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PROJECTIVE VARIETIES WITH NONBIRATIONAL LINEAR PROJECTIONS AND APPLICATIONS

ATSUSHI NOMA

ABSTRACT. We work over an algebraically closed field of characteristic zero. The purpose of this paper is to characterize a nondegenerate projective variety X with a linear projection which induces a nonbirational map to its image. As an application, for smooth X of degree d and codimension e, we prove the "semiampleness" of the $(d-e+1)$th twist of the ideal sheaf. This improves a linear bound of the regularity of smooth projective varieties by Bayer–Mumford–Bertram–Ein–Lazarsfeld, and gives an asymptotic regularity bound.

INTRODUCTION

We work over an algebraically closed field $\Bbbk$ of characteristic zero. Let $X \subseteq \mathbb{P}^N$ be a nondegenerate (i.e., not contained in any hyperplane of $\mathbb{P}^N$) projective variety (i.e., irreducible and reduced) of dimension $n > 0$, codimension e, and degree d. The linear projection from a general point of $\mathbb{P}^N$ induces a morphism of X birational onto its image. If the center of the projection is a special point, this is not true in some cases. Such special projections were originally studied by Segre [23] (see also [1], [4], [12]). The purpose of this paper is to characterize X with such a special center. This study is motivated by the problem of finding out whether X is cut out by hypersurfaces of degree $\leq d-e+1$ (see [22, §3]) as evidence of a regularity conjecture (see [8] and [14] for the regularity conjecture). As applications, for smooth X, we improve a linear bound of the regularity ([2], [3]) and give an asymptotic regularity bound (Theorems 9 and 10).

To be precise, we say that a point $w \in \mathbb{P}^N$ is a *nonbirational center* of X if the linear projection $\pi_w : \mathbb{P}^N \setminus \{w\} \to \mathbb{P}^{N-1}$ induces a nonbirational map of X to its image. By $\mathcal{B}(X)$ we denote the set of all nonbirational centers out of X and by $\mathcal{C}(X)$ that on the smooth locus $\operatorname{Sm} X$ of X:

$$\mathcal{B}(X) := \{v \in \mathbb{P}^N \setminus X | l(\langle v, x\rangle \cap X) \geq 2 \text{ for general } x \in X\},$$
$$\mathcal{C}(X) := \{u \in \operatorname{Sm} X | l(\langle u, x\rangle \cap X) \geq 3 \text{ for general } x \in X\}.$$

Here $l(Z)$ denotes the length of a scheme Z and $\langle\ \rangle$ denotes the linear span of schemes, i.e., the intersection of all hyperplanes containing the schemes. Let $\bar{\mathcal{B}}(X)$ and $\bar{\mathcal{C}}(X)$, respectively, be the closures of $\mathcal{B}(X)$ and $\mathcal{C}(X)$ in $\mathbb{P}^N$. Note that $\mathcal{B}(X) = \bar{\mathcal{B}}(X) \setminus X$ and $\mathcal{C}(X) = \bar{\mathcal{C}}(X) \cap \operatorname{Sm} X$ ([22, (4.1) and (4.2)]).

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The structure of $\bar{\mathcal{B}}(X)$ and $\bar{\mathcal{C}}(X)$ is known essentially due to Segre (see [12] for the positive characteristic case).

Theorem 1 (Segre [23], Calabri–Ciliberto [4], Ballico [1], Noma [22])**.** *Let $X \subseteq \mathbb{P}^N$ be a nondegenerate projective variety of dimension $n > 0$ and codimension $e \geq 2$. Then each irreducible component Z of $\bar{\mathcal{B}}(X)$ or $\bar{\mathcal{C}}(X)$ is linear of $\dim Z \leq n-1$ and the linear span $\langle Z, T_x(X)\rangle$ of Z and the embedded tangent space $T_x(X)$ to X at general $x \in X$ is of dimension $n+1$.*

This means that X with $\mathcal{B}(X) \neq \emptyset$ or $\mathcal{C}(X) \neq \emptyset$ is a codimension-one subvariety of a cone. By taking a resolution of the singularity of the cone, X is the birational image of a divisor of a smooth projective bundle (see Lemma 1.1). The purpose here is to characterize X with $\mathcal{B}(X) \neq \emptyset$ or $\mathcal{C}(X) \neq \emptyset$, by describing the condition for the image of a divisor of a scroll with vertex to have the vertex as a subset of $\mathcal{B}(X)$ or $\mathcal{C}(X)$.

To state our results, we introduce definitions about scrolls with vertex.

Definition 2. Let Λ be an l-dimensional linear subspace of $\mathbb{P}^N$ and let $\mathbb{P}^{\bar{N}}$ ($\bar{N} = N - l - 1$) be a subspace of $\mathbb{P}^N$ disjoint from Λ. Consider the linear projection $\pi_\Lambda : \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{\bar{N}}$ from Λ. By $\tau : \mathbf{F}^\Lambda \to \mathbb{P}^{\bar{N}}$ we denote the $\mathbb{P}^{l+1}$-bundle $\mathbf{F}^\Lambda := \{(x, w) | x \in \langle \Lambda, w\rangle\} \subseteq \mathbb{P}^N \times \mathbb{P}^{\bar{N}}$ over $\mathbb{P}^{\bar{N}}$, which is the family of all $(l+1)$-planes in $\mathbb{P}^N$ containing Λ or the graph of π_Λ. For a smooth projective variety Y with a birational-embedding $\nu : Y \to \mathbb{P}^{\bar{N}}$ (i.e., Y is birational to the image $\nu(Y)$ in $\mathbb{P}^{\bar{N}}$), *the conical scroll with vertex* Λ *over* Y is the pull-back $\tau_Y : \mathbf{F}^\Lambda_Y := \mathbf{F}^\Lambda \times_{\mathbb{P}^{\bar{N}}} Y \to Y$ of τ by ν. In this case, $\mathbf{F}^\Lambda_Y$ has a birational-embedding $\varphi_Y : \mathbf{F}^\Lambda_Y \to \mathbb{P}^N$ induced from the first projection of $\mathbb{P}^N \times Y$ and the subbundle $\tilde{\Lambda}_Y := \Lambda \times Y \subseteq \mathbf{F}^\Lambda_Y$ with projection $\bar{\tau}_Y : \tilde{\Lambda}_Y \to Y$, which is mapped onto Λ by φ_Y. Set $\mathcal{O}_{\mathbf{F}^\Lambda_Y}(1) := \varphi_Y^* \mathcal{O}_{\mathbb{P}^N}(1)$. A projective variety $X \subseteq \mathbb{P}^N$ is called *a birational-divisor of the conical scroll* $\mathbf{F}^\Lambda_Y$ *with vertex* Λ *over* Y if X is birational to some prime divisor $\tilde{X}$ on $\mathbf{F}^\Lambda_Y$ by φ_Y. Moreover X is said to be *of type* $(\mu, \mathcal{L})$ if $\tilde{X} \in |\mathcal{O}_{\mathbf{F}^\Lambda_Y}(\mu) \otimes \tau_Y^* \mathcal{L}|$ for $\mu \in \mathbb{Z}$ and $\mathcal{L} \in \operatorname{Pic} Y$. We call $\tilde{X}$ *the original divisor for* X. We say that ν is *nondegenerate* if $\nu(Y) \subseteq \mathbb{P}^{\bar{N}}$ is nondegenerate.

The first result is the structure of X with $\mathcal{B}(X) \neq \emptyset$, which is almost done in [22]. Conventionally we set $\dim \emptyset = -1$.

Theorem 3. *Let $X \subseteq \mathbb{P}^N$ be a projective variety of dimension $n > 0$ and codimension $e \geq 2$. Let $\Lambda \subseteq \mathbb{P}^N$ be a linear subspace of dimension l $(n-1 \geq l \geq 0)$. Then X is nondegenerate with $\Lambda \subseteq \bar{\mathcal{B}}(X)$, and $\Lambda \not\subseteq X$ if and only if X is a birational-divisor of type $(\mu, \mathcal{O}_Y)$ $(\mu \geq 2)$ on the conical scroll $\mathbf{F}^\Lambda_Y$ with vertex Λ over an $(n-l)$-dimensional smooth projective variety Y with a nondegenerate birational-embedding $\nu : Y \to \mathbb{P}^{\bar{N}}$ $(\bar{N} = N - l - 1)$. Moreover, under these equivalent conditions, the following hold:*

(1) *$\mu = l(X \cap \langle v, x\rangle)$ for general $v \in \Lambda$ and general $x \in X$.*
(2) $\deg X = \mu \cdot \deg \nu(Y)$.
(3) $\Lambda \cap X \subseteq \operatorname{Sing} X$ *and* $\dim \Lambda \cap X = \dim \Lambda - 1$. *In particular,* $\dim \Lambda \leq \dim \operatorname{Sing} X + 1$.
(4) *Λ is an irreducible component of $\bar{\mathcal{B}}(X)$ if and only if $\nu(Y)$ is not a cone.*

The next results are the structure of X with $\mathcal{C}(X) \neq \emptyset$, which is the main purpose of this paper. To this purpose, we divide into two cases by *the partial Gauss map* of

X, that is, a rational map $\gamma|_Z : Z \cap \operatorname{Sm} X \to \mathbb{G}(n, \mathbb{P}^N)$ from a subset $Z \subseteq X$ with $Z \cap \operatorname{Sm} X \neq \emptyset$ to the Grassmannian of n-planes in $\mathbb{P}^N$, mapping y to $T_y(X) \subseteq \mathbb{P}^N$.

First we consider the case when the partial Gauss map on a subset of $\bar{\mathcal{C}}(X)$ is constant.

Theorem 4. *Let $X \subseteq \mathbb{P}^N$ be a projective variety of dimension $n > 0$ and codimension $e \geq 2$. Let $\Lambda \subseteq \mathbb{P}^N$ be a linear subspace of dimension l $(n-1 \geq l \geq 0)$. Suppose that X is nondegenerate with $\Lambda \subseteq \bar{\mathcal{C}}(X)$ and $\Lambda \cap \operatorname{Sm} X \neq \emptyset$ and that the partial Gauss map $\gamma|_\Lambda$ is constant. Then X is a birational-divisor of type $(\mu, \mathcal{L})$ $(\mu \geq 2, \mathcal{L} \in \operatorname{Pic} Y)$ on the conical scroll $\mathbf{F}_Y^\Lambda$ with vertex Λ over an $(n-l)$-dimensional smooth projective variety Y with a nondegenerate birational-embedding $\nu : Y \to \mathbb{P}^{\bar{N}}$ $(\bar{N} = N - l - 1)$ satisfying the following:*

1. *$H^0(Y, \mathcal{L}) \neq 0$, $(\mathcal{L}, \mathcal{O}_Y(1)^{n-l-1}) = 1$ for $\mathcal{O}_Y(1) = \nu^* \mathcal{O}_{\mathbb{P}^{\bar{N}}}(1)$, and $\deg X = \mu \cdot \deg \nu(Y) + 1$;*
2. *$\tilde{X} \cap \tilde{\Lambda}_Y = (g)_0 \times Y + \Lambda \times (w)_0$ as a divisor on $\tilde{\Lambda}_Y$ for some g $(\neq 0) \in H^0(\Lambda, \mathcal{O}_\Lambda(\mu))$ and w $(\neq 0) \in H^0(Y, \mathcal{L})$;*
3. *$(w)_0$ is the sum $\sum_{i=0}^r D_i$ of prime divisors D_i such that $\nu(D_0) = \pi_\Lambda(T_u(X) \setminus \Lambda)$ for general $u \in \Lambda$ and $\nu(D_i) \subsetneq \nu(D_0)$ for all $i \geq 1$ if $r \geq 1$;*
4. *$\mu = l(X \cap \langle u, x \rangle) - 1$ holds for general $u \in \Lambda$ and general $x \in X$;*
5. *$(g)_0 \subseteq \Lambda \cap \operatorname{Sing} X$ as set; in particular, $\dim \Lambda \leq \dim \operatorname{Sing} X + 1$;*
6. *Λ is an irreducible component of $\bar{\mathcal{C}}(X)$ if and only if $\nu(Y)$ is not a cone.*

Theorem 5. *For integers $n > l \geq 0$, let $X \subseteq \mathbb{P}^N$ be a birational-divisor of type $(\mu, \mathcal{L})$ $(\mu \geq 1, \mathcal{L} \in \operatorname{Pic} Y)$ on the conical scroll $\mathbf{F}_Y^\Lambda$ with an l-dimensional linear subspace Λ as vertex over an $(n-l)$-dimensional smooth projective variety Y with a nondegenerate birational-embedding $\nu : Y \to \mathbb{P}^{\bar{N}}$ $(\bar{N} = N - l - 1)$ satisfying (1) and (2) in Theorem 4. If $(w)_0$ is irreducible, then $\Lambda \cap \operatorname{Sm} X = \Lambda \setminus (g)_0 \neq \emptyset$. Consequently, if $(w)_0$ is irreducible and if $\mu \geq 2$, then X is a nondegenerate n-dimensional subvariety of $\mathbb{P}^N$ such that $\Lambda \cap \operatorname{Sm} X = \Lambda \setminus (g)_0 \subseteq \mathcal{C}(X)$ and the partial Gauss map $\gamma|_\Lambda$ is constant.*

Second we consider the case when the partial Gauss map on a component of $\bar{\mathcal{C}}(X)$ is nonconstant. To this purpose, we introduce definitions about a rational scroll with vertex.

Definition 6. For an l-dimensional linear subspace $\Lambda \subseteq \mathbb{P}^N$ and for an ample vector bundle $\mathcal{E}$ of rank $n - l (\geq 1)$ over $\mathbb{P}^1$, the *conical rational scroll* $\mathbf{E}_\mathcal{E}^\Lambda$ with vertex Λ is the projective bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}^{\oplus l+1} \oplus \mathcal{E})$ with birational-embedding $\psi : \mathbf{E}_\mathcal{E}^\Lambda \to \mathbb{P}^N$ defined by a subsystem of $|\mathcal{O}_{\mathbf{E}_\mathcal{E}^\Lambda}(1)|$ such that the subbundle $\tilde{\Lambda}_{\mathbb{P}^1} := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}^{\oplus l+1}) (\subseteq \mathbf{E}_\mathcal{E}^\Lambda)$ maps onto Λ by ψ. Thus $\psi(\mathbf{E}_\mathcal{E}^\Lambda)$ is nondegenerate in $\mathbb{P}^N$ and the cone over $\psi(\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}))$ with vertex Λ. Here $\mathcal{O}_{\mathbf{E}_\mathcal{E}^\Lambda}(1)$ is the tautological line bundle of $\mathbf{E}_\mathcal{E}^\Lambda = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}^{\oplus l+1} \oplus \mathcal{E})$. A projective variety $X \subseteq \mathbb{P}^N$ is a *birational-divisor of the conical rational scroll* $\mathbf{E}_\mathcal{E}^\Lambda$ if X is a birational image of a prime divisor $\tilde{X}$ on $\mathbf{E}_\mathcal{E}^\Lambda$ by the birational-embedding $\psi : \mathbf{E}_\mathcal{E}^\Lambda \to \mathbb{P}^N$. In this case, X is said to be of type (μ, b) if $\tilde{X} \in |\mathcal{O}_{\mathbf{E}_\mathcal{E}^\Lambda}(\mu) \otimes p^* \mathcal{O}_{\mathbb{P}^1}(b)|$ for the projection $p : \mathbf{E}_\mathcal{E}^\Lambda \to \mathbb{P}^1$. We call $\tilde{X}$ *the original divisor for* X.

Theorem 7. *Let $X \subseteq \mathbb{P}^N$ be a projective variety of dimension $n > 0$ and codimension $e \geq 2$. Let $\Lambda \subseteq \mathbb{P}^N$ be a linear subspace of dimension l $(n-1 \geq l \geq 0)$. The following are equivalent:*

(1) *X is nondegenerate and Λ is an irreducible component of $\bar{\mathcal{C}}(X)$ such that the partial Gauss map $\gamma|_\Lambda$ is nonconstant.*
(2) *X is a birational-divisor of type $(\mu, 1)$ $(\mu \geq 2)$ on a conical rational scroll $\mathbf{E}^\Lambda_{\mathcal{E}}$ with vertex Λ and original divisor $\tilde{X}$ such that the intersection $\tilde{X} \cap \tilde{\Lambda}_{\mathbb{P}^1}$ is a nonzero effective divisor of $\tilde{\Lambda}_{\mathbb{P}^1}$ not equal to $(g)_0 \times \mathbb{P}^1 + \Lambda \times (w)_0$ for any $g \in H^0(\Lambda, \mathcal{O}_\Lambda(\mu))$ and any $w \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$.*

Moreover, under these equivalent conditions, we have $l \geq 1$, $\mu = l(X \cap \langle u, x\rangle) - 1$ for general $u \in \Lambda$ and general $x \in X$, $\deg X = \mu c_1(\mathcal{E}) + 1$, *and* $\dim \Lambda \leq \dim \operatorname{Sing} X + 2$.

Theorems 4 and 7 give the structure of smooth X with positive-dimensional $\mathcal{C}(X)$ (Corollary 6.2). As an application, we prove the "semiampleness" of the ideal sheaf for smooth X.

Theorem 8. *Let $X \subseteq \mathbb{P}^N$ be a nondegenerate smooth projective variety of degree d and codimension $e \geq 1$. Let $\sigma : \hat{\mathbb{P}}^N_X \to \mathbb{P}^N$ be the blowing-up of $\mathbb{P}^N$ along X with exceptional divisor E and let A be the divisor of the pull-back of a hyperplane of $\mathbb{P}^N$. Then $\mathcal{O}_{\hat{\mathbb{P}}^N_X}((d-e+1)A - E)$ is semiample, i.e., $\mathcal{O}_{\hat{\mathbb{P}}^N_X}(m((d-e+1)A-E))$ is base-point-free for some $m > 0$.*

Theorem 8 improves a regularity bound for smooth projective varieties ([2] and [3]) and leads to an asymptotic regularity bound. Recall that for an integer m, a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^N$ is said to be *m-regular* in the sense of Castelnuovo–Mumford if $H^i(\mathbb{P}^N, \mathcal{F}(m-i)) = 0$ for the twisted sheaf $\mathcal{F}(m-i) := \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^N}(m-i)$ and for all $i > 0$. A projective variety $X \subseteq \mathbb{P}^N$ is said to be *m-regular* if the ideal sheaf $\mathcal{I}_X$ is m-regular. The regularity $\operatorname{reg}(\mathcal{F})$ is the least integer m for which $\mathcal{F}$ is m-regular.

Theorem 9. *Let $X \subseteq \mathbb{P}^N$ be a nondegenerate smooth projective variety of degree d and codimension $e \geq 1$. Then $H^i(\mathbb{P}^N, \mathcal{I}_X(k)) = 0$ for all $i > 0$ and $k \geq e(d-e+1) - N$. In particular, X is $(e(d-e)+1)$-regular.*

Theorem 10. *Let $X \subseteq \mathbb{P}^N$ be a nondegenerate smooth projective variety of degree d and codimension $e \geq 1$. Let a be a positive integer and let $\mathcal{I}^a_X$ be the ath power of the ideal sheaf of X. Then $H^i(\mathbb{P}^N, \mathcal{I}^a_X(k)) = 0$ for all $i > 0$ and $k \geq (d-e+1)(e+a-1) - N$. In particular, $\mathcal{I}^a_X$ is $\{(d-e)(e+a-1)+a\}$-regular and therefore* $\lim_{a \to +\infty} (\operatorname{reg} \mathcal{I}^a_X / a) \leq d - e + 1$.

The asymptotic regularity bounds are studied by many authors ([6], [7], [20]). In particular, Cutkosky–Ein–Lazarsfeld [6] showed that $\lim_{a\to+\infty}(\operatorname{reg} \mathcal{I}^a_X/a)$ is bounded by *the generating degree* (i.e., the smallest d such that $\mathcal{I}_X(d)$ is generated by global sections). On the other hand, the regularity conjecture ([8], [14, §4]) implies that the generating degree is bounded above by $d - e + 1$ for a projective variety of degree d and codimension e. Hence Theorem 10 supports the conjecture.

This paper is organized as follows. In §1, we summarize some properties of birational-divisors of conical scrolls and prove Theorem 3. In §2, we study conditions for birational-divisors on conical scrolls to be smooth at the general points of vertices. In §3, we prove Theorem 4. In §4, we prove Theorem 5. In §5, we study conditions for birational-divisors on rational conical scrolls to be smooth at the general points of vertices. In §6, we prove Theorem 7. In §7, we deal with the applications of our theorems to regularity problems. In particular, we prove Theorems 8-10.

1. Loci of nonbirational centers: Proof of Theorem 3

Lemma 1.1. *Let $X \subseteq \mathbb{P}^N$ be a projective variety of dimension $n > 0$ and codimension $e \geq 2$. Let $\Lambda \subseteq \mathbb{P}^N$ be a linear subspace of dimension l $(n-1 \geq l \geq 0)$. Let $\bar{X}_\Lambda$ be the closure of $\pi_\Lambda(X \setminus \Lambda)$ for the linear projection $\pi_\Lambda : \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{\bar{N}}$ $(\bar{N} := N - l - 1)$. Then the following are equivalent:*

(a) $\dim\langle T_x(X), \Lambda\rangle = n + 1$ *for general* $x \in X \setminus \Lambda$.

(b) $\dim \bar{X}_\Lambda = n - l$.

(c) $\dim \mathrm{Cone}(\Lambda, \bar{X}_\Lambda) = n+1$ *for the cone* $\mathrm{Cone}(\Lambda, \bar{X}_\Lambda)$ *over* $\bar{X}_\Lambda$ *with vertex* Λ.

(d) *X is a birational-divisor of type $(\mu, \mathcal{L})$ for some $\mu(> 0) \in \mathbb{Z}$ and $\mathcal{L} \in \mathrm{Pic}\, Y$ with $(\mu, \mathcal{L}) \neq (1, \mathcal{O}_Y(-1))$ on the conical scroll $\mathbf{F}^\Lambda_Y$ with vertex Λ over a smooth $(n-l)$-dimensional projective variety Y with a birational-embedding $\nu : Y \to \mathbb{P}^{\bar{N}}$.*

Moreover, under the above equivalent conditions, the following hold:

(1) *For general $x \in X$, $(X \cap \langle\Lambda, x\rangle) \setminus \Lambda$ is an affine (possibly reducible) hypersurface in $\langle\Lambda, x\rangle \setminus \Lambda$ whose closure is a hypersurface of degree μ in $\langle\Lambda, x\rangle$ not containing Λ.*

(2) *If $\Lambda \not\subseteq X$, then $\mu = l(X \cap \langle u, x\rangle)$ for general $u \in \Lambda$ and general $x \in X$.*

(3) *If $\Lambda \subseteq X$ and $\Lambda \cap \mathrm{Sm}\, X \neq \emptyset$, then $\mu = l(X \cap \langle u, x\rangle) - 1$ for general $u \in \Lambda$ and general $x \in X$.*

(4) *Suppose $\mu \geq 2$. Then $X \subseteq \mathbb{P}^N$ is nondegenerate if and only if so is $\bar{X}_\Lambda \subseteq \mathbb{P}^{\bar{N}}$.*

(5) *For a linear subspace $\Lambda' \subseteq \mathbb{P}^N$ containing Λ as a proper subset, $\dim\langle T_x(X), \Lambda'\rangle = n + 1$ for general $x \in X \setminus \Lambda'$ if and only if $\bar{X}_\Lambda \subseteq \mathbb{P}^{\bar{N}}$ is a cone with vertex $\bar{\Lambda}'_\Lambda := \pi_\Lambda(\Lambda' \setminus \Lambda)$.*

Proof. The equivalence (a) $\Leftrightarrow$ (b) follows from $T_{\bar{x}}(\bar{X}_\Lambda) = \pi_\Lambda(\langle T_x(X), \Lambda\rangle \setminus \Lambda)$ for general $x \in X$ and for $\bar{x} := \pi_\Lambda(x) \in \bar{X}_\Lambda$ by the generic smoothness. Since $\dim \mathrm{Cone}(\Lambda, \bar{X}_\Lambda) = l+1+\dim \bar{X}_\Lambda$, (b) and (c) are equivalent. To prove $(c) \implies (d)$, suppose (c). Let $Y \to \bar{X}_\Lambda$ be the resolution of singularity of $\bar{X}_\Lambda$ ([16]) and let $\nu : Y \to \mathbb{P}^{\bar{N}}$ be the composite of $Y \to \bar{X}_\Lambda$ and the inclusion $\bar{X}_\Lambda \subseteq \mathbb{P}^{\bar{N}}$. The conical scroll $\mathbf{F}^\Lambda_Y$ is isomorphic to $\mathrm{Cone}(\Lambda, \bar{X}_\Lambda)$ except on $\tilde{\Lambda}_Y \in |\mathcal{O}_{\mathbf{F}^\Lambda_Y}(1) \otimes \tau^*_Y \mathcal{O}_Y(-1)|$ and on the fibres over the nonisomorphic locus of $Y \to \bar{X}_\Lambda$. Hence its isomorphic locus meets with X and there exists a prime divisor $\tilde{X}$ of $\mathbf{F}^\Lambda_Y$ birational to X. By [15, III, Ex. 12.5], $\tilde{X}$ is a member of $|\mathcal{O}_{\mathbf{F}^\Lambda_Y}(\mu) \otimes \tau^*_Y \mathcal{L}|$ for some $\mu > 0$ and $\mathcal{L} \in \mathrm{Pic}\, Y$ with $(\mu, \mathcal{L}) \neq (1, \mathcal{O}_Y(-1))$. Hence (d) follows. If (d) holds, then $\mathbf{F}^\Lambda_Y$ is birational to $\mathrm{Cone}(\Lambda, \bar{X}_\Lambda)$, and (c) holds.

(1) By assumption, $\tilde{X} \neq \tilde{\Lambda}_Y$ and $\tilde{X} \cap \tilde{\Lambda}_Y$ is a divisor of $\tilde{\Lambda}_Y = \Lambda \times Y$. Let $u \in \Lambda$ be a general point so that $\{u\} \times Y \not\subseteq \tilde{X} \cap \tilde{\Lambda}_Y$. Let $x \in X$ be general points so that $\bar{x}_\Lambda := \pi_\Lambda(x) \in \mathrm{Sm}\, \bar{X}_\Lambda$, $\bar{x}_\Lambda = \nu(y)$ for a unique point $y \in Y$, and $(u, y) \notin \tilde{X} \cap \tilde{\Lambda}_Y$. The intersection $\tilde{X}_y := \tilde{X} \cap \tau_Y^{-1}(y) \subseteq \mathbf{F}^\Lambda_Y$ is a hypersurface of degree μ in $\tau_Y^{-1}(y) \cong \langle\Lambda, x\rangle$ such that $\tilde{X}_y \not\supseteq \Lambda \times \{y\}$. Hence $\tilde{X}_y$ is the closure of $\tilde{X}_y \setminus \Lambda \times \{y\}$ in $\tau_Y^{-1}(y)$. By the generality of x, the induced morphism $\mathbf{F}^\Lambda_Y \setminus \tilde{\Lambda}_Y \to \mathrm{Cone}(\Lambda, \bar{X}_\Lambda) \setminus \Lambda$

from φ_Y is isomorphic along $\tau_Y^{-1}(y)$, and hence $\tilde{X}_y \setminus \Lambda \times \{y\} \cong X \cap \langle \Lambda, x\rangle \setminus \Lambda$. Consequently, the closure $\overline{(X \cap \langle \Lambda, x\rangle) \setminus \Lambda}$ is a hypersurface of degree μ isomorphic to $\tilde{X}_y$.

(2) If $\Lambda \not\subset X$, for general $u \in \Lambda$ and $x \in X$, then $X \cap \Lambda \cap \langle u, x\rangle = \emptyset$; hence $l(X \cap \langle u, x\rangle) = l(X \cap \langle \Lambda, x\rangle \cap \langle u, x\rangle) = l((X \cap \langle \Lambda, x\rangle \setminus \Lambda) \cap \langle u, x\rangle) = \mu$ by (1).

(3) Suppose $\Lambda \subseteq X$ and $\Lambda \cap \operatorname{Sm} X \neq \emptyset$. For general $u \in \Lambda$ and $x \in X$, we may assume $u \in \Lambda \cap \operatorname{Sm} X$ and $x \notin T_u(X)$, i.e., the local length $l_u(X \cap \langle u, x\rangle)$ of $X \cap \langle u, x\rangle$ at u is one. Moreover $u \notin \overline{(X \cap \langle \Lambda, x\rangle) \setminus \Lambda}$ and $X \cap \Lambda \cap \langle u, x\rangle = \{u\}$. Hence, by (1), $l(X \cap \langle u, x\rangle) = l((X \cap \langle \Lambda, x\rangle \setminus \Lambda) \cap \langle u, x\rangle) + l_u(X \cap \langle u, x\rangle) = \mu + 1$.

(4) $X \subseteq \mathbb{P}^N$ is nondegenerate, so is $\bar{X}_\Lambda = \nu(Y) \subseteq \mathbb{P}^{\bar{N}}$. Conversely, suppose X is contained in a hyperplane $H \subseteq \mathbb{P}^N$. For general $x \in X$, H contains the hypersurface $\overline{(X \cap \langle \Lambda, x\rangle) \setminus \Lambda}$ in $\langle \Lambda, x\rangle$ of degree $\mu \geq 2$, and hence $\langle \Lambda, x\rangle$. Thus $\bar{X}_\Lambda$ is degenerate.

(5) To prove the only if part, for general $x \in X \setminus \Lambda'$, assuming $\dim\langle T_x(X), \Lambda'\rangle = n+1$, it suffices to show that $\bar{X}_\Lambda$ contains $\overline{\langle \Lambda', x\rangle}_\Lambda := \overline{\pi_\Lambda(\langle \Lambda', x\rangle \setminus \Lambda)}$. By (1), $\overline{(X \cap \langle \Lambda', x\rangle) \setminus \Lambda'}$ is a hypersurface in $\langle \Lambda', x\rangle$. By the linear projection $\pi_\Lambda : \langle \Lambda', x\rangle \setminus \Lambda \to \overline{\langle \Lambda', x\rangle}_\Lambda$, the hypersurface is mapped onto $\overline{\langle \Lambda', x\rangle}_\Lambda$ or it is a cone with vertex Λ. In the latter, by the generality of x, X is a cone with vertex Λ which contradicts (c). Hence $\overline{\langle \Lambda', x\rangle}_\Lambda \subseteq \bar{X}_\Lambda$. Conversely, for the if part, suppose $\bar{X}_\Lambda \subseteq \mathbb{P}^{\bar{N}}$ is a cone with vertex $\bar{\Lambda}'_\Lambda$. Set $l' := \dim \Lambda'$ and $l'' := \dim \bar{\Lambda}'_\Lambda$. Hence $l' = l + l'' + 1$. The closure $\bar{X}_{\Lambda'}$ of the image of X by the linear projection from Λ' can be seen as the image $\bar{X}_\Lambda$ by the linear projection from $\bar{\Lambda}'_\Lambda$. Hence $\dim \bar{X}_{\Lambda'} = n - l'$. By the first part, $\dim\langle T_x(X), \Lambda'\rangle = n+1$ for general $x \in X \setminus \Lambda'$. $\square$

Proof of Theorem 3. We will prove the first part. Suppose that X is nondegenerate with $\Lambda \subseteq \bar{\mathcal{B}}(X)$ and $\Lambda \not\subseteq X$. By Theorem 1, (a) of Lemma 1.1 holds, and hence, X is a birational-divisor on the conical scroll $\mathbf{F}_Y^\Lambda$ of type $(\mu, \mathcal{L})$ for some $\mu \geq 1$ and $\mathcal{L} \in \operatorname{Pic} Y$. Since $\Lambda \cap \mathcal{B}(X) \neq \emptyset$, by (2) of Lemma 1.1, we have $\mu \geq 2$. By (4) of Lemma 1.1, $\nu(Y) \subseteq \mathbb{P}^{\bar{N}}$ is nondegenerate. It remains to show that $\mathcal{L} \cong \mathcal{O}_Y$. This is proved for $0 < l < n$ in [22, Lemma 4.5]. To prove this for $l = 0$, as in the case $l > 0$, let $G_{\tilde{X}} \in H^0(\mathcal{O}_{\mathbf{F}_Y^\Lambda}(\mu) \otimes \tau_Y^*\mathcal{L})$ be the section defining the original divisor $\tilde{X}$ for X. For $\tilde{\Lambda}_Y = \Lambda \times Y \cong Y$, we have $G_{\tilde{X}}|_{\tilde{\Lambda}_Y} \in H^0(\mathcal{L})$. Moreover $G_{\tilde{X}}|_{\Lambda \times \{y\}}$ is nonzero for any $y \in Y$ since $\Lambda \not\subseteq X$. This means that $\mathcal{L}$ has a nowhere vanishing global section and hence $\mathcal{L} \cong \mathcal{O}_Y$.

Conversely, suppose that X is a birational-divisor of type $(\mu, \mathcal{O}_Y)$ $(\mu \geq 2)$ on $\mathbf{F}_Y^\Lambda$ over an $(n-l)$-dimensional smooth projective variety Y with a nondegenerate birational-embedding $\nu : Y \to \mathbb{P}^{\bar{N}}$. By (4) of Lemma 1.1, X is nondegenerate. The original divisor $\tilde{X} (\subseteq \mathbf{F}_Y^\Lambda)$ for X is not equal to $\tilde{\Lambda}_Y$ and $\tilde{X} \cap \tilde{\Lambda}_Y \in |\mathcal{O}_{\tilde{\Lambda}_Y}(\mu)|$. Since $H^0(\mathcal{O}_{\tilde{\Lambda}_Y}(\mu)) \cong H^0(\mathcal{O}_\Lambda(\mu))$, $X \cap \Lambda$ is codimension one in Λ and $\Lambda \not\subset X$. Consequently $\Lambda \subseteq \bar{\mathcal{B}}(X)$ by (2) of Lemma 1.1.

We will show (1)-(4), supposing X is nondegenerate with $\Lambda \subseteq \bar{\mathcal{B}}(X)$ and $\Lambda \not\subseteq X$. (1) follows from (2) of Lemma 1.1. (2) follows from $\deg X = (\mathcal{O}_{\mathbf{F}_Y^\Lambda}(\mu), \mathcal{O}_{\mathbf{F}_Y^\Lambda}(1)^n) = \mu \cdot \deg \nu(Y)$. (3) is proved in [22, Theorem 4.4]. Finally we will prove (4). If Λ is a proper subset of an irreducible component Λ' of $\bar{\mathcal{B}}(X)$, then Λ' is linear and $\dim\langle T_x(X), \Lambda'\rangle = n+1$ for general $x \in X \setminus \Lambda'$ by Theorem 1, and hence $\bar{X}_\Lambda = \nu(Y)$ is a cone by (5) of Lemma 1.1. Conversely suppose $\bar{X}_\Lambda$ is a cone with vertex $\Lambda'' \subseteq \mathbb{P}^{\bar{N}}$. Set $\Lambda' := \langle \Lambda, \Lambda''\rangle \subseteq \mathbb{P}^N$. For general $x \in X$, $\overline{(X \cap \langle \Lambda', x\rangle) \setminus \Lambda'}$ is a hypersurface in $\langle \Lambda', x\rangle$ by Lemma 1.1. The hypersurface is of degree $\mu(\geq 2)$, since $l(X \cap \langle v, x\rangle) = \mu$ for general $v \in \Lambda$ by (2) of Lemma 1.1 and since $X \cap \Lambda' \cap \langle v, x\rangle = \emptyset$ by the generality of x. Hence $(\Lambda \subset)\Lambda' \subseteq \bar{\mathcal{B}}(X)$. □

2. The structure of projective varieties with nonbirational inner centers

In this section, we find conditions for a birational-divisor of a conical scroll to be smooth at general points of the vertex (Proposition 2.3). We begin with the following proposition.

Proposition 2.1. *Let $X \subseteq \mathbb{P}^N$ be a projective variety of dimension $n > 0$ and codimension $e \geq 2$. Let Λ be an l-dimensional linear subspace of $\mathbb{P}^N$ $(0 \leq l \leq n-1)$. Set $\bar{N} = N - l - 1$. Then the following are equivalent:*

1. *X is nondegenerate with $\Lambda \subseteq \bar{\mathcal{C}}(X)$ and $\Lambda \cap \operatorname{Sm} X \neq \emptyset$.*
2. *X is a birational-divisor of type $(\mu, \mathcal{L})$ $(\mu \geq 2)$ on the conical scroll $\mathbf{F}_Y^\Lambda$ with vertex Λ over an $(n-l)$-dimensional smooth projective variety Y with a nondegenerate birational-embedding $\nu : Y \to \mathbb{P}^{\bar{N}}$ such that $\Lambda \subseteq X$ and $\Lambda \cap \operatorname{Sm} X \neq \emptyset$.*

Moreover, under the condition above, Λ is an irreducible component of $\bar{\mathcal{C}}(X)$ if and only if $\nu(Y)$ is not a cone.

Proof. To prove (1) $\Longrightarrow$ (2), suppose (1). For general $x \in X$, $\dim\langle T_x(X), \Lambda\rangle = n + 1$ by Theorem 1. Hence X is a birational-divisor of type $(\mu, \mathcal{L})$ on the conical scroll $\mathbf{F}_Y^\Lambda$ over an $(n-l)$-dimensional smooth projective variety Y with a birational-embedding $\nu : Y \to \mathbb{P}^{\bar{N}}$ by Lemma 1.1. Moreover $\nu(Y)$ is nondegenerate by (4) of Lemma 1.1. For general $x \in X$ and $u \in \Lambda$, $l(X \cap \langle u, x\rangle) \geq 3$ since $\Lambda \cap \operatorname{Sm} X \subseteq \mathcal{C}(X)$. Hence $\mu \geq 2$ by (3) of Lemma 1.1. Conversely, to prove (2) $\Longrightarrow$ (1), suppose (2). Since $\mu \geq 2$ and $\nu(Y)$ is nondegenerate, X is nondegenerate by (4) of Lemma 1.1. For general $x \in X$ and $u \in \Lambda$, $l(X \cap \langle u, x\rangle) = \mu + 1 \geq 3$ by (3) of Lemma 1.1. This means $\Lambda \cap \operatorname{Sm} X (\neq \emptyset) \subseteq \mathcal{C}(X)$.

We will prove the second part. The if part follows from (5) of Lemma 1.1 as in Theorem 3. To prove the only if part, suppose that $\bar{X}_\Lambda = \nu(Y)$ is a cone with vertex $\Lambda'' \subseteq \mathbb{P}^{\bar{N}}$. We will show $\Lambda' := \langle \Lambda, \Lambda''\rangle \subseteq \bar{\mathcal{C}}(X)$. For general $x \in X$, $\overline{X \cap \langle \Lambda', x\rangle \setminus \Lambda'}$ is a hypersurface in $\langle \Lambda', x\rangle$ by Lemma 1.1, and let m be its degree. Then $m \geq l((X \cap \langle \Lambda', x\rangle \setminus \Lambda') \cap \langle u, x\rangle) = l(X \cap \langle u, x\rangle) - l_u(X \cap \langle u, x\rangle) = \mu \geq 2$ for general $u \in \Lambda$ since the local length $l_u(X \cap \langle u, x\rangle)$ is one (see Lemma 1.1). If $\Lambda' \not\subseteq X$, $m = l(X \cap \langle w, x\rangle)$ for general $w \in \Lambda'$ by (2) of Lemma 1.1, and hence $\Lambda' \subseteq \bar{\mathcal{B}}(X)$ and $\Lambda \subseteq \Lambda' \cap X \subseteq \operatorname{Sing} X$ by (3) of Theorem 3, a contradiction. Consequently $\Lambda' \subseteq X$. Hence $m = l(X \cap \langle w, x\rangle) - 1$ for general $w \in \Lambda'$ by (3) of Lemma 1.1, since $\Lambda' \cap \operatorname{Sm} X (\supseteq \Lambda \cap \operatorname{Sm} X) \neq \emptyset$. This means $\Lambda' \subseteq \bar{\mathcal{C}}(X)$. □

For the remainder of this section, we assume the following conditions.

(2.2). Let $X \subseteq \mathbb{P}^N$ be a nondegenerate projective variety of dimension $n > 0$ which is a birational-divisor of type $(\mu, \mathcal{L})$ $(\mu \geq 1, \mathcal{L} \in \operatorname{Pic} Y)$ on the conical scroll $\mathbf{F}_Y^\Lambda$ with an l-dimensional linear subspace $\Lambda \subseteq \mathbb{P}^N$ $(0 \leq l \leq n-1)$ as vertex over an $(n-l)$-dimensional smooth projective variety Y with a birational-embedding $\nu : Y \to \mathbb{P}^{\bar{N}}$ $(\bar{N} = N - l - 1)$. We keep the notation as in Definition 2. Set $\bar{n} = n - l$. Let $\bar{X}_\Lambda$ be the closure of $\pi_\Lambda(X \setminus \Lambda)$ for the linear projection $\pi_\Lambda : \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{\bar{N}}$.

Proposition 2.3. *Under* (2.2), *suppose that* $\Lambda \subseteq X$ *and* $\Lambda \cap \operatorname{Sm} X \neq \emptyset$. *Then the following hold:*

(1) $(\mathcal{L}, \mathcal{O}_Y(1)^{\bar{n}-1}) = 1$ *and* $H^0(Y, \mathcal{L}) \neq 0$. *Hence* $d = \deg X = \mu \cdot \deg \nu(Y) + 1$.

(2) *The intersection* $\tilde{X} \cap \tilde{\Lambda}_Y$ *is an effective divisor of* $\tilde{\Lambda}_Y$ *containing a prime divisor* $\tilde{D}$ *whose image by the natural morphism* $\tilde{\Lambda}_Y \cong \Lambda \times Y \to \Lambda \times \bar{X}_\Lambda$ *is the closure of* $\amalg_{y \in \Lambda \cap \operatorname{Sm} X} \overline{T_y(X)}_\Lambda$, *where* $\overline{T_y(X)}_\Lambda := \pi_\Lambda(T_y(X) \setminus \Lambda)$.

(3) *The divisor* $\tilde{D}$ *in* (2) *is a unique irreducible component of* $\tilde{X} \cap \tilde{\Lambda}_Y$ *such that* $\varphi_Y(\tilde{D}) = \Lambda$ *and* $\dim \nu(\tau_Y(\tilde{D})) \geq \dim \bar{X}_\Lambda - 1$. *Moreover* $\operatorname{ord}_{\tilde{D}}(\tilde{X} \cap \tilde{\Lambda}_Y) = 1$, *where* $\operatorname{ord}_{\tilde{D}}(\cdot)$ *is the order along* $\tilde{D}$.

To prove Proposition 2.3, we need the following lemma.

Lemma 2.4. *Suppose* $\Lambda \subseteq X$, $\Lambda \cap \operatorname{Sm} X \neq \emptyset$, *and* $\bar{n} = n - l \geq 2$. *Let* $H \subseteq \mathbb{P}^N$ *be a general hyperplane containing* Λ *and let* $\bar{H}_\Lambda$ *be the hyperplane* $\pi_\Lambda(H \setminus \Lambda)$ *in* $\mathbb{P}^{\bar{N}}$. *Then* $Y_{\bar{H}_\Lambda} := Y \times_{\mathbb{P}^{\bar{N}}} \bar{H}_\Lambda$ *is smooth and irreducible,* $\tilde{X}_{\bar{H}_\Lambda} := \tilde{X} \times_{\mathbb{P}^{\bar{N}}} \bar{H}_\Lambda$ *is irreducible and reduced, and* $X_{\bar{H}_\Lambda} := (X \cap H)_{\mathrm{red}}$ *is a nondegenerate projective variety in* H *with* $\Lambda \subseteq X_{\bar{H}_\Lambda}$ *and* $\Lambda \cap \operatorname{Sm} X_{\bar{H}_\Lambda} \neq \emptyset$. *Consequently* $X_{\bar{H}_\Lambda}$ *is a birational-divisor of type* $(\mu, \mathcal{L}|_{Y_{\bar{H}_\Lambda}})$ *on the conical scroll* $\mathbf{F}^\Lambda_{Y_{\bar{H}_\Lambda}}$ *with vertex* Λ *over* $Y_{\bar{H}_\Lambda}$ *whose original divisor is* $\tilde{X}_{\bar{H}_\Lambda}$.

Proof. Since $\bar{H}_\Lambda \subseteq \mathbb{P}^{\bar{N}}$ is general, the reducedness and the smoothness of $\mathbb{P}^{\bar{N}}$-schemes are stable under the pull-back $\bar{H}_\Lambda \to \mathbb{P}^{\bar{N}}$ by Bertini's Theorem in characteristic zero (see [9, (3.4.9)]). Also the irreducibility is stable by the assumption $\dim \bar{X}_\Lambda = \bar{n} \geq 2$ (see [9, (3.4.10)]). Hence $Y_{\bar{H}_\Lambda}$ is smooth and irreducible, and $\tilde{X}_{\bar{H}_\Lambda}$ is irreducible and reduced. Moreover, $Y_{\bar{H}_\Lambda} \to \bar{H}_\Lambda$ is a birational-embedding, and $\tilde{X}_{\bar{H}_\Lambda}$ is birational to $(X \setminus \Lambda)_{\bar{H}_\Lambda}$ by the induced morphism $\varphi_{Y_{\bar{H}_\Lambda}} : \mathbf{F}^\Lambda_{Y_{\bar{H}_\Lambda}} \to \bar{H}_\Lambda$, and hence $X_{\bar{H}_\Lambda} = \varphi_{Y_{\bar{H}_\Lambda}}(\tilde{X}_{\bar{H}_\Lambda})$. This means that $X_{\bar{H}_\Lambda}$ is a birational-divisor of type $(\mu, \mathcal{L}|_{Y_{\bar{H}_\Lambda}})$ on the conical scroll $\mathbf{F}^\Lambda_{Y_{\bar{H}_\Lambda}}$ with original divisor $\tilde{X}_{\bar{H}_\Lambda}$. To see $\Lambda \cap \operatorname{Sm} X_{\bar{H}_\Lambda} \neq \emptyset$, we note, by the generality of H, that $T_y(X) \not\subseteq H$ for general $y \in \Lambda \cap \operatorname{Sm} X$, and hence $X \cap H$ is smooth at y. Therefore $\Lambda \cap \operatorname{Sm} X_{\bar{H}_\Lambda} \neq \emptyset$, since $X_{\bar{H}_\Lambda}$ and $X \cap H$ are equal on the union of $(X \setminus \Lambda) \cap H$ and the locus of points $y \in \Lambda \cap \operatorname{Sm} X$ with $T_y(X) \not\subseteq H$. □

Proof of Proposition 2.3. First we will prove (2). Since X is birational to $\tilde{X}$ and $\dim \tilde{X} = \dim \tilde{\Lambda}_Y$, $\tilde{X} \neq \tilde{\Lambda}_Y$ and $\tilde{X} \cap \tilde{\Lambda}_Y$ is a divisor of $\tilde{\Lambda}_Y$. Consider the blowing-up $\sigma_{X/\Lambda} : \hat{X}_\Lambda \to X$ of X along Λ, which is a closed subset of the blowing-up $\sigma : \hat{\mathbb{P}}^N_\Lambda \to \mathbb{P}^N$ of $\mathbb{P}^N$ along Λ by $\mathcal{I}_{\Lambda/\mathbb{P}^N} \otimes \mathcal{O}_X \to \mathcal{I}_{\Lambda/X}$. Here note that $\hat{\mathbb{P}}^N_\Lambda = \mathbf{F}^\Lambda$ as a closed subscheme of $\mathbb{P}^N \times \mathbb{P}^{\bar{N}}$ and $\sigma = \varphi$. Moreover $\hat{X}_\Lambda$ is a closed subscheme of the pull-back $\mathbf{F}^\Lambda_{\bar{X}_\Lambda} := \mathbf{F}^\Lambda \times_{\mathbb{P}^{\bar{N}}} \bar{X}_\Lambda$ of $\tau : \mathbf{F}^\Lambda \to \mathbb{P}^{\bar{N}}$ by $\bar{X}_\Lambda \to \mathbb{P}^{\bar{N}}$ since $\tau(\hat{X}_\Lambda) = \bar{X}_\Lambda$. Hence $\hat{X}_\Lambda$ is the birational image of $\tilde{X}$ by $\mathbf{F}^\Lambda_Y \to \mathbf{F}^\Lambda_{\bar{X}_\Lambda}$. For $\hat{\Lambda} := \mathbf{F}^\Lambda_{\bar{X}_\Lambda} \times_{\mathbb{P}^N} \Lambda = \Lambda \times \bar{X}_\Lambda$, we have $\hat{\Lambda} \times_{\mathbf{F}^\Lambda_{\bar{X}_\Lambda}} \mathbf{F}^\Lambda_Y = \Lambda \times Y$, and hence the induced morphism $\tilde{X} \cap \tilde{\Lambda}_Y \to \hat{X}_\Lambda \cap \hat{\Lambda}$ is surjective. To obtain $\tilde{D}$, we will show $E_0 := \amalg_{y \in \Lambda \cap \operatorname{Sm} X} \overline{T_y(X)}_\Lambda$ is a subscheme of $\hat{X}_\Lambda \cap \hat{\Lambda}$. In fact, $\tilde{D}$ is an $(n-1)$-dimensional component of $\tilde{X} \cap \tilde{\Lambda}_Y$ dominating E_0 under the surjection. Since $\hat{X}_\Lambda \cap \hat{\Lambda} = \sigma^{-1}_{X/\Lambda}(\Lambda) = \mathbf{Proj} \bigoplus_{k \geq 0} \mathcal{I}^k_{\Lambda/X} / \mathcal{I}^{k+1}_{\Lambda/X}$ and since $\mathcal{I}_{\Lambda/X}/\mathcal{I}^2_{\Lambda/X}$ is locally free on $\Lambda \cap \operatorname{Sm} X$, we have only to show that $\mathbb{P}_{\Lambda \cap \operatorname{Sm} X}(\mathcal{I}_{\Lambda/X}/\mathcal{I}^2_{\Lambda/X}|_{\Lambda \cap \operatorname{Sm} X})$ is E_0.

Set $V := H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ and $W := H^0(\mathbb{P}^N, \mathcal{I}_{\Lambda/\mathbb{P}^N}(1))$. Then $\mathbb{P}^N = \mathbb{P}(V)$ and $\mathbb{P}^{\bar{N}} = \mathbb{P}(W)$, and π_Λ is defined by the evaluation $\varepsilon : W \otimes \mathcal{O}_{\mathbb{P}^N}(1) \to \mathcal{I}_{\Lambda/\mathbb{P}^N}(1)$. Recall that the tangent bundle $\amalg_{x\in X} T_x(X) \subseteq X \times \mathbb{P}^N$ corresponds to the surjection from $V \otimes \mathcal{O}_X$ to the bundle $P^1_X(\mathcal{O}_X(1))$ of the principal part (see [19, Ch. IV.A], [13, Ch. IV, §16]). Hence $\overline{T_y(X)}_\Lambda \subseteq \mathbb{P}^{\bar{N}}$ for each $y \in \Lambda \cap \operatorname{Sm} X$ corresponds to the image of $W \to V \to P^1_X(\mathcal{O}_X(1)) \otimes \Bbbk(y)$. Thus we have to show that the image of $W \otimes \mathcal{O}_\Lambda \to P^1_X(\mathcal{O}_X(1))|_\Lambda$ is $(\mathcal{I}_{\Lambda/X}/\mathcal{I}^2_{\Lambda/X})(1)$. This follows by comparing two of the exact sequences $0 \to (\mathcal{I}_{\Lambda/X}/\mathcal{I}^2_{\Lambda/X})(1) \to P^1_X(\mathcal{O}_X(1))|_\Lambda \to P^1_\Lambda(\mathcal{O}_\Lambda(1)) \to 0$ on $\Lambda \cap \operatorname{Sm} X$ and $0 \to W \otimes \mathcal{O}_\Lambda \to V \otimes \mathcal{O}_\Lambda \to V/W \otimes \mathcal{O}_\Lambda \to 0$, since $V/W \otimes \mathcal{O}_\Lambda \to P^1_\Lambda(\mathcal{O}_\Lambda(1))$ is isomorphic.

We will prove (1) and (3) in case $\dim Y = \bar{n} = 1$. Since $\tilde{X} \cap \tilde{\Lambda}_Y$ is a divisor on $\tilde{\Lambda}_Y$, the birational morphism $\tilde{X} \to X$ is quasi-finite at general points of Λ and hence scheme-theoretically one-to-one at general points of $\Lambda \cap \operatorname{Sm} X \neq \emptyset$ by Zariski's Main Theorem ([15, III.11.4]). Consequently $\tilde{X} \cap \tilde{\Lambda}_Y \to \Lambda$ is generically isomorphic. Since $\tilde{X} \cap \tilde{\Lambda}_Y \in |\mathcal{O}_{\tilde{\Lambda}_Y}(\mu) \otimes \bar{\tau}_Y^* \mathcal{L}|$, this means that $\deg \mathcal{L} = 1$ and $H^0(Y, \mathcal{L}) \neq 0$, which proves (1), and also the uniqueness of the component $\tilde{D}$ in (2), which proves (3).

Now we prove (1) and (3) in case $\dim Y = \bar{n} \geq 2$ by the induction on $\bar{n}$. For a general hyperplane $H \subseteq \mathbb{P}^N$ containing Λ and for $\bar{H}_\Lambda := \pi_\Lambda(H \setminus \Lambda)$, by (2.4), $X_{\bar{H}_\Lambda} := (X \cap H)_{\mathrm{red}}$ is a nondegenerate projective variety in H with $\Lambda \cap \operatorname{Sm} X_{\bar{H}_\Lambda} \neq \emptyset$ which is a birational-divisor of type $(\mu, \mathcal{L}|_{Y_{\bar{H}_\Lambda}})$ on the conical scroll $\mathbf{F}^\Lambda_{Y_{\bar{H}_\Lambda}}$ with vertex Λ over the $(\bar{n}-1)$-dimensional smooth projective variety $Y_{\bar{H}_\Lambda} := Y \times_{\mathbb{P}^{\bar{N}}} \bar{H}_\Lambda$. By the induction, $X_{\bar{H}_\Lambda}$ satisfies (1) and (3). Hence $(\mathcal{L}, \mathcal{O}_Y(1)^{\bar{n}-1}) = (\mathcal{L}|_{Y_{\bar{H}_\Lambda}}, \mathcal{O}_{Y_{\bar{H}_\Lambda}}(1)^{\bar{n}-2}) = 1$. Since $\tilde{X} \cap \tilde{\Lambda}_Y \in |\mathcal{O}_{\tilde{\Lambda}_Y}(\mu) \otimes \bar{\tau}_Y^* \mathcal{L}|$ is a nonzero effective divisor, $\mathcal{L}$ has a nonzero global section, which proves (1).

To prove (3) in case $\bar{n} \geq 2$, note that if D is an irreducible component of $\tilde{X} \cap \tilde{\Lambda}_Y$ with $\dim \nu(\tau_Y(D)) \geq \dim \bar{X}_\Lambda - 1 = \bar{n} - 1 \geq 1$, then the pull-back $D_{\bar{H}_\Lambda} := D \times_{\mathbb{P}^{\bar{N}}} \bar{H}_\Lambda$ is irreducible and reduced. Indeed, by Bertini's Theorem ([9, (3.4.9) and (3.4.10)]), $D_{\bar{H}_\Lambda}$ is irreducible and reduced unless $\dim \nu(\tau_Y(D)) = 1$. If $\dim \nu(\tau_Y(D)) = 1$, then $\bar{n} = 2$ and $D \cong \tau_Y(D) \times \Lambda$, and consequently $D_{\bar{H}_\Lambda} \cong \tau_Y(D)_{\bar{H}_\Lambda} \times \Lambda$ for $\tau_Y(D)_{\bar{H}_\Lambda} := \tau_Y(D) \times_{\mathbb{P}^{\bar{N}}} \bar{H}_\Lambda$, which is reduced by the generality of $\bar{H}_\Lambda$; hence $\deg \tau_Y(D)_{\bar{H}_\Lambda} = 1$ by (3) in case $\bar{n} = 1$. This implies the irreducibility of $D_{\bar{H}_\Lambda}$ for $\dim \nu(\tau_Y(D)) = 1$. To prove the uniqueness of $\tilde{D}$ for $\bar{n} \geq 2$, by the contradiction, we assume that $\tilde{D}'$ ($\neq \tilde{D}$) is another irreducible component of $\tilde{X} \cap \tilde{\Lambda}_Y$ such that $\dim \nu(\tau_Y(\tilde{D}')) \geq \dim \bar{X}_\Lambda - 1$ and $\varphi_Y(\tilde{D}') = \Lambda$. Then $\tilde{D}'_{\bar{H}_\Lambda} \neq \tilde{D}_{\bar{H}_\Lambda}$ satisfy the same property, which contradicts the uniqueness of $\tilde{D}_{\bar{H}_\Lambda}$. The second part is clear since if $\operatorname{ord}_{\tilde{D}}(\tilde{X} \cap \tilde{\Lambda}_Y) > 1$, then $\operatorname{ord}_{\tilde{D}_{\bar{H}_\Lambda}}(\tilde{X}_{\bar{H}_\Lambda} \cap (\tilde{\Lambda}_Y)_{\bar{H}_\Lambda}) > 1$ for $(\tilde{\Lambda}_Y)_{\bar{H}_\Lambda} := (\tilde{\Lambda}_Y) \times_{\mathbb{P}^{\bar{N}}} \bar{H}_\Lambda$. □

We conclude this section by proving the following lemma about the singular locus of X on Λ.

Lemma 2.5. *Under* (2.2), *assume that* $\bar{n} \geq 1$ *and* $l \geq 1$. *If* $u \in \Lambda$ *is a point such that* $\{u\} \times Y \subseteq \tilde{X} \cap \tilde{\Lambda}_Y$ *in* $\tilde{\Lambda}_Y$, *then* $u \in \Lambda \cap \operatorname{Sing} X$.

Proof. To the contrary, suppose $u \in \operatorname{Sm} X$. Take a general hyperplane H containing Λ and set $\bar{H}_\Lambda := \pi_\Lambda(H \setminus \Lambda)$. We may assume $T_u(X) \not\subseteq H$ and $X \cap H$ is smooth at u, and hence $X_{\bar{H}_\Lambda} := (X \cap H)_{\mathrm{red}}$ is smooth at u. If $\bar{n} \geq 2$, replacing X by

$X_{\bar{H}_\Lambda}$ by Lemma 2.4, we may assume $\bar{n} = 1$, since $\tilde{X}_{\bar{H}_\Lambda} = \tilde{X} \times_{\mathbb{P}^{\bar{N}}} \bar{H}_\Lambda$ contains $\{u\} \times Y_{\bar{H}_\Lambda}$. Then H contains a general point $x \in X$ so that, by (1.1), the closure $\overline{(X \cap \langle \Lambda, x\rangle) \setminus \Lambda}$ is a hypersurface in $\langle \Lambda, x\rangle$ not containing Λ, isomorphic to $\tilde{X} \cap \tau_Y^{-1}(\nu^{-1}(\pi_\Lambda(x)))$ by φ_Y. Hence it passes through u by assumption $\{u\} \times Y \subseteq \tilde{X} \cap \tilde{\Lambda}_Y$. Thus $X \cap H$ has two distinct components through u, Λ, and $\overline{(X \cap \langle \Lambda, x\rangle) \setminus \Lambda}$. This contradicts the smoothness of $X \cap H$ at u. □

3. The structure of projective varieties with nonbirational inner centers of constant partial Gauss maps: Proof of Theorem 4

Proof of Theorem 4. By Proposition 2.1, X is a birational-divisor of type $(\mu, \mathcal{L})$ $(\mu \geq 2, \mathcal{L} \in \operatorname{Pic} Y)$ on the conical scroll $\mathbf{F}_Y^\Lambda$ with vertex Λ over a smooth $(n-l)$-dimensional projective variety Y such that (6) holds. Moreover, (1) and (4) hold by Proposition 2.3 and Lemma 1.1.

(2) If an irreducible component of $\tilde{X} \cap \tilde{\Lambda}_Y \in |\mathcal{O}_{\tilde{\Lambda}_Y}(\mu) \otimes \bar{\tau}_Y^* \mathcal{L}|$ dominates both Y and Λ, then it must be the unique divisor $\tilde{D}$ in (3) of Proposition 2.3, but $\tilde{D}$ does not dominate Y by our assumption that $\gamma|_\Lambda$ is constant and (2) of Proposition 2.3, a contradiction. Hence each irreducible component D of $\tilde{X} \cap \tilde{\Lambda}_Y$ is of the form $D = \varphi_Y(D) \times Y$ or $D = \Lambda \times \tau_Y(D)$. This means (2).

(3) For the decomposition $(w)_0 = \sum_{i=0}^r D_i$ into prime divisors D_i on Y, we may assume that the unique divisor $\tilde{D}$ in (3) of Proposition 2.3 is $\bar{\tau}_Y^*(D_0)(= \Lambda \times D_0)$ with $\dim \nu(D_0) = \dim \bar{X}_\Lambda - 1$. It follows from $(\mathcal{L}, \mathcal{O}_Y(1)^{n-l-1}) = 1$ that $\dim \nu(D_i) < \dim \nu(D_0)$ for $i \geq 1$ if $r \geq 1$. Thus we have only to show that $\nu(D_i) \subseteq \nu(D_0)$ for every $i \geq 1$. By contradiction, we assume that $\nu(D_i) \not\subseteq \nu(D_0)$ for some $i \geq 1$. Set $s := \max\{\dim \nu(D_i) | \nu(D_i) \not\subseteq \nu(D_0),\ i \geq 1\}$. By taking general hyperplane sections of $\bar{X}_\Lambda \subseteq \mathbb{P}^{\bar{N}}$ in s-times as in Lemma 2.4, and by (possibly) replacing the decomposition of $(w)_0 \in |\mathcal{L}|$, we may assume that there exists a prime divisor D_{i_0} $(i_0 > 0)$ such that $D_{i_0} \cap D_0 = \emptyset$ and $\nu(D_{i_0})$ is a point which is not contained in $\nu(D_0)$. Hence $(\Lambda \times D_0) \cap (\Lambda \times D_{i_0}) = \emptyset$ in $\tilde{\Lambda}_Y$, and the birational projective morphism $\tilde{X} \to X$ has non-connected fibres at general points of $\Lambda \cap \operatorname{Sm} X$. This is a contradiction by Zariski's Main Theorem ([15, III.11.4]).

(5) If $l = 0$, the assertion is clear since $\Lambda \cap \operatorname{Sing} X = (g)_0 = \emptyset$. If $l \geq 1$, then $(g)_0 \subseteq \Lambda \cap \operatorname{Sing} X$ by Lemma 2.5. Hence $\dim \operatorname{Sing} X \geq \dim \Lambda \cap \operatorname{Sing} X \geq \dim (g)_0 = \dim \Lambda - 1$. □

4. The construction of projective varieties with nonbirational inner centers of constant partial Gauss maps: Proof of Theorem 5

First we describe the section rings of a conical scroll and its birational-divisor.

(4.1). Keep the notation and assumption as in (2.2). Let $T_0, \dots, T_N$ be the homogeneous coordinates of $\mathbb{P}^N$. Let $S = \Bbbk[T_0, \dots, T_N]$ be the homogeneous coordinate ring of $\mathbb{P}^N$. Since we consider the target $\mathbb{P}^{\bar{N}}$ of the linear projection $\pi_\Lambda : \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{\bar{N}}$ from Λ to be a subspace of $\mathbb{P}^N$ disjoint from Λ, we may assume that $H^0(\mathcal{I}_{\mathbb{P}^{\bar{N}}}(1))$ is spanned by $T_0, \dots, T_l$ and $H^0(\mathcal{I}_\Lambda(1))$ is spanned by $T_{l+1}, \dots, T_N$. We may consider that $Z_i := T_i|_\Lambda$ $(i = 0, \dots, l)$ are the homogeneous coordinates of Λ and that $T_i|_\Lambda = 0$ $(i \geq l+1)$. Also we may consider that $T_i|_{\mathbb{P}^{\bar{N}}}$ $(i = l+1, \dots, N)$ are the homogeneous coordinates of $\mathbb{P}^{\bar{N}}$ and that $T_i|_{\mathbb{P}^{\bar{N}}} = 0$ $(i \leq l)$. The surjection

$$H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \otimes \mathcal{O}_{\mathbb{P}^{\bar{N}}} \to H^0(\mathcal{O}_\Lambda(1)) \otimes \mathcal{O}_{\mathbb{P}^{\bar{N}}} \oplus \mathcal{O}_{\mathbb{P}^{\bar{N}}}(1) =: \mathcal{F}$$

induced from the isomorphism $H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \xrightarrow{\sim} H^0(\mathcal{O}_\Lambda(1)) \oplus H^0(\mathcal{O}_{\mathbb{P}^{\bar{N}}}(1))$ corresponds to $\mathbf{F}^\Lambda \subseteq \mathbb{P}^N \times \mathbb{P}^{\bar{N}}$. Let Z_{l+1} be the formal basis of the subbundle $\mathcal{O}_{\mathbb{P}^{\bar{N}}}(1)$ of $\mathcal{F}$ so that $\mathcal{O}_{\mathbb{P}^{\bar{N}}}(1) = \mathcal{O}_{\mathbb{P}^{\bar{N}}}(1)Z_{l+1}$. Let $\mathcal{F}_Y = H^0(\mathcal{O}_\Lambda(1)) \otimes \mathcal{O}_Y \oplus \mathcal{O}_Y(1)Z_{l+1}$ be the pull-back of $\mathcal{F}$ to Y. Hence $\mathbf{F}^\Lambda_Y = \mathbb{P}_Y(\mathcal{F}_Y)$ and $\mathcal{O}_{\mathbf{F}^\Lambda_Y}(1) = \varphi_Y^*\mathcal{O}_{\mathbb{P}^N}(1)$. Let $K(Y)$ be the function field of Y. The section ring $S_{\mathbf{F}^\Lambda_Y} = \bigoplus_{m\geq 0}(S_{\mathbf{F}^\Lambda_Y})_m$ is the graded subring of $K(Y)[Z_0,\dots,Z_{l+1}]$ with $\deg(Z_i) = 1$ $(i = 0,\dots,l+1)$ such that

$$(S_{\mathbf{F}^\Lambda_Y})_m = H^0(\mathbf{F}^\Lambda_Y, \mathcal{O}_{\mathbf{F}^\Lambda_Y}(m)) = H^0(Y, \mathrm{Sym}^m(\mathcal{F}_Y)).$$

For the morphism $\nu : Y \to \mathbb{P}^{\bar{N}}$, we set $f_i := \nu^*(T_i|_{\mathbb{P}^{\bar{N}}}) \in H^0(\mathcal{O}_Y(1)))$ $(i = l+1,\dots,N)$. Then the birational-embedding $\varphi_Y : \mathbf{F}^\Lambda_Y \to \mathbb{P}^N$ is defined by

$$\varphi_Y^*[T_0,\dots,T_N] = [Z_0,\dots,Z_l, f_{l+1}Z_{l+1},\dots,f_N Z_{l+1}] \tag{4.1.1}$$

and the graded homomorphism $\varphi_Y^* : S \to S_{\mathbf{F}^\Lambda_Y}$ is given by (4.1.1). Thus the homogenous coordinate ring R of the cone $\mathrm{Cone}(\Lambda, \nu(Y))$ over $\nu(Y)$ with vertex Λ is given by

$$R = \mathrm{Im}(\varphi_Y^*) = \Bbbk[Z_0,\dots,Z_l, f_{l+1}Z_{l+1},\dots,f_N Z_{l+1}] \subseteq S_{\mathbf{F}^\Lambda_Y}$$

since $\mathrm{Cone}(\Lambda,\nu(Y)) = \varphi_Y(\mathbf{F}^\Lambda_Y)$. Note that $S_{\mathbf{F}^\Lambda_Y}$ is a finitely generated R-module since $\varphi_{Y*}\mathcal{O}_{S_{\mathbf{F}^\Lambda_Y}}$ is a coherent $\mathcal{O}_{\mathrm{Cone}(\Lambda,\bar{X}_\Lambda)}$-module. For each element

$$F \in K(Y)[Z_0,\dots,Z_{l+1}],$$

we define the Z_{l+1}-order of F by

$$\mathrm{ord}_{Z_{l+1}}(F) := \max\{m \geq 0 \mid F = Z_{l+1}^m \cdot Q \text{ for some } Q \in K(Y)[Z_0,\dots,Z_{l+1}]\}.$$

In particular, $\mathrm{ord}_{Z_{l+1}}(0) = +\infty$. If we set

$$M := Z_{l+1}^2 K(Y)[Z_0,\dots,Z_{l+1}] \cap R$$

and consider it a graded submodule by $M_m = M \cap R_m$, then M is the pull-back of the ideal $(T_{l+1},\dots,T_N)^2$ of S by φ_Y^*. For nonnegative integers m and e with $m \geq e \geq 0$, set

$$(S_{\mathbf{F}^\Lambda_Y})_{m,e} := Z_{l+1}^e H^0(Y,\mathcal{O}_Y(e)) \otimes_\Bbbk \Bbbk[Z_0,\dots,Z_l]_{m-e} \quad \text{and}$$
$$R_{m,e} := \Bbbk[f_{l+1}Z_{l+1},\dots,f_N Z_{l+1}]_e \otimes_\Bbbk \Bbbk[Z_0,\dots,Z_l]_{m-e}.$$

Hence $(S_{\mathbf{F}^\Lambda_Y})_m = \bigoplus_{e=0}^m (S_{\mathbf{F}^\Lambda_Y})_{m,e}$, $R_m = \bigoplus_{e=0}^m R_{m,e}$, and $M_m = \bigoplus_{e=2}^m R_{m,e}$. For a nonzero element $p \in (S_{\mathbf{F}^\Lambda_Y})_m$, by p^* we denote the component of p in $(S_{\mathbf{F}^\Lambda_Y})_{m,e}$ for $e = \mathrm{ord}_{Z_{l+1}}(p)$.

Let $G_{\tilde{X}} \in H^0(\mathbf{F}^\Lambda_Y, \mathcal{O}_{\mathbf{F}^\Lambda_Y}(\mu) \otimes \tau_Y^*\mathcal{L})$ be a section with $(G_{\tilde{X}})_0 = \tilde{X}$. For the dual $\mathcal{L}^\vee$ of $\mathcal{L}$, we set

$$J_{\tilde{X}} := \bigoplus_{m\geq 0} B_{m-\mu} \cdot G_{\tilde{X}} \subseteq S_{\mathbf{F}^\Lambda_Y} \quad \text{for} \quad B_m := H^0(\mathbf{F}^\Lambda_Y, \mathcal{O}_{\mathbf{F}^\Lambda_Y}(m) \otimes \tau_Y^*\mathcal{L}^\vee),$$

which have the decompositions $B_m = \bigoplus_{e=0}^m B_{m,e}$ for

$$B_{m,e} := Z_{l+1}^e H^0(Y, \mathcal{O}_Y(e) \otimes \mathcal{L}^\vee) \otimes_\Bbbk \Bbbk[Z_0,\dots,Z_l]_{m-e}.$$

Let $S_{\tilde{X}} = \bigoplus_{m\geq 0} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(m))$ be the section ring of $\tilde{X}$. Then $0 \to J_{\tilde{X}} \to S_{\mathbf{F}^\Lambda_Y} \to S_{\tilde{X}}$ is exact. The homogeneous ideal I_X of $X(= \varphi_Y(\tilde{X})) \subseteq \mathbb{P}^N$ is given by $I_X = (\varphi_Y^*)^{-1}(J_{\tilde{X}})$, since the ideal sheaf $\mathcal{I}_X$ of X is the kernel of $\mathcal{O}_{\mathbb{P}^N} \to \varphi_{Y*}\mathcal{O}_{\tilde{X}}$.

Proof of Theorem 5. By the assumption (2) in Theorem 4, we have $G_{\tilde{X}}|_{\tilde{\Lambda}} = gw$ for $g \in H^0(\Lambda, \mathcal{O}_\Lambda(\mu)) = \Bbbk[Z_0, \ldots, Z_l]_\mu$ and $w \in H^0(Y, \mathcal{L})$, and hence

$$G_{\tilde{X}} = gw - h \tag{4.1.2}$$

for some $h \in H^0(\mathbf{F}_Y^\Lambda, \mathcal{O}_{\mathbf{F}_Y^\Lambda}(\mu) \otimes \tau_Y^* \mathcal{L})$ with $\mathrm{ord}_{Z_{l+1}}(h) \geq 1$. To prove Theorem 5, assuming the divisor $D_0 := (w)_0 \in |\mathcal{L}|$ is irreducible, we have only to show that

$$\Lambda \cap \mathrm{Sm}\, X \supseteq \Lambda \setminus (g)_0 \quad (\neq \emptyset). \tag{4.1.3}$$

In fact, from (4.1.3) it follows that $\Lambda \cap \mathrm{Sm}\, X = \Lambda \setminus (g)_0$ since $(g)_0 \subseteq \Lambda \cap \mathrm{Sing}\, X$ by Lemma 2.5, and hence if $\mu \geq 2$ furthermore, then $\Lambda \setminus (g)_0 \subseteq \mathcal{C}(X)$ by Proposition 2.1 and $\gamma|_\Lambda$ is constant since the prime divisor $\tilde{D} (\subseteq \tilde{X} \cap \tilde{\Lambda}_Y)$ in Proposition 2.3 is $D_0 \times \Lambda (\subseteq \tilde{\Lambda}_Y)$. To prove (4.1.3), we will find homogeneous polynomials defining X which show the smoothness of X along $\Lambda \setminus (g)_0$.

Before starting the proof, for $\mathcal{O}_{D_0}(k) := \mathcal{O}_Y(k)|_{D_0}$, we claim that $H^0(\mathcal{O}_{D_0}(k)) \cong H^0(\mathcal{O}_{\mathbb{P}^{\bar{n}-1}}(k))$ ($\bar{n} = n - l$) for every $k \geq 0$. Indeed, $\nu(D_0) \cong \mathbb{P}^{\bar{n}-1}$ and $\bar{\nu} : D_0 \to \nu(D_0)$ is birational since $(\mathcal{L}, \mathcal{O}_Y(1)^{\bar{n}-1}) = 1$ and D_0 is irreducible. Hence $\bar{\nu}_* \mathcal{O}_{D_0} \cong \mathcal{O}_{\mathbb{P}^{\bar{n}-1}}$ by Zariski's Main Theorem ([15, III.11.4]). Consequently $H^0(\mathcal{O}_{D_0}(k)) \cong H^0(\bar{\nu}_* \mathcal{O}_{D_0}(k)) \cong H^0(\mathcal{O}_{\mathbb{P}^{\bar{n}-1}}(k))$ by the projection formula. Thus we may assume that $f_{l+1}|_{D_0}, \ldots, f_n|_{D_0}$ consist of a basis of $H^0(D_0, \mathcal{O}_{D_0}(1)) \cong H^0(\mathcal{O}_{\mathbb{P}^{\bar{n}-1}}(1))$ and that $f_{n+1}|_{D_0} = \cdots = f_N|_{D_0} = 0$ after a change of the basis. Let $\Bbbk[f_{l+1}, \ldots, f_n]$ be the graded $\Bbbk$-subalgebra of the section ring $\bigoplus_{k \geq 0} H^0(Y, \mathcal{O}_Y(k))$ generated by $f_{l+1}, \ldots, f_n$, which is the polynomial ring with $n - l$ variables. From the exact sequence

$$0 \to H^0(Y, \mathcal{O}_Y(e) \otimes \mathcal{L}^\vee) \overset{\times w}{\to} H^0(Y, \mathcal{O}_Y(e)) \overset{|_{D_0}}{\to} H^0(D_0, \mathcal{O}_{D_0}(e))$$

in which the subspace $\Bbbk[f_{l+1}, \ldots, f_n]_e \subseteq H^0(Y, \mathcal{O}_Y(e))$ spanned by monomials of degree e in $f_{l+1}, \ldots, f_n$ is mapped isomorphically onto $H^0(\mathcal{O}_{D_0}(e))$ by $|_{D_0}$, we have decompositions $H^0(Y, \mathcal{O}_Y(e)) = H^0(Y, \mathcal{O}_Y(e) \otimes \mathcal{L}^\vee) w \oplus \Bbbk[f_{l+1}, \ldots, f_n]_e$ and consequently

$$(S_{\mathbf{F}_Y^\Lambda})_{m,e} = B_{m,e} w \oplus \left(Z_{l+1}^e \Bbbk[f_{l+1}, \ldots, f_n]_e \otimes_\Bbbk \Bbbk[Z_0, \ldots, Z_l]_{m-e} \right), \tag{4.1.4}$$

where the second summand is a subspace of $R_{m,e}$.

To prove (4.1.3), for each $j = n+1, \ldots, N$, we will find a homogenous polynomial

$$T_j g^{k_j}(T_0, \ldots, T_l) + F_j \in I_X \tag{4.1.5}$$

for some integer $k_j > 0$ and $F_j \in (T_{l+1}, \ldots, T_N)^2 \subseteq S$. In fact, (4.1.5) implies that T_j ($j = n+1, \ldots, N$) are defining equations of the tangent space $T_y(X) \subseteq \mathbb{P}^N$ at every $y \in \Lambda \setminus (g)_0$, and hence $\dim T_y(X) \leq n$, which means X is smooth at y.

From now on, we fix j ($j = n+1, \ldots, N$). To obtain (4.1.5), it suffices to show that for some integer $k_j > 0$, there exist $u \in (J_{\tilde{X}})_{k_j \mu + 1}$ and $v \in M_{k_j \mu + 1}$ such that

$$f_j Z_{l+1} g^{k_j} + u + v = 0 \quad \in (S_{\mathbf{F}_Y^\Lambda})_{k_j \mu + 1}. \tag{4.1.6}$$

In fact, if so, we have $f_j Z_{l+1} g^{k_j} + v = -u \in J_{\tilde{X}} \cap R$ and $F_j \in (T_{l+1}, \ldots, T_N)^2 (\subseteq S)$ such that $v = \varphi_Y^*(F_j)$ and hence (4.1.5) holds since $f_j Z_{l+1} g^{k_j} = \varphi_Y^*(T_j g^{k_j}(T_0, \ldots, T_l))$.

To obtain (4.1.6), noting that there exists $q \in B_{1,1}$ such that $f_j Z_{l+1} = qw \in (S_{\mathbf{F}_Y^\Lambda})_{1,1}$ since $f_j|_{D_0} = 0$, and setting $p := qw$, we start with $qG_{\tilde{X}} = pg - qh$ obtained from (4.1.2). Since $qh \in (S_{\mathbf{F}_Y^\Lambda})_{\mu+1}$ in the left-hand side has the decomposition $qh = -v_1 + q_1 w$ for $v_1 \in M_{\mu+1}$ and $q_1 \in B_{\mu+1}$ with $\mathrm{ord}_{Z_{l+1}}(q_1) \geq \mathrm{ord}_{Z_{l+1}}(qh) \geq 2$ by (4.1.4), setting $p_1 := q_1 w \in (S_{\mathbf{F}_Y^\Lambda})_{\mu+1}$ and $u_1 := -qG_{\tilde{X}} \in (J_{\tilde{X}})_{\mu+1}$, we have

$p_1 = pg + u_1 + v_1$. If $p_1 = 0$, we have (4.1.6). If $p_1 \neq 0$, our argument proceeds by the following induction.

Claim. For a positive integer k, suppose that there exist nonzero $p_i \in (S_{\mathbf{F}^\wedge_Y})_{i\mu+1}$ $(i = 1, \ldots, k)$ such that $p_i = q_i w = pg^i + u_i + v_i$ with $u_i \in (J_{\tilde{X}})_{i\mu+1}$, $v_i \in M_{i\mu+1}$, $q_i \in B_{i\mu+1}$, and with $\mathrm{ord}_{Z_{l+1}}(p_1) < \mathrm{ord}_{Z_{l+1}}(p_2) < \cdots < \mathrm{ord}_{Z_{l+1}}(p_k)$. Then there exists $p_{k+1} \in (S_{\mathbf{F}^\wedge_Y})_{(k+1)\mu+1}$ such that

(1) $p_{k+1} = q_{k+1}w = pg^{k+1} + u_{k+1} + v_{k+1}$ for some $q_{k+1} \in B_{(k+1)\mu+1}$, $u_{k+1} \in (J_{\tilde{X}})_{(k+1)\mu+1}$, and $v_{k+1} \in M_{(k+1)\mu+1}$;
(2) $\mathrm{ord}_{Z_{l+1}}(p_{k+1}) > \mathrm{ord}_{Z_{l+1}}(p_k)$; and
(3) each nonzero homogeneous part of p_{k+1} with respect to the Z_{l+1}-order is not contained in $R + Rp_1^* + \cdots + Rp_k^*$.

If the claim is proved, we must have $p_{k_j} = 0$ for some integer $k_j > 0$, i.e., we have $u_{k_j} \in (J_{\tilde{X}})_{(k_j)\mu+1}$, $v_{k_j} \in M_{k_j\mu+1}$, and $q_{k_j} \in B_{k_j\mu+1}$ such that $pg^{k_j} + u_{k_j} + v_{k_j} = 0$, and hence (4.1.6) holds; otherwise, there exists a strictly increasing sequence

$$R \subsetneq R + Rp_1^* \subsetneq R + Rp_1^* + Rp_2^* \subsetneq \cdots$$

of R-submodules of the finite R-module $S_{\mathbf{F}^\wedge_Y}$, which is impossible.

Now we will prove the Claim. From (4.1.2) and $q_k w = p_k$ we have

$$q_k G_{\tilde{X}} = p_k g - q_k h. \tag{4.1.7}$$

We make a *division* of $q_k h \in (S_{\mathbf{F}^\wedge_Y})_{(k+1)\mu+1}$ by $\{p_1, \ldots, p_k\}$: There exist $b_0 \in R_{(k+1)\mu+1}$, $b_i \in R_{(k+1-i)\mu}$ $(i = 1, \ldots, k)$, and $q_{k+1} \in B_{(k+1)\mu+1}$ such that

(i) $q_k h = b_0 + b_1 p_1 + b_2 p_2 + \cdots + b_k p_k + q_{k+1} w$;
(ii) each nonzero homogeneous part of $q_{k+1} w$ with respect to the Z_{l+1}-order is not contained in $R + Rp_1^* + \cdots + Rp_k^*$;
(iii) $\mathrm{ord}_{Z_{l+1}}(b_0), \mathrm{ord}_{Z_{l+1}}(q_{k+1}w) \geq \mathrm{ord}_{Z_{l+1}}(q_k h)$, and $\mathrm{ord}_{Z_{l+1}}(b_i) \geq 1$ $(1 \leq i \leq k)$.

Once we have the division with (i)-(iii), setting $p_{k+1} := q_{k+1} w$, from (4.1.7) and (i) we obtain

$$p_{k+1} (= q_{k+1} w) = p_k g - q_k G_{\tilde{X}} - (b_0 + b_1 p_1 + b_2 p_2 + \cdots + b_k p_k)$$

with (2) and (3). Taking into account that $p_i = pg^i + u_i + v_i$ and setting

$$u_{k+1} := u_k g - q_k G_{\tilde{X}} - (b_1 u_1 + b_2 u_2 + \cdots + b_k u_k) \in (J_{\tilde{X}})_{(k+1)\mu+1} \quad \text{and}$$
$$v_{k+1} := v_k g - b_0 - b_1(pg + v_1) - b_2(pg^2 + v_2) - \cdots - b_k(pg^k + v_k) \in R_{(k+1)\mu+1},$$

we have $p_{k+1} = pg^{k+1} + u_{k+1} + v_{k+1}$. Moreover $v_{k+1} \in M_{(k+1)\mu+1}$ by looking at the Z_{l+1}-order from (iii) and the assumption. Consequently we have the Claim.

To obtain the division, first set $b_0 := 0, b_1 := 0, \ldots, b_k := 0, q_{k+1} := 0$, $r := q_k h$, and $e_i := \mathrm{ord}_{Z_{l+1}}(p_i)$ $(i = 1, \ldots, k)$. While $r \neq 0$, for $e = \mathrm{ord}_{Z_{l+1}}(r)$, do the following process: If $r^* = a_0 + a_1 p_1^* + \cdots + a_k p_k^*$ for some $a_0 \in R_{(k+1)\mu+1,e}$ and $a_i \in R_{(k+1-i)\mu, e-e_i}$ $(i = 1, \ldots, k)$, then add a_i to b_i $(i = 0, \ldots, k)$ and $-(a_0 + a_1 p_1 + \cdots + a_k p_k)$ to r; else (i.e., $r^* \notin R_{(k+1)\mu+1,e} + R_{k\mu, e-e_1} p_1^* + \cdots + R_{\mu, e-e_k} p_k^*$) take $a_0 \in R_{(k+1)\mu+1,e}$ and $c \in B_{(k+1)\mu+1,e}$ such that $r^* = a_0 + cw$ by (4.1.4) and add a_0 to b_0, c to q_{k+1}, and $-r^*$ to r. This process will stop in finite steps, since $\mathrm{ord}_{Z_{l+1}}(r)$ $(\leq (k+1)\mu + 1)$ increases after this process. Then (i) and (ii) hold by the choice of b_i and q_{k+1}. Moreover, the Z_{l+1}-orders of b_0, $b_i p_i$ $(i = 1, \ldots, k)$, and $q_{k+1} w$ are at least that of $q_k h$, and hence $\mathrm{ord}_{Z_{l+1}}(b_i) \geq \mathrm{ord}_{Z_{l+1}}(q_k) - \mathrm{ord}_{Z_{l+1}}(p_i) + \mathrm{ord}_{Z_{l+1}}(h) \geq 1$ $(i = 1, \ldots, k)$, which means (iii). This completes the proof of Theorem 5. □

Example 4.2. For an integer $\bar{n} \geq 1$, we will give two examples of $\bar{n}$-dimensional varieties Y and line bundles $\mathcal{L}$ on Y with birational-embedding $\nu : Y \to \mathbb{P}^{\bar{N}}$ for some $\bar{N}$. For an integer $l \geq 0$ with $n := \bar{n} + l \geq 2$ and $N := \bar{N} + l + 1$, consider $\mathbb{P}^{\bar{N}}$ to be a linear subspace of $\mathbb{P}^N$ and let $\Lambda \subseteq \mathbb{P}^N$ be an l-dimensional linear subspace disjoint from $\mathbb{P}^{\bar{N}}$. Consider the conical scroll $\mathbf{F}_Y^\Lambda$ with birational-embedding $\varphi_Y : \mathbf{F}_Y^\Lambda \to \mathbb{P}^N$. In the both cases, $\dim_{\Bbbk} H^0(Y, \mathcal{L}) = 1$ and $|\mathcal{O}_{\mathbf{F}_Y^\Lambda}(\mu) \otimes \tau_Y^* \mathcal{L}|$ for $\mu \geq 2$ is not a composite with pencil whose base locus is of codimension ≥ 2. A general member $\tilde{X} \in |\mathcal{O}_{\mathbf{F}_Y^\Lambda}(\mu) \otimes \tau_Y^* \mathcal{L}|$ is irreducible and reduced by Bertini's Theorem (see [9, (3.4.10)]) satisfying (1) and (2) in Theorem 4, and $X = \varphi_Y(\tilde{X})$ satisfies $\mathcal{C}(X) \neq \emptyset$.

(1) Let Y' be an $\bar{n}$-dimensional nondegenerate smooth projective variety in $\mathbb{P}^{\bar{N}+1}$ $(\bar{n} \geq 2)$. Let $\sigma : Y \to Y'$ be the blowing up of Y' at a smooth point y not in $\mathcal{C}(Y')$. The linear projection of Y' from y to $\mathbb{P}^{\bar{N}}$ induces a nondegenerate birational-embedding $\nu : Y \to \mathbb{P}^{\bar{N}}$. Let $\mathcal{L}$ be the line bundle on Y of the exceptional divisor of σ. Then $(\mathcal{L}, \mathcal{O}_Y(1)^{\bar{n}-1}) = 1$ and $\dim_{\Bbbk} H^0(Y, \mathcal{L}) = 1$.

(2) Let Y be an $\bar{n}$-dimensional projective bundle over a smooth projective curve C of genus $g \geq 1$ whose tautological line bundle defines a nondegenerate birational-embedding $\nu : Y \to \mathbb{P}^{\bar{N}}$. Let $\mathcal{L}$ be the line bundle on Y associated with a fibre. Then $(\mathcal{L}, \mathcal{O}_Y(1)^{\bar{n}-1}) = 1$ and $\dim_{\Bbbk} H^0(Y, \mathcal{L}) = 1$. A simple case is $Y = C$ and $\mathcal{L}$ to be a line bundle of a point of C.

5. Divisors of conical rational scrolls

In this section, we assume the following conditions and study a prime divisor of a conical rational scroll to be a nondegenerate birational-divisor and to have the nonempty smooth locus on the vertex.

(5.1). Let N, n, l be integers with $e := N - n \geq 2$, $\bar{n} := n - l \geq 1$, and $l \geq 0$. Let $\mathbf{E}_{\mathcal{E}}^\Lambda$ be the conical rational scroll with an l-dimensional linear subspace $\Lambda \subseteq \mathbb{P}^N$ as vertex and with a birational-embedding $\psi : \mathbf{E}_{\mathcal{E}}^\Lambda \to \mathbb{P}^N$ for an ample vector bundle $\mathcal{E}$ of rank $\bar{n}$ over $\mathbb{P}^1$. Keep the notation as in Definition 6. We assume that $\mathcal{E} = \bigoplus_{i=l+1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$ for some positive integers $0 < a_{l+1} \leq \cdots \leq a_n$ and fix formal basis W_i of $\mathcal{O}_{\mathbb{P}^1}^{\oplus l+1} \oplus \mathcal{E}$ so that $\mathcal{O}_{\mathbb{P}^1}^{\oplus l+1} \oplus \mathcal{E} = (\bigoplus_{i=0}^l \mathcal{O}_{\mathbb{P}^1} W_i) \oplus (\bigoplus_{i=l+1}^n \mathcal{O}_{\mathbb{P}^1}(a_i) W_i)$. Let s, t be homogeneous coordinates of $\mathbb{P}^1$. Let $T_0, \ldots, T_N$ be homogeneous coordinates of $\mathbb{P}^N$. Assume that $\Lambda \subseteq \mathbb{P}^N$ is defined by $T_{l+1} = \cdots = T_N = 0$. Since $\Lambda = \psi(\tilde{\Lambda}_{\mathbb{P}^1})$ for $\tilde{\Lambda}_{\mathbb{P}^1} = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}^{\oplus l+1}) \cong \Lambda \times \mathbb{P}^1$, after change of the basis $W_0, \ldots, W_l$, we may assume that $\psi^*(T_0) = W_0, \ldots, \psi^*(T_l) = W_l$, which can be seen as homogeneous coordinates of Λ. Hence the image $\psi(Y)$ of $Y := \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) (\subseteq \mathbf{E}_{\mathcal{E}}^\Lambda)$ is contained in $\mathbb{P}^{\bar{N}} := V_+(T_0, \ldots, T_l)$ $(\bar{N} = N - l - 1)$. Note that $\nu := \psi|_Y : Y \to \mathbb{P}^{\bar{N}}$ is a birational-embedding since ν is a birational-embedding if and only if so is ψ. Also ν is nondegenerate since ψ is defined by a subsystem of $|\mathcal{O}_{\mathbf{E}_{\mathcal{E}}^\Lambda}(1)|$, and hence ψ is nondegenerate.

Let $X \subseteq \mathbb{P}^N$ be the image $\psi(\tilde{X})$ of a prime divisor $\tilde{X} \in |\mathcal{O}_{\mathbf{E}^\Lambda_\mathcal{E}}(\mu) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(b)|$ ($\mu \geq 0$, $b \in \mathbb{Z}$) of $\mathbf{E}^\Lambda_\mathcal{E}$ defined by

$$G_{\tilde{X}} = \sum_{\substack{\mu_0,\dots,\mu_n \geq 0 \\ \mu_0+\cdots+\mu_n=\mu}} g_{\mu_0,\dots,\mu_n} W_0^{\mu_0} \cdots W_n^{\mu_n} \in H^0(\mathbf{E}^\Lambda_\mathcal{E}, \mathcal{O}_{\mathbf{E}^\Lambda_\mathcal{E}}(\mu) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(b))$$

for homogeneous polynomials $g_{\mu_0,\dots,\mu_n} \in \Bbbk[s,t]$ of degree $\mu_{l+1}a_{l+1} + \cdots + \mu_n a_n + b$. Set

$$m_0 := \min\{\mu_{l+1} + \cdots + \mu_n | g_{\mu_0,\dots,\mu_n} \neq 0 \text{ for some } \mu_0, \dots, \mu_l \geq 0\}.$$

Note that $\tilde{\Lambda}_{\mathbb{P}^1} \subseteq \tilde{X}$, namely $G_{\tilde{X}}|_{\tilde{\Lambda}_{\mathbb{P}^1}} = 0$, if and only if m_0 is positive. Let l_0 be the length $l(X \cap \langle u, x\rangle)$ for general $u \in \Lambda$ and general $x \in X$.

Proposition 5.2. *Keep the notation and the assumption as in* (5.1).

(1) *Assume X is nondegenerate and birational to $\tilde{X}$ with $\Lambda \cap \operatorname{Sm} X \neq \emptyset$. Then*
  (a) *$\tilde{\Lambda}_{\mathbb{P}^1} \not\subseteq \tilde{X}$, $m_0 = 0$, $b = 1$, and $\mu \geq 1$; or*
  (b) *$\tilde{\Lambda}_{\mathbb{P}^1} \subseteq \tilde{X}$, $m_0 = 1$, $l = n-2$, $a_{n-1} = 1$, $b = -a_n$, and $\mu \geq 2$.*

(2) *Assume* (a) *in* (1) *holds. Hence $G_{\tilde{X}}|_{\tilde{\Lambda}_{\mathbb{P}^1}}$ is linear in s,t and we may write*

$$G_{\tilde{X}}|_{\tilde{\Lambda}_{\mathbb{P}^1}} = G_1(W_0, \dots, W_l)s + G_2(W_0, \dots, W_l)t \neq 0.$$

Then X is nondegenerate in $\mathbb{P}^N$ and birational to $\tilde{X}$ such that $\Lambda \cap \operatorname{Sm} X = \Lambda \setminus V_+(G_1, G_2) \neq \emptyset$, $\mu = l_0 - 1$, and $\deg X = \mu \cdot c_1(\mathcal{E}) + 1$. If we suppose furthermore that $\mu \geq 2$, then Λ is an irreducible component of $\bar{\mathcal{C}}(X)$. Moreover,
  (i) *if $\deg \operatorname{GCD}(G_1, G_2) = \mu$, then the partial Gauss map $\gamma|_\Lambda$ is constant;*
  (ii) *if $\deg \operatorname{GCD}(G_1, G_2) < \mu$, then the partial Gauss map $\gamma|_\Lambda$ is nonconstant and $\dim \operatorname{Sing} X \geq \dim \Lambda \cap \operatorname{Sing} X = \dim V_+(G_1, G_2) \geq \dim \Lambda - 2$.*

(3) *Assume* (b) *in* (1) *holds. Then X is nondegenerate in $\mathbb{P}^N$ and birational to $\tilde{X}$ such that $\Lambda \cap \operatorname{Sm} X \neq \emptyset$ and $\mu = l_0$. Hence Λ is an irreducible component of $\bar{\mathcal{C}}(X)$ if $\mu \geq 3$. Moreover,*
  (i) *if $a_n > 1$, then the partial Gauss map $\gamma|_\Lambda$ is constant;*
  (ii) *if $a_n = 1$, then X is the birational image of another prime divisor $\tilde{X}_1 \in H^0((\mathbf{E}^\Lambda_\mathcal{E})_1, \mathcal{O}_{(\mathbf{E}^\Lambda_\mathcal{E})_1}(\mu-1) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(1))$ on another conical rational scroll $(\mathbf{E}^\Lambda_\mathcal{E})_1$ with the same vertex Λ and $\mathcal{E}$ but with different $\psi' : (\mathbf{E}^\Lambda_\mathcal{E})_1 \to \mathbb{P}^N$ such that m_0 for $G_{\tilde{X}_1}$ is 0 (equivalently $\tilde{X}_1 \not\supset \tilde{\Lambda}_{\mathbb{P}^1} (\subseteq (\mathbf{E}^\Lambda_\mathcal{E})_1)$).*

To prove (5.2), we consider the conical scroll $\mathbf{F}^\Lambda_Y$ for Λ, $\mathbb{P}^{\bar{N}}$, $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$, and $\nu = \psi|_Y : Y \to \mathbb{P}^{\bar{N}}$, and we will relate $\mathbf{E}^\Lambda_\mathcal{E}$ to $\mathbf{F}^\Lambda_Y$. Keep the notation as in Definition 2 and (4.1) for $\mathbf{F}^\Lambda_Y$. Let $\bar{p} : Y \to \mathbb{P}^1$ be the projection. We fix basis Z_i for the bundle $\mathcal{F}_Y$ as in (4.1).

Lemma 5.3. *Under the assumption* (5.1), *there exists a birational morphism $\sigma : \mathbf{F}^\Lambda_Y \to \mathbf{E}^\Lambda_\mathcal{E}$ such that $\varphi_Y = \psi \circ \sigma$, $\sigma^*\mathcal{O}_{\mathbf{E}^\Lambda_\mathcal{E}}(1) = \mathcal{O}_{\mathbf{F}^\Lambda_Y}(1)$,*

(5.3.1) $\sigma^*W_0 = Z_0, \dots, \sigma^*W_l = Z_l, \sigma^*W_{l+1} = Z_{l+1}W_{l+1}, \dots, \sigma^*W_n = Z_{l+1}W_n.$

If $n - l = 1$, then $Y \cong \mathbb{P}^1$ and σ is an isomorphism. If $n - l \geq 2$, then the exceptional set of σ is $\tilde{\Lambda}_Y$ which is mapped onto $\tilde{\Lambda}_{\mathbb{P}^1}$, and the strict transform $\tilde{X}'$ of $\tilde{X}$ by σ is defined by $G_{\tilde{X}'} \in H^0(\mathcal{O}_{\mathbf{F}^\Lambda_Y}(\mu - m_0) \otimes \tau_Y^*(\mathcal{O}_Y(m_0) \otimes \bar{p}^*\mathcal{O}_{\mathbb{P}^1}(b)))$ such that $\sigma^*G_{\tilde{X}} = Z_{l+1}^{m_0} G_{\tilde{X}'}$.

Proof. The natural homomorphism $\bar{p}^*\mathcal{E} \to \mathcal{O}_Y(1)$ and the isomorphism $\bar{p}^*(\mathcal{O}_{\mathbb{P}^1}^{\oplus l+1}) \cong \mathcal{O}_Y^{\oplus l+1}$ induce the surjection $\varepsilon : \bar{p}^*(\mathcal{O}_{\mathbb{P}^1}^{\oplus l+1} \oplus \mathcal{E}) \to \mathcal{O}_Y^{\oplus l+1} \oplus \mathcal{O}_Y(1)$. Thus we have the corresponding inclusion

$$\mathbf{F}_Y^\Lambda = \mathbb{P}_Y(\mathcal{O}_Y^{\oplus l+1} \oplus \mathcal{O}_Y(1)) \subseteq \mathbb{P}_Y(\bar{p}^*(\mathcal{O}_{\mathbb{P}^1}^{\oplus l+1} \oplus \mathcal{E})) = Y \times_{\mathbb{P}^1} \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}^{\oplus l+1} \oplus \mathcal{E}).$$

From this and the second projection, we have the required morphism σ. By the construction of ε, we have (5.3.1). If $n - l = 1$, then $Y = \mathbb{P}^1$, and hence σ is isomorphic. Suppose $n - l \geq 2$. By looking at each fibre over $\mathbb{P}^1$, we see that σ is a birational morphism and the exceptional set of σ is $\tilde{\Lambda}_Y$. From (5.3.1), we obtain

$$\sigma^* G_{\tilde{X}} = \sum_{\mu_0,\dots,\mu_n} g_{\mu_0\dots\mu_n} W_{l+1}^{\mu_{l+1}} \cdots W_n^{\mu_n} Z_0^{\mu_0} \cdots Z_l^{\mu_l} Z_{l+1}^{\mu_{l+1}+\cdots+\mu_n}.$$

Moreover we recover $G_{\tilde{X}}$ from $\sigma^* G_{\tilde{X}}$ by substituting $W_0, \dots, W_l, 1$ for $Z_0, \dots, Z_l$, Z_{l+1}. Hence if $\sigma^* G_{\tilde{X}}$ is reducible, then $G_{\tilde{X}}$ is reducible or $\sigma^* G_{\tilde{X}}$ is divisible by Z_{l+1}. Since $G_{\tilde{X}}$ is irreducible, $\sigma^* G_{\tilde{X}}/Z_{l+1}^{m_0}$ is irreducible and $\sigma^* G_{\tilde{X}} = Z_{l+1}^{m_0} G_{\tilde{X}'}$. □

Proof of Proposition 5.2. Let $\tilde{X}'$ be the strict transform of $\tilde{X}$ by σ in Lemma 5.3.

(1) By the assumption, X is the birational image of $\tilde{X}' \subseteq \mathbf{F}_Y^\Lambda$ by $\varphi_Y = \psi \circ \sigma$. Note that $\mu - m_0 \geq 1$. In fact, if $\mu - m_0 = 0$, then X is a cone with vertex Λ, which contradicts the assumption that X is nondegenerate of codimension $e \geq 2$ with $\Lambda \cap \operatorname{Sm} X \neq \emptyset$. By (1) of Proposition 2.3 and our assumption,

$$(\mathcal{O}_Y(m_0) \otimes \bar{p}^*\mathcal{O}_{\mathbb{P}^1}(b), \mathcal{O}_Y(1)^{\bar{n}-1}) = m_0 c_1(\mathcal{E}) + b = 1$$

and $H^0(\mathcal{O}_Y(m_0) \otimes \bar{p}^*\mathcal{O}_{\mathbb{P}^1}(b)) \neq 0$. From the latter, we obtain that $m_0 a_n + b \geq 0$. Hence $m_0(c_1(\mathcal{E}) - a_n) \leq 1$. We divide into three cases. When $m_0 = 0$, we have $b = 1$, which is (a). When $m_0 = 1$ and $c_1(\mathcal{E}) = a_n + 1$, we have $a_{n-1} = 1$, $l = n-2$, $b = -a_n$, and $\mu \geq m_0 + 1 = 2$, which is (b). When $m_0 \geq 1$ and $c_1(\mathcal{E}) = a_n$, we have $l = n-1$ and $\tilde{\Lambda}_{\mathbb{P}^1} \subseteq \tilde{X}$, hence $\tilde{\Lambda}_{\mathbb{P}^1} = \tilde{X}$, which contradicts our assumption.

(2) First we will prove that X is nondegenerate in $\mathbb{P}^N$ and birational to $\tilde{X}$. Since ψ is defined a base-point-free subspace of $H^0(\mathcal{O}_{\mathbf{E}_\mathcal{E}^\Lambda}(1))$ and since $H^0(\mathcal{O}_{\mathbf{E}_\mathcal{E}^\Lambda}(1-\mu) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, the pull-back $H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(\mathcal{O}_{\tilde{X}}(1))$ is injective, and hence the image $X = \psi(\tilde{X})$ is nondegenerate in $\mathbb{P}^N$. To prove that X is birational to $\tilde{X}$, we have to show that X is birational to $\tilde{X}'(\subseteq \mathbf{F}_Y^\Lambda)$. Note that $\mathbf{F}_Y^\Lambda$ and $\varphi_Y(\mathbf{F}_Y^\Lambda)$ are isomorphic except for the union of $\tilde{\Lambda}_Y$ and the fibres over the nonisomorphic locus of $Y \to \nu(Y)$. Since the strict transform $\tilde{X}'$ of $\tilde{X}$ by σ is not contained in the exceptional set $\tilde{\Lambda}_Y$ and since $\tilde{X}'$ dominates Y because of $\tilde{X}' \in |\mathcal{O}_{\mathbf{F}_Y^\Lambda}(\mu) \otimes \tau_Y^*\bar{p}^*\mathcal{O}_{\mathbb{P}^1}(b)|$, $\tilde{X}'$ meets the embedding locus of φ_Y, and hence X and $\tilde{X}'$ are birational. Consequently X is a birational-divisor on $\mathbf{F}_Y^\Lambda$ of type $(\mu, \tau_Y^*\bar{p}^*\mathcal{O}_{\mathbb{P}^1}(1))$ and also X is a birational-divisor on $\mathbf{E}_\mathcal{E}^\Lambda$ of type $(\mu, 1)$. Hence $\deg X = (\mathcal{O}_{\mathbf{E}_\mathcal{E}^\Lambda}(1)^n, \mathcal{O}_{\mathbf{E}_\mathcal{E}^\Lambda}(\mu) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(1)) = \mu \cdot c_1(\mathcal{E}) + 1$.

To prove $\Lambda \cap \operatorname{Sm} X = \Lambda \setminus \Lambda_0$ for $\Lambda_0 := V_+(G_1, G_2) \subseteq \Lambda$, first we will show $\operatorname{Sm} X \supseteq \Lambda \setminus \Lambda_0$ by looking at $\tilde{X} \subseteq \mathbf{E}_\mathcal{E}^\Lambda$. Since $G_{\tilde{X}}|\tilde{\Lambda}_{\mathbb{P}^1} = G_1 s + G_2 t$, we see that $\tilde{X}$ is smooth at each point of $\tilde{X} \cap (\tilde{\Lambda}_{\mathbb{P}^1} \setminus \Lambda_0 \times \mathbb{P}^1)$ and that $\tilde{X} \cap \tilde{\Lambda}_{\mathbb{P}^1} \to \Lambda$ is scheme-theoretically one-to-one on $\Lambda \setminus \Lambda_0$. From the latter, we obtain that $\psi|_{\tilde{X}} : \tilde{X} \to \mathbb{P}^N$ is finite and scheme-theoretically one-to-one on $\Lambda \setminus \Lambda_0$ since $(\psi|_{\tilde{X}})^{-1}(\Lambda) = \tilde{X} \cap \tilde{\Lambda}_{\mathbb{P}^1}$. Hence X is isomorphic to $\tilde{X}$ along $\Lambda \setminus \Lambda_0$, and therefore $\operatorname{Sm} X \supseteq \Lambda \setminus \Lambda_0$. Next we will prove $\operatorname{Sing} X \supseteq \Lambda_0$. When $l = 0$, since $G_1 \neq 0$ or $G_2 \neq 0$ by $m_0 = 0$, we have $\Lambda_0 = \emptyset$. When $l \geq 1$, considering X as a birational-divisor on $\mathbf{F}_Y^\Lambda$ of type

$(\mu, \tau_Y^*\bar{p}^*\mathcal{O}_{\mathbb{P}^1}(1))$ and noting that $G_{\tilde{X}'}|\tilde{\Lambda}_Y = G_1(Z_0,\dots,Z_l)s + G_2(Z_0,\dots,Z_l)t$ by Lemma 5.3 and assumption $m_0 = 0$, we have $\Lambda_0 \subseteq \Lambda \cap \operatorname{Sing} X$ by Lemma 2.5.

Since X is a birational-divisor on $\mathbf{F}_Y^\Lambda$ such that $\Lambda \cap \operatorname{Sm} X = \Lambda \setminus \Lambda_0 \neq \emptyset$, by Lemma 1.1, we have $\mu = l_0 - 1$. Hence, if $\mu \geq 2$, then $\Lambda \subseteq \bar{\mathcal{C}}(X)$ and Λ is an irreducible component of $\bar{\mathcal{C}}(X)$ by Proposition 2.1, since the image $\nu(Y)$ of the finite morphism ν is not cone. To prove the last part of (2), set $G_0 := \operatorname{GCD}(G_1, G_2)$. When $l = 0$, Λ is a one-point set, and hence $\deg G_0 = \mu$ and $\gamma|_\Lambda$ is constant. Suppose $l \geq 1$. The unique component $\tilde{D}$ of $\tilde{X}' \cap \tilde{\Lambda}_Y$ dominating Λ in Proposition 2.3 is defined by $\sigma^*((G_1/G_0)s + (G_2/G_0)t)$ on $\tilde{\Lambda}_Y$ in this case, since $\sigma^*((G_1/G_0)s+(G_2/G_0)t)|_{\{x\}\times Y}$ defines the nonempty subset of Y for any point $x \in \Lambda$. Thus $\deg G_0 \neq \mu$ if and only if $\tilde{D}$ dominates Y, namely $\sigma^*((G_1/G_0)s + (G_2/G_0)t)|_{\Lambda\times\{y\}}$ for general $y \in Y$ defines the nonempty subset of Λ. On the other hand, by (2) of Proposition 2.3, $\tilde{D}$ dominates Y if and only if $\gamma|_\Lambda$ is nonconstant. Therefore $\deg G_0 \neq \mu$ if and only if $\gamma|_\Lambda$ is nonconstant. The last inequality in (ii) is clear from $\Lambda \cap \operatorname{Sm} X = \Lambda \setminus \Lambda_0$.

(3) By the same way as in (2), we obtain that X is nondegenerate in $\mathbb{P}^N$ and birational to $\tilde{X}$ in this case. Consequently X is a birational-divisor on $\mathbf{F}_Y^\Lambda$ of type $(\mu - 1, \mathcal{O}_Y(1) \otimes \bar{p}^*\mathcal{O}_{\mathbb{P}^1}(-a_n))$ by Lemma 5.3 since $m_0 = 1$. Moreover

$$G_{\tilde{X}'} = \sigma^* G_{\tilde{X}}/Z_{n-1} = \sum_{\mu_0,\dots,\mu_n} g_{\mu_0\cdots\mu_n} W_{n-1}^{\mu_{n-1}} W_n^{\mu_n} Z_0^{\mu_0}\cdots Z_{n-2}^{\mu_{n-2}} Z_{n-1}^{\mu_{n-1}+\mu_n-1}$$

for homogeneous polynomials $g_{\mu_0\cdots\mu_n} \in \Bbbk[s,t]$ of degree $\mu_{n-1} + \mu_n a_n - a_n$ and $\tilde{X}' \cap \tilde{\Lambda}_Y \in |\mathcal{O}_{\tilde{\Lambda}_Y}(\mu-1) \otimes \bar{\tau}_Y^*(\mathcal{O}_Y(1) \otimes \bar{p}^*\mathcal{O}_{\mathbb{P}^1}(-a_n))|$ with

$$G_{\tilde{X}'}|\tilde{\Lambda}_Y = \sum_{\substack{\mu_0,\dots,\mu_{n-2}\geq 0\\ \mu_0+\cdots+\mu_{n-2}=\mu-1}} (g_{\mu_0\cdots\mu_{n-2}01} W_n + g_{\mu_0\cdots\mu_{n-2}10} W_{n-1}) Z_0^{\mu_0}\cdots Z_{n-2}^{\mu_{n-2}} \ (\neq 0).$$

Here $\deg(g_{\mu_0\cdots\mu_{n-2}01}) = 0$ and $\deg(g_{\mu_0\cdots\mu_{n-2}10}) = 1 - a_n$ if these are nonzero. Thus, to prove the remaining part, we divide into two cases, $a_n > 1$ or $a_n = 1$.

Suppose $a_n > 1$. Then $G_{\tilde{X}'}|\tilde{\Lambda}_Y = gW_n$ for

$$g = \sum_{\substack{\mu_0,\dots,\mu_{n-2}\geq 0\\ \mu_0+\cdots+\mu_{n-2}=\mu-1}} g_{\mu_0\cdots\mu_{n-2}01} Z_0^{\mu_0}\cdots Z_{n-2}^{\mu_{n-2}} \quad \in H^0(\mathcal{O}_\Lambda(\mu-1)).$$

Since the zero of $W_n \in H^0(\mathcal{O}_Y(1)\otimes\bar{p}^*\mathcal{O}_{\mathbb{P}^1}(-a_n))$ is a prime divisor $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(1)W_{n-1})$ of $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(1)W_{n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)W_n)$, from Theorem 5, we obtain that $\Lambda \cap \operatorname{Sm} X = \Lambda \setminus (g)_0 \neq \emptyset$, $\Lambda \subseteq \bar{\mathcal{C}}(X)$, and $\gamma|_\Lambda$ is constant. By (3) of Lemma 1.1, we have $\mu = l_0$. By Proposition 2.1, Λ is an irreducible component of $\bar{\mathcal{C}}(X)$ if $\mu \geq 3$.

Suppose $a_n = 1$. Then we have $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$ with projections $p_i : Y \to \mathbb{P}^1$ and $\mathcal{O}_Y(1) = p_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$. Note that the morphism ψ must be defined by the complete linear system $|\mathcal{O}_{\mathbf{E}_\mathcal{E}^\Lambda}(1)|$ in this case. We consider $\bar{p} = p_2$; namely, the homogeneous coordinates of the second $\mathbb{P}^1$ are s, t and those of the first $\mathbb{P}^1$ are W_{n-1}, W_n. Then $\tilde{X}' \in |\mathcal{O}_{\mathbf{F}_Y^\Lambda}(\mu-1) \otimes \tau_Y^*(p_1^*\mathcal{O}_{\mathbb{P}^1}(1))|$. We take another conical rational scroll $(\mathbf{E}_\mathcal{E}^\Lambda)_1$ with the same vertex Λ and the same ample bundle $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ over different $\mathbb{P}^1$ whose homogeneous coordinates are W_{n-1} and W_n, and with different basis $(\bigoplus_{i=0}^{n-2} \mathcal{O}_{\mathbb{P}^1} W_i) \oplus (\mathcal{O}_{\mathbb{P}^1}(1)s \oplus \mathcal{O}_{\mathbb{P}^1}(1)t)$ and different projection $p' : (\mathbf{E}_\mathcal{E}^\Lambda)_1 \to \mathbb{P}^1$. The morphism $\psi' : (\mathbf{E}_\mathcal{E}^\Lambda)_1 \to \mathbb{P}^N$ is defined by $|\mathcal{O}_{(\mathbf{E}_\mathcal{E}^\Lambda)_1}(1)|$ as ψ. By Lemma 5.3, there is a birational morphism $\sigma' : \mathbf{F}_Y^\Lambda \to (\mathbf{E}_\mathcal{E}^\Lambda)_1$ such that $\sigma'^* W_0 = Z_0, \dots, \sigma'^* W_{n-2} = Z_{n-2}, \sigma'^* s = Z_{n-1}s, \sigma'^* t = Z_{n-1}t$. Then $\psi' \circ \sigma' = \psi \circ \sigma$

and there is a prime divisor $\tilde{X}_1$ on $(\mathbf{E}^{\Lambda}_{\mathcal{E}})_1$ such that ${\sigma'}^*\tilde{X}_1 = \tilde{X}'$. Actually, if we consider $G_{\tilde{X}'}$ as a polynomial $G_{\tilde{X}'}(Z_0, \ldots, Z_{n-1})$, then $G_{\tilde{X}_1} = G_{\tilde{X}'}(W_0, \ldots, W_{n-2}, 1) \in H^0((\mathbf{E}^{\Lambda}_{\mathcal{E}})_1, \mathcal{O}_{(\mathbf{E}^{\Lambda}_{\mathcal{E}})_1}(\mu - 1) \otimes {p'}^*\mathcal{O}_{\mathbb{P}^1}(1))$. Consequently m_0 for $G_{\tilde{X}_1}$ is 0. Hence by (2), we have $\Lambda \cap \operatorname{Sm} X \neq \emptyset$. □

6. The structure of projective varieties with nonbirational inner centers of nonconstant partial Gauss maps

Theorem 7 is a consequence of Proposition 5.2 and the following theorem.

Theorem 6.1. *Let $X \subseteq \mathbb{P}^N$ be a nondegenerate projective variety of dimension $n \geq 1$ and codimension $e \geq 2$ such that $\mathcal{C}(X) \neq \emptyset$. Suppose that the partial Gauss map $\gamma|_\Lambda$ is nonconstant on an l-dimensional irreducible component Λ of $\bar{\mathcal{C}}(X)$, and hence we suppose $n > l \geq 1$. Let l_0 be the length $l(X \cap \langle u, x\rangle)$ for general $u \in \Lambda$ and general $x \in X$. Then X is a birational-divisor of a conical rational scroll $\mathbf{E}^{\Lambda}_{\mathcal{E}}$ with vertex Λ of type $(\mu, 1)$ for $\mu = l_0 - 1$ such that the original divisor $\tilde{X}$ does not contain $\tilde{\Lambda}_{\mathbb{P}^1}$ and $\tilde{X} \cap \tilde{\Lambda}_{\mathbb{P}^1}$ contains a prime divisor of $\tilde{\Lambda}_{\mathbb{P}^1} \cong \Lambda \times \mathbb{P}^1$ dominating both Λ and $\mathbb{P}^1$ by its projections. In particular,* $\dim \operatorname{Sing} X \geq \dim \Lambda \cap \operatorname{Sing} X \geq \dim \Lambda - 2$.

Proof. Let $\pi_\Lambda : \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{\bar{N}}$ ($\bar{N} := N - l - 1$) be the linear projection of $\mathbb{P}^N$ from Λ. We consider the target $\mathbb{P}^{\bar{N}}$ to be a subspace of $\mathbb{P}^N$ disjoint from Λ. By our assumption and by counting the dimension, together with (2) of Proposition 2.3, the closure $\bar{X}_\Lambda$ of the image $\pi_\Lambda(X \setminus \Lambda)$ is the closure of the union $\bigcup_{y \in \Lambda \cap \operatorname{Sm} X} \overline{T_y(X)}_\Lambda$ of the images $\overline{T_y(X)}_\Lambda := \pi_\Lambda(T_y(X) \setminus \Lambda)$ of the tangent spaces to X at $y \in \Lambda \cap \operatorname{Sm} X$. Hence $\operatorname{Cone}(\Lambda, \bar{X}_\Lambda)$ is the closure of $\bigcup_{y \in \Lambda \cap \operatorname{Sm} X} T_y(X)$. We will construct the desingularization of $\bar{X}_\Lambda$ and $\operatorname{Cone}(\Lambda, \bar{X}_\Lambda)$ as projective bundles over $\mathbb{P}^1$. Let $\rho : \Lambda \cap \operatorname{Sm} X \to \mathbb{G} := \mathbb{G}(n - l - 1, \mathbb{P}^{\bar{N}})$ be the morphism to the Grassmannian of $(n-l-1)$-planes of $\mathbb{P}^{\bar{N}}$ defined by $y \mapsto \overline{T_y(X)}_\Lambda$. By assumption, $\dim \rho(\Lambda \cap \operatorname{Sm} X) \geq 1$. Let L be a general line in Λ so that the closure of $\rho(L \cap \operatorname{Sm} X)$ is a rational curve, say $C \subseteq \mathbb{G}$. Let $\eta : \tilde{C}(\cong \mathbb{P}^1) \to C$ be the normalization and let $\mathcal{E}$ be the pull-back of the universal quotient bundle on $\mathbb{G}$ to $\tilde{C}$. We claim that $\mathcal{E}$ is ample or, more strongly, $\Lambda' := \bigcap_{y \in L \cap \operatorname{Sm} X} \overline{T_y(X)}_\Lambda$ is empty. Indeed, if $\Lambda' \neq \emptyset$, then $\bar{X}_\Lambda$ is a cone with vertex Λ', and hence Λ is a proper subset of an irreducible component of $\bar{\mathcal{C}}(X)$ by Proposition 2.1, which contradicts our assumption. Moreover the natural morphism $\nu : \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \to \mathbb{P}^{\bar{N}}$ induces a birational morphism $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \to \bar{X}_\Lambda$, since η is birational and $\bar{X}_\Lambda = \nu(\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}))$ is nondegenerate in $\mathbb{P}^{\bar{N}}$ (see [17, Lemma 1.1]). Hence we have the conical rational scroll $\mathbf{E}^{\Lambda}_{\mathcal{E}}$ for Λ and $\mathcal{E}$ with birational-embedding $\psi : \mathbf{E}^{\Lambda}_{\mathcal{E}} = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}^{\oplus l+1} \oplus \mathcal{E}) \to \mathbb{P}^N$ induced from ν. By the construction, ψ induces a birational morphism $\psi_{\mathbf{E}^{\Lambda}_{\mathcal{E}}} : \mathbf{E}^{\Lambda}_{\mathcal{E}} \to \operatorname{Cone}(\Lambda, \bar{X}_\Lambda)$. Moreover the isomorphic locus of $\psi_{\mathbf{E}^{\Lambda}_{\mathcal{E}}}$ meets X, since $\psi_{\mathbf{E}^{\Lambda}_{\mathcal{E}}}$ is isomorphic on $\operatorname{Cone}(\Lambda, U) \setminus \Lambda$ for the isomorphic locus $U \subseteq \bar{X}_\Lambda$ of $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \to \bar{X}_\Lambda$ (see Lemma 5.3). Let $\tilde{X}$ be the prime divisor on $\mathbf{E}^{\Lambda}_{\mathcal{E}}$ birational to X by ψ. Since $\Lambda \cap \operatorname{Sm} X \neq \emptyset$ and $\gamma|_\Lambda$ is nonconstant, by Proposition 5.2, possibly after replacing $\tilde{X} \subseteq \mathbf{E}^{\Lambda}_{\mathcal{E}}$ in case (ii) of Proposition 5.2 (3), X is a birational-divisor of type $(\mu, 1)$ on $\mathbf{E}^{\Lambda}_{\mathcal{E}}$ such that $\tilde{X} \not\supseteq \tilde{\Lambda}_{\mathbb{P}^1}$ and the divisor $\tilde{X} \cap \tilde{\Lambda}_{\mathbb{P}^1}$ of $\tilde{\Lambda}_{\mathbb{P}^1} \cong \Lambda \times \mathbb{P}^1$ defined by $G_{\tilde{X}}|_{\tilde{\Lambda}_{\mathbb{P}^1}}$ contains a prime divisor dominating Λ and $\mathbb{P}^1$. In particular, the inequality holds. □

Proof of Theorem 7. First suppose (1) holds. By Theorem 6.1, X is a birational-divisor of a conical rational scroll $\mathbf{E}^{\Lambda}_{\mathcal{E}}$ with vertex Λ of type $(\mu, 1)$ for $\mu = l_0 - 1$ such that $\tilde{X} \not\supseteq \tilde{\Lambda}_{\mathbb{P}^1}$ and $\tilde{X} \cap \tilde{\Lambda}_{\mathbb{P}^1}$ contains a prime divisor of $\tilde{\Lambda}_{\mathbb{P}^1} \cong \Lambda \times \mathbb{P}^1$ dominating both Λ and $\mathbb{P}^1$ by its projections. Hence (2) holds. Moreover $\deg X = \mu \cdot c_1(\mathcal{E}) + 1$ and $\dim \Lambda \leq \dim \operatorname{Sing} X + 2$. Conversely if (2) holds, by (2) of Proposition 5.2, we have (1). □

Corollary 6.2. *Let $X \subseteq \mathbb{P}^N$ be a nondegenerate projective variety of dimension $n \geq 2$, codimension $e \geq 2$, and degree d. Assume that X is smooth and $\mathcal{C}(X)(=\bar{\mathcal{C}}(X))$ has an irreducible component Λ of $\dim \Lambda \geq 1$. Let l_0 be the length $l(X \cap \langle u, x\rangle)$ for general $u \in \Lambda$ and general $x \in X$. Then Λ is a line, X is a birational-divisor of type $(\mu, 1)$ for $\mu = l_0 - 1 \geq 2$ on a conical rational scroll $\mathbf{E}^{\Lambda}_{\mathcal{E}}$ with a birational-embedding $\psi : \mathbf{E}^{\Lambda}_{\mathcal{E}} \to \mathbb{P}^N$ and with original divisor $\tilde{X}$, and X is isomorphic to $\tilde{X}$ by ψ. In particular, $h^1(\mathcal{O}_X) = 0$, and if $n \geq 3$, $\operatorname{Pic} X \cong \mathbb{Z}^2$. Moreover $(\mathcal{I}_X/\mathcal{I}_X^2)(d - e + 1)|_{\Lambda}$ is an ample vector bundle on Λ.*

Proof. By Theorem 1, Λ is linear. By Theorem 4, the partial Gauss map $\gamma|_{\Lambda}$ is nonconstant since $\Lambda \cap \operatorname{Sm} X = \Lambda$ and $\dim \Lambda \geq 1$. By Theorem 6.1, Λ is a line and X is a birational-divisor of a conical rational scroll $\mathbf{E}^{\Lambda}_{\mathcal{E}}$ of type $(\mu, 1)$ for $\mu = l_0 - 1 \geq 2$ with the original divisor $\tilde{X}$. By (2) of Proposition 5.2, $\tilde{X} \cap \tilde{\Lambda}_{\mathbb{P}^1} \to \Lambda$ is isomorphic and $\tilde{X} \to X$ is finite. Thus $\tilde{X}$ and X are isomorphic by Zariski's Main Theorem ([15, III.11.4]). By a Lefschetz-type Theorem (see for example [10, p. 55]), an ample divisor $\tilde{X} \cong X$ of $\mathbf{E}^{\Lambda}_{\mathcal{E}}$ satisfies the required condition. To see the last part, consider the bundle $P^1_X(\mathcal{O}_X(1))$ of principal part of X which fits into the exact sequence

$$(6.3.1) \qquad 0 \to (\mathcal{I}_X/\mathcal{I}_X^2)(1) \to V \otimes \mathcal{O}_X \xrightarrow{\alpha} P^1_X(\mathcal{O}_X(1)) \to 0$$

for $V := H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ and gives the tangent bundle $\amalg_{x \in X} T_x(X) = \mathbb{P}(P^1_X(\mathcal{O}_X(1))) \subseteq \mathbb{P}^N \times X$ (see [13, Ch. IV §16], [19, Ch. IV. A)]). As in (6.1), for the linear projection $\pi_\Lambda : \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{N-2}$ from Λ, let $\tilde{\rho} : \Lambda \to \mathbb{P}^1$ be the morphism induced from the morphism $\rho : \Lambda \to \mathbb{G} := \mathbb{G}(n-2, \mathbb{P}^{N-2})$, $\rho(y) = [\pi_\Lambda(T_y(X) \setminus \Lambda)]$ by taking the normalization $\eta : \mathbb{P}^1 = \tilde{C} \to C$ of the image $C = \rho(\Lambda)$, i.e., $\rho = i_C \circ \eta \circ \tilde{\rho}$ for the inclusion $i_C : C \hookrightarrow \mathbb{G}$. By the construction of $\mathbf{E}^{\Lambda}_{\mathcal{E}}$, the quotient $\beta : V \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{E}$ defining $\psi : \mathbf{E}^{\Lambda}_{\mathcal{E}} \to \mathbb{P}^N$ is obtained from α so that we have $\tilde{\rho}^*\beta = \alpha|_\Lambda$. Hence, for the kernel $\mathcal{K}$ of β, we have $\tilde{\rho}^*\mathcal{K} \cong (\mathcal{I}_X/\mathcal{I}_X^2)(1)|_\Lambda$. Note that $\mathcal{K} = \bigoplus_{i=1}^{e} \mathcal{O}_{\mathbb{P}^1}(-b_i)$ for some $b_i > 0$ with $\sum_{i=1}^{e} b_i = c_1(\mathcal{E})$, since ψ is defined by β and $\psi(\mathbf{E}^{\Lambda}_{\mathcal{E}})$ is nondegenerate. To see $\tilde{\rho}$, we claim that *for each $\tilde{y} \in \tilde{\Lambda}_{\mathbb{P}^1} \cap \tilde{X}$ and for $y := \psi(\tilde{y}) \in \Lambda$, $T_y(X) \subseteq \mathbb{P}^N$ is the image $\psi(F) \subseteq \mathbb{P}^N$ of the fiber $F := p^{-1}(p(\tilde{y}))$ for the projection $p : \mathbf{E}^{\Lambda}_{\mathcal{E}} \to \mathbb{P}^1$, i.e., $\tilde{\rho}(y) = p(\tilde{y})$ for general $\tilde{y} \in \tilde{\Lambda}_{\mathbb{P}^1} \cap \tilde{X}$. This implies that $\tilde{\rho}$ is finite of degree μ since $\tilde{X} \cap \tilde{\Lambda}_{\mathbb{P}^1}$ is a divisor of $\tilde{\Lambda}_{\mathbb{P}^1}$ of type $(\mu, 1)$.* To prove the claim, since both are linear in $\mathbb{P}^N$ through $y \in \Lambda$, we will show that the Zariski tangent space $\Theta_y(X)(\subseteq \Theta_y(\mathbb{P}^N))$ is equal to the image of $\Theta_{\tilde{y}}(F)$ by the differential $d\psi_{\tilde{y}} : \Theta_{\tilde{y}}(\mathbf{E}^{\Lambda}_{\mathcal{E}}) \to \Theta_y(\mathbb{P}^N)$ of ψ at $\tilde{y}$. Since $\tilde{X}$ and X are isomorphic by ψ, $\Theta_{\tilde{y}}(\tilde{X})$ is mapped isomorphically to $\Theta_y(X)$ by $d\psi_{\tilde{y}}$. On the other hand, as a vector space, $\Theta_{\tilde{y}}(\mathbf{E}^{\Lambda}_{\mathcal{E}})$ is spanned by $\Theta_{\tilde{y}}(F)$ and $\Theta_{\tilde{y}}(\psi^{-1}_{\tilde{\Lambda}_{\mathbb{P}^1}}(y))$ for $\psi_{\tilde{\Lambda}_{\mathbb{P}^1}} : \tilde{\Lambda}_{\mathbb{P}^1} \to \Lambda$. Moreover, $\Theta_{\tilde{y}}(F)$ is mapped injectively by $d\psi_{\tilde{y}}$ in $\Theta_y(\mathbb{P}^N)$ since F is embedded in $\mathbb{P}^N$, and $\Theta_{\tilde{y}}(\psi^{-1}_{\tilde{\Lambda}_{\mathbb{P}^1}}(y))$ is the kernel of $d\psi_{\tilde{y}}$. Therefore $\Theta_y(X) = d\psi_{\tilde{y}}(\Theta_{\tilde{y}}(F))$.

By the claim, $(\mathcal{I}_X/\mathcal{I}_X^2)(1)|_\Lambda \cong \tilde{\rho}^*\mathcal{K} \cong \bigoplus_{i=1}^e \mathcal{O}_{\mathbb{P}^1}(-\mu b_i)$. To show the ampleness of $(\mathcal{I}_X/\mathcal{I}_X^2)(d-e+1)|_\Lambda$, we have to prove $d-e-\mu b_i \geq 1$. Since $\sum_{i=1}^e \mu b_i = \mu c_1(\mathcal{E}) = d-1$ by Proposition 5.2, and since $\mu b_i \geq \mu \geq 2$, we have $\max\{\mu b_i | i = 1,\dots,e\} \leq d-1-2(e-1)$. Hence

$$d-e-\mu b_i \geq d-e-\max\{\mu b_i|i=1,\dots,e\} \geq d-e-(d-2e+1) = e-1 \geq 1.$$

□

7. Applications to the Castelnuovo-Mumford regularity: Proof of Theorems 8, 9 and 10

Proof of Theorem 8. When $e=1$, the assertion is clear. So we assume $e \geq 2$. Set $\mathcal{L} := \mathcal{O}_{\hat{\mathbb{P}}^N}((d-e+1)A-E)$. Let $\tilde{\varepsilon} : H^0(\hat{\mathbb{P}}^N_X, \mathcal{L}) \otimes \mathcal{O}_{\hat{\mathbb{P}}^N} \to \mathcal{L}$ and let $\varepsilon : H^0(\mathbb{P}^N, \mathcal{I}_X(d-e+1)) \otimes \mathcal{O}_{\mathbb{P}^N} \to \mathcal{I}_X(d-e+1)$ be the evaluation maps. Since $\tilde{\varepsilon}$ is the composite of $\sigma^*\varepsilon$ and the natural surjection $\sigma^*\mathcal{I}_X(d-e+1) \to \mathcal{L}$, we have $\operatorname{Supp}\operatorname{Coker}(\tilde{\varepsilon}) \subseteq \sigma^{-1}(\operatorname{Supp}\operatorname{Coker}(\varepsilon))$. On the other hand, by (2) of Theorem 2 in [22] and our assumption, $\operatorname{Supp}\operatorname{Coker}(\varepsilon) \subseteq \mathcal{B}(X) \cup \mathcal{C}(X)$. Hence $\operatorname{Supp}\operatorname{Coker}(\tilde{\varepsilon}) \subseteq \sigma^{-1}(\mathcal{B}(X)) \cup \sigma^{-1}(\mathcal{C}(X))$. To see the semiampleness of $\mathcal{L}$, by Zariski–Fujita's Theorem [11] (see also Remark 2.1.32 [21, I. p. 132]), it is enough to show that $\mathcal{L}$ is ample on each irreducible component of $\sigma^{-1}(\mathcal{C}(X)) \cup \sigma^{-1}(\mathcal{B}(X))$. Each component of $\mathcal{B}(X)$ is a point away from X by (3) of Theorem 3, and hence each irreducible component of $\sigma^{-1}(\mathcal{B}(X))$ is a point of $\hat{\mathbb{P}}^N_X$ and $\mathcal{L}$ is ample on it. Before looking at the ampleness on each irreducible component of $\sigma^{-1}(\mathcal{C}(X))$, we note that $E = \sigma^{-1}(X)$ is the projective bundle $\mathbb{P}_X(\mathcal{I}_X/\mathcal{I}_X^2)$ with projection $\sigma_E := \sigma|_E : \mathbb{P}_X(\mathcal{I}_X/\mathcal{I}_X^2) \to X$ and with $\mathcal{O}_{\mathbb{P}_X(\mathcal{I}_X/\mathcal{I}_X^2)}(1) \cong \mathcal{O}_{\hat{\mathbb{P}}^N}(-E) \otimes \mathcal{O}_E$ (see [15, Ch. II 8.24]). Hence

$$\mathcal{L} \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}_X(\mathcal{I}_X/\mathcal{I}_X^2)}(1) \otimes \sigma_E^*\mathcal{O}_X(d-e+1) \cong \mathcal{O}_{\mathbb{P}_X((\mathcal{I}_X/\mathcal{I}_X^2)(d-e+1))}(1).$$

If Q is a 0-dimensional component of $\mathcal{C}(X)$, then $\sigma^{-1}(Q) \cong \mathbb{P}^{e-1}$ and $\mathcal{L} \otimes \mathcal{O}_{\sigma^{-1}(Q)} \cong \mathcal{O}_{\mathbb{P}^{e-1}}(1)$ is ample. If Λ is a positive-dimensional component of $\mathcal{C}(X)$, then Λ is a line and $(\mathcal{I}_X/\mathcal{I}_X^2)(d-e+1)|_\Lambda$ is ample by Corollary 6.2, and hence $\mathcal{L} \otimes \mathcal{O}_{\sigma^{-1}(\Lambda)}$ is ample. □

Proof of Theorem 9. If $e=1$, the conclusion is clear. Hence we assume $e \geq 2$. By Theorem 8, the proof follows from the argument of [3] as follows (see also [21, I. (4.3.15), p. 259]). Let $\sigma : \hat{\mathbb{P}}^N_X \to \mathbb{P}^N$ be the blowing-up of $\mathbb{P}^N$ along X with exceptional divisor E and let A be the divisor of the pull-back of a hyperplane section of $\mathbb{P}^N$. For each integer $\ell \geq 1$, $D_\ell := e((d-e+1)A-E)+\ell A$ is a nef and big divisor by Theorem 8. Since $K_{\hat{\mathbb{P}}^N_X} + D_\ell = (e(d-e+1)+\ell-N-1)A-E$ for the canonical divisor $K_{\hat{\mathbb{P}}^N_X}$ of $\hat{\mathbb{P}}^N_X$, by the Kawamata–Viehweg Vanishing Theorem ([18], [23]; see also [21, I. Theorem 4.3.1]), $H^i(\hat{\mathbb{P}}^N_X, \mathcal{O}_{\hat{\mathbb{P}}^N_X}((e(d-e+1)+\ell-N-1)A-E)) = 0$ for all $i>0$. Hence $H^i(\mathbb{P}^N, \mathcal{I}_X(e(d-e+1)-N+(\ell-1))) = 0$ for all $i>0$ (see for example [21, I (4.3.16), p. 259]). This together with the consequence of the Grothendieck vanishing theorem, $H^i(\mathbb{P}^N, \mathcal{I}_X(m)) = 0$ for all $i \geq n+2$ and $m \geq -N$, implies that X is $(e(d-e)+1)$-regular. □

Corollary 7.1. *Let $X \subseteq \mathbb{P}^N$ be a nondegenerate smooth projective variety of dimension n, codimension e, and degree d. Set*

$$e_0 := \begin{cases} e & \text{if } e < n+1 \text{ or} \\ & \text{if } e \geq n+1 \text{ and } |\frac{d}{2} - e| \geq |\frac{d}{2} - (n+1)|, \\ n+1 & \text{if } e \geq n+1 \text{ and } |\frac{d}{2} - e| \leq |\frac{d}{2} - (n+1)|. \end{cases}$$

Then X is $(e_0(d-e_0)+1)$-regular.

Proof. We have only to show that if $e \geq n+1$, for each k with $n+1 \leq k \leq e$, X is $(k(d-k)+1)$-regular. This is because the quadratic function $f(k) := k(d-k)+1$ on k $(n+1 \leq k \leq e)$ has the minimum value $f(e_0)$ at $k = e_0$. Assume $e > n+1$. For each k with $n+1 \leq k < e$, by taking a general linear projection $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^{n+k}$ we have $X' := \pi(X) \subseteq \mathbb{P}^{n+k}$, which is isomorphic to X by π. Hence X' is $(k(d-k)+1)$-regular for $n+1 \leq k \leq e$. Set $t := k(d-k)+1$. In this case, $H^1(\mathcal{I}_{X'/\mathbb{P}^{n+k}}(t-1)) = 0$ implies $H^1(\mathcal{I}_{X/\mathbb{P}^N}(t-1)) = 0$. This is because if $H^1(\mathcal{I}_{X'/\mathbb{P}^{n+k}}(t-1)) = 0$, then $H^0(\mathcal{O}_{\mathbb{P}^{n+k}}(t-1)) \to H^0(\mathcal{O}_{X'}(t-1))$ is surjective, and hence so is $H^0(\mathcal{O}_{\mathbb{P}^N}(t-1)) \to H^0(\mathcal{O}_X(t-1))$. Moreover for $2 \leq i \leq n+1$, $H^i(\mathcal{I}_{X/\mathbb{P}^N}(t-i)) = 0$ if and only if $H^i(\mathcal{I}_{X'/\mathbb{P}^{n+k}}(t-i)) = 0$, since $H^i(\mathcal{I}_{X/\mathbb{P}^N}(t-i)) = 0$ if and only if $H^{i-1}(\mathcal{O}_X(t-i)) = 0$, and the same is true for $X' \subseteq \mathbb{P}^{n+k}$. Thus X is also $(k(d-k)+1)$-regular for $n+1 \leq k \leq e$, as required. □

Remark 7.2. Our result slightly improves [3] (see also [5]): [3] proved under the same assumption as in Theorem 9, that X is $(c(d-1)+1)$-regular for $c := \min\{e, n+1\}$. It is easy to see that $c(d-1)+1 \geq c(d-c)+1 \geq e_0(d-e_0)+1$.

Proof of Theorem 10. Keep the same notation as in the proof of Theorem 9, but for D_ℓ we set $D_\ell := (e+a-1)((d-e+1)A - E) + \ell A$ for $\ell \geq 1$ instead. Then D_ℓ is nef and big, and $H^i(\hat{\mathbb{P}}^N_X, \mathcal{O}_{\hat{\mathbb{P}}^N_X}(K_{\hat{\mathbb{P}}^N_X} + D_\ell)) = 0$ for all $i > 0$ by the Kawamata–Viehweg vanishing theorem, and hence $H^i(\mathbb{P}^N, \mathcal{I}^a_X((e+a-1)(d-e+1) - N + (\ell-1))) = 0$ for all $i > 0$ and $\ell \geq 1$. By the same way as in Theorem 9, $\mathcal{I}^a_X$ is $\{(d-e)(e+a-1)+a)\}$-regular. □

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Faculty of Engineering Sciences, Department of Mathematics, Yokohama National University, Yokohama 240-8501 Japan

E-mail address: `noma@ynu.ac.jp`