq \Lambda$, $(W_j/W_{j-1})_\mu = 0$.*

Such a filtration is called a *local composition series* of W at Λ. Suppose W is in category $\mathcal{O}$ and $\mu \in \mathfrak{H}^*$. Fix $\Lambda \in \mathfrak{H}^*$ such that $\mu \geq \Lambda$ and construct a local composition series of W at Λ. Denote by $[W : L(\mu)]$ the number of times μ occurs among $\{\Lambda_j \mid j \in J\}$. In fact $[W : L(\mu)]$ is independent of the filtration furnished by Proposition 2.4 and the choice of Λ. This number is called the *multiplicity* of $L(\mu)$ in W.

The main results in this paper apply to $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, so we describe how some of the constructions in this section apply to this case and fix notation that we will use for $\mathfrak{sl}(2, \mathbb{C})$. We take the usual basis $\{h, e, f\}$ of $\mathfrak{sl}(2, \mathbb{C})$ satisfying the bracket relations

$$[h, e] = 2e,\ [h, f] = -2f,\ [e, f] = h,$$

and we take the symmetric invariant bilinear form on $\mathfrak{sl}(2, \mathbb{C})$ to be given by

$$\langle e, f\rangle = \frac{1}{2}\langle h, h\rangle = 1,\ \langle e, e\rangle = \langle f, f\rangle = \langle e, h\rangle = \langle f, h\rangle = 0.$$

Then $\widetilde{\mathfrak{sl}(2, \mathbb{C})}$ has the Cartan subalgebra

$$\mathfrak{H} = \mathbb{C}h \oplus \mathbb{C}\mathbf{k} \oplus \mathbb{C}\mathbf{d},$$

and we have $h_0 = -h + \mathbf{k}$ and $h_1 = h$. We also consider

$$\mathfrak{H}^* = \mathbb{C}\alpha \oplus \mathbb{C}\mathbf{k}' \oplus \mathbb{C}\mathbf{d}',$$

where $\{\alpha/2, \mathbf{k}', \mathbf{d}'\}$ is the basis of $\mathfrak{H}^*$ dual to $\{h, \mathbf{k}, \mathbf{d}\}$ The simple roots of $\widehat{\mathfrak{sl}(2, \mathbb{C})}$ are $\alpha_0 = -\alpha + \mathbf{d}'$ and $\alpha_1 = \alpha$. We may also take $\rho = \frac{\alpha}{2} + 2\mathbf{k}'$ since then $\rho(h_0) = \rho(h_1) = 1$. The Weyl group generators r_0 and r_1 act on $\mathfrak{H}^*$ via

$$\begin{aligned} r_0(\alpha) &= \alpha - \alpha(-h + \mathbf{k})(-\alpha + \mathbf{d}') = -\alpha + 2\mathbf{d}'; && r_1(\alpha) = -\alpha;\\ r_0(\mathbf{k}') &= \mathbf{k}' - \mathbf{k}'(-h + \mathbf{k})(-\alpha + \mathbf{d}') = \alpha + \mathbf{k}' - \mathbf{d}'; && r_1(\mathbf{k}') = \mathbf{k}';\\ r_0(\mathbf{d}') &= \mathbf{d}'; && r_1(\mathbf{d}') = \mathbf{d}'. \end{aligned} \tag{2.2}$$

When $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, the elements of P_1 have the form

$$\Lambda = n\frac{\alpha}{2} + \ell\mathbf{k}' + h\mathbf{d}',$$

where $n \in \mathbb{N}$. We will typically use $M(n, \ell)$, or $M(n)$ if ℓ is understood, to denote the finite-dimensional irreducible $\widehat{\mathfrak{g}}_0 \oplus \mathbb{C}\mathbf{d}$-module $M(\Lambda)$ of highest weight Λ. Thus we typically use $V^{M(n)}$ to refer to the generalized Verma module for $\widehat{\mathfrak{sl}(2, \mathbb{C})}$ (or $\widetilde{\mathfrak{sl}(2, \mathbb{C})}$) generated by $M(\Lambda)$. Note that as an $\mathfrak{sl}(2, \mathbb{C})$-module, $M(n)$ is the irreducible $\mathfrak{sl}(2, \mathbb{C})$-module of dimension $n + 1$.

3. Irreducibility of maximal proper submodules in generalized Verma modules for $\widehat{\mathfrak{sl}(2,\mathbb{C})}$

In this section, we prove that for certain weights Λ of $\widehat{\mathfrak{sl}(2,\mathbb{C})}$, the maximal proper submodule of $V^{M(\Lambda)}$ is irreducible. We use a formula of Rocha-Caridi and Wallach for multiplicities of irreducible modules in Verma modules as well as the resolutions of standard modules by generalized Verma modules of Garland and Lepowsky.

We start by recalling the definition of a Coxeter system, examples of which are provided by the Weyl groups of Kac-Moody Lie algebras and the simple reflections which generate them.

Definition 3.1. A *Coxeter system* is a pair $(\mathcal{W}, S)$ where $\mathcal{W}$ is a group and S is a set of involutions in $\mathcal{W}$ such that $\mathcal{W}$ has a presentation of the form

$$W = \langle S \mid (st)^{m(s,t)} \rangle.$$

Here $m(s,t)$ denotes the order of the element st in $\mathcal{W}$ and in the presentation for $\mathcal{W}$, (s,t) ranges over all pairs in $S \times S$ such that $m(s,t) \neq \infty$. We further require the set S to be finite. The group $\mathcal{W}$ is a *Coxeter group* and S is a *fundamental set of generators* of $\mathcal{W}$.

Definition 3.2. For a Coxeter system $(\mathcal{W}, S)$, a *reduced expression* for $\sigma \in \mathcal{W}$ in terms of the generators S is an expression $\sigma = s_1 \cdots s_t$, where $s_i \in S$, for which t is minimal; we say t is the *length* of σ. The *Bruhat order* on $\mathcal{W}$ is defined by $\sigma \leq \tau$ for $\sigma, \tau \in \mathcal{W}$ if a reduced expression for σ in terms of the generators S occurs as a subword of a reduced expression for τ.

The following multiplicity result for $\widetilde{\mathfrak{sl}(2,\mathbb{C})}$ is Corollary 4.14(i) in [RW2], which applies more generally to any rank 2 symmetrizable Kac-Moody Lie algebra.

Theorem 3.3. *For Λ a dominant integral weight of $\widetilde{\mathfrak{sl}(2,\mathbb{C})}$ and x, y in the Weyl group of $\widetilde{\mathfrak{sl}(2,\mathbb{C})}$,*

$$[V^{x(\Lambda+\rho)-\rho} : L(y(\Lambda+\rho)-\rho)] = \begin{cases} 1 & \text{if } x \leq y; \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.4. Theorem 3.3 can be generalized to any symmetrizable Kac-Moody Lie algebra $\mathfrak{G}$ using Kazhdan-Lusztig polynomials. In [KL], Kazhdan and Lusztig constructed polynomials $P_{x,y}(q) \in \mathbb{Z}[q]$, where x and y are elements of some Coxeter group $\mathcal{W}$, using the Hecke algebra associated to $\mathcal{W}$. If $\mathcal{W}$ is the Weyl group of $\mathfrak{G}$, Deodhar, Gabber, and Kac conjectured ([DGK]) that

$$[V^{x(\Lambda+\rho)-\rho} : L(y(\Lambda+\rho)-\rho)] = P_{x,y}(1) \tag{3.1}$$

for any $\Lambda \in P^+$ and $x, y \in \mathcal{W}$. This conjecture was proved by Kashiwara-Tanisaki ([Ka], [KaT]) and Casian ([C]). In general, $P_{x,y}(q) = 0$ unless $x \leq y$, and if the Coxeter group $\mathcal{W}$ has rank 2, $P_{x,y}(q) = 1$ for $x \leq y$. Thus in the case that $\mathfrak{G}$ has rank 2, (3.1) reduces to Theorem 3.3.

To obtain multiplicity results for generalized Verma modules for $\widetilde{\mathfrak{sl}(2,\mathbb{C})}$, we need the following analogue for $\widehat{\mathfrak{sl}(2,\mathbb{C})}$ of Proposition 2.1 in [Le2].

Proposition 3.5. *Let $\Lambda \in P_1$ and let v_0 be a highest weight vector of the Verma module V^Λ. Then there is an $\widehat{\mathfrak{sl}(2,\mathbb{C})}$-module exact sequence*

$$0 \longrightarrow V^{r_1(\Lambda+\rho)-\rho} \xrightarrow{\xi} V^\Lambda \xrightarrow{\eta} V^{M(\Lambda)} \longrightarrow 0,$$

where η takes v_0 to a highest weight vector generating $V^{M(\Lambda)}$, and ξ takes a highest weight vector generating $V^{r_1(\Lambda+\rho)-\rho}$ to a non-zero multiple of the highest weight vector $f^{\Lambda(h)+1} \cdot v_0 \in V^\Lambda$.

Proof. The fact that ξ is injective follows from [RW1] (cf. [V]). We only need to prove that Ker η is the $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}(2,\mathbb{C})}$-submodule of V^λ generated by $f^{\Lambda(h)+1} \cdot v_\Lambda$; we denote this module by K^Λ.

Let X be the $\widehat{\mathfrak{g}}_0$-submodule of V^Λ generated by v_Λ. Then by a classical theorem of Harish-Chandra (see for instance Section 21.4 of [Hu]), the kernel of the surjection $\eta|_X : X \to M(\Lambda)$ is the $\widehat{\mathfrak{g}}_0$-submodule of X generated by $f^{\lambda(h)+1} \cdot v_\Lambda$; here we regard $M(\Lambda)$ in the natural way as a $\widehat{\mathfrak{g}}_0$-submodule of $V^{M(\Lambda)}$. Thus $K^\Lambda \subseteq$ Ker η.

On the other hand, there are $\widehat{\mathfrak{g}}_0$-module maps $M(\Lambda) \to X/\text{Ker }(\eta|_X) \to V^\Lambda/K^\Lambda$ whose composition is non-zero, and so we have a natural $\widehat{\mathfrak{g}}_0$-module injection $\iota : M(\Lambda) \to V^\Lambda/K^\Lambda$. The image of this injection is clearly annihilated by $\widehat{\mathfrak{g}}_+$, so that ι extends to a $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}(2,\mathbb{C})}$-module map $V^{M(\Lambda)} \to V^\Lambda/K^\Lambda$, by the universal property of generalized Verma modules. Since $V^{M(\Lambda)} = V^\Lambda/\text{Ker } \eta$, we see that Ker $\eta \subseteq K^\Lambda$. □

We now combine Theorem 3.3 and Proposition 3.5 to obtain:

Corollary 3.6. *Let $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{sl}(2,\mathbb{C})}$, and let $\Lambda \in P^+$, $x, y \in \mathcal{W}$ be such that $x(\Lambda+\rho) - \rho$, $y(\Lambda + \rho) - \rho \in P_1$. Then*

$$[V^{M(x(\Lambda+\rho)-\rho)} : L(y(\Lambda+\rho)-\rho)] = \begin{cases} 1 & \text{if } x \leq y,\ r_1x \not\leq y; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For the case $x \not\leq y$, multiplicity 0 follows easily from Theorem 3.3. Thus suppose $x \leq y$, so that $[V^{x(\Lambda+\rho)-\rho} : L(y(\Lambda+\rho)-\rho)] = 1$. Since by Proposition 3.5 $V^{M(x(\Lambda+\rho)-\rho)} \cong V^{x(\Lambda+\rho)-\rho}/V^{r_1x(\Lambda+\rho)-\rho}$, we can get a local composition series of $V^{x(\Lambda+\rho)-\rho}$ at $y(\Lambda+\rho)-\rho$ by joining together local composition series of $V^{r_1x(\Lambda+\rho)-\rho}$ and $V^{M(x(\Lambda+\rho)-\rho)}$ at $y(\Lambda+\rho)-\rho$. If $r_1x \leq y$, then by Theorem 3.3, $L(y(\Lambda+\rho)-\rho)$ appears once in any local composition series for $V^{r_1x(\Lambda+\rho)-\rho}$, and so it cannot appear in a local composition series for $V^{M(x(\Lambda+\rho)-\rho)}$. On the other hand, if $r_1x \not\leq y$, then by Theorem 3.3 $L(y(\Lambda+\rho)-\rho)$ does not appear in any local composition series for $V^{r_1x(\Lambda+\rho)-\rho}$, and so it must appear once in any local composition series for $V^{M(x(\Lambda+\rho)-\rho)}$. □

Suppose that Λ is a dominant integral weight of $\widehat{\mathfrak{sl}(2,\mathbb{C})}$. We now recall the $\widehat{\mathfrak{sl}(2,\mathbb{C})}$-module resolution for the standard module $L(\Lambda)$ from [GL]. Let $\mathcal{W}^1$ be the subset of the Weyl group $\mathcal{W}$ consisting of those $w \in \mathcal{W}$ such that $w^{-1}\alpha_1$ is a positive root. Then we can easily see that

$$\mathcal{W}^1 = \{(r_0r_1)^n r_0, (r_0r_1)^n \mid n \in \mathbb{N}\}.$$

Let $\mathcal{W}^1_j$ be the set of elements of $\mathcal{W}^1$ of length j, for $j \in \mathbb{N}$. Then the set $\mathcal{W}^1_j$ only consists of one element w_j, that is,

$$w_j = \begin{cases} (r_0r_1)^n r_0 & \text{if} \quad j = 2n+1; \\ (r_0r_1)^n & \text{if} \quad j = 2n. \end{cases}$$

The next theorem gives an $\widehat{\mathfrak{sl}(2,\mathbb{C})}$-module resolution for $L(\Lambda)$ which follows from Theorem 8.7 in ([GL]).

Theorem 3.7. *There is an $\widehat{\mathfrak{sl}(2,\mathbb{C})}$-module resolution for $L(\Lambda)$ as follows:*

$$\cdots \to V^{M(w_j(\Lambda+\rho)-\rho)} \overset{d_j}{\to} V^{M(w_{j-1}(\Lambda+\rho)-\rho)}$$

$$\overset{d_{j-1}}{\to} \cdots \overset{d_2}{\to} V^{M(r_0(\Lambda+\rho)-\rho)} \overset{d_1}{\to} V^{M(\Lambda)} \overset{\Pi_\Lambda}{\to} L(\Lambda) \to 0,$$

where d_j is an $\widehat{\mathfrak{sl}(2,\mathbb{C})}$-module map from $V^{M(w_j(\Lambda+\rho)-\rho)}$ to $V^{M(w_{j-1}(\Lambda+\rho)-\rho)}$ for $j \geq 0$.

Now we prove the main theorem of this section.

Theorem 3.8. *In the setting of Theorem 3.7, $\mathrm{Im}\, d_j$ is an irreducible and maximal proper submodule of $V^{M(w_{j-1}(\Lambda+\rho)-\rho)}$. More precisely, we have*

$$\mathrm{Im}\ d_j \;=\; J(w_{j-1}(\Lambda+\rho)-\rho) \;\cong\; L(w_j(\Lambda+\rho)-\rho)$$

for $j \in \mathbb{Z}_+$.

Proof. It follows from Theorem 2 in [KK] (cf. Corollary 4.13 in [RW2]) that if Λ is a dominant integral weight of $\widehat{\mathfrak{sl}(2,\mathbb{C})}$ and $L(\mu)$ appears in a local composition series for $V^{M(w(\Lambda+\rho)-\rho)}$ where $w \in W^1$, then $\mu = w'(\Lambda+\rho)-\rho \in P_1$ for some $w' \in W$. Then by Corollary 3.6, the generalized Verma module $V^{M(w_{j-1}(\Lambda+\rho)-\rho)}$ only contains highest weight vectors of weight $w_{j-1}(\Lambda+\rho)-\rho$ and weight $w_j(\Lambda+\rho)-\rho$. Since Im d_j is generated by a highest weight vector of weight $w_j(\Lambda+\rho)-\rho$ in $V^{M(w_{j-1}(\Lambda+\rho)-\rho)}$, any non-zero proper submodule of Im d_j would provide a highest weight vector of weight other than $w_{j-1}(\Lambda+\rho)-\rho$ or $w_j(\Lambda+\rho)-\rho$ in $V^{M(w_{j-1}(\Lambda+\rho)-\rho)}$. Hence $\mathrm{Im}\, d_j$ has no non-trivial submodule and is irreducible for any $j \geq 1$. Also, notice that since

$$V^{M(w_{j-1}(\Lambda+\rho)-\rho)}/\mathrm{Im}\ d_j = V^{M(w_{j-1}(\Lambda+\rho)-\rho)}/\mathrm{Ker}\ d_{j-1} \cong \mathrm{Im}\, d_{j-1}$$

is irreducible, Im d_j has to be maximal in $V^{M(w_{j-1}(\Lambda+\rho)-\rho)}$. □

Remark 3.9. From Theorem 3.8, we obtain the following exact sequence:

$$0 \longrightarrow L(w_{j+1}(\Lambda+\rho)-\rho) \longrightarrow V^{M(w_j(\Lambda+\rho)-\rho)} \longrightarrow L(w_j(\Lambda+\rho)-\rho) \longrightarrow 0$$

for $j \in \mathbb{N}$.

4. Vertex operator algebras and modules from affine Lie algebras

In this section $\mathfrak{g}$ is any finite-dimensional simple Lie algebra over $\mathbb{C}$. We recall the vertex algebraic structure on generalized Verma modules for $\widehat{\mathfrak{g}}$; see for example [LL] for more details.

We use the definition of a vertex operator algebra $(V = \coprod_{n\in\mathbb{Z}} V_{(n)}, Y, \mathbf{1}, \omega)$ from [LL], where the grading on V is by conformal weight. We also use the definition of a module $(W = \coprod_{h\in\mathbb{C}} W_{(n)}, Y_W)$ for a vertex operator algebra V from [LL]. Note that we will typically drop the subscript from the module vertex operator Y_W if the module W is understood. If V is any vertex operator algebra and W is any V-module, the *contragredient* of W is the V-module

$$W' = \coprod_{h\in\mathbb{C}} W^*_{(h)}$$

with vertex operator map given by

$$\langle Y_{W'}(v,x)w', w\rangle = \langle w', Y_W(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between a vector space and its dual. See [FHL] for the proof that this gives a V-module structure on W'.

We fix a level $\ell \in \mathbb{C}$. When $\ell \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, the generalized Verma module $V^{M(0,\ell)}$ (induced from the one-dimensional $\mathfrak{g}$-module $\mathbb{C}\mathbf{1}$), has the structure of a vertex operator algebra with vacuum $\mathbf{1}$ and vertex operator map determined by

$$Y(g(-1)\mathbf{1}, x) = g(x) = \sum_{n\in\mathbb{Z}} g(n)x^{-n-1} \tag{4.1}$$

for $g \in \mathfrak{g}$. The conformal vector ω of $V^{M(0,\ell)}$ is given by

$$\omega = \frac{1}{2(\ell+h^\vee)}\sum_{i=1}^{\dim\mathfrak{g}} u_i(-1)u_i(-1)\mathbf{1},$$

where $\{u_i\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the form $\langle\cdot,\cdot\rangle$.

For any dominant integral weight λ of $\mathfrak{g}$, the generalized Verma module $V^{M(\lambda,\ell)}$ is a $V^{M(0,\ell)}$-module with vertex operator map also determined by (4.1). The conformal weight grading on $V^{M(\lambda,\ell)}$ is given by

$$\operatorname{wt} g_1(-n_1)\cdots g_k(-n_k)u = n_1+\ldots+n_k+\frac{\langle\lambda,\lambda+2\rho_{\mathfrak{g}}\rangle}{2(\ell+h^\vee)}$$

for $g_i \in \mathfrak{g}$, $n > 0$, and $u \in M(\lambda,\ell)$, where $\rho_{\mathfrak{g}}$ is half the sum of the positive roots of $\mathfrak{g}$. We observe that $g(n)$ decreases weight by n, and so

$$[L(0), g(n)] = -ng(n) \tag{4.2}$$

for any $g \in \mathfrak{g}$ and $n \in \mathbb{Z}$. In particular, $g(0)$ preserves weights, so each weight space of $V^{M(\lambda,\ell)}$ is a (finite-dimensional) $\mathfrak{g}$-module.

Remark 4.1. Note that (4.2) implies that $V^{M(\lambda,\ell)}$ has a natural $\widetilde{\mathfrak{g}}$-module structure on which $\mathbf{d}$ acts as $-L(0)$.

Remark 4.2. For convenience, we will shift the grading of any graded subspace X of a $V^{M(0,\ell)}$-module W as follows: if the lowest conformal weight of W is some $h \in \mathbb{C}$, then we define $X(n) = X \cap W_{(n+h)}$, so that X is $\mathbb{N}$-graded:

$$X = \coprod_{n\geq 0} X(n).$$

We remark that any $\widehat{\mathfrak{g}}$-submodule of $V^{M(\lambda)}$ is a $V^{M(0)}$-module as well, and vice versa. Thus any quotient of $V^{M(\lambda)}$ is a $V^{M(0)}$-module. In particular, the maximal proper submodule $J(\lambda)$ is a $V^{M(0)}$-submodule, and the irreducible quotient $L(\lambda)$ is a $V^{M(0)}$-module. Since the weight spaces of $V^{M(\lambda)}$ are finite-dimensional $\mathfrak{g}$-modules and since finite-dimensional $\mathfrak{g}$-modules are completely reducible, there is a graded $\mathfrak{g}$-module $K(\lambda)$ such that $V^{M(\lambda)} = K(\lambda) \oplus J(\lambda)$ (but $K(\lambda)$ certainly need not be a $\widehat{\mathfrak{g}}$-module). We use P_K and P_J to refer to the projections from $V^{M(\lambda)}$ to $K(\lambda)$ and $J(\lambda)$, respectively; these are $\mathfrak{g}$-module homomorphisms, but not generally $\widehat{\mathfrak{g}}$-module homomorphisms. There is a $\mathfrak{g}$-module isomorphism between $K(\lambda)$ and $L(\lambda)$ which sends $u \in K(\lambda)$ to $u + J(\lambda)$.

The contragredient of a $V^{M(0)}$-module W is also a $\widehat{\mathfrak{g}}$-module, with the action of $\widehat{\mathfrak{g}}$ given by

$$\langle g(n)w', w\rangle = -\langle w', g(-n)w\rangle$$

for $g \in \mathfrak{g}$, $n \in \mathbb{Z}$, $w' \in W'$, and $w \in W$. Note that this is not the $\widehat{\mathfrak{g}}$-module structure on W' viewed as a subspace of the dual module W^*. The module W' is in category $\mathcal{O}$ (see for example [MP]).

Since the goal of this paper is to construct intertwining operators among $V^{M(0)}$-modules, we recall the definition of intertwining operator among a triple of modules for a vertex operator algebra from [FHL]. For a general vector space W, we use $W\{x\}$ to denote the vector space of formal series of the form $\sum_{n\in\mathbb{C}} w_n x^n$, $w_n \in W$.

Definition 4.3. Let W_1, W_2 and W_3 be modules for a vertex operator algebra V. An *intertwining operator* of type $\binom{W_3}{W_1\, W_2}$ is a linear map

$$\begin{aligned} \mathcal{Y}: W_1 \otimes W_2 &\to W_3\{x\}, \\ w_{(1)} \otimes w_{(2)} &\mapsto \mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n\in\mathbb{C}} (w_{(1)})_n w_{(2)} x^{-n-1} \in W_3\{x\} \end{aligned}$$

satisfying the following conditions:

(1) *Lower truncation*: For any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $n \in \mathbb{C}$,

$$(w_{(1)})_{n+m} w_{(2)} = 0 \quad \text{for} \quad m \in \mathbb{N} \text{ sufficiently large.}$$

(2) The *Jacobi identity*:

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_{W_3}(v, x_1)\mathcal{Y}(w_{(1)}, x_2)w_{(2)} \\ &\quad - x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right)\mathcal{Y}(w_{(1)}, x_2)Y_{W_2}(v, x_1)w_{(2)} \\ &\quad = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}(Y_{W_1}(v, x_0)w_{(1)}, x_2)w_{(2)} \end{aligned}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.

(3) The $L(-1)$-*derivative property:* for any $w_{(1)} \in W_1$,

$$\mathcal{Y}(L(-1)w_{(1)}, x) = \frac{d}{dx}\mathcal{Y}(w_{(1)}, x).$$

Remark 4.4. We denote the vector space of intertwining operators of type $\binom{W_3}{W_1\, W_2}$ by $\mathcal{V}^{W_3}_{W_1 W_2}$ and the corresponding *fusion rule* is given by $\mathcal{N}^{W_3}_{W_1 W_2} = \dim \mathcal{V}^{W_3}_{W_1 W_2}$.

Remark 4.5. Taking the coefficient of x_0^{-1} in the Jacobi identity for intertwining operators yields the *commutator formula*

$$\begin{aligned} &Y_{W_3}(v, x_1)\mathcal{Y}(w_{(1)}, x_2) - \mathcal{Y}(w_{(1)}, x_2)Y_{W_2}(v, x_1) \\ &\quad = \mathrm{Res}_{x_0}\, x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}(Y_{W_1}(v, x_0)w_{(1)}, x_2) \end{aligned}$$

for $v \in V$ and $w_{(1)} \in W_1$. Similarly, taking the coefficient of x_1^{-1} and then the coefficient of x_0^{-n-1} for $n \in \mathbb{Z}$ yields the *iterate formula*

$$\begin{aligned} &\mathcal{Y}(v_n w_{(1)}, x_2) \\ &\quad = \mathrm{Res}_{x_1} \left((x_1 - x_2)^n Y_{W_3}(v, x_1)\mathcal{Y}(w_{(1)}, x_2) - (-x_2 + x_1)^n \mathcal{Y}(w_{(1)}, x_2)Y_{W_2}(v, x_1)\right) \end{aligned}$$

for $v \in V$ and $w_{(1)} \in W_1$. The commutator and iterate formulas for v and $w_{(1)}$ together are equivalent to the Jacobi identity for v and $w_{(1)}$ (see [LL] for the special case where $\mathcal{Y}$ is a module vertex operator).

Remark 4.6. We recall from [FHL] that if W_1, W_2, and W_3 are three V-modules with conformal weights contained in $h_1 + \mathbb{N}$, $h_2 + \mathbb{N}$, and $h_3 + \mathbb{N}$, respectively, then any $\mathcal{Y} \in \mathcal{V}_{W_1 W_2}^{W_3}$ can be written

$$\mathcal{Y}(u, x)v = \sum_{m \in \mathbb{Z}} \mathcal{Y}_m(u)v\, x^{-m+h_3-h_1-h_2}, \tag{4.3}$$

where for $u \in W_1(k) = (W_1)_{(h_1+k)}$ and $v \in W_2(n) = (W_2)_{(h_2+n)}$, $\mathcal{Y}_m(u)v \in W_3(k+n-m)$. In particular, $\mathcal{Y}_0$ maps $W_1(0) \otimes W_2(0)$ into $W_3(0)$.

5. Invariant pairings between generalized Verma modules

In this section we continue to allow $\mathfrak{g}$ to be any finite-dimensional simple Lie algebra over $\mathbb{C}$, and we work with $\widehat{\mathfrak{g}}$-modules of a fixed level ℓ. One of the main tools we will use to prove our main theorems on intertwining operators among $\widehat{\mathfrak{sl}(2,\mathbb{C})}$-modules is a bilinear pairing between generalized Verma modules $V^{M(\lambda)}$ and $V^{M(\lambda^*)}$, where as $\mathfrak{g}$-modules $M(\lambda^*) \cong M(\lambda)^*$. More precisely, we want a bilinear map

$$\langle \cdot, \cdot \rangle_M : V^{M(\lambda^*)} \times V^{M(\lambda)} \to \mathbb{C}$$

that is *invariant* in the sense that

$$\langle g(m)u, v\rangle_M = -\langle u, g(-m)v\rangle_M \tag{5.1}$$

for $g \in \mathfrak{g}$, $m \in \mathbb{Z}$, $u \in V^{M(\lambda^*)}$, and $v \in V^{M(\lambda)}$. Note that we have a (non-degenerate) such pairing between $(V^{M(\lambda)})'$ and $V^{M(\lambda)}$, but $(V^{M(\lambda)})'$ is not isomorphic to $V^{M(\lambda^*)}$.

Remark 5.1. An invariant bilinear pairing between the generalized Verma modules $V^{M(\lambda)}$ and $V^{M(\lambda^*)}$ can be induced from the Shapovalov pairing between the corresponding Verma modules (see for example [MP] for information on Shapovalov forms). For convenience, we shall give direct proofs of all the results that we need here, in the context of generalized Verma modules.

Remark 5.2. Some of the results in this section when $\lambda = 0$ have also been proved in a vertex algebraic setting in [Li1].

Remark 5.3. We will use the notation $\langle \cdot, \cdot \rangle_M$ to denote the invariant bilinear pairing between generalized Verma modules $V^{M(\lambda^*)}$ and $V^{M(\lambda)}$, and we will now reserve the notation $\langle \cdot, \cdot \rangle$ to denote the pairing between a module and its contragredient, or the form on $\mathfrak{g}$.

To start with, we observe that there is an involution (anti-automorphism) of $\widehat{\mathfrak{g}}$ determined by $g(m) \mapsto -g(-m)$ and $\mathbf{k} \mapsto \mathbf{k}$. This extends to an involution θ of $U(\widehat{\mathfrak{g}})$ that interchanges $U(\widehat{\mathfrak{g}}_+)$ and $U(\widehat{\mathfrak{g}}_-)$. We can then define a bilinear pairing between $V^{M(\lambda)} \cong U(\widehat{\mathfrak{g}}_-) \otimes M(\lambda)$ and $V^{M(\lambda^*)}$ via

$$\langle u, y \otimes v\rangle_M = \langle P(\theta(y)u), v\rangle$$

for $u \in V^{M(\lambda^*)}$, $y \in U(\widehat{\mathfrak{g}}_-)$, and $v \in M(\lambda)$, where P denotes projection to the lowest conformal weight space $M(\lambda^*)$ and $\langle \cdot, \cdot \rangle$ is any non-degenerate $\mathfrak{g}$-invariant pairing between $M(\lambda^*) \cong M(\lambda)^*$ and $M(\lambda)$.

Proposition 5.4. *The bilinear form $\langle \cdot, \cdot \rangle_M$ is invariant.*

Proof. Suppose $u \in V^{M(\lambda^*)}$, $y \in U(\widehat{\mathfrak{g}}_-)$, and $v \in M(\lambda)$. For $g \in \mathfrak{g}$ and $m > 0$, we have

$$\begin{aligned} \langle g(m)u, y \otimes v \rangle_M &= \langle \theta(y)\theta(-g(-m))u, v \rangle_M \\ &= \langle \theta(-g(-m)y)u, v \rangle_M \\ &= \langle u, -g(-m)y \otimes v \rangle_M, \end{aligned}$$

as desired. For $m = 0$, we observe that for any $y \in U(\widehat{\mathfrak{g}}_\pm)$, $[g(0), y] \in U(\widehat{\mathfrak{g}}_\pm)$ as well. Thus, using the $\mathfrak{g}$-module actions on $M(\lambda)$ and $M(\lambda^*)$, and the fact that $g(0)$ commutes with P (since $g(0)$ preserves conformal weight), we obtain:

$$\begin{aligned} \langle g(0)u, y \otimes v \rangle_M &= \langle \theta(y)\theta(-g(0))u, v \rangle_M \\ &= \langle \theta(-g(0))\theta(y)u, v \rangle_M + \langle [\theta(y), \theta(-g(0))]u, v \rangle_M \\ &= \langle P(g(0)\theta(y)u), v \rangle + \langle \theta([-g(0), y])u, v \rangle_M \\ &= \langle g \cdot P(\theta(y)u), v \rangle + \langle u, [-g(0), y] \otimes v \rangle_M \\ &= \langle P(\theta(y)u), -g \cdot v \rangle + \langle u, -g(0)y \otimes v \rangle_M + \langle u, y \otimes g \cdot v \rangle_M \\ &= \langle u, -g(0)y \otimes v \rangle. \end{aligned}$$

Now, for $m < 0$, we will prove invariance by induction on the degree of y in $y \otimes v \in V^{M(\lambda)}$. For the base case $y = 1$, we have

$$\langle g(m)u, v \rangle_M = 0 = \langle u, -g(-m)v \rangle_M$$

since $P(g(m)u) = 0$ and $g(-m)v = 0$. Now suppose invariance holds for any $y \otimes v$ with the degree of y less than some k. It is enough to show that

$$\langle g(m)u, h(-n)y \otimes v \rangle_M = \langle u, -g(-m)h(-n)y \otimes v \rangle_M$$

when $h \in \mathfrak{g}$, $n > 0$, and $n + \deg y = k$. Thus:

$$\begin{aligned} &\langle g(m)u, h(-n)y \otimes v \rangle_M \\ &\quad = -\langle h(n)g(m)u, y \otimes v \rangle_M \\ &\quad = -\langle g(m)h(n)u, y \otimes v \rangle_M - \langle ([h, g](n + m) + n\langle h, g \rangle \delta_{n+m,0}\ell)u, y \otimes v \rangle_M \\ &\quad = \langle h(n)u, g(-m)y \otimes v \rangle_M + \langle u, ([h, g](-n - m) - n\langle h, g \rangle \delta_{n+m,0}\ell)y \otimes v \rangle_M \\ &\quad = -\langle u, h(-n)g(-m)y \otimes v \rangle_M + \langle u, [h(-n), g(-m)]y \otimes v \rangle_M \\ &\quad = \langle u, -g(-m)h(-n)y \otimes v \rangle_M, \end{aligned}$$

completing the proof of invariance. □

Proposition 5.5. *If $u \in V^{M(\lambda^*)}(m)$, $v \in V^{M(\lambda)}(n)$ and $m \neq n$, then $\langle u, v \rangle_M = 0$.*

Proof. If $u = y \otimes u'$ and $v = y' \otimes v'$ where $u' \in M(\lambda^*)$, $v' \in M(\lambda)$, $\deg y = m$ and $\deg y' = n$, then

$$\langle u, v \rangle_M = \langle y \otimes u', y' \otimes v' \rangle_M = \langle P(\theta(y')y \otimes u'), v' \rangle_M.$$

This is non-zero only if $\theta(y')y \otimes u' \in V^{M(\lambda^*)}(0)$, which only happens when $\deg y' = n = m = \deg y$. □

Proposition 5.6. *The left and right radicals of $\langle\cdot,\cdot\rangle_M$ are the maximal proper submodules $J(\lambda^*)$ and $J(\lambda)$, respectively.*

Proof. Since $\langle\cdot,\cdot\rangle_M$ is non-degenerate on the lowest weight spaces, and since the left and right radicals are $\widehat{\mathfrak{g}}$-modules by invariance, the left radical is contained in $J(\lambda^*)$ and the right radical is contained in $J(\lambda)$. On the other hand, suppose $v \in J(\lambda)(n)$. Since $n > 0$, $V^{M(\lambda^*)}(n)$ is spanned by vectors of the form $y \otimes u$ where $u \in M(\lambda^*)$ and $y \in U(\widehat{\mathfrak{g}}_-)$ has positive degree. Then

$$\langle y \otimes u, v\rangle_M = \langle u, \theta(y)v\rangle_M = 0$$

because $\theta(y)v$ is in the trivial intersection of $J(\lambda)$ with $V^{M(\lambda)}(0)$. Thus v is orthogonal to $V^{M(\lambda^*)}(n)$ as well as to every other weight space of $V^{M(\lambda^*)}$ by Proposition 5.5. This means $J(\lambda)$ is contained in the right radical of $\langle\cdot,\cdot\rangle_M$. Similarly, $J(\lambda^*)$ is contained in the left radical of $\langle\cdot,\cdot\rangle_M$, proving the proposition. □

Using Proposition 5.6, we see that $\langle\cdot,\cdot\rangle_M$ induces a well-defined bilinear pairing between any two quotients of $V^{M(\lambda)}$ and $V^{M(\lambda^*)}$. In particular, the pairing is non-degenerate between $L(\lambda)$ and $L(\lambda^*)$. This means that $L(\lambda^*) \cong L(\lambda)'$. Notice also that the restriction of $\langle\cdot,\cdot\rangle_M$ to $K(\lambda^*) \times K(\lambda)$ is non-degenerate.

Let $J(\lambda)^\perp$ denote the set of functionals $u' \in (V^{M(\lambda)})'$ such that $\langle u', v\rangle = 0$ for all $v \in J(\lambda)$. It is easy to see that $J(\lambda)^\perp$ is a $\widehat{\mathfrak{g}}$-module.

Proposition 5.7. *As $\widehat{\mathfrak{g}}$-modules, $J(\lambda)^\perp$ is isomorphic to $L(\lambda^*)$. Furthermore, there is a $\mathfrak{g}$-module isomorphism $\Phi : J(\lambda)^\perp \to K(\lambda^*)$ such that*

$$\langle\Phi(u'), v\rangle_M = \langle u', v\rangle$$

for any $u' \in J(\lambda)^\perp$ and $v \in V^{M(\lambda)}$.

Proof. We first define a map $\phi : L(\lambda^*) \to (V^{M(\lambda)})'$ by

$$\langle\phi(u + J(\lambda^*)), v\rangle = \langle u, v\rangle_M$$

for $u \in V^{M(\lambda^*)}$ and $v \in V^{M(\lambda)}$. This map is well-defined and injective because $J(\lambda^*)$ is the left radical of $\langle\cdot,\cdot\rangle_M$. Furthermore, ϕ is a $\widehat{\mathfrak{g}}$-homomorphism because for $g \in \mathfrak{g}$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned}\langle g(n)\cdot\phi(u + J(\lambda^*)), v\rangle &= -\langle\phi(u + J(\lambda^*)), g(-n)v\rangle = -\langle u, g(-n)v\rangle_M\\ &= \langle g(n)u, v\rangle_M = \langle\phi(g(n)(u + J(\lambda^*))), v\rangle,\end{aligned}$$

by the definition of the $\widehat{\mathfrak{g}}$-module action on $(V^{M(\lambda)})'$. We observe that the image of ϕ is contained in $J(\lambda)^\perp$ because $J(\lambda)$ is the right radical of $\langle\cdot,\cdot\rangle_M$, and that ϕ preserves conformal weights because it is a $\widehat{\mathfrak{g}}$-homomorphism. Now, the weight spaces of $J(\lambda)^\perp$ have the same dimension as the weight spaces of $L(\lambda)$ since $L(\lambda) = V^{M(\lambda)}/J(\lambda)$, and the weight spaces of $L(\lambda)$ have the same dimension as the weight spaces of $L(\lambda^*)$ since $L(\lambda^*) \cong L(\lambda)'$. Since the weight spaces are finite dimensional, this means that ϕ is surjective onto $J(\lambda)^\perp$, and we have a $\widehat{\mathfrak{g}}$-isomorphism $\phi^{-1} : J(\lambda)^\perp \to L(\lambda^*)$.

We also define a map $\psi : L(\lambda^*) \to K(\lambda^*)$ by $\psi(u + J(\lambda^*)) = P_K(u)$ for $u \in V^{M(\lambda^*)}$. This is well-defined and injective because $J(\lambda^*)$ is the kernel of P_K and ψ is a $\mathfrak{g}$-homomorphism because P_K is a $\mathfrak{g}$-homomorphism. Also, ψ is surjective because the weight spaces of $L(\lambda^*)$ and $K(\lambda^*)$ have the same dimensions.

We then define the $\mathfrak{g}$-homomorphism $\Phi = \psi \circ \phi^{-1} : J(\lambda)^\perp \to K(\lambda^*)$. If $u' = \phi(u + J(\lambda^*)) \in J(\lambda)^\perp$, then

$$\begin{aligned}\langle \Phi(u'), v\rangle_M &= \langle \psi(u + J(\lambda^*)), v\rangle_M = \langle P_K(u), v\rangle_M \\ &= \langle u, v\rangle_M = \langle \phi(u + J(\lambda^*)), v\rangle = \langle u', v\rangle\end{aligned}$$

for any $v \in V^{M(\lambda)}$, as desired. □

Example 5.8. When $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, we have $\lambda^* = \lambda$ for any $\lambda = r\alpha/2$ where $r \in \mathbb{N}$. Thus we have constructed a $\widehat{\mathfrak{g}}$-invariant bilinear form on $V^{M(r)} = V^{M(r\alpha/2)}$ for any $r \in \mathbb{N}$, and Φ gives a $\mathfrak{g}$-module isomorphism from $J(r)^\perp \subseteq (V^{M(r)})'$ to $K(r)$ determined by

$$\langle \Phi(u'), v\rangle_M = \langle u', v\rangle \tag{5.2}$$

for any $u' \in J(r)^\perp$, $v \in V^{M(r)}$.

6. The main theorems on intertwining operators among generalized Verma modules

In this section we present our main theorems for intertwining operators among generalized Verma modules for $\widehat{\mathfrak{sl}(2,\mathbb{C})}$. We shall be interested in intertwining operators among modules for the generalized Verma module vertex operator algebra $V^{M(0,\ell)}$ for some fixed $\ell \in \mathbb{N}$. Recalling the notation of (4.3), we will prove:

Theorem 6.1. *Suppose $M(p)$, $M(q)$, and $M(r)$ for $p, q, r \in \mathbb{N}$ are finite-dimensional irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules with highest weights p, q, and r respectively and the conditions of Theorem 6.3 below hold. Then the linear map*

$$\begin{aligned}\mathcal{V}^{V^{M(r)}}_{V^{M(p)}\,V^{M(q)}} &\longrightarrow \mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(M(p) \otimes M(q), M(r)) \\ \mathcal{Y} &\longmapsto \mathcal{Y}_0|_{M(p)\otimes M(q)}\end{aligned}$$

is a linear isomorphism.

In order to prove this theorem, we will need a result on extending an intertwining operator map from lowest weight spaces to entire modules. It is similar to Tsuchiya and Kanie's nuclear democracy theorem [TK] and Li's generalized nuclear democracy theorem (Theorem 4.12 in [Li2]); see also the main theorem in [Li3]. Since the proof requires lengthy but standard calculations using formal calculus, we defer it to the appendix. Note that the following theorem applies to any finite-dimensional complex simple Lie algebra $\mathfrak{g}$, not merely $\mathfrak{sl}(2,\mathbb{C})$:

Theorem 6.2. *Suppose $M(\lambda_1)$ and $M(\lambda_2)$ are finite-dimensional irreducible $\mathfrak{g}$-modules with highest weights λ_1 and λ_2, respectively, W is a $V^{M(0)}$-module, and*

$$\mathcal{Y} : M(\lambda_1) \otimes M(\lambda_2) \to W\{x\}$$

is a linear map, given by $u \otimes v \mapsto \mathcal{Y}(u, x)v$ for $u \in M(\lambda_1)$ and $v \in M(\lambda_2)$, satisfying

$$[g(n), \mathcal{Y}(u, x)] = x^n \mathcal{Y}(g(0)u, x) \tag{6.1}$$

for any $g \in \mathfrak{g}$, $n \geq 0$, and

$$[L(0), \mathcal{Y}(u, x)] = x\frac{d}{dx}\mathcal{Y}(u, x) + \mathcal{Y}(L(0)u, x). \tag{6.2}$$

Then $\mathcal{Y}$ has a unique extension to an intertwining operator

$$\mathcal{Y} : V^{M(\lambda_1)} \otimes V^{M(\lambda_2)} \to W\{x\}.$$

Recalling (4.3), we see that a map $\mathcal{Y} : M(\lambda_1) \otimes M(\lambda_2) \to V^{M(\lambda_3)}\{x\}$ satisfying (6.1) and (6.2) is equivalent to a sequence of maps

$$\mathcal{Y}_m : M(\lambda_1) \otimes M(\lambda_2) \to V^{M(\lambda_3)}$$

for each $m \in \mathbb{Z}$ such that $\mathcal{Y}_m(u)v \in V^{M(\lambda_3)}(-m)$ and

$$[g(n), \mathcal{Y}_m(u)]v = \mathcal{Y}_{m+n}(g(0)u)v \tag{6.3}$$

for any $u \in M(\lambda_1)$, $v \in M(\lambda_2)$, $m \in \mathbb{Z}$, $g \in \mathfrak{g}$, and $n \geq 0$. Of course, we take $V^{M(\lambda_3)}(m) = 0$ for $m < 0$ so that $\mathcal{Y}_m = 0$ for $m > 0$.

In order to construct maps satisfying (6.3), we will take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ once again:

Theorem 6.3. *Suppose $M(p)$, $M(q)$, and $M(r)$ for $p, q, r \in \mathbb{N}$ are finite-dimensional irreducible $\mathfrak{sl}(2, \mathbb{C})$-modules with highest weights p, q, and r, respectively, and the maximal proper submodule $J(r) \subseteq V^{M(r)}$ is irreducible and isomorphic to some $L(r')$. Suppose moreover that $J(r')$ is irreducible and isomorphic to some $L(r'')$, and that there are no non-zero $\mathfrak{sl}(2, \mathbb{C})$-homomorphisms from $M(p) \otimes M(q)$ to $M(r')$ or $M(r'')$. Then given $f \in \mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(M(p) \otimes M(q), M(r))$, there are for $m \in \mathbb{Z}$ unique maps $\mathcal{Y}_m : M(p) \otimes M(q) \to V^{M(r)}(-m)$ such that $\mathcal{Y}_0 = f$ and (6.3) holds.*

Remark 6.4. The irreducibility of the maximal proper submodules $J(r)$ and $J(r')$ are natural conditions to consider in light of Theorem 3.8. Moreover, we shall see examples in Section 8 for which there are non-zero homomorphisms $M(p) \otimes M(q) \to M(r')$ and for which the assertion of Theorem 6.3 fails to hold.

Proof of Theorem 6.3. We take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$-modules $M(p)$ and $M(q)$ and a generalized Verma module $V^{M(r)}$. We assume that $J(r)$ is irreducible and isomorphic to some $L(r')$ and that $J(r')$ is irreducible and isomorphic to some $L(r'')$. Since $J(r) \cong L(r')$, we may equip $J(r)$ with the non-degenerate invariant bilinear form $\langle \cdot, \cdot \rangle_J$ induced by the invariant form on $V^{M(r')}$ constructed in the previous section. Suppose the lowest weight space of $J(r)$ is contained in $V^{M(r)}(M)$ for some $M > 0$. Note that in the identification $J(r) \cong L(r')$, the weight space $J(r)(n)$ for any $n \geq M$ is then identified with $L(r')(n - M)$ (recall Remark 4.2).

In order to prove the theorem, take a $\mathfrak{g}$-homomorphism $f : M(p) \otimes M(q) \to M(r)$. We need to construct unique maps

$$\mathcal{Y}_m : M(p) \otimes M(q) \to V^{M(r)}(-m)$$

for all $m \in \mathbb{Z}$ such that for any $u \in M(p)$, $v \in M(q)$,

$$\mathcal{Y}_0(u)v = f(u \otimes v) \tag{6.4}$$

and

$$[g(n), \mathcal{Y}_m(u)]v = \mathcal{Y}_{m+n}(g(0)u)v \tag{6.5}$$

for $g \in \mathfrak{g}$ and $n \geq 0$.

For the uniqueness assertion, suppose we have maps $\mathcal{Y}_m$ satisfying (6.4) and (6.5). We write $\mathcal{Y}_m = \mathcal{Y}_m^K + \mathcal{Y}_m^J$ where $\mathcal{Y}_m^K = P_K \circ \mathcal{Y}_m$ and $\mathcal{Y}_m^J = P_J \circ \mathcal{Y}_m$. Then it is clear that $\mathcal{Y}_m^K = 0$ for $m > 0$ and $\mathcal{Y}_m^J = 0$ for $m > -M$, and that $\mathcal{Y}_0^K = f$. We also claim that $\mathcal{Y}_{-M}^J = 0$. Indeed, by the $n = 0$ case of (6.5), each $\mathcal{Y}_m$ is a $\mathfrak{g}$-module homomorphism from $M(p) \otimes M(q)$ to $V^{M(r)}(-m)$, and because P_K and P_J are $\mathfrak{g}$-module homomorphisms, so is each $\mathcal{Y}_m^K$ and $\mathcal{Y}_m^J$. In particular, $\mathcal{Y}_{-M}^J$ is a $\mathfrak{g}$-module homomorphism from $M(p) \otimes M(q)$ to $M(r')$, the lowest weight space of

$J(r)$. However, by the assumptions of the theorem, the only such homomorphism is 0.

Now, for any $m \in \mathbb{Z}$, $n > 0$, $g \in \mathfrak{g}$, $u \in M(p)$, and $v \in M(q)$, (6.5) implies that

$$\begin{aligned}\mathcal{Y}^K_{-m+n}(g(0)u)v+\mathcal{Y}^J_{-m+n}(g(0)u)v &= \mathcal{Y}_{-m+n}(g(0)u)v = g(n)\mathcal{Y}_{-m}(u)v\\ &= g(n)\mathcal{Y}^K_{-m}(u)v + g(n)\mathcal{Y}^J_{-m}(u)v\\ &= P_K(g(n)\mathcal{Y}^K_{-m}(u)v) + P_J(g(n)\mathcal{Y}^K_{-m}(u)v) + g(n)\mathcal{Y}^J_{-m}(u)v. \qquad (6.6)\end{aligned}$$

Thus if we suppose $m > 0$, $0 < n \leq m$ and $w \in V^{M(r)}(m-n)$, the invariance of the form $\langle\cdot,\cdot\rangle_M$ on $V^{M(r)}$, the fact that $J(r)$ is the radical of the form, and the projection of (6.6) to $K(r)$ imply that

$$\begin{aligned}\langle\mathcal{Y}^K_{-m}(u)v, g(-n)w\rangle_M &= -\langle g(n)\mathcal{Y}^K_{-m}(u)v, w\rangle_M\\ &= -\langle P_K(g(n)\mathcal{Y}^K_{-m}(u)v), w\rangle_M\\ &= -\langle\mathcal{Y}^K_{-m+n}(g(0)u)v, w\rangle_M.\end{aligned}$$

Since $\langle\cdot,\cdot\rangle_M$ is non-degenerate on $K(r)$ and since $K(r)(m)$ is spanned by linear combinations of certain vectors of the form $g(-n)w$ where $0 < n \leq m$ and $w \in V^{M(r)}(m-n)$, we see that for $m > 0$, $\mathcal{Y}^K_{-m}$ is uniquely determined by $\mathcal{Y}^K_{-m+n}$ for $0 < n \leq m$. Thus $\mathcal{Y}^K_{-m}$ for $m > 0$ is uniquely determined by $\mathcal{Y}^K_0 = f$.

Similarly, if we suppose $m > M$, $0 < n \leq m - M$, and $w \in J(r)(m-n)$, we must have

$$\begin{aligned}\langle\mathcal{Y}^J_{-m}(u)v, g(-n)w\rangle_J &= -\langle g(n)\mathcal{Y}^J_{-m}(u)v, w\rangle_J\\ &= -\langle\mathcal{Y}^J_{-m+n}(g(0)u)v, w\rangle_J + \langle P_J(g(n)\mathcal{Y}^K_{-m}(u)v), w\rangle_J.\end{aligned}$$

Since $\langle\cdot,\cdot\rangle_J$ is non-degenerate on $J(r)$ and since vectors of the form $g(-n)w$ for $0 < n \leq m - M$ and $w \in J(r)(m-n)$ span $J(r)(m)$, we see as before that $\mathcal{Y}^J_{-m}$ for $m > M$ is determined by $\mathcal{Y}^J_{-M} = 0$ and by $\mathcal{Y}^K_{-m}$. This proves the uniqueness assertion of the theorem, and it remains to show that the above recursive formulas for $\mathcal{Y}^K_{-m}$ and $\mathcal{Y}^J_{-m}$ do in fact determine well-defined linear maps with the required properties.

Since $M(r) \cong M(r)^*$, the lowest weight space of $(V^{M(r)})'$ is isomorphic to $M(r)$. Thus we can view f as a homomorphism from $M(p)\otimes M(q)$ into $(V^{M(r)})'(0)$. Now we would like to define

$$\bar{\mathcal{Y}}^K_{-m} : M(p)\otimes M(q) \to (V^{M(r)})'(m)$$

for any $m \geq 0$ recursively using the formulas:

$$\langle\bar{\mathcal{Y}}^K_0(u)v, w\rangle = \langle f(u\otimes v), w\rangle \qquad (6.7)$$

for $u \in M(p)$, $v \in M(q)$, and $w \in M(r)$, and for $m > 0$,

$$\langle\bar{\mathcal{Y}}^K_{-m}(u)v, g(-n)w\rangle = -\langle\bar{\mathcal{Y}}^K_{-m+n}(g(0)u)v, w\rangle, \qquad (6.8)$$

where $g \in \mathfrak{g}$, $0 < n \leq m$, and $w \in V^{M(r)}(m-n)$.

Lemma 6.5. *Equations* (6.7) *and* (6.8) *determine unique well-defined linear maps* $\bar{\mathcal{Y}}^K_{-m} : M(p)\otimes M(q) \to (V^{M(r)})'(m)$ *for* $m \geq 0$.

Proof. The uniqueness follows because $V^{M(r)}(m)$ is spanned by vectors of the form $g(-n)w$ for $g \in \mathfrak{g}$, $0 < n \leq m$, and $w \in V^{M(r)}(m-n)$, so we need to check well-definedness. In fact, (6.7) and (6.8) a priori define maps from $M(p)\otimes M(q)$

into the graded dual of $T(\widehat{\mathfrak{g}}_-) \otimes M(r)$, where T denotes the tensor algebra. Since $V^{M(r)} \cong U(\widehat{\mathfrak{g}}_-) \otimes M(r)$ linearly, we need to show that for $u \in M(p)$, $v \in M(q)$, and $m \geq 0$,

$$\langle \bar{\mathcal{Y}}^K_{-m}(u)v, w\rangle = 0 \tag{6.9}$$

for $w \in T(\widehat{\mathfrak{g}}_-) \otimes M(r)$ of the form

$$w_1(g_1(-n_1)g_2(-n_2) - g_2(-n_2)g_1(-n_1) - [g_1, g_2](-n_1 - n_2))w_2,$$

where $w_1 \in T(\widehat{\mathfrak{g}}_-)$, $w_2 \in T(\widehat{\mathfrak{g}}_-) \otimes M(r)$, $g_1, g_2 \in \mathfrak{g}$, and $n_1, n_2 > 0$. We can prove (6.9) by induction on the degree of w_1. For the base case we take $w_1 = 1$ (and we may also take w_2 to be homogeneous of degree $m - n_1 - n_2$) to obtain

$$\begin{aligned}
&\langle \bar{\mathcal{Y}}^K_{-m}(u)v, [g_1(-n_1), g_2(-n_2)]w_2\rangle \\
&\quad = -\langle \bar{\mathcal{Y}}^K_{-m+n_1}(g_1(0)u)v, g_2(-n_2)w_2\rangle + \langle \bar{\mathcal{Y}}^K_{-m+n_2}(g_2(0)u)v, g_1(-n_1)w_2\rangle \\
&\quad = \langle \bar{\mathcal{Y}}^K_{-m+n_1+n_2}(g_2(0)g_1(0)u)v, w_2\rangle - \langle \bar{\mathcal{Y}}^K_{-m+n_1+n_2}(g_1(0)g_2(0)u)v, w_2\rangle \\
&\quad = -\langle \bar{\mathcal{Y}}^K_{-m+n_1+n_2}([g_1, g_2](0)u)v, w_2\rangle \\
&\quad = \langle \bar{\mathcal{Y}}^K_{-m}(u)v, [g_1, g_2](-n_1 - n_2)w_2\rangle,
\end{aligned}$$

as desired. Then the inductive step follows immediately because (6.8) implies that if (6.9) holds for all $u \in M(p)$, $v \in M(q)$, and $m \geq 0$, and for some specific $w \in T(\widehat{\mathfrak{g}}_-) \otimes M(r)$, then (6.9) continues to hold when w is replaced by $g(-n)w$ for any $g \in \mathfrak{g}$ and $n > 0$. □

Now that we know each $\bar{\mathcal{Y}}^K_{-m}$ maps into $(V^{M(r)})'(m)$, we can show that each $\bar{\mathcal{Y}}^K_{-m}$ is a $\mathfrak{g}$-module homomorphism:

Lemma 6.6. *For any $m \geq 0$, we have*

$$[g(0), \bar{\mathcal{Y}}^K_{-m}(u)]v = \bar{\mathcal{Y}}^K_{-m}(g(0)u)v$$

for $g \in \mathfrak{g}$, $u \in M(p)$, and $v \in M(q)$.

Proof. We prove by induction on m. The conclusion is true for $m = 0$ because $\bar{\mathcal{Y}}^K_0 = f$ is a $\mathfrak{g}$-module homomorphism. For $m > 0$, we have

$$\begin{aligned}
&\langle [g(0), \bar{\mathcal{Y}}^K_{-m}(u)]v, h(-n)w\rangle \\
&\quad = -\langle \bar{\mathcal{Y}}^K_{-m}(u)v, g(0)h(-n)w\rangle - \langle \bar{\mathcal{Y}}^K_{-m}(u)g(0)v, h(-n)w\rangle \\
&\quad = -\langle \bar{\mathcal{Y}}^K_{-m}(u)v, h(-n)g(0)w\rangle - \langle \bar{\mathcal{Y}}^K_{-m}(u)v, [g, h](-n)w\rangle \\
&\qquad + \langle \bar{\mathcal{Y}}^K_{-m+n}(h(0)u)g(0)v, w\rangle \\
&\quad = -\langle g(0)\bar{\mathcal{Y}}^K_{-m+n}(h(0)u)v, w\rangle + \langle \bar{\mathcal{Y}}^K_{-m+n}([g, h](0)u)v, w\rangle \\
&\qquad + \langle \bar{\mathcal{Y}}^K_{-m+n}(h(0)u)g(0)v, w\rangle \\
&\quad = -\langle [g(0), \bar{\mathcal{Y}}^K_{-m+n}(h(0)u)]v, w\rangle + \langle \bar{\mathcal{Y}}^K_{-m+n}([g(0), h(0)]u)v, w\rangle \\
&\quad = -\langle \bar{\mathcal{Y}}^K_{-m+n}(g(0)h(0)u)v, w\rangle + \langle \bar{\mathcal{Y}}^K_{-m+n}(g(0)h(0)u)v, w\rangle \\
&\qquad - \langle \bar{\mathcal{Y}}^K_{-m+n}(h(0)g(0)u)v, w\rangle \\
&\quad = \langle \bar{\mathcal{Y}}^K_{-m}(g(0)u)v, h(-n)w\rangle.
\end{aligned}$$

Since $V^{M(r)}(m)$ is spanned by vectors of the form $h(-n)w$, for $h \in \mathfrak{g}$, $0 < n \leq m$ and $w \in V^{M(r)}(m - n)$, the conclusion follows. □

Lemma 6.7. *For any $m \geq 0$, $u \in M(p)$, and $v \in M(q)$, $\bar{\mathcal{Y}}^K_{-m}(u)v \in J(r)^\perp$.*

Proof. By the previous lemma, $\bar{\mathcal{Y}}^K_{-m}$ is a $\mathfrak{g}$-module homomorphism from $M(p) \otimes M(q)$ to $(V^{M(r)})'(m)$. Also, as a $\mathfrak{g}$-module, the lowest weight space of $J(r) \cong L(r')$ is $M(r')$. Thus $\bar{\mathcal{Y}}^K_{-m}$ defines a $\mathfrak{g}$-module homomorphism f_m from $M(p) \otimes M(q)$ to $M(r')^* \cong M(r')$ via

$$\langle f_m(u \otimes v), w\rangle = \langle \bar{\mathcal{Y}}^K_{-m}(u)v, w\rangle$$

for $u \in M(p)$, $v \in M(q)$, and $w \in M(r')$. But the assumptions of Theorem 6.3 imply that there are no non-zero $\mathfrak{g}$-module homomorphisms from $M(p) \otimes M(q)$ to $M(r')$. Thus

$$\langle \bar{\mathcal{Y}}^K_{-m}(u)v, w\rangle = 0$$

for any $w \in M(r') = J(r')(M)$.

Now, if $w \in J(r)$ is a homogeneous vector such that $\langle \bar{\mathcal{Y}}^K_{-m}(u)v, w\rangle = 0$ for any $u \in M(p)$, $v \in M(q)$, and $m \geq 0$, then for any $g \in \mathfrak{g}$ and $n > 0$ such that $g(-n)w \in J(r)(m)$, we have

$$\langle \bar{\mathcal{Y}}^K_{-m}(u)v, g(-n)w\rangle = -\langle \bar{\mathcal{Y}}^K_{-m+n}(g(0)u)v, w\rangle = 0.$$

If $n > 0$ and $g(-n)w \notin J(r)(m)$, then $\langle \bar{\mathcal{Y}}^K_{-m}(u)v, g(-n)w\rangle = 0$ automatically. Since $J(r) \cong L(r')$, $J(r)$ is generated as a $\widehat{\mathfrak{g}}_-$-module by its lowest weight space $M(r')$. Thus $\langle \bar{\mathcal{Y}}^K_{-m}(u)v, w\rangle = 0$ for all $w \in J(r)$, and $\bar{\mathcal{Y}}^K_{-m}(u)v \in J(r)^\perp$. □

Now, by the previous lemma, we can use the $\mathfrak{g}$-module isomorphism $\Phi : J(r)^\perp \to K(r)$ given by Proposition 5.7 to identify $\bar{\mathcal{Y}}^K_{-m}(u)v$ with $\mathcal{Y}^K_{-m}(u)v = \Phi(\bar{\mathcal{Y}}^K_{-m}(u)v) \in K(r)$. By (5.2), (6.7), and (6.8), we have that $\mathcal{Y}^K_0(u)v = f(u \otimes v)$ and for $m > 0$,

$$\langle \mathcal{Y}^K_{-m}(u)v, g(-n)w\rangle_M = -\langle \mathcal{Y}^K_{-m+n}(g(0)u)v, w\rangle_M$$

for $u \in M(p)$, $v \in M(q)$ and $g(-n)w \in V^{M(r)}(m)$. Moreover, since Φ is a $\mathfrak{g}$-module homomorphism, we also have for $m \geq 0$

$$[g(0), \mathcal{Y}^K_{-m}(u)]v = \mathcal{Y}^K_{-m}(g(0)u)v \tag{6.10}$$

for $g \in \mathfrak{g}$, $u \in M(p)$, and $v \in M(q)$.

For $m < 0$, we set $\mathcal{Y}^K_{-m} = 0$. Then we have:

Lemma 6.8. *If $n > 0$, $P_K(g(n)\mathcal{Y}^K_{-m}(u)v) = \mathcal{Y}^K_{-m+n}(g(0)u)v$ for $u \in M(p)$, $v \in M(q)$, $g \in \mathfrak{g}$, and any $m \in \mathbb{Z}$.*

Proof. If $n > m$, both sides are zero. Otherwise, if $0 < n \leq m$ (so that $m > 0$), for any $w \in V^{M(r)}(m - n)$ we have

$$\begin{aligned}\langle P_K(g(n)\mathcal{Y}^K_{-m}(u)v), w\rangle_M &= \langle g(n)\mathcal{Y}^K_{-m}(u)v, w\rangle_M\\ &= -\langle \mathcal{Y}^K_{-m}(u)v, g(-n)w\rangle_M\\ &= \langle \mathcal{Y}^K_{-m+n}(g(0)u)v, w\rangle_M.\end{aligned}$$

Since $\langle \cdot, \cdot\rangle_M$ is non-degenerate on $K(r)$, and since both $P_K(g(n)\mathcal{Y}^K_{-m}(u)v)$ and $\mathcal{Y}^K_{-m+n}(g(0)u)v$ are in $K(r)(m - n)$, they are equal. □

Next, we want to construct maps $\bar{\mathcal{Y}}^J_{-m} : M(p) \otimes M(q) \to (V^{M(r')})'(m - M)$ for $m \geq M$. As before, we would like to define these maps recursively: $\bar{\mathcal{Y}}^J_{-M} = 0$ and for $m > M$,

$$\langle \bar{\mathcal{Y}}^J_{-m}(u)v, g(-n)w\rangle = \langle P_J(g(n)\mathcal{Y}^K_{-m}(u)v), w + J(r')\rangle_J - \langle \bar{\mathcal{Y}}^J_{-m+n}(g(0)u)v, w\rangle \tag{6.11}$$

for $u \in M(p)$, $v \in M(q)$, $g \in \mathfrak{g}$, $0 < n \leq m - M$, and $w \in V^{M(r')}(m - M - n)$; for the first term on the right, we are identifying the quotient $V^{M(r')}/J(r')$ with $J(r) \subseteq V^{M(r)}$.

Remark 6.9. Since we will need $\bar{\mathcal{Y}}^J_{-M}$ to be a $\mathfrak{g}$-homomorphism from $M(p) \otimes M(q)$ into $(V^{M(r')})'(0) \cong M(r')^* \cong M(r')$, and since by assumption there are no non-zero such homomorphisms, we are required to set $\bar{\mathcal{Y}}^J_{-M} = 0$.

As with the $\bar{\mathcal{Y}}^K_{-m}$, equation (6.11) defines a priori a map from $M(p) \otimes M(q)$ into the graded dual of $T(\widehat{\mathfrak{g}}_-) \otimes M(r')$, and to show that the $\bar{\mathcal{Y}}^J_{-m}$ define maps into $(V^{M(r')})'(m - M)$, we need to show that

$$\langle \bar{\mathcal{Y}}^J_{-m}(u)v, w\rangle = 0 \tag{6.12}$$

for all $u \in M(p)$, $v \in M(q)$, and $w \in T(\widehat{\mathfrak{g}}_-) \otimes M(r')$ of the form

$$w_1(g_1(-n_1)g_2(-n_2) - g_2(-n_2)g_1(-n_1) - [g_1, g_2](-n_1 - n_2))w_2,$$

where $w_1 \in T(\widehat{\mathfrak{g}}_-)$, $w_2 \in T(\widehat{\mathfrak{g}}_-) \otimes M(r')$, $g_1, g_2 \in \mathfrak{g}$, and $n_1, n_2 > 0$. For the case $w_1 = 1$, we have

$$\begin{aligned}
&\langle \bar{\mathcal{Y}}^J_{-m}(u)v, [g_1(-n_1), g_2(-n_2)]w_2\rangle \\
&\quad = \langle P_J(g_1(n_1)\mathcal{Y}^K_{-m}(u)v), g_2(-n_2)(w_2 + J(r'))\rangle_J - \langle \bar{\mathcal{Y}}^J_{-m+n_1}(g_1(0)u)v, g_2(-n_2)w_2\rangle \\
&\qquad - \langle P_J(g_2(n_2)\mathcal{Y}^K_{-m}(u)v), g_1(-n_1)(w_2 + J(r'))\rangle_J \\
&\qquad + \langle \bar{\mathcal{Y}}^J_{-m+n_2}(g_2(0)u)v, g_1(-n_1)w_2\rangle \\
&\quad = -\langle g_2(n_2)P_J(g_1(n_1)\mathcal{Y}^K_{-m}(u)v), w_2 + J(r')\rangle_J \\
&\qquad - \langle P_J(g_2(n_2)\mathcal{Y}^K_{-m+n_1}(g_1(0)u)v), w_2 + J(r')\rangle_J \\
&\qquad + \langle \bar{\mathcal{Y}}^J_{-m+n_1+n_2}(g_2(0)g_1(0)u)v, w_2\rangle + \langle g_1(n_1)P_J(g_2(n_2)\mathcal{Y}^K_{-m}(u)v), w_2 + J(r')\rangle_J \\
&\qquad + \langle P_J(g_1(n_1)\mathcal{Y}^K_{-m+n_2}(g_2(0)u)v), w_2 + J(r')\rangle_J \\
&\qquad - \langle \bar{\mathcal{Y}}^J_{-m+n_1+n_2}(g_1(0)g_2(0)u)v, w_2\rangle \\
&\quad = -\langle P_J(g_2(n_2)P_J(g_1(n_1)\mathcal{Y}^K_{-m}(u)v)) \\
&\qquad + P_J(g_2(n_2)P_K(g_1(n_1)\mathcal{Y}^K_{-m}(u)v)), w_2 + J(r')\rangle_J \\
&\qquad + \langle P_J(g_1(n_1)P_J(g_2(n_2)\mathcal{Y}^K_{-m}(u)v)) \\
&\qquad + P_J(g_1(n_1)P_K(g_2(n_2)\mathcal{Y}^K_{-m}(u)v)), w_2 + J(r')\rangle_J \\
&\qquad - \langle \bar{\mathcal{Y}}^J_{-m+n_1+n_2}([g_1(0), g_2(0)]u)v, w_2\rangle \\
&\quad = \langle P_J([g_1(n_1), g_2(n_2)]\mathcal{Y}^K_{-m}(u)v), w_2 + J(r')\rangle_J \\
&\qquad - \langle \bar{\mathcal{Y}}^J_{-m+n_1+n_2}([g_1, g_2](0)u)v, w_2\rangle \\
&\quad = \langle \bar{\mathcal{Y}}^J_{-m}(u)v, [g_1, g_2](-n_1 - n_2)w_2\rangle,
\end{aligned}$$

where we have used (6.11), the invariance of $\langle\cdot,\cdot\rangle_J$, the fact that $J(r)$ is a $\widehat{\mathfrak{g}}$-module, and Lemma 6.8. Then (6.12) for general w follows easily from (6.11) (note that the first term on the right of (6.11) is zero for any w containing a factor of $g_1(-n_1)g_2(-n_2) - g_2(-n_2)g_1(-n_1) - [g_1, g_2](-n_1 - n_2)$ since such a w equals zero in $V^{M(r')} \cong U(\widehat{\mathfrak{g}}_-) \otimes M(r')$).

Lemma 6.10. *For any $m \geq M$, we have*

$$[g(0), \bar{\mathcal{Y}}^J_{-m}(u)]v = \bar{\mathcal{Y}}^J_{-m}(g(0)u)v$$

for $g \in \mathfrak{g}$, $u \in M(p)$, and $v \in M(q)$.

Proof. We prove by induction on m. For $m = M$, the result is true because both sides of the equation are zero. Now for $m > M$, using the induction hypothesis, the fact that P_J is a $\mathfrak{g}$-module homomorphism, and (6.10), we have for $h \in \mathfrak{g}$, $0 < n \leq m - M$, and $w \in V^{M(r')}(m - M - n)$:

$$\begin{aligned}
\langle [g(0), \bar{\mathcal{Y}}^J_{-m}(u)]v,\, h(-n)w\rangle &= -\langle \bar{\mathcal{Y}}^J_{-m}(u)v, g(0)h(-n)w\rangle - \langle \bar{\mathcal{Y}}^J_{-m}(u)g(0)v, h(-n)w\rangle \\
&= -\langle \bar{\mathcal{Y}}^J_{-m}(u)v, h(-n)g(0)w\rangle - \langle \bar{\mathcal{Y}}^J_{-m}(u)v, [g,h](-n)w\rangle \\
&\quad + \langle \bar{\mathcal{Y}}^J_{-m+n}(h(0)u)g(0)v, w\rangle - \langle P_J(h(n)\mathcal{Y}^K_{-m}(u)g(0)v), w + J(r')\rangle_J \\
&= -\langle g(0)\bar{\mathcal{Y}}^J_{-m+n}(h(0)u)v, w\rangle + \langle g(0)P_J(h(n)\mathcal{Y}^K_{-m}(u)v), w + J(r')\rangle_J \\
&\quad + \langle \bar{\mathcal{Y}}^J_{-m+n}([g,h](0)u)v, w\rangle - \langle P_J([g,h](n)\mathcal{Y}^K_{-m}(u)v), w + J(r')\rangle_J \\
&\quad + \langle \bar{\mathcal{Y}}^J_{-m+n}(h(0)u)g(0)v, w\rangle - \langle P_J(h(n)\mathcal{Y}^K_{-m}(u)g(0)v), w + J(r')\rangle_J \\
&= -\langle [g(0), \bar{\mathcal{Y}}^J_{-m+n}(h(0)u)]v, w\rangle + \langle \bar{\mathcal{Y}}^J_{-m+n}([g(0), h(0)]u)v, w\rangle \\
&\quad + \langle P_J(g(0)h(n)\mathcal{Y}^K_{-m}(u)v - [g(0), h(n)]\mathcal{Y}^K_{-m}(u)v), w + J(r')\rangle_J \\
&\quad - \langle P_J(h(n)\mathcal{Y}^K_{-m}(u)g(0)v), w + J(r')\rangle_J \\
&= -\langle \bar{\mathcal{Y}}^J_{-m+n}(g(0)h(0)u)v, w\rangle + \langle \bar{\mathcal{Y}}^J_{-m+n}(g(0)h(0)u)v, w\rangle \\
&\quad - \langle \bar{\mathcal{Y}}^J_{-m+n}(h(0)g(0)u)v, w\rangle + \langle P_J(h(n)[g(0), \mathcal{Y}^K_{-m}(u)]v), w + J(r')\rangle_J \\
&= -\langle \bar{\mathcal{Y}}^J_{-m+n}(h(0)g(0)u)v, w\rangle + \langle P_J(h(n)\mathcal{Y}^K_{-m}(g(0)u)v), w + J(r')\rangle_J \\
&= \langle \bar{\mathcal{Y}}^J_{-m}(g(0)u)v, h(-n)w\rangle.
\end{aligned}$$

Since $V^{M(r')}(m-M)$ is spanned by the vectors $h(-n)w$ where $h \in \mathfrak{g}$, $0 < n \leq m-M$ and $w \in V^{M(r')}(m - M - n)$, the result follows. ☐

Lemma 6.11. *For any $m \geq M$, $u \in M(p)$, and $v \in M(q)$, $\bar{\mathcal{Y}}^J_{-m}(u)v \in J(r')^\perp$.*

Proof. The previous lemma implies that $\bar{\mathcal{Y}}^J_{-m}$ determines a $\mathfrak{g}$-module homomorphism from $M(p) \otimes M(q)$ to $M(r'')^* \cong M(r'')$, where $M(r'')$ is the lowest weight space of $J(r') \cong L(r'')$; this homomorphism is necessarily zero, just as in the proof of Lemma 6.7. Thus

$$\langle \bar{\mathcal{Y}}^J_{-m}(u)v, w\rangle = 0$$

for any $w \in M(r'') \subseteq J(r')$.

If $w \in J(r')$ is a homogeneous vector such that $\langle \bar{\mathcal{Y}}^J_{-m}(u)v, w\rangle = 0$ for any $u \in M(p)$, $v \in M(q)$, and $m \geq M$, then for any $g \in \mathfrak{g}$ and $n > 0$ such that $g(-n)w \in J(r')(m - M)$,

$$\langle \bar{\mathcal{Y}}^J_{-m}(u)v, g(-n)w\rangle = -\langle \bar{\mathcal{Y}}^J_{-m+n}(g(0)u)v, w\rangle + \langle P_J(g(n)\mathcal{Y}^K_{-m}(u)v), w + J(r')\rangle_J = 0$$

because $w + J(r') = 0$. If $g(-n)w \notin J(r')(m - M)$, then $\langle \bar{\mathcal{Y}}^J_{-m}(u)v, g(-n)w\rangle = 0$ automatically. As in the proof of Lemma 6.7, this proves the lemma. ☐

By this last lemma, $\bar{\mathcal{Y}}^J_{-m}(u)v$ for $m \geq M$ defines an element in $(V^{M(r')}/J(r'))' \cong L(r')' \cong J(r)'$. Since the non-degenerate form $\langle \cdot, \cdot \rangle_J$ identifies $J(r)'$ with $J(r)$,

$\bar{\mathcal{Y}}^J_{-m}(u)v$ induces a unique element $\mathcal{Y}^J_{-m}(u)v \in J(r)(m) \cong L(r')(m-M)$ such that $\mathcal{Y}^J_{-M}(u)v = 0$ and for $m > M$

$$\langle \mathcal{Y}^J_{-m}(u)v, g(-n)w\rangle_J = \langle P_J(g(n)\mathcal{Y}^K_{-m}(u)v), w\rangle_J - \langle \mathcal{Y}^J_{-m+n}(g(0)u)v, w\rangle_J$$

for any $g \in \mathfrak{g}$, $0 < n \leq m - M$, and $w \in J(r)(m-n)$; also

$$[g(0), \mathcal{Y}^J_{-m}(u)]v = \mathcal{Y}^J_{-m}(g(0)u)v. \tag{6.13}$$

We define $\mathcal{Y}^J_{-m}(u)v = 0$ for $m < M$, and so we have:

Lemma 6.12. *For any $m \in \mathbb{Z}$ and $n > 0$,*

$$P_J(g(n)\mathcal{Y}^K_{-m}(u)v) + g(n)\mathcal{Y}^J_{-m}(u)v = \mathcal{Y}^J_{-m+n}(g(0)u)v,$$

where $g \in \mathfrak{g}$, $u \in M(p)$, and $v \in M(q)$.

Proof. If $n > m - M$, both sides are zero. Otherwise, if $0 < n \leq m - M$ (so that $m > M$), then for any $w \in J(r)(m-n)$,

$$\begin{aligned}\langle g(n)\mathcal{Y}^J_{-m}(u)v, w\rangle_J &= -\langle \mathcal{Y}^J_{-m}(u)v, g(-n)w\rangle_J \\ &= \langle \mathcal{Y}^J_{-m+n}(g(0)u)v, w\rangle_J - \langle P_J(g(n)\mathcal{Y}^K_{-m}(u)v), w\rangle_J.\end{aligned}$$

Since $\langle \cdot, \cdot \rangle_J$ is non-degenerate on $J(r)(m-n)$, the result follows. □

Finally, we can define $\mathcal{Y}_m(u)v = \mathcal{Y}^K_m(u)v + \mathcal{Y}^J_m(u)v$ for any $m \in \mathbb{Z}$. We have $\mathcal{Y}_0(u)v = f(u \otimes v)$, and

$$[g(0), \mathcal{Y}_m(u)]v = \mathcal{Y}_m(g(0)u)v \tag{6.14}$$

by (6.10) and (6.13) (or by the fact that both sides are zero if $m > 0$). Also, by Lemmas 6.8 and 6.12, for $n > 0$,

$$\begin{aligned}[g(n), \mathcal{Y}_m(u)]v &= g(n)\mathcal{Y}_m(u)v \\ &= g(n)\mathcal{Y}^K_m(u)v + g(n)\mathcal{Y}^J_m(u)v \\ &= P_K(g(n)\mathcal{Y}^K_m(u)v) + P_J(g(n)\mathcal{Y}^K_m(u)v) + g(n)\mathcal{Y}^J_m(u)v \\ &= \mathcal{Y}^K_{m+n}(g(0)u)v + \mathcal{Y}^J_{m+n}(g(0)u)v \\ &= \mathcal{Y}_{m+n}(g(0)u)v.\end{aligned}$$

Thus we have proven the existence of the desired maps $\mathcal{Y}_m$, completing the proof of the theorem. □

Now using Theorems 6.2 and 6.3, we can complete the proof of Theorem 6.1.

Proof. If $\mathcal{Y} \in \mathcal{V}^{V^{M(r)}}_{V^{M(p)}\,V^{M(q)}}$, then (6.3) implies that $\mathcal{Y}_0|_{M(p)\otimes M(q)}$ gives an $\mathfrak{sl}(2,\mathbb{C})$-module homomorphism $f_{\mathcal{Y}}$ into $M(r)$. Conversely, Theorems 6.2 and 6.3 give us a unique intertwining operator $\mathcal{Y}_f$ for each $f \in \mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(M(p) \otimes M(q), M(r))$ such that $(\mathcal{Y}_f)_0(u)v = f(u \otimes v)$ for $u \otimes v \in M(p) \otimes M(q)$. So $f_{\mathcal{Y}_f} = f$, and the uniqueness assertions in Theorems 6.2 and 6.3 guarantee that $\mathcal{Y}_{f_{\mathcal{Y}}} = \mathcal{Y}$. Thus we have a linear isomorphism between $\mathcal{V}^{V^{M(r)}}_{V^{M(p)}\,V^{M(q)}}$ and $\mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(M(p) \otimes M(q), M(r))$. □

7. Intertwining operators of type $\binom{L(r)}{V^{M(p)}\ L(q)}$

In this section we continue to take $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ and continue to fix a level $\ell \in \mathbb{N}$. We prove that under certain circumstances, an intertwining operator of type $\binom{V^{M(r)}}{V^{M(p)}\ V^{M(q)}}$ descends to an intertwining operator of type $\binom{L(r)}{V^{M(p)}\ L(q)}$.

Theorem 7.1. *Assume the conditions of Theorem* 6.3 *hold and that in addition $J(q)$ is generated by its lowest weight space $M(q')$ and that there are no non-zero $\mathfrak{g}$-homomorphisms from $M(p) \otimes M(r)$ to $M(q')$. Then there is a natural linear isomorphism $\mathcal{V}^{L(r)}_{V^{M(p)}\ L(q)} \cong \mathcal{V}^{V^{M(r)}}_{V^{M(p)}\ V^{M(q)}}$.*

Proof. We want to show that if $\mathcal{Y}$ is an intertwining operator of type $\binom{V^{M(r)}}{V^{M(p)}\ V^{M(q)}}$, then we obtain a well-defined intertwining operator $\mathcal{Y}^L$ of type $\binom{L(r)}{V^{M(p)}\ L(q)}$ defined by

$$\mathcal{Y}^L(u,x)(v + J(q)) = \mathcal{Y}(u,x)v + J(r). \tag{7.1}$$

Then the linear map $\mathcal{Y} \mapsto \mathcal{Y}^L$ will be our desired isomorphism. It will be injective because both $\mathcal{Y}$ and $\mathcal{Y}^L$ define the same $\mathfrak{g}$-homomorphism

$$\mathcal{Y}_0 = \mathcal{Y}_0^L : M(p) \otimes M(q) \to M(r),$$

and by Theorem 6.1, $\mathcal{Y}$ is uniquely determined by this homomorphism. Moreover, the map will be surjective because an argument similar to but simpler than the uniqueness argument in the proof of Theorem 6.3 shows that any intertwining operator $\mathcal{Y}^L$ of type $\binom{L(r)}{V^{M(p)}\ L(q)}$ is also determined by such a homomorphism $M(p)\otimes M(q) \to M(r)$, and so the fusion rule $\mathcal{N}^{L(r)}_{V^{M(p)}\ L(q)}$ is no more than $\mathcal{N}^{V^{M(r)}}_{V^{M(p)}\ V^{M(q)}}$.

Because $\mathcal{Y}$ is an intertwining operator, $\mathcal{Y}^L$ will satisfy lower truncation, the Jacobi identity, and the $L(-1)$-derivative property, provided that it is well defined. To prove well-definedness, we need to show that if $u \in V^{M(p)}$ and $v \in J(q)$, then $\mathcal{Y}(u,x)v \in J(r)\{x\}$.

Lemma 7.2. *The subspace W of $V^{M(r)}$ spanned by the coefficients of products of the form $\mathcal{Y}(u,x)v$ where $u \in V^{M(p)}$ and $v \in J(q)$ is a $\widehat{\mathfrak{g}}$-submodule of $V^{M(r)}$.*

Proof. If $g \in \mathfrak{g}$, $n \in \mathbb{Z}$, $u \in V^{M(p)}$, and $v \in J(q)$, then from the commutator formula,

$$g(n)\mathcal{Y}(u,x)v = \mathcal{Y}(u,x)g(n)v + \sum_{i\geq 0}\binom{n}{i}x^{n-i}\mathcal{Y}(g(i)u,x)v.$$

Since all coefficients on the right side are in W, the result follows. □

Lemma 7.3. *The intersection of W with the lowest weight space of $V^{M(r)}$ is trivial.*

Proof. If we use $\langle\cdot,\cdot\rangle_M$ to denote a non-zero $\widehat{\mathfrak{g}}$-invariant bilinear form on $V^{M(r)}$ as in Proposition 5.4, then it is enough to show that

$$\langle\mathcal{Y}(u,x)v, w\rangle_M = 0 \tag{7.2}$$

for any $u \in V^{M(p)}$, $v \in J(q)$, and $w \in M(r)$, since $\langle\cdot,\cdot\rangle_M$ gives a non-degenerate bilinear form on $M(r)$. We first assume $u \in M(p)$ and $v \in M(q')$ (the lowest weight space of $J(q)$). Then for each $n \in \mathbb{C}$, we have a map $F_n : M(p)\otimes M(r) \to M(q')^* \cong M(q')$ given by

$$\langle F_n(u\otimes w), v\rangle = \langle\mathcal{Y}_n(u)v, w\rangle_M$$

for $u \in M(p)$, $v \in M(q')$, and $w \in M(r)$. These maps are $\mathfrak{g}$-homomorphisms because if $g \in \mathfrak{g}$,

$$\begin{aligned}\langle g \cdot F_n(u \otimes w), v\rangle &= -\langle F_n(u \otimes w), g(0)v\rangle = -\langle \mathcal{Y}_n(u)g(0)v, w\rangle_M \\ &= -\langle g(0)\mathcal{Y}_n(u)v, w\rangle_M + \langle \mathcal{Y}_n(g(0)u)v, w\rangle_M \\ &= \langle \mathcal{Y}_n(u)v, g(0)w\rangle_M + \langle \mathcal{Y}_n(g(0)u)v, w\rangle_M \\ &= \langle F_n(g(0)u \otimes w + u \otimes g(0)w), v\rangle.\end{aligned}$$

But by assumption there are no non-zero homomorphisms from $M(p) \otimes M(r)$ to $M(q')$, so therefore

$$\langle \mathcal{Y}(u, x)v, w\rangle_M = 0$$

when $u \in M(p)$ and $v \in M(q')$. Now we suppose that (7.2) holds when $u \in M(p)$ and $w \in M(r)$ for some $v \in J(q)$ and show that it also holds for $g(-n)v$ where $g \in \mathfrak{g}$ and $n > 0$; this will show that (7.2) holds for all $v \in J(q)$ since $J(q)$ is generated as a $\widehat{\mathfrak{g}}_-$-module by its lowest weight space. Thus:

$$\langle \mathcal{Y}(u, x)g(-n)v, w\rangle_M = \langle g(-n)\mathcal{Y}(u, x)v, w\rangle_M - \langle \mathcal{Y}(g(0)u, x)v, w\rangle_M = 0$$

since $g(-n)\mathcal{Y}(u, x)v$ has no component in the lowest weight space since $n > 0$. Finally, we must prove that if (7.2) holds for any $v \in J(q)$, $w \in M(r)$, and for some $u \in V^{M(p)}$, then it also holds for $g(-n)u$ when $g \in \mathfrak{g}$ and $n > 0$:

$$\begin{aligned}\langle \mathcal{Y}(g(-n)u, x)v, w\rangle_M &= \mathrm{Res}_{x_1}(x_1 - x)^{-n}\langle g(x_1)\mathcal{Y}(u, x)v, w\rangle_M \\ &\quad - \mathrm{Res}_{x_1}(-x + x_1)^{-n}\langle \mathcal{Y}(u, x)g(x_1)v, w\rangle_M \\ &= \sum_{i\geq 0}\binom{-n}{i}(-x)^i\langle g(-n-i)\mathcal{Y}(u, x)v, w\rangle_M + 0 = 0\end{aligned}$$

by the assumptions on u and since $n + i > 0$. □

Thus we have shown that W is a submodule of $V^{M(r)}$ such that $W \cap M(r) = 0$. This means that W is a proper submodule contained in $J(r)$, and so $\mathcal{Y}(u, x)v \subseteq J(r)\{x\}$ for any $u \in V^{M(p)}$, $v \in J(q)$, completing the proof of the theorem. □

Remark 7.4. Applying skew-symmetry (see for example [FHL]) to Theorem 7.1 immediately yields a corresponding theorem on intertwining operators of type $\binom{L(r)}{L(p)\ V^{M(q)}}$.

8. Examples and counterexamples

In this section we fix a level $\ell \in \mathbb{N}$ for $\widehat{\mathfrak{sl}(2, \mathbb{C})}$ and use Theorems 6.1 and 7.1 to exhibit new intertwining operators among $V^{M(0)}$-modules. We also show by counterexample that the conditions of Theorem 6.3 are necessary in order for the conclusions to hold.

First we need to determine explicitly which generalized Verma modules appear in the Garland-Lepowsky resolutions of standard $\widehat{\mathfrak{sl}(2, \mathbb{C})}$-modules. Recalling notation from Sections 2 and 3, we consider the action of the subset $\mathcal{W}^1$ of the Weyl group on dominant integral weights of $\widehat{\mathfrak{sl}(2, \mathbb{C})}$, which have the form

$$\Lambda = n\frac{\alpha}{2} + \ell\mathbf{k}'$$

for $\ell \in \mathbb{N}$ and $0 \leq n \leq \ell$. (Note that for the purposes of studying $\widehat{\mathfrak{sl}(2, \mathbb{C})}$-modules, the coefficient of $\mathbf{d}'$ in Λ is not important.) We recall the action of the Weyl group generators r_0 and r_1 on $\mathfrak{H}^*$ from (2.2) and note that since $\mathbf{d}'$ is fixed by $\mathcal{W}$, the action of $\mathcal{W}$ on $\mathfrak{H}^*$ projects to an action of $\mathcal{W}$ on $\mathbb{C}\alpha \oplus \mathbb{C}\mathbf{k}'$. Using (2.2) above, it is easy to prove by induction on m:

Lemma 8.1. *Suppose $\Lambda = n\frac{\alpha}{2} + \ell\mathbf{k}'$ is a dominant integral weight of $\widehat{\mathfrak{sl}(2,\mathbb{C})}$. Then for $m \geq 0$, we have*

$$(r_0 r_1)^m r_0 (\Lambda + \rho) - \rho = (2((m+1)(\ell+2) - 1) - n)\,\frac{\alpha}{2} + \ell\mathbf{k}',$$

and for $m \geq 1$ we have

$$(r_0 r_1)^m (\Lambda + \rho) - \rho = (2m(\ell+2) + n)\,\frac{\alpha}{2} + \ell\mathbf{k}'.$$

Using the fact that $w_j \in \mathcal{W}^1$ is given by $(r_0r_1)^m r_0$ if $j = 2m+1$ is odd and by $(r_0r_1)^m$ if $j = 2m$ is even, we obtain:

Proposition 8.2. *If $\Lambda = n\frac{\alpha}{2} + \ell\mathbf{k}'$ is a dominant integral weight of $\widehat{\mathfrak{sl}(2,\mathbb{C})}$, then for $j \geq 0$ we have*

$$w_j(\Lambda+\rho) - \rho = \left((\ell+2)j + \frac{\ell}{2}(1-(-1)^j) + (-1)^j n\right)\frac{\alpha}{2} + \ell\mathbf{k}'.$$

Thus the resolution of $L(\Lambda)$ given by Theorem 3.7 is

$$(8.1)\qquad \cdots \to V^{M((\ell+2)j+\ell(1-(-1)^j)/2+(-1)^j n,\, \ell)} \xrightarrow{d_j} \cdots \xrightarrow{d_2} V^{M(2(\ell+1)-n,\,\ell)} \xrightarrow{d_1} V^{M(\Lambda)} \xrightarrow{\Pi_\Lambda} L(\Lambda) \to 0.$$

For $j \in \mathbb{N}$ and $0 \leq n \leq \ell$, we use the notation

$$(8.2)\qquad m(j,n) = (\ell+2)j + \frac{\ell}{2}(1-(-1)^j) + (-1)^j n.$$

We can now apply Theorem 6.1 to the case that $V^{M(r)}$ appears in the resolution of a level ℓ standard $\widehat{\mathfrak{sl}(2,\mathbb{C})}$-module given by Theorem 3.7. In particular:

Theorem 8.3. *Suppose $p, q \geq 0$ and*

$$m(j+1,n), m(j+2,n) \notin \{p+q, p+q-2, \ldots, |p-q|\}.$$

Then there is a linear isomorphism

$$\mathcal{V}_{V^{M(p)}\,V^{M(q)}}^{V^{M(m(j,n))}} \cong \mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(M(p)\otimes M(q), M(m(j,n))).$$

In particular,

$$\mathcal{N}_{V^{M(p)}\,V^{M(q)}}^{V^{M(m(j,n))}} = \begin{cases} 1 & \text{if} \quad m(j,n) \in \{p+q, p+q-2, \ldots, |p-q|\}, \\ 0 & \text{if} \quad m(j,n) \notin \{p+q, p+q-2, \ldots, |p-q|\}. \end{cases}$$

Proof. By Theorem 3.8 and the resolution (8.1), $J(m(j,n))$ is irreducible and isomorphic to $L(m(j+1,n))$, and $J(m(j+1,n))$ is irreducible and isomorphic to $L(m(j+2,n))$. Thus the conclusions follow from Theorem 6.1 and the fact that as an $\mathfrak{sl}(2,\mathbb{C})$-module,

$$(8.3)\qquad M(p)\otimes M(q) \cong M(p+q) \oplus M(p+q-2) \oplus \ldots \oplus M(|p-q|)$$

for any $p, q \geq 0$. □

Example 8.4. Suppose we take $p = 1$, $q = m(j, n)$, and $r = m(j, n + 1)$ for some $j \geq 0$ and $0 \leq n \leq \ell - 1$. Since

$$m(j, n + 1) = m(j, n) \pm 1,$$

there is a one-dimensional space of $\mathfrak{sl}(2, \mathbb{C})$-homomorphisms from $M(1) \otimes M(m(j, n))$ into $M(m(j, n+1))$. Moreover, one can check directly from the definition of $m(j, n)$ that

$$m(j, n) + 1 < m(j + 1, n + 1), m(j + 2, n + 1).$$

Thus we can conclude from Theorem 8.3 that

$$\mathcal{N}_{V^{M(1)}\, V^{M(m(j,n))}}^{V^{M(m(j,n+1))}} = 1.$$

When $j = n = 0$, these are simply the intertwining operators of type $\binom{V^{M(1)}}{V^{M(1)}\, V^{M(0)}}$, and they are obtained from multiples of the vertex operator of $V^{M(0)}$ acting on $V^{M(1)}$ by skew-symmetry (see for example [FHL]). However, when either j or n is positive, these intertwining operators are new.

We can also apply Theorem 7.1 to the case that $r = m(j, n)$ and $q = m(j', n')$ for some $j, j' \geq 0$ and $0 \leq n, n' \leq \ell$:

Theorem 8.5. *Suppose the conditions of Theorem* 8.3 *hold and that in addition*

$$m(j' + 1, n') \notin \{p + m(j, n), p + m(j, n) - 2, \ldots, |p - m(j, n)|\}.$$

Then as vector spaces, $\mathcal{V}_{V^{M(p)}\, V^{M(m(j',n'))}}^{V^{M(m(j,n))}} \cong \mathcal{V}_{V^{M(p)}\, L(m(j',n'))}^{L(m(j,n))}$. *In particular,*

$$\mathcal{N}_{V^{M(p)}\, L(m(j',n'))}^{L(m(j,n))} = \begin{cases} 1 & \text{if} \quad m(j, n) \in \{p + m(j', n'), p + m(j', n') - 2, \ldots, |p - m(j', n')|\}, \\ 0 & \text{if} \quad m(j, n) \notin \{p + m(j', n'), p + m(j', n') - 2, \ldots, |p - m(j', n')|\}. \end{cases}$$

Proof. The theorem follows immediately from Theorems 8.3 and 7.1. To apply Theorem 7.1, we recall from Theorem 3.8 and (8.1) that $J(m(j', n'))$ is irreducible and thus generated by its lowest weight space $M(m(j' + 1, n'))$. □

Example 8.6. As in Example 8.4, we consider $p = 1$, $q = m(j, n)$, and $r = m(j, n + 1)$ for some $j \geq 0$ and $0 \leq n \leq \ell - 1$. From Example 8.4, we know that $\mathcal{N}_{V^{M(1)}\, V^{M(m(j,n))}}^{V^{M(m(j,n+1))}} = 1$. It can be checked directly from the definition (8.2) of $m(j, n)$ that

$$m(j + 1, n) > m(j, n + 1) + 1.$$

Thus by Theorem 8.5,

$$\mathcal{N}_{V^{M(1)}\, L(m(j,n))}^{L(m(j,n+1))} = 1$$

as well. Note that when $j = n = 0$, these intertwining operators can be obtained by skew-symmetry from intertwining operators of type $\binom{L(1)}{L(0)\, V^{M(1)}}$. Since $L(1)$ is a module for the simple vertex operator algebra $L(0)$ when $\ell \geq 1$ (see for example [LL]), a basis for the one-dimensional space of intertwining operators of type $\binom{L(1)}{L(0)\, V^{M(1)}}$ is $Y_{L(1)} \circ (1_{L(0)} \otimes \Pi_1)$ where Π_1 denotes the canonical projection from $V^{M(1)}$ to $L(1)$. As in Example 8.4, the intertwining operators discussed in this example are new when j or n is positive.

We conclude this section by showing with a counterexample that the conclusions of Theorem 6.3 may not hold if the conditions do not hold. Given a level $\ell \in \mathbb{N}$, we take $M(p) = M(q) = M(\ell+1)$ and $M(r) = M(0) = \mathbb{C}\mathbf{1}$. There is a one-dimensional space of $\mathfrak{sl}(2,\mathbb{C})$-module homomorphisms $M(\ell+1)\otimes M(\ell+1) \to M(0)$ spanned by a non-degenerate invariant bilinear form on $M(\ell+1)$. From (8.1), the maximal proper submodule of $V^{M(0)}$ is generated by $M(2\ell+2)$. Thus the conditions of Theorem 6.3 do not hold because there is a one-dimensional space of $\mathfrak{sl}(2,\mathbb{C})$-module homomorphisms $M(\ell+1)\otimes M(\ell+1) \to M(2\ell+2)$.

Now suppose $f : M(\ell+1)\otimes M(\ell+1) \to M(0)$ is an $\mathfrak{sl}(2,\mathbb{C})$-module homomorphism and we have maps

$$\mathcal{Y}_m : M(\ell+1)\otimes M(\ell+1) \to V^{M(0)}(-m)$$

for $m \in \mathbb{Z}$ such that $\mathcal{Y}_0 = f$ and (6.3) holds. Since we know (see for example [FZ] or [LL]) that $e(-1)^{\ell+1}\mathbf{1}$ generates the maximal proper submodule $J(0)$, Proposition 5.6 implies

$$\langle \mathcal{Y}_{-\ell-1}(u)v, e(-1)^{\ell+1}\mathbf{1}\rangle_M = 0$$

for $u, v \in M(\ell+1)$. By (6.3), this means

$$(-1)^{\ell+1}\langle \mathcal{Y}_0(e(0)^{\ell+1}u)v, \mathbf{1}\rangle_M = (-1)^{\ell+1}\langle f((e^{\ell+1}\cdot u)\otimes v), \mathbf{1}\rangle_M = 0.$$

If we take $u = v$ to be a lowest weight vector in $M(\ell+1)$, so that $e^{\ell+1}\cdot u$ is a (non-zero) highest weight vector, we see that f must be zero. Thus we cannot construct maps $\mathcal{Y}_m$ satisfying (6.3) for non-zero f.

Interestingly, we do still have a one-dimensional space of intertwining operators of type $\binom{V^{M(0)}}{V^{M(\ell+1)}\,V^{M(\ell+1)}}$, but the image of these operators is contained in the maximal proper submodule $J(0)$. To construct them, note that Theorem 6.1 implies that there is a one-dimensional space of intertwining operators of type $\binom{V^{M(2\ell+2)}}{V^{M(\ell+1)}\,V^{M(\ell+1)}}$ induced by the one-dimensional space of $\mathfrak{sl}(2,\mathbb{C})$-module homomorphisms

$$M(\ell+1)\otimes M(\ell+1) \to M(2\ell+2).$$

Then if $\mathcal{Y} \in \mathcal{V}^{V^{M(2\ell+2)}}_{V^{M(\ell+1)}\,V^{M(\ell+1)}}$ is non-zero, so is $d_1 \circ \mathcal{Y} \in \mathcal{V}^{V^{M(0)}}_{V^{M(\ell+1)}\,V^{M(\ell+1)}}$, where d_1 is as in (8.1).

9. Appendix: Proof of Theorem 6.2

In this section $\mathfrak{g}$ is an arbitrary finite-dimensional complex simple Lie algebra, and we fix a level $\ell \in \mathbb{N}$. We will heavily use results from formal calculus, especially delta function properties, in our proof of Theorem 6.2. These results will typically be used without comment; see [FLM], Chapters 2 and 8, and [LL], Chapter 2, for details on formal calculus.

We now proceed with the proof of Theorem 6.2; we recall that $M(\lambda_1)$ and $M(\lambda_2)$ are irreducible finite-dimensional $\mathfrak{g}$-modules and W is a $V^{M(0)}$-module. We assume that we have a map

$$\mathcal{Y} : M(\lambda_1)\otimes M(\lambda_2) \to W\{x\}$$

satisfying (6.1) and (6.2). We need to show that $\mathcal{Y}$ has a unique extension to an intertwining operator of type $\binom{W}{V^{M(\lambda_1)}\,V^{M(\lambda_2)}}$.

First we extend $\mathcal{Y}$ to a map on $M(\lambda_1)\otimes V^{M(\lambda_2)}$. Assuming that $\mathcal{Y}(u,x)v$ has been defined for $u \in M(\lambda_1)$ and some $v \in V^{M(\lambda_2)}$, we must define

$$\mathcal{Y}(u,x)h(-n)v = h(-n)\mathcal{Y}(u,x)v - x^{-n}\mathcal{Y}(h(0)u,x)v \tag{9.1}$$

for any $h \in \mathfrak{g}$, $n > 0$, in order for the commutator formula to hold. We need to verify that this formula determines a unique well-defined map

$$\mathcal{Y} : M(\lambda_1) \otimes V^{M(\lambda_2)} \to W\{x\}.$$

Uniqueness follows because $V^{M(\lambda_2)} \cong U(\widehat{\mathfrak{g}}_-) \otimes M(\lambda_2)$ linearly. To verify that $\mathcal{Y}$ is well defined, note that, similar to the definition of the $\bar{\mathcal{Y}}^K_{-m}$ and $\bar{\mathcal{Y}}^J_{-m}$ in the proof of Theorem 6.3, (9.1) defines a priori a map from $M(\lambda_1) \otimes (T(\widehat{\mathfrak{g}}_-) \otimes M(\lambda_2))$ into $W\{x\}$, where T denotes the tensor algebra. Thus we need to show that

$$\mathcal{Y}(u,x)v_1(h_1(-n_1)h_2(-n_2) - h_2(-n_2)h_1(-n_1) - [h_1,h_2](-n_1-n_2))v_2 = 0 \tag{9.2}$$

for $u \in M(\lambda_1)$, $h_1, h_2 \in \mathfrak{g}$, $n_1, n_2 > 0$, $v_1 \in T(\widehat{\mathfrak{g}}_-)$ and $v_2 \in T(\widehat{\mathfrak{g}}_-) \otimes M(\lambda_2)$. For the case $v_1 = 1$, we have

$$\begin{aligned}
&\mathcal{Y}(u,x)h_1(-n_1)h_2(-n_2)v_2\\
&\quad = h_1(-n_1)\mathcal{Y}(u,x)h_2(-n_2)v_2 - x^{-n_1}\mathcal{Y}(h_1(0)u,x)h_2(-n_2)v_2\\
&\quad = h_1(-n_1)h_2(-n_2)\mathcal{Y}(u,x)v_2 - x^{-n_2}h_1(-n_1)\mathcal{Y}(h_2(0)u,x)v_2\\
&\qquad - x^{-n_1}h_2(-n_2)\mathcal{Y}(h_1(0)u,x)v_2 + x^{-n_1-n_2}\mathcal{Y}(h_2(0)h_1(0)u,x)v_2,
\end{aligned}$$

and we have a similar expression for $\mathcal{Y}(u,x)h_2(-n_2)h_1(-n_1)v_2$ obtained by exchanging h_1 with h_2 and n_1 with n_2. Then

$$\begin{aligned}
&\mathcal{Y}(u,x)[h_1(-n_1),h_2(-n_2)]v_2\\
&\quad = [h_1(-n_1),h_2(-n_2)]\mathcal{Y}(u,x)v_2 + x^{-n_1-n_2}\mathcal{Y}([h_2(0),h_1(0)]u,x)v_2\\
&\quad = [h_1,h_2](-n_1-n_2)\mathcal{Y}(u,x)v_2 - x^{-n_1-n_2}\mathcal{Y}([h_1,h_2](0)u,x)v_2\\
&\quad = \mathcal{Y}(u,x)][h_1,h_2](-n_1-n_2)v_2,
\end{aligned}$$

as desired. Then (9.2) for all v_1 follows easily from (9.1) by induction on the degree of v_1.

We next show that this $\mathcal{Y}$ is lower truncated. Suppose the lowest conformal weights of $V^{M(\lambda_1)}$, $V^{M(\lambda_2)}$ and W are h_1, h_2, and h_3 respectively. (We may assume without loss of generality that all conformal weights of W are congruent mod $\mathbb{Z}$ to some lowest weight h_3.) Then by (6.2), if $v \in M(\lambda_2)$,

$$\mathcal{Y}(u,x)v = \sum_{m\in\mathbb{Z}} \mathcal{Y}_m(u)v\, x^{-m-h},$$

where $h = h_1 + h_2 - h_3$, and $\mathcal{Y}_m(u)v \in W(-m)$. Thus truncation holds when $v \in M(\lambda_2)$, and if we assume that it holds for any $u \in M(\lambda_1)$ and some particular $v \in V^{M(\lambda_2)}$, then the definition (9.1) shows that $\mathcal{Y}(u,x)h(-n)v$ is also lower truncated. This proves lower truncation since $V^{M(\lambda_2)}$ is generated by $M(\lambda_2)$ as $\widehat{\mathfrak{g}}_-$-module.

We now verify the commutator formula

$$[g(m),\mathcal{Y}(u,x)]v = x^m\mathcal{Y}(g(0)u,x)v \tag{9.3}$$

for any $g \in \mathfrak{g}$ and $m \in \mathbb{Z}$. By the definition this holds for $m < 0$, and for $m \geq 0$ it holds for $v \in M(\lambda_2)$ by (6.1). Thus it is enough to suppose that (9.3) holds for some $v \in V^{M(\lambda_2)}$ and then show that it also holds for $h(-n)v$ where $h \in \mathfrak{g}$ and $n > 0$:

$$\begin{aligned}
[g(m),\mathcal{Y}(u,x)]h(-n)v &= g(m)\mathcal{Y}(u,x)h(-n)v - \mathcal{Y}(u,x)g(m)h(-n)v\\
&= g(m)h(-n)\mathcal{Y}(u,x)v - g(m)x^{-n}\mathcal{Y}(h(0)u,x)v - \mathcal{Y}(u,x)g(m)h(-n)v\\
&= h(-n)g(m)\mathcal{Y}(u,x)v + ([g,h](m-n) + m\langle g,h\rangle\delta_{m-n,0}\ell)\mathcal{Y}(u,x)v
\end{aligned}$$

$$\begin{aligned}
&- x^{-n}\mathcal{Y}(h(0)u,x)g(m)v - x^{m-n}\mathcal{Y}(g(0)h(0)u,x)v - \mathcal{Y}(u,x)g(m)h(-n)v\\
&= h(-n)\mathcal{Y}(u,x)g(m)v + h(-n)x^m\mathcal{Y}(g(0)u,x)v + \mathcal{Y}(u,x)[g(m),h(-n)]v\\
&\quad + x^{m-n}\mathcal{Y}([g(0),h(0)]u,x)v\\
&\quad - x^{-n}\mathcal{Y}(h(0)u,x)g(m)v - x^{m-n}\mathcal{Y}(g(0)h(0)u,x)v\\
&\quad - \mathcal{Y}(u,x)g(m)h(-n)v\\
&= (h(-n)\mathcal{Y}(u,x) - \mathcal{Y}(u,x)h(-n) - x^{-n}\mathcal{Y}(h(0)u,x))g(m)v\\
&\quad + h(-n)x^m\mathcal{Y}(g(0)u,x)v - x^{m-n}\mathcal{Y}(h(0)g(0)u,x)v\\
&= x^m\mathcal{Y}(g(0)u,x)h(-n)v.
\end{aligned}$$

We also need to prove the $L(0)$-commutator formula

$$[L(0),\mathcal{Y}(u,x)]v = x\frac{d}{dx}\mathcal{Y}(u,x)v + \mathcal{Y}(L(0)u,x)v$$

for any $u \in M(\lambda_1)$ and $v \in V^{M(\lambda_2)}$. The formula holds for $v \in M(\lambda_2)$ by (6.2), so it is enough to assume that it holds for some $v \in V^{M(\lambda_2)}$ and then show that it holds for any $h(-n)v$:

$$\begin{aligned}
&\left(L(0) - x\frac{d}{dx}\right)\mathcal{Y}(u,x)h(-n)v\\
&\quad = \left(L(0) - x\frac{d}{dx}\right)(h(-n)\mathcal{Y}(u,x)v - x^{-n}\mathcal{Y}(h(0)u,x)v)\\
&\quad = h(-n)\left(L(0) - x\frac{d}{dx}\right)\mathcal{Y}(u,x)v - x^{-n}\left(L(0) - x\frac{d}{dx}\right)\mathcal{Y}(h(0)u,x)v\\
&\qquad + [L(0),h(-n)]\mathcal{Y}(u,x)v - nx^{-n}\mathcal{Y}(h(0)u,x)v\\
&\quad = h(-n)(\mathcal{Y}(L(0)u,x)v + \mathcal{Y}(u,x)L(0)v + n\mathcal{Y}(u,x)v)\\
&\qquad - x^{-n}(\mathcal{Y}(L(0)h(0)u,x)v + \mathcal{Y}(h(0)u,x)L(0)v + n\mathcal{Y}(h(0)u,x)v)\\
&\quad = \mathcal{Y}(L(0)u,x)h(-n)v + \mathcal{Y}(u,x)h(-n)L(0)v + n\mathcal{Y}(u,x)h(-n)v\\
&\quad = \mathcal{Y}(L(0)u,x)h(-n)v + \mathcal{Y}(u,x)L(0)h(-n)v. \qquad (9.4)
\end{aligned}$$

We also want to verify that $\mathcal{Y}$ satisfies the following iterate formula for $u \in M(\lambda_1)$, $v \in V^{M(\lambda_2)}$, $g \in \mathfrak{g}$, and $m \geq 0$:

$$\begin{aligned}
\mathcal{Y}(g(m)u,x) &= \mathrm{Res}_{x_1}((x_1-x)^m g(x_1)\mathcal{Y}(u,x) - (-x+x_1)^m\mathcal{Y}(u,x)g(x_1)\\
&= \mathrm{Res}_{x_1}(x_1-x)^m[g(x_1),\mathcal{Y}(u,x)]. \qquad (9.5)
\end{aligned}$$

Now, since

$$\begin{aligned}
[g(x_1),\mathcal{Y}(u,x)] &= \sum_{n\in\mathbb{Z}}[g(n),\mathcal{Y}(u,x)]x_1^{-n-1}\\
&= \sum_{n\in\mathbb{Z}} x^n x_1^{-n-1}\mathcal{Y}(g(0)u,x) = x_1^{-1}\delta\left(\frac{x}{x_1}\right)\mathcal{Y}(g(0)u,x)
\end{aligned}$$

and

$$(x_1-x)^m x_1^{-1}\delta\left(\frac{x}{x_1}\right) = (x-x)^m x_1^{-1}\delta\left(\frac{x}{x_1}\right)$$

if $m \geq 0$, we have that

$$\mathrm{Res}_{x_1}(x_1-x)^m[g(x_1),\mathcal{Y}(u,x)] = \left\{\begin{array}{lll} \mathcal{Y}(g(0)u,x) & \text{if} & m=0,\\ 0 & \text{if} & m>0, \end{array}\right.$$

as desired since $g(m)u = 0$ for $m > 0$.

Now that we have extended $\mathcal{Y}$ to $M(\lambda_1) \otimes V^{M(\lambda_2)}$, we want to extend $\mathcal{Y}$ to $V^{M(\lambda_1)} \otimes V^{M(\lambda_2)}$: assuming that $\mathcal{Y}(u,x)$ has been defined and is lower truncated for some particular $u \in V^{M(\lambda_1)}$, we must define

$$\mathcal{Y}(h(-n)u,x) = \mathrm{Res}_{x_1}((x_1-x)^{-n}h(x_1)\mathcal{Y}(u,x) - (-x+x_1)^{-n}\mathcal{Y}(u,x)h(x_1)) \tag{9.6}$$

for any $h \in \mathfrak{g}$ and $n > 0$, in order for the iterate formula to hold. This expression is a well-defined operator on $V^{M(\lambda_2)}$ because both $\mathcal{Y}(u,x)$ and $h(x_1)$ are lower truncated when acting on $V^{M(\lambda_2)}$. To show that (9.6) yields a well-defined map first on $(T(\widehat{\mathfrak{g}}_-) \otimes M(\lambda_1)) \otimes V^{M(\lambda_2)}$, we need to show that it is lower truncated. In fact, the lower truncation of $\mathcal{Y}(u,x)$ shows that the first term in (9.6) is also lower truncated. As for the second term, for any $v \in V^{M(\lambda_2)}$,

$$\mathrm{Res}_{x_1}(-x+x_1)^{-n}\mathcal{Y}(u,x)h(x_1)v = \sum_{i\geq 0}\binom{-n}{i}(-x)^{-n-i}\mathcal{Y}(u,x)h(i)v.$$

Since $h(i)v = 0$ for i sufficiently large, the sum is finite and lower truncation follows from the lower truncation of $\mathcal{Y}(u,x)$.

Next, to show that $\mathcal{Y}$ is a well-defined map on $V^{M(\lambda_1)} \otimes V^{M(\lambda_2)}$, we need to show that

$$\mathcal{Y}(u_1(h_1(-n_1)h_2(-n_2) - h_2(-n_2)h_1(-n_1) - [h_1,h_2](-n_1-n_2))u_2,x) = 0 \tag{9.7}$$

for $u_1 \in T(\widehat{\mathfrak{g}}_-)$, $u_2 \in T(\widehat{\mathfrak{g}}_-) \otimes M(\lambda_1)$, $h_1, h_2 \in \mathfrak{g}$ and $n_1, n_2 \geq 0$. For the case $u_1 = 1$, we have

$$\begin{aligned}
&\mathcal{Y}(h_1(-n_1)h_2(-n_2)u_2,x)\\
&\quad= \mathrm{Res}_{x_1}\big((x_1-x)^{-n_1}h_1(x_1)\mathcal{Y}(h_2(-n_2)u_2,x)\\
&\qquad\qquad -(-x+x_1)^{-n_1}\mathcal{Y}(h_2(-n_2)u_2,x)h_1(x_1)\big)\\
&\quad= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}(x_1-x)^{-n_1}(x_2-x)^{-n_2}h_1(x_1)h_2(x_2)\mathcal{Y}(u_2,x)\\
&\qquad - \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}(x_1-x)^{-n_1}(-x+x_2)^{-n_2}h_1(x_1)\mathcal{Y}(u_2,x)h_2(x_2)\\
&\qquad - \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}(-x+x_1)^{-n_1}(x_2-x)^{-n_2}h_2(x_2)\mathcal{Y}(u_2,x)h_1(x_1)\\
&\qquad + \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}(-x+x_1)^{-n_1}(-x+x_2)^{-n_2}\mathcal{Y}(u_2,x)h_2(x_2)h_1(x_1),
\end{aligned}$$

and we have a similar expression for $\mathcal{Y}(h_2(-n_2)h_1(-n_1)u_2,x)$ where we exchange h_1 with h_2, n_1 with n_2, and x_1 with x_2. Combining the two expressions, we get

$$\begin{aligned}
&\mathcal{Y}([h_1(-n_1),h_2(-n_2)]u_2,x)\\
&\quad= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}(x_1-x)^{-n_1}(x_2-x)^{-n_2}[h_1(x_1),h_2(x_2)]\mathcal{Y}(u_2,x)\\
&\qquad - \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}(-x+x_1)^{-n_1}(-x+x_2)^{-n_2}\mathcal{Y}(u_2,x)[h_1(x_1),h_2(x_2)]\\
&\quad= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}(x_1-x)^{-n_1}(x_2-x)^{-n_2}\\
&\qquad\quad \cdot\left(x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)[h_1,h_2](x_1) + \langle h_1,h_2\rangle\ell\frac{\partial}{\partial x_2}\left(x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\right)\right)\mathcal{Y}(u_2,x)\\
&\qquad - \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}(-x+x_1)^{-n_1}(-x+x_2)^{-n_2}\\
&\qquad\quad \cdot\mathcal{Y}(u_2,x)\left(x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)[h_1,h_2](x_1) + \langle h_1,h_2\rangle\ell\frac{\partial}{\partial x_2}\left(x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\right)\right)\\
&\quad= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\cdot\big((x_1-x)^{-n_1-n_2}[h_1,h_2](x_1)\mathcal{Y}(u_2,x)\\
&\qquad -(-x+x_1)^{-n_1-n_2}\mathcal{Y}(u_2,x)[h_1,h_2](x_1)\big)\\
&- \langle h_1,h_2\rangle\ell\,\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}(x_1-x)^{-n_1}(-n_2)(x_2-x)^{-n_2-1}x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\mathcal{Y}(u_2,x)\\
&+ \langle h_1,h_2\rangle\ell\,\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}(-x+x_1)^{-n_1}(-n_2)(-x+x_2)^{-n_2-1}x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\mathcal{Y}(u_2,x)\\
&= \mathrm{Res}_{x_1}\big((x_1-x)^{-n_1-n_2}[h_1,h_2](x_1)\mathcal{Y}(u_2,x)\\
&\qquad -(-x+x_1)^{-n_1-n_2}\mathcal{Y}(u_2,x)[h_1,h_2](x_1)\big)\\
&+ n_2\langle h_1,h_2\rangle\ell\,\mathrm{Res}_{x_1}\left((x_1-x)^{-n_1-n_2-1}-(-x+x_1)^{-n_1-n_2-1}\right)\mathcal{Y}(u_2,x)\\
&= \mathcal{Y}([h_1,h_2](-n_1-n_2)u_2,x)+0,
\end{aligned}
$$

where we have used the generating function form of the affine Lie algebra commutation relations (see [FLM], Proposition 2.3.1), δ-function substitution properties, the fact that the residue of a derivative is zero, and the fact that $n_1+n_2>0$. Then (9.7) for general v_1 follows easily from (9.6) by induction on the degree of v_1. This shows that $\mathcal{Y}$ is a well-defined map on $V^{M(\lambda_1)}\otimes V^{M(\lambda_2)}$.

We next want to prove the $L(0)$-commutator formula (6.2) for any $u\in V^{M(\lambda_1)}$. Since it holds for $u\in M(\lambda_1)$ by (9.4), it is enough to assume it holds for $u\in V^{M(\lambda_1)}$ and then show that it holds for $g(-n)u$ as well, where $g\in\mathfrak{g}$ and $n>0$. In the following calculation, we use the commutation formula

$$[L(0),g(x)] = x\frac{d}{dx}g(x)+g(x)$$

as well as the fact that the residue of a derivative is zero:

(9.8)

$$
\begin{aligned}
&\left(L(0)-x\frac{d}{dx}\right)\mathcal{Y}(g(-n)u,x)\\
&= \mathrm{Res}_{x_1}\left(L(0)-x\frac{\partial}{\partial x}\right)\big((x_1-x)^{-n}g(x_1)\mathcal{Y}(u,x)-(-x+x_1)^{-n}\mathcal{Y}(u,x)g(x_1)\big)\\
&= \mathrm{Res}_{x_1}\Bigg((x_1-x)^{-n}g(x_1)\left(L(0)-x\frac{\partial}{\partial x}\right)\mathcal{Y}(u,x)\\
&\qquad -(-x+x_1)^{-n}\left(L(0)-x\frac{\partial}{\partial x}\right)\mathcal{Y}(u,x)g(x_1)\Bigg)\\
&- \mathrm{Res}_{x_1}nx\big((x_1-x)^{-n-1}g(x_1)\mathcal{Y}(u,x)-(-x+x_1)^{-n-1}\mathcal{Y}(u,x)g(x_1)\big)\\
&+ \mathrm{Res}_{x_1}(x_1-x)^{-n}[L(0),g(x_1)]\mathcal{Y}(u,x)\\
&= \mathrm{Res}_{x_1}\big((x_1-x)^{-n}g(x_1)\mathcal{Y}(L(0)u,x)-(-x+x_1)^{-n}\mathcal{Y}(L(0)u,x)g(x_1)\big)\\
&+ \mathrm{Res}_{x_1}\big((x_1-x)^{-n}g(x_1)\mathcal{Y}(u,x)L(0)-(-x+x_1)^{-n}\mathcal{Y}(u,x)L(0)g(x_1)\big)\\
&+ \mathrm{Res}_{x_1}n\big((x_1-x)^{-n}g(x_1)\mathcal{Y}(u,x_1)-(-x+x_1)^{-n}\mathcal{Y}(u,x)g(x_1)\big)\\
&- \mathrm{Res}_{x_1}nx_1\big((x_1-x)^{-n-1}g(x_1)\mathcal{Y}(u,x_1)-(-x+x_1)^{-n-1}\mathcal{Y}(u,x)g(x_1)\big)\\
&+ \mathrm{Res}_{x_1}(x_1-x)^{-n}[L(0),g(x_1)]\mathcal{Y}(u,x)\\
&= \mathcal{Y}(g(-n)L(0)u,x)+\mathcal{Y}(g(-n),x)L(0)+n\mathcal{Y}(g(-n)u,x)\\
&+ \mathrm{Res}_{x_1}\left(x_1\frac{\partial}{\partial x_1}\left((x_1-x)^{-n}\right)g(x_1)\mathcal{Y}(u,x)-x_1\frac{\partial}{\partial x_1}\left((-x+x_1)^{-n}\right)\mathcal{Y}(u,x)g(x_1)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathrm{Res}_{x_1}\left((x_1-x)^{-n}[L(0),g(x_1)]\mathcal{Y}(u,x)-(-x+x_1)^{-n}\mathcal{Y}(u,x)[L(0),g(x_1)]\right)\\
&=\mathcal{Y}(L(0)g(-n)u,x)+\mathcal{Y}(g(-n),x)L(0)\\
&+\mathrm{Res}_{x_1}\left(\frac{\partial}{\partial x_1}\left(x_1(x_1-x)^{-n}\right)g(x_1)\mathcal{Y}(u,x)-\frac{\partial}{\partial x_1}\left(x_1(-x+x_1)^{-n}\right)\mathcal{Y}(u,x)g(x_1)\right)\\
&-\mathrm{Res}_{x_1}\left((x_1-x)^{-n}g(x_1)\mathcal{Y}(u,x)-(-x+x_1)^{-n}\mathcal{Y}(u,x)g(x_1)\right)\\
&+\mathrm{Res}_{x_1}\left(x_1(x_1-x)^{-n}\frac{\partial}{\partial x_1}g(x_1)\mathcal{Y}(u,x)-x_1(-x+x_1)^{-n}\mathcal{Y}(u,x)\frac{\partial}{\partial x_1}g(x_1)\right)\\
&+\mathrm{Res}_{x_1}\left((x_1-x)^{-n}g(x_1)\mathcal{Y}(u,x)-(-x+x_1)^{-n}\mathcal{Y}(u,x)g(x_1)\right)\\
&=\mathcal{Y}(L(0)g(-n)u,x)+\mathcal{Y}(g(-n),x)L(0).
\end{aligned}
$$

Now we prove the Jacobi identity

$$
x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)g(x_1)\mathcal{Y}(u,x_2)-x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\mathcal{Y}(u,x_2)g(x_1)
=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(g(x_0)u,x_2)
$$

for any $g\in\mathfrak{g}$ and $u\in V^{M(\lambda_1)}$. Recalling Remark 4.5, this is equivalent to proving the iterate formula

$$
\mathcal{Y}(g(m)u,x_2)=\mathrm{Res}_{x_1}\left((x_1-x_2)^m g(x_1)\mathcal{Y}(u,x_2)-(-x_2+x_1)^m\mathcal{Y}(u,x_2)g(x_1)\right)
$$

for any $m\in\mathbb{Z}$ and the commutator formula

$$
[g(x_1),\mathcal{Y}(u,x_2)]=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(g(x_0)u,x_2).
$$

By (9.3), (9.5), and the definition (9.6), the commutator and iterate formulas hold for $u\in M(\lambda_1)$, so it suffices to show that if they hold for $u\in V^{M(\lambda_1)}$, then they also hold for $h(-n)u$ where $h\in\mathfrak{h}$ and $n>0$. We prove the iterate formula first, using the assumptions on u and the generating function form of the affine Lie algebra commutation relations:

$$
\begin{aligned}
&\mathcal{Y}(g(m)h(-n)u,x_2)\\
&=\mathcal{Y}(h(-n)g(m)u,x_2)+\mathcal{Y}(([g,h](m-n)+m\langle g,h\rangle\ell\delta_{m-n,0})u,x_2)\\
&=\mathrm{Res}_{y_1}((y_1-x_2)^{-n}h(y_1)\mathcal{Y}(g(m)u,x_2)-(-x_2+y_1)^{-n}\mathcal{Y}(g(m)u,x_2)h(y_1))\\
&\quad+\mathcal{Y}(([g,h](m-n)+m\langle g,h\rangle\ell\delta_{m-n,0})u,x_2)\\
&=\mathrm{Res}_{y_1}\mathrm{Res}_{x_1}(y_1-x_2)^{-n}\left((x_1-x_2)^m h(y_1)g(x_1)\mathcal{Y}(u,x_2)\right.\\
&\qquad\left.-(-x_2+x_1)^m h(y_1)\mathcal{Y}(u,x_2)g(x_1)\right)\\
&\quad-\mathrm{Res}_{y_1}\mathrm{Res}_{x_1}(-x_2+y_1)^{-n}\left((x_1-x_2)^m g(x_1)\mathcal{Y}(u,x_2)h(y_1)\right.\\
&\qquad\left.-(-x_2+x_1)^m\mathcal{Y}(u,x_2)g(x_1)h(y_1)\right)\\
&\quad+\mathcal{Y}(([g,h](m-n)+m\langle g,h\rangle\ell\delta_{m-n,0})u,x_2)\\
&=\mathrm{Res}_{x_1}\mathrm{Res}_{y_1}(x_1-x_2)^m g(x_1)\left((y_1-x_2)^{-n}h(y_1)\mathcal{Y}(u,x_2)\right.\\
&\qquad\left.-(-x_2+y_1)^{-n}\mathcal{Y}(u,x_2)h(y_1)\right)\\
&\quad-\mathrm{Res}_{x_1}\mathrm{Res}_{y_1}(x_1-x_2)^m(y_1-x_2)^{-n}[g(x_1),h(y_1)]\mathcal{Y}(u,x_2)\\
&\quad-\mathrm{Res}_{x_1}\mathrm{Res}_{y_1}(-x_2+x_1)^m\left((y_1-x_2)^{-n}h(y_1)\mathcal{Y}(u,x_2)\right.\\
&\qquad\left.-(-x_2+y_1)^{-n}\mathcal{Y}(u,x_2)h(y_1)\right)g(x_1)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathrm{Res}_{x_1}\mathrm{Res}_{y_1}(-x_2+x_1)^m(-x_2+y_1)^{-n}\mathcal{Y}(u,x_2)[g(x_1),h(y_1)]\\
&+\mathcal{Y}(([g,h](m-n)+m\langle g,h\rangle\ell\delta_{m-n,0})u,x_2)\\
=\ &\mathrm{Res}_{x_1}((x_1-x_2)^m g(x_1)\mathcal{Y}(h(-n)u,x_2)-(-x_2+x_1)^m\mathcal{Y}(h(-n)u,x_2)g(x_1))\\
&-\mathrm{Res}_{x_1}\mathrm{Res}_{y_1}(x_1-x_2)^m(y_1-x_2)^{-n}y_1^{-1}\delta\left(\frac{x_1}{y_1}\right)[g,h](x_1)\mathcal{Y}(u,x_2)\\
&-\mathrm{Res}_{x_1}\mathrm{Res}_{y_1}(x_1-x_2)^m(y_1-x_2)^{-n}\langle g,h\rangle\ell\frac{\partial}{\partial y_1}\left(y_1^{-1}\delta\left(\frac{x_1}{y_1}\right)\right)\mathcal{Y}(u,x_2)\\
&+\mathrm{Res}_{x_1}\mathrm{Res}_{y_1}(-x_2+x_1)^m(-x_2+y_1)^{-n}y_1^{-1}\delta\left(\frac{x_1}{y_1}\right)\mathcal{Y}(u,x_2)[g,h](x_1)\\
&+\mathrm{Res}_{x_1}\mathrm{Res}_{y_1}(-x_2+x_1)^m(-x_2+y_1)^{-n}\langle g,h\rangle\ell\frac{\partial}{\partial y_1}\left(y_1^{-1}\delta\left(\frac{x_1}{y_1}\right)\right)\mathcal{Y}(u,x_2)\\
&+\mathcal{Y}(([g,h](m-n)+m\langle g,h\rangle\ell\delta_{m-n,0})u,x_2))\\
=\ &\mathrm{Res}_{x_1}((x_1-x_2)^m g(x_1)\mathcal{Y}(h(-n)u,x_2)-(-x_2+x_1)^m\mathcal{Y}(h(-n)u,x_2)g(x_1))\\
&-\mathrm{Res}_{x_1}((x_1-x_2)^{m-n}[g,h](x_1)\mathcal{Y}(u,x_2)-(-x_2+x_1)^{m-n}\mathcal{Y}(u,x_2)[g,h](x_1))\\
&+\mathrm{Res}_{x_1}(x_1-x_2)^m\frac{\partial}{\partial x_1}(x_1-x_2)^{-n}\langle g,h\rangle\ell\,\mathcal{Y}(u,x_2)\\
&-\mathrm{Res}_{x_1}(-x_2+x_1)^m\frac{\partial}{\partial x_1}(-x_2+x_1)^{-n}\langle g,h\rangle\ell\,\mathcal{Y}(u,x_2)\\
&+\mathcal{Y}(([g,h](m-n)+m\langle g,h\rangle\ell\delta_{m-n,0})u,x_2))\\
=\ &\mathrm{Res}_{x_1}((x_1-x_2)^m g(x_1)\mathcal{Y}(h(-n)u,x_2)-(-x_2+x_1)^m\mathcal{Y}(h(-n)u,x_2)g(x_1))\\
&-\mathrm{Res}_{x_1}(n(x_1-x_2)^{m-n-1}\langle g,h\rangle\ell\,\mathcal{Y}(u,x_2))+m\langle g,h\rangle\delta_{m-n,0}\ell\,\mathcal{Y}(u,x_2)\\
=\ &\mathrm{Res}_{x_1}((x_1-x_2)^m g(x_1)\mathcal{Y}(h(-n)u,x_2)-(-x_2+x_1)^m\mathcal{Y}(h(-n)u,x_2)g(x_1)),
\end{aligned}
$$

which proves the iterate formula for $h(-n)u$.

Now we need to prove the commutator formula for $h(-n)u$. We use the fact that the Jacobi identity holds for u and so

$$
\begin{aligned}
(x_1-x_2)^m g(x_1)\mathcal{Y}(u,x_2)-(-x_2+x_1)^m\mathcal{Y}(u,x_2)g(x_1)&\\
=\mathrm{Res}_{x_0}x_0^m x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(g(x_0)u,x_2)&
\end{aligned}
$$

for any $g\in\mathfrak{g}$ and $m\in\mathbb{Z}$:

$$
\begin{aligned}
&[g(x_1),\mathcal{Y}(h(-n)u,x_2)]=\mathrm{Res}_{y_1}(y_1-x_2)^{-n}[g(x_1),h(y_1)\mathcal{Y}(u,x_2)]\\
&\quad-\mathrm{Res}_{y_1}(-x_2+y_1)^{-n}[g(x_1),\mathcal{Y}(u,x_2)h(y_1)]\\
&=\mathrm{Res}_{y_1}\big((y_1-x_2)^{-n}[g(x_1),h(y_1)]\mathcal{Y}(u,x_2)+(y_1-x_2)^{-n}h(y_1)[g(x_1),\mathcal{Y}(u,x_2)]\big)\\
&\quad-\mathrm{Res}_{y_1}\big((-x_2+y_1)^{-n}[g(x_1),\mathcal{Y}(u,x_2)]h(y_1)\\
&\qquad\qquad+(-x_2+y_1)^{-n}\mathcal{Y}(u,x_2)[g(x_1),h(y_1)]\big)\\
&=\mathrm{Res}_{y_1}(y_1-x_2)^{-n}\left(y_1^{-1}\delta\left(\frac{x_1}{y_1}\right)[g,h](x_1)+\langle g,h\rangle\ell\frac{\partial}{\partial y_1}\left(y_1^{-1}\delta\left(\frac{x_1}{y_1}\right)\right)\right)\mathcal{Y}(u,x_2)\\
&\quad-\mathrm{Res}_{y_1}(-x_2+y_1)^{-n}\mathcal{Y}(u,x_2)\left(y_1^{-1}\delta\left(\frac{x_1}{y_1}\right)[g,h](x_1)\right.
\end{aligned}
$$

$$+\langle g,h\rangle\ell\frac{\partial}{\partial y_1}\left(y_1^{-1}\delta\left(\frac{x_1}{y_1}\right)\right)\Bigg)$$
$$+\operatorname{Res}_{x_0}\operatorname{Res}_{y_1}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)(y_1-x_2)^{-n}h(y_1)\mathcal{Y}(g(x_0)u,x_2)$$
$$-\operatorname{Res}_{x_0}\operatorname{Res}_{y_1}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)(-x_2+y_1)^{-n}\mathcal{Y}(g(x_0)u,x_2)h(y_1)$$
$$=(x_1-x_2)^{-n}[g,h](x_1)\mathcal{Y}(u,x_2)-(-x_2+x_1)^{-n}\mathcal{Y}(u,x_2)[g,h](x_1)$$
$$-\langle g,h\rangle\ell\frac{\partial}{\partial x_1}\left((x_1-x_2)^{-n}-(-x_2+x_1)^{-n}\right)\mathcal{Y}(u,x_2)$$
$$+\operatorname{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(h(-n)g(x_0)u,x_2)$$
$$=\operatorname{Res}_{x_0}\Bigg(x_0^{-n}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}([g,h](x_0)u,x_2)\Bigg)$$
$$+n\langle g,h\rangle\ell\frac{(-1)^n}{n!}\frac{\partial^n}{\partial x_1^n}\left(x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\right)\mathcal{Y}(u,x_2)$$
$$+\operatorname{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(g(x_0)h(-n)u,x_2)$$
$$-\operatorname{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\sum_{m\in\mathbb{Z}}x_0^{-m-1}\mathcal{Y}\left(([g,h](m-n)+m\delta_{m-n,0}\langle g,h\rangle\ell)u,x_2\right)$$
$$=\operatorname{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(g(x_0)h(-n)u,x_2).$$

We have thus proved that the Jacobi identity holds for $g(-1)\mathbf{1}\in V^{M(0)}$, where $g\in\mathfrak{g}$, and any $v\in V^{M(\lambda_1)}$, and it is obvious that the Jacobi identity holds for $\mathbf{1}$ and any $v\in V^{M(\lambda_1)}$. To prove the Jacobi identity for any $u\in V^{M(0)}$ and $v\in V^{M(\lambda_1)}$, it suffices to assume that it holds for some specific $u\in V^{M(0)}$ and then show that it holds for $g(-n)u$ for any $g\in\mathfrak{g}$ and $n>0$. Observing throughout that each formal expression is well-defined, we obtain

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(g(-n)u,x_1)\mathcal{Y}(v,x_2)-x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\mathcal{Y}(v,x_2)Y(g(-n)u,x_1)$$
$$=\operatorname{Res}_{y_1}(y_1-x_1)^{-n}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)g(y_1)Y(u,x_1)\mathcal{Y}(v,x_2)$$
$$-\operatorname{Res}_{y_1}(-x_1+y_1)^{-n}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)g(y_1)\mathcal{Y}(v,x_2)$$
$$-\operatorname{Res}_{y_1}(y_1-x_1)^{-n}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\mathcal{Y}(v,x_2)g(y_1)Y(u,x_1)$$
$$+\operatorname{Res}_{y_1}(-x_1+y_1)^{-n}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\mathcal{Y}(v,x_2)Y(u,x_1)g(y_1).\tag{9.9}$$

We analyze the first and third terms on the right side first: they become

$$\operatorname{Res}_{y_1}(y_1-x_1)^{-n}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)[g(y_1),\mathcal{Y}(u,x_2)]Y(u,x_1)$$
$$+\operatorname{Res}_{y_1}(y_1-x_1)^{-n}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)g(y_1)\mathcal{Y}(Y(u,x_0)v,x_2)$$

$$
\begin{aligned}
&= \mathrm{Res}_{y_1}\mathrm{Res}_{y_2}(y_1-x_1)^{-n}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\\
&\qquad\qquad \cdot x_2^{-1}\delta\left(\frac{y_1-y_2}{x_2}\right)\mathcal{Y}(g(y_2)v,x_2)Y(u,x_1)\\
&\quad + \mathrm{Res}_{y_1}(y_1-x_2-x_0)^{-n}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)g(y_1)\mathcal{Y}(Y(u,x_0)v,x_2)\\
&= \mathrm{Res}_{y_1}\mathrm{Res}_{y_2}(x_2+y_2-x_1)^{-n}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\\
&\qquad\qquad \cdot x_2^{-1}\delta\left(\frac{y_1-y_2}{x_2}\right)\mathcal{Y}(g(y_2)v,x_2)Y(u,x_1)\\
&\quad + x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\\
&\qquad \cdot \sum_{i\geq 0}\binom{-n}{i}(-x_0)^i\,\mathrm{Res}_{y_1}(y_1-x_2)^{-n-i}g(y_1)\mathcal{Y}(Y(u,x_0)v,x_2)\\
&= \mathrm{Res}_{y_2}(-x_0+y_2)^{-n}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\mathcal{Y}(g(y_2)v,x_2)Y(u,x_1)\\
&\quad + x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\\
&\qquad \cdot \sum_{i\geq 0}\binom{-n}{i}(-x_0)^i\,\mathrm{Res}_{y_1}(y_1-x_2)^{-n-i}g(y_1)\mathcal{Y}(Y(u,x_0)v,x_2).
\end{aligned} \tag{9.10}
$$

We next analyze the second and fourth terms on the right side of (9.9), which become

$$
\begin{aligned}
&- \mathrm{Res}_{y_1}(-x_0-x_2+y_1)^{-n}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)g(y_1)\mathcal{Y}(v,x_2)\\
&\quad + \mathrm{Res}_{y_1}(-x_1+y_1)^{-n}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\mathcal{Y}(v,x_2)Y(u,x_1)g(y_1)\\
&= -x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)\sum_{i\geq 0}\binom{-n}{i}(-x_0)^{-n-i}\\
&\qquad\qquad \cdot \mathrm{Res}_{y_1}(y_1-x_2)^i g(y_1)\mathcal{Y}(v,x_2)\\
&\quad + \mathrm{Res}_{y_1}(-x_1+y_1)^{-n}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\mathcal{Y}(v,x_2)Y(u,x_1)g(y_1)\\
&= -x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)\sum_{i\geq 0}\binom{-n}{i}(-x_0)^{-n-i}\\
&\qquad\qquad \cdot \mathrm{Res}_{y_1}(-x_2+y_1)^i\mathcal{Y}(v,x_2)g(y_1)\\
&\quad - x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\sum_{i\geq 0}\binom{-n}{i}(-x_0)^{-n-i}Y(u,x_1)\mathcal{Y}(g(i)v,x_2)\\
&\quad + \mathrm{Res}_{y_1}(-x_1+y_1)^{-n}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\mathcal{Y}(v,x_2)Y(u,x_1)g(y_1)\\
&= -\mathrm{Res}_{y_1}(-x_0-x_2+y_1)^{-n}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)\mathcal{Y}(v,x_2)g(y_1)
\end{aligned}
$$

$$
\begin{aligned}
&- x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\mathrm{Res}_{y_2}(-x_0+y_2)^{-n}Y(u,x_1)\mathcal{Y}(g(y_2)v,x_2)\\
&+ \mathrm{Res}_{y_1}(-x_1+y_1)^{-n}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\mathcal{Y}(v,x_2)Y(u,x_1)g(y_1)\\
=& -x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\mathrm{Res}_{y_2}(-x_0+y_2)^{-n}Y(u,x_1)\mathcal{Y}(g(y_2)v,x_2)\\
&- \mathrm{Res}_{y_1}(-x_1+y_1)^{-n}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(Y(u,x_0)v,x_2)g(y_1)\\
=& -x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\mathrm{Res}_{y_2}(-x_0+y_2)^{-n}Y(u,x_1)\mathcal{Y}(g(y_2)v,x_2)\\
&- \mathrm{Res}_{y_1}(-x_2+y_1-x_0)^{-n}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(Y(u,x_0)v,x_2)g(y_1)\\
=& -x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\mathrm{Res}_{y_2}(-x_0+y_2)^{-n}Y(u,x_1)\mathcal{Y}(g(y_2)v,x_2)\\
&- x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\sum_{i\geq 0}\binom{-n}{i}(-x_0)^i\\
&\qquad\cdot \mathrm{Res}_{y_1}(-x_2+y_1)^{-n-i}\mathcal{Y}(Y(u,x_0)v,x_2)g(y_1).
\end{aligned}
\tag{9.11}
$$

If we add (9.10) and (9.11) back together, we get

$$
\begin{aligned}
&- x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathrm{Res}_{y_2}(-x_0+y_2)^{-n}\mathcal{Y}(Y(u,x_0)g(y_2)v,x_2)\\
&+ x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\sum_{i\geq 0}\binom{-n}{i}(-x_0)^i\,\mathrm{Res}_{y_1}(y_1-x_2)^{-n-i}g(y_1)\mathcal{Y}(Y(u,x_0)v,x_2)\\
&- x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\sum_{i\geq 0}\binom{-n}{i}(-x_0)^i\,\mathrm{Res}_{y_1}(-x_2+y_1)^{-n-i}\mathcal{Y}(Y(u,x_0)v,x_2)g(y_1)\\
=& -x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathrm{Res}_{y_2}(-x_0+y_2)^{-n}\mathcal{Y}(Y(u,x_0)g(y_2)v,x_2)\\
&+ x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\sum_{i\geq 0}\binom{-n}{i}(-x_0)^i\mathcal{Y}(g(-n-i)Y(u,x_0)v,x_2)\\
=& -x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathrm{Res}_{y_2}(-x_0+y_2)^{-n}\mathcal{Y}(Y(u,x_0)g(y_2)v,x_2)\\
&+ x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathrm{Res}_{y_2}(y_2-x_0)^{-n}\mathcal{Y}(g(y_2)Y(u,x_0)v,x_2)\\
=& \, x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(Y(g(-n)u,x_0)v,x_2).
\end{aligned}
$$

This completes the proof of the Jacobi identity.

Finally, to prove that $\mathcal{Y}$ is an intertwining operator, we need to prove the $L(-1)$-derivative property. From the Jacobi identity, or specifically from the commutator formula, we have for any $u \in V^{M(\lambda_1)}$ that

$$[L(0),\mathcal{Y}(u,x)] = x\mathcal{Y}(L(-1)u,x) + \mathcal{Y}(L(0)u,x).$$

On the other hand, we proved in (9.8) that

$$[L(0), \mathcal{Y}(u,x)] = x\frac{d}{dx}\mathcal{Y}(u,x) + \mathcal{Y}(L(0)u,x).$$

It follows that

$$\mathcal{Y}(L(-1)u,x) = \frac{d}{dx}\mathcal{Y}(u,x),$$

as desired. This completes the proof that $\mathcal{Y}$ is an intertwining operator extending the original $\mathcal{Y} : M(\lambda_1) \otimes M(\lambda_2) \to W\{x\}$. The uniqueness assertion of the theorem follows from the observation that the extensions of $\mathcal{Y}$ from $M(\lambda_1) \otimes M(\lambda_2)$ to $M(\lambda_1) \otimes V^{M(\lambda_2)}$ and then from $M(\lambda_1) \otimes V^{M(\lambda_2)}$ to $V^{M(\lambda_1)} \otimes V^{M(\lambda_2)}$ are uniquely determined by the commutator and iterate formulas, respectively.

Remark 9.1. If we take $\lambda_1 = \lambda_2 = 0$ and $W = V^{M(0)}$, with $\mathcal{Y}(\mathbf{1},x)\mathbf{1} = \mathbf{1}$, we see that the argument here proves the Jacobi identity for $V^{M(0)}$.

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