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NONDIVERGENCE PARABOLIC EQUATIONS IN WEIGHTED VARIABLE EXPONENT SPACES

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ABSTRACT. We prove the global Calderón-Zygmund estimates for second order parabolic equations in nondivergence form in weighted variable exponent Lebesgue spaces. We assume that the associated variable exponent is log-Hölder continuous, the weight is of a certain Muckenhoupt class with respect to the variable exponent, the coefficients of the equation are the functions of small bonded mean oscillation, and the underlying domain is a $C^{1,1}$-domain.

1. INTRODUCTION

This paper is devoted to the study of the following Dirichlet problem for a second order parabolic equation in nondivergence form:

$$\begin{cases} u_t - a_{ij}D_{ij}u = f & \text{in} \quad \Omega_T, \\ u = 0 & \text{on} \quad \partial_p\Omega_T. \end{cases} \tag{1.1}$$

Here Ω_T stands for the space-time cylinder $\Omega \times (0,T]$ over a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, and its parabolic boundary is denoted by $\partial_p\Omega_T := (\partial\Omega \times [0,T]) \cup (\Omega \times \{t=0\})$. The coefficient matrix $\mathbf{A} = (a_{ij}) : \mathbb{R}^{n+1} \to \mathbb{R}^{n\times n}$ is assumed to be measurable and symmetric and satisfies the uniform parabolicity condition; i.e., there exists a positive constant Λ, called the parabolicity constant, such that

$$\Lambda^{-1}|\eta|^2 \leq \langle \mathbf{A}(z)\eta,\eta\rangle \leq \Lambda|\eta|^2 \quad \forall \eta\in\mathbb{R}^n \text{ and a.e. } z=(x,t)\in\mathbb{R}^{n+1}. \tag{1.2}$$

For the problem (1.1), we prove the Calderón-Zygmund type estimates in the weighted variable exponent Lebesgue spaces like

$$\begin{aligned} &\|u_t\|_{L^{p(\cdot)}(\Omega_T,w)} + \|u\|_{L^{p(\cdot)}(\Omega_T,w)} \\ &\qquad + \|Du\|_{L^{p(\cdot)}(\Omega_T,w)} + \|D^2u\|_{L^{p(\cdot)}(\Omega_T,w)} \leq c\|f\|_{L^{p(\cdot)}(\Omega_T,w)}, \end{aligned} \tag{1.3}$$

for any log-Hölder continuous function $p(\cdot) : \mathbb{R}^{n+1} \to (1,\infty)$ with

$$1 < \inf_{z\in\mathbb{R}^{n+1}} p(z) \leq \sup_{z\in\mathbb{R}^{n+1}} p(z) < \infty, \tag{1.4}$$

for any weight w belonging to $A_{p(\cdot)}$ class, and for some constant $c > 0$ independent of u and f under possibly a minimal regularity requirement on the coefficient

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matrix **A**. We will introduce the definitions of log-Hölder continuity, $A_{p(\cdot)}$ class, and the weighted variable exponent space in the next section. The estimate (1.3) ultimately implies the weighted $L^{p(\cdot)}$ solvability of the equation (1.1) with the implication

$$f \in L^{p(\cdot)}(\Omega_T, w) \implies u_t, D^2 u \in L^{p(\cdot)}(\Omega_T, w). \tag{1.5}$$

The main points in this paper are the variable exponent $p(\cdot)$ and the weight w. Note that if $p(\cdot) \equiv p$ then the weighted variable exponent space is the classical weighted Lebesgue space and if $w \equiv 1$, then it is the variable exponent Lebesgue space. We aim to establish a weighted $L^{p(\cdot)}$ regularity theory that is a natural generalization of the classical L^p regularity theory. The Calderón-Zygmund estimates for linear equations have been extensively studied since the celebrated work [11]. In particular, Chiarenza, Frasca, and Longo [13] obtained $W^{2,p}$-estimates for solutions to nondivergence elliptic equations with discontinuous coefficients of vanishing mean oscillation (VMO) type, and Bramanti and Cerutti [6] extended this result to nondivergence parabolic equations.

Over the past decades there has been much investigation into the variable exponent spaces [14–18, 20, 30, 33] and related partial differential equations [1, 2, 5, 8, 22]. The physical motivation for such research is concerned with the modeling of various phenomena in physics, engineering, and other fields, such as electrorheological fluids [28, 29], elastic mechanics [33], the thermister problem [34], and image restoration [12]. Moreover, the weighted variable exponent Lebesgue spaces have been actively studied; see [25–27, 31] and the references therein. Especially, one of main research interests for the weighted Lebesgue spaces has been to find the necessary and sufficient condition on weights to ensure boundedness of the maximal operator. Quite recently, Diening and Hästö [15, 18], in turn, characterized the class of weights for which the maximal operator is bounded on the weighted variable exponent Lebesgue spaces, that is, the $A_{p(\cdot)}$ class which is a generalization of the classical Muckenhoupt class.

Recently some results of L^p theory have been generalized to the variable exponent spaces. Diening, Lengeler, and Růžička [19] obtained $W^{2,p(\cdot)}$-estimates and $W^{1,p(\cdot)}$-estimates for the Poisson equation $\Delta u = f$. The authors derived $W^{1,p(\cdot)}$-estimates for divergence linear elliptic equations with possibly measurable coefficients in a nonsmooth domain [9] and $W^{2,p(\cdot)}$-estimates for nondivergence linear elliptic equations with coefficients of bounded mean oscillation (BMO) type in a $C^{1,1}$ domain [7]. On the other hand, as far as we know, there are no results either of weighted $L^{p(\cdot)}$ estimates even for elliptic equations or of $L^{p(\cdot)}$ estimates for parabolic equations, even for the heat equation $u_t - \Delta u = f$.

We point out that the approach in this paper is different from the classical one which uses representation formulas in terms of singular integral operators and commutators, as in the previous papers [6, 11, 13, 19]. In addition, our approach does not employ any maximal function that has been frequently used in L^p theory. Our method is influenced by the so-called *large-M-inequality* principle, which was first introduced by Acerbi and Mingione [3] in order to prove the Calderón-Zygmund type estimates for parabolic systems of p-Laplacian type. We first derive local interior and boundary *a priori* estimates. To do this, we apply a certain stopping time argument to find a suitable Vitali type covering of the upper-levels

$$\left\{ z \in \Omega_T \cap Q : |u_t|^{\gamma_0 \frac{p(z)}{p^-}} + |D^2 u|^{\gamma_0 \frac{p(z)}{p^-}} > \lambda \right\},$$

for sufficiently large numbers λ, where $\gamma_0 > 1$ is to be selected as a suitable constant satisfying $\gamma_0 \leq \inf p(z)$ and $p^- := \inf_{z \in Q} p(z)$. We then estimate the weighted measures of these upper-level sets by taking advantage of the comparison estimates in the classical Lebesgue space L^{γ_0}, the log-Hölder continuity of the variable exponent $p(\cdot)$, and the properties of $A_{p(\cdot)}$ class. The desired estimate (1.3) follows by standard flattening and covering arguments along with an appropriate approximation procedure. We point out that in this procedure, we need to control the term $\|Du\|_{L^{p(\cdot)}(\Omega_T,w)}$. For the particular case when $p(\cdot) \equiv p$ and $w \equiv 1$, i.e., the classical Lebesgue space, this term can be easily controlled by $\|u\|_{L^p(\Omega_T)}$ and $\|D^2 u\|_{L^p(\Omega_T)}$ from the interpolation inequality for the Sobolev space $W^{2,p}(\Omega)$. For the case of the weighted variable exponent Lebesgue space, however, it is not easy to do in a similar way as in the constant exponent case, because the exponent $p(\cdot)$ and the weight w depend on the t variable. To overcome this difficulty, we instead use a certain compactness argument, which will be indicated in the last section.

The remainder of this paper is organized as follows. In the next section we introduce some notation, the definitions of log-Hölder continuity, $A_{p(\cdot)}$ class and weighted variable exponent spaces, and the main assumption on the coefficient matrix to state the main results. In Section 3 we further discuss the properties of the $A_{p(\cdot)}$ class and the weighted variable exponent spaces. In Section 4 the comparison estimates are mainly provided in L^q spaces with $1 < q < \infty$. In Section 5 we derive local interior and boundary *a priori* weighted $W^{2,1}_{p(\cdot)}$-estimates. The proof of our main result, Theorem 2.5, is presented in Section 6.

2. Main results

We first introduce some standard notation and definitions that will be used throughout the paper. The variable in $\mathbb{R}^{n+1}$ is termed $z = (x,t)$ for the spatial variables $x = (x', x_n) = (x_1, \cdots, x_{n-1}, x_n) \in \mathbb{R}^n$ and the time variable $t \in \mathbb{R}$. For a function $g : U \subset \mathbb{R}^{n+1} \to \mathbb{R}$, we denote the spatial gradient of g by $Dg = (D_1 g, \cdots, D_n g)$, the spatial Hessian of g by $D^2 g = (D_{ij} g)$, where $D_i g = D_{x_i} g = \frac{\partial g}{\partial x_i}$, $D_{ij} g = D_{x_i x_j} g = \frac{\partial^2 g}{\partial x_i \partial x_j}$ for $i,j = 1, \cdots, n$, while the time derivative of g by $g_t = D_t g = \frac{\partial g}{\partial t}$. As usual, the parabolic distance d_p between two points $\xi = (y,s), \tilde{\xi} = (\tilde{y}, \tilde{s}) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ is denoted by

$$d_p(\xi, \tilde{\xi}) := \max \left\{ |y - \tilde{y}|, \sqrt{|s - \tilde{s}|} \right\},$$

where $|\cdot|$ is the Euclidean norm. In this paper, we shall use a parabolic cylinder of the form

$$Q_r(\xi) = Q_r(y,s) := B_r(y) \times \left(s - r^2, s + r^2 \right)$$

with center $\xi = (y,s) \in \mathbb{R}^{n+1}$ and radius $r > 0$, where $B_r(y) = \{x \in \mathbb{R}^n : |x - y| < r\}$ is the open ball in $\mathbb{R}^n$ with center y and radius r. Its parabolic boundary is denoted by

$$\partial_p Q_r(\xi) = \partial_p Q_r(y,s) = \left(\partial B_r(y) \times \left[s - r^2, s + r^2 \right) \right) \cup \left(B_r(y) \times \left\{ t = s - r^2 \right\} \right).$$

For the sake of simplicity, we abbreviate $B_r^+(y) = B_r(y) \cap \{x_n > 0\}$, $B_r = B_r(0)$, and $B_r^+ = B_r^+(0)$. We also write $Q_r = B_r \times \left(-r^2, r^2\right)$, $Q_r^+ = B_r^+ \times \left(-r^2, r^2\right)$. In our further considerations, we shall use the notation $T_r = Q_r \cap \{x_n = 0\}$ and $T_r(y,s) = T_r + (y,s)$. Furthermore, we shall employ parabolic cubes of the form

$$C_r(\xi) = C_r(y,s) := \{x \in \mathbb{R}^n : |x_i - y_i| < r,\ i = 1, \dots, n\} \times (s - r^2, s + r^2)$$

for $\xi = (y,s) \in \mathbb{R}^{n+1}$ and $r > 0$.

For a vector valued function $\mathbf{f} : U \to \mathbb{R}^N$, $N \geq 1$, we denote $\bar{\mathbf{f}}_U$ by the integral average of $\mathbf{f}$ on U, that is,

$$\bar{\mathbf{f}}_U = \fint_U \mathbf{f}(z)dz = \frac{1}{|U|}\int_U \mathbf{f}(z)dz.$$

We consider the *variable exponent* $p(z) = p(x,t) = p(\cdot) : \mathbb{R}^{n+1} \to \mathbb{R}$ with

$$1 < \gamma_1 := \inf_{z\in\mathbb{R}^{n+1}} p(z) \leq \sup_{z\in\mathbb{R}^{n+1}} p(z) =: \gamma_2 < \infty, \tag{2.1}$$

for some constants γ_1 and γ_2, and its conjugate exponent $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$. Let $w : \mathbb{R}^{n+1} \to (0,\infty)$ be a locally integrable function, which is called a *weight.* For $U \subset \mathbb{R}^{n+1}$, we define the *weighted variable exponent Lebesgue space* $L^{p(\cdot)}(U,w)$ to be the set of all measurable functions $g : U \to \mathbb{R}$ such that the *modular*

$$\varrho_{p(\cdot),w}(g) := \int_U |g|^{p(z)} w(z)\, dz$$

is finite. Then $L^{p(\cdot)}(U,w)$ becomes a Banach space equipped with the following *Luxemburg norm*:

$$\|g\|_{L^{p(\cdot)}(U,w)} := \inf\left\{\lambda > 0 : \varrho_{p(\cdot),w}\left(\frac{g}{\lambda}\right) \leq 1\right\}. \tag{2.2}$$

If $w \equiv 1$, we simply write $L^{p(\cdot)}(U) = L^{p(\cdot)}(U,1)$, which is the usual variable exponent Lebesgue space. On the other hand, if the variable exponent $p(\cdot)$ is constant, i.e., $p(\cdot) \equiv p$, then the space $L^{p(\cdot)}(U,w)$ coincides with the weighted Lebesgue space $L^p(U,w)$; i.e., its norm becomes the classical norm of the space $L^p(U,w)$ as follows:

$$\|g\|_{L^p(U,w)} = \left(\int_U |g|^p w(z)\, dz\right)^{\frac{1}{p}}.$$

We now present crucial conditions on the variable exponent $p(\cdot)$ and the weight w.

Definition 2.1. We say that $p(\cdot) : \mathbb{R}^{n+1} \to \mathbb{R}$ is *log-Hölder continuous*, denoted by $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^{n+1})$, if

$$|p(\xi) - p(\tilde{\xi})| \leq \frac{c_{LH}}{\log\left(e + 1/|\xi - \tilde{\xi}|\right)} \tag{2.3}$$

and

$$|p(\xi) - p_\infty| \leq \frac{c_{LH}}{\log(e + |\xi|)},$$

for all $\xi, \tilde{\xi} \in \mathbb{R}^{n+1}$ and for some $p_\infty \in \mathbb{R}$ and $c_{LH} = c_{LH}(p(\cdot)) > 0$. Here, c_{LH} is called the *log-Hölder constant* of $p(\cdot)$.

In particular, if $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^{n+1})$ satisfies (2.1), we write $p(\cdot) \in \mathcal{P}^{\log}_{\pm}(\mathbb{R}^{n+1})$.

Hereafter, we abbreviate $\mathcal{P}^{\log}_{\pm} := \mathcal{P}^{\log}_{\pm}(\mathbb{R}^{n+1})$ for the sake of simplicity. We point out that the condition (2.3) implies that

$$\theta(r)\log\left(\frac{1}{r}\right) \le M, \quad \text{for all } 0<r<\infty, \tag{2.4}$$

where $\theta(\cdot):[0,\infty)\to[0,2\gamma_2]$ with $\theta(0)=0$ is the modulus of continuity of $p(\cdot)$ with respect to the parabolic distance d_p such that

$$\theta(r) := \sup\left\{|p(\xi)-p(\tilde{\xi})| : d_p(\xi,\tilde{\xi}) \le r \text{ and } \xi,\tilde{\xi}\in\mathbb{R}^{n+1}\right\} \tag{2.5}$$

and the constant $M>0$ depends only on c_{LH} and γ_2. Indeed, if $d_p(\xi,\tilde{\xi}) = \tilde{r} \le r \le 1$, we have $|\xi-\tilde{\xi}| \le \tilde{r}\sqrt{1+\tilde{r}^2} \le \sqrt{2}\tilde{r}$. Then a direct computation yields

$$\begin{aligned} |p(\xi)-p(\tilde{\xi})|\log\left(\frac{1}{r}\right) &\le |p(\xi)-p(\tilde{\xi})|\log\left(\frac{1}{\tilde{r}}\right) \le |p(\xi)-p(\tilde{\xi})|\log\left(\frac{\sqrt{2}}{|\xi-\tilde{\xi}|}\right)\\ &\le |p(\xi)-p(\tilde{\xi})|\log\left(e+\frac{1}{|\xi-\tilde{\xi}|}\right) + |p(\xi)-p(\tilde{\xi})|\log\sqrt{2}, \end{aligned}$$

which implies that

$$\theta(r)\log\left(\frac{1}{r}\right) \le c_{LH} + 2\gamma_2\log\sqrt{2} \quad \text{for all } 0<r\le 1.$$

In addition, from (2.5) we have

$$|p(\xi)-p(\tilde{\xi})| \le \theta(d_p(\xi,\tilde{\xi})). \tag{2.6}$$

Definition 2.2. For $U\subset\mathbb{R}^{n+1}$, we say that the weight w is of $A_{p(\cdot)}(U)$ *class*, denoted by $w\in A_{p(\cdot)}(U)$, if

$$[w]_{A_{p(\cdot)}(U)} := \sup_{C\subset U} |C|^{-p_C}\|w\|_{L^1(C)}\|w^{-1}\|_{L^{p'(\cdot)/p(\cdot)}(C)} < \infty,$$

where C is any parabolic cube and p_C is the harmonic average of $p(\cdot)$ over C denoted by

$$p_C := \left(\fint_C p(z)^{-1}\,dz\right)^{-1}.$$

In particular, when $U=\mathbb{R}^{n+1}$, we simply write $A_{p(\cdot)} = A_{p(\cdot)}(\mathbb{R}^{n+1})$.

Here, $[w]_{A_{p(\cdot)}(U)}$ is called the $A_{p(\cdot)}$-constant of w and $\|\cdot\|_{L^{p'(\cdot)/p(\cdot)}(C)}$ is defined by (2.2) with $p(\cdot)$ replaced by $p'(\cdot)/p(\cdot)$. Note that $p'(\cdot)/p(\cdot)$ might be less than one and, in this case, $\|\cdot\|_{L^{p'(\cdot)/p(\cdot)}(C)}$ is not a norm but is only a quasi-norm. When $p(\cdot)$ is constant, i.e., $p(\cdot)\equiv p$, the $A_{p(\cdot)}(U)$ class is the ordinary Muckenhoupt class $A_p(U)$, and we have

$$[w]_{A_p(U)} = \sup_{C\subset U}\left(\fint_C w(z)\,dz\right)\left(\fint_C w(z)^{-\frac{1}{p-1}}\,dz\right)^{p-1},$$

which is the classical definition of the A_p-constant of w.

Remark 2.3. We adopted parabolic cubes instead of usual cubes in the definition of $A_{p(\cdot)}$ class. This is suitable for our problem dealing with parabolic equations. We also note that the weight $w\in A_{p(\cdot)}$ satisfies the doubling property in the same way as the classical Muckenhoupt weight. On the other hand, we still used the Euclidean distance in the definition of log-Hölder continuity.

We suppose $p(\cdot) \in \mathcal{P}^{\log}_{\pm}$ and $w \in A_{p(\cdot)}$ and recall the bounded domain $\Omega_T = \Omega \times (0,T]$. The *parabolic weighted variable exponent Sobolev space* $W^{2,1}_{p(\cdot)}(\Omega_T, w)$ is defined as

$$W^{2,1}_{p(\cdot)}(\Omega_T, w) := \left\{ g \in L^{p(\cdot)}(\Omega_T, w) : |Dg|, |D^2 g|, g_t \in L^{p(\cdot)}(\Omega_T, w) \right\},$$

endowed with the norm

$$\|g\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} = \|g\|_{L^{p(\cdot)}(\Omega_T,w)} + \|Dg\|_{L^{p(\cdot)}(\Omega_T,w)} + \|D^2 g\|_{L^{p(\cdot)}(\Omega_T,w)} + \|g_t\|_{L^{p(\cdot)}(\Omega_T,w)},$$

where we abbreviate $\|Dg\|_{L^{p(\cdot)}(\Omega_T,w)} := \||Dg|\|_{L^{p(\cdot)}(\Omega_T,w)}$ and $\|D^2 g\|_{L^{p(\cdot)}(\Omega_T,w)} := \||D^2 g|\|_{L^{p(\cdot)}(\Omega_T,w)}$ for the sake of convenience. We also define $W^{2,1}_{p(\cdot)}(Q_r(\xi), w)$ and $W^{2,1}_{p(\cdot)}(Q^+_r, w)$ for parabolic cylinders $Q_r(\xi)$ and Q^+_r in the same way. In addition, we denote

$$\overset{\circ}{W}{}^{2,1}_{p(\cdot)}(\Omega_T, w) = \left\{ g \in W^{2,1}_{p(\cdot)}(\Omega_T, w) : g = 0 \text{ on } \partial_p \Omega_T \right\}.$$

We remark that the log-Hölder continuity is considered as an unavoidable condition, because given the variable exponent $p(\cdot)$ with this condition, the properties of the classical Lebesgue and Sobolev spaces, such as Sobolev embeddings, Poincaré's inequality, and the boundedness of singular integral operators are valid in variable exponent Lebesgue and Sobolev spaces. We further discuss weighted variable exponent spaces in the next section.

The following is our principal assumption on the coefficient matrix $\mathbf{A}$.

Definition 2.4. For $\delta, R > 0$, we say that the coefficient matrix $\mathbf{A} = (a_{ij})$ is (δ, R)-*vanishing* if

$$[\mathbf{A}]_R := \sup_{0<r\le R} \sup_{\xi \in \mathbb{R}^{n+1}} \fint_{Q_r(\xi)} |\mathbf{A}(z) - \overline{\mathbf{A}}_{Q_r(\xi)}| \, dz \le \delta. \tag{2.7}$$

The above condition means that $\mathbf{A}$ has a small bounded mean oscillation (BMO) seminorm. We note that Bramanti and Cerutti [6] showed that if $f \in L^p(\Omega_T)$ for any constant p with $1 < p < \infty$, there exists a unique *strong solution* u, i.e., a function $u \in W^{2,1}_p(\Omega_T)$ which satisfies the equation (1.1) almost everywhere in Ω_T and $u \equiv 0$ on $\partial_p \Omega_T$ in the trace sense, whose coefficient matrix belongs to the class of functions of VMO type. This result can be naturally extended to the same equation whose coefficient matrix is (δ, R)-vanishing for some sufficiently small $\delta > 0$ and any $R > 0$.

Our main result in this paper is the following.

Theorem 2.5. *Let $p(\cdot) \in \mathcal{P}^{\log}_{\pm}$ with the log-Hölder constant c_{LH} and the modulus of continuity $\theta(\cdot)$, and $w \in A_{p(\cdot)}$. Assume $\partial\Omega \in C^{1,1}$ and $f \in L^{p(\cdot)}(\Omega_T, w)$. Then there is a small $\delta = \delta(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, w, \partial\Omega) > 0$ so that if $\mathbf{A}$ is (δ, R)-vanishing for some $R > 0$, the problem* (1.1) *has a unique strong solution $u \in \overset{\circ}{W}{}^{2,1}_{p(\cdot)}(\Omega_T, w)$, and we have the estimate*

$$\|u\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} \le c \|f\|_{L^{p(\cdot)}(\Omega_T,w)}, \tag{2.8}$$

for some positive constant $c = c(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, \theta(\cdot), w, R, \Omega, T)$.

Thanks to the linearity of the equation (1.1), we have a direct consequence of the above theorem as follows.

Corollary 2.6. *Let $p(\cdot) \in \mathcal{P}^{\log}_{\pm}$ with the log-Hölder constant c_{LH} and the modulus of continuity $\theta(\cdot)$, and $w \in A_{p(\cdot)}$. Assume that $\partial\Omega \in C^{1,1}$, $f \in L^{p(\cdot)}(\Omega_T, w)$, and $\phi \in W^{2,1}_{p(\cdot)}(\Omega_T, w)$. Then there is a small $\delta = \delta(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_w, \partial\Omega) > 0$ so that if $\mathbf{A}$ is (δ, R)-vanishing for some $R > 0$, the problem*

$$\begin{cases} u_t - a_{ij} D_{ij} u &= f \quad \text{in} \quad \Omega_T, \\ u &= \phi \quad \text{on} \quad \partial_p \Omega_T \end{cases}$$

has a unique solution $u \in W^{2,1}_{p(\cdot)}(\Omega_T, w)$ with $u - \phi \in \overset{\circ}{W}{}^{2,1}_{p(\cdot)}(\Omega_T, w)$, and we have the estimate

$$\|u\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} \leq c \left(\|f\|_{L^{p(\cdot)}(\Omega_T,w)} + \|\phi\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} \right), \tag{2.9}$$

for some positive constant $c = c(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, w, \Omega, R, T)$.

3. Muckenhoupt classes and weighted variable exponent spaces

3.1. A_p **class.** We introduce the properties of weights belonging to A_p class for $1 < p < \infty$. For their proofs, we refer to [10, 23, 32]. Let us define

$$w(E) := \int_E w(z)\, dz,$$

for a measurable set $E \subset \mathbb{R}^{n+1}$, and let $U \subset \mathbb{R}^{n+1}$ be an open set. We first remark that $u \in A_p(U)$ if and only if there exists $c \geq 1$ such that

$$\left(\fint_C g\, dz \right)^p \leq \frac{c}{w(C)} \int_C g^p w(z)\, dz, \tag{3.1}$$

for all nonnegative measurable functions g and all parabolic cubes $C \subset U$. In particular, the smallest constant c satisfying the inequality (3.1) is equal to $[w]_{A_p(U)}$.

Lemma 3.1. *Let $w \in A_p(U)$ for some $1 < p < \infty$.*

(1) *There exist positive constants ν_1 and $d_1 \geq 1$ depending only on n, p and $[w]_{A_p(U)}$ such that*

$$\left(\fint_C w(z)^{1+\nu_1}\, dz \right)^{\frac{1}{1+\nu_1}} \leq d_1 \fint_C w(z)\, dz$$

for all parabolic cubes $C \subset U$.

(2) *We have*

$$\frac{1}{[w]_{A_p(U)}} \left(\frac{|E|}{|C|} \right)^p \leq \frac{w(E)}{w(C)} \leq d_1 \left(\frac{|E|}{|C|} \right)^{\frac{\nu_1}{1+\nu_1}}$$

for all parabolic cubes $C \subset U$ and all measurable subsets E of C, where ν_1 and d_1 have been determined in (1).

(3) *There exist $\epsilon_1 \in (0, p-1)$ and $\tilde{d}_1 \geq 1$ depending only on n, p and $[w]_{A_p(U)}$ such that $w \in A_{p-\epsilon_1}(U)$ with $[w]_{A_{p-\epsilon_1}(U)} \leq \tilde{d}_1$.*

Remark 3.2. In view of the proofs of Theorem 9.2.2, Theorem 9.2.5, and Corollary 9.2.6 in [23], we see that the constants $\nu_1, \epsilon_1, d_1, \tilde{d}_1$ depend continuously on the values p and $[w]_{A_p(U)}$, respectively.

From Lemma 3.1 and Remark 3.2, we have the following lemma.

Lemma 3.3. *Let $1 < \gamma_1 \le \gamma_2 < \infty$ and $A_0 \ge 1$.*

(1) *There exist positive constants ν_0 and $\tilde{d}_0$ depending only on n, γ_1, γ_2 and A_0 such that for any $p \in [\gamma_1, \gamma_2]$ and any weight $w \in A_p(U)$ with $[w]_{A_p(U)} \le A_0$, we have*
$$\frac{1}{A_0}\left(\frac{|E|}{|C|}\right)^{\gamma_2} \le \frac{w(E)}{w(C)} \le d_0\left(\frac{|E|}{|C|}\right)^{\nu_0}$$
for all parabolic cubes $C \subset U$ and all measurable subsets E of C.

(2) *There exist $\epsilon_0 \in (0, \gamma_1 - 1)$ and $\tilde{d}_0 \ge 1$ depending only on n, γ_1, γ_2 and A_0 such that for any $p \in [\gamma_1, \gamma_2]$ and any weight $w \in A_p(U)$ with $[w]_{A_p(U)} \le A_0$, we have $w \in A_{p-\epsilon_0}(U)$ with $[w]_{A_{p-\epsilon_0}(U)} \le \tilde{d}_0$.*

3.2. **Weighted variable exponent Lebesgue spaces.** We recall basic properties for weighted variable exponent Lebesgue spaces. The results in the following lemma can be found in [17, Chapter 2] by letting $\varphi(x,t) = t^{p(x)}w(x)$.

Lemma 3.4. *Let $p(\cdot) : \mathbb{R}^{n+1} \to (1,\infty)$ satisfy (2.1) and let w be a weight.*

(1) *Norm-modular unit ball property:*
$$\|g\|_{L^{p(\cdot)}(U,w)} \le 1 \quad \Longleftrightarrow \quad \varrho_{p(\cdot),w}(g) \le 1. \tag{3.2}$$

(2) *Relationship between norm and modular:*
$$\begin{aligned}\min&\left\{\left(\varrho_{p(\cdot),w}(g)\right)^{\frac{1}{\gamma_1}}, \left(\varrho_{p(\cdot),w}(g)\right)^{\frac{1}{\gamma_2}}\right\}\\ &\le \|g\|_{L^{p(\cdot)}(U,w)} \le \max\left\{\left(\varrho_{p(\cdot),w}(g)\right)^{\frac{1}{\gamma_1}}, \left(\varrho_{p(\cdot),w}(g)\right)^{\frac{1}{\gamma_2}}\right\}.\end{aligned} \tag{3.3}$$

(3) *Hölder's inequality: For $q(\cdot) : \mathbb{R}^{n+1} \to (1,\infty)$, let $\frac{1}{s(\cdot)} := \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}$. Then we have*
$$\|fg\|_{L^{s(\cdot)}(U,w)} \le 2\|f\|_{L^{p(\cdot)}(U,w)}\|g\|_{L^{q(\cdot)}(U,w)}. \tag{3.4}$$

(4) *$C_0^\infty(U)$ is dense in $L^{p(\cdot)}(U,w)$.*

(5) *$L^{p'(\cdot)}(U, w^{-1/(p(\cdot)-1)})$ is isomorphic to the dual space $(L^{p(\cdot)}(U,w))^*$ of the space $L^{p(\cdot)}(U,w)$ in the sense that for $g \in L^{p'(\cdot)}(U, w^{-1/(p(\cdot)-1)})$, we define $J_g \in (L^{p(\cdot)}(U,w))^*$ by*
$$J_g(f) := \int_U fg\,dz.$$
In particular, there exists $c = c(\gamma_1, \gamma_2, w) \ge 1$ such that
$$\frac{1}{c}\|g\|_{L^{p'(\cdot)}(U,w^{-1/(p(\cdot)-1)})} \le \|J_g\|_{(L^{p(\cdot)}(U,w))^*} \le c\|g\|_{L^{p'(\cdot)}(U,w^{-1/(p(\cdot)-1)})}.$$

We next show two properties of $A_p(\cdot)$ class. The first one is duality, and the second one is monotonicity. Similar results can be found in [15, 18]. In contrast with [15, 18], however, we adopted parabolic cubes in the definition of $A_{p(\cdot)}$ class, Definition 2.2.

Lemma 3.5. *Let $p(\cdot) \in \mathcal{P}_\pm^{\log}$ and let $U \subset \mathbb{R}^{n+1}$ be bounded. Then we have the relation that*
$$w \in A_{p(\cdot)}(U) \quad \Longleftrightarrow \quad w^{-1/(p(\cdot)-1)} \in A_{p'(\cdot)}(U).$$

Proof. It suffices to show that $w \in A_{p(\cdot)}(U)$ implies $w^{-1/(p(\cdot)-1)} \in A_{p'(\cdot)}(U)$, since this means that this reverse is also valid. Suppose $w \in A_{p(\cdot)}(U)$. Then we will show that

$$\begin{aligned}
&[w^{-1/(p(\cdot)-1)}]_{A_{p'(\cdot)}(U)} \\
&= \sup_{C\subset U} |C|^{-(p')_C} \|w^{-1/(p(\cdot)-1)}\|_{L^1(C)} \|w^{1/(p(\cdot)-1)}\|_{L^{p(\cdot)/p'(\cdot)}(C)} < \infty.
\end{aligned} \tag{3.5}$$

Here $(p')_C$ is the harmonic mean of p' in C and $(p')_C = (p_C)'$. We first compute $|C|^{-(p')_C}$:

$$|C|^{-(p')_C} = |C|^{-p_C/(p_C-1)} = |C|^{-p_C/(p_C^+-1)} |C|^{-p_C(p_C^+-p_C)/\{(p_C^+-1)(p_C-1)\}}.$$

If $|C| \ge 1$ we have

$$|C|^{-(p')_C} \le |C|^{-p_C/(p_C^+-1)},$$

and if $|C| = |C_r(\xi)| = (2r)^{n+2} \le 1$ we have from (2.4) that

$$|C|^{-p_C(p_C^+-p_C)/\{(p_C^+-1)(p_C-1)\}} \le \left(\frac{1}{2r}\right)^{\theta(2\sqrt{2}r)(n+2)\gamma_2/(\gamma_1-1)^2} \le c$$

and so

$$|C|^{-(p')_C} \le c|C|^{-p_C/(p_C^+-1)}.$$

As for $\|w^{-1/(p(\cdot)-1)}\|_{L^1(C)}$ and $\|w^{1/(p(\cdot)-1)}\|_{L^{p(\cdot)/p'(\cdot)}(C)}$, we estimate by (3.3) that

$$\begin{aligned}
\|w^{-1/(p(\cdot)-1)}\|_{L^1(C)} &= \left(\int_C w^{-1/(p(z)-1)}\,dz\right)^{(p_C^+-1)/(p_C^+-1)} \\
&\le \max\left\{w(U)^{(\gamma_2-\gamma_1)(\gamma_1-1)}, 1\right\} \|w^{-1}\|_{L^{p'(\cdot)/p(\cdot)}(C)}^{1/(p_C^+-1)}
\end{aligned}$$

and

$$\|w^{1/(p(\cdot)-1)}\|_{L^{p(\cdot)/p'(\cdot)}(C)} \le \max\left\{w(U)^{(\gamma_2-\gamma_1)/(\gamma_1-1)^2}, 1\right\} \|w\|_{L^1(C)}^{1/(p_C^+-1)}.$$

Therefore, we have

$$|C|^{-(p')_C} \|w^{-1/(p(\cdot)-1)}\|_{L^1(C)} \|w^{1/(p(\cdot)-1)}\|_{L^{p(\cdot)/p'(\cdot)}(C)} \le c[w]_{A_{p(\cdot)}}^{1/(p_C^+-1)} \le c[w]_{A_{p(\cdot)}}^{1/(\gamma_1-1)},$$

which implies (3.5). □

Lemma 3.6. *Let $p(\cdot), q(\cdot) \in \mathcal{P}_{\pm}^{\log}$ with $1 < \gamma_1 \le p(\cdot) \le \gamma_2 < \infty$ and $1 < \gamma_3 \le p(\cdot) \le \gamma_4 < \infty$. If $p(\cdot) \le q(\cdot)$ and $U \subset \mathbb{R}^{n+1}$ are bounded, then there exists $c_m \ge 1$ depending only on $n, \gamma_1, \gamma_2, \gamma_3, \gamma_4$ and the log-Hölder constants of $p(\cdot)$ and $q(\cdot)$ such that*

$$[w]_{A_{q(\cdot)}(U)} \le c_m \max\left\{|U|^{\gamma_2-\gamma_1+\gamma_4-\gamma_3}, 1\right\} [w]_{A_{p(\cdot)}(U)}.$$

In particular, if $q(\cdot)$ is a constant function, then the constant c_m depends only on n, γ_1, γ_2 and the log-Hölder constant of $p(\cdot)$.

Proof. For a parabolic cube $C = C_r(\xi) \subset U$, we first observe from (3.3) and (3.4) that

$$\begin{aligned}
\|w^{-1}\|_{L^{q'(\cdot)/q(\cdot)}(C)} &\le 2\|w^{-1}\|_{L^{p'(\cdot)/p(\cdot)}(C)} \|1\|_{L^{s(\cdot)}(C)} \\
&\le 2\|w^{-1}\|_{L^{p'(\cdot)/p(\cdot)}(C)} \max\left\{|C|^{1/s_C^+}, |C|^{1/s_C^-}\right\},
\end{aligned}$$

where $s(\cdot)^{-1} = q(\cdot) - p(\cdot)$. Note that if $|C| \geq 1$ we see that

$$\max\left\{|C|^{1/s_C^+}, |C|^{1/s_C^-}\right\} |C|^{p_C - q_C} \leq |C|^{p_C^+ - p_C^- + q_C^+ - q_C^-} \leq |U|^{\gamma_2 - \gamma_1 + \gamma_2 - \gamma_1},$$

and if $|C| = (2r)^{n+2} \leq 1$ we see that

$$\begin{aligned} \max\left\{|C|^{1/s_C^+}, |C|^{1/s_C^-}\right\} |C|^{p_C - q_C} &\leq |C|^{-(p_C^+ - p_C^-) - (q_C^+ - q_C^-)} \\ &\leq \left(\frac{1}{2r}\right)^{(\theta_p(2\sqrt{2}r) + \theta_q(2\sqrt{2}r))(n+2)} \leq c, \end{aligned}$$

where θ_p and θ_q are the modulus of constant of $p(\cdot)$ and $q(\cdot)$, respectively. Therefore, we have

$$\begin{aligned} &|C|^{-q_C} \|w\|_{L^1(C)} \|w^{-1}\|_{L^{q'(\cdot)/q(\cdot)}(C)} \\ &\quad \leq \max\left\{|C|^{1/s_C^+}, |C|^{1/s_C^-}\right\} |C|^{p_C - q_C} |C|^{-p_C} \|w\|_{L^1(C)} \|w^{-1}\|_{L^{p'(\cdot)/p(\cdot)}(C)} \\ &\quad \leq c \max\left\{|U|^{\gamma_2 - \gamma_1 + \gamma_4 - \gamma_3}, 1\right\} |C|^{-p_C} \|w\|_{L^1(C)} \|w^{-1}\|_{L^{p'(\cdot)/p(\cdot)}(C)}. \end{aligned}$$

This implies the desired result. □

The following lemma plays a crucial role in Sections 5 and 6.

Lemma 3.7. *Let $p(\cdot) \in \mathcal{P}_{\pm}^{\log}$, let $w \in A_{p(\cdot)}$, and let $U \subset \mathbb{R}^{n+1}$ be bounded. There exist $\tilde{\gamma}_0 = \tilde{\gamma}_0(n, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) \in (1, \gamma_1)$ and $c = c(n, \gamma_1, \gamma_2, c_{LH}, w, U) > 0$ such that*

$$\|f\|_{L^{\tilde{\gamma}_0}(U)} \leq c\|f\|_{L^{p(\cdot)}(U,w)}. \tag{3.6}$$

Proof. We extend f from U to $\mathbb{R}^{n+1}$ by zero. Let $C = C_r(\xi)$ be a parabolic cube. We first note that if $|C| \leq 1$ we have from Lemma 3.6 that $w \in A_q(C)$ with $[w]_q(U) \leq c_m[w]_{p(\cdot)}$ for all $q \geq p(\cdot)$ in C and for some $c_m = c_m(n, \gamma_1, \gamma_2, c_{LH})$. Therefore, since $p_C^+ \geq p(\cdot)$ in C and $\gamma_1 \leq p_C^+ \leq \gamma_2$, applying (2) of Lemma 3.3 to $A_0 = c_m[w]_{A_{p(\cdot)}}$, there exists $\epsilon_0 = \epsilon_0(n, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) \in (0, \gamma_1 - 1)$ such that $w \in A_{p_C^+ - \epsilon_0}(C)$ for all parabolic cubes C with $|C| \leq 1$, where $[w]_{A_{p_C^+ - \epsilon_0}(C)}$ depends only on $n, \gamma_1, \gamma_2, c_{LH}$ and $[w]_{A_{p(\cdot)}}$.

We now consider parabolic cubes C such that

$$|C| = (2r)^{n+1} \leq 1 \quad \text{and} \quad \theta(2\sqrt{2}r) \leq \frac{\epsilon_0}{4}. \tag{3.7}$$

Note that $p_C^+ - \epsilon_0/2 \leq p(\cdot)$ in C. Moreover, we infer from (3.3) and (3.4) that

$$\begin{aligned} \|f\|_{L^{p_C^+ - \epsilon_0/2}(C,w)} &\leq 2\|1\|_{L^{(1/(p_C^+ - \epsilon_0/2) - 1/p(\cdot))^{-1}}(C,w)} \|f\|_{L^{p(\cdot)}(C,w)} \\ &\leq 2 \max\left\{w(C)^{\frac{1}{p_C^+ - \epsilon_0/2} - \frac{1}{p_C^+}}, w(C)^{\frac{1}{p_C^+ - \epsilon_0/2} - \frac{1}{p_C^-}}\right\} \|f\|_{L^{p(\cdot)}(C,w)} \\ &\leq 2 \max\left\{w(C)^{-\frac{1}{p_C^+}}, w(C)^{-\frac{1}{p_C^-}}\right\} w(C)^{\frac{1}{p_C^+ - \epsilon_0/2}} \|f\|_{L^{p(\cdot)}(C,w)} \\ &\leq 2 \max\left\{w(C)^{-\frac{1}{\gamma_1}}, 1\right\} w(C)^{\frac{1}{p_C^+ - \epsilon_0/2}} \|f\|_{L^{p(\cdot)}(C,w)} \end{aligned}$$

and from (3.1) that

$$\left(\fint_C |f|^{\frac{p_C^+-\epsilon_0/2}{p_C^+-\epsilon_0}}\,dz\right)^{p_C^+-\epsilon_0} \le \frac{[w]_{Ap_C^+-\epsilon_0(C)}}{w(C)}\int_C |f|^{p_C^+-\epsilon_0/2} w(z)\,dz.$$

Consequently, letting

$$\tilde{\gamma}_0 := \frac{\gamma_2-\epsilon_0/2}{\gamma_2-\epsilon_0} = 1+\frac{\epsilon_0}{2(\gamma_2-\epsilon_0)} \tag{3.8}$$

we have

$$\begin{aligned}\left(\fint_C |f|^{\tilde{\gamma}_0}\,dz\right)^{\frac{1}{\tilde{\gamma}_0}} &\le \left(\fint_C |f|^{\frac{p_C^+-\epsilon_0/2}{p_C^+-\epsilon_0}}\,dz\right)^{\frac{p_C^+-\epsilon_0}{p_C^+-\epsilon_0/2}}\\ &\le 2[w]_{Ap_C^+-\epsilon_0(C)}^{\frac{1}{p_C^+-\epsilon_0/2}} \max\left\{w(C)^{-\frac{1}{\gamma_1}},1\right\}\|f\|_{L^{p(\cdot)}(C,w)},\end{aligned}$$

and hence,

$$\|f\|_{L^{\tilde{\gamma}_0}(C)} \le 2[w]_{Ap_C^+-\epsilon_0(C)}^{\frac{1}{p_C^+-\epsilon_0/2}} \max\left\{w(C)^{-\frac{1}{\gamma_1}},1\right\}\|f\|_{L^{p(\cdot)}(C,w)}, \tag{3.9}$$

for all C satisfying (3.7). By a standard covering argument, the desired estimate (3.6) follows from the previous estimate (3.9). $\square$

4. Comparison estimates

We start this section by recalling the interior and boundary *a priori* $W_q^{2,1}$-estimates and the global $W_q^{2,1}$-estimates in a $C^{1,1}$ domain that have been proved in [6].

Lemma 4.1. *Let $1<q<\infty$. There exist a small $\delta=\delta(\Lambda,n,q)>0$ and $c=c(\Lambda,n,q)>0$ such that the following hold for any fixed $r>0$:*

(i) (*Interior estimates*) *If* $\mathbf{A}$ *is $(\delta,2r)$-vanishing and $f\in L^q(Q_{2r})$, then for any strong solution $u\in W_q^{2,1}(Q_{2r})$ of*

$$u_t - a_{ij}D_{ij}u = f \quad \text{in } Q_{2r},$$

we have the estimate

$$\|u_t\|_{L^q(Q_r)} + \|D^2u\|_{L^q(Q_r)} \le c\left(\|f\|_{L^q(Q_{2r})}+\frac{1}{r^2}\|u\|_{L^q(Q_{2r})}\right).$$

(ii) (*Boundary estimates*) *If* $\mathbf{A}$ *is $(\delta,2r)$-vanishing and $f\in L^q(Q_{2r}^+)$, then for any strong solution $u\in W_q^{2,1}(Q_{2r}^+)$ of*

$$\begin{cases} u_t - a_{ij}D_{ij}u = f & \text{in } Q_{2r}^+,\\ u = 0 & \text{on } T_{2r},\end{cases}$$

we have the estimate

$$\|u_t\|_{L^q(Q_r^+)} + \|D^2u\|_{L^q(Q_r^+)} \le c\left(\|f\|_{L^q(Q_{2r}^+)}+\frac{1}{r^2}\|u\|_{L^q(Q_{2r}^+)}\right).$$

Moreover, let $\partial\Omega \in C^{1,1}$. Then there exists a small $\delta = \delta(\Lambda, n, q, \partial\Omega) > 0$ such that if $f \in L^q(\Omega_T)$ and $\mathbf{A}$ is (δ, R)-vanishing for some $R > 0$, then the problem (1.1) has a unique strong solution $u \in \overset{\circ}{W}{}^{2,1}_q(\Omega_T)$, and we have the estimate

$$\|u\|_{W^{2,1}_q(\Omega_T)} \le c\|f\|_{L^q(\Omega_T)}, \tag{4.1}$$

for some $c = c(n, \Lambda, q, \Omega, R, T) > 0$.

We next derive the comparison estimates in L^q spaces with $1 < q < \infty$ by using a compactness argument. These estimates play crucial roles in the proofs of the interior and boundary *a priori* $W^{2,1}_{p(\cdot)}$-estimates in Section 4. In what follows, we denote by c any positive constant depending only on n, Λ and q, which may vary from line to line.

We first prove the Poincaré type inequalities in Sobolev space $W^{2,1}_q$.

Lemma 4.2. *For any $1 < q < \infty$, let $h \in W^{2,1}_q(Q_4)$. Then there is a positive constant c depending only on q and n so that*

$$\fint_{Q_4} |h - \overline{h}_{Q_4} - \overline{(Dh)}_{Q_4} \cdot x|^q dz \le c \fint_{Q_4} \left(|h_t|^q + |D^2 h|^q\right) dz. \tag{4.2}$$

Proof. We argue by contradiction. Suppose that (4.2) is not true. Then there exists a sequence $\{h_k\}_{k=1}^\infty$ in $W^{2,1}_q(Q_4)$ satisfying

$$\fint_{Q_4} |h_k - \overline{h_k}_{Q_4} - \overline{(Dh_k)}_{Q_4} \cdot x|^q dz > k \fint_{Q_4} \left(|(h_k)_t|^q + |D^2 h_k|^q\right) dz. \tag{4.3}$$

By normalization, we may assume that

$$\fint_{Q_4} |h_k - \overline{h_k}_{Q_4} - \overline{(Dh_k)}_{Q_4} \cdot x|^q dz = 1,$$

and then the inequality (4.3) implies that

$$\fint_{Q_4} \left(|(h_k)_t|^q + |D^2 h_k|^q\right) dz < \frac{1}{k}.$$

Now let us consider $\tilde{h}_k := h_k - \overline{h_k}_{Q_4} - \overline{(Dh_k)}_{Q_4} \cdot x$. Then it is easy to check that

$$\fint_{Q_4} \tilde{h}_k dz = \fint_{Q_4} D\tilde{h}_k dz = 0, \tag{4.4}$$

$$\fint_{Q_4} |\tilde{h}_k|^q dz = 1, \quad \text{and} \quad \fint_{Q_4} \left(|(\tilde{h}_k)_t|^q + |D^2 \tilde{h}_k|^q\right) dz < \frac{1}{k} \le 1. \tag{4.5}$$

In addition, we use the interpolation inequality (see [4, Theorem 5.2]) for each time slice $B_4 \times \{t\}$ with $t \in [-4^2, 4^2]$ to obtain

$$\begin{aligned} \int_{Q_4} |D\tilde{h}_k|^q dz &\le \int_{[-4^2,4^2]} c\left(\int_{B_4} |\tilde{h}_k|^q dx + \int_{B_4} |D^2\tilde{h}_k|^q dx\right) dt \\ &= c\left(\int_{Q_4} |\tilde{h}_k|^q dz + \int_{Q_4} |D^2\tilde{h}_k|^q dz\right), \end{aligned} \tag{4.6}$$

and in turn, it follows from (4.5) that

$$\int_{Q_4} |D\tilde{h}_k|^q dz \le c. \tag{4.7}$$

In view of (4.5) and (4.7), we then see that $\{\tilde{h}_k\}_{k=1}^{\infty}$ is bounded in $W_q^{2,1}(Q_4)$. Therefore there exist a subsequence of $\{\tilde{h}_k\}_{k=1}^{\infty}$, which we still denote by $\{\tilde{h}_k\}_{k=1}^{\infty}$, and a function $\tilde{h} \in W_q^{2,1}(Q_4)$ such that

$$\begin{cases} \tilde{h}_k \rightharpoonup \tilde{h} & \text{weakly in } W_q^{2,1}(Q_4), \\ \tilde{h}_k \to \tilde{h} & \text{strongly in } L^q(Q_4), \end{cases} \quad \text{as} \ \ k \to \infty.$$

Then we infer from (4.5) that

$$\fint_{Q_4} |\tilde{h}|^q dz = 1 \ \text{ and } \ \tilde{h}_t = D^2\tilde{h} = 0. \tag{4.8}$$

So we can write $\tilde{h} = c_1 \cdot x + c_2$ for some constants $c_1 \in \mathbb{R}^n$ and $c_2 \in \mathbb{R}$. However, it follows from (4.4) that

$$c_1 = \fint_{Q_4} D\tilde{h} dz = 0 \ \text{ and } \ c_2 = \fint_{Q_4} \tilde{h} dz = 0.$$

In turn, we see that $\tilde{h} = 0$ in Q_4, which is a contradiction to the first equality in (4.8). □

Lemma 4.3. *For any $1 < q < \infty$, let $h \in W_q^{2,1}(Q_4^+)$ with $h = 0$ on T_4. Then there is a constant c depending only on q and n so that*

$$\fint_{Q_4^+} |h - \overline{(D_n h)}_{Q_4^+} x_n|^q dz \le c \fint_{Q_4^+} \left(|h_t|^q + |D^2 h|^q\right) dz. \tag{4.9}$$

Proof. Suppose that (4.9) is not true. Then there exists a sequence $\{h_k\}_{k=1}^{\infty}$ in $W_q^{2,1}(Q_4^+)$ with $h_k = 0$ on T_4 such that

$$\fint_{Q_4^+} |h_k - \overline{(D_n h_k)}_{Q_4^+} x_n|^q dz > k \fint_{Q_4^+} \left(|(h_k)_t|^q + |D^2 h_k|^q\right) dz. \tag{4.10}$$

By normalization, we may assume that

$$\fint_{Q_4^+} |h_k - \overline{(D_n h_k)}_{Q_4^+} x_n|^q dz = 1.$$

Then the inequality (4.10) yields

$$\fint_{Q_4^+} \left(|(h_k)_t|^q + |D^2 h_k|^q\right) dz < \frac{1}{k}.$$

Setting $\tilde{h}_k := h_k - \overline{(D_n h_k)}_{Q_4^+} x_n$, we then easily see that

$$\fint_{Q_4^+} |\tilde{h}_k|^q dz = 1 \quad \text{and} \quad \fint_{Q_4^+} \left(|(\tilde{h}_k)_t|^q + |D^2 \tilde{h}_k|^q\right) dz < \frac{1}{k} \le 1. \tag{4.11}$$

In a way analagous to how (4.7) was deduced, we can infer from (4.11) instead of (4.4) that

$$\int_{Q_4^+} |D\tilde{h}_k|^q dz \le c. \tag{4.12}$$

We also know that

$$\fint_{Q_4^+} D_n \tilde{h}_k dz = \fint_{Q_4^+} \left(D_n h_k - \overline{(D_n h_k)}_{Q_4^+}\right) dz = 0. \tag{4.13}$$

From (4.11) and (4.12), we see that $\{\tilde{h}_k\}_{k=1}^{\infty}$ is bounded in $W_q^{2,1}(Q_4^+)$, and so there exist a subsequence of $\{\tilde{h}_k\}_{k=1}^{\infty}$, which we still denote by $\{\tilde{h}_k\}_{k=1}^{\infty}$, and a function $\tilde{h} \in W_q^{2,1}(Q_4^+)$ with $\tilde{h} = 0$ on T_4 such that

$$\begin{cases} \tilde{h}_k \rightharpoonup \tilde{h} & \text{weakly in } W_q^{2,1}(Q_4^+), \\ \tilde{h}_k \to \tilde{h} & \text{strongly in } L^q(Q_4^+), \end{cases} \quad \text{as} \quad k \to \infty.$$

For $1 \leq i \leq n-1$, since $D_i\tilde{h}_k = 0$ on T_4, we apply the standard Poincaré inequality for each time slice of Q_4^+ in order to discover that

$$\fint_{Q_4^+} \left|D_i\tilde{h}_k\right|^q dz \leq c\fint_{Q_4^+} |D^2\tilde{h}_k|^q dz = c\fint_{Q_4^+} |D^2 h_k|^q dz < \frac{c}{k} \longrightarrow 0$$

as $k \to \infty$, which implies that

$$D_i\tilde{h} = 0.$$

Furthermore, it is easy to check from (4.11) that

$$\fint_{Q_4^+} |\tilde{h}|^q dz = 1 \quad \text{and} \quad \tilde{h}_t = D^2\tilde{h} = 0. \tag{4.14}$$

So we can write $\tilde{h} = c_1 x_n + c_2$ for some constants $c_1, c_2 \in \mathbb{R}$. However, since $\tilde{h} = 0$ on T_4, we have $c_2 = 0$, and then by (4.13) we see that

$$c_1 = \fint_{Q_4^+} D_n\tilde{h} dz = 0.$$

Therefore, we finally have $\tilde{h} = 0$ in Q_4^+, which is a contradiction to the first equality in (4.14). This completes the proof. □

Let us now derive the following comparison estimates.

Lemma 4.4. *Let $1 < q < \infty$. Assume that $\mathbf{B} = (b_{ij}) : \mathbb{R}^{n+1} \to \mathbb{R}^{n\times n}$ satisfies the uniform parabolicity condition* (1.2)*. For any $\epsilon \in (0, 1)$, there is $\delta = \delta(\epsilon, n, \Lambda, q) > 0$ such that the following hold:*

If $\mathbf{B}$ is $(\delta, 4)$-vanishing and $h \in W_q^{2,1}(Q_4)$ is a solution of

$$h_t - b_{ij}D_{ij}h = g \quad \text{in } Q_4 \tag{4.15}$$

satisfying

$$\fint_{Q_4} \left(|h_t|^q + |D^2h|^q\right) dz \leq 1 \quad \text{and} \quad \fint_{Q_4} |g|^q dz \leq \delta,$$

then there exist a constant matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})$ with $\|\overline{\mathbf{B}}_{Q_4} - \tilde{\mathbf{B}}\|_{L^\infty(\mathbb{R}^{n+1})} \leq \epsilon$ and a solution $v \in W_q^{2,1}(Q_4)$ of

$$v_t - \tilde{b}_{ij}D_{ij}v = 0 \quad \text{in } Q_4 \tag{4.16}$$

satisfying

$$\fint_{Q_4} \left(|v_t|^q + |D^2v|^q\right) dz \leq 1 \tag{4.17}$$

and

$$\int_{Q_4} |h - \overline{h}_{Q_4} - (\overline{Dh})_{Q_4} \cdot x - v|^q dz \leq \epsilon.$$

Proof. We argue by contradiction. If not, there exist $\epsilon_0 > 0$, $h_l \in W_q^{2,1}(Q_4)$, $g_l \in L^q(Q_4)$, and $\mathbf{B}_l = (b_{ij}^l) : \mathbb{R}^{n+1} \to \mathbb{R}^{n\times n}$, where $l = 1, 2, \ldots$, such that $\mathbf{B}_l$ is uniformly parabolic with the parabolicity constant Λ satisfying $[\mathbf{B}_l]_4 \le \frac{1}{l}$, which implies that

$$\fint_{Q_4} |\mathbf{B}_l - \overline{\mathbf{B}_l}_{Q_4}| dz \le \frac{1}{l} \tag{4.18}$$

and $h_l \in W_q^{2,1}(Q_4)$ is a solution of

$$(h_l)_t - b_{ij}^l D_{ij} h_l = g_l \quad \text{in } Q_4$$

satisfying

$$\fint_{Q_4} \left(|(h_l)_t|^q + |D^2 h_l|^q\right) dz \le 1 \;\text{ and }\; \fint_{Q_4} |g_l|^q dz \le \frac{1}{l}, \tag{4.19}$$

but

$$\int_{Q_4} |h_l - \overline{h_l}_{Q_4} - (\overline{Dh_l})_{Q_4} \cdot x - v|^q dz > \epsilon_0, \tag{4.20}$$

for any constant matrix $\tilde{\mathbf{B}}$ with $\|\overline{\mathbf{B}}_{Q_4} - \tilde{\mathbf{B}}\|_{L^\infty(\mathbb{R}^{n+1})} \le \epsilon_0$ and any solution $v \in W_q^{2,1}(Q_4)$ of (4.16) with (4.17).

By virtue of the uniform parabolicity on $\mathbf{B}_l$ and (4.18), we infer that

$$\fint_{Q_4} |\mathbf{B}_l - \overline{\mathbf{B}_l}_{Q_4}|^{q'} dz \le (2\Lambda)^{q'-1} \fint_{Q_4} |\mathbf{B}_l - \overline{\mathbf{B}_l}_{Q_4}| dz \le \frac{(2\Lambda)^{q'-1}}{l},$$

where $q' = \frac{q}{q-1}$. On the other hand, it is clear that $\{\overline{\mathbf{B}_l}_{Q_4}\}_{l=1}^\infty$ is bounded in $\mathbb{R}^{n\times n}$, and so it has a subsequence, which is still denoted by $\{\overline{\mathbf{B}_l}_{Q_4}\}$, such that

$$\overline{\mathbf{B}_l}_{Q_4} \longrightarrow \mathbf{B}_0 \;\text{ in }\; \mathbb{R}^{n\times n} \quad \text{as } \; l \to \infty, \tag{4.21}$$

for some constant matrix $\mathbf{B}_0 = (b_{ij}^0)$. Therefore it follows that

$$\mathbf{B}_l \longrightarrow \mathbf{B}_0 \;\text{ in }\; L^{q'}(Q_4) \quad \text{as } \; l \to \infty. \tag{4.22}$$

Let us now consider $v_l := h_l - \overline{h_l}_{Q_4} - (\overline{Dh_l})_{Q_4} \cdot x$. Then we see from Lemma 4.2 that

$$\fint_{Q_4} |v_l|^q dz \le c \fint_{Q_4} \left(|(v_l)_t|^q + |D^2 v_l|^q\right) dz = c \fint_{Q_4} \left(|(h_l)_t|^q + |D^2 h_l|^q\right) dz \le c, \tag{4.23}$$

where the last inequality follows from (4.19). Moreover, in an analogous way to (4.6), the interpolation inequality leads us to get

$$\int_{Q_4} |Dv_l|^q dz \le c \left(\int_{Q_4} |v_l|^q dz + \int_{Q_4} |D^2 v_l|^q dz \right),$$

and then it follows from (4.23) that

$$\int_{Q_4} |Dv_l|^q dz \le c. \tag{4.24}$$

Therefore, in view of (4.23) and (4.24), we see that $\{v_l\}_{l=1}^{\infty}$ is bounded in $W_q^{2,1}(Q_4)$, and so there exist a subsequence of $\{v_l\}_{l=1}^{\infty}$, which is still denoted by $\{v_l\}_{l=1}^{\infty}$, and a function $v_0 \in W_q^{2,1}(Q_4)$ such that

$$\begin{cases} v_l \rightharpoonup v_0 & \text{weakly in } W_q^{2,1}(Q_4), \\ v_l \to v_0 & \text{strongly in } L^q(Q_4), \end{cases} \quad \text{as } \ l \to \infty. \tag{4.25}$$

From (4.19), (4.22), and (4.25), we observe that $v_0 \in W_q^{2,1}(Q_4)$ is a solution of

$$(v_0)_t - b_{ij}^0 D_{ij} v_0 = 0 \quad \text{in} \quad Q_4$$

satisfying

$$\fint_{Q_4} \left(|(v_0)_t|^q + |D^2 v_0|^q \right) dz \le \liminf_{l\to\infty} \fint_{Q_4} \left(|(v_l)_t|^q + |D^2 v_l|^q \right) dz \le 1. \tag{4.26}$$

However, it is a contradiction to (4.20). This completes the proof. □

Corollary 4.5. *Under the hypotheses and conclusion of Lemma* 4.4*, we have*

$$\fint_{Q_1} \left(|(h-v)_t|^q + |D^2(h-v)|^q \right) dz \le \epsilon.$$

Proof. From the assumptions of Lemma 4.4, we see that

$$\fint_{Q_4} |g|^q dz \le \delta \quad \text{and} \quad \fint_{Q_4} |\mathbf{B} - \overline{\mathbf{B}}_{Q_4}| dz \le \delta. \tag{4.27}$$

Apply Lemma 4.4 with any $\kappa > 0$ in place of ϵ in order to find a constant matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})$ with $\|\overline{\mathbf{B}}_{Q_4} - \tilde{\mathbf{B}}\|_{L^\infty(\mathbb{R}^{n+1})} \le \kappa$ and a solution $v \in W^{2,q}(Q_4)$ of (4.16) such that

$$\fint_{Q_4} \left(|v_t|^q + |D^2 v|^q \right) dz \le 1 \quad \text{and} \quad \int_{Q_4} |h - \overline{h}_{Q_4} - (\overline{Dh})_{Q_4} \cdot x - v|^q dz \le \kappa \tag{4.28}$$

by taking $\delta = \delta(\kappa, n, \Lambda, q) > 0$ sufficiently small. Then we use the local estimates on derivatives of solutions to the equation (4.16) (see Theorem 9 in [21, page 61]) to obtain

$$\|v_t\|_{L^\infty(Q_2)}^q + \|D^2 v\|_{L^\infty(Q_2)}^q \le c \fint_{Q_4} \left(|v_t|^q + |D^2 v|^q \right) dz \le c. \tag{4.29}$$

Setting $\tilde{h} := h - \overline{h}_{Q_4} - (\overline{Dh})_{Q_4} \cdot x - v$, one can readily see that $\tilde{h} \in W_q^{2,1}(Q_4)$ is a solution of

$$\tilde{h}_t - b_{ij} D_{ij} \tilde{h} = g + (b_{ij} - \tilde{b}_{ij}) D_{ij} v \quad \text{in } Q_4.$$

Then Lemma 4.1 gives

$$\fint_{Q_1} \left(|\tilde{h}_t|^q + |D^2 \tilde{h}|^q \right) dz \le c \left(\fint_{Q_2} \left| g + (b_{ij} - \tilde{b}_{ij}) D_{ij} v \right|^q dz + \fint_{Q_2} |\tilde{h}|^q dz \right) \tag{4.30}$$

if we take $\delta = \delta(\kappa, n, \Lambda, q) > 0$ sufficiently small.

In view of (4.27)-(4.30) and (1.2), we consequently deduce that

$$\begin{aligned}
&\fint_{Q_1}\left(|(h-v)_t|^q+\left|D^2(h-v)\right|^q\right)dz=\fint_{Q_1}\left(|\tilde h_t|^q+|D^2\tilde h|^q\right)dz\\
&\le c\left(\fint_{Q_2}\left|g+(b_{ij}-\tilde b_{ij})D_{ij}v\right|^q dz+\fint_{Q_2}|\tilde h|^q dz\right)\\
&\le c\left(\fint_{Q_4}|g|^q dz+\|D^2v\|^q_{L^\infty(Q_2)}\fint_{Q_4}\left|\mathbf{B}-\tilde{\mathbf{B}}\right|^q dz+\kappa\right)\\
&\le c\left(\fint_{Q_4}|g|^q dz+2^{q-1}\fint_{Q_4}\left(\left|\mathbf{B}-\overline{\mathbf{B}}_{Q_4}\right|^q+\left|\overline{\mathbf{B}}_{Q_4}-\tilde{\mathbf{B}}\right|^q\right)dz+\kappa\right)\\
&\le c\left(\delta+(4\Lambda)^{q-1}\fint_{Q_4}\left|\mathbf{B}-\overline{\mathbf{B}}_{Q_4}\right|dz+2^{q-1}\kappa^q+\kappa\right)\\
&\le c(\delta+\kappa),
\end{aligned}$$

where the elementary inequality $(a+b)^\beta\le 2^{\beta-1}(a^\beta+b^\beta)$ for any $a,b>0$ and $\beta\ge 1$ has been used in the third inequality. Hence, the proof is completed by choosing $\kappa=\kappa(\epsilon,n,\Lambda,q)>0$ and $\delta=\delta(\epsilon,n,\Lambda,q)>0$ small enough so that $c(\delta+\kappa)<\epsilon$. □

The following is the flat boundary version of Lemma 4.4, which will be proved by the same argument as in Lemma 4.4 with Lemma 4.3 instead of Lemma 4.2.

Lemma 4.6. *Let $1<q<\infty$. Assume that $\mathbf{B}=(b_{ij}):\mathbb{R}^{n+1}\to\mathbb{R}^{n\times n}$ satisfies the uniform parabolicity condition* (1.2)*. For any $\epsilon\in(0,1)$, there is $\delta=\delta(\epsilon,n,\Lambda,q)>0$ such that the following hold:*

If $\mathbf{B}$ is $(\delta,4)$-vanishing and $h\in W^{2,1}_q(Q_4^+)$ is a solution of

$$\left\{\begin{array}{rcll} h_t-b_{ij}D_{ij}h &=& g & \text{in}\quad Q_4^+,\\ h &=& 0 & \text{on}\quad T_4\end{array}\right. \tag{4.31}$$

satisfying

$$\fint_{Q_4^+}|h_t|^q+|D^2h|^q dz\le 1\quad\text{and}\quad\fint_{Q_4^+}|g|^q dz\le\delta,$$

then there exist a constant matrix $\tilde{\mathbf{B}}=(\tilde b_{ij})$ with $\|\overline{\mathbf{B}}_{Q_4^+}-\tilde{\mathbf{B}}\|_{L^\infty(\mathbb{R}^{n+1})}\le\epsilon$ and a solution $v\in W^{2,1}_q(Q_4^+)$ of

$$\left\{\begin{array}{rcll} v_t-\tilde b_{ij}D_{ij}v &=& 0 & \text{in}\quad Q_4^+,\\ v &=& 0 & \text{on}\quad T_4\end{array}\right. \tag{4.32}$$

satisfying

$$\fint_{Q_4^+}|v_t|^q+|D^2v|^q dz\le 1 \tag{4.33}$$

and

$$\int_{Q_4^+}|h-(\overline{D_nh})_{Q_4^+}x_n-v|^q dz\le\epsilon.$$

Proof. We argue by contradiction. If not, there exist $\epsilon_0>0$, $h_l\in W^{2,1}_q(Q_4^+)$, $g_l\in L^q(Q_4^+)$, and $\mathbf{B}_l=(b^l_{ij}):\mathbb{R}^{n+1}\to\mathbb{R}^{n\times n}$, where $l=1,2,\dots$, such that $\mathbf{B}_l$

is uniformly parabolic with the parabolicity constant Λ satisfying $[\mathbf{B}_l]_4 \le \frac{1}{l}$, and $h_l \in W_q^{2,1}(Q_4^+)$ is a solution of

$$\begin{cases} (h_l)_t - b_{ij}^l D_{ij} h_l = g_l & \text{in} \quad Q_4^+, \\ h_l = 0 & \text{on} \quad T_4 \end{cases}$$

satisfying

$$\fint_{Q_4^+} |(h_l)_t|^q + |D^2 h_l|^q dz \le 1 \;\text{ and }\; \fint_{Q_4^+} |g_l|^q dz \le \frac{1}{l}, \tag{4.34}$$

but

$$\int_{Q_4^+} |h_l - (\overline{D_n h_l})_{Q_4^+} x_n - v|^q dz > \epsilon_0, \tag{4.35}$$

for any constant matrix $\tilde{\mathbf{B}}$ with $\|\overline{\mathbf{B}}_{Q_4^+} - \tilde{\mathbf{B}}\|_{L^\infty(\mathbb{R}^{n+1})} \le \epsilon_0$ and any solution $v \in W_q^{2,1}(Q_4^+)$ of (4.32) satisfying (4.33).

From the condition $[\mathbf{B}_l]_4 \le \frac{1}{l}$, a simple computation gives

$$\begin{aligned} \fint_{Q_4^+} |\mathbf{B}_l - \overline{\mathbf{B}_l}_{Q_4^+}| dz &\le 2\fint_{Q_4} |\mathbf{B}_l - \overline{\mathbf{B}_l}_{Q_4}| dz + |\overline{\mathbf{B}_l}_{Q_4^+} - \overline{\mathbf{B}_l}_{Q_4}| \\ &\le 2\fint_{Q_4} |\mathbf{B}_l - \overline{\mathbf{B}_l}_{Q_4}| dz + 2\fint_{Q_4} |\mathbf{B}_l - \overline{\mathbf{B}_l}_{Q_4}| dz \le \frac{4}{l}. \end{aligned} \tag{4.36}$$

By the same argument as in (4.22) along with (4.36), we deduce that

$$\mathbf{B}_l \longrightarrow \mathbf{B}_0 \;\text{ in }\; L^{q'}(Q_4^+) \;\text{ as }\; l \to \infty \;\text{ (up to subsequence)}, \tag{4.37}$$

for some constant matrix $\mathbf{B}_0 = (b_{ij}^0)$.

We now set $v_l = h_l - (\overline{D_n h_l})_{Q_4^+} x_n$. It is clear that $v_l \in W_q^{2,1}(Q_4^+)$ with $v_l = 0$ on T_4, and then Lemma 4.3 implies that

$$\begin{aligned} \fint_{Q_4^+} |v_l|^q dz &\le c\fint_{Q_4^+} |(v_l)_t|^q + |D^2 v_l|^q dz \\ &= c\fint_{Q_4^+} |(h_l)_t|^q + |D^2 h_l|^q dz \;\le\; c, \end{aligned} \tag{4.38}$$

where the last inequality comes from (4.34). In an analogous way to (4.24) with (4.38) in place of (4.23), we also have

$$\int_{Q_4^+} |Dv_l|^q dz \le c\left(\int_{Q_4^+} |v_l|^q dz + \int_{Q_4^+} |D^2 v_l|^q dz\right) \le c. \tag{4.39}$$

In turn, it follows from (4.34), (4.38), and (4.39) that $\{v_l\}_{l=1}^\infty$ is bounded in $W_q^{2,1}(Q_4^+)$. Then there exist a subsequence of $\{v_l\}_{l=1}^\infty$, which is still denoted by $\{v_l\}_{l=1}^\infty$, and a function $v_0 \in W_q^{2,1}(Q_4^+)$ such that

$$\begin{cases} v_l \rightharpoonup v_0 & \text{weakly in } W_q^{2,1}(Q_4^+), \\ v_l \to v_0 & \text{strongly in } L^q(Q_4^+), \end{cases} \quad \text{as } \; l \to \infty. \tag{4.40}$$

By (4.34), (4.37), and (4.40), it is easy to check that $v_0 \in W_q^{2,1}(Q_4^+)$ is a solution of

$$\begin{cases} (v_0)_t - b_{ij}^0 D_{ij} v_0 = 0 & \text{in} \quad Q_4^+, \\ v_0 = 0 & \text{on} \quad T_4 \end{cases}$$

satisfying

$$\fint_{Q_4^+} |(v_0)_t|^q + |D^2 v_0|^q dz \le \liminf_{l\to\infty} \fint_{Q_4^+} |(v_l)_t|^q + |D^2 v_l|^q dz \le 1. \tag{4.41}$$

However, this is a contradiction to (4.35). This completes the proof. □

Corollary 4.7. *Under the hypotheses and conclusion of Lemma* 4.6*, we have*

$$\fint_{Q_1^+} |(h-v)_t|^q + |D^2(h-v)|^q dz \le \epsilon.$$

Proof. We proceed as in Corollary 4.5 with Lemma 4.6 in place of Lemma 4.4. □

5. Local estimates

In this section, we establish interior and boundary *a priori* weighted $W^{2,1}_{p(\cdot)}$-estimates, which are a core part of the proof of our main result, Theorem 2.5.

The following is the main theorem in this section.

Theorem 5.1. *Let $p(\cdot) \in \mathcal{P}^{\log}_{\pm}$ with* (2.1)*, let the log-Hölder constant $c_{LH} > 0$, and let the modulus of continuity be $\theta(\cdot)$, and suppose $w \in A_{p(\cdot)}$. Then there exists a small $\rho_0 = \rho_0(n, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) \in (0,1)$ such that the following hold:*
For any $\rho \in (0, \rho_0]$, there exists a small $\delta = \delta(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) \in (0,1)$ such that:

(i) (*Interior estimates*) *If $\mathbf{A}$ is $(\delta, 4\rho)$-vanishing and $f \in L^{p(\cdot)}(Q_{4\rho}, w)$, then for any solution $u \in W^{2,1}_{p(\cdot)}(Q_{4\rho}, w)$ of*

$$u_t - a_{ij} D_{ij} u = f \quad \text{in } Q_{4\rho},$$

we have

$$\begin{aligned} &\|u_t\|_{L^{p(\cdot)}(Q_\rho, w)} + \|D^2 u\|_{L^{p(\cdot)}(Q_\rho, w)} \\ &\qquad \le c\rho^{-\frac{(n+2)\gamma_2}{\gamma_1}} \left(\|f\|_{L^{p(\cdot)}(Q_{4\rho}, w)} + \frac{1}{\rho^2} \|u\|_{L^{p(\cdot)}(Q_{4\rho}, w)} \right) \end{aligned} \tag{5.1}$$

for some $c = c(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 1$.

(ii) (*Boundary estimates*) *If $\mathbf{A}$ is $(\delta, 4\rho)$-vanishing and $f \in L^{p(\cdot)}(Q^+_{4\rho}, w)$, then for any solution $u \in W^{2,1}_{p(\cdot)}(Q^+_{4\rho}, w)$ of*

$$\begin{cases} u_t - a_{ij} D_{ij} u = f & \text{in } Q^+_{4\rho}, \\ u = 0 & \text{on } T_{4\rho}, \end{cases} \tag{5.2}$$

we have

$$\begin{aligned} &\|u_t\|_{L^{p(\cdot)}(Q^+_\rho, w)} + \|D^2 u\|_{L^{p(\cdot)}(Q^+_\rho, w)} \\ &\qquad \le c\rho^{-\frac{(n+2)\gamma_2}{\gamma_1}} \left(\|f\|_{L^{p(\cdot)}(Q^+_{4\rho}, w)} + \frac{1}{\rho^2} \|u\|_{L^{p(\cdot)}(Q^+_{4\rho}, w)} \right) \end{aligned} \tag{5.3}$$

for some $c = c(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 1$.

Since the proof of the interior estimate (5.1) in Theorem 5.1 is analogous to that of the boundary estimate (5.3) in Theorem 5.1, we shall only establish the boundary estimate (5.3). We divide the proof of the boundary case into several subsections.

5.1. **Setting and notation.** We first take ρ_0 as follows. Recall that

$$\epsilon_0 = \epsilon_0(n, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) \in (0, \gamma_1 - 1)$$

determined in (2) of Lemma 3.3 with $A_0 = c_m[w]_{A_{p(\cdot)}}$. Without loss of generality, we assume that

$$\epsilon_0 \leq \frac{2\gamma_2}{3}. \tag{5.4}$$

Then we take $\rho_0 > 0$ to be the largest number satisfying

$$\rho_0 \leq \frac{1}{8}, \quad |C_{4\rho_0}| \leq 1, \quad \text{and} \quad \theta(8\sqrt{2}\rho_0) \leq \min\left\{\frac{\gamma_1\epsilon_0}{2\gamma_2 - \epsilon_0}, \frac{\epsilon_0}{4}, 1\right\}. \tag{5.5}$$

From now on, we fix $\rho \leq \rho_0$ and suppose that $\mathbf{A}$ is $(\delta, 4\rho)$-vanishing, where $\delta > 0$ will be determined later; see Remark 5.4. Setting

$$p^- := \inf_{z \in Q^+_{2\rho}} p(z) \quad \text{and} \quad p^+ := \sup_{z \in Q^+_{2\rho}} p(z)$$

and recalling $\tilde{\gamma}_0 = 1 + \frac{\epsilon_0}{2(\gamma_2 - \epsilon_0)} < \gamma_1$ in (3.8) of Lemma 3.7, define

$$\gamma_0 := \frac{1 + \tilde{\gamma}_0}{2} = 1 + \frac{\epsilon_0}{4(\gamma_2 - \epsilon_0)} = \frac{4\gamma_2 - 3\epsilon_0}{4(\gamma_2 - \epsilon_0)}. \tag{5.6}$$

Then we see that

$$1 < \gamma_0 < \tilde{\gamma}_0 < \gamma_1 \leq p^- \leq p^+ \leq \gamma_2 < +\infty.$$

Moreover, using the restriction $\theta(4\rho) \leq \theta(4\rho_0) \leq \min\left\{\frac{\gamma_1\epsilon_0}{2\gamma_2 - \epsilon_0}, \frac{\epsilon_0}{4}, 1\right\}$ as in (5.5), along with (5.4), we obtain that

$$\frac{\gamma_0 p(z)}{p^-} \leq \gamma_0\left(1 + \frac{\theta(4\rho)}{\gamma_1}\right) \leq \gamma_0\left(1 + \frac{\epsilon_0}{2\gamma_2 - \epsilon_0}\right) = \tilde{\gamma}_0 \quad \text{for } z \in Q^+_{2\rho} \tag{5.7}$$

and

$$\begin{aligned} p^+ - \epsilon_0 &= p^+ - \frac{4(\gamma_2 - \epsilon_0)p^-}{4\gamma_2 - 3\epsilon_0} + \frac{p^-}{\gamma_0} - \epsilon_0 = p^+ - p^- + \frac{\epsilon_0 p^-}{4\gamma_2 - 3\epsilon_0} + \frac{p^-}{\gamma_0} - \epsilon_0 \\ &\leq \theta(4\rho) + \frac{\epsilon_0\gamma_2}{4\gamma_2 - 3\epsilon_0} + \frac{p^-}{\gamma_0} - \epsilon_0 \leq \frac{\epsilon_0}{4} + \frac{\epsilon_0}{2} + \frac{p^-}{\gamma_0} - \epsilon_0 \\ &< \frac{p^-}{\gamma_0}. \end{aligned} \tag{5.8}$$

To simplify the proof of (5.3), we assume that

$$\|f\|_{L^{p(\cdot)}(Q^+_{4\rho}, w)} \leq 1 \quad \text{and} \quad \|u\|_{L^{p(\cdot)}(Q^+_{4\rho}, w)} \leq \rho^2, \tag{5.9}$$

and then show that

$$\|u_t\|_{L^{p(\cdot)}(Q^+_{\rho}, w)} + \|D^2u\|_{L^{p(\cdot)}(Q^+_{\rho}, w)} \leq c\rho^{-\frac{(n+2)\gamma_2}{\gamma_1}} \tag{5.10}$$

for some $c = c(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 1$. In fact, by virtue of the standard normalization argument, defining

$$\tilde{u} := \frac{u}{\|f\|_{L^{p(\cdot)}(Q^+_{4\rho}, w)} + \frac{1}{\rho^2}\|u\|_{L^{p(\cdot)}(Q^+_{4\rho}, w)}}$$

and

$$\tilde{f} := \frac{f}{\|f\|_{L^{p(\cdot)}(Q^+_{4\rho}, w)} + \frac{1}{\rho^2}\|u\|_{L^{p(\cdot)}(Q^+_{4\rho}, w)}},$$

for u and f given in Theorem 5.1(ii), we have that

$$\|\tilde{f}\|_{L^{p(\cdot)}(Q^+_{4\rho},w)} \leq 1, \quad \|\tilde{u}\|_{L^{p(\cdot)}(Q^+_{4\rho},w)} \leq \rho^2$$

and $\tilde{u}$ is a solution of

$$\begin{cases} \tilde{u}_t - a_{ij}D_{ij}\tilde{u} &= \tilde{f} \quad \text{in} \quad Q^+_{4\rho}, \\ \tilde{u} &= 0 \quad \text{on} \quad T_{4\rho}. \end{cases}$$

Then (5.10) implies that

$$\|\tilde{u}_t\|_{L^{p(\cdot)}(Q^+_{\rho},w)} + \|D^2\tilde{u}\|_{L^{p(\cdot)}(Q^+_{\rho},w)} \leq c\rho^{-\frac{(n+2)\gamma_2}{\gamma_1}},$$

which means the desired estimate (5.3). Therefore, from now on, we prove the estimate (5.10), instead of (5.3), under the additional assumption (5.9).

We remark that in view of Lemma 3.7, especially (3.9), we have from (5.9) and the restriction $|C_{4\rho}| \leq |C_{4\rho_0}| \leq 1$ and $\theta(8\sqrt{2}\rho) \leq \theta(8\sqrt{2}\rho_0) \leq \frac{\epsilon_0}{4}$ in (5.5) that

$$\|f\|_{L^{\tilde{\gamma}_0}(Q^+_{4\rho})} \leq c \;\text{ and }\; \|u\|_{L^{\tilde{\gamma}_0}(Q^+_{4\rho})} \leq c\rho^2 \tag{5.11}$$

for some $c = c(n, \gamma_1, \gamma_2, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 0$. Therefore, recalling (ii) of Lemma 4.1 we see that

$$\|u_t\|_{L^{\tilde{\gamma}_0}(Q^+_{2\rho})} + \|D^2u\|_{L^{\tilde{\gamma}_0}(Q^+_{2\rho})} \leq c,$$

and hence, it follows from (3.3) that

$$\int_{Q^+_{2\rho}} |u_t|^{\tilde{\gamma}_0}\,dz + \int_{Q^+_{2\rho}} |D^2u|^{\tilde{\gamma}_0}\,dz \leq c \tag{5.12}$$

for some $c = c(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, \theta(\cdot), [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 0$.

Hereafter, in this section, we denote by the letter c any positive constant depending only on $n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}$, and $w(Q_{4\rho})$, and it is possibly varying from line to line.

5.2. **Covering argument.** Let us define

$$\lambda_0 := \fint_{Q^+_{2\rho}} \left[|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}\left(|f|^{\frac{\gamma_0 p(z)}{p^-}} + 1 \right) \right] dz > 1. \tag{5.13}$$

We choose any s_1, s_2 with $1 \leq s_1 < s_2 \leq 2$, and for $\lambda > 0$, define the upper-level set

$$E(\lambda) := \left\{ z \in Q^+_{s_1\rho} : |u_t(z)|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u(z)|^{\frac{\gamma_0 p(z)}{p^-}} > \lambda \right\}. \tag{5.14}$$

Using a stopping time argument and the Vitali covering lemma, we will find an appropriate covering of the upper-level set $E(\lambda)$, where λ is large enough so that

$$\lambda \geq A\lambda_0, \;\text{ where }\; A := \left(\frac{240}{s_2 - s_1} \right)^{n+2}. \tag{5.15}$$

For each $\xi \in E(\lambda)$, define a continuous function $\Phi_\xi : (0, (s_2 - s_1)\rho] \to [0, \infty)$ by

$$\Phi_\xi(\tau) := \fint_{Q^+_\tau(\xi)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(z)}{p^-}} \right) dz.$$

Note that $Q^+_\tau(\xi) \subset Q^+_{s_2\rho} \subset Q^+_{2\rho}$ for $\tau \in (0,(s_2-s_1)\rho)$. Then, for any $\xi \in E(\lambda)$ and any $\tau \in \left[\frac{(s_2-s_1)\rho}{120}, (s_2-s_1)\rho\right]$, we have

$$\begin{aligned}\Phi_\xi(\tau) &= \fint_{Q^+_\tau(\xi)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(z)}{p^-}}\right) dz \\ &\le \frac{|Q^+_{2\rho}|}{|Q^+_\tau(\xi)|} \fint_{Q^+_{2\rho}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(z)}{p^-}}\right) dz \\ &= \left(\frac{2\rho}{\tau}\right)^{n+2} \fint_{Q^+_{2\rho}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(z)}{p^-}}\right) dz \\ &< \left(\frac{240}{s_2-s_1}\right)^{n+2} \fint_{Q^+_{2\rho}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(z)}{p^-}}\right) dz \\ &< A\lambda_0 \le \lambda,\end{aligned}$$

where the inequalities in the last two lines come from (5.13) and (5.15). On the other hand, Lebesgue's differentiation theorem leads us to obtain

$$\lim_{\tau\to 0}\Phi_\xi(\tau) = \lim_{\tau\to 0} \fint_{Q^+_\tau(\xi)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(z)}{p^-}}\right) dz > \lambda,$$

for almost every $\xi \in E(\lambda)$. Hence, for almost every $\xi \in E(\lambda)$, there exists

$$\tau_\xi \in \left(0, \frac{(s_2-s_1)\rho}{120}\right)$$

such that

$$\Phi_\xi(\tau_\xi) = \lambda \quad \text{and} \quad \Phi_\xi(\tau) < \lambda, \quad \text{for all } \tau \in (\tau_\xi, (s_2-s_1)\rho].$$

According to the Vitali covering lemma, we consequently find $\xi^k \in E(\lambda)$ and $\tau_k := \tau_{\xi^k} \in \left(0, \frac{(s_2-s_1)\rho}{120}\right)$, $k = 1, 2, \ldots$, such that the family of parabolic cylinders $\left\{Q^+_{\tau_k}(\xi^k)\right\}_{k=1}^\infty$ is mutually disjoint and satisfies the relation

$$E(\lambda) \subset \bigcup_{k=1}^\infty Q^+_{5\tau_k}(\xi^k) \subset Q^+_{s_2\rho}, \tag{5.16}$$

except a Lebesgue measure zero set. Note that for each k we have

$$\Phi_{\xi^k}(\tau_k) = \lambda \quad \text{and} \quad \Phi_{\xi^k}(\tau) < \lambda, \quad \text{for all } \tau \in (\tau_k, (s_2-s_1)\rho]. \tag{5.17}$$

Lemma 5.2. *Under the above settings, we have for each $k = 1, 2, \ldots$,*

$$\begin{aligned} w(C_{\tau_k}(\xi^k)) &\le \frac{2c_a}{\lambda^{p^+-\epsilon_0}} \\ &\times \left[\int_{Q^+_{\tau_k}(\xi^k)\cap\{|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}} > \frac{\lambda}{4c_a}\}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{p^+-\epsilon_0} w(z)\,dz \right. \\ &\left. + \int_{Q^+_{\tau_k}(\xi^k)\cap\{|f|^{\frac{\gamma_0 p(z)}{p^-}} > \frac{\lambda\delta}{4c_a}\}} \left(\delta^{-1}|f|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{p^+-\epsilon_0} w(z)\,dz \right], \end{aligned} \tag{5.18}$$

for some $c_a = c_a(n, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) > 1$.

Proof. By the first equality of (5.17), we have

$$\lambda \leq \frac{2^{n+1}}{|C_{\tau_k}(\xi^k)|} \int_{Q^+_{\tau_k}(\xi^k)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(z)}{p^-}} \right) dz.$$

Since $w \in A_{p^+-\epsilon_0}(Q_{4\rho})$, we apply (3.1) to obtain

$$\begin{aligned} w(C_{\tau_k}(\xi^k)) \quad \leq \quad & \frac{c_a}{\lambda^{p^+-\epsilon_0}} \Bigg[\int_{Q^+_{\tau_k}(\xi^k)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} \right)^{p^+-\epsilon_0} w(z)\,dz \\ & + \int_{Q^+_{\tau_k}(\xi^k)} \left(\delta^{-1} |f|^{\frac{\gamma_0 p(z)}{p^-}} \right)^{p^+-\epsilon_0} w(z)\,dz \Bigg], \end{aligned} \tag{5.19}$$

for some $c_a = c_a(n, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) > 1$. Note that

$$\begin{aligned} & \int_{Q^+_{\tau_k}(\xi^k)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} \right)^{p^+-\epsilon_0} w(z)\,dz \\ & \leq \int_{Q^+_{\tau_k}(\xi^k) \cap \{|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} > \frac{\lambda}{4c_a}\}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} \right)^{p^+-\epsilon_0} w(z)\,dz \\ & \quad + \frac{\lambda^{p^+-\epsilon_0}}{4c_a} w(C_{\tau_k}(\xi^k)) \end{aligned}$$

and

$$\begin{aligned} & \int_{Q^+_{\tau_k}(\xi^k)} \left(\delta^{-1} |f|^{\frac{\gamma_0 p(z)}{p^-}} \right)^{p^+-\epsilon_0} w(z)\,dz \\ & \leq \int_{Q^+_{\tau_k}(\xi^k) \cap \{|f|^{\frac{\gamma_0 p(z)}{p^-}} > \frac{\lambda\delta}{4c_a}\}} \left(\delta^{-1} |f|^{\frac{\gamma_0 p(z)}{p^-}} \right)^{p^+-\epsilon_0} w(z)\,dz + \frac{\lambda^{p^+-\epsilon_0}}{4c_a} w(C_{\tau_k}(\xi^k)). \end{aligned}$$

Therefore, inserting the above two inequalities into (5.19), we conclude the desired estimate (5.18). □

Now, we seek comparison estimates on each cylinder $Q_{5\tau_k}(\xi^k)$. We first divide the covers $Q_{5\tau_k}(\xi^k)$, $k = 1, 2, \ldots$, into two cases: the *interior case* $B_{20\tau_k}(y^k) \subset B^+_{s_2\rho}$ and the *boundary case* $B_{20\tau_k}(y^k) \not\subset B^+_{s_2\rho}$, i.e., $B_{20\tau_k}(y^k) \cap \{x \in \mathbb{R}^n : x_n < 0\} \neq \emptyset$, where $\xi^k := (y^k, s^k)$. In particular, for the boundary case, we can find a point $\tilde{\xi}^k := (\tilde{y}^k, s^k)$ where $\tilde{y}^k \in B_{s_2\rho}(0) \cap \{x \in \mathbb{R}^n : x_n = 0\}$ satisfying $|y^k - \tilde{y}^k| < 20\tau_k$.

Lemma 5.3. *Under the above settings, the following hold:*

(a) (*Interior case*) *If* $B_{20\tau_k}(y^k) \subset B^+_{s_2\rho}$, *we have*

$$\fint_{Q_{20\tau_k}(\xi^k)} \left(|u_t|^{\gamma_0} + |D^2 u|^{\gamma_0} \right) dz \leq c_0 \lambda^{\frac{p^-}{p_k^+}} \text{ and } \fint_{Q_{20\tau_k}(\xi^k)} |f|^{\gamma_0} dz \leq c_0 \lambda^{\frac{p^-}{p_k^+}} \delta^{\frac{\gamma_1}{\gamma_2}}, \tag{5.20}$$

for some $c_0 = c_0(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 1$. *Moreover, for any* $\epsilon \in (0,1)$, *there exist* $\delta = \delta(\epsilon, n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) > 0$ *and* $v_k \in W^{2,1}_{\gamma_0}(Q_{20\tau_k}(\xi^k)) \cap W^{2,1}_{\infty}(Q_{5\tau_k}(\xi^k))$ *such that*

$$\fint_{Q_{5\tau_k}(\xi^k)} \left(|(u - v_k)_t|^{\gamma_0} + |D^2(u - v_k)|^{\gamma_0} \right) dz \leq \epsilon c_0 \lambda^{\frac{p^-}{p_k^+}} \tag{5.21}$$

and

$$\|(v_k)_t\|^{\gamma_0}_{L^\infty\left(Q_{5\tau_k}(\xi^k)\right)} + \|D^2 v_k\|^{\gamma_0}_{L^\infty\left(Q_{5\tau_k}(\xi^k)\right)} \le c_1 \lambda^{\frac{p_k^-}{p_k^+}}, \tag{5.22}$$

for some $c_1 = c_1(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 1$.

(b) (*Boundary case*) *If* $B_{20\tau_k}(y^k) \not\subset B^+_{s_2\rho}$, *we have*

$$\fint_{Q^+_{100\tau_k}(\tilde{\xi}^k)} \left(|u_t|^{\gamma_0} + |D^2 u|^{\gamma_0}\right) dz \le c_2 \lambda^{\frac{p_k^-}{p_k^+}} \ \ \text{and} \ \fint_{Q^+_{100\tau_k}(\tilde{\xi}^k)} |f|^{\gamma_0} dz \le c_2 \lambda^{\frac{p_k^-}{p_k^+}} \delta^{\frac{\gamma_1}{\gamma_2}}, \tag{5.23}$$

for some $c_2 = c_2(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 1$. *Moreover, for any* $\epsilon \in (0,1)$, *there exist* $\delta = \delta(\epsilon, n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) > 0$ *and* $v_k \in W^{2,1}_{\gamma_0}(Q^+_{100\tau_k}(\tilde{\xi}^k)) \cap W^{2,1}_\infty(Q^+_{25\tau_k}(\tilde{\xi}^k))$ *such that*

$$\fint_{Q^+_{25\tau_k}(\tilde{\xi}^k)} \left(|(u-v_k)_t|^{\gamma_0} + |D^2(u-v_k)|^{\gamma_0}\right) dz \le \epsilon c_2 \lambda^{\frac{p_k^-}{p_k^+}} \tag{5.24}$$

and

$$\|(v_k)_t\|^{\gamma_0}_{L^\infty\left(Q^+_{25\tau_k}(\tilde{\xi}^k)\right)} + \|D^2 v_k\|^{\gamma_0}_{L^\infty\left(Q^+_{25\tau_k}(\tilde{\xi}^k)\right)} \le c_3 \lambda^{\frac{p_k^-}{p_k^+}}, \tag{5.25}$$

for some $c_3 = c_3(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 1$.

Proof. Let us first consider the interior case (a) $B_{20\tau_k}(y^k) \subset B^+_{s_2\rho}$. One can easily see that

$$20\tau_k \le (s_2 - s_1)\rho \le \rho_0 \ \text{ and } \ B_{20\tau_k}(y^k) \subset B^+_{s_2\rho}.$$

For the sake of simplicity, we write

$$p_k^- := \inf_{z \in Q_{20\tau_k}(\xi^k)} p(z) \ \text{ and } \ p_k^+ := \sup_{z \in Q_{20\tau_k}(\xi^k)} p(z), \tag{5.26}$$

and then it follows from (2.6) that

$$p_k^+ - p_k^- \le \theta(40\tau_k). \tag{5.27}$$

From (5.5) we know that $40\tau_k \le 1$, $\theta(40\tau_k) \le 1$ and $|Q_{20\tau_k}| \le 1$. Using these facts, along with (5.12) and (5.27), we deduce that

$$\begin{aligned} &\left[\fint_{Q_{20\tau_k}(\xi^k)} \left(|u_t|^{\gamma_0} + |D^2 u|^{\gamma_0}\right) dz\right]^{p_k^+ - p_k^-} \\ &\le \left[\frac{1}{|Q_{20\tau_k}(\xi^k)|} \int_{Q^+_{2\rho}} \left(|u_t|^{\tilde{\gamma}_0} + |D^2 u|^{\tilde{\gamma}_0} + 2\right) dz\right]^{p_k^+ - p_k^-} \\ &\le c \left(\frac{1}{|Q_{20\tau_k}(\xi^k)|}\right)^{\theta(40\tau_k)} \le c \left(\frac{1}{40\tau_k}\right)^{(n+2)\theta(40\tau_k)} \le c, \end{aligned} \tag{5.28}$$

where the last inequality comes from (2.4). In an analogous way to (5.28), we can obtain from (2.4), (5.11), and (5.27) that

$$\left(\fint_{Q_{20\tau_k}(\xi^k)} |f|^{\gamma_0} dz\right)^{p_k^+ - p_k^-} \le c.$$

According to Hölder's inequality with facts $\gamma_1 \le p_k^+$ and $p^- \le p_k^-$, we then infer from (5.17) and (5.28) that

$$\begin{aligned}\fint_{Q_{20\tau_k}(\xi^k)} \left(|u_t|^{\gamma_0} + |D^2u|^{\gamma_0}\right) dz \quad &\le \quad c\left[\fint_{Q_{20\tau_k}(\xi^k)} \left(|u_t|^{\gamma_0} + |D^2u|^{\gamma_0}\right) dz\right]^{\frac{p_k^-}{p_k^+}} \\ &\le \ c\left[\fint_{Q_{20\tau_k}(\xi^k)} \left(|u_t|^{\frac{\gamma_0 p_k^-}{p^-}} + |D^2u|^{\frac{\gamma_0 p_k^-}{p^-}}\right) dz\right]^{\frac{p^-}{p_k^+}} \\ &\le \ c\left[\fint_{Q_{20\tau_k}(\xi^k)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}}\right) dz + 2\right]^{\frac{p^-}{p_k^+}} \le c\lambda^{\frac{p^-}{p_k^+}},\end{aligned}$$

and moreover, using the same argument as above, we also deduce that

$$\begin{aligned}\fint_{Q_{20\tau_k}(\xi^k)} |f|^{\gamma_0} dz \quad &\le \quad c\left(\fint_{Q_{20\tau_k}(\xi^k)} |f|^{\frac{\gamma_0 p(z)}{p^-}} dz + 1\right)^{\frac{p^-}{p_k^+}} \\ &\le \quad c(\delta\lambda + 1)^{\frac{p^-}{p_k^+}} \ \le \ c\lambda^{\frac{p^-}{p_k^+}} \delta^{\frac{\gamma_1}{\gamma_2}},\end{aligned}$$

where the last inequality comes from the fact that $1 < \delta\lambda_0 < \delta\lambda$, which is induced by (5.13) and (5.15). In turn, the desired estimate (5.20) follows.

We now rescale $Q_{20\tau_k}(\xi^k)$ to Q_4 by setting

$$h_k(\tilde z) := \frac{u\left(5\tau_k(\tilde x - y^k), 25\tau_k^2(\tilde t - s^k)\right)}{25\tau_k^2 \left(c_0 \lambda^{\frac{p^-}{p_k^+}}\right)^{\frac{1}{\gamma_0}}}, \quad g_k(\tilde z) := \frac{f\left(5\tau_k(\tilde x - y^k), 25\tau_k^2(\tilde t - s^k)\right)}{\left(c_0 \lambda^{\frac{p^-}{p_k^+}}\right)^{\frac{1}{\gamma_0}}},$$

and

$$(b_{ij}^k(\tilde z)) := \mathbf{B}_k(\tilde z) := \mathbf{A}\left(5\tau_k(\tilde x - y^k), 25\tau_k^2(\tilde t - s^k)\right),$$

for $\tilde z := (\tilde x, \tilde t) \in Q_4$. By a straightforward calculation, one can check from (1.2), the $(\delta, 4\rho)$-vanishing condition of $\mathbf{A}$, and the above resulting estimates (5.20) that $\mathbf{B}_k$ also satisfies (1.2) with $\mathbf{A}(z)$ replaced by $\mathbf{B}_k(\tilde z)$:

$$[\mathbf{B}_k]_4 \le \delta, \quad \fint_{Q_4} \left(|(h_k)_t|^{\gamma_0} + |D^2 h_k|^{\gamma_0}\right) dz \le 1, \quad \text{and} \quad \fint_{Q_4} |g_k|^{\gamma_0} dz \le \delta^{\frac{\gamma_1}{\gamma_2}}.$$

Besides, $h_k \in W^{2,1}_{p(\cdot)}(Q_4, w) \subset W^{2,1}_{\gamma_0}(Q_4)$ is a solution of

$$(h_k)_t - b_{ij}^k D_{ij} h = g_k \quad \text{in } Q_4. \tag{5.29}$$

Therefore, applying Lemma 4.4 and Corollary 4.5 to the equation (5.29) with $\mathbf{B}$, q, and δ replaced by $\mathbf{B}_k$, γ_0, and $\delta^{\frac{\gamma_1}{\gamma_2}}$, respectively, we obtain that there exist a constant matrix $\tilde{\mathbf{B}}_k = (\tilde b_{ij}^k)$ and a solution $\tilde v_k \in W^{2,1}_{\gamma_0}(Q_4)$ of

$$(\tilde v_k)_t - \tilde b_{ij}^k D_{ij} \tilde v_k = 0 \quad \text{in } Q_4$$

satisfying

$$\fint_{Q_1} \left(|(h_k - \tilde v_k)_t|^{\gamma_0} + |D^2(h_k - \tilde v_k)|^{\gamma_0}\right) dz \le \epsilon$$

and

$$\fint_{Q_4} \left(|(\tilde{v}_k)_t|^{\gamma_0} + |D^2 \tilde{v}_k|^{\gamma_0} \right) dz \le 1$$

by choosing sufficiently small $\delta = \delta(n, \Lambda, \gamma_1, \gamma_2, \theta(\cdot)) > 0$. Moreover, we also have

$$\|(\tilde{v}_k)_t\|^{\gamma_0}_{L^\infty(Q_1)} + \|D^2 \tilde{v}_k\|^{\gamma_0}_{L^\infty(Q_1)} \le c.$$

Therefore, letting

$$v_k(z) := 25\tau_k^2 \left(c_0 \lambda^{\frac{p^-}{p_k^+}} \right)^{\frac{1}{\gamma_0}} \tilde{v}_k \left(y^k + \frac{1}{5\tau_k} x, s^k + \frac{1}{25\tau_k^2} t \right)$$

for all $z := (x,t) \in Q_{20\tau_k}(\xi^k)$, we conclude that v_k is in $W^{2,1}_{\gamma_0}(Q_{20\tau_k}(\xi^k)) \cap W^{2,1}_{\infty}(Q_{5\tau_k}(\xi^k))$ and satisfies the estimates (5.21) and (5.22).

Next we deal with the boundary case (b) $B_{20\tau_k}(y^k) \not\subset B^+_{s_2\rho}$. Note that $|y^k - \tilde{y}^k| < 20\tau_k$. From the fact that $120\tau_k \le (s_2 - s_1)\rho \le \rho_0$, it is clear that

$$B_{5\tau_k}(y^k) \subset B^+_{25\tau_k}(\tilde{y}^k) \subset B^+_{100\tau_k}(\tilde{y}^k) \subset B^+_{120\tau_k}(y^k) \subset B^+_{s_2\rho}. \tag{5.30}$$

We abbreviate

$$p_k^- := \inf_{z \in Q^+_{100\tau_k}(\tilde{\xi}^k)} p(z) \quad \text{and} \quad p_k^+ := \sup_{z \in Q^+_{100\tau_k}(\tilde{\xi}^k)} p(z). \tag{5.31}$$

We also get from (2.6) that

$$p_k^+ - p_k^- \le \theta(200\tau_k).$$

We recall (5.17) to discover that

$$\fint_{Q^+_{120\tau_k}(\xi^k)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} \right) dz \le \lambda \quad \text{and} \quad \fint_{Q^+_{120\tau_k}(\xi^k)} |f|^{\frac{\gamma_0 p(z)}{p^-}} dz \le \delta\lambda.$$

By means of (5.30), we then obtain

$$\fint_{Q^+_{100\tau_k}(\tilde{\xi}^k)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} \right) dz \le 2^{n+2}\lambda$$

and

$$\fint_{Q^+_{100\tau_k}(\tilde{\xi}^k)} |f|^{\frac{\gamma_0 p(z)}{p^-}} dz \le 2^{n+2}\delta\lambda.$$

Using an analogous argument to the above interior case (a) by taking into account (5.31), the previous two estimates, Lemma 4.6, and Corollary 4.7, in place of (5.26), (5.17), Lemma 4.4, and Corollary 4.5, respectively, we can derive the desired estimates (5.23) and find the desired v_k satisfying (5.24) and (5.25). □

5.3. **The proof of** (5.10)**.** For constants c_1 and c_3 given in Lemma 5.3, let us set

$$K := (2^{\gamma_0 - 1} c_4)^{\frac{\gamma_2}{\gamma_1}} \quad \text{where} \quad c_4 := \max\{c_1, c_3\}. \tag{5.32}$$

Recalling the upper-level set (5.14), an elementary calculus yields

$$\begin{aligned}
&\int_{Q^+_{s_1\rho}} \left(|u_t|^{p(z)} + |D^2u|^{p(z)}\right) w(z)\,dz \\
&\qquad \le c\int_{Q^+_{s_1\rho}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{\frac{p^-}{\gamma_0}} w(z)\,dz \\
&\qquad = \frac{c\,p^-}{\gamma_0} K^{\frac{p^-}{\gamma_0}} \int_0^\infty \lambda^{\frac{p^-}{\gamma_0}-1} w(E(K\lambda))\,d\lambda \\
&\qquad \le c\left(\int_0^{A\lambda_0} \lambda^{\frac{p^-}{\gamma_0}-1} w(E(K\lambda))\,d\lambda + \int_{A\lambda_0}^\infty \lambda^{\frac{p^-}{\gamma_0}-1} w(E(K\lambda))\,d\lambda\right) \\
&\qquad \le c\left((A\lambda_0)^{\frac{p^-}{\gamma_0}} w(Q^+_{s_1\rho}) + \int_{A\lambda_0}^\infty \lambda^{\frac{p^-}{\gamma_0}-1} w(E(K\lambda))\,d\lambda\right) =: c(I_1 + I_2).
\end{aligned} \tag{5.33}$$

Taking into account the definitions of λ_0, A, and K in (5.13)), (5.15), and (5.32), we deduce from (5.7), (5.11), and (5.12) that

$$\begin{aligned}
I_1 &\le \frac{c\,w(Q^+_{s_1\rho})}{(s_2-s_1)^{\frac{(n+2)p^-}{\gamma_0}}} \left[⨍_{Q^+_{2\rho}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}}\right) dz \right. \\
&\qquad\qquad \left. + \frac{1}{\delta} ⨍_{Q^+_{2\rho}} \left(|f|^{\frac{\gamma_0 p(z)}{p^-}} + 1\right) dz \right]^{\frac{p^-}{\gamma_0}} \\
&\le \frac{c\,w(Q_{4\rho})}{(s_2-s_1)^{\frac{(n+2)\gamma_2}{\gamma_0}} |Q_{2\rho}|^{\frac{p^-}{\gamma_0}}} \left[\int_{Q^+_{2\rho}} \left(|u_t|^{\tilde\gamma_0} + |D^2u|^{\tilde\gamma_0}\right) dz \right. \\
&\qquad\qquad \left. + \frac{1}{\delta} \int_{Q^+_{2\rho}} \left(|f|^{\tilde\gamma_0} + 1\right) dz \right]^{\frac{p^-}{\gamma_0}} \\
&\le \frac{c\left(1+\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}} |Q_\rho|^{-\gamma_2}}{(s_2-s_1)^{\frac{(n+2)\gamma_2}{\gamma_0}}}.
\end{aligned} \tag{5.34}$$

Now we compute I_2. We start by estimating $w(E(K\lambda))$ for $\lambda \ge A\lambda_0$. We recall the covering $\left\{Q^+_{5\tau_k}(\xi^k)\right\}_{k=1}^\infty$ of $E(\lambda)$ in Section 5.2. Since $K \ge 1$, we see that $E(K\lambda) \subset E(\lambda)$, and so it follows that

$$\begin{aligned}
w(E(K\lambda)) &\le \sum_{k=1}^\infty w\left(\left\{ z \in Q^+_{5\tau_k}(\xi^k) : |u_t(z)|^{\frac{\gamma_0 p(z)}{p^-}} + \left|D^2u(z)\right|^{\frac{\gamma_0 p(z)}{p^-}} > K\lambda \right\}\right) \\
&\le \sum_{k=1}^\infty w\left(\left\{ z \in Q^+_{5\tau_k}(\xi^k) : |u_t(z)|^{\gamma_0} + \left|D^2u(z)\right|^{\gamma_0} > (K\lambda)^{\frac{p^-}{p(z)}} \right\}\right).
\end{aligned} \tag{5.35}$$

In order to estimate the sum of measures of the level sets on the right-hand side of (5.35), we should consider two cases, the interior case $B_{20\tau_k}(y^k) \subset B^+_{s_2\rho}$ and the boundary case $B_{20\tau_k}(y^k) \not\subset B^+_{s_2\rho}$.

For the interior case $B_{20\tau_k}(y^k) \subset B^+_{s_2\rho}$, which means $Q^+_{5\tau_k}(\xi^k) = Q_{5\tau_k}(\xi^k)$, we infer from (5.21), (5.22), (5.26), (5.32), and the elementary inequality $(a+b)^\beta \leq 2^{\beta-1}(a^\beta + b^\beta)$ for any $a, b > 0$ and $\beta \geq 1$ that

$$
\begin{aligned}
&\left|\left\{z \in Q^+_{5\tau_k}(\xi^k) : |u_t(z)|^{\gamma_0} + \left|D^2u(z)\right|^{\gamma_0} > (K\lambda)^{\frac{p^-}{p(z)}}\right\}\right| \\
&\quad \leq \left|\left\{z \in Q_{5\tau_k}(\xi^k) : |(u-v_k)_t(z)|^{\gamma_0} + \left|D^2(u-v_k)(z)\right|^{\gamma_0} > c_1\lambda^{\frac{p^-}{p_k^+}}\right\}\right| \\
&\qquad + \left|\left\{z \in Q_{5\tau_k}(\xi^k) : |(v_k)_t(z)|^{\gamma_0} + \left|D^2v_k(z)\right|^{\gamma_0} > c_1\lambda^{\frac{p^-}{p_k^+}}\right\}\right| \\
&\quad \leq \left(c_1\lambda^{\frac{p^-}{p_k^+}}\right)^{-1} \int_{Q_{5\tau_k}(\xi^k)} \left(|(u-v_k)_t|^{\gamma_0} + \left|D^2(u-v_k)\right|^{\gamma_0}\right) dz \leq \frac{\epsilon c_0}{c_1} \left|C_{5\tau_k}(\xi^k)\right|.
\end{aligned}
$$

Then (1) of Lemma 3.3 allows us to discover that

$$
\begin{aligned}
w\left(\left\{z \in Q^+_{5\tau_k}(\xi^k) : |u_t(z)|^{\gamma_0} + \left|D^2u(z)\right|^{\gamma_0} > (K\lambda)^{\frac{p^-}{p(z)}}\right\}\right) &\leq c\epsilon^{\nu_0} w(C_{5\tau_k}(\xi^k)) \\
&\leq c\epsilon^{\nu_0} w(C_{\tau_k}(\xi^k)). \qquad (5.36)
\end{aligned}
$$

Similarly, for the boundary case $B_{20\tau_k}(y^k) \not\subset B^+_{s_2\rho}$, it follows from (5.24), (5.25), (5.30), and (5.31) that

$$
\begin{aligned}
&\left|\left\{z \in Q^+_{5\tau_k}(\xi^k) : |u_t(z)|^{\gamma_0} + \left|D^2u(z)\right|^{\gamma_0} > (K\lambda)^{\frac{p^-}{p(z)}}\right\}\right| \\
&\quad \leq \left|\left\{z \in Q^+_{25\tau_k}(\tilde{\xi}^k) : |(u-v_k)_t(z)|^{\gamma_0} + \left|D^2(u-v_k)(z)\right|^{\gamma_0} > c_3\lambda^{\frac{p^-}{p_k^+}}\right\}\right| \\
&\qquad + \left|\left\{z \in Q^+_{25\tau_k}(\tilde{\xi}^k) : |(v_k)_t(z)|^{\gamma_0} + \left|D^2v_k(z)\right|^{\gamma_0} > c_3\lambda^{\frac{p^-}{p_k^+}}\right\}\right| \\
&\quad \leq \left(c_3\lambda^{\frac{p^-}{p_k^+}}\right)^{-1} \int_{Q^+_{25\tau_k}(\tilde{\xi}^k)} \left(|(u-v)_t|^{\gamma_0} + \left|D^2(u-v)\right|^{\gamma_0}\right) dz \leq \frac{\epsilon c_2}{c_3} \left|C_{25\tau_k}(\tilde{\xi}^k)\right|,
\end{aligned}
$$

and then we apply (1) of Lemma 3.3 to find that

$$
\begin{aligned}
w\left(\left\{z \in Q^+_{5\tau_k}(\xi^k) : |u_t(z)|^{\gamma_0} + \left|D^2u(z)\right|^{\gamma_0} > (K\lambda)^{\frac{p^-}{p(z)}}\right\}\right) &\leq c\epsilon^{\nu_0} w(C_{25\tau_k}(\tilde{\xi}^k)) \\
&\leq c\epsilon^{\nu_0} w(C_{\tau_k}(\tilde{\xi}^k)). \qquad (5.37)
\end{aligned}
$$

Inserting (5.36) and (5.37) into (5.35), we eventually obtain from (5.18) that

$$\begin{aligned} w(E(K\lambda)) &\leq c\epsilon^{\nu_0} \sum_{k=1}^{\infty} w(C_{\tau_k}(\xi^k)) \\ &\leq \frac{c\epsilon^{\nu_0}}{\lambda^{p^+-\epsilon_0}} \sum_{k=1}^{\infty} \left[\int_{Q^+_{\tau_k}(\xi^k)\cap\{|f|^{\frac{\gamma_0 p(z)}{p^-}} > \frac{\lambda\delta}{4c_a}\}} \left(\frac{|f|^{\frac{\gamma_0 p(z)}{p^-}}}{\delta} \right)^{p^+-\epsilon_0} w(z)\, dz \right. \\ &\quad \left. + \int_{Q^+_{\tau_k}(\xi^k)\cap\{|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} > \frac{\lambda}{4c_a}\}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} \right)^{p^+-\epsilon_0} w(z)\, dz \right] \\ &\leq \frac{c\epsilon^{\nu_0}}{\lambda^{p^+-\epsilon_0}} \\ &\quad \times \left[\int_{Q^+_{S_{2\rho}}\cap\{|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} > \frac{\lambda}{4c_a}\}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} \right)^{p^+-\epsilon_0} w(z)\, dz \right. \\ &\quad \left. + \int_{Q^+_{s_{2\rho}}\cap\{|f|^{\frac{\gamma_0 p(z)}{p^-}} > \frac{\lambda\delta}{4c_a}\}} \left(\frac{|f|^{\frac{\gamma_0 p(z)}{p^-}}}{\delta} \right)^{p^+-\epsilon_0} w(z)\, dz \right]. \end{aligned} \tag{5.38}$$

Accordingly, this estimate (5.38) leads us to discover that

$$\begin{aligned} I_2 &= \int_{A\lambda_0}^{\infty} \lambda^{\frac{p^-}{\gamma_0}-1} w(E(K\lambda))\, d\lambda \\ &\leq c\epsilon^{\nu_0} \int_0^{\infty} \lambda^{\frac{p^-}{\gamma_0}-(p^+-\epsilon_0)-1} \\ &\times \left[\int_{Q^+_{S_{2\rho}}\cap\{|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} > \frac{\lambda}{4c_a}\}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} \right)^{p^+-\epsilon_0} w(z)\, dz \right] d\lambda \\ &\quad + c\epsilon^{\nu_0} \int_0^{\infty} \lambda^{\frac{p^-}{\gamma_0}-(p^+-\epsilon_0)-1} \int_{Q^+_{s_{2\rho}}\cap\{|f|^{\frac{\gamma_0 p(z)}{p^-}} > \frac{\lambda\delta}{4c_a}\}} \left(\frac{|f|^{\frac{\gamma_0 p(z)}{p^-}}}{\delta} \right)^{p^+-\epsilon_0} w(z)\, dz d\lambda. \end{aligned}$$

Then applying the basic identity

$$\int_U |g(z)|^q\, w(z)\, dz = (q-\tilde{q}) \int_0^{\infty} \lambda^{q-\tilde{q}-1} \int_{\{z\in U:|g(x)|>\lambda\}} |g(z)|^{\tilde{q}}\, w(z)\, dz d\lambda$$

for $q > \tilde{q} \geq 1$, together with (5.8) and the additional assumption (5.9), we deduce that

$$\begin{aligned} I_2 &\leq c\epsilon^{\nu_0} \left\{ \int_{Q^+_{s_{2\rho}}} \left(|u_t|^{p(z)} + |D^2 u|^{p(z)} \right) w(z)\, dz + \left(\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}} \int_{Q^+_{s_{2\rho}}} |f|^{p(z)} w(z)\, dz \right\} \\ &\leq c\epsilon^{\nu_0} \int_{Q^+_{s_{2\rho}}} \left(|u_t|^{p(z)} + |D^2 u|^{p(z)} \right) w(z) dz + c\left(\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}}. \end{aligned} \tag{5.39}$$

Therefore, combining (5.33), (5.34), and (5.39), we arrive at

$$\int_{Q^+_{s_1\rho}} \left(|u_t|^{p(z)} + |D^2u|^{p(z)}\right) w(z)\,dz \le c_5\epsilon^{\nu_0} \int_{Q^+_{s_2\rho}} \left(|u_t|^{p(z)} + |D^2u|^{p(z)}\right) w(z)\,dz$$
$$+\frac{c\left(1+\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}} |Q_\rho|^{-\gamma_2}}{(s_2-s_1)^{\frac{(n+2)\gamma_2}{\gamma_0}}} + c\left(\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}},$$

for some $c_5 = c_5(n,\Lambda,\gamma_1,\gamma_2,c_{LH},[w]_{A_{p(\cdot)}},w(Q_{4\rho})) > 0$. At this stage, we take $\epsilon = \epsilon(n,\Lambda,\gamma_1,\gamma_2,c_{LH},[w]_{A_{p(\cdot)}},w(Q_{4\rho})) > 0$ small enough so that

$$0 < c_5\epsilon^{\nu_0} \le \frac{1}{2} \tag{5.40}$$

to establish

$$\int_{Q^+_{s_1\rho}} \left(|u_t|^{p(z)} + |D^2u|^{p(z)}\right) w(z)\,dz$$
$$\le \frac{1}{2}\int_{Q^+_{s_2\rho}} \left(|u_t|^{p(z)} + |D^2u|^{p(z)}\right) w(z)\,dz + \frac{c\,|Q_\rho|^{-\gamma_2}}{(s_2-s_1)^{\frac{(n+2)\gamma_2}{\gamma_0}}} + c.$$

Since s_1 and s_2 with $1 \le s_1 < s_2 \le 2$ are arbitrary, we apply the standard iteration lemma [24, Lemma 4.3] to conclude that

$$\int_{Q^+_\rho} \left(|u_t|^{p(z)} + |D^2u|^{p(z)}\right) w(z)\,dz \le c\,|Q_\rho|^{-\gamma_2} + c \le c_6\rho^{-(n+2)\gamma_2} \tag{5.41}$$

for some $c_6 = c_6(n,\Lambda,\gamma_1,\gamma_2,c_{LH},[w]_{A_{p(\cdot)}},w(Q_{4\rho})) > 1$. By virtue of (3.3), we consequently obtain the desired estimate (5.10). This completes the proof.

Remark 5.4. From the choice of $\epsilon > 0$ in (5.40), one can select $\delta > 0$ depending only on $n,\Lambda,\gamma_1,\gamma_2,[w]_{A_{p(\cdot)}}$, and $w(Q_{4\rho})$.

6. Global estimates

The proof of our main result, Theorem 2.5, proceeds in three steps. In the first step we show that it suffices to derive the estimate (2.8) only for the solutions u of (1.1) belonging to $W^{2,1}_{p(\cdot)}(\Omega_T,w)$. Then in the next two steps, by using standard covering and flattening arguments, we obtain the *a priori* estimate (2.8) from the interior and boundary *a priori* weighted estimates that have been established in the previous section. In what follows, we denote by c a universal constant being dependent only on $n,\Lambda,\gamma_1,\gamma_2,\theta(\cdot),w,\Omega$, and R, and possibly varying from line to line.

6.1. **Approximation.** We first suppose that we have the *a priori* estimate; that is, the estimate (2.8) holds for any $W^{2,1}_{p(\cdot)}(\Omega_T,w)$-solution of the problem (1.1). To get rid of this *a priori* assumption, we show that the solution u of the problem (1.1) can be suitably approximated by solutions u_k, $k=1,2,\dots$, in $W^{2,1}_{p(\cdot)}(\Omega_T,w)$ to regular equations.

Given $\mathbf{A} = (a_{ij})$, we choose a sequence $\{\mathbf{A}^k\}_{k=1}^\infty = \{(a^k_{ij})\}_{k=1}^\infty$ of smooth matrix functions satisfying the uniform parabolicity condition with the parabolicity constant Λ and (δ,R)-vanishing property, which converges to $\mathbf{A} = (a_{ij})$ in $L^\alpha(\Omega_T)$ for each $1 < \alpha < \infty$. For instance we may define $(a^k_{ij}) := (a_{ij} * \varphi_{1/k})$, where $\varphi_{1/k}(x) := k^n\varphi(kx)$ and φ is a standard mollification function. On the other hand, for given $f \in L^{p(\cdot)}(\Omega_T, w)$, we also find a sequence $\{f_k\}_{k=1}^{\infty}$ of smooth functions in $C_0^{\infty}(\Omega_T)$ converging to f in $L^{p(\cdot)}(\Omega_T, w)$ and satisfying that

$$\|f_k\|_{L^{p(\cdot)}(\Omega_T,w)} \leq \|f\|_{L^{p(\cdot)}(\Omega_T,w)} + 1 \quad \text{for all } k = 1, 2, \dots. \tag{6.1}$$

Since $w \in A_{p(\cdot)} \subset A_{\gamma_2+1}$ and $w^{-1/(p(\cdot)-1)} \in A_{p'(\cdot)} \subset A_{\gamma_1/(\gamma_1-1)+1}$, by Lemmas 3.5 and 3.6, we note that in view of (1) of Lemma 3.1, there exist positive constants ν_1 and $\tilde{\nu}_1$ such that $w \in L^{1+\nu_1}(\mathbb{R}^{n+1})$ and $w^{-1/(p(\cdot)-1)} \in L^{1+\tilde{\nu}_1}(\mathbb{R}^{n+1})$. Therefore, we have that for $g \in L^{\frac{\gamma_2(1+\nu_1)}{\nu_1}}(\Omega_T)$,

$$\int_{\Omega_T} |g|^{\gamma_2+1} w \, dz \leq \left(\int_{\Omega_T} |g|^{\frac{(\gamma_2+1)(1+\nu_1)}{\nu_1}} \, dz \right)^{\frac{\nu_1}{1+\nu_1}} \left(\int_{\Omega_T} w^{1+\nu_1} \, dz \right)^{\frac{1}{1+\nu_1}},$$

from which together with (3.4) one can find $q_1 = \frac{(\gamma_2+1)(1+\nu_1)}{\nu_1} \in (\gamma_2 + 1, \infty)$ such that

$$L^{q_1}(\Omega_T) \hookrightarrow L^{\gamma_2+1}(\Omega_T, w) \hookrightarrow L^{p(\cdot)}(\Omega_T, w). \tag{6.2}$$

In the same argument, there exists $q_2 \in (\gamma_1/(\gamma_1 - 1) + 1, \infty)$ such that

$$L^{q_2}(\Omega_T) \hookrightarrow L^{\gamma_1/(\gamma_1-1)+1}(\Omega_T, w^{-1/(p(\cdot)-1)}) \hookrightarrow L^{p'(\cdot)}(\Omega_T, w^{-1/(p(\cdot)-1)}). \tag{6.3}$$

Since $\mathbf{A}^k$ and f_k are smooth, according to [6, Theorem 4.3], there exists the unique solution $u_k \in W^{2,1}_{q_1}(\Omega_T)$ of

$$\begin{cases} (u_k)_t - a^k_{ij} D_{ij} u_k &= f_k \quad \text{in} \quad \Omega_T, \\ u_k &= 0 \quad \text{on} \quad \partial\Omega_T. \end{cases} \tag{6.4}$$

We then see from (6.2) that $u_k \in W^{2,1}_{p(\cdot)}(\Omega_T, w)$. Hence, by the *a priori* assumption we have the estimate

$$\|u_k\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} \leq c \|f_k\|_{L^{p(\cdot)}(\Omega_T,w)}.$$

Moreover, it follows from (6.1) that

$$\|u_k\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} \leq c \|f_k\|_{L^{p(\cdot)}(\Omega_T,w)} \leq c \left(\|f\|_{L^{p(\cdot)}(\Omega_T,w)} + 1 \right), \tag{6.5}$$

where c is independent of k, and so $\{u_k\}_{k=1}^{\infty}$ is bounded in $W^{2,1}_{p(\cdot)}(\Omega_T, w)$. Therefore, there exist a subsequence, which is still denoted by $\{u_k\}_{k=1}^{\infty}$, and a function $u_0 \in W^{2,1}_{p(\cdot)}(\Omega_T, w)$ such that

$$u_k \rightharpoonup u_0 \quad \text{weakly in } W^{2,1}_{p(\cdot)}(\Omega_T, w).$$

On the other hand, for the sequence $\{\mathbf{A}^k\}$, we see from (6.3) that

$$\mathbf{A}^k \to \mathbf{A} \text{ strongly in } L^{p'(\cdot)}(\Omega_T, w^{-1/(p(\cdot)-1)}) = (L^{p(\cdot)}(\Omega_T, w))^*.$$

Hence, taking into account the convergence properties of a^k_{ij}, f_k, and u_k, we conclude that $u_0 \in W^{2,1}_{p(\cdot)}(\Omega_T, w)$ is a solution of (1.1). The uniqueness of strong solutions of (1.1) directly follows from Lemma 3.7 and [6, Theorem 4.3].

6.2. **Flattening and covering.** In this subsection, we assume that the strong solution u of (1.1) satisfies that

$$u \in W^{2,1}_{p(\cdot)}(\Omega_T, w) \tag{6.6}$$

and then prove

$$\begin{aligned} &\|u_t\|_{L^{p(\cdot)}(\Omega_T,w)} + \|D^2u\|_{L^{p(\cdot)}(\Omega_T,w)} \\ &\qquad \le c\left(\|f\|_{L^{p(\cdot)}(\Omega_T,w)} + \|u\|_{L^{p(\cdot)}(\Omega_T,w)} + \|Du\|_{L^{p(\cdot)}(\Omega_T,w)}\right). \end{aligned} \tag{6.7}$$

In fact, it suffices to show that

$$\|u_t\|_{L^{p(\cdot)}(\Omega_T,w)} + \|D^2u\|_{L^{p(\cdot)}(\Omega_T,w)} \le c, \tag{6.8}$$

under the additional assumption that

$$\|f\|_{L^{p(\cdot)}(\Omega_T,w)} + \|u\|_{L^{p(\cdot)}(\Omega_T,w)} + \|Du\|_{L^{p(\cdot)}(\Omega_T,w)} \le 1. \tag{6.9}$$

First, we extend the solution u and the function f in (1.1) to $\Omega_T^* := \Omega \times (-T, 2T)$ by letting $u(x,t) = f(x,t) = 0$ for $-T < t < 0$ and $u(x,t) = u(x, 2T-t)$, $f(x,t) = f(x, 2T-t)$ for $T < t < 2T$ and redefine the coefficient matrix $\mathbf{A}(x,t)$ by

$$\mathbf{A}(x,t) = \begin{cases} (a_{ij}(x,t)) & \text{in} \quad \mathbb{R}^n \times (-\infty, T], \\ (a_{ij}(x, 2T-t)) & \text{in} \quad \mathbb{R}^n \times (T, \infty). \end{cases}$$

Then the extended function f is obviously in $L^{p(\cdot)}(\Omega_T^*, w)$, and the redefined $\mathbf{A}$ satisfies the uniform parabolicity condition with the parabolicity constant Λ and $(4\delta, R)$-vanishing property. Furthermore, it is clear that $w \in A_{p(\cdot)}$, and we observe that u is in $W^{2,1}_{p(\cdot)}(\Omega_T^*, w)$ and solves

$$\begin{cases} u_t - a_{ij}D_{ij}u &= f \quad \text{in} \quad \Omega_T^*, \\ \qquad\qquad u &= 0 \quad \text{on} \quad \partial_p\Omega_T^*. \end{cases}$$

From the additional assumption (6.9), we also have that

$$\begin{aligned} &\|f\|_{L^{p(\cdot)}(\Omega_T^*,w)} + \|u\|_{L^{p(\cdot)}(\Omega_T^*,w)} + \|Du\|_{L^{p(\cdot)}(\Omega_T^*,w)} \\ &\qquad \le 2\left(\|f\|_{L^{p(\cdot)}(\Omega_T,w)} + \|u\|_{L^{p(\cdot)}(\Omega_T,w)} + \|Du\|_{L^{p(\cdot)}(\Omega_T,w)}\right) \le 2. \end{aligned} \tag{6.10}$$

Now, let us fix any point $\xi = (y,s) = (y', y_n, s) \in \partial\Omega \times [0,T]$. From the boundary regularity assumption that $\partial\Omega \in C^{1,1}$, there exist $r > 0$ and a $C^{1,1}$ function $\mu = \mu(x') : \mathbb{R}^{n-1} \to \mathbb{R}$ in a new spatial coordinate system with origin at y, which is obtained by a translation and a rotation from the original one and will still be defined by the x-coordinate system, such that

$$\Omega \cap B_r(0) = \{x \in B_r(0) : x_n > \mu(x')\}, \tag{6.11}$$

$$\mu(0) = 0, \quad \nabla_{x'}\mu(0) = 0, \quad \text{and} \quad \|\nabla^2_{x'}\mu\|_{L^\infty(\mathbb{R}^{n-1})} < \infty. \tag{6.12}$$

Note that (6.11) is also valid for all $\tilde{r} < r$ as well as r, and hence we further assume that $r < \min\{T, R\}$.

In order to flatten out the boundary near the origin by changing coordinates, we define

$$\begin{cases} \tilde{x}_i = x_i & =: \varphi^i(x) \quad \text{if } i = 1, 2, \dots, n-1, \\ \tilde{x}_n = x_n - \mu(x') & =: \varphi^n(x) \end{cases} \tag{6.13}$$

and write $\tilde{x} = \varphi(x)$. Setting $\psi := \varphi^{-1}$, we see that $x = \psi(\tilde{x})$. Then we let $\tilde{\mathbf{A}}(\tilde{x},\tilde{t}) = (\tilde{a}_{lm}(\tilde{x}, s+\tilde{t})) = [\nabla\varphi(\psi(\tilde{x}))]\cdot \mathbf{A}(\psi(\tilde{x}), s+\tilde{t})\cdot[\nabla\varphi(\psi(\tilde{x}))]^T$, and $\tilde{p}(\tilde{x},\tilde{t}) = p(\psi(\tilde{x}), s+\tilde{t})$. Note that $\tilde{\mathbf{A}}$ is uniformly parabolic with the parabolicity constant Λ. On the other hand, $\tilde{p}$ satisfies that $\gamma_1 \le \tilde{p}(\cdot) \le \gamma_2$ and

$$\begin{aligned}\left|\tilde{p}(\tilde{\xi}^1) - \tilde{p}(\tilde{\xi}^2)\right| &\le \theta\left(d_p((\psi(\tilde{y}^1), s+\tilde{s}^1), (\psi(\tilde{y}^2), s+\tilde{s}^2))\right)\\ &\le \theta\left((\|\nabla\psi\|_{L^\infty}+1)d_p(\tilde{\xi}^1,\tilde{\xi}^2)\right) =: \tilde{\theta}\left(d_p(\tilde{\xi}^1,\tilde{\xi}^2)\right),\end{aligned}$$

where $\tilde{\xi}^1 := (\tilde{y}^1,\tilde{s}^1),\ \tilde{\xi}^2 := (\tilde{y}^2,\tilde{s}^2) \in \mathbb{R}^{n+1}$, and $\tilde{\theta}(\rho) := \theta\left((\|\nabla\psi\|_{L^\infty}+1)\rho\right)$, and hence there holds

$$\tilde{\theta}(\rho)\log\left(\frac{1}{\rho}\right) \le \tilde{M} \quad \text{for all } 0<\rho<\infty,$$

for some constant $\tilde{M} = \tilde{M}(\mu, M) = \tilde{M}(\mu,\gamma_2,c_{LH}) > 0$.

We now choose $\rho = \rho(\rho_0, r,\mu) > 0$ so small that $Q^+_{4\rho} \subset \varphi(\Omega\cap B_r(0))\times(-r^2,r^2)$ with $\rho\le\rho_0$ in the $(\tilde{x},\tilde{t})$-coordinate system, where ρ_0 is given by (5.5), and define

$$\tilde{u}(\tilde{x},\tilde{t}) := u\left(\psi(\tilde{x}), s+\tilde{t}\right) \quad\text{and}\quad \tilde{w}(\tilde{x},\tilde{t}) := w\left(\psi(\tilde{x}), s+\tilde{t}\right) \quad \text{for } (\tilde{x},\tilde{t})\in Q^+_{4\rho}.$$

Then we deduce that $\tilde{u}$ is in $W^{2,1}_{\tilde{p}(\cdot)}(Q^+_{4\rho},\tilde{w})$ and solve

$$\left\{\begin{array}{rcll}\tilde{u}_{\tilde{t}} - \tilde{a}_{lm}D_{\tilde{x}_l\tilde{x}_m}\tilde{u} &=& \tilde{f} & \text{in}\quad Q^+_{4\rho},\\ \tilde{u} &=& 0 & \text{on}\quad T_{4\rho},\end{array}\right. \tag{6.14}$$

where

$$\tilde{f}(\tilde{x},\tilde{t}) = f(\psi(\tilde{x}), s+\tilde{t}) + a_{ij}(\psi(\tilde{x}), s+\tilde{t})\varphi^l_{x_ix_j}(\psi(\tilde{x}))D_{\tilde{x}_l}\tilde{u}.$$

From the assumption $\partial\Omega \in C^{1,1}$, we can see that $\tilde{w}\in A_{\tilde{p}(\cdot)}$ with $[\tilde{w}]_{A_{\tilde{p}(\cdot)}} \le c(n,\gamma_1,\gamma_2,c_{LH},[w]_{A_{p(\cdot)}},\mu)$. Moreover, a direct computation yields

$$\begin{aligned}[\tilde{\mathbf{A}}]_{4\rho} &\le c\left([\mathbf{A}]_R + \|\nabla_{x'}\mu\|_{L^\infty(B'_r(0))} + \|\nabla_{x'}\mu\|^2_{L^\infty(B'_r(0))}\right)\\ &\le c\left(\delta + \|\nabla_{x'}\mu\|_{L^\infty(B'_r(0))} + \|\nabla_{x'}\mu\|^2_{L^\infty(B'_r(0))}\right)\\ &\le c\left(\delta + r\|\nabla^2_{x'}\mu\|_{L^\infty(B'_r(0))} + r^2\|\nabla^2_{x'}\mu\|^2_{L^\infty(B'_r(0))}\right)\\ &\le c\left(\delta + r + r^2\right),\end{aligned}$$

where we used the third inequality in (6.12) for the last inequality.

Taking into account the conditions on f, $\mathbf{A}$, and $\partial\Omega$, it is also clear that $\tilde{f}\in L^{\tilde{p}(\cdot)}\left(Q^+_{4\rho},\tilde{w}\right)$ with the estimate

$$\|\tilde{f}\|_{L^{\tilde{p}(\cdot)}\left(Q^+_{4\rho},\tilde{w}\right)} \le c(\mu)\left(\|f\left(\psi(\tilde{x}), s+\tilde{t}\right)\|_{L^{\tilde{p}(\cdot)}\left(Q^+_{4\rho},\tilde{w}\right)} + \|D\tilde{u}\|_{L^{\tilde{p}(\cdot)}\left(Q^+_{4\rho},\tilde{w}\right)}\right), \tag{6.15}$$

where $c(\mu)$ is a constant depending only on n,Λ, and μ.

In turn, all the hypotheses of Theorem 5.1(ii) are fulfilled with respect to the above equation (6.14) by taking $\delta = \delta(n,\Lambda,\gamma_1,\gamma_2,c_{LH},\theta(\cdot),[w]_{A_{p(\cdot)}}, w(Q_{4\rho}),\mu) > 0$ and $r = r(n,\Lambda,\gamma_1,\gamma_2,\theta(\cdot),R,T,[w]_{A_{p(\cdot)}},\mu) > 0$ sufficiently small, and hence, Theorem 5.1(ii) gives

$$\|\tilde{u}_{\tilde{t}}\|_{L^{\tilde{p}(\cdot)}\left(Q^+_{\rho},\tilde{w}\right)} + \|D^2\tilde{u}\|_{L^{\tilde{p}(\cdot)}\left(Q^+_{\rho},\tilde{w}\right)} \le c\left(\|\tilde{f}\|_{L^{\tilde{p}(\cdot)}\left(Q^+_{4\rho},\tilde{w}\right)} + \|\tilde{u}\|_{L^{\gamma_1}\left(Q^+_{4\rho},\tilde{w}\right)}\right).$$

In view of (2.2) and (6.13), the change of variables from $(\tilde{x}, \tilde{t})$ to (x, t) finally yields from the previous estimate and (6.10) that

$$\begin{aligned}
&\|u_t\|_{L^{p(\cdot)}(V_\xi,w)} + \|D^2 u\|_{L^{p(\cdot)}(V_\xi,w)} \\
&\quad\le c\left(\|f\|_{L^{p(\cdot)}(U_\xi,w)} + \|u\|_{L^{\gamma_1}(U_\xi,w)} + \|Du\|_{L^{p(\cdot)}(U_\xi,w)}\right) \\
&\quad\le c\left(\|f\|_{L^{p(\cdot)}(Q_r(\xi),w)} + \|u\|_{L^{p(\cdot)}(Q_r(\xi),w)} + \|Du\|_{L^{p(\cdot)}(Q_r(\xi),w)}\right) \\
&\quad\le c\left(\|f\|_{L^{p(\cdot)}(\Omega_T^*,w)} + \|u\|_{L^{p(\cdot)}(\Omega_T^*,w)} + \|Du\|_{L^{p(\cdot)}(\Omega_T^*,w)}\right) \le c, \qquad (6.16)
\end{aligned}$$

where $V_\xi := \psi(B_\rho^+) \times \left(s - \rho^2, s + \rho^2\right)$ and $U_\xi := \psi(B_{4\rho}^+) \times \left(s - (4\rho)^2, s + (4\rho)^2\right)$.

Thanks to the compactness of $\overline{\Omega_T}$, we can cover it with a finite number of sets $V_{\xi^1}, V_{\xi^2}, \ldots, V_{\xi^N}$ for some points $\xi^j \in \partial\Omega \times (0, T)$, $j = 1, 2, \ldots, N$, as above, and $V \Subset \Omega_T^*$ such that $\Omega_T \subset V \cup \left(\bigcup_{j=1}^N V_{\xi^j}\right)$. On the other hand, applying a standard covering argument, it follows from (5.1), along with (6.10), that

$$\|u_t\|_{L^{p(\cdot)}(V,w)} + \|D^2 u\|_{L^{p(\cdot)}(V,w)} \le c\left(\|f\|_{L^{p(\cdot)}(\Omega_T^*,w)} + \|u\|_{L^{p(\cdot)}(\Omega_T^*,w)}\right) \le c. \qquad (6.17)$$

Consequently, by summing the estimates (6.16) for $\xi = \xi^1, \xi^2, \ldots, \xi^N$, together with (6.17), we obtain (6.8).

6.3. **Elimination of lower order terms.** From (6.7), we have

$$\|u\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} \le c\left(\|f\|_{L^{p(\cdot)}(\Omega_T,w)} + \|u\|_{L^{p(\cdot)}(\Omega_T,w)} + \|Du\|_{L^{p(\cdot)}(\Omega_T,w)}\right). \qquad (6.18)$$

It only remains to drop the last two terms on the right-hand side of the previous estimate in order to arrive at the desired estimate (2.8). To deal with this, we argue by contradiction. If the estimate (2.8) is false, then there exist sequences $\{u_k\}_{k=1}^\infty$ and $\{f_k\}_{k=1}^\infty$ such that u_k is a solution of

$$\begin{cases} (u_k)_t - a_{ij} D_{ij} u_k = f_k & \text{in} \quad \Omega_T, \\ u_k = 0 & \text{on} \quad \partial\Omega_T \end{cases}$$

satisfying

$$\|u_k\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} > k\|f_k\|_{L^{p(\cdot)}(\Omega_T,w)}, \qquad (6.19)$$

for any $k = 1, 2, 3, \ldots$. By a usual normalization argument, we may assume that

$$\|u_k\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} = 1. \qquad (6.20)$$

Then (6.19) and (6.20) turn into

$$\|f_k\|_{L^{p(\cdot)}(\Omega_T,w)} < \frac{1}{k} \to 0 \quad \text{as} \quad k \to \infty. \qquad (6.21)$$

Furthermore, by (3.6), (4.1) with $q = \tilde{\gamma}_0$, and (6.21), we deduce that

$$\|u_k\|_{W^{2,1}_{\tilde{\gamma}_0}(\Omega_T)} \le c\|f_k\|_{L^{\tilde{\gamma}_0}(\Omega_T)} \le c\|f_k\|_{L^{p(\cdot)}(\Omega_T,w)} \to 0 \quad \text{as} \quad k \to \infty,$$

which implies that there exists a subsequence of $\{u_k\}_{l=1}^\infty$, still say $\{u_k\}_{k=1}^\infty$, such that $\lim_{k\to\infty} |u_k(z)| = \lim_{k\to\infty} |Du_k(z)| = 0$ for almost every $z \in \Omega_T$. Then Lebesgue's dominant convergence theorem along with (6.20) yields that

$$\int_{\Omega_T} |u_k|^{p(z)} w(z)\, dz, \ \int_{\Omega_T} |Du_k|^{p(z)} w(z)\, dz \to 0 \quad \text{as} \quad k \to \infty,$$

which means that

$$\|u_k\|_{L^{p(\cdot)}(\Omega_T,w)},\ \|Du_k\|_{L^{p(\cdot)}(\Omega_T,w)} \to 0 \quad \text{as} \quad k \to \infty.$$

However, from the above result and (6.18), we discover that

$$1 \le c\left(\|f_k\|_{L^{p(\cdot)}(\Omega_T,w)} + \|u_k\|_{L^{p(\cdot)}(\Omega_T,w)} + \|Du_k\|_{L^{p(\cdot)}(\Omega_T,w)}\right) \to 0 \quad \text{as} \quad k \to \infty.$$

This contradiction establishes the desired estimates (2.8).

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