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SPECIALIZATION OF NONSYMMETRIC MACDONALD POLYNOMIALS AT $t = \infty$ AND DEMAZURE SUBMODULES OF LEVEL-ZERO EXTREMAL WEIGHT MODULES

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ABSTRACT. In this paper, we give a representation-theoretic interpretation of the specialization $E_{w_\circ\lambda}(q,\infty)$ of the nonsymmetric Macdonald polynomial $E_{w_\circ\lambda}(q,t)$ at $t = \infty$ in terms of the Demazure submodule $V_{w_\circ}^-(\lambda)$ of the level-zero extremal weight module $V(\lambda)$ over a quantum affine algebra of an arbitrary untwisted type. Here, λ is a dominant integral weight, and $w_\circ$ denotes the longest element in the finite Weyl group W. Also, for each $x \in W$, we obtain a combinatorial formula for the specialization $E_{x\lambda}(q,\infty)$ at $t = \infty$ of the nonsymmetric Macdonald polynomial $E_{x\lambda}(q,t)$ and also a combinatorical formula for the graded character $\operatorname{gch} V_x^-(\lambda)$ of the Demazure submodule $V_x^-(\lambda)$ of $V(\lambda)$. Both of these formulas are described in terms of quantum Lakshmibai-Seshadri paths of shape λ.

1. INTRODUCTION

Symmetric Macdonald polynomials with two parameters q and t were introduced by Macdonald [M1] as a family of orthogonal symmetric polynomials, which includes as special or limiting cases almost all the classical families of orthogonal symmetric polynomials. This family of polynomials is characterized in terms of the double affine Hecke algebra (DAHA) introduced by Cherednik ([Ch1], [Ch2]). In fact, there exists another family of orthogonal polynomials, called nonsymmetric Macdonald polynomials, which are simultaneous eigenfunctions of Y-operators acting on the polynomial representation of the DAHA; by "symmetrizing" nonsymmetric Macdonald polynomials, we obtain symmetric Macdonald polynomials (see [M]).

Based on the characterization above of nonsymmetric Macdonald polynomials, Ram-Yip [RY] obtained a combinatorial formula expressing symmetric or nonsymmetric Macdonald polynomials associated to an arbitrary untwisted affine root system. This formula is described in terms of alcove walks, which are certain strictly combinatorial objects. In addition, Orr-Shimozono [OS] refined the Ram-Yip formula above and generalized it to an arbitrary affine root system (including the twisted case). Also, they specialized their formula at $t = 0$, $t = \infty$, $q = 0$, and $q = \infty$.

As for representation-theoretic interpretations of the specialization of symmetric or nonsymmetric Macdonald polynomials at $t = 0$, we know the following. Ion [I] proved that for a dominant integral weight λ and an element x of a finite Weyl group W, the specialization $E_{x\lambda}(q,0)$ of the nonsymmetric Macdonald polynomial $E_{x\lambda}(q,t)$ at $t = 0$ is equal to the graded character of a certain Demazure

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submodule of an integrable, irreducible highest weight module over an affine Lie algebra of untwisted simply-laced type or twisted non-simply-laced type. Afterward, Lenart-Naito-Sagaki-Schilling-Shimozono [LNSSS2] proved that for a dominant integral weight λ, the set $\mathrm{QLS}(\lambda)$ of all quantum Lakshmibai-Seshadri (QLS) paths of shape λ provides a realization of the crystal basis of a special quantum Weyl module over a quantum affine algebra $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$ (without degree operator) of an arbitrary untwisted type, and also proved that its graded character equals the specialization $E_{w_\circ\lambda}(q,0)$ at $t=0$, where $w_\circ$ denotes the longest element of W. Here a QLS path is obtained from an affine level-zero Lakshmibai-Seshadri (LS) path through the projection $\mathbb{R}\otimes_{\mathbb{Z}} P_{\mathrm{aff}} \to \mathbb{R}\otimes_{\mathbb{Z}} P$, which factors the null root δ of an affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}$, and is described in terms of (the parabolic version of) the quantum Bruhat graph, introduced by Brenti-Fomin-Postnikov [BFP]. The set of QLS paths is endowed with an affine crystal structure in a way similar to the one for the set of ordinary LS paths introduced by Littelmann [L]. Moreover, Lenart-Naito-Sagaki-Schilling-Shimozono [LNSSS3] obtained a formula for the specialization $E_{x\lambda}(q,0)$, $x\in W$, at $t=0$ in an arbitrary untwisted affine type, which is described in terms of QLS paths of shape λ, and proved that the specialization $E_{x\lambda}(q,0)$ is just the graded character of a certain Demazure-type submodule of the special quantum Weyl module. The crucial ingredient in the proof of this result is a graded character formula obtained in [NS3] for the Demazure submodule $V_e^-(\lambda)$ of the level-zero extremal weight module $V(\lambda)$ of extremal weight λ over a quantum affine algebra $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$, where e is the identity element of W. More precisely, in [NS3], Naito and Sagaki proved that the graded character $\mathrm{gch}\, V_e^-(\lambda)$ of $V_e^-(\lambda)\subset V(\lambda)$ is equal to $\left(\prod_{i\in I}\prod_{r=1}^{m_i}(1-q^{-r})\right)^{-1} E_{w_\circ\lambda}(q^{-1},0)$, where $\lambda=\sum_{i\in I} m_i\varpi_i$ is a dominant integral weight, with ϖ_i, $i\in I$, the fundamental weights. The graded character $\mathrm{gch}\, V_e^-(\lambda)$ is obtained from the ordinary character of $V_e^-(\lambda)$ by replacing e^{δ} by q, with δ the null root of the affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}$.

The purpose of this paper is to give a representation-theoretic interpretation of the specialization $E_{w_\circ\lambda}(q,\infty)$ of the nonsymmetric Macdonald polynomial $E_{w_\circ\lambda}(q,t)$ at $t=\infty$ in terms of the Demazure submodule $V_{w_\circ}^-(\lambda)$ of $V(\lambda)$; here we remark that $V_{w_\circ}^-(\lambda)\subset V_e^-(\lambda)$. More precisely, we prove the following theorem.

Theorem A (= Theorem 5.1.2). *Let $\lambda=\sum_{i\in I} m_i\varpi_i$ be a dominant integral weight. Then, the graded character* $\mathrm{gch}\, V_{w_\circ}^-(\lambda)$ *of the Demazure submodule $V_{w_\circ}^-(\lambda)$ of $V(\lambda)$ is equal to*

$$\left(\prod_{i\in I}\prod_{r=1}^{m_i}(1-q^{-r})\right)^{-1} E_{w_\circ\lambda}(q,\infty).$$

In order to prove Theorem A, we first rewrite the Orr-Shimozono formula for the specialization $E_{x\lambda}(q,\infty)$ for $x\in W$ (originally described in terms of quantum alcove walks) in terms of QLS paths by use of an explicit bijection sending quantum alcove walks to QLS paths that preserves weights and degrees; in some ways, this bijection generalizes a similar one in [LNSSS2]. In particular, for $x=w_\circ$, the Orr-Shimozono formula rewritten in terms of QLS paths states that

$$(*)\qquad E_{w_\circ\lambda}(q,\infty)=\sum_{\psi\in\mathrm{QLS}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_{w_\circ\lambda}(\psi)},$$

where $\mathrm{QLS}(\lambda)$ is the set of all QLS paths of shape λ, and for $\psi \in \mathrm{QLS}(\lambda)$, $\deg_{w_\circ\lambda}(\psi)$ is a certain nonpositive integer, which is explicitly described in terms of the quantum Bruhat graph; see §3.2 for details.

Next, using the explicit realization, obtained in [INS], of the crystal basis $\mathcal{B}(\lambda)$ of $V(\lambda)$ by semi-infinite LS paths of shape λ, we compute the graded character $\mathrm{gch}\, V_x^-(\lambda)$ of the Demazure submodule $V_x^-(\lambda)$ for $x \in W$ and prove the following theorem.

Theorem B (= Theorem 5.1.1)**.** *Let $\lambda = \sum_{i\in I} m_i \varpi_i$ be a dominant integral weight and x an element of the finite Weyl group W. Then, the graded character $\mathrm{gch}\, V_x^-(\lambda)$ of $V_x^-(\lambda)$ is equal to*

$$\left(\prod_{i\in I}\prod_{r=1}^{m_i}(1-q^{-r})\right)^{-1} \sum_{\psi\in\mathrm{QLS}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_{x\lambda}(\psi)}.$$

The proof of Theorem B is based on the fact that by factoring the null root δ of $\mathfrak{g}_{\mathrm{aff}}$, we obtain a surjective strict morphism of crystals from the set of all semi-infinite LS paths of shape λ onto $\mathrm{QLS}(\lambda)$. By combining the special case $x = w_\circ$ of Theorem B with equation $(*)$ above, we obtain Theorem A.

Finally, for $x \in W$, we define a certain (finite-dimensional) quotient module $V_x^-(\lambda)/X_x^-(\lambda)$ of $V_x^-(\lambda)$ and prove that its graded character $\mathrm{gch}\,(V_x^-(\lambda)/X_x^-(\lambda))$ is equal to $\sum_{\psi\in\mathrm{QLS}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_{x\lambda}(\psi)}$. Hence it follows that under the specialization $e^\delta = q = 1$, all the modules $V_x^-(\lambda)/X_x^-(\lambda)$, $x \in W$, have the same character; in particular, they have the same dimension. Also, in the case $x = w_\circ$, we have $\mathrm{gch}\,\big(V_{w_\circ}^-(\lambda)/X_{w_\circ}^-(\lambda)\big) = E_{w_\circ\lambda}(q,\infty)$. Note that in the case $x = e$, the quotient module $V_e^-(\lambda)/X_e^-(\lambda)$ is just the one in [NS3, §7.2], and hence we have $\mathrm{gch}\,(V_e^-(\lambda)/X_e^-(\lambda)) = E_{w_\circ\lambda}(q^{-1},0)$ (see [LNSSS3, §3] and [NS3, §6.4]). Based on these results together with [Kat, Theorem 5.1] for the classical limit, we can think of the quotient modules $V_x^-(\lambda)/X_x^-(\lambda)$, $x \in W$, as a quantum analog of "generalized Weyl modules" introduced in [FM].

This paper is organized as follows. In Section 2, we fix our notation and recall some basic facts about the (parabolic) quantum Bruhat graph. Also, we briefly review the Orr-Shimozono formula for the specialization $E_{x\lambda}(q,\infty)$ at $t = \infty$ for $x \in W$. In Section 3, we prove equation $(*)$ above or, more generally, Theorem 3.2.7. This theorem gives the description of the specialization $E_{x\lambda}(q,\infty)$ at $t = \infty$ for $x \in W$ in terms of QLS paths of shape λ. In Section 4, we compute the graded character $\mathrm{gch}\, V_x^-(\lambda)$ for an arbitrary $x \in W$ and prove Theorem B. By combining the special case $x = w_\circ$ of Theorem B with equation $(*)$, we obtain Theorem A. Finally, for $x \in W$, we define a certain (finite-dimensional) quotient module $V_x^-(\lambda)/X_x^-(\lambda)$ of $V_x^-(\lambda)$ and compute its graded character. In the special case $x = w_\circ$, we obtain the equality $\mathrm{gch}\,\big(V_{w_\circ}^-(\lambda)/X_{w_\circ}^-(\lambda)\big) = E_{w_\circ\lambda}(q,\infty)$.

2. (Parabolic) quantum Bruhat graph and Orr-Shimozono formula

2.1. **(Parabolic) quantum Bruhat graph.** Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, I the vertex set for the Dynkin diagram of $\mathfrak{g}$, $\{\alpha_i\}_{i\in I}$ (resp., $\{\alpha_i^\vee\}_{i\in I}$) the set of all simple roots (resp., coroots) of $\mathfrak{g}$, $\mathfrak{h} = \bigoplus_{i\in I}\mathbb{C}\alpha_i^\vee$ a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{h}^* = \bigoplus_{i\in I}\mathbb{C}\alpha_i$ the dual space of $\mathfrak{h}$, and $\mathfrak{h}^*_\mathbb{R} = \bigoplus_{i\in I}\mathbb{R}\alpha_i$ the real form of $\mathfrak{h}^*$. The canonical pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$ is denoted by $\langle\cdot,\cdot\rangle : \mathfrak{h}^*\times\mathfrak{h}\to\mathbb{C}$. Let $Q = \sum_{i\in I}\mathbb{Z}\alpha_i \subset \mathfrak{h}^*_\mathbb{R}$ denote the root lattice of $\mathfrak{g}$, $Q^\vee = \sum_{i\in I}\mathbb{Z}\alpha_i^\vee \subset \mathfrak{h}_\mathbb{R}$ the coroot lattice of $\mathfrak{g}$, and $P = \sum_{i \in I} \mathbb{Z}\varpi_i \subset \mathfrak{h}^*_{\mathbb{R}}$ the weight lattice of $\mathfrak{g}$, where the ϖ_i, $i \in I$, are the fundamental weights for $\mathfrak{g}$, i.e., $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $i, j \in I$. We set $P^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0}\varpi_i$ and call an element λ of P^+ a dominant weight. Let us denote by Δ the set of all roots and by Δ^+ (resp., Δ^-) the set of all positive (resp., negative) roots. Also, let $W := \langle s_i \mid i \in I \rangle$ be the Weyl group of $\mathfrak{g}$, where s_i, $i \in I$, are the simple reflections acting on $\mathfrak{h}^*$ and on $\mathfrak{h}$:

$$s_i \nu = \nu - \langle \nu, \alpha_i^\vee \rangle \alpha_i, \quad \nu \in \mathfrak{h}^*,$$
$$s_i h = h - \langle \alpha_i, h \rangle \alpha_i^\vee, \quad h \in \mathfrak{h}.$$

We denote the identity element and the longest element of W by e and $w_\circ$, respectively. If $\alpha \in \Delta$ is written as $\alpha = w\alpha_i$ for $w \in W$ and $i \in I$, then we define $\alpha^\vee$ to be $w\alpha_i^\vee$; note that $s_\alpha = s_{\alpha^\vee} = ws_iw^{-1}$. For $u \in W$, the length of u is denoted by $\ell(u)$, which equals $\#(\Delta^+ \cap u^{-1}\Delta^-)$.

Definition 2.1.1 ([BFP, Definition 6.1])**.** The quantum Bruhat graph, denoted by QBG, is the directed graph with vertex set W whose directed edges are labeled by positive roots as follows. For $u, v \in W$, and $\beta \in \Delta^+$, an arrow $u \xrightarrow{\beta} v$ is an edge of QBG if the following hold:

(1) $v = us_\beta$, and
(2) either (2a): $\ell(v) = \ell(u) + 1$ or (2b): $\ell(v) = \ell(u) - 2\langle \rho, \beta^\vee \rangle + 1$,

where $\rho := \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$. An edge satisfying (2a) (resp., (2b)) is called a Bruhat (resp., quantum) edge.

Remark 2.1.2. The quantum Bruhat graph defined above is a "right-handed" version, while the one defined in [BFP] is a "left-handed" version. We remark that the results of [BFP] used in this paper (such as Proposition 2.1.4) are unaffected by this difference (cf. [Po]).

For an edge $u \xrightarrow{\beta} v$ of QBG, we set

$$\mathrm{wt}(u \to v) := \begin{cases} 0 & \text{if } u \xrightarrow{\beta} v \text{ is a Bruhat edge,} \\ \beta^\vee & \text{if } u \xrightarrow{\beta} v \text{ is a quantum edge.} \end{cases}$$

Also, for $u, v \in W$, we take a shortest directed path $u = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} x_r = v$ in QBG and set

$$\mathrm{wt}(u \Rightarrow v) := \mathrm{wt}(x_0 \to x_1) + \cdots + \mathrm{wt}(x_{r-1} \to x_r) \in Q^\vee.$$

We know from [Po, Lemma 1 (2),(3)] that this definition does not depend on the choice of a shortest directed path from u to v in QBG. For a dominant weight $\lambda \in P^+$, we set $\mathrm{wt}_\lambda(u \Rightarrow v) := \langle \lambda, \mathrm{wt}(u \Rightarrow v) \rangle$ and call it the λ-weight of a directed path from u to v in QBG.

Lemma 2.1.3. *If $x \xrightarrow{\beta} y$ is a Bruhat (resp., quantum) edge of* QBG*, then $yw_\circ \xrightarrow{-w_\circ\beta} xw_\circ$ is also a Bruhat (resp., quantum) edge of* QBG*.*

Proof. This follows easily from equalities $\ell(y) - \ell(x) = \ell(xw_\circ) - \ell(yw_\circ)$ and $\langle \rho, -w_\circ\beta^\vee \rangle = \langle \rho, \beta^\vee \rangle$. ◻

Let $w \in W$. We take (and fix) reduced expressions $w = s_{i_1} \cdots s_{i_p}$ and $w_\circ w^{-1} = s_{i_{-q}} \cdots s_{i_0}$. Note that

$$w_\circ = s_{i_{-q}} \cdots s_{i_0} s_{i_1} \cdots s_{i_p}$$

is also a reduced expression for the longest element $w_\circ$. Now we set

$$\beta_k := s_{i_p} \cdots s_{i_{k+1}} \alpha_{i_k}, \quad -q \le k \le p; \tag{2.1}$$

we have $\{\beta_{-q}, \ldots, \beta_0, \ldots, \beta_p\} = \Delta^+$. Then we define a total order $\prec$ on Δ^+ by

$$\beta_{-q} \prec \beta_{-q+1} \prec \cdots \prec \beta_p. \tag{2.2}$$

Note that this total order is a weak reflection order in the sense of Definition 3.1.2 below.

Proposition 2.1.4 ([BFP, Theorem 6.4]). *Let u and v be elements in W.*

(1) *There exists a unique directed path from u to v in* QBG *for which the edge labels are strictly increasing* (*resp., strictly decreasing*) *in the total order $\prec$ above.*

(2) *The unique label-increasing* (*resp., label-decreasing*) *path*

$$u = u_0 \xrightarrow{\gamma_1} u_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} u_r = v$$

from u to v in QBG *is a shortest directed path from u to v. Moreover, it is lexicographically minimal* (*resp., lexicographically maximal*) *among all shortest directed paths from u to v*; *namely, for an arbitrary shortest directed path*

$$u = u'_0 \xrightarrow{\gamma'_1} u'_1 \xrightarrow{\gamma'_2} \cdots \xrightarrow{\gamma'_r} u'_r = v$$

from u to v in QBG, *there exists $1 \le j \le r$ such that $\gamma_j \prec \gamma'_j$* (*resp., $\gamma_j \succ \gamma'_j$*), *and $\gamma_k = \gamma'_k$ for $1 \le k \le j-1$.*

For a subset $S \subset I$, we set $W_S := \langle s_i \mid i \in S \rangle$; notice that S may be the empty set $\emptyset$. We denote the longest element of W_S by $w_\circ(S)$. Also, we set $\Delta_S := Q_S \cap \Delta$, where $Q_S := \sum_{i \in S} \mathbb{Z}\alpha_i$, and then $\Delta_S^+ := \Delta_S \cap \Delta^+$, $\Delta_S^- := \Delta_S \cap \Delta^-$. Let W^S denote the set of all minimal-length coset representatives for the cosets in W/W_S. For $w \in W$, we denote the minimal-length coset representative of the coset wW_S by $\lfloor w \rfloor$, and for a subset $U \subset W$, we set $\lfloor U \rfloor := \{\lfloor w \rfloor \mid w \in U\} \subset W^S$.

Definition 2.1.5 ([LNSSS1, §4.3]). The parabolic quantum Bruhat graph, denoted by QBG^S, is the directed graph with vertex set W^S whose directed edges are labeled by positive roots in $\Delta^+ \setminus \Delta_S^+$ as follows. For $u, v \in W^S$ and $\beta \in \Delta^+ \setminus \Delta_S^+$, an arrow $u \xrightarrow{\beta} v$ is an edge of QBG^S if the following hold:

(1) $v = \lfloor u s_\beta \rfloor$, and

(2) either (2a): $\ell(v) = \ell(u) + 1$ or (2b): $\ell(v) = \ell(u) - 2\langle \rho - \rho_S, \beta^\vee \rangle + 1$,

where $\rho_S := \frac{1}{2}\sum_{\alpha \in \Delta_S^+} \alpha$. An edge satisfying (2a) (resp., (2b)) is called a Bruhat (resp., quantum) edge.

For an edge $u \xrightarrow{\beta} v$ in QBG^S, we set

$$\mathrm{wt}^S(u \to v) := \begin{cases} 0 & \text{if } u \xrightarrow{\beta} v \text{ is a Bruhat edge,} \\ \beta^\vee & \text{if } u \xrightarrow{\beta} v \text{ is a quantum edge.} \end{cases}$$

Also, for $u, v \in W^S$, we take a shortest directed path $\mathbf{p} : u = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} x_r = v$ in QBG^S (such a path always exists by [LNSSS1, Lemma 6.12]) and set

$$\mathrm{wt}^S(\mathbf{p}) := \mathrm{wt}^S(x_0 \to x_1) + \cdots + \mathrm{wt}^S(x_{r-1} \to x_r) \in Q^\vee.$$

We know from [LNSSS1, Proposition 8.1] that if $\mathbf{q}$ is another shortest directed path from u to v in QBG^S, then $\mathrm{wt}^S(\mathbf{p}) - \mathrm{wt}^S(\mathbf{q}) \in Q_S^\vee := \sum_{i \in S} \mathbb{Z}\alpha_i^\vee$.

Now, we take and fix an arbitrary dominant weight $\lambda \in P^+$ and set

$$S = S_\lambda := \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}.$$

By the remark just above, for $u, v \in W^S$, the value $\langle \lambda, \mathrm{wt}^S(\mathbf{p}) \rangle$ does not depend on the choice of a shortest directed path $\mathbf{p}$ from u to v in QBG^S; this value is called the λ-weight of a directed path from u to v in QBG^S. Moreover, we know from [LNSSS2, Lemma 7.2] that the value $\langle \lambda, \mathrm{wt}^S(\mathbf{p}) \rangle$ is equal to the value $\mathrm{wt}_\lambda(x \Rightarrow y) = \langle \lambda, \mathrm{wt}(x \Rightarrow y) \rangle$ for all $x \in uW_S$ and $y \in vW_S$.

Definition 2.1.6 ([LNSSS2, §3.2])**.** Let $\lambda \in P^+$ be a dominant weight and $\sigma \in \mathbb{Q} \cap [0,1]$, and set $S = S_\lambda$. We denote by $\mathrm{QBG}_{\sigma\lambda}$ (resp., $\mathrm{QBG}_{\sigma\lambda}^S$) the subgraph of QBG (resp., QBG^S) with the same vertex set but having only the edges $u \xrightarrow{\beta} v$ with $\sigma\langle \lambda, \beta^\vee \rangle \in \mathbb{Z}$.

Lemma 2.1.7 ([LNSSS2, Lemma 6.2])**.** *Let $\sigma \in \mathbb{Q} \cap [0,1]$; notice that σ may be* 1. *If $u \xrightarrow{\beta} v$ is an edge of $\mathrm{QBG}_{\sigma\lambda}$, then there exists a directed path from $\lfloor u \rfloor$ to $\lfloor v \rfloor$ in $\mathrm{QBG}_{\sigma\lambda}^S$.*

Also, for $u, v \in W$, let $\ell(u \Rightarrow v)$ denote the length of a shortest directed path in QBG from u to v. For $w \in W$, as in [BFP], we define the w-tilted Bruhat order $\leq_w$ on W as follows: for $u, v \in W$,

$$u \leq_w v \overset{\mathrm{def}}{\Leftrightarrow} \ell(w \Rightarrow v) = \ell(w \Rightarrow u) + \ell(u \Rightarrow v).$$

We remark that the w-tilted Bruhat order on W is a partial order with the unique minimal element w.

Lemma 2.1.8 ([LNSSS1, Theorem 7.1], [LNSSS2, Lemma 6.6])**.** *Let $u, v \in W^S$ and $w \in W_S$.*

(1) *There exists a unique minimal element in the coset vW_S in the uw-tilted Bruhat order $\leq_{uw}$. We denote it by $\min(vW_S, \leq_{uw})$.*
(2) *There exists a unique directed path from uw to some $x \in vW_S$ in* QBG *whose edge labels are increasing in the total order $\prec$ on Δ^+, defined in* (2.2)*, and lie in $\Delta^+ \setminus \Delta_S^+$. This path ends with $\min(vW_S, \leq_{uw})$.*
(3) *Let $\sigma \in \mathbb{Q} \cap [0,1]$, and let $\lambda \in P$ be a dominant weight. If there exists a directed path from u to v in $\mathrm{QBG}_{\sigma\lambda}^S$, then the directed path in part* (2) *is in* $\mathrm{QBG}_{\sigma\lambda}$.

2.2. **Orr-Shimozono formula.** In this subsection, we review a formula [OS, Proposition 5.4] for the specialization of nonsymmetric Macdonald polynomials at $t = \infty$.

Let $\widetilde{\mathfrak{g}}$ denote the finite-dimensional simple Lie algebra whose root datum is dual to that of $\mathfrak{g}$; the set of simple roots is $\{\alpha_i^\vee\}_{i \in I} \subset \mathfrak{h}$, and the set of simple coroots is $\{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$. We denote the set of all roots of $\widetilde{\mathfrak{g}}$ by $\widetilde{\Delta} = \{\alpha^\vee \mid \alpha \in \Delta\}$ and the set of all positive (resp., negative) roots of $\widetilde{\mathfrak{g}}$ by $\widetilde{\Delta}^+$ (resp., $\widetilde{\Delta}^-$). Also, for a subset $S \subset I$, we set

$$\widetilde{Q}_S := \sum_{i \in S} \mathbb{Z}\alpha_i^\vee, \quad \widetilde{\Delta}_S := \widetilde{\Delta} \cap \widetilde{Q}_S, \quad \widetilde{\Delta}_S^+ = \widetilde{\Delta}_S \cap \widetilde{\Delta}^+, \quad \widetilde{\Delta}_S^- = \widetilde{\Delta}_S \cap \widetilde{\Delta}^-.$$

We consider the untwisted affinization of the root datum of $\widetilde{\mathfrak{g}}$. Let us denote by $\widetilde{\Delta}_{\mathrm{aff}}$ the set of all real roots and by $\widetilde{\Delta}_{\mathrm{aff}}^+$ (resp., $\widetilde{\Delta}_{\mathrm{aff}}^-$) the set of all positive (resp., negative) real roots. Then we have $\widetilde{\Delta}_{\text{aff}} = \{\alpha^\vee + a\widetilde{\delta} \mid \alpha \in \Delta, a \in \mathbb{Z}\}$, with $\widetilde{\delta}$ the null root. We set $\alpha_0^\vee := \widetilde{\delta} - \varphi^\vee$, where $\varphi \in \Delta$ denotes the highest short root, and set $I_{\text{aff}} := I \sqcup \{0\}$. Then, $\{\alpha_i^\vee\}_{i \in I_{\text{aff}}}$ is the set of all simple roots. Also, for $\beta \in \mathfrak{h} \oplus \mathbb{C}\widetilde{\delta}$, we define $\deg(\beta) \in \mathbb{C}$ and $\overline{\beta} \in \mathfrak{h}$ by

$$\beta = \overline{\beta} + \deg(\beta)\widetilde{\delta}. \tag{2.3}$$

We denote the Weyl group of $\widetilde{\mathfrak{g}}$ by $\widetilde{W}$; we identify $\widetilde{W}$ and W through the identification of the simple reflections of the same index for each $i \in I$. For $\nu \in \mathfrak{h}^*$, let $t(\nu)$ denote the translation in $\mathfrak{h}^*$: $t(\nu)\gamma = \gamma + \nu$ for $\gamma \in \mathfrak{h}^*$. The corresponding affine Weyl group and the extended affine Weyl group are defined by $\widetilde{W}_{\text{aff}} := t(Q) \rtimes W$ and $\widetilde{W}_{\text{ext}} := t(P) \rtimes W$, respectively. Also, we define $s_0 : \mathfrak{h}^* \to \mathfrak{h}^*$ by $\nu \mapsto \nu - (\langle \nu, \varphi^\vee \rangle - 1)\varphi$. Then, $\widetilde{W}_{\text{aff}} = \langle s_i \mid i \in I_{\text{aff}} \rangle$; note that $s_0 = t(\varphi)s_\varphi$. The extended affine Weyl group $\widetilde{W}_{\text{ext}}$ acts on $\mathfrak{h} \oplus \mathbb{C}\widetilde{\delta}$ as linear transformations and on $\mathfrak{h}^*$ as affine transformations: for $v \in W$, $t(\nu) \in t(P)$,

$$vt(\nu)(\overline{\beta} + r\widetilde{\delta}) = v\overline{\beta} + (r - \langle \nu, \overline{\beta} \rangle)\widetilde{\delta}, \quad \overline{\beta} \in \mathfrak{h}, r \in \mathbb{C},$$
$$vt(\nu)\gamma = v\nu + v\gamma, \qquad \gamma \in \mathfrak{h}^*.$$

An element $u \in \widetilde{W}_{\text{ext}}$ can be written as

$$u = t(\text{wt}(u))\text{dir}(u), \tag{2.4}$$

where $\text{wt}(u) \in P$ and $\text{dir}(u) \in W$, according to the decomposition $\widetilde{W}_{\text{ext}} = t(P) \rtimes W$. For $w \in \widetilde{W}_{\text{ext}}$, we denote the length of w by $\ell(w)$, which equals $\#\left(\widetilde{\Delta}^+_{\text{aff}} \cap w^{-1}\widetilde{\Delta}^-_{\text{aff}}\right)$. Also, we set $\Omega := \{w \in \widetilde{W}_{\text{ext}} \mid \ell(w) = 0\}$.

For $\mu \in P$, we denote the shortest element in the coset $t(\mu)W$ by $m_\mu \in \widetilde{W}_{\text{ext}}$. In the following, we fix $\mu \in P$ and take a reduced expression $m_\mu = us_{\ell_1} \cdots s_{\ell_L} \in \widetilde{W}_{\text{ext}} = \Omega \ltimes \widetilde{W}_{\text{aff}}$, where $u \in \Omega$ and $\ell_1, \ldots, \ell_L \in I_{\text{aff}}$.

For each $J = \{j_1 < j_2 < j_3 < \cdots < j_r\} \subset \{1, \ldots, L\}$, we define an alcove path $p_J^{\text{OS}} = \left(m_\mu = z_0^{\text{OS}}, z_1^{\text{OS}}, \ldots, z_r^{\text{OS}}; \beta_{j_1}^{\text{OS}}, \ldots, \beta_{j_r}^{\text{OS}}\right)$ as follows: we set $\beta_k^{\text{OS}} := s_{\ell_L} \cdots s_{\ell_{k+1}} \alpha_{\ell_k}^\vee \in \widetilde{\Delta}^+_{\text{aff}}$ for $1 \le k \le L$, and set

$$\begin{aligned} z_0^{\text{OS}} &:= m_\mu, \\ z_1^{\text{OS}} &:= m_\mu s_{\beta_{j_1}^{\text{OS}}}, \\ z_2^{\text{OS}} &:= m_\mu s_{\beta_{j_1}^{\text{OS}}} s_{\beta_{j_2}^{\text{OS}}}, \\ &\vdots \\ z_r^{\text{OS}} &:= m_\mu s_{\beta_{j_1}^{\text{OS}}} \cdots s_{\beta_{j_r}^{\text{OS}}}. \end{aligned}$$

Also, following [OS, §3.3], we set $\text{B}(e; m_\mu) := \left\{p_J^{\text{OS}} \mid J \subset \{1, \ldots, L\}\right\}$ and $\text{end}(p_J^{\text{OS}}) := z_r^{\text{OS}} \in \widetilde{W}_{\text{ext}}$. Then we define $\overleftarrow{\text{QB}}(e; m_\mu)$ to be the following subset of $\text{B}(e; m_\mu)$:

$$\left\{ p_J^{\text{OS}} \in \text{B}(e; m_\mu) \;\middle|\; \text{dir}(z_i^{\text{OS}}) \xleftarrow{-\left(\overline{\beta_{j_{i+1}}^{\text{OS}}}\right)^\vee} \text{dir}(z_{i+1}^{\text{OS}}) \text{ is an edge of QBG}, \; 0 \le i \le r-1 \right\}. \tag{2.5}$$

Remark 2.2.1 ([M, (2.4.7)]). If $j \in \{1, \dots, L\}$, then $-\left(\overline{\beta_j^{\mathrm{OS}}}\right)^\vee \in \Delta^+$.

For $p_J^{\mathrm{OS}} \in \overleftarrow{\mathrm{QB}}(e; m_\mu)$, we define $\mathrm{qwt}^*(p_J^{\mathrm{OS}})$ as follows. Let $J^+ \subset J$ denote the set of all indices $j_i \in J$ for which $\mathrm{dir}(z_{i-1}^{\mathrm{OS}}) \xleftarrow{-\left(\overline{\beta_{j_i}^{\mathrm{OS}}}\right)^\vee} \mathrm{dir}(z_i^{\mathrm{OS}})$ is a quantum edge. Then we set

$$\mathrm{qwt}^*(p_J^{\mathrm{OS}}) := \sum_{j \in J^+} \beta_j^{\mathrm{OS}}.$$

For $\mu \in P$, we denote by $E_\mu(q,t)$ the nonsymmetric Macdonald polynomial and by $E_\mu(q,\infty)$ the specialization $\lim_{t\to\infty} E_\mu(q,t)$ at $t = \infty$. This specialization is studied in [CO] in untwisted simply-laced types and twisted non-simply-laced types.

We know the following formula for the specialization $E_\mu(q,\infty)$ at $t = \infty$.

Proposition 2.2.2 ([OS, Proposition 5.4])**.** *Let $\mu \in P$. Then,*

$$E_\mu(q,\infty) = \sum_{p_J^{\mathrm{OS}} \in \overleftarrow{\mathrm{QB}}(e; m_\mu)} q^{-\deg(\mathrm{qwt}^*(p_J^{\mathrm{OS}}))} e^{\mathrm{wt}(\mathrm{end}(p_J^{\mathrm{OS}}))}.$$

3. Orr-Shimozono formula in terms of qls paths

3.1. **Weak reflection orders.** Let $\lambda \in P^+$ be a dominant weight, $\mu \in W\lambda$, and set $S := S_\lambda = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$. We denote by $v(\mu) \in W^S$ the minimal-length coset representative for the coset $\{w \in W \mid w\lambda = \mu\}$ in W/W_S. We have $\ell(v(\mu)w) = \ell(v(\mu)) + \ell(w)$ for all $w \in W_S$. In particular, we have $\ell(v(\mu)w_\circ(S)) = \ell(v(\mu)) + \ell(w_\circ(S))$. When $\mu = \lambda_- := w_\circ\lambda$, it is clear that $w_\circ \in \{w \in W \mid w\lambda = \lambda_-\}$. Since $w_\circ$ is the longest element of W, we have

$$w_\circ = v(\lambda_-)w_\circ(S) \tag{3.1}$$

and $\ell(v(\lambda_-)w_\circ(S)) = \ell(v(\lambda_-)) + \ell(w_\circ(S))$; note that $v(\lambda_-) = w_\circ w_\circ(S) = \lfloor w_\circ \rfloor$. The following lemma follows from [M, Chap. 2].

Lemma 3.1.1.

(1) $\mathrm{dir}(m_\mu) = v(\mu)v(\lambda_-)^{-1}$ *and* $\ell(\mathrm{dir}(m_\mu)) + \ell(v(\mu)) = \ell(v(\lambda_-))$; *hence*

$$m_\mu = t(\mu)v(\mu)v(\lambda_-)^{-1}. \tag{3.2}$$

(2) $v(\mu)v(\lambda_-)^{-1}w_\circ = v(\mu)w_\circ(S)$.

(3) $\left(v(\lambda_-)v(\mu)^{-1}\right)m_\mu = m_{\lambda_-}$, *and* $\ell(v(\lambda_-)v(\mu)^{-1}) + \ell(m_\mu) = \ell(m_{\lambda_-})$.

(4) $\ell(v(\lambda_-)v(\mu)^{-1}) + \ell(v(\mu)) = \ell(v(\lambda_-))$.

In this subsection, we give a particular reduced expression for m_{λ_-} $(= t(\lambda_-)$ by (3.2)) and then study some of its properties.

First of all, we recall the notion of a weak reflection order on Δ^+.

Definition 3.1.2. A total order $\prec$ on Δ^+ is called a weak reflection order on Δ^+ if it satisfies the following condition: if $\alpha, \beta, \gamma \in \Delta^+$ with $\gamma^\vee = \alpha^\vee + \beta^\vee$, then $\alpha \prec \gamma \prec \beta$ or $\beta \prec \gamma \prec \alpha$.

The following result is well-known (see [Pa, Theorem on p. 662] for example).

Proposition 3.1.3. *For a total order $\prec$ on Δ^+, the following are equivalent*:

(1) *the order $\prec$ is a weak reflection order*;

(2) *there exists a (unique) reduced expression $w_\circ = s_{i_1} \cdots s_{i_N}$ for $w_\circ$ such that $s_{i_N} \cdots s_{i_{k+1}} \alpha_{i_k} \prec s_{i_N} \cdots s_{i_{j+1}} \alpha_{i_j}$ for $1 \le k < j \le N$.*

Next, we recall from [Pa, pp. 661–662] the notion and some properties of a weak reflection order on a finite subset of $\widetilde{\Delta}^+_{\mathrm{aff}}$. We remark that arguments in [Pa] also work in the general setting of Kac-Moody algebras.

Definition 3.1.4. Let T be a finite subset of $\widetilde{\Delta}^+_{\mathrm{aff}}$ and $\prec'$ a total order on T. We say that the order $\prec'$ is a weak reflection order on T if it satisfies the following conditions:

(1) if $\theta_1, \theta_2 \in T$ satisfy $\theta_1 \prec' \theta_2$ and $\theta_1 + \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}}$, then $\theta_1 + \theta_2 \in T$ and $\theta_1 \prec' \theta_1 + \theta_2 \prec' \theta_2$;

(2) if $\theta_1, \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}}$ satisfy $\theta_1 + \theta_2 \in T$, then $\theta_1 \in T$ and $\theta_1 + \theta_2 \prec' \theta_1$, or $\theta_2 \in T$ and $\theta_1 + \theta_2 \prec' \theta_2$.

We remark that there does not necessarily exist a weak reflection order on an arbitrary finite subset of $\widetilde{\Delta}^+_{\mathrm{aff}}$.

Proposition 3.1.5. *Let T be a finite subset of $\widetilde{\Delta}^+_{\mathrm{aff}}$ and $\prec'$ a weak reflection order on T. We write T as $\{\gamma_1 \prec' \gamma_2 \prec' \cdots \prec' \gamma_p\}$. Then there exists $w \in \widetilde{W}_{\mathrm{aff}}$ such that $\widetilde{\Delta}^+_{\mathrm{aff}} \cap w^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} = T$. Moreover, there exists a (unique) reduced expression $w = s_{\ell_1} \cdots s_{\ell_p}$ for w such that $s_{\ell_p} \cdots s_{\ell_{j+1}} \alpha^\vee_{\ell_j} = \gamma_j$ for $1 \le j \le p$.*

The converse of Proposition 3.1.5 also holds.

Proposition 3.1.6. *Let $w \in \widetilde{W}_{\mathrm{aff}}$, and let $w = s_{\ell_1} \cdots s_{\ell_p}$ be a reduced expression. We set a $\gamma_j := s_{\ell_p} \cdots s_{\ell_{j+1}} \alpha^\vee_{\ell_j}$ for $1 \le j \le p$, and define a total order $\prec'$ on $\widetilde{\Delta}^+_{\mathrm{aff}} \cap w^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$ as follows: for $1 \le j, k \le p$, $\gamma_j \prec' \gamma_k \overset{\mathrm{def}}{\Leftrightarrow} j < k$. Then, the total order $\prec'$ is a weak reflection order on $\widetilde{\Delta}^+_{\mathrm{aff}} \cap w^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$.*

Remark 3.1.7. Let

$$\begin{aligned} v(\lambda_-) &= s_{i_1} \cdots s_{i_M}, \\ w_\circ(S) &= s_{i_{M+1}} \cdots s_{i_N}, \\ w_\circ &= s_{i_1} \cdots s_{i_M} s_{i_{M+1}} \cdots s_{i_N} \end{aligned}$$

be reduced expressions for $v(\lambda_-)$, $w_\circ(S)$, and $w_\circ = v(\lambda_-) w_\circ(S)$, respectively, where $S = S_\lambda = \{i \in I \mid \langle \lambda, \alpha^\vee_i \rangle = 0\}$. Recall that $w_\circ(S)$ is the longest element of W_S. We set $\beta_j := s_{i_N} \cdots s_{i_{j+1}} \alpha_{i_j}$, $1 \le j \le N$. By Proposition 3.1.3, we have $\Delta^+ \setminus \Delta^+_S = \{\beta_1 \prec \beta_2 \prec \cdots \prec \beta_M\}$ and $\Delta^+_S = \{\beta_{M+1} \prec \beta_{M+2} \prec \cdots \prec \beta_N\}$, where $\prec$ is the weak reflection order on Δ^+ determined by the reduced expression above for $w_\circ$. In particular, we have

$$\theta_1 \prec \theta_2 \text{ for } \theta_1 \in \Delta^+ \setminus \Delta^+_S \text{ and } \theta_2 \in \Delta^+_S. \tag{3.3}$$

Conversely, if a weak reflection order on Δ^+ satisfies (3.3), then the reduced expression $w_\circ = s_{\ell_1} \cdots s_{\ell_N}$ for $w_\circ$ corresponding to this weak reflection order is given by concatenating a reduced expression for $v(\lambda_-)$ with a reduced expression for $w_\circ(S)$. Moreover, if we alter a reduced expression for $w_\circ(S)$ with a reduced expression for $v(\lambda_-)$ unchanged, then the restriction to $\Delta^+ \setminus \Delta^+_S$ of the weak reflection order on

Δ^+ does not change. Thus, the restriction to $\Delta^+ \setminus \Delta^+_S$ of the weak reflection order on Δ^+ satisfying (3.3) depends only on a reduced expression for $v(\lambda_-)$.

First let us take a reduced expression $v(\lambda_-) = s_{i_1} \cdots s_{i_M}$ and a weak reflection order $\prec$ on Δ^+ such that the restriction to $\Delta^+ \setminus \Delta^+_S$ of this weak reflection order $\prec$ is determined by the reduced expression $v(\lambda_-) = s_{i_1} \cdots s_{i_M}$ as in Remark 3.1.7. Also, we define an injective map Φ by

$$\begin{aligned} \Phi : \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}^-_{\mathrm{aff}} &\to \mathbb{Q}_{\geq 0} \times (\Delta^+ \setminus \Delta^+_S), \\ \beta = \overline{\beta} + \deg(\beta)\widetilde{\delta} &\mapsto \left(\frac{\langle \lambda_-, \overline{\beta} \rangle - \deg(\beta)}{\langle \lambda_-, \overline{\beta} \rangle}, w_\circ \overline{\beta}^\vee \right). \end{aligned}$$

Note that $\langle \lambda_-, \overline{\beta} \rangle > 0$, $\langle \lambda_-, \overline{\beta} \rangle - \deg(\beta) \geq 0$, and $w_\circ \overline{\beta}^\vee \in \Delta^+ \setminus \Delta^+_S$ since we know from [M, (2.4.7) (i)] that

$$\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}^-_{\mathrm{aff}} = \{ \alpha^\vee + a\tilde{\delta} \mid \alpha \in \Delta^-, 0 < a \leq \langle \lambda_-, \alpha^\vee \rangle \}. \tag{3.4}$$

We now consider the lexicographic order $<$ on $\mathbb{Q}_{\geq 0} \times (\Delta^+ \setminus \Delta^+_S)$ induced by the usual total order on $\mathbb{Q}_{\geq 0}$ and the restriction to $\Delta^+ \setminus \Delta^+_S$ of the weak reflection order $\prec$ on Δ^+; that is, for $(a, \alpha), (b, \beta) \in \mathbb{Q}_{\geq 0} \times (\Delta^+ \setminus \Delta^+_S)$,

$$(a, \alpha) < (b, \beta) \text{ if and only if } a < b, \text{ or } a = b \text{ and } \alpha \prec \beta.$$

Then we denote by $\prec'$ the total order on $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}^-_{\mathrm{aff}}$ induced by the lexicographic order on $\mathbb{Q}_{\geq 0} \times (\Delta^+ \setminus \Delta^+_S)$ through the map Φ, and write $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}^-_{\mathrm{aff}}$ as $\{\gamma_1 \prec' \cdots \prec' \gamma_L\}$.

Proposition 3.1.8. *Keep the notation and setting above. Then, there exists a unique reduced expression $m_{\lambda_-} = u s_{\ell_1} \cdots s_{\ell_L}$ for m_{λ_-}, $u \in \Omega$, $\{\ell_1, \ldots, \ell_L\} \subset I_{\mathrm{aff}}$, such that $\beta^{\mathrm{OS}}_j \left(= s_{\ell_L} \cdots s_{\ell_{j+1}} \alpha^\vee_{\ell_j} \right) = \gamma_j$ for $1 \leq j \leq L$.*

Proof. We will show that the total order $\prec'$ is a weak reflection order on $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}^-_{\mathrm{aff}}$.

We check condition (1) in Definition 3.1.4. Assume that $\theta_1, \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}^-_{\mathrm{aff}}$ satisfy $\theta_1 \prec' \theta_2$ and $\theta_1 + \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}}$. Then it is clear that $\theta_1 + \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}^-_{\mathrm{aff}}$.

Consider the case that the first component of $\Phi(\theta_1)$ is less than that of $\Phi(\theta_2)$ (i.e., $\frac{\langle \lambda_-, \overline{\theta_1} \rangle - \deg(\theta_1)}{\langle \lambda_-, \overline{\theta_1} \rangle} < \frac{\langle \lambda_-, \overline{\theta_2} \rangle - \deg(\theta_2)}{\langle \lambda_-, \overline{\theta_2} \rangle}$). In this case, the first component of $\Phi(\theta_1 + \theta_2)$ is equal to $\frac{\langle \lambda_-, \overline{\theta_1 + \theta_2} \rangle - \deg(\theta_1 + \theta_2)}{\langle \lambda_-, \overline{\theta_1 + \theta_2} \rangle}$, which lies between the first components of $\Phi(\theta_1)$ and $\Phi(\theta_2)$. Hence we have $\Phi(\theta_1) < \Phi(\theta_1 + \theta_2) < \Phi(\theta_2)$.

Consider the case that the first component of $\Phi(\theta_1)$ is equal to that of $\Phi(\theta_2)$. In this case, we have $w_\circ \overline{\theta_1}^\vee \prec w_\circ \overline{\theta_2}^\vee$, where $\prec$ is the restriction to $\Delta^+ \setminus \Delta^+_S$ of the weak reflection order on Δ^+. Note that the first component of $\Phi(\theta_1 + \theta_2)$ is equal to $\frac{\langle \lambda_-, \overline{\theta_1 + \theta_2} \rangle - \deg(\theta_1 + \theta_2)}{\langle \lambda_-, \overline{\theta_1 + \theta_2} \rangle}$, which is equal to both of the first components of $\Phi(\theta_1)$ and $\Phi(\theta_2)$. Moreover, since $\theta_1 + \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}^-_{\mathrm{aff}}$, we have $w_\circ \left(\overline{\theta_1 + \theta_2} \right)^\vee \in \Delta^+ \setminus \Delta^+_S$. It follows from the definition of the weak reflection order $\prec$ on Δ^+ that $w_\circ \overline{\theta_1}^\vee \prec w_\circ \left(\overline{\theta_1 + \theta_2} \right)^\vee \prec w_\circ \overline{\theta_2}^\vee$. Hence we have $\Phi(\theta_1) < \Phi(\theta_1 + \theta_2) < \Phi(\theta_2)$. Thus, the total order $\prec'$ satisfies condition (1).

We check condition (2) in Definition 3.1.4. If $\theta_1, \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \setminus m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$ and $\theta_1+\theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}}$, then it is clear that $\theta_1+\theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \setminus m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$. Hence we may assume that $\theta_1 \in \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$ and $\theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \setminus m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$; indeed, if $\theta_1, \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$, then the assertion is obvious by condition (1). Since $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} = \{\alpha^\vee + a\tilde{\delta} \mid \alpha \in \Delta^-, 0 < a \le \langle \lambda_-, \alpha^\vee\rangle\}$, we have $0 < \deg(\theta_1) \le \langle \lambda_-, \overline{\theta_1}\rangle$ and $0 < \deg(\theta_1+\theta_2) \le \langle \lambda_-, \overline{\theta_1+\theta_2}\rangle$. Also, since $\theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \setminus m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$, we find that $\langle \lambda_-, \overline{\theta_2}\rangle < 0 \le \deg(\theta_2)$, $\deg(\theta_2) > \langle \lambda_-, \overline{\theta_2}\rangle \ge 0$, or $\langle \lambda_-, \overline{\theta_2}\rangle = \deg(\theta_2) = 0$. If $0 > \deg(\theta_2)$, then we have $\theta_2 \in \widetilde{\Delta}^-_{\mathrm{aff}}$, a contradiction.

In the case that $\langle \lambda_-, \overline{\theta_2}\rangle < 0 \le \deg(\theta_2)$, the first component of $\Phi(\theta_1+\theta_2)$, which is $\frac{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle - \deg(\theta_1+\theta_2)}{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle}$, satisfies the inequalities

$$\begin{aligned}\frac{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle - \deg(\theta_1+\theta_2)}{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle} &\le \frac{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle - \deg(\theta_1)}{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle} \\ &= 1 - \frac{\deg(\theta_1)}{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle} < 1 - \frac{\deg(\theta_1)}{\langle \lambda_-, \overline{\theta_1}\rangle} \\ &= \frac{\langle \lambda_-, \overline{\theta_1}\rangle - \deg(\theta_1)}{\langle \lambda_-, \overline{\theta_1}\rangle}.\end{aligned}$$

Therefore, we deduce that the first component of $\Phi(\theta_1+\theta_2)$ is less than that of $\Phi(\theta_1)$, and hence $\Phi(\theta_1+\theta_2) < \Phi(\theta_1)$.

In the case that $\deg(\theta_2) > \langle \lambda_-, \overline{\theta_2}\rangle \ge 0$, the first component of $\Phi(\theta_1+\theta_2)$ satisfies the inequalities

$$\begin{aligned}\frac{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle - \deg(\theta_1+\theta_2)}{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle} &= \frac{\big(\langle \lambda_-, \overline{\theta_1}\rangle - \deg(\theta_1)\big) + \big(\langle \lambda_-, \overline{\theta_2}\rangle - \deg(\theta_2)\big)}{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle} \\ &< \frac{\big(\langle \lambda_-, \overline{\theta_1}\rangle - \deg(\theta_1)\big)}{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle} \le \frac{\langle \lambda_-, \overline{\theta_1}\rangle - \deg(\theta_1)}{\langle \lambda_-, \overline{\theta_1}\rangle}.\end{aligned}$$

Therefore, we deduce that the first component of $\Phi(\theta_1+\theta_2)$ is less than that of $\Phi(\theta_1)$, and hence that $\Phi(\theta_1+\theta_2) < \Phi(\theta_1)$.

In the case that $\langle \lambda_-, \overline{\theta_2}\rangle = \deg(\theta_2) = 0$, the first component of $\Phi(\theta_1+\theta_2)$ is equal to that of $\Phi(\theta_1)$. Moreover, since $\langle \lambda_-, \overline{\theta_2}\rangle = \langle \lambda, w_\circ \overline{\theta_2}\rangle = 0$, we have $w_\circ \overline{\theta_2}^\vee \in \Delta^+_S$. Therefore, by (3.3), we see that $w_\circ (\overline{\theta_1+\theta_2})^\vee \prec w_\circ \overline{\theta_2}^\vee$. It follows from the definition of the weak reflection order on Δ^+ that $w_\circ \overline{\theta_1}^\vee \prec w_\circ (\overline{\theta_1+\theta_2})^\vee \prec w_\circ \overline{\theta_2}^\vee$, and hence that $\Phi(\theta_1+\theta_2) < \Phi(\theta_1)$.

Thus, we conclude that $\prec'$ satisfies condition (2), and the total order $\prec'$ is a weak reflection order on $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$.

Now, by Proposition 3.1.5, there exists $w \in \widetilde{W}_{\mathrm{aff}}$ such that $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} = \widetilde{\Delta}^+_{\mathrm{aff}} \cap w^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$, and there exists a reduced expression $w = s_{\ell_1} \cdots s_{\ell_L}$, $\{\ell_1, \dots, \ell_L\} \subset I_{\mathrm{aff}}$ for w such that $\gamma_j = s_{\ell_L} \cdots s_{\ell_{j+1}} \alpha^\vee_{\ell_j}$ for $1 \le j \le L$. Since $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} = \widetilde{\Delta}^+_{\mathrm{aff}} \cap w^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$, it follows from [M, (2.2.6)] that there exists $u \in \Omega$ such that $uw = m_{\lambda_-}$. Thus, we obtain a reduced expression $m_{\lambda_-} = us_{\ell_1} \cdots s_{\ell_L}$ for m_{λ_-}, with $\gamma_j = s_{\ell_L} \cdots s_{\ell_{j+1}} \alpha^\vee_{\ell_j} = \beta^{\mathrm{OS}}_j$ for $1 \le j \le L$. This completes the proof of the proposition. □

By Remark 3.1.7, the restriction to $\Delta^+ \setminus \Delta_S^+$ of a weak reflection order on Δ^+ satisfying (3.3) corresponds bijectively to a reduced expression $v(\lambda_-) = s_{i_1} \cdots s_{i_M}$ for $v(\lambda_-)$. Hence, by Proposition 3.1.8, we can take a reduced expression $m_{\lambda_-} = us_{\ell_1} \cdots s_{\ell_L}$ for m_{λ_-} corresponding to each reduced expression $v(\lambda_-) = s_{i_1} \cdots s_{i_M}$ for $v(\lambda_-)$. Conversely, as seen in Lemma 3.1.10, from the reduced expression $m_{\lambda_-} = us_{\ell_1} \cdots s_{\ell_L}$ for m_{λ_-}, we obtain a reduced expression for $v(\lambda_-)$, which is identical to the original reduced expression $v(\lambda_-) = s_{i_1} \cdots s_{i_M}$ (see Lemma 3.1.10 below).

In the remainder of this subsection, we fix reduced expressions $v(\lambda_-) = s_{i_1} \cdots s_{i_M}$ and $w_\circ(S) = s_{i_{M+1}} \cdots s_{i_N}$, and use the weak reflection order $\prec$ on Δ^+ (which satisfies (3.3)) determined by these reduced expressions for $v(\lambda_-)$ and $w_\circ(S)$. Also, we use the total order $\prec'$ on $\widetilde{\Delta}_{\mathrm{aff}}^+ \cap m_{\lambda_-}^{-1} \widetilde{\Delta}_{\mathrm{aff}}^-$ defined just before Proposition 3.1.8, and take a reduced expression $m_{\lambda_-} = us_{\ell_1} \cdots s_{\ell_L}$ for m_{λ_-} given by Proposition 3.1.8.

Recall that $\beta_k^{\mathrm{OS}} = s_{\ell_L} \cdots s_{\ell_{k+1}} \alpha_{\ell_k}^\vee$ for $1 \leq k \leq L$. We set $a_k := \deg(\beta_k^{\mathrm{OS}}) \in \mathbb{Z}_{>0}$. Since $\widetilde{\Delta}_{\mathrm{aff}}^+ \cap m_{\lambda_-}^{-1} \widetilde{\Delta}_{\mathrm{aff}}^- = \{\beta_1^{\mathrm{OS}}, \ldots, \beta_L^{\mathrm{OS}}\}$, we see by (3.4) that $0 < a_k \leq \langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle$. Also, for $1 \leq j \leq L$, we set $\beta_k^{\mathrm{L}} := us_{\ell_1} \cdots s_{\ell_{k-1}} \alpha_{\ell_k}^\vee$ and $b_k := \deg(\beta_k^{\mathrm{L}}) \in \mathbb{Z}_{\geq 0}$. Then we have $\{\beta_k^{\mathrm{L}} \mid 1 \leq k \leq L\} = \widetilde{\Delta}_{\mathrm{aff}}^+ \cap m_{\lambda_-} \widetilde{\Delta}_{\mathrm{aff}}^- = \{\alpha^\vee + a\tilde{\delta} \mid \alpha \in \Delta^+, 0 \leq a < -\langle \lambda_-, \alpha^\vee \rangle\}$ (see [M, (2.4.7) (ii)]).

Remark 3.1.9. For $1 \leq k \leq L$, we have

$$\begin{aligned} -t(\lambda_-)\beta_k^{\mathrm{OS}} &= -(us_{\ell_1} \cdots s_{\ell_L})(s_{\ell_L} \cdots s_{\ell_{k+1}} \alpha_{\ell_k}^\vee) = -us_{\ell_1} \cdots s_{\ell_{k-1}} s_{\ell_k} \alpha_{\ell_k}^\vee \\ &= -us_{\ell_1} \cdots s_{\ell_{k-1}} (-\alpha_{\ell_k}^\vee) = us_{\ell_1} \cdots s_{\ell_{k-1}} \alpha_{\ell_k}^\vee = \beta_k^{\mathrm{L}} = \overline{\beta_k^{\mathrm{L}}} + b_k \tilde{\delta}. \end{aligned}$$

From this, together with $-t(\lambda_-)\beta_k^{\mathrm{OS}} = -\overline{\beta_k^{\mathrm{OS}}} - (a_k - \langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle)\tilde{\delta}$, we obtain $\overline{\beta_k^{\mathrm{L}}} = -\overline{\beta_k^{\mathrm{OS}}}$ and $\langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle - a_k = b_k$.

Lemma 3.1.10. *Keep the notation and setting above. Since $us_{\ell_k} = s_{i'_k} u$ for some $i'_k \in I_{\mathrm{aff}}$, $1 \leq k \leq M$, we can rewrite the reduced expression $us_{\ell_1} \cdots s_{\ell_L}$ for m_{λ_-} as $s_{i'_1} \cdots s_{i'_M} us_{\ell_{M+1}} \cdots s_{\ell_L}$. Then, $s_{i'_1} \cdots s_{i'_M}$ is a reduced expression for $v(\lambda_-)$, and $us_{\ell_{M+1}} \cdots s_{\ell_L}$ is a reduced expression for m_λ. Moreover, $i_k = i'_k$ for $1 \leq k \leq M$.*

Proof. First we show that $\{\beta_k^{\mathrm{L}} \mid 1 \leq k \leq M\} = -w_\circ \left(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+\right)$. Since $\{\beta_j^{\mathrm{OS}} \mid 1 \leq j \leq L\} = \{\alpha^\vee + a\tilde{\delta} \mid \alpha \in \Delta^-, 0 < a \leq \langle \lambda_-, \alpha^\vee \rangle\}$, we see that the minimum value of the first components of $\Phi(\beta_k^{\mathrm{OS}})$, i.e., $\frac{\langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle - a_k}{\langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle}$ for $1 \leq k \leq L$, is equal to 0. Since $\Phi(\beta_1^{\mathrm{OS}}) < \Phi(\beta_2^{\mathrm{OS}}) < \cdots < \Phi(\beta_L^{\mathrm{OS}})$, where $<$ denotes the lexicographic order on $\mathbb{Q}_{\geq 0} \times (\Delta^+ \setminus \Delta_S^+)$, there exists a positive integer M' such that the first component of $\Phi(\beta_k^{\mathrm{OS}})$ is equal to 0 for $1 \leq k \leq M'$ and greater than 0 for $M' + 1 \leq k \leq L$. Since $\beta_k^{\mathrm{L}} = \overline{\beta_k^{\mathrm{L}}} + b_k \tilde{\delta}$ and $\langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle - a_k = b_k$ by Remark 3.1.9, we deduce that the first component of $\Phi(\beta_k^{\mathrm{OS}})$ is equal to 0 if and only if $\beta_k^{\mathrm{L}} = \overline{\beta_k^{\mathrm{L}}} \in \widetilde{\Delta}^+$. In this case, we have $\langle \lambda, -w_\circ \beta_k^{\mathrm{L}} \rangle = \langle \lambda_-, -\beta_k^{\mathrm{L}} \rangle \overset{\text{Remark 3.1.9}}{=} \langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle > 0$, and hence $\beta_k^{\mathrm{L}} \in -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$. Therefore, we obtain $\{\beta_k^{\mathrm{L}} \mid 1 \leq k \leq L\} \cap -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+) = \{\beta_k^{\mathrm{L}} \mid 1 \leq k \leq M'\} \subset -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$. Also, because $\{\beta_k^{\mathrm{L}} \mid 1 \leq k \leq L\} = \widetilde{\Delta}_{\mathrm{aff}}^+ \cap m_{\lambda_-} \widetilde{\Delta}_{\mathrm{aff}}^- = \{\alpha^\vee + a\tilde{\delta} \mid \alpha \in \Delta^+, 0 \leq a < -\langle \lambda_-, \alpha^\vee \rangle\} \supset -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$, we deduce that $\{\beta_k^{\mathrm{L}} \mid 1 \leq k \leq M'\} = -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$. Since $\#(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+) = M$, it follows that $M = M'$, and hence $\{\beta_k^{\mathrm{L}} \mid 1 \leq k \leq M\} = -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$.

We show that $i'_k \in I$ for $1 \le k \le M$. We set $\zeta_k^\vee := s_{i'_1} \cdots s_{i'_{k-1}} \alpha_{i'_k}^\vee$ for $1 \le k \le M$. Since $u\alpha_{\ell_k}^\vee = \alpha_{i'_k}^\vee$, we have

$$\beta_k^{\mathrm{L}} = us_{\ell_1} \cdots s_{\ell_{k-1}} \alpha_{\ell_k}^\vee = s_{i'_1} \cdots s_{i'_{k-1}} u\alpha_{\ell_k}^\vee = s_{i'_1} \cdots s_{i'_{k-1}} \alpha_{i'_k}^\vee = \zeta_k^\vee.$$

Hence it follows that $\{\zeta_k^\vee \mid 1 \le k \le M\} = \{\beta_k^{\mathrm{L}} \mid 1 \le k \le M\} = -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$. If there exists $k \in \{1, \dots, M\}$ such that $i'_k = 0$, then, by choosing the minimum of such k's, we obtain $\zeta_k^\vee = s_{i'_1} \cdots s_{i'_{k-1}} \alpha_{i'_k}^\vee \notin \widetilde{\Delta}^+$, contrary to the equality $\{\zeta_k^\vee \mid 1 \le k \le M\} = -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$. Therefore, we have $i'_k \in I$ for $1 \le k \le M$.

Next, we show that $s_{i'_1} \cdots s_{i'_M}$ is a reduced expression for $v(\lambda_-)$ and $us_{\ell_{M+1}} \cdots s_{\ell_L}$ is a reduced expression for m_λ. Since $s_{\ell_1} \cdots s_{\ell_M}$ is a reduced expression, so is $s_{i'_1} \cdots s_{i'_M}$. Therefore, there exist $i'_{M+1}, \dots, i'_N \in I$ such that $w_\circ = s_{i'_1} \cdots s_{i'_M} s_{i'_{M+1}} \cdots s_{i'_N}$ is a reduced expression for $w_\circ$. Because $s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_{k+1}} \alpha_{i'_k}^\vee = -w_\circ \beta_k^{\mathrm{L}}$, $1 \le k \le M$, by using the reduced expression above for $w_\circ$, we obtain

$$\widetilde{\Delta}^+ = \{-w_\circ \beta_1^{\mathrm{L}}, \dots, -w_\circ \beta_M^{\mathrm{L}}, s_{i'_N} \cdots s_{i'_{M+2}} \alpha_{i'_{M+1}}^\vee, \dots, \alpha_{i'_N}^\vee\}.$$

Here, $\{\beta_k^{\mathrm{L}} \mid 1 \le k \le M\} = -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$ implies $\{s_{i'_N} \cdots s_{i'_{M+2}} \alpha_{i'_{M+1}}^\vee, \dots, \alpha_{i'_N}^\vee\} = \widetilde{\Delta}_S^+$. From this by descending induction on $M+1 \le k \le N$, we deduce that $i'_{M+1}, \dots, i'_N \in S$ and $s_{i'_{M+1}} \cdots s_{i'_N}$ is an element of W_S; note that the length of this element is equal to $N - M$, which is the cardinality of $\widetilde{\Delta}_S^+$. Therefore, $s_{i'_{M+1}} \cdots s_{i'_N}$ is the longest element $w_\circ(S)$ of W_S, and hence $s_{i'_1} \cdots s_{i'_M} = w_\circ w_\circ(S) = v(\lambda_-)$, which is a reduced expression for $v(\lambda_-)$. Moreover, because $m_{\lambda_-} = v(\lambda_-) m_\lambda$ with $\ell(m_{\lambda_-}) = \ell(v(\lambda_-)) + \ell(m_\lambda)$ by Lemma 3.1.1(3) for the case $\mu = \lambda$, $m_\lambda = v(\lambda_-)^{-1} m_{\lambda_-} = us_{\ell_{M+1}} \cdots s_{\ell_L}$ is a reduced expression for m_λ.

Finally, we show that $i_k = i'_k$ for $1 \le k \le M$. Since $M = M'$ as shown above,

$$\Phi(\beta_k^{\mathrm{OS}}) = \left(\frac{\langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle - a_k}{\langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle}, w_\circ \left(\overline{\beta_k^{\mathrm{OS}}} \right)^\vee \right) = \left(0, w_\circ \left(\overline{\beta_k^{\mathrm{OS}}} \right)^\vee \right)$$

for $1 \le k \le M$ by the definition of Φ, and

$$\begin{aligned} w_\circ \left(\overline{\beta_k^{\mathrm{OS}}} \right)^\vee = -w_\circ \left(\overline{\beta_k^{\mathrm{L}}} \right)^\vee &= -w_\circ \zeta_k = -s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_1} s_{i'_1} \cdots s_{i'_{k-1}} \alpha_{i'_k} \\ &= s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_{k+1}} \alpha_{i'_k} \end{aligned}$$

by Remark 3.1.9. Thus, for $1 \le k < j \le M$, we have $s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_{k+1}} \alpha_{i'_k} \prec s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_{j+1}} \alpha_{i'_j}$, where the order $\prec$ is the fixed weak reflection order on Δ^+ defined just before Proposition 3.1.8. Here we recall from Remark 3.1.7 that $\beta_k = s_{i_N} \cdots s_{i_{k+1}} \alpha_{i_k}$, $1 \le k \le N$. Because

$$\{\beta_k \mid 1 \le k \le M\} = \{s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_{k+1}} \alpha_{i'_k} \mid 1 \le k \le M\} = \Delta^+ \setminus \Delta_S^+,$$

it follows from the definition of the weak reflection order $\prec$ on Δ^+ together with (3.3) that

$$\begin{aligned} \{\beta_1 \prec \cdots \prec \beta_M\} &= \left\{ s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_2} \alpha_{i'_1} \prec \cdots \prec s_{i'_N} \cdots s_{i'_{M+1}} \alpha_{i'_M} \right\} \\ &= \Delta^+ \setminus \Delta_S^+. \end{aligned}$$

Therefore, noting that $\beta_k = s_{i_N} \cdots s_{i_{k+1}} \alpha_{i_k}$ for $1 \le k \le N$, we obtain

$$s_{i_N} \cdots s_{i_{k+1}} \alpha_{i_k} = s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_{k+1}} \alpha_{i'_k}, \quad \text{for } 1 \le k \le M. \tag{3.5}$$

By substituting the equalities $s_{i_{M+1}} \cdots s_{i_N} = w_\circ(S) = s_{i'_{M+1}} \cdots s_{i'_N}$ into (3.5), we have $s_{i_M} \cdots s_{i_{k+1}} \alpha_{i_k} = s_{i'_M} \cdots s_{i'_{k+1}} \alpha_{i'_k}$ for $1 \le k \le M$. In particular, when $k = M$, we have $\alpha_{i_M} = \alpha_{i'_M}$, which implies that $i_M = i'_M$. If $i_j = i'_j$ for $k+1 \le j \le M$, then it follows from $s_{i_M} \cdots s_{i_{k+1}} \alpha_{i_k} = s_{i'_M} \cdots s_{i'_{k+1}} \alpha_{i'_k}$ that $\alpha_{i_k} = \alpha_{i'_k}$, and hence $i_k = i'_k$. Thus, by descending induction on k, we deduce that $i_k = i'_k$ for $1 \le k \le M$. □

Remark 3.1.11 ([LNSSS2, §6.1]). For $1 \le k \le L$, we set

$$d_k := \frac{\langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle - a_k}{\langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle} = \frac{b_k}{\langle -\lambda_-, \overline{\beta_k^{\mathrm{L}}} \rangle};$$

the second equality follows from Remark 3.1.9; here d_k is just the first component of $\Phi(\beta_k^{\mathrm{OS}}) \in \mathbb{Q}_{\ge 0} \times (\Delta^+ \setminus \Delta_S^+)$. For $1 \le k, j \le L$, $\Phi(\beta_k^{\mathrm{OS}}) < \Phi(\beta_j^{\mathrm{OS}})$ if and only if $k < j$, and hence we have

$$0 \le d_1 \le \cdots \le d_L \lneqq 1. \tag{3.6}$$

Lemma 3.1.12. *If* $1 \le k < j \le L$ *and* $d_k = d_j$, *then* $w_\circ\left(\overline{\beta_k^{\mathrm{OS}}}\right)^\vee \prec w_\circ\left(\overline{\beta_j^{\mathrm{OS}}}\right)^\vee$.

Proof. By the definitions, we obtain $\Phi(\beta_k^{\mathrm{OS}}) = \left(d_k, w_\circ\left(\overline{\beta_k^{\mathrm{OS}}}\right)^\vee\right)$ and $\Phi(\beta_j^{\mathrm{OS}}) = \left(d_j, w_\circ\left(\overline{\beta_j^{\mathrm{OS}}}\right)^\vee\right)$. Since $d_k = d_j$ and $\Phi(\beta_k^{\mathrm{OS}}) < \Phi(\beta_j^{\mathrm{OS}})$, we have $w_\circ\left(\overline{\beta_k^{\mathrm{OS}}}\right)^\vee \prec w_\circ\left(\overline{\beta_j^{\mathrm{OS}}}\right)^\vee$. □

3.2. **Orr-Shimozono formula in terms of QLS paths.** Let $\lambda \in P^+$ be a dominant weight, and set $S = S_\lambda = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$.

Definition 3.2.1 ([LNSSS2, Definition 3.1]). A pair $\psi = (w_1, w_2, \dots, w_s; \sigma_0, \sigma_1, \dots, \sigma_s)$ of a sequence $w_1, \dots, w_s$ of elements in W^S such that $w_k \ne w_{k+1}$ for $1 \le k \le s-1$ and an increasing sequence $0 = \sigma_0 < \cdots < \sigma_s = 1$ of rational numbers is called a quantum Lakshmibai-Seshadri (QLS) path of shape λ if

(C) for every $1 \le i \le s-1$, there exists a directed path from w_{i+1} to w_i in $\mathrm{QBG}^S_{\sigma_i \lambda}$.

Let $\mathrm{QLS}(\lambda)$ denote the set of all QLS paths of shape λ.

Remark 3.2.2. We know from [LNSSS4, Definition 3.2.2 and Theorem 4.1.1] that condition (C) can be replaced by

(C)' for every $1 \le i \le s-1$, there exists a directed path from w_{i+1} to w_i in $\mathrm{QBG}^S_{\sigma_i \lambda}$ that is also a shortest directed path from w_{i+1} to w_i in QBG^S.

For $\psi = (w_1, w_2, \dots, w_s; \sigma_0, \sigma_1, \dots, \sigma_s) \in \mathrm{QLS}(\lambda)$, we set

$$\mathrm{wt}(\psi) := \sum_{i=0}^{s-1} (\sigma_{i+1} - \sigma_i) w_{i+1} \lambda,$$

and we define a map $\kappa : \mathrm{QLS}(\lambda) \to W^S$ by $\kappa(\psi) := w_s$. Also, for $\mu \in W\lambda$, we define the degree of ψ at μ by

$$\deg_\mu(\psi) := -\sum_{i=1}^{s} \sigma_i \mathrm{wt}_\lambda(w_{i+1} \Rightarrow w_i);$$

here we set $w_{s+1} := v(\mu)$. Note that by Remark 3.2.2, $\sigma_i \mathrm{wt}_\lambda(w_{i+1} \Rightarrow w_i) \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq s-1$. Also, $\sigma_s = 1$ for $i = s$ by the definition of a QLS path. Hence it follows that $\deg_\mu(\psi) \in \mathbb{Z}_{\leq 0}$.

Now, we define a subset $\mathrm{EQB}(w)$ of W for each $w \in W$. Let $w = s_{i_1} \cdots s_{i_p}$ be a reduced expression for w. For each $J = \{j_1 < j_2 < j_3 < \cdots < j_r\} \subset \{1, \ldots, p\}$, we define

$$p_J := (w = z_0, \ldots, z_r; \beta_{j_1}, \ldots, \beta_{j_r})$$

as follows: we set $\beta_k := s_{i_p} \cdots s_{i_{k+1}}(\alpha_{i_k}) \in \Delta^+$ for $1 \leq k \leq p$, and set

$$\begin{aligned}
z_0 &:= w = s_{i_1} \cdots s_{i_p},\\
z_1 &:= w s_{\beta_{j_1}} = s_{i_1} \cdots s_{i_{j_1 - 1}} s_{i_{j_1+1}} \cdots s_{i_p} = s_{i_1} \cdots \breve{s_{i_{j_1}}} \cdots s_{i_p},\\
z_2 &:= w s_{\beta_{j_1}} s_{\beta_{j_2}} = s_{i_1} \cdots s_{i_{j_1 - 1}} s_{i_{j_1+1}} \cdots s_{i_{j_2 - 1}} s_{i_{j_2+1}} \cdots s_{i_p} = s_{i_1} \cdots \breve{s_{i_{j_1}}} \cdots \breve{s_{i_{j_2}}} \cdots s_{i_p},\\
&\ \vdots\\
z_r &:= w s_{\beta_{j_1}} \cdots s_{\beta_{j_r}} = s_{i_1} \cdots \breve{s_{i_{j_1}}} \cdots \breve{s_{i_{j_r}}} \cdots s_{i_p},
\end{aligned}$$

where the symbol $\breve{\cdot}$ indicates a term to be omitted; also, we set $\mathrm{end}(p_J) := z_r$. Then we define $\mathrm{B}(w) := \{p_J \mid J \subset \{1, \ldots, p\}\}$ and

$$\mathrm{QB}(w) := \{p_J \in \mathrm{B}(w) \mid z_i \xrightarrow{\beta_{j_{i+1}}} z_{i+1} \text{ is an edge of QBG for all } 0 \leq i \leq r-1\}.$$

We remark that J may be the empty set $\emptyset$; in this case, $\mathrm{end}(p_\emptyset) = w$.

Remark 3.2.3. We identify elements in $\mathrm{QB}(w)$ with directed paths in QBG. More precisely, for $p_J = (w = z_0, \ldots, z_r; \beta_{j_1}, \ldots \beta_{j_r}) \in \mathrm{QB}(w)$, we write

$$p_J = (w = z_0, \ldots, z_r; \beta_{j_1}, \ldots \beta_{j_r}) = \left(w = z_0 \xrightarrow{\beta_{j_1}} \cdots \xrightarrow{\beta_{j_r}} z_r\right).$$

Remark 3.2.4. Let $w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z$ be a directed path in QBG. Then we see that

$$1 \leq j_1 < j_2 < \cdots < j_r \leq p \Leftrightarrow \left(w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z\right) \in \mathrm{QB}(w).$$

Also, it follows from Proposition 2.1.4(1) that the map $\mathrm{end} : \mathrm{QB}(w) \to W$ is injective.

By using the map $\mathrm{end} : \mathrm{B}(w) \to W$ defined above, we set $\mathrm{EQB}(w) := \mathrm{end}(\mathrm{QB}(w))$.

Proposition 3.2.5. *The set* $\mathrm{EQB}(w)$ *is independent of the choice of a reduced expression for* w.

Proof. Let us take two reduced expressions for w:

$$\mathbf{I} : w = s_{i_1} \cdots s_{i_p} \text{ and } \mathbf{K} : w = s_{k_1} \cdots s_{k_p}.$$

In this proof, let $\mathrm{EQB}(w)_{\mathbf{I}}$ (resp., $\mathrm{EQB}(w)_{\mathbf{K}}$) denote the set $\mathrm{EQB}(w)$ associated to $\mathbf{I}$ (resp., $\mathbf{K}$).

It suffices to show that $\mathrm{EQB}(w)_{\mathbf{I}} \subset \mathrm{EQB}(w)_{\mathbf{K}}$. From the two reduced expressions above for w, we obtain the following two reduced expressions for $w_\circ$:

$$w_\circ = s_{i_{-q}} \cdots s_{i_0} s_{i_1} \cdots s_{i_p}, \tag{3.7}$$

$$w_\circ = s_{i_{-q}} \cdots s_{i_0} s_{k_1} \cdots s_{k_p}. \tag{3.8}$$

Using the reduced expression (3.7) (resp., (3.8)), we define β_m (resp., γ_m), $-q \le m \le p$, as in (2.1). Then we have

$$\{\beta_{-q}, \dots, \beta_p\} = \{\gamma_{-q}, \dots, \gamma_p\} = \Delta^+, \tag{3.9}$$

$$\{\beta_1, \dots, \beta_p\} = \{\gamma_1, \dots, \gamma_p\} = \Delta^+ \cap w^{-1}\Delta^-. \tag{3.10}$$

Let $z \in \mathrm{EQB}(w)_{\mathbf{I}}$, and

$$p_J = \left(w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z \right) \in \mathrm{QB}(w)_{\mathbf{I}}. \tag{3.11}$$

Recall from Remark 3.2.4 that $1 \le j_1 \le \cdots \le j_r \le p$. It follows from Proposition 2.1.4(1) that there exists a unique shortest directed path in QBG,

$$w = y_0 \xrightarrow{\gamma_{n_1}} y_1 \xrightarrow{\gamma_{n_2}} \cdots \xrightarrow{\gamma_{n_r}} y_r = z, \tag{3.12}$$

with $-q \le n_1 < n_2 < \cdots < n_r \le p$; this is a label-increasing directed path with respect to the weak reflection order defined by $\gamma_{-q} \prec \cdots \prec \gamma_p$. To prove that $z \in \mathrm{EQB}(w)_{\mathbf{K}}$, it suffices to show that $1 \le n_1$. It follows from (3.9) that for $1 \le u \le r$, there exists $-q \le t_u \le p$ such that $\beta_{t_u} = \gamma_{n_u}$. Therefore, by (3.12),

$$w = y_0 \xrightarrow{\beta_{t_1}} y_1 \xrightarrow{\beta_{t_2}} \cdots \xrightarrow{\beta_{t_r}} y_r = z$$

is a directed path in QBG. We see from Proposition 2.1.4(2) that this path is greater than or equal to the path (3.11) in the lexicographic order with respect to the edge labels. In particular, we have $t_1 \ge j_1 \ge 1$. Since $\gamma_{n_1} = \beta_{t_1} \in \Delta^+ \cap w^{-1}\Delta^-$, we deduce that $n_1 \ge 1$ by (3.10). This implies that $\mathrm{EQB}(w)_{\mathbf{I}} \subset \mathrm{EQB}(w)_{\mathbf{K}}$. □

Let $\mu \in W\lambda$. Recall that $v(\mu) \in W^S$ is the minimal-length coset representative for the coset $\{w \in W \mid w\lambda = \mu\}$. We set

$$\mathrm{QLS}^{\mu,\infty}(\lambda) := \{\psi \in \mathrm{QLS}(\lambda) \mid \kappa(\psi) \in \lfloor \mathrm{EQB}(v(\mu)w_\circ(S)) \rfloor\}.$$

Remark 3.2.6. If $w = w_\circ$, then we have $\mathrm{EQB}(w_\circ) = W$ by Proposition 2.1.4(1), since in this case, we can use all the positive roots as an edge label. If $\mu = \lambda_- = w_\circ\lambda$, then $v(\mu)w_\circ(S) = w_\circ$ by (3.1), and hence $\lfloor \mathrm{EQB}(v(\mu)w_\circ(S)) \rfloor = W^S$. Therefore, we have $\mathrm{QLS}^{w_\circ\lambda,\infty}(\lambda) = \mathrm{QLS}(\lambda)$.

With the notation above, we set

$$\mathrm{gch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda) := \sum_{\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_\mu(\psi)}.$$

The following is the main result of this section.

Theorem 3.2.7. *Let $\lambda \in P^+$ be a dominant weight, and $\mu \in W\lambda$. Then,*

$$E_\mu(q, \infty) = \mathrm{gch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda).$$

3.3. **Proof of Theorem 3.2.7.** Let $\lambda \in P^+$ be a dominant weight, $\mu \in W\lambda$, and set $S := S_\lambda = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$. In this subsection, in order to prove Theorem 3.2.7, we give a bijection

$$\Xi : \overleftarrow{\mathrm{QB}}(e; m_\mu) \to \mathrm{QLS}^{\mu,\infty}(\lambda)$$

that preserves weights and degrees.

We fix reduced expressions

$$v(\lambda_-)v(\mu)^{-1} = s_{i_1}\cdots s_{i_K},$$

$$v(\mu) = s_{i_{K+1}}\cdots s_{i_M}, \tag{3.13}$$

$$w_\circ(S) = s_{i_{M+1}}\cdots s_{i_N} \tag{3.14}$$

for $v(\lambda_-)v(\mu)^{-1}$, $v(\mu)$, and $w_\circ(S)$, respectively; recall that $\lambda_- = w_\circ\lambda$. Then, by Lemma 3.1.1(4), $v(\lambda_-) = s_{i_1}\cdots s_{i_M}$ is a reduced expression for $v(\lambda_-)$. As in §3.1, we use the weak reflection order $\prec$ on Δ^+ introduced in Remark 3.1.7 (which satisfies (3.3)) determined by the reduced expressions above for $v(\lambda_-)$ and $w_\circ(S)$. Also, we use the total order $\prec'$ on $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$ defined just before Proposition 3.1.8 and take the reduced expression $m_{\lambda_-} = us_{\ell_1}\cdots s_{\ell_L}$ for m_{λ_-} given by Proposition 3.1.8; recall that $us_{\ell_k} = s_{i_k}u$ for $1 \le k \le M$. It follows from Lemma 3.1.1(3) that $\big(v(\mu)v(\lambda_-)^{-1}\big)\, m_{\lambda_-} = m_\mu$ and $-\ell(v(\mu)v(\lambda_-)^{-1}) + \ell(m_{\lambda_-}) = \ell(m_\mu)$. Moreover, we see that

$$\begin{aligned}\big(v(\mu)v(\lambda_-)^{-1}\big)\, m_{\lambda_-} &= (s_{i_K}\cdots s_{i_1})\, us_{\ell_1}\cdots s_{\ell_L}\\ &\overset{\text{Lemma 3.1.10}}{=} us_{\ell_K}\cdots s_{\ell_1}s_{\ell_1}\cdots s_{\ell_L} = us_{\ell_{K+1}}\cdots s_{\ell_L},\end{aligned}$$

and hence $m_\mu = us_{\ell_{K+1}}\cdots s_{\ell_L}$ is a reduced expression for m_μ. In particular, when $\mu = \lambda$ (note that $v(\lambda) = e$), $m_\lambda = us_{\ell_{M+1}}\cdots s_{\ell_L}$ is a reduced expression for m_λ.

Also, recall from Remark 3.1.7 and the beginning of §3.1 that $\beta_k = s_{i_N}\cdots s_{i_{k+1}}\alpha_{i_k}$, $1 \le k \le N$, and $\beta^{\mathrm{OS}}_k = s_{\ell_L}\cdots s_{\ell_{k+1}}\alpha^\vee_{\ell_k}$, $1 \le k \le L$.

Remark 3.3.1. Keep the notation above. We have

$$\begin{aligned}\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} &= \{\beta^{\mathrm{OS}}_1, \dots, \beta^{\mathrm{OS}}_L\},\\ \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\mu}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} &= \{\beta^{\mathrm{OS}}_{K+1}, \dots, \beta^{\mathrm{OS}}_L\},\\ \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} &= \{\beta^{\mathrm{OS}}_{M+1}, \dots, \beta^{\mathrm{OS}}_L\}.\end{aligned}$$

In particular, we have $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} \subset \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\mu}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} \subset \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$.

Lemma 3.3.2 ([M, (2.4.7) (i)])**.** *If we denote by ς the characteristic function of Δ^-, i.e.,*

$$\varsigma(\gamma) := \begin{cases} 0 & \text{if } \gamma \in \Delta^+,\\ 1 & \text{if } \gamma \in \Delta^-,\end{cases}$$

then

$$\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\mu}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} = \{\alpha^\vee + a\widetilde{\delta} \mid \alpha \in \Delta^-, 0 < a < \varsigma(v(\mu)v(\lambda_-)^{-1}\alpha) + \langle \lambda, w_\circ\alpha^\vee\rangle\}.$$

Remark 3.3.3. Let $\gamma_1, \gamma_2, \dots, \gamma_r \in \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\mu}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$, and define a sequence $\big(y_0, y_1, \dots, y_r; \gamma_1, \gamma_2, \dots, \gamma_r\big)$ by $y_0 = m_\mu$, and $y_i = y_{i-1}s_{\gamma_i}$ for $1 \le i \le r$. Then, the sequence $(y_0, y_1, \dots, y_r; \gamma_1, \gamma_2, \dots, \gamma_r)$ is an element of $\overleftarrow{\mathrm{QB}}(e; m_\mu)$ if and only if the following conditions hold:

(1) $\gamma_1 \prec' \gamma_2 \prec' \cdots \prec' \gamma_r$, where the order $\prec'$ is the weak reflection order on $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\mu}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$ introduced at the beginning of §3.3;

(2) $\mathrm{dir}(y_{i-1}) \xleftarrow{-(\overline{\gamma_i})^\vee} \mathrm{dir}(y_i)$ is an edge of QBG for $1 \le i \le r$.

In the following, we define a map $\Xi : \overleftarrow{\mathrm{QB}}(e; m_\mu) \to \mathrm{QLS}^{\mu,\infty}(\lambda)$. Let p_J^{OS} be an arbitrary element of $\overleftarrow{\mathrm{QB}}(e; m_\mu)$ of the form

$$p_J^{\mathrm{OS}} = \left(m_\mu = z_0^{\mathrm{OS}}, z_1^{\mathrm{OS}}, \dots, z_r^{\mathrm{OS}}; \beta_{j_1}^{\mathrm{OS}}, \beta_{j_2}^{\mathrm{OS}}, \dots, \beta_{j_r}^{\mathrm{OS}}\right) \in \overleftarrow{\mathrm{QB}}(e; m_\mu),$$

with $J = \{j_1 < \cdots < j_r\} \subset \{K+1, \dots, L\}$. We set $x_k := \mathrm{dir}(z_k^{\mathrm{OS}})$, $0 \le k \le r$. Then, by the definition of $\overleftarrow{\mathrm{QB}}(e; m_\mu)$,

$$v(\mu)v(\lambda_-)^{-1} \overset{\text{Lemma 3.1.1}}{=} x_0 \xleftarrow{-\left(\overline{\beta_{j_1}^{\mathrm{OS}}}\right)^\vee} x_1 \xleftarrow{-\left(\overline{\beta_{j_2}^{\mathrm{OS}}}\right)^\vee} \cdots \xleftarrow{-\left(\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^\vee} x_r \tag{3.15}$$

is a directed path in QBG. We take $0 = u_0 \le u_1 < \cdots < u_{s-1} < u_s = r$ and $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_{s-1} < 1 = \sigma_s$ in such a way that (see (3.6))
(3.16)

$$\underbrace{0 = d_{j_1} = \cdots = d_{j_{u_1}}}_{=\sigma_0} < \underbrace{d_{j_{u_1+1}} = \cdots = d_{j_{u_2}}}_{=\sigma_1} < \cdots < \underbrace{d_{j_{u_{s-1}+1}} = \cdots = d_{j_r}}_{=\sigma_{s-1}} < 1 = \sigma_s;$$

note that $d_{j_1} > 0$ if and only if $u_1 = 0$. We set $w'_p := x_{u_p}$ for $0 \le p \le s-1$ and $w'_s := x_r$. Then, by taking a subsequence of (3.15), we obtain the following directed path in QBG for each $0 \le p \le s-1$:

$$w'_p = x_{u_p} \xleftarrow{-\left(\overline{\beta_{j_{u_p+1}}^{\mathrm{OS}}}\right)^\vee} x_{u_p+1} \xleftarrow{-\left(\overline{\beta_{j_{u_p+2}}^{\mathrm{OS}}}\right)^\vee} \cdots \xleftarrow{-\left(\overline{\beta_{j_{u_{p+1}}}^{\mathrm{OS}}}\right)^\vee} x_{u_{p+1}} = w'_{p+1}.$$

Multiplying the vertices in this directed path on the right by $w_\circ$, we obtain the following directed path in QBG for each $0 \le p \le s-1$ (see Lemma 2.1.3):
(3.17)

$$w_p =: w'_p w_\circ = x_{u_p} w_\circ \xrightarrow{w_\circ\left(\overline{\beta_{j_{u_p+1}}^{\mathrm{OS}}}\right)^\vee} \cdots \xrightarrow{w_\circ\left(\overline{\beta_{j_{u_{p+1}}}^{\mathrm{OS}}}\right)^\vee} x_{u_{p+1}} w_\circ = w'_{p+1} w_\circ := w_{p+1}.$$

Note that the edge labels of this directed path are increasing in the weak reflection order $\prec$ on Δ^+ introduced at the beginning of §3.3 (see Lemma 3.1.12), and lie in $\Delta^+ \setminus \Delta_S^+$; this property will be used to give the inverse to Ξ. Because

$$(1-\sigma_p)\langle \lambda, w_\circ \overline{\beta_{j_u}^{\mathrm{OS}}} \rangle = (1-d_{j_u})\langle \lambda, w_\circ \overline{\beta_{j_u}^{\mathrm{OS}}} \rangle = -\frac{a_{j_u}}{\langle \lambda_-, -\overline{\beta_{j_u}^{\mathrm{OS}}} \rangle} \langle \lambda_-, \overline{\beta_{j_u}^{\mathrm{OS}}} \rangle = a_{j_u} \in \mathbb{Z}$$

for $u_p + 1 \le u \le u_{p+1}$, $0 \le p \le s-1$, we find that (3.17) is a directed path in $\mathrm{QBG}_{(1-\sigma_p)\lambda}$ for $0 \le p \le s-1$. Therefore, by Lemma 2.1.7, there exists a directed path in $\mathrm{QBG}^S_{(1-\sigma_p)\lambda}$ from $\lfloor w_p \rfloor$ to $\lfloor w_{p+1} \rfloor$, where $S = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$. Also, we claim that $\lfloor w_p \rfloor \ne \lfloor w_{p+1} \rfloor$ for $1 \le p \le s-1$. Suppose, for a contradiction, that $\lfloor w_p \rfloor = \lfloor w_{p+1} \rfloor$ for some p. Then, $w_p W_S = w_{p+1} W_S$, and hence $\min(w_{p+1} W_S, \le_{w_p}) = \min(w_p W_S, \le_{w_p}) = w_p$. Recall that the directed path (3.17) is a path in QBG from w_p to w_{p+1} whose edge labels are increasing and lie in $\Delta^+ \setminus \Delta_S^+$. By Lemma 2.1.8(1), (2), the directed path (3.17) is a shortest path in QBG from w_p to $\min(w_{p+1} W_S, \le_{w_p}) = \min(w_p W_S, \le_{w_p}) = w_p$, which implies that the length of the directed path (3.17) is equal to 0. Therefore, $\{j_{u_p+1}, \dots, j_{u_{p+1}}\} = \emptyset$, and hence $u_p = u_{p+1}$, which contradicts the fact that $u_p < u_{p+1}$.

Thus we obtain

$$\psi := (\lfloor w_s \rfloor, \lfloor w_{s-1} \rfloor, \dots, \lfloor w_1 \rfloor; 1-\sigma_s, \dots, 1-\sigma_0) \in \mathrm{QLS}(\lambda). \tag{3.18}$$

We now define $\Xi(p_J^{\mathrm{OS}}) := \psi$.

Lemma 3.3.4. *Keep the notation and setting above, and let* $s_{i_{K+1}} \cdots s_{i_M} s_{i_{M+1}} \cdots s_{i_N}$ *be a reduced expression for* $v(\mu)w_\circ(S)$ *obtained by concatenating* (3.13) *and* (3.14). *Then,* $\lfloor w_1 \rfloor \in \lfloor \mathrm{EQB}(v(\mu)w_\circ(S)) \rfloor$. *Hence we obtain a map* $\Xi : \overleftarrow{\mathrm{QB}}(e; m_\mu) \to \mathrm{QLS}^{\mu,\infty}(\lambda)$.

Proof. Since it is clear that $v(\mu) \in \lfloor \mathrm{EQB}(v(\mu)w_\circ(S)) \rfloor$, we may assume that $\lfloor w_1 \rfloor \neq v(\mu)$.

Since $z_0^{\mathrm{OS}} = m_\mu$, we have $w_0' = x_0 = \mathrm{dir}(z_0^{\mathrm{OS}}) = v(\mu)v(\lambda_-)^{-1}$. It follows that $w_0 = w_0' w_\circ = \left(v(\mu)v(\lambda_-)^{-1}\right) w_\circ \overset{\text{Lemma 3.1.1(2)}}{=} v(\mu)w_\circ(S)$. If $u_1 = 0$, then we obtain $w_1 = w_0 = v(\mu)w_\circ(S)$, contrary to the assumption that $\lfloor w_1 \rfloor \neq v(\mu)$. Hence it follows that $u_1 \geq 1$. This implies that $j_{u_1} \leq M$ by the definition of u_1 in (3.16) and the proof of Lemma 3.1.10. Thus, we obtain $K+1 \leq j_1 < j_2 < \cdots < j_{u_1} \leq M$.

Now, consider the directed path (3.17) in the case $p = 0$. This is a (nontrivial) directed path in QBG from $w_0 = v(\mu)w_\circ(S)$ to w_1 whose edge labels are increasing in the weak reflection order $\prec$ on Δ^+ introduced at the beginning of §3.3. Because these edge labels are $w_\circ \left(\overline{\beta_{j_k}^{\mathrm{OS}}}\right)^\vee = \beta_{j_k} = s_{i_N} \cdots s_{i_{j_k+1}} \alpha_{i_{j_k}}$ for $1 \leq k \leq u_1$ (the first equality follows from the proof of Lemma 3.1.10), it follows from the fact that $K+1 \leq j_1 < j_2 < \cdots < j_{u_1} \leq M$ and Remark 3.2.4 (recall that we take a reduced expression for $w_\circ$ given by concatenating the reduced expressions for $v(\lambda_-)v(\mu)^{-1}$ and $v(\mu)w_\circ(S)$) that $w_1 \in \mathrm{EQB}(v(\mu)w_\circ(S))$. Hence $\lfloor w_1 \rfloor \in \lfloor \mathrm{EQB}(v(\mu)w_\circ(S)) \rfloor$. □

Proposition 3.3.5. *The map* $\Xi : \overleftarrow{\mathrm{QB}}(e; m_\mu) \to \mathrm{QLS}^{\mu,\infty}(\lambda)$ *is bijective.*

Proof. Let us give the inverse to Ξ. Take an arbitrary $\psi = (y_1, \ldots, y_s; \tau_0, \ldots, \tau_s) \in \mathrm{QLS}^{\mu,\infty}(\lambda)$. By convention, we set $y_{s+1} = v(\mu) \in W^S$. We define the elements v_p, $1 \leq p \leq s+1$, by $v_{s+1} = v(\mu)w_\circ(S)$ and $v_p = \min(y_p W_S, \leq_{v_{p+1}})$ for $1 \leq p \leq s$.

Because there exists a directed path in $\mathrm{QBG}^S_{\tau_p\lambda}$ from y_{p+1} to y_p for $1 \leq p \leq s-1$, we see from Lemma 2.1.8(2), (3) that there exists a unique directed path

$$v_p \xleftarrow{-w_\circ\gamma_{p,1}} \cdots \xleftarrow{-w_\circ\gamma_{p,t_p}} v_{p+1} \tag{3.19}$$

in $\mathrm{QBG}_{\tau_p\lambda}$ from v_{p+1} to v_p whose edge labels $-w_\circ\gamma_{p,t_p}, \ldots, -w_\circ\gamma_{p,1}$ are increasing in the weak reflection order $\prec$ and lie in $\Delta^+ \setminus \Delta_S^+$ for $1 \leq p \leq s-1$. We remark that this is also true for $p = s$, since $\tau_s = 1$. Multiplying the vertices in this directed path on the right by $w_\circ$, we obtain by Lemma 2.1.3 the following directed paths:

$$v_{p,0} =: v_p w_\circ \xrightarrow{\gamma_{p,1}} v_{p,1} \xrightarrow{\gamma_{p,2}} \cdots \xrightarrow{\gamma_{p,t_p}} v_{p+1}w_\circ := v_{p,t_p}, \quad 1 \leq p \leq s.$$

Concatenating these paths for $1 \leq p \leq s$, we obtain the following directed path:

$$\begin{aligned} v_{1,0} &\xrightarrow{\gamma_{1,1}} \cdots \xrightarrow{\gamma_{1,t_1}} v_{1,t_1} = v_{2,0} \xrightarrow{\gamma_{2,1}} \cdots \xrightarrow{\gamma_{s-2,t_{s-2}}} v_{s-2,t_{s-2}} = v_{s-1,0} \xrightarrow{\gamma_{s-1,1}} \\ &\cdots \xrightarrow{\gamma_{s-1,t_{s-1}}} v_{s-1,t_{s-1}} = v_{s,0} \xrightarrow{\gamma_{s,1}} \cdots \xrightarrow{\gamma_{s,t_s}} v_{s,t_s} = v_{s+1,0} = v(\mu)v(\lambda_-)^{-1} \end{aligned} \tag{3.20}$$

in QBG. Now, for $1 \leq p \leq s$ and $1 \leq m \leq t_p$, we set $d_{p,m} := 1 - \tau_p \in \mathbb{Q} \cap [0,1)$, $a_{p,m} := (d_{p,m} - 1)\langle \lambda_-, \gamma_{p,m}^\vee \rangle$, and $\widetilde{\gamma}_{p,m} := a_{p,m}\tilde{\delta} - \gamma_{p,m}^\vee$.

Claim 1. $\widetilde{\gamma}_{p,m} \in \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_\mu^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$.

Proof of Claim 1. Since $\tau_p > 0$, and since the path (3.19) is a directed path in $\mathrm{QBG}_{\tau_p\lambda}$ whose edge labels are increasing and lie in $\Delta^+ \setminus \Delta_S^+$, we obtain $a_{p,m} = -\tau_p\langle \lambda_-, \gamma_{p,m}^\vee \rangle = \tau_p\langle \lambda, -w_\circ\gamma_{p,m}^\vee \rangle \in \mathbb{Z}_{>0}$.

We will show that $a_{p,m} < \varsigma(v(\mu)v(\lambda_-)^{-1}(-\gamma_{p,m})) + \langle \lambda, w_\circ\left(-\gamma_{p,m}^\vee\right)\rangle$. Here we note that the inequality $\langle \lambda, w_\circ\left(-\gamma_{p,m}^\vee\right)\rangle = -\langle \lambda_-, \gamma_{p,m}^\vee\rangle \geq -\tau_p\langle \lambda_-, \gamma_{p,m}^\vee\rangle = a_{p,m}$ holds, with equality if and only if $p = s$. Hence it suffices to consider the case $p = s$. In the case $p = s$, the path (3.19) is the unique directed path in QBG from $v(\mu)w_\circ(S) = v_{s+1}$ to v_s whose edge labels are increasing and lie in $\Delta^+ \setminus \Delta_S^+$. Also, since $\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)$ and $\kappa(\psi) = y_s = \lfloor v_s \rfloor$, we find that there exists $v_s' \in \mathrm{EQB}(v(\mu)w_\circ(S))$ such that $\lfloor v_s' \rfloor = y_s$. By the definition of $\mathrm{EQB}(v(\mu)w_\circ(S))$, there exists a unique directed path in QBG from $v(\mu)w_\circ(S)$ to v_s' whose edge labels are increasing; we see from (3.3) that this directed path is obtained as the concatenation of the following two directed paths: the one whose edge labels lie in $\Delta^+\setminus\Delta_S^+$, and the one whose edge labels lie in Δ_S^+. Therefore, by removing all the edges whose labels lie in Δ_S^+ from the path above, we obtain a directed path in QBG from $v(\mu)w_\circ(S)$ to some $v_s'' \in y_s W_S \cap \mathrm{EQB}(v(\mu)w_\circ(S))$ whose edge labels are increasing and lie in $\Delta^+ \setminus \Delta_S^+$. Here, since $\lfloor v_s \rfloor = \lfloor v_s'' \rfloor$ and $v_s = \min(y_s W_S, \leq_{v(\mu)w_\circ(S)})$, Lemma 2.1.8(2) shows that $v_s = v_s''$. Hence we have $v_s \in \mathrm{EQB}(v(\mu)w_\circ(S))$. Moreover, by the definition of $\mathrm{EQB}(v(\mu)w_\circ(S))$, the edge labels $-w_\circ\gamma_{s,1}, \dots, -w_\circ\gamma_{s,t_s}$ in the given directed path in QBG from $v(\mu)w_\circ(S) = v_{s+1}$ to v_s are elements of $\Delta^+ \cap (v(\mu)w_\circ(S))^{-1}\Delta^-$, and hence $v(\mu)w_\circ(S)\,(-w_\circ\gamma_{s,m}) \overset{\text{Lemma 3.1.1(2)}}{=} v(\mu)v(\lambda_-)^{-1}(-\gamma_{s,m}) \in \Delta^-$. Therefore, in the case $p = s$, we have $\varsigma(v(\mu)v(\lambda_-)^{-1}(-\gamma_{s,m})) = 1$. Thus we have shown that $a_{s,m} = \langle \lambda, w_\circ\left(-\gamma_{s,m}^\vee\right)\rangle < \varsigma(v(\mu)v(\lambda_-)^{-1}(-\gamma_{s,m})) + \langle \lambda, w_\circ\left(-\gamma_{s,m}^\vee\right)\rangle$. Hence we conclude that $\widetilde{\gamma}_{p,m} \in \widetilde{\Delta}_{\mathrm{aff}}^+ \cap m_\mu^{-1}\widetilde{\Delta}_{\mathrm{aff}}^-$ by Lemma 3.3.2. This proves Claim 1.

Claim 2.

(1) *We have*

$$\widetilde{\gamma}_{s,t_s} \prec' \cdots \prec' \widetilde{\gamma}_{s,1} \prec' \widetilde{\gamma}_{s-1,t_{s-1}} \prec' \cdots \prec' \widetilde{\gamma}_{1,1},$$

where $\prec'$ *denotes the weak reflection order on* $\widetilde{\Delta}_{\mathrm{aff}}^+ \cap m_{\lambda_-}^{-1}\widetilde{\Delta}_{\mathrm{aff}}^-$ *introduced at the beginning of* §3.3*; hence we can choose* $J' = \{j_1', \dots, j_{r'}'\} \subset \{K+1, \dots, L\}$ *in such a way that*

$$\left(\beta_{j_1'}^{\mathrm{OS}}, \dots, \beta_{j_{r'}'}^{\mathrm{OS}}\right) = \left(\widetilde{\gamma}_{s,t_s}, \dots, \widetilde{\gamma}_{s,1}, \widetilde{\gamma}_{s-1,t_{s-1}}, \dots, \widetilde{\gamma}_{1,1}\right).$$

(2) *Let* $1 \leq k \leq r'$*, and take* $1 \leq p \leq s$*,* $0 < m \leq t_p$ *such that*

$$\left(\beta_{j_1'}^{\mathrm{OS}} \prec' \cdots \prec' \beta_{j_k'}^{\mathrm{OS}}\right) = \left(\widetilde{\gamma}_{s,t_s} \prec' \cdots \prec' \widetilde{\gamma}_{p,m}\right).$$

Then, $\mathrm{dir}(z_k^{\mathrm{OS}}) = v_{p,m-1}$*. Moreover,* $\mathrm{dir}(z_{k-1}^{\mathrm{OS}}) \xleftarrow{-\left(\overline{\beta_{j_k'}^{\mathrm{OS}}}\right)^\vee} \mathrm{dir}(z_k^{\mathrm{OS}})$ *is an edge of* QBG.

Proof of Claim 2. (1) It suffices to show the following:
(i) for $1 \leq p \leq s$ and $1 < m \leq t_p$, we have $\widetilde{\gamma}_{p,m} \prec' \widetilde{\gamma}_{p,m-1}$;
(ii) for $2 \leq p \leq s$, we have $\widetilde{\gamma}_{p,1} \prec' \widetilde{\gamma}_{p-1,t_{p-1}}$.
(i) Because $\frac{\langle \lambda_-, -\gamma_{p,m}^\vee\rangle - a_{p,m}}{\langle \lambda_-, -\gamma_{p,m}^\vee\rangle} = d_{p,m}$ and $\frac{\langle \lambda_-, -\gamma_{p,m-1}^\vee\rangle - a_{p,m-1}}{\langle \lambda_-, -\gamma_{p,m-1}^\vee\rangle} = d_{p,m-1}$, we have

$$\Phi(\widetilde{\gamma}_{p,m}) = (d_{p,m}, -w_\circ\gamma_{p,m}),$$
$$\Phi(\widetilde{\gamma}_{p,m-1}) = (d_{p,m-1}, -w_\circ\gamma_{p,m-1}).$$

Therefore, the first component of $\Phi(\widetilde{\gamma}_{p,m})$ is equal to that of $\Phi(\widetilde{\gamma}_{p,m-1})$ since $d_{p,m} = 1 - \tau_p = d_{p,m-1}$. Moreover, since $-w_\circ\gamma_{p,m} \prec -w_\circ\gamma_{p,m-1}$, we have $\Phi(\widetilde{\gamma}_{p,m}) < \Phi(\widetilde{\gamma}_{p,m-1})$. This implies that $\widetilde{\gamma}_{p,m} \prec' \widetilde{\gamma}_{p,m-1}$ by Proposition 3.1.8.

(ii) The proof of (ii) is similar to that of (i). The first components of $\Phi(\widetilde{\gamma}_{p,1})$ and $\Phi(\widetilde{\gamma}_{p-1,t_{p-1}})$ are $d_{p,1}$ and $d_{p-1,t_{p-1}}$, respectively. Since $d_{p,1} = 1 - \tau_p < 1 - \tau_{p-1} = d_{p-1,t_{p-1}}$, we have $\Phi(\widetilde{\gamma}_{p,1}) < \Phi(\widetilde{\gamma}_{p-1,t_{p-1}})$. This implies that $\widetilde{\gamma}_{p,1} \prec' \widetilde{\gamma}_{p-1,t_{p-1}}$.

(2) We proceed by induction on k. Since $\mathrm{dir}(z_0^{\mathrm{OS}}) = \mathrm{dir}(m_\mu) = v(\mu)v(\lambda_-)^{-1}$ and $\beta_{j'_1}^{\mathrm{OS}} = \widetilde{\gamma}_{s,t_s}$, we have $\mathrm{dir}(z_1^{\mathrm{OS}}) = \mathrm{dir}(z_0^{\mathrm{OS}})s_{-\overline{\beta_{j'_1}^{\mathrm{OS}}}} = v(\mu)v(\lambda_-)^{-1}s_{\gamma_{s,t_s}} = v_{s,t_s-1}$. Hence the assertion holds in the case $k = 1$.

Assume that $\mathrm{dir}(z_{k-1}^{\mathrm{OS}}) = v_{p,m}$ for $0 \lneqq m \le t_p$. Here we remark that $v_{p,m-1}$ is the predecessor of $v_{p,m}$ in the directed path (3.20) since $0 \le m-1 \le t_{p-1}$. Hence we have $\mathrm{dir}(z_k^{\mathrm{OS}}) = \mathrm{dir}(z_{k-1}^{\mathrm{OS}})s_{-\overline{\beta_{j'_k}^{\mathrm{OS}}}} = v_{p,m}s_{\gamma_{p,m}} \overset{(3.20)}{=} v_{p,m-1}$. Also, since (3.20) is a directed path in QBG, $v_{p,m} = \mathrm{dir}(z_{k-1}^{\mathrm{OS}}) \xleftarrow{-\left(\overline{\beta_{j'_k}^{\mathrm{OS}}}\right)^\vee} \mathrm{dir}(z_k^{\mathrm{OS}}) = v_{p,m-1}$ is an edge of QBG. This proves Claim 2.

Since $J' = \{j_1, \dots, j'_{r'}\} \subset \{K+1, \dots, L\}$, we can define an element $p_{J'}^{\mathrm{OS}}$ to be $\left(m_\mu = z_0^{\mathrm{OS}}, z_1^{\mathrm{OS}}, \dots, z_{r'}^{\mathrm{OS}}; \beta_{j'_1}^{\mathrm{OS}}, \beta_{j'_2}^{\mathrm{OS}}, \dots, \beta_{j'_{r'}}^{\mathrm{OS}}\right)$, where $z_0^{\mathrm{OS}} = m_\mu$, $z_k^{\mathrm{OS}} = z_{k-1}^{\mathrm{OS}}s_{\beta_{j'_k}^{\mathrm{OS}}}$ for $1 \le k \le r'$. It follows from Remark 3.3.3 and Claim 2 that $p_{J'}^{\mathrm{OS}} \in \overleftarrow{\mathrm{QB}}(e; m_\mu)$. Hence we can define a map $\Theta : \mathrm{QLS}^{\mu,\infty}(\lambda) \to \overleftarrow{\mathrm{QB}}(e; m_\mu)$ by $\Theta(\psi) := p_{J'}^{\mathrm{OS}}$.

It remains to show that the map Θ is the inverse to the map Ξ, i.e., the following two claims.

Claim 3. *For* $\psi = (y_1, \dots, y_s; \tau_0, \dots, \tau_s) \in \mathrm{QLS}(\lambda)$, *we have* $\Xi \circ \Theta(\psi) = \psi$.

Claim 4. *For* $p_J^{\mathrm{OS}} = \left(m_\mu = z_0^{\mathrm{OS}}, z_1^{\mathrm{OS}}, \dots, z_r^{\mathrm{OS}}; \beta_{j_1}^{\mathrm{OS}}, \beta_{j_2}^{\mathrm{OS}}, \dots, \beta_{j_r}^{\mathrm{OS}}\right) \in \overleftarrow{\mathrm{QB}}(e; m_\mu)$, *we have* $\Theta \circ \Xi(p_J^{\mathrm{OS}}) = p_J^{\mathrm{OS}}$.

Proof of Claim 3. We set $\Theta(\psi) = p_{J'}^{\mathrm{OS}}$, with $J' = \{j'_1, \dots, j'_r\}$. In the following description of $\Theta(\psi) = p_{J'}^{\mathrm{OS}}$, we employ the notation u_p, σ_p, w'_p, and w_p used in the definition of $\Xi(p_J^{\mathrm{OS}})$.

For $1 \le k \le r'$, if we set $\beta_{j'_k}^{\mathrm{OS}} = \widetilde{\gamma}_{p,m}$ with $m > 0$, then we have $d_{j'_k} = 1 + \frac{\deg(\beta_{j'_k}^{\mathrm{OS}})}{\langle \lambda_-, -\beta_{j'_k}^{\mathrm{OS}} \rangle} = 1 + \frac{\deg(\widetilde{\gamma}_{p,m})}{\langle \lambda_-, -\widetilde{\gamma}_{p,m} \rangle} = 1 + \frac{a_{p,m}}{\langle \lambda_-, \gamma_{p,m}^\vee \rangle} = d_{p,m}$. Therefore, the sequence (3.16) determined by $\Theta(\psi) = p_{J'}^{\mathrm{OS}}$ is

$$\begin{aligned}(3.21)\qquad &\underbrace{0 = d_{s,t_s} = \cdots = d_{s,1}}_{=1-\tau_s} < \underbrace{d_{s-1,t_{s-1}} = \cdots = d_{s-1,1}}_{=1-\tau_{s-1}} < \cdots < \underbrace{d_{1,t_1} = \cdots = d_{1,1}}_{=1-\tau_1} \\ &< 1 = 1 - \tau_0.\end{aligned}$$

Because the sequence (3.21) of rational numbers is just the sequence (3.16) for $\Theta(\psi) = p_{J'}^{\mathrm{OS}}$, we deduce that $\beta_{j'_{u_p}}^{\mathrm{OS}} = \widetilde{\gamma}_{s-p+1,1}$ for $1 \le p \le s$, and $\sigma_p = 1 - \tau_{s-p}$ for $0 \le p \le s$. Therefore, we have $w'_p = \mathrm{dir}(z_{u_p}^{\mathrm{OS}}) = v_{s-p+1,0}$ and $w_p = v_{s-p+1,0}w_\circ = v_{s-p+1}$. Since $\lfloor w_p \rfloor = \lfloor v_{s-p+1} \rfloor = y_{s-p+1}$, we conclude that $\Xi \circ \Theta(\psi) = (\lfloor w_s \rfloor, \dots, \lfloor w_1 \rfloor; 1 - \sigma_s, \dots, 1 - \sigma_0) = (y_1, \dots, y_s; \tau_0, \dots, \tau_s) = \psi$. This proves Claim 3.

Proof of Claim 4. We set $\psi := \Xi(p_J^{\mathrm{OS}})$,and write it as $\psi = (y_1, \ldots, y_s; \tau_0, \ldots, \tau_s)$, where $y_p = \lfloor w_{s+1-p} \rfloor$ for $1 \le p \le s$ and $\tau_p = 1 - \sigma_{s-p}$ for $0 \le p \le s$ in the notation of (3.18) (and the comment preceding it). Also, in the following description of $\Xi(p_J^{\mathrm{OS}}) = \psi$, we employ the notation $v_{p,m}$, $d_{p,m}$, $a_{p,m}$, $\gamma_{p,m}$, $\widetilde{\gamma}_{p,m}$, and J' used in the definition of $\Theta(\psi)$.

Recall that $w_0 = v(\mu)w_\circ(S) = v_{s+1}$. For $0 \le p \le s-1$,

$$v_{s-p+1} \xrightarrow{-w_\circ \gamma_{s-p,t_{s-p}}} \cdots \xrightarrow{-w_\circ \gamma_{s-p,1}} v_{s-p}$$

is a directed path in QBG whose edge labels are increasing and lie in $\Delta^+ \setminus \Delta_S^+$ (see (3.19)). Now we can show by induction on p that $w_p = v_{s-p+1}$ for $1 \le p \le s$. Indeed, if $w_p = v_{s-p+1}$, then both of the paths above and the path (3.17) start from w_p and end with some element in $w_{p+1}W_S = v_{s-p}W_S$ (this equality follows from the definition of v_{s-p}) and have increasing edge labels lying in $\Delta^+ \setminus \Delta_S^+$. Therefore, by Lemma 2.1.8 (2), we deduce that the ends of these two paths are identical, and hence that $w_{p+1} = v_{s-p}$. Moreover, since these two paths are identical, so are the edge labels of them:

$$\left(w_\circ \left(\overline{\beta_{j_{u_p+1}}^{\mathrm{OS}}} \right)^\vee \prec \cdots \prec w_\circ \left(\overline{\beta_{j_{u_{p+1}}}^{\mathrm{OS}}} \right)^\vee \right) = \left(-w_\circ \gamma_{s-p,t_{s-p}} \prec \cdots \prec -w_\circ \gamma_{s-p,1} \right)$$

for $0 \le p \le s-1$. From the above, we have $u_{p+1} - u_p = t_{s-p}$ and $-\left(\overline{\beta_{j_{u_p+k}}^{\mathrm{OS}}} \right)^\vee = \gamma_{s-p,t_{s-p}-k+1}$ for $0 \le p \le s-1$, $1 \le k \le t_{s-p}$. Because $\sigma_p = d_{j_{u_p+1}} = \cdots = d_{j_{u_{p+1}}}$ for $0 \le p \le s-1$, $1 - \sigma_p = \tau_{s-p}$ for $0 \le p \le s$, and $1 - \tau_{s-p} = d_{s-p,1} = \cdots = d_{s-p,t_{s-p}}$ for $0 \le p \le s-1$, we see that for $1 \le k \le t_{s-p}$,

$$\begin{aligned} \beta_{j_{u_p+k}}^{\mathrm{OS}} &= \overline{\beta_{j_{u_p+k}}^{\mathrm{OS}}} + a_{j_{u_p+k}} \widetilde{\delta} \\ &= \overline{\beta_{j_{u_p+k}}^{\mathrm{OS}}} - (d_{j_{u_p+k}} - 1) \langle \lambda_-, \overline{\beta_{j_{u_p+k}}^{\mathrm{OS}}} \rangle \widetilde{\delta} \\ &= -\gamma_{s-p,t_{s-p}-k+1}^\vee + (d_{s-p,t_{s-p}-k+1} - 1) \langle \lambda_-, \gamma_{s-p,t_{s-p}-k+1}^\vee \rangle \widetilde{\delta} \\ &= -\gamma_{s-p,t_{s-p}-k+1}^\vee + a_{s-p,t_{s-p}-k+1} \widetilde{\delta} \\ &= \widetilde{\gamma}_{s-p,t_{s-p}-k+1}. \end{aligned}$$

Therefore, we have

$$\left(\beta_{j_{u_p+1}}^{\mathrm{OS}} \prec' \cdots \prec' \beta_{j_{u_{p+1}}}^{\mathrm{OS}} \right) = \left(\widetilde{\gamma}_{s-p,t_{s-p}} \prec' \cdots \prec' \widetilde{\gamma}_{s-p,1} \right), \quad 0 \le p \le s-1.$$

Concatenating the sequences above for $0 \le p \le s-1$, we obtain

$$\begin{aligned} \left(\beta_{j_1}^{\mathrm{OS}} \prec' \cdots \prec' \beta_{j_r}^{\mathrm{OS}} \right) &= \left(\widetilde{\gamma}_{s,t_s} \prec' \cdots \prec' \widetilde{\gamma}_{s,1} \prec' \widetilde{\gamma}_{s-1,t_{s-1}} \prec' \cdots \prec' \widetilde{\gamma}_{1,1} \right) \\ &= \left(\beta_{j'_1}^{\mathrm{OS}} \prec' \cdots \prec' \beta_{j'_{r'}}^{\mathrm{OS}} \right). \end{aligned}$$

Hence the set J' determined by $\Xi(p_J^{\mathrm{OS}}) = \psi$ is identical to J. Thus we conclude that $\Theta \circ \Xi(p_J^{\mathrm{OS}}) = p_{J'}^{\mathrm{OS}} = p_J^{\mathrm{OS}}$. This proves Claim 4.

This completes the proof of Proposition 3.3.5. □

We recall from (2.3) and (2.4) that $\deg(\beta)$ is defined by $\beta = \overline{\beta} + \deg(\beta)\widetilde{\delta}$ for $\beta \in \mathfrak{h} \oplus \mathbb{C}\widetilde{\delta}$, and $\mathrm{wt}(u) \in P$ and $\mathrm{dir}(u)$ are defined by $u = t(\mathrm{wt}(u))\mathrm{dir}(u)$ for $u \in \widetilde{W}_{\mathrm{ext}} = t(P) \rtimes W$.

Proposition 3.3.6. *The bijection* $\Xi : \overleftarrow{\mathrm{QB}}(e; m_\mu) \to \mathrm{QLS}^{\mu,\infty}(\lambda)$ *satisfies the following*:

(1) $\mathrm{wt}(\mathrm{end}(p_J^{\mathrm{OS}})) = \mathrm{wt}(\Xi(p_J^{\mathrm{OS}}))$;
(2) $\deg(\mathrm{qwt}^*(p_J^{\mathrm{OS}})) = -\deg_\mu(\Xi(p_J^{\mathrm{OS}}))$.

Proof. We proceed by induction on $\#J$.

If $J = \emptyset$, then it is obvious that $\deg(\mathrm{qwt}^*(p_J^{\mathrm{OS}})) = \deg_\mu(\Xi(p_J^{\mathrm{OS}})) = 0$ and $\mathrm{wt}(\mathrm{end}(p_J^{\mathrm{OS}}))$
$= \mathrm{wt}(\Xi(p_J^{\mathrm{OS}})) = \mu$, since $\Xi(p_J^{\mathrm{OS}}) = (v(\mu)w_\circ(S); 0, 1)$.

Let $J = \{j_1 < j_2 < \cdots < j_r\}$, and set $K := J \setminus \{j_r\}$; assume that $\Xi(p_K^{\mathrm{OS}})$ is of the form: $\Xi(p_K^{\mathrm{OS}}) = (\lfloor w_s \rfloor, \lfloor w_{s-1} \rfloor, \dots, \lfloor w_1 \rfloor; 1-\sigma_s, \dots, 1-\sigma_0)$. In the following, we employ the notation w_p, $0 \le p \le s$, used in the definition of the map Ξ. Note that $\mathrm{dir}(p_K^{\mathrm{OS}}) = w_s w_\circ$ and $w_0 = v(\mu)w_\circ(S)$ by the definition of Ξ. Also, observe that if $d_{j_r} = d_{j_{r-1}} = \sigma_{s-1}$, then $\{d_{j_1} \le \cdots \le d_{j_{r-1}} \le d_{j_r}\} = \{d_{j_1} \le \cdots \le d_{j_{r-1}}\}$, and if $d_{j_r} > d_{j_{r-1}} = \sigma_{s-1}$, then $\{d_{j_1} \le \cdots \le d_{j_{r-1}} \le d_{j_r}\} = \{d_{j_1} \le \cdots \le d_{j_{r-1}} < d_{j_r}\}$. From these, we deduce that

$$\Xi(p_J^{\mathrm{OS}}) = \begin{cases} (\lfloor w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \rfloor, \lfloor w_{s-1} \rfloor, \dots, \lfloor w_1 \rfloor; 1-\sigma_s, 1-\sigma_{s-1}, \dots, 1-\sigma_0) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } d_{j_r} = d_{j_{r-1}} = \sigma_{s-1}, & \\ (\lfloor w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \rfloor, \lfloor w_s \rfloor, \lfloor w_{s-1} \rfloor, \dots, \lfloor w_1 \rfloor; 1-\sigma_s, 1-d_{j_r}, & \\ \qquad 1-\sigma_{s-1}, \dots, 1-\sigma_0) \qquad\qquad\qquad \text{if } d_{j_r} > d_{j_{r-1}} = \sigma_{s-1}. & \end{cases}$$

For the induction step, it suffices to show the following claims.

Claim 1.

(1) *We have*

$$\mathrm{wt}(\Xi(p_J^{\mathrm{OS}})) = \mathrm{wt}(\Xi(p_K^{\mathrm{OS}})) + a_{j_r} w_s w_\circ \left(-\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^\vee.$$

(2) *We have*

$$\deg_\mu(\Xi(p_J^{\mathrm{OS}})) = \deg_\mu(\Xi(p_K^{\mathrm{OS}})) - \chi_r a_{j_r},$$

where $\chi_r := 0$ *(resp.,* $\chi_r := 1$*) if* $w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \leftarrow w_s$ *is a Bruhat (resp., quantum) edge.*

Claim 2.

(1) *We have*

$$\mathrm{wt}(\mathrm{end}(p_J^{\mathrm{OS}})) = \mathrm{wt}(\mathrm{end}(p_K^{\mathrm{OS}})) + a_{j_r} w_s w_\circ \left(-\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^\vee.$$

(2) *We have*

$$\deg(\mathrm{qwt}^*(p_J^{\mathrm{OS}})) = \deg(\mathrm{qwt}^*(p_K^{\mathrm{OS}})) + \chi_r a_{j_r}.$$

Proof of Claim 1. (1) If $d_{j_r} = d_{j_{r-1}} = \sigma_{s-1}$, then we compute:

$$\begin{aligned}
\mathrm{wt}(\Xi(p_J^{\mathrm{OS}})) &= (\sigma_s - \sigma_{s-1})\lfloor w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \rfloor \lambda + \sum_{p=1}^{s-1} (\sigma_p - \sigma_{p-1}) \lfloor w_p \rfloor \lambda \\
&= (\sigma_s - \sigma_{s-1}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda + \sum_{p=1}^{s-1} (\sigma_p - \sigma_{p-1}) w_p \lambda \\
&= \sum_{p=1}^{s} (\sigma_p - \sigma_{p-1}) w_p \lambda + (\sigma_s - \sigma_{s-1}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda - (\sigma_s - \sigma_{s-1}) w_s \lambda \\
&\overset{d_{j_r} = \sigma_{s-1},\ \sigma_s = 1}{=} \sum_{p=1}^{s} (\sigma_p - \sigma_{p-1}) w_p \lambda + (1 - d_{j_r}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda - (1 - d_{j_r}) w_s \lambda.
\end{aligned}$$

If $d_{j_r} > d_{j_{r-1}} = \sigma_{s-1}$, then we compute:

$$\begin{aligned}
\mathrm{wt}(\Xi(p_J^{\mathrm{OS}})) &= (\sigma_s - d_{j_r})\lfloor w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \rfloor \lambda + (d_{j_r} - \sigma_{s-1}) \lfloor w_s \rfloor \lambda + \sum_{p=1}^{s-1} (\sigma_p - \sigma_{p-1}) \lfloor w_p \rfloor \lambda \\
&= (\sigma_s - d_{j_r}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda + (d_{j_r} - \sigma_{s-1}) w_s \lambda + \sum_{p=1}^{s-1} (\sigma_p - \sigma_{p-1}) w_p \lambda \\
&= \sum_{p=1}^{s} (\sigma_p - \sigma_{p-1}) w_p \lambda - (\sigma_s - \sigma_{s-1}) w_s \lambda \\
&\quad + (\sigma_s - d_{j_r}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda + (d_{j_r} - \sigma_{s-1}) w_s \lambda \\
&= \sum_{p=1}^{s} (\sigma_p - \sigma_{p-1}) w_p \lambda + (\sigma_s - d_{j_r}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda - (\sigma_s - d_{j_r}) w_s \lambda \\
&\overset{\sigma_s = 1}{=} \sum_{p=1}^{s} (\sigma_p - \sigma_{p-1}) w_p \lambda + (1 - d_{j_r}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda - (1 - d_{j_r}) w_s \lambda.
\end{aligned}$$

In both cases above, since

$$\mathrm{wt}(\Xi(p_K^{\mathrm{OS}})) = \sum_{p=1}^{s} (\sigma_p - \sigma_{p-1}) \lfloor w_p \rfloor \lambda = \sum_{p=1}^{s} (\sigma_p - \sigma_{p-1}) w_p \lambda,$$

and since

$$\begin{aligned}
&(1 - d_{j_r}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda - (1 - d_{j_r}) w_s \lambda \\
&\quad = -(1 - d_{j_r}) w_s \langle \lambda, w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}} \rangle w_\circ \left(\overline{\beta_{j_r}^{\mathrm{OS}}} \right)^\vee \\
&\quad \overset{\text{Remark 3.1.11}}{=} -\frac{a_{j_r}}{\langle \lambda_-, \overline{\beta_{j_r}^{\mathrm{OS}}} \rangle} \langle \lambda_-, \overline{\beta_{j_r}^{\mathrm{OS}}} \rangle w_s w_\circ \left(\overline{\beta_{j_r}^{\mathrm{OS}}} \right)^\vee \\
&\quad = a_{j_r} w_s w_\circ \left(-\overline{\beta_{j_r}^{\mathrm{OS}}} \right)^\vee,
\end{aligned}$$

it follows that

$$\begin{aligned}\mathrm{wt}(\Xi(p_J^{\mathrm{OS}})) &= \sum_{p=1}^{s}(\sigma_p-\sigma_{p-1})w_p\lambda+(1-d_{j_r})w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}}\lambda-(1-d_{j_r})w_s\lambda\\ &= \mathrm{wt}(\Xi(p_K^{\mathrm{OS}}))+a_{j_r}w_s w_\circ\left(-\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^{\vee}.\end{aligned}$$

(2) From the relation between p_J^{OS} and p_K^{OS}, and from the definition of $\overleftarrow{\mathrm{QB}}(e;m_\mu)$, we find that $w_s w_\circ s_{-\overline{\beta_{j_r}^{\mathrm{OS}}}} \xrightarrow{-\left(\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^{\vee}} w_s w_\circ$ is an edge of QBG. Hence, by Lemma 2.1.3, $w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}} \xleftarrow{w_\circ\left(\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^{\vee}} w_s$ is an edge of QBG.

If $d_{j_r}=d_{j_{r-1}}=\sigma_{s-1}$, then by the definition of $\deg_\mu$ (along with [LNSSS2, Lemma 7.2]), we see that

(3.22)

$$\begin{aligned}\deg_\mu(\Xi(p_J^{\mathrm{OS}})) = &-\sum_{p=0}^{s-2}(1-\sigma_p)\mathrm{wt}_\lambda(\lfloor w_{p+1}\rfloor\Leftarrow\lfloor w_p\rfloor)\\ &-(1-d_{j_r})\mathrm{wt}_\lambda(\lfloor w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}}\rfloor\Leftarrow\lfloor w_{s-1}\rfloor)\\ =&-\sum_{p=0}^{s-2}(1-\sigma_p)\mathrm{wt}_\lambda(w_{p+1}\Leftarrow w_p)-(1-d_{j_r})\mathrm{wt}_\lambda(w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}}\Leftarrow w_{s-1}).\end{aligned}$$

Here, $w_0=v(\mu)w_\circ(S)$ as mentioned in the proof of Lemma 3.3.4, so that $\lfloor w_0\rfloor=v(\mu)$. Since $d_{j_r}=d_{j_{r-1}}=\sigma_{s-1}$, we have $w_\circ\left(\overline{\beta_{j_{r-1}}^{\mathrm{OS}}}\right)^{\vee}\prec w_\circ\left(\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^{\vee}$ by Lemma 3.1.12. Because the (unique) label-increasing directed path in QBG from w_{s-1} to w_s has the final edge label $w_\circ\left(\overline{\beta_{j_{r-1}}^{\mathrm{OS}}}\right)^{\vee}$, by concatenating this directed path from w_{s-1} to w_s with $w_s \xrightarrow{w_\circ\left(\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^{\vee}} w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}}$, we obtain a label-increasing (hence shortest) directed path from w_{s-1} to $w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}}$ passing through w_s. Therefore, we deduce that

$$\mathrm{wt}_\lambda(w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}}\Leftarrow w_{s-1})=\mathrm{wt}_\lambda(w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}}\leftarrow w_s)+\mathrm{wt}_\lambda(w_s\Leftarrow w_{s-1}). \tag{3.23}$$

It follows from (3.22) and (3.23) that

$$\deg_\mu(\Xi(p_J^{\mathrm{OS}}))=-\sum_{p=0}^{s-1}(1-\sigma_p)\mathrm{wt}_\lambda(w_{p+1}\Leftarrow w_p)-(1-d_{j_r})\mathrm{wt}_\lambda(w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}}\leftarrow w_s).$$

If $d_{j_r}>d_{j_{r-1}}=\sigma_{s-1}$, then by the definition of $\deg_\mu$ (along with [LNSSS2, Lemma 7.2]), we see that

$$\deg_\mu(\Xi(p_J^{\mathrm{OS}}))=-\sum_{p=0}^{s-1}(1-\sigma_p)\mathrm{wt}_\lambda(w_{p+1}\Leftarrow w_p)-(1-d_{j_r})\mathrm{wt}_\lambda(w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}}\leftarrow w_s),$$

where $w_0 = v(\mu)w_\circ(S)$. Also, by the definition of $\deg_\mu$ (along with [LNSSS2, Lemma 7.2]), we have

$$\deg_\mu(\Xi(p_K^{\mathrm{OS}})) = -\sum_{p=0}^{s-1}(1-\sigma_p)\mathrm{wt}_\lambda(w_{p+1} \Leftarrow w_p),$$

where $w_0 = v(\mu)w_\circ(S)$.

In both cases above, we deduce that

$$\deg_\mu(\Xi(p_J^{\mathrm{OS}})) = \deg_\mu(\Xi(p_K^{\mathrm{OS}})) - (1-d_{j_r})\mathrm{wt}_\lambda(w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}} \leftarrow w_s).$$

If $w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}} \leftarrow w_s$ is a Bruhat edge, then we have $\mathrm{wt}_\lambda(w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}} \leftarrow w_s) = 0$. If $w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}} \leftarrow w_s$ is a quantum edge, then we have $\mathrm{wt}_\lambda(w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}} \leftarrow w_s) = \langle \lambda, w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}\rangle$. Note that

$$(1-d_{j_r})\langle \lambda, w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}\rangle \overset{\text{Remark 3.1.11}}{=} \frac{a_{j_r}}{\langle \lambda_-, \overline{\beta_{j_r}^{\mathrm{OS}}}\rangle}\langle \lambda_-, \overline{\beta_{j_r}^{\mathrm{OS}}}\rangle = a_{j_r}.$$

Therefore, in both cases, we have $\deg_\mu(\Xi(p_J^{\mathrm{OS}})) = \deg_\mu(\Xi(p_K^{\mathrm{OS}})) - \chi_r a_{j_r}$, and Claim 1(2) is proved.

Proof of Claim 2. Let us prove part (1). Note that $\mathrm{end}(p_J^{\mathrm{OS}}) = \mathrm{end}(p_K^{\mathrm{OS}})s_{\beta_{j_r}^{\mathrm{OS}}}$ and that

$$\mathrm{end}(p_K^{\mathrm{OS}}) = t(\mathrm{wt}(\mathrm{end}(p_K^{\mathrm{OS}})))\mathrm{dir}(\mathrm{end}(p_K^{\mathrm{OS}})) = t(\mathrm{wt}(\mathrm{end}(p_K^{\mathrm{OS}})))w_s w_\circ;$$

the second equality follows from the comment at the beginning of the proof of Proposition 3.3.6. Also, we have $s_{\beta_{j_r}^{\mathrm{OS}}} = s_{a_{j_r}\tilde{\delta}+\overline{\beta_{j_r}^{\mathrm{OS}}}} = t\left(a_{j_r}\left(-\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^\vee\right)s_{\overline{\beta_{j_r}^{\mathrm{OS}}}}$. Combining these, we obtain

$$\begin{aligned}\mathrm{end}(p_J^{\mathrm{OS}}) &= \left(t(\mathrm{wt}(\mathrm{end}(p_K^{\mathrm{OS}})))w_s w_\circ\right)\left(t\left(a_{j_r}\left(-\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^\vee\right)s_{\overline{\beta_{j_r}^{\mathrm{OS}}}}\right)\\ &= t\left(\mathrm{wt}(\mathrm{end}(p_K^{\mathrm{OS}})) + a_{j_r}w_s w_\circ\left(-\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^\vee\right)w_s w_\circ s_{\overline{\beta_{j_r}^{\mathrm{OS}}}},\end{aligned}$$

and hence

$$\mathrm{wt}(\mathrm{end}(p_J^{\mathrm{OS}})) = \mathrm{wt}(\mathrm{end}(p_K^{\mathrm{OS}})) + a_{j_r}w_s w_\circ\left(-\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^\vee.$$

Let us prove part (2). Since $\mathrm{dir}(\mathrm{end}(p_K^{\mathrm{OS}})) = w_s w_\circ$, we have $\mathrm{dir}(\mathrm{end}(p_J^{\mathrm{OS}})) = w_s w_\circ s_{\overline{\beta_{j_r}^{\mathrm{OS}}}}$. If $w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}} \xleftarrow{w_\circ\left(\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^\vee} w_s$ is a Bruhat edge, then it follows from Lemma 2.1.3 that $w_s w_\circ s_{-\overline{\beta_{j_r}^{\mathrm{OS}}}} \xrightarrow{-\left(\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^\vee} w_s w_\circ$ is also a Bruhat edge. Hence we obtain $J^+ = K^+$. This implies that $\deg(\mathrm{qwt}^*(p_J^{\mathrm{OS}})) = \deg(\mathrm{qwt}^*(p_K^{\mathrm{OS}}))$. If $w_s s_{w_\circ\overline{\beta_{j_r}^{\mathrm{OS}}}} \xleftarrow{w_\circ\left(\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^\vee} w_s$ is a quantum edge, then it follows from Lemma 2.1.3 that $w_s w_\circ s_{-\overline{\beta_{j_r}^{\mathrm{OS}}}} \xrightarrow{-\left(\overline{\beta_{j_r}^{\mathrm{OS}}}\right)^\vee} w_s w_\circ$ is also a quantum edge. Hence we obtain $J^+ = K^+ \sqcup \{j_r\}$. This implies that $\deg(\mathrm{qwt}^*(p_J^{\mathrm{OS}})) = \deg(\mathrm{qwt}^*(p_K^{\mathrm{OS}})) + \deg(\beta_{j_r}^{\mathrm{OS}}) = \deg(\mathrm{qwt}^*(p_K^{\mathrm{OS}})) + a_{j_r}$. Therefore, in both cases, we have $\deg(\mathrm{qwt}^*(p_J^{\mathrm{OS}})) = \deg(\mathrm{qwt}^*(p_K^{\mathrm{OS}})) + \chi_r a_{j_r}$, and Claim 2(2) is proved.

This completes the proof of Proposition 3.3.6. □

Proof of Theorem 3.2.7. We know from Proposition 2.2.2 that

$$E_\mu(q,\infty) = \sum_{p_J^{\mathrm{OS}} \in \overleftarrow{\mathrm{QB}}(e;m_\mu)} e^{\mathrm{wt}(\mathrm{end}(p_J^{\mathrm{OS}}))} q^{-\deg(\mathrm{qwt}^*(p_J^{\mathrm{OS}}))}.$$

Therefore, it follows from Propositions 3.3.5 and 3.3.6 that

$$E_\mu(q,\infty) = \sum_{\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_\mu(\psi)}.$$

Hence we conclude that $E_\mu(q,\infty) = \mathrm{gch}_\mu\, \mathrm{QLS}^{\mu,\infty}(\lambda)$, as desired. □

4. Demazure submodules of level-zero extremal weight modules

4.1. **Untwisted affine root data.** Let $\mathfrak{g}_{\mathrm{aff}}$ be the untwisted affine Lie algebra over $\mathbb{C}$ associated to the finite-dimensional simple Lie algebra $\mathfrak{g}$, and let $\mathfrak{h}_{\mathrm{aff}} = \left(\bigoplus_{j\in I_{\mathrm{aff}}} \mathbb{C}\alpha_j^\vee\right) \oplus \mathbb{C}D$ be its Cartan subalgebra, where $\left\{\alpha_j^\vee\right\}_{j\in I_{\mathrm{aff}}} \subset \mathfrak{h}_{\mathrm{aff}}$ is the set of simple coroots, with $I_{\mathrm{aff}} = I \sqcup \{0\}$, and $D \in \mathfrak{h}_{\mathrm{aff}}$ is the degree operator. We denote by $\left\{\alpha_j\right\}_{j\in I_{\mathrm{aff}}} \subset \mathfrak{h}_{\mathrm{aff}}^*$ the set of simple roots, and by $\Lambda_j \in \mathfrak{h}_{\mathrm{aff}}^*$, $j \in I_{\mathrm{aff}}$, the fundamental weights. Note that $\langle \alpha_j, D\rangle = \delta_{j,0}$ and $\langle \Lambda_j, D\rangle = 0$ for $j \in I_{\mathrm{aff}}$, where $\langle\cdot\,,\cdot\rangle : \mathfrak{h}_{\mathrm{aff}}^* \times \mathfrak{h}_{\mathrm{aff}} \to \mathbb{C}$ denotes the canonical pairing between $\mathfrak{h}_{\mathrm{aff}}$ and $\mathfrak{h}_{\mathrm{aff}}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{\mathrm{aff}}, \mathbb{C})$. Also, let $\delta = \sum_{j\in I_{\mathrm{aff}}} a_j\alpha_j \in \mathfrak{h}_{\mathrm{aff}}^*$ and $c = \sum_{j\in I_{\mathrm{aff}}} a_j^\vee \alpha_j^\vee \in \mathfrak{h}_{\mathrm{aff}}$ denote the null root and the canonical central element of $\mathfrak{g}_{\mathrm{aff}}$, respectively. Here we note that $\mathfrak{h}_{\mathrm{aff}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}D$. If we regard an element $\lambda \in \mathfrak{h}^*$ as an element of $\mathfrak{h}_{\mathrm{aff}}^*$ by $\langle \lambda, c\rangle = \langle \lambda, D\rangle = 0$, then we have $\varpi_i = \Lambda_i - a_i^\vee \Lambda_0$ for $i \in I$. We take a weight lattice P_{aff} for $\mathfrak{g}_{\mathrm{aff}}$ as follows: $P_{\mathrm{aff}} = \left(\bigoplus_{j\in I_{\mathrm{aff}}} \mathbb{Z}\Lambda_j\right) \oplus \mathbb{Z}\delta \subset \mathfrak{h}_{\mathrm{aff}}^*$, and set $Q_{\mathrm{aff}} := \bigoplus_{j\in I_{\mathrm{aff}}} \mathbb{Z}\alpha_j$.

Remark 4.1.1. We should warn the reader that the root datum of the affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}$ is not necessarily dual to that of the untwisted affine Lie algebra associated to $\widetilde{\mathfrak{g}}$ in §2.2, though the root datum of $\widetilde{\mathfrak{g}}$ is dual to that of $\mathfrak{g}$. In particular, for the index $0 \in I_{\mathrm{aff}}$, the simple coroot $\alpha_0^\vee = c - \theta^\vee$, with $\theta \in \Delta^+$ the highest root of $\mathfrak{g}$, does not agree with the simple root $\widetilde{\delta} - \varphi^\vee$ in §2.2, which is denoted by $\alpha_0^\vee$ there.

The Weyl group W_{aff} of $\mathfrak{g}_{\mathrm{aff}}$ is defined to be the subgroup $\langle s_j \mid j \in I_{\mathrm{aff}}\rangle \subset \mathrm{GL}(\mathfrak{h}_{\mathrm{aff}}^*)$ generated by the simple reflections s_j associated to α_j for $j \in I_{\mathrm{aff}}$, with length function $\ell : W_{\mathrm{aff}} \to \mathbb{Z}_{\geq 0}$ and identity element $e \in W_{\mathrm{aff}}$. For $\xi \in Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$, let $t(\xi) \in W_{\mathrm{aff}}$ denote the translation in $\mathfrak{h}_{\mathrm{aff}}^*$ by ξ (see [Kac, §6.5]). Then we know from [Kac, Proposition 6.5] that $\left\{t(\xi) \mid \xi \in Q^\vee\right\}$ forms an abelian normal subgroup of W_{aff} such that $t(\xi)t(\zeta) = t(\xi+\zeta)$, $\xi, \zeta \in Q^\vee$, and $W_{\mathrm{aff}} = W \ltimes \left\{t(\xi) \mid \xi \in Q^\vee\right\}$. We denote by Δ_{aff} the set of real roots, i.e., $\Delta_{\mathrm{aff}} := \left\{x\alpha_j \mid x \in W_{\mathrm{aff}},\, j \in I_{\mathrm{aff}}\right\}$, and by $\Delta_{\mathrm{aff}}^+ \subset \Delta_{\mathrm{aff}}$ the set of positive real roots. We know from [Kac, Proposition 6.3] that

$$\Delta_{\mathrm{aff}} = \left\{\alpha + n\delta \mid \alpha \in \Delta,\, n \in \mathbb{Z}\right\},$$
$$\Delta_{\mathrm{aff}}^+ = \Delta^+ \sqcup \left\{\alpha + n\delta \mid \alpha \in \Delta,\, n \in \mathbb{Z}_{>0}\right\}.$$

For $\beta \in \Delta_{\mathrm{aff}}$, we denote by $\beta^\vee \in \mathfrak{h}_{\mathrm{aff}}$ the dual root of β and by $s_\beta \in W_{\mathrm{aff}}$ the reflection with respect to β. Note that if $\beta \in \Delta_{\mathrm{aff}}$ is of the form $\beta = \alpha + n\delta$ with $\alpha \in \Delta$ and $n \in \mathbb{Z}$, then $s_\beta = s_\alpha t(n\alpha^\vee)$.

4.2. **Peterson's coset representatives.** Let S be a subset of I. Following [Pe] (see also [LS, §10]), we set:

$$Q_S^\vee := \sum_{i \in S} \mathbb{Z}\alpha_i^\vee, \tag{4.1}$$

$$(\Delta_S)_{\mathrm{aff}} := \{\alpha + n\delta \mid \alpha \in \Delta_S, n \in \mathbb{Z}\} \subset \Delta_{\mathrm{aff}}, \tag{4.2}$$

$$(\Delta_S)_{\mathrm{aff}}^+ := (\Delta_S)_{\mathrm{aff}} \cap \Delta_{\mathrm{aff}}^+ = \Delta_S^+ \sqcup \{\alpha + n\delta \mid \alpha \in \Delta_S,\, n \in \mathbb{Z}_{>0}\}, \tag{4.3}$$

$$(W_S)_{\mathrm{aff}} := W_S \ltimes \{t(\xi) \mid \xi \in Q_S^\vee\} = \langle s_\beta \mid \beta \in (\Delta_S)_{\mathrm{aff}}^+ \rangle, \tag{4.4}$$

$$(W^S)_{\mathrm{aff}} := \{x \in W_{\mathrm{aff}} \mid x\beta \in \Delta_{\mathrm{aff}}^+ \text{ for all } \beta \in (\Delta_S)_{\mathrm{aff}}^+\}. \tag{4.5}$$

Then we know the following from [Pe] (see also [LS, Lemma 10.6]).

Proposition 4.2.1. *For each $x \in W_{\mathrm{aff}}$, there exist a unique $x_1 \in (W^S)_{\mathrm{aff}}$ and a unique $x_2 \in (W_S)_{\mathrm{aff}}$ such that $x = x_1x_2$.*

We define a (surjective) map $\Pi^S : W_{\mathrm{aff}} \to (W^S)_{\mathrm{aff}}$ by $\Pi^S(x) := x_1$ if $x = x_1x_2$ with $x_1 \in (W^S)_{\mathrm{aff}}$ and $x_2 \in (W_S)_{\mathrm{aff}}$.

Lemma 4.2.2 ([Pe]; see also [LS, Proposition 10.10])**.**

1. *$\Pi^S(w) = \lfloor w \rfloor$ for every $w \in W$.*
2. *$\Pi^S(xt(\xi)) = \Pi^S(x)\Pi^S(t(\xi))$ for every $x \in W_{\mathrm{aff}}$ and $\xi \in Q^\vee$.*

An element $\xi \in Q^\vee$ is said to be S-adjusted if $\langle \gamma, \xi \rangle \in \{-1, 0\}$ for all $\gamma \in \Delta_S^+$ (see [LNSSS1, Lemma 3.8]). Let $Q^{\vee, S\text{-ad}}$ denote the set of S-adjusted elements.

Lemma 4.2.3 ([INS, Lemma 2.3.5])**.**

1. *For each $\xi \in Q^\vee$, there exists a unique $\phi_S(\xi) \in Q_S^\vee$ such that $\xi + \phi_S(\xi) \in Q^{\vee, S\text{-ad}}$. In particular, $\xi \in Q^{\vee, S\text{-ad}}$ if and only if $\phi_S(\xi) = 0$.*
2. *For each $\xi \in Q^\vee$, the element $\Pi^S(t(\xi)) \in (W^S)_{\mathrm{aff}}$ is of the form $\Pi^S(t(\xi)) = z_\xi t(\xi + \phi_S(\xi))$ for a specific element $z_\xi \in W_S$. Also, $\Pi^S(wt(\xi)) = \lfloor w \rfloor z_\xi t(\xi + \phi_S(\xi))$ for every $w \in W$ and $\xi \in Q^\vee$.*
3. *We have*

$$(W^S)_{\mathrm{aff}} = \{wz_\xi t(\xi) \mid w \in W^S,\, \xi \in Q^{\vee, S\text{-ad}}\}. \tag{4.6}$$

Remark 4.2.4. (1) Let $\xi, \zeta \in Q^\vee$. If $\xi \equiv \zeta \mod Q_S^\vee$, i.e., $\xi - \zeta \in Q_S^\vee$, then $\Pi^S(t(\xi)) = \Pi^S(t(\zeta))$ since $t(\xi - \zeta) \in (W_S)_{\mathrm{aff}}$. Hence we see by Lemma 4.2.3 (2) that $\xi + \phi_S(\xi) = \zeta + \phi_S(\zeta)$ and $z_\xi = z_\zeta$. In particular, $z_{\xi+\phi_S(\xi)} = z_\xi$ for every $\xi \in Q^\vee$.

(2) Let $x = wz_\xi t(\xi) \in (W^S)_{\mathrm{aff}}$, with $w \in W^S$ and $\xi \in Q^{\vee, S\text{-ad}}$; note that $\Pi^S(x) = x$. Then it follows from Lemma 4.2.2 (2) that for every $\zeta \in Q^\vee$,

$$x\Pi^S(t(\zeta)) = \Pi^S(x)\Pi^S(t(\zeta)) = \Pi^S(xt(\zeta)) \in (W^S)_{\mathrm{aff}}. \tag{4.7}$$

4.3. **Parabolic semi-infinite Bruhat graph.** In this subsection, we prove some technical lemmas, which we use later.

Definition 4.3.1 ([Pe])**.** Let $x \in W_{\mathrm{aff}}$, and write it as $x = wt(\xi)$ for $w \in W$ and $\xi \in Q^\vee$. Then we define the semi-infinite length $\ell^{\frac{\infty}{2}}(x)$ of x by $\ell^{\frac{\infty}{2}}(x) := \ell(w) + 2\langle \rho, \xi \rangle$, where $\rho = (1/2)\sum_{\alpha \in \Delta^+} \alpha$.

Let us fix a subset S of I.

Definition 4.3.2. (1) We define the (parabolic) semi-infinite Bruhat graph SiBG^S to be the Δ^+_{aff}-labeled, directed graph with vertex set $(W^S)_{\mathrm{aff}}$ and Δ^+_{aff}-labeled, directed edges of the following form: $x \xrightarrow{\beta} s_\beta x$ for $x \in (W^S)_{\mathrm{aff}}$ and $\beta \in \Delta^+_{\mathrm{aff}}$, where $s_\beta x \in (W^S)_{\mathrm{aff}}$ and $\ell^{\frac{\infty}{2}}(s_\beta x) = \ell^{\frac{\infty}{2}}(x) + 1$.

(2) The semi-infinite Bruhat order is a partial order $\preceq$ on $(W^S)_{\mathrm{aff}}$ defined as follows: for $x, y \in (W^S)_{\mathrm{aff}}$, we write $x \preceq y$ if there exists a directed path from x to y in SiBG^S; also, we write $x \prec y$ if $x \preceq y$ and $x \neq y$.

Let $[\,\cdot\,] = [\,\cdot\,]_{I\setminus S} : Q^\vee \twoheadrightarrow Q^\vee_{I\setminus S}$ denote the projection from $Q^\vee$ onto $Q^\vee_{I\setminus S}$ with kernel $Q^\vee_S$. Also, for $\xi, \zeta \in Q^\vee$, we write

$$\xi \geq \zeta \;\text{ if }\; \xi - \zeta \in Q^{\vee,+} := \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i^\vee. \tag{4.8}$$

The next lemma follows from [NS3, Remark 2.3.3].

Lemma 4.3.3. *Let $u, v \in W^S$, $\xi, \zeta \in Q^{\vee, S\text{-ad}}$, and $\beta \in \Delta^+_{\mathrm{aff}}$. If $uz_\zeta t(\zeta) \xrightarrow{\beta} vz_\xi t(\xi)$ in SiBG^S, then $[\xi] \geq [\zeta]$.*

Lemma 4.3.4. *Let $x \in W^S$, and $\xi, \zeta \in Q^{\vee, S\text{-ad}}$. Then, $xz_\xi t(\xi) \succeq xz_\zeta t(\zeta)$ if and only if $[\xi] \geq [\zeta]$.*

Proof. The "only if" part is obvious by Lemma 4.3.3. We show the "if" part by induction on $\ell(x)$. If $\ell(x) = 0$, i.e., $x = e$, then the assertion $z_\xi t(\xi) \succeq z_\zeta t(\zeta)$ follows from [INS, Lemma 6.2.1] (with $a = 1$, and J replaced by S). Assume now that $\ell(x) > 0$, and take $i \in I$ such that $\ell(s_i x) = \ell(x) - 1$; note that $s_i x \in W^S$ and $-x^{-1}\alpha_i \in \Delta^+ \setminus \Delta^+_S$. By the induction hypothesis, we have $s_i x z_\xi t(\xi) \succeq s_i x z_\zeta t(\zeta)$. If we take a dominant weight $\lambda \in P^+$ such that $S_\lambda = \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \} = S$, then we see that

$$\langle s_i x z_\xi t(\xi)\lambda, \alpha_i^\vee \rangle = \langle s_i x z_\zeta t(\zeta)\lambda, \alpha_i^\vee \rangle = \langle s_i x \lambda, \alpha_i^\vee \rangle > 0.$$

Therefore, we deduce from [NS3, Lemma 2.3.6 (3)] that $xz_\xi t(\xi) \succeq xz_\zeta t(\zeta)$, as desired. □

Lemma 4.3.5. *Let $x, y \in (W^S)_{\mathrm{aff}}$ and $\beta \in \Delta^+_{\mathrm{aff}}$ be such that $x \xrightarrow{\beta} y$ in SiBG^S. Then, $\Pi^S(xt(\xi)) \xrightarrow{\beta} \Pi^S(yt(\xi))$ in SiBG^S for every $\xi \in Q^\vee$. Therefore, if $x, y \in (W^S)_{\mathrm{aff}}$ satisfy $x \preceq y$, then $\Pi^S(xt(\xi)) \preceq \Pi^S(yt(\xi))$.*

Proof. We see from (4.7) that $\Pi^S(xt(\xi)) = x\Pi^S(t(\xi))$ and $\Pi^S(yt(\xi)) = y\Pi^S(t(\xi))$. Since $y = s_\beta x$ by the assumption, we obtain $\Pi^S(yt(\xi)) = s_\beta \Pi^S(xt(\xi))$. Hence it suffices to show that

$$\ell^{\frac{\infty}{2}}(\Pi^S(yt(\xi))) = \ell^{\frac{\infty}{2}}(\Pi^S(xt(\xi))) + 1. \tag{4.9}$$

We write $x \in (W^S)_{\mathrm{aff}}$ as $x = wz_\zeta t(\zeta)$, with $w \in W^S$ and $\zeta \in Q^{\vee, S\text{-ad}}$ (see (4.6)). Then we see from [INS, Lemma A.2.1 and (A.2.1)] that

$$\begin{aligned} \ell^{\frac{\infty}{2}}(\Pi^S(xt(\xi))) &= \ell(w) + 2\langle \rho - \rho_S, \zeta + \xi \rangle \\ &= \ell(w) + 2\langle \rho - \rho_S, \zeta \rangle + 2\langle \rho - \rho_S, \xi \rangle \\ &= \ell^{\frac{\infty}{2}}(\Pi^S(x)) + 2\langle \rho - \rho_S, \xi \rangle \\ &= \ell^{\frac{\infty}{2}}(x) + 2\langle \rho - \rho_S, \xi \rangle. \end{aligned}$$

Similarly, we see that $\ell^{\frac{\infty}{2}}(\Pi^S(yt(\xi))) = \ell^{\frac{\infty}{2}}(y) + 2\langle \rho - \rho_S, \xi \rangle$. Since $\ell^{\frac{\infty}{2}}(y) = \ell^{\frac{\infty}{2}}(x) + 1$ by the assumption, we obtain (4.9), as desired. □

Let $x, y \in W^S$, and take a shortest directed path

$$\mathbf{p} : x = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} x_2 \xrightarrow{\gamma_3} \cdots \xrightarrow{\gamma_p} x_p = y$$

from x to y in QBG^S. Recall from §2.1 that the weight $\mathrm{wt}^S(\mathbf{p})$ of this directed path is defined to be

$$\mathrm{wt}^S(\mathbf{p}) = \sum_{\substack{1 \le k \le p \\ x_{k-1} \xrightarrow{\gamma_k} x_k \text{ is} \\ \text{a quantum edge}}} \gamma_k^\vee \in Q^{\vee,+}.$$

We set

$$\xi_{x,y} := \mathrm{wt}^S(\mathbf{p}) + \phi_S(\mathrm{wt}^S(\mathbf{p})) \in Q^{\vee,\, S\text{-ad}} \tag{4.10}$$

in the notation of Lemma 4.2.3 (1). We now claim that $\xi_{x,y}$ does not depend on the choice of a shortest directed path $\mathbf{p}$ from x to y in QBG^S. Indeed, let $\mathbf{p}'$ be another directed path from x to y in QBG^S. We know from [LNSSS1, Proposition 8.1] that $\mathrm{wt}^S(\mathbf{p}) = \mathrm{wt}^S(\mathbf{p}') \mod Q_S^\vee$. Therefore, by Remark 4.2.4(1), we obtain $\mathrm{wt}^S(\mathbf{p}) + \phi_S(\mathrm{wt}^S(\mathbf{p})) = \mathrm{wt}^S(\mathbf{p}') + \phi_S(\mathrm{wt}^S(\mathbf{p}'))$. This proves the claim.

Lemma 4.3.6. *Let $x, y \in W^S$. Then we have $yz_{\xi_{x,y}}t(\xi_{x,y}) \succeq x$.*

Proof. We proceed by induction on the length p of a shortest directed path from x to y in QBG^S. If $p = 0$, i.e., $x = y$, then $\xi_{x,y} = \xi_{x,x} = 0$, and hence $z_{\xi_{x,y}} = t(\xi_{x,y}) = e$. Thus the assertion of the lemma is obvious. Assume now that $p > 0$, and let

$$\mathbf{p} : x = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_p} x_p = y$$

be a shortest directed path from x to y in QBG^S. Then we deduce from [INS, Proposition A.1.2] that $x \xrightarrow{\beta} s_\beta x$ in SiBG^S (in particular, $s_\beta x \succeq x$), where

$$\beta := \begin{cases} x_0\gamma_1 & \text{if } x = x_0 \xrightarrow{\gamma_1} x_1 \text{ is a Bruhat edge,} \\ x_0\gamma_1 + \delta & \text{if } x = x_0 \xrightarrow{\gamma_1} x_1 \text{ is a quantum edge.} \end{cases}$$

Note that

$$s_\beta x = s_\beta x_0 = \begin{cases} x_1 & \text{if } x = x_0 \xrightarrow{\gamma_1} x_1 \text{ is a Bruhat edge,} \\ x_1 t(\gamma_1^\vee) & \text{if } x = x_0 \xrightarrow{\gamma_1} x_1 \text{ is a quantum edge.} \end{cases}$$

In the case that $x = x_0 \xrightarrow{\gamma_1} x_1$ is a quantum edge, we have $x_1 t(\gamma_1^\vee) = s_\beta x \in (W^S)_{\mathrm{aff}}$, which implies, by (4.6) and the fact that $x_1 \in W^S$, that

$$\gamma_1^\vee \in Q^{\vee,\, S\text{-ad}} \text{ and } z_{\gamma_1^\vee} = e. \tag{4.11}$$

Assume first that $x = x_0 \xrightarrow{\gamma_1} x_1$ is a Bruhat edge. Note that $\mathbf{p}' : x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_p} x_p = y$ is a shortest directed path from x_1 to y in QBG^S. Since $\mathrm{wt}^S(\mathbf{p}) = \mathrm{wt}^S(\mathbf{p}')$ by the definition, we deduce that $\xi_{x,y} = \xi_{x_1,y}$. Also, by the induction hypothesis, we have $yz_{\xi_{x_1,y}}t(\xi_{x_1,y}) \succeq x_1$. Combining these, we obtain $yz_{\xi_{x,y}}t(\xi_{x,y}) = yz_{\xi_{x_1,y}}t(\xi_{x_1,y}) \succeq x_1 = s_\beta x \succeq x$, as desired.

Next, assume that $x = x_0 \xrightarrow{\gamma_1} x_1$ is a quantum edge; we have $\mathrm{wt}^S(\mathbf{p}) = \mathrm{wt}^S(\mathbf{p}') + \gamma_1^\vee$, which implies that $\xi_{x,y} \equiv \xi_{x_1,y} + \gamma_1^\vee \mod Q_S^\vee$. We compute

$$\begin{aligned} yz_{\xi_{x,y}}t(\xi_{x,y}) &= y\Pi^S(t(\xi_{x,y})) \quad \text{by Lemma 4.2.3 (2)} \\ &= y\Pi^S(t(\xi_{x_1,y})t(\xi_{x,y} - \xi_{x_1,y})) \\ &= y\Pi^S(t(\xi_{x_1,y}))\Pi^S(t(\xi_{x,y} - \xi_{x_1,y})) \quad \text{by Lemma 4.2.2 (2)} \\ &= yz_{\xi_{x_1,y}}t(\xi_{x_1,y})\Pi^S(t(\xi_{x,y} - \xi_{x_1,y})). \end{aligned}$$

Since $\xi_{x,y} \equiv \xi_{x_1,y} + \gamma_1^\vee \mod Q_S^\vee$, we see from Remark 4.2.4 (1) and (4.11) that $\Pi^S(t(\xi_{x,y} - \xi_{x_1,y})) = t(\gamma_1^\vee)$. Therefore, using the induction hypothesis $yz_{\xi_{x_1,y}}t(\xi_{x_1,y}) \succeq x_1$ and Lemma 4.3.5, we deduce that

$$\begin{aligned} \underbrace{yz_{\xi_{x,y}}t(\xi_{x,y})}_{\in (W^S)_{\mathrm{aff}}} &= \big(yz_{\xi_{x_1,y}}t(\xi_{x_1,y})\big)t(\gamma_1^\vee) = \Pi^S\big(\big(yz_{\xi_{x_1,y}}t(\xi_{x_1,y})\big)t(\gamma_1^\vee)\big) \succeq \Pi^S(x_1 t(\gamma_1^\vee)) \\ &= \Pi^S(s_\beta x) = s_\beta x \succeq x. \end{aligned}$$

This proves the lemma. □

Lemma 4.3.7. *Let $x, y \in W^S$ and $\zeta \in Q^{\vee, S\text{-ad}}$. If $yz_\zeta t(\zeta) \succeq x$, then $[\zeta] \geq [\xi_{x,y}]$.*

Proof. We set

$$\widetilde{s}_j := \begin{cases} s_j & \text{if } j \neq 0, \\ s_\theta & \text{if } j = 0, \end{cases} \qquad \text{and} \qquad \widetilde{\alpha}_j := \begin{cases} \alpha_j & \text{if } j \neq 0, \\ -\theta & \text{if } j = 0. \end{cases}$$

We know from [LNSSS1, Lemma 6.12] that there exist a sequence $x = x_0, x_1, \ldots, x_n = e$ of elements of W^S and a sequence $i_1, \ldots, i_n \in I_{\mathrm{aff}} = I \sqcup \{0\}$ such that

$$x = x_0 \xrightarrow{x_0^{-1}\widetilde{\alpha}_{i_1}} x_1 \xrightarrow{x_1^{-1}\widetilde{\alpha}_{i_2}} \cdots \xrightarrow{x_{n-1}^{-1}\widetilde{\alpha}_{i_n}} x_n = e \quad \text{in } \mathrm{QBG}^S.$$

Note that $x_{k-1}^{-1}\widetilde{\alpha}_{i_k} \in \Delta^+ \setminus \Delta_S^+$ for all $1 \leq k \leq n$. We prove the assertion of the lemma by induction on n.

Assume first that $n = 0$, i.e., $x = e$. Because $y \in W^S$ is greater than or equal to e in the (ordinary) Bruhat order, there exists a directed path $\mathbf{p}$ from e to y in QBG^S whose edges are all Bruhat edges (see, e.g., [BB, Theorem 2.5.5]). Since $\mathrm{wt}^S(\mathbf{p}) = 0$, we obtain $\xi_{e,y} = \mathrm{wt}^S(\mathbf{p}) + \phi_S(\mathrm{wt}^S(\mathbf{p})) = 0$. Also, if $yz_\zeta t(\zeta) \succeq x = e = ez_0t(0)$, then it follows from Lemma 4.3.3 that $[\zeta] \geq [0] = [\xi_{e,y}]$, which proves the assertion in the case $n = 0$.

Assume next that $n > 0$; we set $i := i_1$ for simplicity of notation. Then, $x^{-1}\widetilde{\alpha}_i = x_0^{-1}\widetilde{\alpha}_i \in \Delta^+ \setminus \Delta_S^+$, and the assertion of the lemma holds for $x_1 = \widetilde{s}_i x_0 = \widetilde{s}_i x$ by the induction hypothesis.

Case 1. Assume that $y^{-1}\widetilde{\alpha}_i \in (-\Delta^+) \cup \Delta_S^+$. We deduce by [LNSSS1, Lemma 7.7 (3)] that

$$\xi_{\widetilde{s}_i x,y} \equiv \xi_{x,y} - \delta_{i,0}x^{-1}\widetilde{\alpha}_i^\vee \mod Q_S^\vee. \tag{4.12}$$

Assume first that $i \neq 0$. Let $\zeta \in Q^{\vee, S\text{-ad}}$ be such that $yz_\zeta t(\zeta) \succeq x$. Because $x^{-1}\alpha_i \in \Delta^+ \setminus \Delta_S^+$ and $y^{-1}\alpha_i \in (-\Delta^+) \cup \Delta_S^+$, we see from [INS, Lemma 4.1.6 (2)] that $yz_\zeta t(\zeta) \succeq s_i x = \widetilde{s}_i x$. Therefore, by the induction hypothesis, we obtain $[\zeta] \geq [\xi_{\widetilde{s}_i x,y}] \overset{(4.12)}{=} [\xi_{x,y}]$.

Assume next that $i = 0$. Let $\zeta \in Q^\vee$ be such that $yz_\zeta t(\zeta) \succeq x$. Because $x^{-1}\widetilde{\alpha}_0 = -x^{-1}\theta$ (= the finite part $\overline{x^{-1}\alpha_0}$ of $x^{-1}\alpha_0$) $\in \Delta^+ \setminus \Delta_S^+$ and $y^{-1}\widetilde{\alpha}_0 = -y^{-1}\theta$ (= the finite part $\overline{y^{-1}\alpha_0}$ of $y^{-1}\alpha_0$) $\in (-\Delta^+) \cup \Delta_S^+$, we see from [INS, Lemma 4.1.6 (2)] that

$$yz_\zeta t(\zeta) \succeq s_0 x = s_\theta x t(-x^{-1}\theta^\vee) = \underbrace{\widetilde{s}_0 x}_{=x_1} t(x^{-1}\widetilde{\alpha}_0^\vee).$$

Therefore, by Lemma 4.3.5,

$$\begin{aligned}\Pi^S(yz_\zeta t(\zeta - x^{-1}\widetilde{\alpha}_0^\vee)) &= \Pi^S\big((yz_\zeta t(\zeta))t(-x^{-1}\widetilde{\alpha}_0^\vee)\big)\\ &\succeq \Pi^S(\widetilde{s}_0 x t(x^{-1}\widetilde{\alpha}_0^\vee)t(-x^{-1}\widetilde{\alpha}_0^\vee)) = \Pi^S(\widetilde{s}_0 x)\\ &= \Pi^S(x_1) = x_1 = \widetilde{s}_0 x.\end{aligned}$$

If we write the left-hand side of this inequality as $\Pi^S(yz_\zeta t(\zeta - x^{-1}\widetilde{\alpha}_0^\vee)) = yz_{\zeta'}t(\zeta')$ for some $\zeta' \in Q^{\vee,\,S\text{-ad}}$ (see Lemma 4.2.3 (2)), then we have $\zeta' \equiv \zeta - x^{-1}\widetilde{\alpha}_0^\vee \mod Q_S^\vee$. Also, by the induction hypothesis, we have $[\zeta'] \geq [\xi_{\widetilde{s}_0 x, y}]$. Combining these, we obtain

$$[\zeta] = [\zeta' + x^{-1}\widetilde{\alpha}_0^\vee] \geq [\xi_{\widetilde{s}_0 x, y} + x^{-1}\widetilde{\alpha}_0^\vee] \overset{(4.12)}{=} [\xi_{x,y}],$$

as desired.

Case 2. Assume that $y^{-1}\widetilde{\alpha}_i \in \Delta^+ \setminus \Delta_S^+$. By [LNSSS1, Lemma 7.7 (4)], we have

$$\xi_{\widetilde{s}_i x, \lfloor \widetilde{s}_i y \rfloor} \equiv \xi_{x,y} - \delta_{i,0}x^{-1}\widetilde{\alpha}_i^\vee + \delta_{i,0}y^{-1}\widetilde{\alpha}_i^\vee \mod Q_S^\vee. \tag{4.13}$$

Assume first that $i \neq 0$; note that $\widetilde{s}_i y = s_i y \in W^S$ (see, e.g., [LNSSS1, Proposition 5.10]). Let $\zeta \in Q^\vee$ be such that $yz_\zeta t(\zeta) \succeq x$. Because $x^{-1}\alpha_i \in \Delta^+ \setminus \Delta_S^+$ and $y^{-1}\alpha_i \in \Delta^+ \setminus \Delta_S^+$, we see that

$$\widetilde{s}_i yz_\zeta t(\zeta) = s_i yz_\zeta t(\zeta) \succeq s_i x = \widetilde{s}_i x \quad \text{by [NS3, Lemma 2.3.6 (3)]}.$$

Therefore, by the induction hypothesis, we obtain $[\zeta] \geq [\xi_{\widetilde{s}_i x, \widetilde{s}_i y}] \overset{(4.13)}{=} [\xi_{x,y}]$.

Assume next that $i = 0$. Let $\zeta \in Q^\vee$ be such that $yz_\zeta t(\zeta) \succeq x$. Because $x^{-1}\widetilde{\alpha}_0 = -x^{-1}\theta$ (= the finite part $\overline{x^{-1}\alpha_0}$ of $x^{-1}\alpha_0$) $\in \Delta^+ \setminus \Delta_S^+$ and $y^{-1}\widetilde{\alpha}_0 = -y^{-1}\theta$ (= the finite part $\overline{y^{-1}\alpha_0}$ of $y^{-1}\alpha_0$) $\in \Delta^+ \setminus \Delta_S^+$, we see from [NS3, Lemma 2.3.6 (3)] that $s_0 yz_\zeta t(\zeta) \succeq s_0 x$. Therefore, by Lemma 4.3.5, we have

$$\Pi^S\big((s_0 yz_\zeta t(\zeta))t(-x^{-1}\widetilde{\alpha}_0^\vee)\big) \succeq \Pi^S\big((s_0 x)t(-x^{-1}\widetilde{\alpha}_0^\vee)\big).$$

Here we have

$$\Pi^S\big((s_0 x)t(-x^{-1}\widetilde{\alpha}_0^\vee)\big) = \Pi^S\big((\widetilde{s}_0 x t(x^{-1}\widetilde{\alpha}_0^\vee))t(-x^{-1}\widetilde{\alpha}_0^\vee)\big) = \widetilde{s}_0 x = x_1.$$

Also, using Lemma 4.2.3 (2), we compute

$$\begin{aligned}&\Pi^S\big((s_0 yz_\zeta t(\zeta))t(-x^{-1}\widetilde{\alpha}_0^\vee)\big) = \Pi^S(s_0 yz_\zeta t(\zeta - x^{-1}\widetilde{\alpha}_0^\vee))\\ &\quad = \Pi^S(s_0 yz_\zeta)\Pi^S(t(\zeta - x^{-1}\widetilde{\alpha}_0^\vee)) = \Pi^S(s_0 y)\Pi^S(t(\zeta - x^{-1}\widetilde{\alpha}_0^\vee))\\ &\quad = \Pi^S(\widetilde{s}_0 y t(y^{-1}\widetilde{\alpha}_0^\vee))\Pi^S(t(\zeta - x^{-1}\widetilde{\alpha}_0^\vee)) = \Pi^S(\widetilde{s}_0 y t(y^{-1}\widetilde{\alpha}_0^\vee)t(\zeta - x^{-1}\widetilde{\alpha}_0^\vee))\\ &\quad = \Pi^S(\widetilde{s}_0 y t(\zeta + y^{-1}\widetilde{\alpha}_0^\vee - x^{-1}\widetilde{\alpha}_0^\vee)).\end{aligned}$$

If we write this element as $\Pi^S\big((s_0 yz_\zeta t(\zeta))t(-x^{-1}\widetilde{\alpha}_0^\vee)\big) = \lfloor s_0 y \rfloor z_{\zeta''}t(\zeta'')$ for some $\zeta'' \in Q^{\vee,\,S\text{-ad}}$ (see Lemma 4.2.3 (2)), we see that $\zeta'' \equiv \zeta + y^{-1}\widetilde{\alpha}_0^\vee - x^{-1}\widetilde{\alpha}_0^\vee \mod Q_S^\vee$.

In addition, by the induction hypothesis, we have $[\zeta''] \ge [\xi_{\widetilde{s}_0 x, \lfloor \widetilde{s}_0 y \rfloor}]$. Combining these, we obtain

$$\begin{aligned}[\zeta] &= [\zeta'' - y^{-1}\widetilde{\alpha}_0^{\vee} + x^{-1}\widetilde{\alpha}_0^{\vee}] \\ &\ge [\xi_{\widetilde{s}_0 x, \lfloor \widetilde{s}_0 y \rfloor} - y^{-1}\widetilde{\alpha}_0^{\vee} + x^{-1}\widetilde{\alpha}_0^{\vee}] \overset{(4.13)}{=} [\xi_{x,y}],\end{aligned}$$

as desired. This completes the proof of the lemma.

□

4.4. **Semi-infinite Lakshmibai-Seshadri paths.** Let $\lambda \in P^+$ be a dominant weight; we set $S := S_\lambda = \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \} \subset I$.

Definition 4.4.1. For a rational number $0 < \sigma \le 1$, define $\mathrm{SiBG}(\lambda\,;\sigma)$ to be the subgraph of SiBG^S with the same vertex set but having only the edges of the form $x \xrightarrow{\beta} y$ with $\sigma\langle x\lambda, \beta^\vee\rangle \in \mathbb{Z}$; note that $\mathrm{SiBG}(\lambda\,;1) = \mathrm{SiBG}^S$.

Definition 4.4.2. A semi-infinite Lakshmibai-Seshadri (SiLS for short) path of shape λ is, by definition, a pair $\eta = (x_1 \succ \cdots \succ x_s\,;\, 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1)$ of a (strictly) decreasing sequence $x_1 \succ \cdots \succ x_s$ of elements in $(W^S)_{\mathrm{aff}}$ and an increasing sequence $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers such that there exists a directed path from x_{u+1} to x_u in $\mathrm{SiBG}(\lambda\,;\sigma_u)$ for all $u = 1, 2, \ldots, s-1$. We denote by $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ the set of all SiLS paths of shape λ.

Following [INS, §3.1] (see also [NS3, §2.4]), we endow the set $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ with a crystal structure with weights in P_{aff} by the root operators e_i, f_i, $i \in I_{\mathrm{aff}}$, and the map $\mathrm{wt} : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to P_{\mathrm{aff}}$ defined by

$$\mathrm{wt}(\eta) := \sum_{u=1}^{s} (\sigma_u - \sigma_{u-1}) x_u \lambda \in P_{\mathrm{aff}} \quad \text{for } \eta = (x_1, \ldots, x_s\,;\,\sigma_0, \sigma_1, \ldots, \sigma_s) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda). \tag{4.14}$$

Let $\mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$ denote the set of all connected components of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$, and let $\mathbb{B}_0^{\frac{\infty}{2}}(\lambda) \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$ denote the connected component of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ containing $\eta_e := (e\,;0, 1) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$.

Also, we define a surjective map $\mathrm{cl} : (W^S)_{\mathrm{aff}} \twoheadrightarrow W^S$ by

$$\mathrm{cl}(x) = w \quad \text{if } x = wz_\xi t(\xi), \text{ with } w \in W^S \text{ and } \xi \in Q^{\vee, S\text{-ad}},$$

and for $\eta = (x_1, \ldots, x_s\,;\,\sigma_0, \sigma_1, \ldots, \sigma_s) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$, we set

$$\mathrm{cl}(\eta) := (\mathrm{cl}(x_1), \ldots, \mathrm{cl}(x_s)\,;\,\sigma_0, \sigma_1, \ldots, \sigma_s),$$

where, for each $1 \le p < q \le s$ such that $\mathrm{cl}(x_p) = \cdots = \mathrm{cl}(x_q)$, we drop $\mathrm{cl}(x_p), \ldots, \mathrm{cl}(x_{q-1})$ and $\sigma_p, \ldots, \sigma_{q-1}$. We know from [NS3, §6.2] that $\mathrm{cl}(\eta) \in \mathrm{QLS}(\lambda)$. Thus we obtain a map $\mathrm{cl} : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to \mathrm{QLS}(\lambda)$.

Remark 4.4.3. Recall that $\psi_e := (e\,;0, 1) \in \mathrm{QLS}(\lambda)$. We see from the definition that an element in $\mathrm{cl}^{-1}(\psi_e)$ is of the form

$$(z_{\xi_1} t(\xi_1), z_{\xi_2} t(\xi_2), \ldots, z_{\xi_{s-1}} t(\xi_{s-1}), z_{\xi_s} t(\xi_s)\,;\,\sigma_0, \sigma_1, \ldots, \sigma_{s-1}, \sigma_s) \tag{4.15}$$

for some $s \ge 1$ and $\xi_1, \xi_2, \ldots, \xi_s \in Q^{\vee, S\text{-ad}}$.

The final direction of $\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is defined to be

$$\kappa(\eta) := x_s \in (W^S)_{\mathrm{aff}} \quad \text{if } \eta = (x_1, \ldots, x_s ; \sigma_0, \sigma_1, \ldots, \sigma_s). \tag{4.16}$$

Then, for $x \in (W^S)_{\mathrm{aff}}$, we set

$$\mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) := \{\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda) \mid \kappa(\eta) \succeq x\}. \tag{4.17}$$

The next lemma follows from [INS, Lemma 7.1.4].

Lemma 4.4.4. *Let $\eta \in \mathbb{B}^{\frac{\infty}{2}}_0(\lambda)$, and let X be a monomial in root operators such that $\eta = X\eta_e$. Assume that $\eta_0 \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is of the form (4.15). Then, $\kappa(X\eta_0) = \kappa(\eta)\kappa(\eta_0)$.*

Now, we recall from §3.2 the degree function $\deg_\lambda : \mathrm{QLS}(\lambda) \to \mathbb{Z}_{\le 0}$ for the case $\mu = \lambda$. We know the following lemma from [NS3, Lemma 6.2.3].

Lemma 4.4.5. *For each $\psi \in \mathrm{QLS}(\lambda)$, there exists a unique $\eta_\psi \in \mathbb{B}^{\frac{\infty}{2}}_0(\lambda)$ such that $\mathrm{cl}(\eta_\psi) = \psi$ and $\kappa(\eta_\psi) \in W^S$.*

Let $\psi \in \mathrm{QLS}(\lambda)$. We know from [NS3, (6.2.5)] that $\mathrm{wt}(\eta_\psi)$ is of the form

$$\mathrm{wt}(\eta_\psi) = \underbrace{\lambda - \gamma}_{=\mathrm{wt}(\psi)} + K\delta \quad \text{for some } \gamma \in Q^+ \text{ and } K \in \mathbb{Z}_{\le 0}. \tag{4.18}$$

Also, we know from [LNSSS2, Corollary 4.8] (see also the comment after [NS3, (6.2.5)]) that

$$K = -\sum_{u=1}^{s-1} \sigma_u \mathrm{wt}_\lambda(w_{u+1} \Rightarrow w_u) = \deg_\lambda(\psi) \tag{4.19}$$

for $\psi = (w_1, \ldots, w_s ; \sigma_0, \sigma_1, \ldots, \sigma_s) \in \mathrm{QLS}(\lambda)$. Here we should note that in the definition of $\deg_\lambda(\psi)$, $w_{s+1} = v(\lambda) = e$, and hence that $\mathrm{wt}_\lambda(w_{s+1} \Rightarrow w_s) = \mathrm{wt}_\lambda(e \Rightarrow w_s) = 0$.

Let us write a dominant weight $\lambda \in P^+$ as $\lambda = \sum_{i\in I} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\ge 0}$ for $i \in I$, and define $\overline{\mathrm{Par}(\lambda)}$ (resp., $\mathrm{Par}(\lambda)$) to be the set of I-tuples $\boldsymbol{\rho} = (\rho^{(i)})_{i\in I}$ of partitions such that $\rho^{(i)}$ is a partition of length less than or equal to m_i (resp., strictly less than m_i) for each $i \in I$. A partition of length less than 0 is understood to be the empty partition $\emptyset$; note that $\mathrm{Par}(\lambda) \subset \overline{\mathrm{Par}(\lambda)}$. Also, for $\boldsymbol{\rho} = (\rho^{(i)})_{i\in I} \in \overline{\mathrm{Par}(\lambda)}$, we set $|\boldsymbol{\rho}| := \sum_{i\in I} |\rho^{(i)}|$, where for a partition $\chi = (\chi_1 \ge \chi_2 \ge \cdots \ge \chi_m)$, we set $|\chi| := \chi_1 + \cdots + \chi_m$. Following [INS, (3.2.2)], we endow the set $\mathrm{Par}(\lambda)$ with a crystal structure with weights in P_{aff}; note that $\mathrm{wt}(\boldsymbol{\rho}) = -|\boldsymbol{\rho}|\delta$.

Proposition 4.4.6. *Keep the notation above.*

(1) *Each connected component $C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$ of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ contains a unique element of the form*

$$\eta^C = (z_{\xi_1}t(\xi_1), z_{\xi_2}t(\xi_2), \ldots, z_{\xi_{s-1}}t(\xi_{s-1}), e ; \sigma_0, \sigma_1, \ldots, \sigma_{s-1}, \sigma_s) \tag{4.20}$$

for some $s \ge 1$ and $\xi_1, \xi_2, \ldots, \xi_{s-1} \in Q^{\vee, S\text{-ad}}$ (see [INS, Proposition 7.1.2]).

(2) *There exists a bijection $\Theta : \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda)) \to \mathrm{Par}(\lambda)$ such that $\mathrm{wt}(\eta^C) = \lambda - |\Theta(C)|\delta$ (see [INS, Proposition 7.2.1 and its proof]).*

(3) *Let $C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda)$. Then, there exists an isomorphism $C \xrightarrow{\sim} \{\Theta(C)\} \otimes \mathbb{B}^{\frac{\infty}{2}}_0(\lambda)$ of crystals that maps η^C to $\Theta(C) \otimes \eta_e$. Consequently, $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is isomorphic as a crystal to $\mathrm{Par}(\lambda) \otimes \mathbb{B}^{\frac{\infty}{2}}_0(\lambda)$ (see [INS, Proposition 3.2.4 and its proof]).*

4.5. **Extremal weight modules.** In this and the next subsection, we mainly follow the notation of [NS3, §4 and §5]; we use the symbol "v" for the quantum parameter in order to distinguish it from $q = e^{\delta}$. Let $\lambda \in P^+$ be a dominant weight. We denote by $V(\lambda)$ the extremal weight module of extremal weight λ over a quantum affine algebra $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$. This is the integrable $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$-module generated by a single element v_λ with the defining relation that v_λ is an "extremal weight vector" of weight λ (for details, see [Kas1, §8] and [Kas2, §3]). We know from [Kas1, Proposition 8.2.2] that $V(\lambda)$ has a crystal basis $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ with global basis $\{G(b) \mid b \in \mathcal{B}(\lambda)\}$. Denote by u_λ the element of $\mathcal{B}(\lambda)$ such that $G(u_\lambda) = v_\lambda \in V(\lambda)$, and by $\mathcal{B}_0(\lambda)$ the connected component of $\mathcal{B}(\lambda)$ containing u_λ.

Let $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}}) \subset U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$ denote a quantum affine algebra without the degree operator. We know the following from [Kas2] (see also [NS3, §5.2]):

(i) for each $i \in I$, there exists a $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$-module automorphism $z_i : V(\varpi_i) \to V(\varpi_i)$ that maps v_{ϖ_i} to $v_{\varpi_i}^{[1]} := G(u_{\varpi_i}^{[1]})$, where $u_{\varpi_i}^{[1]} \in \mathcal{B}(\varpi_i)$ is a (unique) element of weight $\varpi_i + \delta$;

(ii) the map $z_i : V(\varpi_i) \to V(\varpi_i)$ induces a bijection $z_i : \mathcal{B}(\varpi_i) \to \mathcal{B}(\varpi_i)$ that maps u_{ϖ_i} to $u_{\varpi_i}^{[1]}$; this map commutes with the Kashiwara operators e_j, f_j, $j \in I_{\mathrm{aff}}$, on $\mathcal{B}(\varpi_i)$.

Let us write a dominant weight $\lambda \in P^+$ as $\lambda = \sum_{i \in I} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$ for $i \in I$. We fix an arbitrary total ordering on I, and then set $\widetilde{V}(\lambda) := \bigotimes_{i \in I} V(\varpi_i)^{\otimes m_i}$. By [BN, eq. (4.8) and Corollary 4.15], there exists a $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$-module embedding $\Phi_\lambda : V(\lambda) \hookrightarrow \widetilde{V}(\lambda)$ that maps v_λ to $\widetilde{v}_\lambda := \bigotimes_{i \in I} v_{\varpi_i}^{\otimes m_i}$. Also, for each $i \in I$ and $1 \leq k \leq m_i$, we define $z_{i,k}$ to be the $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$-module automorphism of $\widetilde{V}(\lambda)$ that acts as z_i only on the k-th factor of $V(\varpi_i)^{\otimes m_i}$ in $\widetilde{V}(\lambda)$ and as the identity map on the other factors of $\widetilde{V}(\lambda)$; these $z_{i,k}$'s, $i \in I$, $1 \leq k \leq m_i$, commute with each other. Now, for $\boldsymbol{\rho} = (\rho^{(i)})_{i \in I} \in \overline{\mathrm{Par}(\lambda)}$, we set

$$s_{\boldsymbol{\rho}}(z^{-1}) := \prod_{i \in I} s_{\rho^{(i)}}(z_{i,1}^{-1}, \ldots, z_{i,m_i}^{-1}). \tag{4.21}$$

Here, for a partition $\rho = (\rho_1 \geq \cdots \geq \rho_{m-1} \geq 0)$ of length less than $m \in \mathbb{Z}_{\geq 1}$, $s_\rho(x) = s_\rho(x_1, \ldots, x_m)$ denotes the Schur polynomial in the variables $x_1, \ldots, x_m$ corresponding to the partition ρ. We can easily show (see [NS3, §7.3]) that $s_{\boldsymbol{\rho}}(z^{-1})(\operatorname{Img} \Phi_\lambda) \subset \operatorname{Img} \Phi_\lambda$ for each $\boldsymbol{\rho} = (\rho^{(i)})_{i \in I} \in \overline{\mathrm{Par}(\lambda)}$. Hence we can define a $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$-module homomorphism $z_{\boldsymbol{\rho}} : V(\lambda) \to V(\lambda)$ in such a way that the following diagram commutes:

$$\begin{array}{ccc} V(\lambda) & \xrightarrow{\Phi_\lambda} & \widetilde{V}(\lambda) \\ {\scriptstyle z_{\boldsymbol{\rho}}}\downarrow & & \downarrow{\scriptstyle s_{\boldsymbol{\rho}}(z^{-1})} \\ V(\lambda) & \xrightarrow{\Phi_\lambda} & \widetilde{V}(\lambda). \end{array} \tag{4.22}$$

Note that $z_{\boldsymbol{\rho}} v_\lambda = S_{\boldsymbol{\rho}}^{-} v_\lambda$ in the notation of [BN] (and [NS3]). The map $z_{\boldsymbol{\rho}} : V(\lambda) \to V(\lambda)$ induces a $\mathbb{C}$-linear map $z_{\boldsymbol{\rho}} : \mathcal{L}(\lambda)/\mathsf{v}\mathcal{L}(\lambda) \to \mathcal{L}(\lambda)/\mathsf{v}\mathcal{L}(\lambda)$; this map commutes with Kashiwara operators. It follows from [BN, p. 371] that

$$\mathcal{B}(\lambda) = \{z_{\boldsymbol{\rho}} b \mid \boldsymbol{\rho} \in \mathrm{Par}(\lambda),\ b \in \mathcal{B}_0(\lambda)\}; \tag{4.23}$$

for $\boldsymbol{\rho} \in \mathrm{Par}(\lambda)$, we set

$$u^{\boldsymbol{\rho}} := z_{\boldsymbol{\rho}} u_\lambda \in \mathcal{B}(\lambda). \tag{4.24}$$

Remark 4.5.1. We see from [BN, Theorem 4.16 (ii)] (see also the argument after [NS3, (7.3.8)]) that $z_{\boldsymbol{\rho}} G(b) = G(z_{\boldsymbol{\rho}} b)$ for $b \in \mathcal{B}_0(\lambda)$ and $\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)}$.

4.6. **Demazure submodules.** Let $\lambda \in P^+$ be a dominant weight. For each $x \in W_{\mathrm{aff}}$, we set

$$V_x^-(\lambda) := U_{\mathsf{v}}^-(\mathfrak{g}_{\mathrm{aff}}) S_x^{\mathrm{norm}} v_\lambda \subset V(\lambda), \tag{4.25}$$

where $S_x^{\mathrm{norm}} v_\lambda$ denotes the extremal weight vector of weight $x\lambda$ (see, e.g., [NS3, (3.2.1)]), and $U_{\mathsf{v}}^-(\mathfrak{g}_{\mathrm{aff}})$ is the negative part of $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$. Since $V_x^-(\lambda) = V_{\Pi^S(x)}^-(\lambda)$ for $x \in W_{\mathrm{aff}}$ by [NS3, Lemma 4.1.2], we consider Demazure submodules $V_x^-(\lambda)$ only for $x \in (W^S)_{\mathrm{aff}}$ in what follows. We know from [Kas3, §2.8] and [NS3, §4.1] that $V_x^-(\lambda)$ is "compatible" with the global basis of $V(\lambda)$; namely, there exists a subset $\mathcal{B}_x^-(\lambda) \subset \mathcal{B}(\lambda)$ such that

$$V_x^-(\lambda) = \bigoplus_{b \in \mathcal{B}_x^-(\lambda)} \mathbb{C}(\mathsf{v}) G(b) \subset V(\lambda) = \bigoplus_{b \in \mathcal{B}(\lambda)} \mathbb{C}(\mathsf{v}) G(b). \tag{4.26}$$

We know the following theorem from [INS, Theorem 3.2.1] and [NS3, Theorem 4.2.1].

Theorem 4.6.1. *Let $\lambda \in P^+$ be a dominant weight. There exists an isomorphism $\Psi_\lambda : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ of crystals such that*

(a) $\Psi_\lambda(u^{\boldsymbol{\rho}}) = \eta^{\Theta^{-1}(\boldsymbol{\rho})}$ *for all* $\boldsymbol{\rho} \in \mathrm{Par}(\lambda)$ (*in particular,* $\Psi_\lambda(u_\lambda) = \eta_e$);

(b) $\Psi_\lambda(\mathcal{B}_x^-(\lambda)) = \mathbb{B}_{\succeq x}^{\frac{\infty}{2}}(\lambda)$ *for all* $x \in (W^S)_{\mathrm{aff}}$.

4.7. **Affine Weyl group action.** Let $\mathcal{B}$ be a regular crystal for $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$ in the sense of [Kas2, §2.2] (or [Kas1, p. 389]); in particular, as a crystal for $U_{\mathsf{v}}(\mathfrak{g}) \subset U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$, it decomposes into a disjoint union of ordinary highest weight crystals. By [Kas1, §7], the Weyl group W_{aff} acts on $\mathcal{B}$ by

$$s_j \cdot b := \begin{cases} f_j^n b & \text{if } n := \langle \mathrm{wt} b,\, \alpha_j^\vee \rangle \geq 0, \\ e_j^{-n} b & \text{if } n := \langle \mathrm{wt} b,\, \alpha_j^\vee \rangle \leq 0 \end{cases} \tag{4.27}$$

for $b \in \mathcal{B}$ and $j \in I_{\mathrm{aff}}$. Here we note that $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is a regular crystal for $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$ for a dominant weight $\lambda \in P^+$.

Remark 4.7.1 ([NS3, Remark 3.5.2]). Recall from Remark 4.4.3 that every element $\eta \in \mathrm{cl}^{-1}(\psi_e)$ is of the form (4.15). Then, for each $x \in W_{\mathrm{aff}}$,

$$x \cdot \eta = \bigl(\Pi^S(x z_{\xi_1} t(\xi_1)),\, \ldots,\, \Pi^S(x z_{\xi_s} t(\xi_s))\,;\, \sigma_0, \sigma_1, \ldots, \sigma_s\bigr), \tag{4.28}$$

where $S = S_\lambda = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$. In particular, we see by (4.28) and the uniqueness of η^C that $\eta = (z_{\xi_s} t(\xi_s)) \cdot \eta^C$, with $C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$ the connected component containing the η.

Remark 4.7.2. Let $\boldsymbol{\rho} = (\rho^{(i)})_{i \in I} \in \overline{\mathrm{Par}(\lambda)}$. Denote by $c_i \in \mathbb{Z}_{\geq 0}$, $i \in I$, the number of columns of length m_i in the Young diagram corresponding to the partition $\rho^{(i)}$, and set $\xi := \sum_{i \in I} c_i \alpha_i^\vee \in Q^{\vee,+}$; note that $c_i = 0$ for all $i \in S$. Also, for $i \in I$, let $\varrho^{(i)}$ denote the partition corresponding to the Young diagram obtained from that of $\rho^{(i)}$ by removing all columns of length m_i (i.e., the first c_i columns), and set $\boldsymbol{\varrho} := (\varrho^{(i)})_{i \in I}$; note that $\boldsymbol{\varrho} \in \mathrm{Par}(\lambda)$. Then we deduce from [BN, Lemma 4.14 and its proof] that

$$z_{\boldsymbol{\rho}} u_\lambda = t(\xi) \cdot (z_{\boldsymbol{\varrho}} u_\lambda) = t(\xi) \cdot u^{\boldsymbol{\varrho}}. \tag{4.29}$$

5. Graded character formulas for Demazure submodules and their certain quotients

5.1. Graded character formula for Demazure submodules. Fix a dominant weight $\lambda \in P^+$; recall that $S = S_\lambda = \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \}$.

Because every weight space of the Demazure submodule $V_x^-(\lambda)$ corresponding to $x \in W^S = W \cap (W^S)_{\text{aff}}$ is finite-dimensional, we can define the (ordinary) character $\operatorname{ch} V_x^-(\lambda)$ of $V_x^-(\lambda)$ by

$$\operatorname{ch} V_x^-(\lambda) := \sum_{\beta \in Q_{\text{aff}}} \dim V_x^-(\lambda)_{\lambda-\beta}\, e^{\lambda-\beta},$$

where $V_x^-(\lambda)_{\lambda-\beta}$ denotes the $(\lambda-\beta)$-weight space of $V_x^-(\lambda)$. Here we recall that an element $\beta \in Q_{\text{aff}}$ can be written uniquely in the form $\beta = \gamma + k\delta$ for $\gamma \in Q$ and $k \in \mathbb{Z}$. If we set $q := e^\delta$, then $e^{\lambda-\beta} = e^{\lambda-\gamma} q^{-k}$. Now we define the graded character $\operatorname{gch} V_x^-(\lambda)$ of $V_x^-(\lambda)$ to be

$$\operatorname{gch} V_x^-(\lambda) := \sum_{\gamma \in Q,\, k \in \mathbb{Z}} \dim V_x^-(\lambda)_{\lambda-\gamma-k\delta}\, e^{\lambda-\gamma} q^{-k},$$

which is obtained from the ordinary character $\operatorname{ch} V_x^-(\lambda)$ by replacing e^δ with q.

Theorem 5.1.1. *Keep the notation and setting above. Let $\lambda = \sum_{i \in I} m_i \varpi_i \in P^+$, and $x \in W^S$. The graded character $\operatorname{gch} V_x^-(\lambda)$ of $V_x^-(\lambda)$ can be expressed as*

$$\operatorname{gch} V_x^-(\lambda) = \left(\prod_{i \in I} \prod_{r=1}^{m_i} (1 - q^{-r})^{-1} \right) \sum_{\psi \in \text{QLS}(\lambda)} e^{\text{wt}(\psi)} q^{\deg_{x\lambda}(\psi)}. \tag{5.1}$$

By combining the special case $x = \lfloor w_\circ \rfloor \in W^S$ of Theorem 5.1.1 with the special case $\mu = w_\circ \lambda$ of Theorem 3.2.7, we obtain the following theorem. Recall from Remark 3.2.6 that $\text{QLS}^{w_\circ\lambda,\infty}(\lambda) = \text{QLS}(\lambda)$.

Theorem 5.1.2. *Let $\lambda \in P^+$ be a dominant weight of the form $\lambda = \sum_{i \in I} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$, $i \in I$. Then, the graded character $\operatorname{gch} V_{w_\circ}^-(\lambda)$ is equal to*

$$\left(\prod_{i \in I} \prod_{r=1}^{m_i} (1 - q^{-r})^{-1} \right) E_{w_\circ\lambda}(q, \infty).$$

Remark 5.1.3 ([NS3, Theorem 6.1.1]). We know from [LNSSS2, Theorem 7.9] that

$$P_\lambda(q^{-1}, 0) = \sum_{\psi \in \text{QLS}(\lambda)} e^{\text{wt}(\psi)} q^{\deg_\lambda(\psi)},$$

where $P_\lambda(q^{-1}, 0)$ is the specialization of the symmetric Macdonald polynomial $P_\lambda(q^{-1}, t)$ at $t = 0$. Also, by [LNSSS2, Lemma 7.7], we have $E_{w_\circ\lambda}(q^{-1}, 0) = P_\lambda(q^{-1}, 0)$. Therefore, it follows from the special case $x = e$ of Theorem 5.1.1 that the graded character $\operatorname{gch} V_e^-(\lambda)$ is equal to

$$\left(\prod_{i \in I} \prod_{r=1}^{m_i} (1 - q^{-r})^{-1} \right) E_{w_\circ\lambda}(q^{-1}, 0).$$

Note that we have $V_{w_\circ}^-(\lambda) \subset V_e^-(\lambda)$ by [NS3, Corollary 5.2.5].

5.2. **Proof of Theorem 5.1.1.** We see from Theorem 4.6.1 that

$$\operatorname{ch} V_x^-(\lambda) = \sum_{\eta \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)} e^{\mathrm{wt}(\eta)}.$$

Since

$$\mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) = \bigsqcup_{\psi \in \mathrm{QLS}(\lambda)} \big(\mathrm{cl}^{-1}(\psi) \cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)\big),$$

we deduce that

$$\operatorname{ch} V_x^-(\lambda) = \sum_{\psi \in \mathrm{QLS}(\lambda)} \underbrace{\Bigg(\sum_{\eta \in \mathrm{cl}^{-1}(\psi) \cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)} e^{\mathrm{wt}(\eta)} \Bigg)}_{(*)}. \tag{5.2}$$

In order to obtain the graded character formula (5.1) for $V_x^-(\lambda)$, we will compute the sum $(*)$ of the terms $e^{\mathrm{wt}(\eta)}$ over all $\eta \in \mathrm{cl}^{-1}(\psi) \cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$ for each $\psi \in \mathrm{QLS}(\lambda)$. Let $\psi \in \mathrm{QLS}(\lambda)$, and take $\eta_\psi \in \mathbb{B}^{\frac{\infty}{2}}_0(\lambda)$ as in Lemma 4.4.5. Let X be a monomial in root operators such that $\eta_\psi = X\eta_e$, where $\eta_e = (e\,;\,0,1)$. We see by [NS3, Lemma 6.2.2] that

$$\mathrm{cl}^{-1}(\psi) = \big\{ X(t(\zeta) \cdot \eta^C) \mid C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda)),\ \zeta \in Q^\vee \big\}; \tag{5.3}$$

for the definition of η^C, see (4.20). We claim that

$$\mathrm{cl}^{-1}(\psi) \cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) = \left\{ X(t(\zeta) \cdot \eta^C) \;\middle|\; \begin{array}{l} C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda)), \\ \zeta \in Q^\vee,\ [\zeta] \geq [\xi_{x,\kappa(\psi)}] \end{array} \right\}. \tag{5.4}$$

We first show the inclusion $\subset$. Let $\eta \in \mathrm{cl}^{-1}(\psi) \cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$, and write it as $\eta = X(t(\zeta) \cdot \eta^C)$ for some $C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$ and some $\zeta \in Q^\vee$ (see (5.3)). Also, we set $y := \kappa(\psi) = \kappa(\eta_\psi) \in W^S$. We see by (4.28) that $t(\zeta) \cdot \eta^C$ is of the form (4.15), with $\kappa(t(\zeta) \cdot \eta^C) = \Pi^S(t(\zeta)) = z_\zeta t(\zeta + \phi_S(\zeta))$. Therefore, we deduce from Lemma 4.4.4 that $\kappa(X(t(\zeta) \cdot \eta^C)) = \kappa(\eta_\psi)\kappa(t(\zeta) \cdot \eta^C) = y z_\zeta t(\zeta + \phi_S(\zeta))$. Since $\eta = X(t(\zeta) \cdot \eta^C) \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$ by the assumption, we have $y z_\zeta t(\zeta + \phi_S(\zeta)) \succeq x$. Hence it follows from Lemma 4.3.7 that $[\zeta] = [\zeta + \phi_S(\zeta)] \geq [\xi_{x,y}] = [\xi_{x,\kappa(\psi)}]$. Thus, η is contained in the set on the right-hand side of (5.4).

For the opposite inclusion $\supset$, let $C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$, and let $\zeta \in Q^\vee$ be such that $[\zeta] \geq [\xi_{x,\kappa(\psi)}]$. It is obvious by (5.3) that $X(t(\zeta) \cdot \eta^C) \in \mathrm{cl}^{-1}(\psi)$. Hence it suffices to show that $X(t(\zeta) \cdot \eta^C) \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$. The same argument as above shows that $\kappa(X(t(\zeta) \cdot \eta^C)) = y z_\zeta t(\zeta + \phi_S(\zeta))$, with $y := \kappa(\psi) \in W^S$. Therefore, we see that

$$\begin{aligned} \kappa(X(t(\zeta) \cdot \eta^C)) = y z_\zeta t(\zeta + \phi_S(\zeta)) &\succeq y z_{\xi_{x,y}} t(\xi_{x,y}) \quad \text{by Lemma 4.3.4} \\ &\succeq x \quad \text{by Lemma 4.3.6}, \end{aligned}$$

which implies that $X(t(\zeta) \cdot \eta^C) \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$. This proves (5.4).

Let $C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$, and write $\Theta(C) \in \mathrm{Par}(\lambda)$ as $\Theta(C) = (\rho^{(i)})_{i \in I}$, with $\rho^{(i)} = (\rho^{(i)}_1 \geq \cdots \geq \rho^{(i)}_{m_i - 1})$ for each $i \in I$. Also, let $\zeta \in Q^\vee$ be such that $[\zeta] \geq$

$[\xi_{x,\kappa(\psi)}]$, and write the difference $[\zeta] - [\xi_{x,\kappa(\psi)}] \in Q^{\vee,+}$ as

$$[\zeta] - [\xi_{x,\kappa(\psi)}] = \sum_{i \in I} c_i \alpha_i^\vee;$$

note that $c_i = 0$ for all $i \in S$. Now, for each $i \in I$, we set $c_i + \rho^{(i)} := (c_i + \rho_1^{(i)} \geq \cdots \geq c_i + \rho_{m_i-1}^{(i)} \geq c_i)$, which is a partition of length less than or equal to m_i, and then set

$$(c_i)_{i \in I} + \Theta(C) := (c_i + \rho^{(i)})_{i \in I} \in \overline{\mathrm{Par}(\lambda)}. \tag{5.5}$$

Noting that $\langle \lambda, Q_S^\vee \rangle = \{0\}$, we compute:

$$\begin{aligned}
\mathrm{wt}(t(\zeta) \cdot \eta^C) &= t(\zeta)(\mathrm{wt}(\eta^C)) \\
&= t(\zeta)\big(\lambda - |(\rho^{(i)})_{i \in I}|\delta\big) \quad \text{by Proposition 4.4.6 (2)} \\
&= \lambda - \langle \lambda, \zeta \rangle \delta - |(\rho^{(i)})_{i \in I}|\delta \\
&= \lambda - \langle \lambda, \xi_{x,\kappa(\psi)} \rangle \delta - \left\langle \lambda, \sum_{i \in I} c_i \alpha_i^\vee \right\rangle \delta - |(\rho^{(i)})_{i \in I}|\delta \\
&= \lambda - \mathrm{wt}_\lambda\big(x \Rightarrow \kappa(\psi)\big)\delta - \left(\sum_{i \in I} m_i c_i \right) \delta - |(\rho^{(i)})_{i \in I}|\delta \\
&= \mathrm{wt}(\eta_e) - \mathrm{wt}_\lambda\big(x \Rightarrow \kappa(\psi)\big)\delta - |(c_i + \rho^{(i)})_{i \in I}|\delta.
\end{aligned}$$

From this computation, together with (4.18), we deduce that

$$\begin{aligned}
\mathrm{wt}(X(t(\zeta) \cdot \eta^C)) &= \mathrm{wt}(X\eta_e) - \mathrm{wt}_\lambda\big(x \Rightarrow \kappa(\psi)\big)\delta - |(c_i + \rho^{(i)})_{i \in I}|\delta \\
&= \mathrm{wt}(\eta_\psi) - \mathrm{wt}_\lambda\big(x \Rightarrow \kappa(\psi)\big)\delta - |(c_i + \rho^{(i)})_{i \in I}|\delta \\
&= \mathrm{wt}(\psi) + \Big(\deg_\lambda(\psi) - \mathrm{wt}_\lambda\big(x \Rightarrow \kappa(\psi)\big)\Big)\delta - |(c_i + \rho^{(i)})_{i \in I}|\delta.
\end{aligned} \tag{5.6}$$

Because $\deg_\lambda(\psi) - \mathrm{wt}_\lambda\big(x \Rightarrow \kappa(\psi)\big) = \deg_{x\lambda}(\psi)$ by the definitions of $\deg_{x\lambda}(\psi)$ and $\deg_\lambda(\psi)$, we obtain

$$\mathrm{wt}(X(t(\zeta) \cdot \eta^C)) = \mathrm{wt}(\psi) + \big(\deg_{x\lambda}(\psi) - |(c_i + \rho^{(i)})_{i \in I}|\big)\delta.$$

Summarizing, we find that for each $\psi \in \mathrm{QLS}(\lambda)$,

$$\begin{aligned}
\sum_{\eta \in \mathrm{cl}^{-1}(\psi) \cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)} e^{\mathrm{wt}(\eta)} &\overset{(5.4)}{=} \sum_{\substack{C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda)) \\ \zeta \in Q^\vee,\, [\zeta] \geq [\xi_{x,\kappa(\psi)}]}} e^{\mathrm{wt}(X(t(\zeta) \cdot \eta^C))} \\
&= e^{\mathrm{wt}(\psi)} e^{\deg_{x\lambda}(\psi)\delta} \sum_{\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)}} x^{-|\boldsymbol{\rho}|\delta} \overset{e^\delta = q}{=} e^{\mathrm{wt}(\psi)} q^{\deg_{x\lambda}(\psi)} \sum_{\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)}} q^{-|\boldsymbol{\rho}|} \\
&= e^{\mathrm{wt}(\psi)} q^{\deg_{x\lambda}(\psi)} \prod_{i \in I} \prod_{r=1}^{m_i} (1 - q^{-r})^{-1}.
\end{aligned}$$

Substituting this into (5.2), we finally obtain (5.1). This completes the proof of Theorem 5.1.1. □

5.3. **Graded character formula for certain quotients of Demazure submodules.** Let $\lambda \in P^+$ be a dominant weight; recall that $S = S_\lambda = \bigl\{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \bigr\}$.

For each $x \in W^S = W \cap (W^S)_{\mathrm{aff}}$, we set

$$X_x^-(\lambda) := \sum_{\substack{\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \\ \boldsymbol{\rho} \neq (\emptyset)_{i \in I}}} U_{\mathsf{v}}^-(\mathfrak{g}_{\mathrm{aff}}) S_x^{\mathrm{norm}} z_{\boldsymbol{\rho}} v_\lambda = \sum_{\substack{\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \\ \boldsymbol{\rho} \neq (\emptyset)_{i \in I}}} z_{\boldsymbol{\rho}} \Bigl(V_x^-(\lambda) \Bigr); \tag{5.7}$$

for the definition of $z_{\boldsymbol{\rho}} : V(\lambda) \to V(\lambda)$, see (4.22).

For $\psi \in \mathrm{QLS}(\lambda)$, we take and fix a monomial X_ψ in root operators such that $X_\psi \eta_e = \eta_\psi$, and set

$$\eta_\psi \cdot t(\xi) := X_\psi (t(\xi) \cdot \eta_e) \quad \text{for } \xi \in Q^\vee.$$

Remark 5.3.1. Note that $t(\xi) \cdot \eta_e = (\Pi^S(t(\xi))\,;\, 0,\, 1)$ (see (4.28)). We deduce from [INS, Lemma 7.1.4] that if $\eta_\psi = X_\psi \eta_e$ is of the form $\eta_\psi = (x_1, \ldots, x_s\,;\, \sigma_0, \sigma_1, \ldots, \sigma_s)$, then

$$\eta_\psi \cdot t(\xi) = X_\psi (t(\xi) \cdot \eta_e) = (x_1 \Pi^S(t(\xi)), \ldots, x_s \Pi^S(t(\xi))\,;\, \sigma_0, \sigma_1, \ldots, \sigma_s).$$

In particular, the element $\eta_\psi \cdot t(\xi)$ does not depend on the choice of X_ψ. Also, since $x_u \Pi^S(t(\xi))\lambda = x_u \lambda - \langle \lambda, \xi \rangle \delta$ for all $1 \le u \le s$, we see by (4.14) that

$$\begin{aligned} \mathrm{wt}(\eta_\psi \cdot t(\xi)) &= \mathrm{wt}(\eta_\psi) - \langle \lambda, \xi \rangle \delta \\ &\overset{(4.18)}{=} \mathrm{wt}(\psi) + \bigl(\deg_\lambda(\psi) - \langle \lambda, \xi \rangle \bigr) \delta \end{aligned} \tag{5.8}$$

and that

$$\mathrm{cl}(\eta_\psi \cdot t(\xi)) = \psi. \tag{5.9}$$

Theorem 5.3.2. *Keep the notation and setting above. For each $x \in W^S$, there exists a subset $\mathcal{B}(X_x^-(\lambda))$ of $\mathcal{B}(\lambda)$ such that*

$$X_x^-(\lambda) = \bigoplus_{b \in \mathcal{B}(X_x^-(\lambda))} \mathbb{C}(\mathsf{v}) G(b). \tag{5.10}$$

Moreover, under the isomorphism $\Psi_\lambda : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ of crystals in Theorem 4.6.1, *the subset $\mathcal{B}(X_x^-(\lambda)) \subset \mathcal{B}(\lambda)$ is mapped to the following subset of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$:*

$$\mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) \setminus \bigl\{ \eta_\psi \cdot t(\xi_{x,\kappa(\psi)}) \mid \psi \in \mathrm{QLS}(\lambda) \bigr\}. \tag{5.11}$$

From Theorem 5.3.2, we immediately obtain the following corollary; cf. [NS3, Theorem 6.1.1 combined with Proposition 6.2.4] for the case $x = e$.

Corollary 5.3.3. *For each $x \in W^S$, there holds the equality*

$$\mathrm{gch}(V_x^-(\lambda)/X_x^-(\lambda)) = \sum_{\psi \in \mathrm{QLS}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_{x\lambda}(\psi)}. \tag{5.12}$$

By combining the special case $x = \lfloor w_\circ \rfloor \in W^S$ of Corollary 5.3.3 with the special case $\mu = w_\circ \lambda$ of Theorem 3.2.7, we obtain the equality

$$\mathrm{gch}(V_{w_\circ}^-(\lambda)/X_{w_\circ}^-(\lambda)) = E_{w_\circ \lambda}(q, \infty).$$

Remark 5.3.4. We recall from Remark 5.1.3 that

$$E_{w_\circ\lambda}(q^{-1},0) = \sum_{\psi\in\mathrm{QLS}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_\lambda(\psi)}.$$

Hence it follows from the special case $x = e$ of Corollary 5.3.3 that

$$\mathrm{gch}(V_e^-(\lambda)/X_e^-(\lambda)) = E_{w_\circ\lambda}(q^{-1},0);$$

cf. [LNSSS3, Theorem 35]. Here we have $V_{w_\circ}^-(\lambda) \subset V_e^-(\lambda)$, as mentioned in Remark 5.1.3. However, we can easily show that $X_e^-(\lambda)\cap V_{w_\circ}^-(\lambda) \supsetneq X_{w_\circ}^-(\lambda)$ (except for some trivial cases). Therefore, there is no inclusion relation between the quotient modules $V_{w_\circ}^-(\lambda)/X_{w_\circ}^-(\lambda)$ and $V_e^-(\lambda)/X_e^-(\lambda)$. This can also be observed from the comparison of some explicit computations of $E_{w_\circ\lambda}(q^{-1},0)$ and $E_{w_\circ\lambda}(q,\infty)$.

5.4. **Proof of Theorem 5.3.2.**

Lemma 5.4.1 (cf. (4.23))**.** *Let $x \in W^S$. Then, we have*

$$\mathcal{B}_x^-(\lambda) = \big\{ z_{\boldsymbol\rho} b \mid \boldsymbol\rho \in \mathrm{Par}(\lambda),\ b \in \mathcal{B}_x^-(\lambda)\cap\mathcal{B}_0(\lambda)\big\}. \tag{5.13}$$

Moreover, for every $\boldsymbol\rho \in \overline{\mathrm{Par}(\lambda)}$ and $b \in \mathcal{B}_x^-(\lambda)\cap\mathcal{B}_0(\lambda)$, the element $z_{\boldsymbol\rho} b$ is contained in $\mathcal{B}_x^-(\lambda)$.

Proof. We first prove the inclusion $\supset$. Let $b \in \mathcal{B}_x^-(\lambda)\cap\mathcal{B}_0(\lambda)$, and write it as $b = Xu_\lambda$ for a monomial X in Kashiwara operators. For $\boldsymbol\rho \in \mathrm{Par}(\lambda)$, we have $z_{\boldsymbol\rho} b = Xz_{\boldsymbol\rho}u_\lambda = Xu^{\boldsymbol\rho}$ since $z_{\boldsymbol\rho}$ commutes with Kashiwara operators (see §4.5). Now we set $\eta := \Psi_\lambda(b)$ and $\eta' := \Psi_\lambda(z_{\boldsymbol\rho} b)$, where $\Psi_\lambda : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is the isomorphism of crystals in Theorem 4.6.1. Then, we have $\eta = X\eta_e$ and $\eta' = X\Psi_\lambda(u^{\boldsymbol\rho}) = X\eta^C$, with $C := \Theta^{-1}(\boldsymbol\rho) \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$. Therefore, noting that $\kappa(\eta^C) = e$, we deduce from Lemma 4.4.4 that $\kappa(\eta') = \kappa(\eta)\kappa(\eta^C) = \kappa(\eta)$. Also, since $b \in \mathcal{B}_x^-(\lambda)$, it follows that $\kappa(\eta) \succeq x$, and hence $\kappa(\eta') = \kappa(\eta) \succeq x$. Hence we obtain $\eta' \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$, which implies that $z_{\boldsymbol\rho} b \in \mathcal{B}_x^-(\lambda)$.

Next we prove the opposite inclusion $\subset$. Let $b' \in \mathcal{B}_x^-(\lambda)$, and write it as $b' = z_{\boldsymbol\rho} b$ for some $\boldsymbol\rho \in \mathrm{Par}(\lambda)$ and $b \in \mathcal{B}_0(\lambda)$ (see (4.23)); we need to show that $b \in \mathcal{B}_x^-(\lambda)$. We set $\eta := \Psi_\lambda(b) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ and $\eta' := \Psi_\lambda(b') \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$. Then, the same argument as above shows that $\kappa(\eta) = \kappa(\eta') \succeq x$. Hence we obtain $\eta \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$, which implies that $b \in \mathcal{B}_x^-(\lambda)$.

For the second assertion, let $\boldsymbol\rho = (\rho^{(i)})_{i\in I} \in \overline{\mathrm{Par}(\lambda)}$ and $b \in \mathcal{B}_x^-(\lambda)\cap\mathcal{B}_0(\lambda)$; remark that

$$z_{\boldsymbol\rho} b \in \mathcal{B}_x^-(\lambda) \iff \Psi_\lambda(z_{\boldsymbol\rho} b) \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) \iff \kappa(\Psi_\lambda(z_{\boldsymbol\rho} b)) \succeq x.$$

We write b as $b = Xu_\lambda$ for a monomial X in Kashiwara operators. Also, define $\boldsymbol\varrho := (\varrho^{(i)})_{i\in I} \in \mathrm{Par}(\lambda)$ and $\xi := \sum_{i\in I} c_i\alpha_i^\vee \in Q^{\vee,+}$ as in Remark 4.7.2. Then it follows that $z_{\boldsymbol\rho} b = z_{\boldsymbol\rho} Xu_\lambda = Xz_{\boldsymbol\rho}u_\lambda \overset{(4.29)}{=} X(t(\xi)\cdot u^{\boldsymbol\varrho})$. If we set $C := \Theta^{-1}(\boldsymbol\varrho) \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$, then we have

$$\Psi_\lambda(z_{\boldsymbol\rho} b) = \Psi_\lambda\big(X(t(\xi)\cdot u^{\boldsymbol\varrho})\big) = X\big(t(\xi)\cdot\Psi_\lambda(u^{\boldsymbol\varrho})\big) = X\big(t(\xi)\cdot\eta^C\big).$$

Note that $t(\xi)\cdot\eta^C$ is of the form (4.15) with $\kappa(t(\xi)\cdot\eta^C) = \Pi^S(t(\xi))$ by Remark 4.7.1 and the fact that $\kappa(\eta^C) = e$. Therefore, we see from Lemma 4.4.4 that

$$\kappa(\Psi_\lambda(z_{\boldsymbol\rho} b)) = \kappa(X(t(\xi)\cdot\eta^C)) = \kappa(X\eta_e)\Pi^S(t(\xi)). \tag{5.14}$$

Here we recall that $\kappa(X\eta_e) \succeq x$ since $b \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda)$. Also, recall that $\xi \in Q^{\vee,+}$. From these, we deduce that

$$\begin{aligned}\kappa(\Psi_\lambda(z_{\boldsymbol{\rho}} b)) &= \kappa(X\eta_e)\Pi^S(t(\xi)) \succeq \kappa(X\eta_e) \quad \text{by Lemma 4.3.4} \\ &\succeq x.\end{aligned}$$

This proves the lemma. □

Proof of Theorem 5.3.2. We will prove that if we set

$$\mathcal{B} := \left\{ z_{\boldsymbol{\rho}} b \mid \boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \setminus (\emptyset)_{i\in I},\, b \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda) \right\} \subset \mathcal{B}(\lambda), \tag{5.15}$$

then

$$X_x^-(\lambda) = \bigoplus_{b\in\mathcal{B}} \mathbb{C}(\mathsf{v}) G(b). \tag{5.16}$$

We first show the inclusion $\supset$ in (5.16). Let $\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \setminus (\emptyset)_{i\in I}$ and $b \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda)$. We see from Remark 4.5.1 that $G(z_{\boldsymbol{\rho}} b) = z_{\boldsymbol{\rho}} G(b)$. Since $G(b) \in V_x^-(\lambda)$ and

$$X_x^-(\lambda) = \sum_{\substack{\boldsymbol{\rho}\in\overline{\mathrm{Par}(\lambda)} \\ \boldsymbol{\rho}\neq(\emptyset)_{i\in I}}} z_{\boldsymbol{\rho}}\Big(V_x^-(\lambda)\Big)$$

by the definition, we have $G(z_{\boldsymbol{\rho}} b) = z_{\boldsymbol{\rho}} G(b) \in X_x^-(\lambda)$. Thus we have shown the inclusion $\supset$ in (5.16). Next we show the opposite inclusion $\subset$ in (5.16). Since $\left\{G(b) \mid b \in \mathcal{B}_x^-(\lambda)\right\}$ is a $\mathbb{C}(\mathsf{v})$-basis of $V_x^-(\lambda)$, we deduce from (5.7) that

$$X_x^-(\lambda) = \mathrm{Span}_{\mathbb{C}(\mathsf{v})}\left\{ z_{\boldsymbol{\rho}} G(b) \mid \boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \setminus (\emptyset)_{i\in I},\, b \in \mathcal{B}_x^-(\lambda)\right\}. \tag{5.17}$$

Let $\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \setminus (\emptyset)_{i\in I}$ and $b \in \mathcal{B}_x^-(\lambda)$. By Lemma 5.4.1, we can write the b as $b = z_{\boldsymbol{\rho}'} b'$ for some $\boldsymbol{\rho}' \in \mathrm{Par}(\lambda)$ and $b' \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda)$. It follows that $z_{\boldsymbol{\rho}} b = z_{\boldsymbol{\rho}} z_{\boldsymbol{\rho}'} b'$. Because $z_{\boldsymbol{\rho}}$ and $z_{\boldsymbol{\rho}'}$ are defined to be a certain product of Schur polynomials (see (4.21)), the element $z_{\boldsymbol{\rho}} z_{\boldsymbol{\rho}'}$ can be expressed as

$$z_{\boldsymbol{\rho}} z_{\boldsymbol{\rho}'} = \sum_{\substack{\boldsymbol{\rho}''\in\overline{\mathrm{Par}(\lambda)} \\ |\boldsymbol{\rho}''|=|\boldsymbol{\rho}|+|\boldsymbol{\rho}'|}} n_{\boldsymbol{\rho}''} z_{\boldsymbol{\rho}''}, \quad \text{with } n_{\boldsymbol{\rho}''} \in \mathbb{Z};$$

here we remark that $|\boldsymbol{\rho}| + |\boldsymbol{\rho}'| \geq 1$ since $\boldsymbol{\rho} \neq (\emptyset)_{i\in I}$. Therefore, we deduce that

$$\begin{aligned} z_{\boldsymbol{\rho}} G(b) &= z_{\boldsymbol{\rho}} G(z_{\boldsymbol{\rho}'} b') = z_{\boldsymbol{\rho}} z_{\boldsymbol{\rho}'} G(b') \\ &= \sum_{\substack{\boldsymbol{\rho}''\in\overline{\mathrm{Par}(\lambda)} \\ |\boldsymbol{\rho}''|=|\boldsymbol{\rho}|+|\boldsymbol{\rho}'|}} n_{\boldsymbol{\rho}''} G(z_{\boldsymbol{\rho}''} b') \in \bigoplus_{b\in\mathcal{B}} \mathbb{C}(\mathsf{v}) G(b).\end{aligned}$$

From this, together with (5.17), we obtain the inclusion $X_x^-(\lambda) \subset \bigoplus_{b\in\mathcal{B}} \mathbb{C}(\mathsf{v}) G(b)$ in (5.16). Thus, we obtain (5.16), as desired. In what follows, we write $\mathcal{B}(X_x^-(\lambda))$ for the subset $\mathcal{B} \subset \mathcal{B}(\lambda)$ in (5.15).

Furthermore, we will prove that

$$\Psi_\lambda\big(\mathcal{B}(X_x^-(\lambda))\big) = \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) \setminus \left\{ \eta_\psi \cdot t(\xi_{x,\kappa(\psi)}) \mid \psi \in \mathrm{QLS}(\lambda)\right\}.$$

For this purpose, it suffices to show that for each $\psi \in \mathrm{QLS}(\lambda)$,

$$\mathrm{cl}^{-1}(\psi) \cap \Psi_\lambda\big(\mathcal{B}(X_x^-(\lambda))\big) = \Big(\mathrm{cl}^{-1}(\psi) \cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)\Big) \setminus \left\{ \eta_\psi \cdot t(\xi_{x,\kappa(\psi)})\right\}. \tag{5.18}$$

Let $\psi \in \mathrm{QLS}(\lambda)$; recall that X_ψ is a monomial in root operators such that $\eta_\psi = X_\psi \eta_e$. Then we know from (5.4) that

$$\begin{aligned}&\mathrm{cl}^{-1}(\psi) \cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)\\&\quad = \bigl\{ X_\psi(t(\zeta) \cdot \eta^C) \mid C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda)),\, \zeta \in Q^\vee,\, [\zeta] \ge [\xi_{x,\kappa(\psi)}] \bigr\}.\end{aligned}$$

We first show the inclusion $\supset$ in (5.18). Let η be an element in the set on the right-hand side of (5.18), and write it as $\eta = X_\psi(t(\zeta) \cdot \eta^C)$ for some $C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$ and $\zeta \in Q^\vee$ such that $[\zeta] \ge [\xi_{x,\kappa(\psi)}]$. We write the difference $[\zeta] - [\xi_{x,\kappa(\psi)}] \in Q^{\vee,+}$ as $[\zeta] - [\xi_{x,\kappa(\psi)}] = \sum_{i \in I} c_i \alpha_i^\vee$ with $c_i \in \mathbb{Z}_{\ge 0}$ for $i \in I$ (note that $c_i = 0$ for all $i \in S$), and define $\boldsymbol{\rho} := (c_i)_{i \in I} + \Theta(C) \in \overline{\mathrm{Par}(\lambda)}$ as in (5.5). We claim that $\boldsymbol{\rho} \ne (\emptyset)_{i \in I}$. Suppose, for a contradiction, that $\boldsymbol{\rho} = (\emptyset)_{i \in I}$. Then we have $\Theta(C) = (\emptyset)_{i \in I}$ and $c_i = 0$ for all $i \in I$, and hence

$$\begin{aligned}\eta &= X_\psi(t(\zeta) \cdot \eta^C) = X_\psi(t(\zeta) \cdot \eta_e) = X_\psi(\Pi^S(t(\zeta))\,;\, 0,\, 1)\\&= X_\psi(\Pi^S(t(\xi_{x,\kappa(\psi)}))\,;\, 0,\, 1) \quad \text{since } [\zeta] = [\xi_{x,\kappa(\psi)}]\\&= X_\psi(t(\xi_{x,\kappa(\psi)}) \cdot \eta_e) = \eta_\psi \cdot t(\xi_{x,\kappa(\psi)}),\end{aligned}$$

which contradicts the assumption that η is an element in the set on the right-hand side of (5.18). Thus we obtain $\boldsymbol{\rho} \ne (\emptyset)_{i \in I}$. Now, we set

$$b := \Psi_\lambda^{-1}(\eta_\psi \cdot t(\xi_{x,\kappa(\psi)})) = \Psi_\lambda^{-1}\bigl(X_\psi(t(\xi_{x,\kappa(\psi)}) \cdot \eta_e)\bigr) \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda);$$

note that $\eta_\psi \cdot t(\xi_{x,\kappa(\psi)}) \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$ by (5.4) and that $b = X_\psi(t(\xi_{x,\kappa(\psi)}) \cdot u_\lambda)$. Then we see by (5.15) that $z_{\boldsymbol{\rho}} b \in \mathcal{B}(X_x^-(\lambda))$. Also, we have

$$\begin{aligned}z_{\boldsymbol{\rho}} b &= z_{\boldsymbol{\rho}}\bigl(X_\psi(t(\xi_{x,\kappa(\psi)}) \cdot u_\lambda)\bigr) = X_\psi\bigl(t(\xi_{x,\kappa(\psi)}) \cdot (z_{\boldsymbol{\rho}} u_\lambda)\bigr)\\&= X_\psi\bigl(t(\xi_{x,\kappa(\psi)}) \cdot t([\zeta] - [\xi_{x,\kappa(\psi)}]) \cdot u^{\Theta(C)}\bigr) \quad \text{by Remark 4.7.2}\\&= X_\psi(t(\zeta + \gamma) \cdot u^{\Theta(C)}) \quad \text{for some } \gamma \in Q_S^\vee\\&= X_\psi(t(\zeta) \cdot u^{\Theta(C)}).\end{aligned}$$

Therefore, $\Psi_\lambda(z_{\boldsymbol{\rho}} b) = X_\psi(t(\zeta) \cdot \eta^C) = \eta$, which implies that η is contained in $\Psi_\lambda(\mathcal{B}(X_x^-(\lambda)))$. Thus we have shown the inclusion $\supset$ in (5.18).

Next we show the opposite inclusion $\subset$ in (5.18). Since $\mathcal{B}(X_x^-(\lambda)) \subset \mathcal{B}_x^-(\lambda)$, it follows that

$$\mathrm{cl}^{-1}(\psi) \cap \Psi_\lambda\bigl(\mathcal{B}(X_x^-(\lambda))\bigr) \subset \mathrm{cl}^{-1}(\psi) \cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda).$$

Hence it suffices to show that $\eta_\psi \cdot t(\xi_{x,\kappa(\psi)}) \notin \Psi_\lambda\bigl(\mathcal{B}(X_x^-(\lambda))\bigr)$. Suppose, for a contradiction, that there exists $b' \in \mathcal{B}(X_x^-(\lambda))$ such that $\Psi_\lambda(b') = \eta_\psi \cdot t(\xi_{x,\kappa(\psi)})$. By (5.15), we can write it as $b' = z_{\boldsymbol{\rho}} b$ for some $\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \setminus (\emptyset)_{i \in I}$ and $b \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda)$. We set $\eta := \Psi_\lambda^{-1}(b) \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) \cap \mathbb{B}^{\frac{\infty}{2}}_0(\lambda)$ and write $\kappa(\eta) \in (W^S)_{\mathrm{aff}}$ as $\kappa(\eta) = y z_\xi t(\xi)$ for some $y \in W^S$ and $\xi \in Q^{\vee,\, S\text{-ad}}$. Then, $\kappa(\eta) = y z_\xi t(\xi) \succeq x$ since $\eta \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$, and hence

$$[\xi] \ge [\xi_{x,y}] \qquad \text{by Lemma 4.3.7.} \tag{5.19}$$

Let us write b as $b = Y u_\lambda$ for some monomial Y in Kashiwara operators (note that $\eta = Y \eta_e$), and define $\zeta = \sum_{i \in I} c_i \alpha_i^\vee \in Q^{\vee,+}$ and $\boldsymbol{\varrho} = (\varrho^{(i)})_{i \in I} \in \mathrm{Par}(\lambda)$ in such a way that $\boldsymbol{\rho} = (c_i)_{i \in I} + \boldsymbol{\varrho}$ (see Remark 4.7.2 and (5.5)); note that $c_i = 0$ for all $i \in S$. Then, by (4.29), we have

$$b' = z_{\boldsymbol{\rho}} b = z_{\boldsymbol{\rho}} Y u_\lambda = Y z_{\boldsymbol{\rho}} u_\lambda = Y(t(\zeta) \cdot u^{\boldsymbol{\varrho}}).$$

Therefore, we see that

$$\begin{aligned} &\eta_\psi \cdot t(\xi_{x,\kappa(\psi)}) = \Psi_\lambda(b') = \Psi_\lambda\big(Y(t(\zeta)\cdot u^{\boldsymbol{\varrho}})\big) = Y(t(\zeta)\cdot \eta^C), \\ &\text{with } C := \Theta^{-1}(\boldsymbol{\varrho}) \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda)). \end{aligned} \tag{5.20}$$

Since $\eta_\psi \cdot t(\xi_{x,\kappa(\psi)}) = X_\psi(t(\xi_{x,\kappa(\psi)}) \cdot \eta_e) \in \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$, it follows that $\eta^C = \eta_e$, and hence $\boldsymbol{\varrho} = (\emptyset)_{i\in I}$. Hence we obtain $\eta_\psi \cdot t(\xi_{x,\kappa(\psi)}) = Y(t(\zeta)\cdot\eta_e)$. Since $t(\zeta)\cdot\eta_e = (\Pi^S(t(\zeta))\,;\,0,\,1)$, we see from Lemma 4.4.4 that $\kappa(Y(t(\zeta)\cdot\eta_e)) = \kappa(\eta)\kappa(t(\zeta)\cdot\eta_e) = yz_\xi t(\xi)\Pi^S(t(\zeta))$. Similarly, we see that $\kappa(\eta_\psi \cdot t(\xi_{x,\kappa(\psi)})) = \kappa(\psi)\Pi^S(t(\xi_{x,\kappa(\psi)}))$. Combining these equalities, we obtain $\kappa(\psi)\Pi^S(t(\xi_{x,\kappa(\psi)})) = yz_\xi t(\xi)\Pi^S(t(\zeta))$, and hence ($y = \kappa(\psi)$ and) $[\zeta+\xi] = [\xi_{x,\kappa(\psi)}]$. Since $[\xi] \geq [\xi_{x,y}]$ by (5.19) and $\zeta \in Q^{\vee,+}$, it follows that ($[\xi] = [\xi_{x,y}]$ and) $[\zeta] = 0$, which implies that $c_i = 0$ for all $i \in I \setminus S$. Recall that $c_i = 0$ for all $i \in S$ by the definition. Therefore, we conclude that $\boldsymbol{\rho} = (c_i)_{i\in I} + \boldsymbol{\varrho} = (\emptyset)_{i\in I}$; this contradicts our assumption that $\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \setminus (\emptyset)_{i\in I}$. Thus we have shown the inclusion $\subset$ in (5.18). This completes the proof of Theorem 5.3.2. □

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