TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 4, April 2018, Pages 2907–2953
http://dx.doi.org/10.1090/tran/7363
Article electronically published on November 14, 2017

HIGGS BUNDLES FOR REAL GROUPS AND THE HITCHIN–KOSTANT–RALLIS SECTION

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Abstract. We consider the moduli space of polystable L-twisted G-Higgs bundles over a compact Riemann surface X, where G is a real reductive Lie group and L is a holomorphic line bundle over X. Evaluating the Higgs field on a basis of the ring of polynomial invariants of the isotropy representation defines the Hitchin map. This is a map to an affine space whose dimension is determined by L and the degrees of the polynomials in the basis. In this paper, we construct a section of this map and identify the connected components of the moduli space containing the image. This section factors through the moduli space for G_{split}, a split real subgroup of G. Our results generalize those by Hitchin, who considered the case when L is the canonical line bundle of X and G is complex. In this case, the image of the section is related to the Hitchin–Teichmüller components of the moduli space of representations of the fundamental group of X in G_{split}, a split real form of G. The construction involves the notion of a maximal split subgroup of a real reductive Lie group and builds on results by Kostant and Rallis.

Contents

1. Introduction

Let G be a **real reductive Lie group**. Following Knapp [25], by this we mean a tuple (G, H, θ, B), where $H \subset G$ is a maximal compact subgroup, $\theta\colon \mathfrak{g} \to \mathfrak{g}$ is a Cartan involution, and B is a non-degenerate bilinear form on $\mathfrak{g}$, which is $\mathrm{Ad}(G)$- and θ-invariant, satisfying natural compatibility conditions. We will also need the notion of a real strongly reductive Lie group (see Definition 3.1 for details). The Cartan involution θ gives a decomposition (the Cartan decomposition)

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

Received by the editors February 8, 2017.
2010 *Mathematics Subject Classification.* Primary 14H60; Secondary 53C07, 58D29.
The second author was supported by an FPU grant from the Ministerio de Educación.

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into its ± 1-eigenspaces, where $\mathfrak{h}$ is the Lie algebra of H. The group H acts linearly on $\mathfrak{m}$ through the adjoint representation of G — this is the isotropy representation that we complexify to obtain a representation (also referred to as isotropy representation) $\iota\colon H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$.

Let X be a compact Riemann surface and let L be a holomorphic line bundle over X. An L-twisted G-Higgs bundle on X is a pair (E, φ), where E is a holomorphic principal $H^{\mathbb{C}}$-bundle over X and φ is a holomorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes L$, where $E(\mathfrak{m}^{\mathbb{C}}) = E \times_{H^{\mathbb{C}}} \mathfrak{m}^{\mathbb{C}}$ is the $\mathfrak{m}^{\mathbb{C}}$-bundle associated to E via the isotropy representation. The section φ is called the Higgs field. Two L-twisted G-Higgs bundles (E, φ) and (E', φ') are isomorphic if there is an isomorphism $f\colon E \to E'$ such that $\varphi = f^*\varphi'$ where f^* is the obvious induced map. When L is the canonical line bundle K of X we obtain the familiar theory of G-Higgs bundles. When G is compact the Higgs field is identically zero and an L-twisted G-Higgs bundle is simply a principal $G^{\mathbb{C}}$-bundle. When G is complex $G = H^{\mathbb{C}}$ and the isotropy representation coincides with the adjoint representation of G. This is the situation originally considered by Hitchin in [22, 23], for $L = K$. It is worth pointing out that consideration of the theory for an arbitrary line bundle L is indeed relevant, as illustrated for example in the works [2, 30]. In fact, even in the study of G-Higgs bundles for $L = K$ one may end up with a different twisting, as in the case of maximal Toledo invariant G-Higgs bundles (see [5]).

There is a notion of stability which depends on an element α of the centre of $\mathfrak{h}$. This element is fixed by the topology of the bundle, except in the case when G/H is a Hermitian symmetric space. In this situation α is a continuous parameter, which varies in a way governed by the Milnor–Wood inequality (see [5]). Let $\mathcal{M}_L^\alpha(G)$ be the moduli space of isomorphism classes of α-polystable L-twisted G-Higgs bundles. We will omit the subindex L when $L = K$. We will also omit the superindex α when $\alpha = 0$.

To study this moduli space one considers the Hitchin map

$$h_L : \mathcal{M}_L^\alpha(G) \to B_L(G)$$

defined by evaluating the Higgs field on a basis of the ring of polynomial $H^{\mathbb{C}}$-invariants of the isotropy representation. Here, $B_L(G) \cong H^0(X, \bigoplus_{i=1}^a L^{m_i})$ is the Hitchin base, where a is the real rank of the group and m_i are the exponents of G (see Section 6 for a more intrinsic definition of this map and the definition of exponents). Again we will omit the subindex L in h_L and $B_L(G)$ when $L = K$. As a first step to analyse the Hitchin map, in this paper, we construct a section under certain conditions. This generalizes the construction given by Hitchin, who considered a complex Lie group G and $L = K$ [24]. In this case the image of the section is related to the Hitchin components of the moduli space of representations of the fundamental group of X in G_{split}, a split real form of the complex group G. In fact, in relation to this, our construction is indeed very natural since we can start directly with the moduli space $\mathcal{M}(G_{\mathrm{split}})$ instead of $\mathcal{M}(G)$ and construct the section for the Hitchin map for G_{split} instead of that for G, which by construction lies in $\mathcal{M}(G_{\mathrm{split}})$. It is important to point out that $B_L(G) = B_L(G_{\mathrm{split}})$.

Sections 2 and 3 establish the Lie theoretical results necessary for the sequel. Section 2 is essentially introductory: we recall the Cartan theory for reductive complex Lie algebras in Section 2.1. Section 2.2 reviews the construction of the maximal split subalgebra $\widehat{\mathfrak{g}}$ of any real reductive Lie algebra $\mathfrak{g}$, due to Kostant–Rallis [29].

In Section 3 we study real reductive Lie groups following Knapp's definition ([25, Chap. VII]). We extend classical structural results in Lie theory, such as closedness of reductivity by involutions (Proposition 2.3), or basic results used in the Cartan theory of groups (Proposition 3.6). All of this is done in Section 3.1. Let (G, H, θ, B) be a real reductive Lie group in the sense of Definition 3.1. The main aim of Section 3.2 is to study the interplay between involutions ι of G and the fixed point subgroup G^ι, as well as the relations with adjoint groups and normalizing subgroups. The main result in this direction is Proposition 3.17, which specialises to real forms of complex reductive Lie groups in Corollary 3.18. All of these results are essential for Sections 5 and 6. Section 3.3 deals with the construction of a maximal split subgroup

$$(\widehat{G}_0, \widehat{H}_0, \widehat{\theta}, \widehat{B}) \leq (G, H, \theta, B)$$

(see Definition 3.24 and Proposition 3.25). We use results by Borel and Tits [7,8] to study the connections between the topology of both groups (Corollary 3.31), which will be used in Section 7.

Section 4 generalizes part of the work of Kostant and Rallis [29] to our context. More precisely, given $\mathfrak{g}$ the reductive Lie algebra of a reductive Lie group G, consider its Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h} = \mathrm{Lie}(H)$ for some maximal compact subgroup $H \leq G$. We study the Chevalley morphism $\chi : \mathfrak{m}^{\mathbb{C}} \to \mathfrak{m}^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$ and in particular the existence of a section of this morphism (see Theorem 4.9). We hereby note the prominent role of real forms of quasi-split type in the whole theory (see Lemma 4.13(2)).

We recall the basics on moduli spaces of Higgs bundles in Section 5, following [18]. The results in this section are not original, with the exception perhaps of Proposition 5.9.

The main result of this paper is in Section 6, where we generalize Hitchin's construction of a section of the Hitchin map [24]. This yields Theorem 6.13, which reads as follows.

Theorem. *Let (G, H, θ, B) be a strongly reductive Lie group, and let $(\widehat{G}_0, \widehat{H}_0, \widehat{\theta}, \widehat{B})$ be its maximal connected split subgroup. Let $L \to X$ be a line bundle with degree $d_L \geq 2g - 2$. Let $\alpha \in i\mathfrak{z}(\mathfrak{so}(2))$ be such that $\rho'(\alpha) \in \mathfrak{z}(\mathfrak{h})$, where $\rho' : \mathfrak{so}(2) \to \mathfrak{h}$ is given by* (39)*. Then, the choice of a square root of L determines N inequivalent sections of the map*

$$h_L : \mathcal{M}_L^{\rho'(\alpha)}(G) \to B_L(G).$$

Here, N is the number of cosets in $\mathrm{Ad}(G)^\theta/\mathrm{Ad}(H)$.

Each such section s_G satisfies the following:

1. *If G is quasi-split, $s_G(B_L(G))$ is contained in the stable locus of $\mathcal{M}_L^{\rho'(\alpha)}(G)$, and in the smooth locus if $Z(G) = Z_G(\mathfrak{g})$ and $d_L \geq 2g - 2$.*
2. *If G is not quasi-split, the image of the section is contained in the strictly polystable locus.*
3. *For arbitrary groups, the Higgs field is everywhere regular.*
4. *If $\rho'(\alpha) \in i\mathfrak{z}\left(\widehat{\mathfrak{h}}\right)$, the section factors through $\mathcal{M}_L^{\rho'(\alpha)}(\widehat{G}_0)$. This is in particular the case if $\alpha = 0$.*
5. *If $G_{\mathrm{split}} < G^{\mathbb{C}}$ is the split real form of a complex reductive Lie group, $K = L$ and $\alpha = 0$, s_G is the factorisation of the Hitchin section through $\mathcal{M}(G_{\mathrm{split}})$.*

We will refer to a section defined as above as a **Hitchin–Kostant–Rallis**, abbreviated HKR, section.

Due to the degree of generality in which we have chosen to work, we need to develop the theory with new tools. A remarkable fact is that the image of the section need not be smooth, even when the group is connected, of adjoint type, and the twisting is the canonical bundle. This differs from the complex group case studied by Hitchin in [24] and is due to the fact that split groups are quasi-split (see Proposition 6.10 and Corollary 6.16). After some analysis in Section 6.1 of the representation theory involved (note the differences with the complex case pointed out in Corollary 4.15), we move in Section 6.2 to study the basic case: the HKR section for $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles. The latter is then used in Section 6.3 to produce a G-Higgs bundle, which will be deformed to yield a section of the Hitchin map. The analysis is done in Section 6.4. We use the results in this section to prove in Proposition 6.20 that for quasi-split groups G, the image of the section covers a connected component of the moduli space if and only if the real group is split. We include in Section 6.5 a geometric interpretation of the algebraic notion of regularity.

The topological type of the elements in the image of the HKR section is studied in Section 7. We study the Hermitian and non-Hermitian cases separately. In the first case, an answer is given in Proposition 7.2. In the second case, however, the answer depends on the topological type of elements of the Hitchin section for the maximal split subgroup. We deal with this in Proposition 7.1.

2. Reductive Lie algebras and maximal split subalgebras

A **reductive Lie algebra** over a field k is a Lie algebra $\mathfrak{g}$ over k whose adjoint representation is completely reducible. Semisimple Lie algebras are reductive. It is well known that any reductive Lie algebra decomposes as a direct sum

$$\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{z}(\mathfrak{g}),$$

where $\mathfrak{g}_{ss} = [\mathfrak{g},\mathfrak{g}]$ is a semisimple Lie subalgebra (the semisimple part of $\mathfrak{g}$) and $\mathfrak{z}(\mathfrak{g})$ is the centre of $\mathfrak{g}$, thus an abelian subalgebra.

We will focus on Lie algebras over the real and complex numbers and the relation between them. As a first example, note that any complex reductive Lie algebra $\mathfrak{g}^{\mathbb{C}}$ with its underlying real structure $\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}}$ is a real reductive Lie algebra. On the other hand, given a real reductive Lie algebra $\mathfrak{g}$, its complexification $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex reductive Lie algebra.

2.1. **Real forms of complex Lie algebras.** A **real form** $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ of a complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is the subalgebra of fixed points of an antilinear involution $\sigma \in \mathrm{Aut}_2\left(\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}}\right)$, where $\mathrm{Aut}_2\left(\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}}\right)$ denotes the subset of order two automorphisms of the real Lie algebra underlying $\mathfrak{g}^{\mathbb{C}}$. Equivalently, it is a real subalgebra $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ such that the natural homomorphism of $\mathbb{C}$-algebras $\mathfrak{g} \otimes \mathbb{C} \to \mathfrak{g}^{\mathbb{C}}$ is an isomorphism.

Any real Lie algebra $\mathfrak{g}$ is a real form of its complexification $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ with associated involution $\mathfrak{g}^{\mathbb{C}} \cong_{\mathbb{R}} \mathfrak{g} \oplus \mathfrak{g} \ni (X,Y) \mapsto (X,-Y)$. Also, given a complex reductive Lie algebra $\mathfrak{g}^{\mathbb{C}}$, one can obtain it as a real form of $\mathfrak{g}^{\mathbb{C}} \otimes \mathbb{C}$ by choosing a maximal compact subalgebra $\mathfrak{u} \subset \mathfrak{g}^{\mathbb{C}}$ (i.e., a real subalgebra whose adjoint group is compact). Let $\tau \in \mathrm{Aut}_{\mathbb{R}}\left(\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}}\right)$ be the antilinear involution defining $\mathfrak{u}$. Then,

considering $\mathfrak{g}^{\mathbb{C}} \otimes \mathbb{C} \cong \mathfrak{g}^{\mathbb{C}} \oplus \mathfrak{g}^{\mathbb{C}}$, define on it the antilinear involution

$$\tau^{\mathbb{C}}(x, y) := (\tau(x), -\tau(y)),$$

whose subalgebra of fixed points is isomorphic to $\mathfrak{u} \oplus i\mathfrak{u} \cong \left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}}$.

Two real forms $\mathfrak{g}$ and $\mathfrak{g}'$ of $\mathfrak{g}^{\mathbb{C}}$ (defined respectively by antilinear involutions $\sigma,\ \sigma' \in \mathrm{Aut}_{\mathbb{R}}(\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}})$) are Cartan isomorphic, denoted by $\sigma \sim_c \sigma'$, if there exists $\varphi \in \mathrm{Aut}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}})$ such that $\sigma' \circ \varphi = \varphi \circ \sigma$.

We will consider the stronger equivalence condition, which we will denote by $\sigma \sim_i \sigma'$, if furthermore φ can be chosen inside the group of inner automorphisms of the Lie algebra $\mathrm{Int}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}})$.

It is well known (see for example [31, Sec. 3]) that there exists a correspondence between isomorphism classes (under equivalence $\sim_c$ or $\sim_i$) of real forms of a complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and orbits of $\mathbb{C}$-linear involutions (under $\mathrm{Int}(\mathfrak{g}^{\mathbb{C}})$, resp. $\mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$) of $\mathfrak{g}^{\mathbb{C}}$. This correspondence is obtained by composing the involution defining the real form with a commuting involution defining a compact form. Both forms are then said to be compatible.

Proposition 2.1. *Given a complex reductive Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and a compact real form $\mathfrak{u}$ of $\mathfrak{g}^{\mathbb{C}}$, there is a* 1-1 *correspondence between conjugacy classes under $\sim_i$ of real forms compatible with $\mathfrak{u}$ and conjugacy classes under $\sim_i$ of linear automorphisms $\theta : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$.*

Proof. We note first that involutions of a Lie algebra leave the semisimple part and the centre invariant. This, together with Theorem 3.2 in [31], implies that it is enough to prove the proposition for abelian Lie algebras, that is, vector spaces.

Let $\mathfrak{g}^{\mathbb{C}}$ be an abelian Lie algebra of dimension n. A choice of basis allows us to identify it with $\mathbb{C}^n$. A real form $\mathfrak{g}$ is a real subspace of dimension n, which is the set of fixed points of the reflection with respect to $\mathfrak{g}$. Note that the only compact real form is $(i\mathbb{R})^n \subset \mathbb{C}^n$, as if $v_1, \ldots, v_n$ are the real vectors expanding the subspaces, then exponentiation of any vector that is not purely imaginary contains a spiral which is non-compact (as real forms of $\mathbb{C}$ are in correspondence with real vectorial lines in $\mathbb{C} \cong \mathbb{R}^2$ which exponentiate to U(1) or spirals; the case of $\mathbb{R}$ corresponds to the degenerate spiral).

Now, the only real form compatible with $(i\mathbb{R})^n$ is a direct sum of copies of $\mathbb{R}$ and $i\mathbb{R}$. On the other hand, involutions compatible with $\sigma : (z_1, \ldots, z_n) \mapsto -(\overline{z_1}, \ldots, \overline{z_n})$ are combinations of complex conjugation and multiplication by ± 1 on the factors and transpositions, which composed with σ yield all possible linear involutions of $\mathbb{C}^n$, that is, transpositions and multiplication by ± 1. □

Remark 2.2. Proposition 2.1 classifies real forms of an abelian Lie algebra up to $\sim_i$ equivalence. Note that the result does not depend on the choice of a compact form, as neither does the result for semisimple algebras, and the compact form of the centre is unique, but we are forced to consider compatible real forms. If we considered real forms up to outer isomorphism, then the compact form and the split one would be identified.

An involution of a real reductive Lie algebra $\mathfrak{g}$ defining a maximal compact form is called a **Cartan involution**. The decomposition of $\mathfrak{g}$ into $(+1)$- and (-1)-eigenspaces is a **Cartan decomposition**. Any such has the form

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{1}$$

satisfying the relations

$$[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}, \qquad [\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{h}, \qquad [\mathfrak{h},\mathfrak{m}] \subseteq \mathfrak{m}.$$

In particular, we have an action $\iota : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m})$ induced by the adjoint action of $\mathfrak{g}$ on itself, which is called the infinitesimal **isotropy representation**.

Proposition 2.3. *The class of reductive Lie algebras is closed by taking fixed points of involutions.*

Proof. By the preceding discussion, it is enough to prove the statement for simple Lie algebras, as any extension of a simple Lie algebra by a central subalgebra is reductive, and all reductive Lie algebras are a direct sum of algebras of this kind. Now, any Lie algebra $\mathfrak{g}$ is a real form of its complexification $\mathfrak{g}^{\mathbb{C}}$. Given ι and involution of $\mathfrak{g}$, we may extend it to a $\mathbb{C}$-linear involution of $\mathfrak{g}^{\mathbb{C}}$. Then, the Cartan theory for semisimple Lie algebras and Proposition 3.21 imply that $(\mathfrak{g}^{\mathbb{C}})^{\iota} = \mathfrak{h}^{\mathbb{C}}$ for some compact Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. But [31, I.11] implies that $\mathfrak{h}$ is reductive. □

Remark 2.4. The above proves that fixed points of involutions of simple Lie algebras are reductive; however one cannot expect them to be semisimple. For example, the maximal compact subalgebra $\mathfrak{u}(2) \subset \mathfrak{sp}(4,\mathbb{R})$ is fixed by the Cartan involution and is reductive, but not simple or semisimple.

2.2. **Maximal split subalgebras and restricted root systems.** Let $\mathfrak{g}$ be a real reductive Lie algebra with a Cartan involution θ decomposing $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Given a maximal subalgebra $\mathfrak{a} \subset \mathfrak{m}$ it follows from the definitions that it must be abelian, and one can easily prove that its elements are semisimple and diagonalizable over the real numbers (cf. [25, Chap. VI]. Note that Knapp proves it for semisimple Lie algebras, but for reductive Lie algebras it suffices to use invariance of the centre and the semisimple part of $[\mathfrak{g}^{\mathbb{C}},\mathfrak{g}^{\mathbb{C}}]$) under the Cartan involution. Any such subalgebra is called a **maximal anisotropic** Cartan subalgebra of $\mathfrak{g}$. By extension, its complexification $\mathfrak{a}^{\mathbb{C}}$ is called a **maximal anisotropic** Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ (with respect to $\mathfrak{g}$). A maximal anisotropic Cartan subalgebra $\mathfrak{a}$ can be completed to a θ-equivariant **Cartan subalgebra** of $\mathfrak{g}$, namely, a subalgebra whose complexification is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Indeed, define

$$\mathfrak{d} = \mathfrak{t} \oplus \mathfrak{a}, \tag{2}$$

where $\mathfrak{t} \subset \mathfrak{c}_{\mathfrak{h}}(\mathfrak{a}) := \{x \in \mathfrak{h} \ : \ [x,\mathfrak{a}] = 0\}$ is a maximal abelian subalgebra ([25], Proposition 6.47). Cartan subalgebras of this kind (and their complexifications) are called **maximally split**.

The dimension of maximal anisotropic Cartan subalgebras of a real reductive Lie algebra $\mathfrak{g}$ is called the **the real (or split) rank of** $\mathfrak{g}$. This number measures the degree of compactness of real forms: indeed, a real form is compact (that is, its adjoint group is compact) if and only if $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) = 0$. On the other hand, a real form is defined to be **split** if $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) = \mathrm{rk}\mathfrak{g}^{\mathbb{C}}$. Note that the split rank depends on the involution θ associated with the real form when $\mathfrak{g}$ is not semisimple.

The restriction to $\mathfrak{a}$ of the adjoint representation of $\mathfrak{g}$ yields a decomposition of $\mathfrak{g}$ into $\mathfrak{a}$-eigenspaces

$$\mathfrak{g} = \bigoplus_{\lambda \in \Lambda(\mathfrak{a})} \mathfrak{g}_{\lambda},$$

where $\Lambda(\mathfrak{a}) \subset \mathfrak{a}^*$ is called the set of **restricted roots** of $\mathfrak{g}$ with respect to $\mathfrak{a}$. The set $\Lambda(\mathfrak{a})$ forms a root system (see [25, Chap. II, Sec. 5]), which may not be reduced

(that is, there may be roots whose double is also a root). The name restricted roots is due to the following fact: extending restricted roots by $\mathbb{C}$-linearity, we obtain $\Lambda(\mathfrak{a}^{\mathbb{C}}) \subset (\mathfrak{a}^{\mathbb{C}})^*$, also called restricted roots. Now, take a maximally split θ-invariant Cartan subalgebra $\mathfrak{d} \subset \mathfrak{g}$ as in (2), and let $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{d}^{\mathbb{C}})$ be the corresponding set of roots. Then, restricted roots are restrictions of roots. In fact, a root $\gamma \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{d}^{\mathbb{C}})$ decomposes as

$$\gamma = \lambda + i\beta, \tag{3}$$

where λ is the extension by complex linearity of an element in $\mathfrak{a}^*$ and β is the extension by complex linearity of an element $\mathfrak{t}^*$. This implies $\gamma|_{\mathfrak{a}^{\mathbb{C}}} = \lambda|_{\mathfrak{a}^{\mathbb{C}}}$. We can decompose $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{d}^{\mathbb{C}}) = \Delta_i \cup \Delta_r \cup \Delta_c$, where

$$\begin{aligned} \Delta_i &= \{\gamma \in \Delta \ : \ \gamma|_{\mathfrak{a}^{\mathbb{C}}} \equiv 0\}, \\ \Delta_r &= \{\gamma \in \Delta \ : \ \gamma|_{\mathfrak{t}^{\mathbb{C}}} \equiv 0\}, \\ \Delta_c &= \Delta \setminus (\Delta_i \cup \Delta_r) \end{aligned} \tag{4}$$

are respectively called **imaginary, real,** and **complex** roots.

In [29], Kostant and Rallis give a procedure to construct a θ-invariant subalgebra $\widehat{\mathfrak{g}} \subset \mathfrak{g}$ such that $\widehat{\mathfrak{g}} \subset (\widehat{\mathfrak{g}})^{\mathbb{C}}$ is a split real form whose Cartan subalgebra is $\mathfrak{a}$ and such that $\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{m}$. Their construction relies on the following notion.

Definition 2.5. A **three-dimensional subalgebra** (TDS) $\mathfrak{s}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ is the image of an injective morphism $\mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}^{\mathbb{C}}$. A TDS is called **normal** if $\dim \mathfrak{s}^{\mathbb{C}} \cap \mathfrak{h}^{\mathbb{C}} = 1$ and $\dim \mathfrak{s}^{\mathbb{C}} \cap \mathfrak{m}^{\mathbb{C}} = 2$. It is called **principal** if it is generated by elements $\{e, f, x\}$, where e and f are nilpotent regular elements in $\mathfrak{m}^{\mathbb{C}}$ (cf. Definition 4.6), and $x \in \mathfrak{h}^{\mathbb{C}}$ is semisimple. A set of generators satisfying such relations is called a **normal basis** or **normal triple**.

Definition 2.6. A subalgebra $\widehat{\mathfrak{g}} \subset \mathfrak{g}$ generated by $\mathfrak{a}$ and $\mathfrak{s}^{\mathbb{C}} \cap \mathfrak{g}$, where $\mathfrak{s}^{\mathbb{C}}$ is a principal normal TDS invariant by the involution defining $\mathfrak{g}$ inside $\mathfrak{g}^{\mathbb{C}}$, is called a **maximal split subalgebra**.

Maximal split subalgebras can be constructed very explicitly; for this, consider the following reduced system of roots:

$$\widehat{\Lambda}(\mathfrak{a}) = \{\lambda \in \Lambda(\mathfrak{a}) \mid \lambda/2 \notin \Lambda(\mathfrak{a})\}. \tag{5}$$

Let $\{\lambda_1, \dots, \lambda_a\} = \Sigma(\mathfrak{a}) \subset \Lambda(\mathfrak{a})$ be a system of simple restricted roots (cf. [25, Chap. VI]), which is also a system of simple roots for $\widehat{\Lambda}(\mathfrak{a})$. Let $h_i \in \mathfrak{a}$ be the dual to λ_i with respect to some θ and $\mathrm{Ad}(\exp(\mathfrak{g}))$-invariant bilinear form B satisfying that B is negative definite on $\mathfrak{h}$ and positive definite on $\mathfrak{m}$. Strictly speaking, in [29] Kostant and Rallis take B to be the Cartan–Killing form on $\mathfrak{g}$; however, the above assumptions are enough to obtain the necessary results quoted here. Now, for each $\lambda_i \in \Sigma(\mathfrak{a})$ choose $y_i \in \mathfrak{g}_{\lambda_i}$. We have

$$[y_i, \theta y_i] = b_i h_i,$$

where $b_i = B(y_i, \theta y_i)$. Indeed, $[y_i, \theta y_i] \in \mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}]$, so it is enough to prove that $B([y_i, \theta y_i], x) = B(y_i, \theta y_i)\lambda_i(x)$ for all $x \in \mathfrak{a}$, which is a simple calculation.

Consider

$$z_i = \frac{2}{\lambda_i(h_i) b_i} \theta y_i, \qquad w_i = [y_i, z_i] = \frac{2}{\lambda_i(h_i)} h_i.$$

TABLE 1. Maximal split subalgebras

Type	$\mathfrak{g}$	$\widehat{\mathfrak{g}}$
AI	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{R})$
AII	$\mathfrak{su}^*(2n)$	$\mathfrak{sl}(n,\mathbb{R})$
AIII	$\mathfrak{su}(p,q),\ p<q$	$\mathfrak{so}(p,p+1)$
	$\mathfrak{su}(p,p)$	$\mathfrak{sp}(2p,\mathbb{R})$
BI	$\mathfrak{so}(2p,2q+1),\ p\leq q$	$\mathfrak{so}(2p,2p+1)$
CI	$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{sp}(2n,\mathbb{R})$
CII	$\mathfrak{sp}(p,q)\ p<q$	$\mathfrak{so}(p,p+1)$
	$\mathfrak{sp}(p,p)$	$\mathfrak{sp}(2p,\mathbb{R})$
BDI	$\mathfrak{so}(p,q)\ p+q=2n, p<q$	$\mathfrak{so}(p,p+1)$
DI	$\mathfrak{so}(p,p)$	$\mathfrak{so}(p,p)$
DII	$\mathfrak{so}^*(4p+2)\ p<q$	$\mathfrak{so}(p,p+1)$
	$\mathfrak{so}^*(4p)$	$\mathfrak{sp}(2p,\mathbb{R})$

Type	$\mathfrak{g}$	$\widehat{\mathfrak{g}}$
EI	$\mathfrak{e}_{6(6)}$	$\mathfrak{e}_{6(6)}$
EII	$\mathfrak{e}_{6(2)}$	$\mathfrak{f}_{4(4)}$
EIII	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(3,2)$
EIV	$\mathfrak{e}_{6(-26)}$	$\mathfrak{sl}(3,\mathbb{R})$
EV	$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{7(7)}$
EVI	$\mathfrak{e}_{7(-5)}$	$\mathfrak{f}_{4(4)}$
EVII	$\mathfrak{e}_{7(-25)}$	$\mathfrak{sp}(6,\mathbb{R})$
EVIII	$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_{8(8)}$
EIX	$\mathfrak{e}_{8(-24)}$	$\mathfrak{f}_{4(4)}$
FI	$\mathfrak{f}_{4(4)}$	$\mathfrak{f}_{4(4)}$
FII	$\mathfrak{f}_{4(-20)}$	$\mathfrak{sl}(2,\mathbb{R})$
G	$\mathfrak{g}_{2(2)}$	$\mathfrak{g}_{2(2)}$

We have the following (Proposition 23 in [29]).

Proposition 2.7. *Let $\mathfrak{g}\subset\mathfrak{g}^{\mathbb{C}}$ be a real form, and let σ be the antilinear involution of $\mathfrak{g}^{\mathbb{C}}$ defining $\mathfrak{g}$. Let $\widehat{\mathfrak{g}}$ be the subalgebra generated by all the y_i, z_i, and w_i as above, and let $\mathfrak{c}_{\mathfrak{m}}(\mathfrak{a})$ be the centraliser of $\mathfrak{a}$ in $\mathfrak{m}$. Let $\widehat{\mathfrak{g}}^{\mathbb{C}}=\widehat{\mathfrak{g}}\otimes\mathbb{C}$. Then:*

1. *$\widehat{\mathfrak{g}}^{\mathbb{C}}$ is a σ- and θ-invariant reductive subalgebra of $\mathfrak{g}^{\mathbb{C}}$. We thus have $\widehat{\mathfrak{g}}^{\mathbb{C}}=\widehat{\mathfrak{h}}^{\mathbb{C}}\oplus\widehat{\mathfrak{m}}^{\mathbb{C}}$ where $\widehat{\mathfrak{h}}^{\mathbb{C}}=\mathfrak{h}^{\mathbb{C}}\cap\widehat{\mathfrak{g}}^{\mathbb{C}}$, $\widehat{\mathfrak{m}}^{\mathbb{C}}=\mathfrak{m}^{\mathbb{C}}\cap\widehat{\mathfrak{g}}^{\mathbb{C}}$.*
2. *$\widehat{\mathfrak{g}}\subset\mathfrak{g}$ is a maximal split subalgebra as in Definition 2.6. Moreover, the subsystem $\widehat{\Lambda}(\mathfrak{a}^{\mathbb{C}})\subset\Lambda(\mathfrak{a}^{\mathbb{C}})$ as defined in (5) is the root system of $\widehat{\mathfrak{g}}^{\mathbb{C}}$ with respect to $\mathfrak{a}^{\mathbb{C}}$.*

Since $\widehat{\Lambda}(\mathfrak{a}^{\mathbb{C}})$ is a reduced root system, we can uniquely assign to it a complex semisimple Lie algebra $\widehat{\mathfrak{g}}^{\mathbb{C}}$. In [4] Araki gives the details necessary to obtain $\widehat{\mathfrak{g}}^{\mathbb{C}}$ (or its Dynkin diagram) from the Satake diagram of $\mathfrak{g}$ whenever the latter is a simple Lie algebra. The advantage of Araki's procedure is that it allows identifying the isomorphism class of $\widehat{\mathfrak{g}}$ easily. However, unlike Kostant and Rallis's method, it does not provide the embedding $\widehat{\mathfrak{g}}\hookrightarrow\mathfrak{g}$. See [4] for details. We include Table 1 with all real forms of the simple Lie algebras.

Remark 2.8. Let $\mathfrak{g}^{\mathbb{C}}$ be a complex reductive Lie algebra, and let $\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}}$ be its underlying real reductive algebra. Then, the maximal split subalgebra of $\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}}$ is isomorphic to the split real form $\mathfrak{g}_{\text{split}}$ of $\mathfrak{g}^{\mathbb{C}}$. It is clearly split within its complexification and it is maximal within $\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}}$ with this property, which can be easily checked by identifying $\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}} \cong \mathfrak{g}_{\text{split}} \oplus i\mathfrak{g}_{\text{split}}$.

3. Reductive Lie groups and maximal split subgroups

3.1. **Real reductive Lie groups.** Following Knapp [25, VII.2], we define reductivity of a Lie group as follows.

Definition 3.1. A **real reductive group** is a 4-tuple (G, H, θ, B) where:

(1) G is a real Lie group with reductive Lie algebra $\mathfrak{g}$.
(2) $H < G$ is a maximal compact subgroup.
(3) θ is a Lie algebra involution of $\mathfrak{g}$ inducing an eigenspace decomposition
$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$
where $\mathfrak{h} = \text{Lie}(H)$ is the $(+1)$-eigenspace for the action of θ and $\mathfrak{m}$ is the (-1)-eigenspace.
(4) B is a θ- and $\text{Ad}(G)$-invariant non-degenerate bilinear form, with respect to which $\mathfrak{h} \perp_B \mathfrak{m}$ and B is negative definite on $\mathfrak{h}$ and positive definite on $\mathfrak{m}$.
(5) The multiplication map $H \times \exp(\mathfrak{m}) \to G$ is a diffeomorphism.

Furthermore if (G, H, θ, B) satisfies

(SR) G acts by inner automorphisms on the complexification $\mathfrak{g}^{\mathbb{C}}$ of its Lie algebra via the adjoint representation,

then the group will be called **strongly reductive**.

Remark 3.2. Note that the definition of Knapp [25, VII.2] differs from ours in two ways: on the one hand, he assumes (SR) in the definition of reductivity. Since we will cite his results, we will need to pay attention to which of them really use this hypothesis. On the other hand, he does not assume H to be maximal, just compact. Maximality in fact results from the polar decomposition.

Remark 3.3. If $G^{\mathbb{C}}$ satisfies condition (SR) in Definition 3.1, then, by definition, $\text{Ad}(G^{\mathbb{C}})$ is equal to $\text{Ad}(\mathfrak{g}^{\mathbb{C}})$, the connected component of $\text{Aut}(\mathfrak{g}^{\mathbb{C}})$.

Given a Lie group G with reductive Lie algebra $\mathfrak{g}$, the extra data (H, θ, B) defining a reductive structure will be referred to as **Cartan data** for G.

A morphism of reductive Lie groups $(G', H', \theta', B') \to (G, H, \theta, B)$ is a morphism of Lie groups $G' \to G$ which respects the corresponding Cartan data in the obvious way. In particular, a reductive Lie subgroup of a reductive Lie group (G, H, θ, B) is a reductive Lie group (G', H', θ', B') such that $G' \leq G$ is a Lie subgroup and the Cartan data (H', θ', B') is obtained by intersection and restriction.

Remark 3.4. When the group G is semisimple, letting B be the Killing form, the rest of the Cartan data is fully determined by the choice of a maximal compact subgroup H. In this case, we omit the Cartan data from the notation.

Lemma 3.5. *Let G be a semisimple Lie group with maximal compact subgroup $H \leq G$. Then, $Z(G) \leq Z(H)$, and equality holds if G is complex.*

Proof. From Corollary 7.26(2) in [25], we have that $Z(G) = Z_H(G)e^{i\mathfrak{z}_{\mathfrak{m}}(\mathfrak{g})}$, as the quoted result does not use (SR) in Definition 3.1, but semisimplicity implies that $\mathfrak{z}_{\mathfrak{m}}(\mathfrak{g}) = 0$, so $Z(G) \leq Z(H)$. Now, if G is a complex group, given that $G = He^{i\mathfrak{h}}$ and that $Z(H) \subset Z_H(\mathfrak{h}) = Z_H(i\mathfrak{h})$, we have that $Z(H)$ centralises the identity component G_0. Since any connected component of G is of the form hG_0 for some $h \in H$, it follows that $Z(H)$ centralises all connected components, and so also G. □

3.2. **Real forms of complex reductive Lie groups.** A great variety of examples of real reductive Lie groups is provided by real forms of complex reductive Lie groups. Recall that a real form G of a complex Lie group $G^{\mathbb{C}}$ is the group of fixed points of an antiholomorphic involution $\sigma : G^{\mathbb{C}} \to G^{\mathbb{C}}$.

Some of the results in this section are common knowledge, but due to the lack of known references covering the general case we include them in this section. Similar results are also proved in [19].

The following proposition proves real forms of some complex reductive Lie groups inherit a reductive group structure from their complexification.

Proposition 3.6. *Let $(G^{\mathbb{C}}, U, \tau, B)$ be a connected complex reductive Lie group, and let σ be an antilinear involution of $G^{\mathbb{C}}$ defining $G = \left(G^{\mathbb{C}}\right)^{\sigma}$. Then, on $G^{\mathbb{C}}$, there exists an involution conjugate by an inner element $\sigma' = \mathrm{Ad}_g \circ \sigma \circ \mathrm{Ad}_{g^{-1}}$ such that $G' = gGg^{-1}$ can be endowed with Cartan data (H', θ', B'), making it a reductive subgroup of $(G^{\mathbb{C}}, U, \tau, B)$ in the sense of Definition* 3.1.

Proof. By Proposition 2.1 and the fact that all maximal compact Lie subalgebras are conjugate, at the level of the Lie algebras there is an inner conjugate of $d\sigma$ that commutes with τ, say $(d\sigma)' = \mathrm{Ad}_g \circ d\sigma \circ \mathrm{Ad}_{g^{-1}}$. We notice that $(d\sigma)' = d\sigma'$ where $\sigma' = \mathrm{Ad}_g \circ \sigma \circ \mathrm{Ad}_{g^{-1}}$. So $U_0 = \exp(\mathfrak{u})$ is σ'-invariant.

All of this implies that the polar decomposition of $G^{\mathbb{C}}$ for a choice of Cartan data (U, τ, B) induces one for $G' = \mathrm{Ad}(g)(G)$. Indeed, G' is diffeomorphic to $H' \times \exp \mathfrak{m}' = G^{\mathbb{C}^{\sigma'}}$, where $H' = U^{\sigma'}$, $\exp \mathfrak{m}' = \exp \mathfrak{u}^{\sigma'}$, as any $g \in G'$ can be written as $g = ue^V$ for $u \in U$, $V \in i\mathfrak{u}$, and it must be

$$u^{\sigma} e^{\sigma' V} = ue^V \iff u^{-1}u^{\sigma} = e^{-\sigma V} e^V \in U \cap \exp i\mathfrak{u} = \{1\}.$$

So $G' \cong H' \times \exp \mathfrak{m}'$.

Non-degeneracy of $B|_{\mathfrak{g}}$ follows easily: for any element $X \in \mathfrak{g}$ there exists $Y = Y_1 + iY_2 \in \mathfrak{g}^{\mathbb{C}}$ such that $0 \neq B(X, Y) = B(X, Y_1) + iB(X, Y_2)$. In particular $B(X, Y_1) \neq 0$. Clearly $\mathfrak{h}' \perp_B \mathfrak{m}'$, and all the other properties of Definition 3.1 are straightforward to check. □

Remark 3.7. Proposition 3.6 is well known for semisimple Lie groups (see for example Theorem 4.3.2 in [20]).

Corollary 3.8. *Let $G^{\mathbb{C}}$ be a connected complex reductive Lie group. Then, there exists a correspondence between $G^{\mathbb{C}}$-conjugacy classes of real forms $G < G^{\mathbb{C}}$ and holomorphic involutions of $G^{\mathbb{C}}$ up to conjugation by* $\mathrm{Ad}(G)$.

Proof. It follows from Proposition 3.6 by noticing that a choice of Cartan data is determined up to conjugation (except for the metric B, which plays no role, so we can ignore it), and the indeterminacy in the choice of the antiholomorphic involution yielding a given real form too. To see the latter, assume σ and σ' are two different involutions of $G^{\mathbb{C}}$ with the same fixed point subgroup G. Then, since

$\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{g} \oplus i\mathfrak{g}$, the differentials are the same $d\sigma = d\sigma'$. This means that σ and σ' act the same way on the identity component $(G^{\mathbb{C}})_0$, which is the group itself. □

The following important fact is a consequence of Proposition 3.6.

Proposition 3.9. *Let G' and $G^{\mathbb{C}}$ be as in Proposition* 3.6. *We abuse notation by calling both σ' and τ the involutions defining G' and U and their differentials. Then the composition $\sigma'\tau = \tau\sigma'$ defines a holomorphic involution of $G^{\mathbb{C}}$ which lifts the extension of θ to $\mathfrak{g}^{\mathbb{C}}$ by complex linearity, and so we will abuse notation and denote $\theta := \tau\sigma$ for the holomorphic involution of $G^{\mathbb{C}}$. Note that in particular, this holomorphic involution lifts θ to G.*

Proposition 3.9 is relevant at a conceptual level: it tells us that antilinear involution of a connected complex reductive Lie group can be chosen to respect the Cartan data. This motivates the following definition, covering also the case of non-compact groups.

Definition 3.10. Let $(G^{\mathbb{C}}, U, \tau, B)$ be a complex reductive Lie group. We define a real form $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B)$ to be a real reductive subgroup such that $G < G^{\mathbb{C}}$ is a real form. This implies in particular that the involution σ defining G commutes with τ.

There are more reductive real subgroups of a complex reductive Lie group than real forms; some of these are related to real forms, as in the following example.

Example 3.11. Consider $\mathrm{SL}(2,\mathbb{R}) < \mathrm{SL}(2,\mathbb{C})$, which is a real form with associated involution σ given by complex conjugation. But its normalizer inside $\mathrm{SL}(2,\mathbb{C})$, say $N := N_{\mathrm{SL}(2,\mathbb{C})}(\mathrm{SL}(2,\mathbb{R}))$, is not. Reductivity of this group is shown in Corollary 3.18. We just recall here some basic facts.

The group N is generated by $\mathrm{SL}(2,\mathbb{R})$ and the element

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

so that it fits into an exact sequence

$$1 \to \mathrm{SL}(2,\mathbb{R}) \to N \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

The importance of these normalizing subgroups will be made clear in Section 5.

More generally, one may produce a real subgroup from a real form $G < G^{\mathbb{C}}$ defined by σ as follows.

Definition 3.12. Given a complex or real Lie group G and an involution $\iota : G \to G$ (holomorphic or antiholomorphic), we define

$$G_\iota = \{g \in G \ : \ g^{-1}g^\iota \in Z(G)\}.$$

Remark 3.13. Note that $G_\iota \subset N_G(G^\iota)$, as $Z(G) \subset Z_G(G^\iota)$.

With the above definition, $(G^{\mathbb{C}})_\sigma$ is a subgroup which is not necessarily a real form.

Example 3.14. With the notation of Example 3.11, for $G = \mathrm{SL}(2,\mathbb{R})$, we have that $(G^{\mathbb{C}})_\sigma = N$, which is not a real form.

The above example generalises to all semisimple Lie groups.

Lemma 3.15. *Let $G < G^{\mathbb{C}}$ be a real form of a complex semisimple Lie group defined by the involution σ. Then:*

(1) $Z(G^{\mathbb{C}})=Z_{G^{\mathbb{C}}}(G)$.
(2) $Z(G)=Z(G^{\mathbb{C}})^{\sigma}$.

Proof. By Corollary IV.4.22 in [25], $G^{\mathbb{C}} \hookrightarrow \mathrm{Aut}(V_{\mathbb{C}}) \subset \mathrm{End}(V_{\mathbb{C}})$ is a matrix group, so $G^{\mathbb{C}}$ is contained in the complex subspace spanned by G inside $\mathrm{End}(V_{\mathbb{C}})$. This implies that $Z_{G^{\mathbb{C}}}(G) \subset Z(G^{\mathbb{C}})$. The other inclusion is trivial, which proves (1).

As for (2), by (1), $Z(G^{\mathbb{C}})^{\sigma} = Z_{G^{\mathbb{C}}}(G)^{\sigma} = Z_G(G) = Z(G)$. □

Lemma 3.16. *If $G < G^{\mathbb{C}}$ is a real form of a complex semisimple Lie group, then $(G^{\mathbb{C}})_{\sigma} = N_{G^{\mathbb{C}}}(G)$.*

Proof. We easily see that $N_{G^{\mathbb{C}}}(G) = \{g \in G^{\mathbb{C}} \ : \ g^{-1}g^{\sigma} \in Z_{G^{\mathbb{C}}}(G)\}$, as $g \in N_{G^{\mathbb{C}}}(G)$ is equivalent to $\sigma(gfg^{-1}) = gfg^{-1}$ for all $f \in G$, which is in turn equivalent to $g^{-1}g^{\sigma}f(g^{\sigma})^{-1}g = f$, i.e., $g^{-1}g^{\sigma} \in Z_{G^{\mathbb{C}}}(G)$.

Now, by (1) in Lemma 3.15 above $Z_{G^{\mathbb{C}}}(G) = Z(G^{\mathbb{C}})$. Substituting this in the expression for N we get the equality we wanted. □

We next study the existence of a reductive structure of G_ι and apply it to the case $(G^{\mathbb{C}})_{\sigma}$, which we then compare with $N_{G^{\mathbb{C}}}(G)$.

Proposition 3.17. *Let (G, H, θ, B) be a reductive Lie group.*

(1) *Assume G is connected, and let ι be an involution of G. Then, a conjugate $H' := \mathrm{Ad}(g)(H)$ of H and its corresponding involution θ' provide Cartan data that induces Cartan data on G_ι by restriction and intersection.*
(2) *When G is not necessarily connected, if ι is an involution of (G, H, θ, B) (namely, ι leaves each component of the Cartan data invariant), then G_ι is θ stable and $(G_\iota, (G_\iota)^{\theta}, \theta, B)$ is a reductive subgroup whose Lie algebra is $\mathfrak{g}_\iota = \mathfrak{g}_+ \oplus \mathfrak{z}(\mathfrak{g})_-$ (where $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ is the decomposition of $\mathfrak{g}$ into the ± 1 ι-eigenspaces, and likewise for $\mathfrak{z}(\mathfrak{g})$).*
(3) *Let $\mathrm{Ad}_G : G \to \mathrm{Aut}(G)$ be the adjoint representation, and define the action $\iota \curvearrowright \mathrm{Aut}(G)$ by $\varphi^{\iota}(g) = \iota(\varphi(\iota(g)))$. Then, G_ι is the preimage by Ad_G of $\mathrm{Ad}_G(G)^{\iota}$.*
(4) *With the hypothesis of (2), consider $N = N_G(G^{\iota})$. If $Z_G(\mathfrak{g}^{\iota}) = Z_G(G^{\iota})$, then $(N, N^{\theta}, \theta, B)$ is a reductive subgroup whose Lie algebra is also $\mathfrak{g}_\iota$.*
(5) *If $Z_G(G^{\iota}) = Z(G)$, then $G_\iota = N$.*
(6) *We have*

$$\big(\mathrm{Ad}_{\mathfrak{g}}(N), \mathrm{Ad}_{\mathfrak{g}}(N)^{\theta}, \mathrm{ad}_{\mathfrak{g}}(\theta), \mathrm{ad}_{\mathfrak{g}}(B)\big) = \big(\mathrm{Ad}_{\mathfrak{g}}(G_\iota), \mathrm{Ad}_{\mathfrak{g}}(G_\iota)^{\theta}, \mathrm{ad}_{\mathfrak{g}}(\theta), \mathrm{ad}_{\mathfrak{g}}(B)\big)\,,$$

where $\mathrm{Ad}_{\mathfrak{g}} : G \to \mathrm{Aut}(\mathfrak{g})$ is the adjoint representation.

Proof. To prove (1), we first need to prove a conjugate of H is ι-invariant. The proof is the same as in Proposition 3.6 (with the difference that we conjugate the Cartan data rather than ι). Once this has been done, if we prove (2), the remaining part of (1) follows.

For the proof of (2), note that the fact that ι is an involution of the whole reductive structure implies that each datum is left invariant by ι. In particular, the maximal compact subgroup of G_ι is $H \cap G_\iota = (G_\iota)^{\theta}$. The polar decomposition follows from Corollary 7.26(2) in [25], just noticing that its proof does not use condition (SR) in Definition 3.1. Indeed, according to this result $Z(G) = Z_H(G)e^{\mathfrak{z}_{\mathfrak{m}}(\mathfrak{g})}$, so that

if $g = he^V$ is the polar decomposition of an element $g \in G_\iota$, then $h^{-1}h^\iota \in Z_H(G)$, $V - \iota V \in \mathfrak{z}_{\mathfrak{m}}(\mathfrak{g})$, namely $h \in (G_\iota)^\theta$, $V \in \mathfrak{m} \cap \mathfrak{g}_\iota$. Reductivity of $\mathfrak{g}_\iota$ will follow once we prove its decomposition, as reductivity is closed by taking fixed points of involutions (Proposition 2.3) and extensions by central abelian subalgebras.

Now, $X \in \mathfrak{g}_\iota \iff X - \iota X \in \mathfrak{z}(\mathfrak{g})$. Let $Y \in \mathfrak{z}(\mathfrak{g})$ be such that $X = \iota X + Y (*)$. Then $\iota X = X + \iota Y$, which substituting yields $X = X + Y + \iota Y$. Namely, $Y \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_-$. Now let $X = X_+ + X_-$, with $X_\pm \in \mathfrak{g}_\pm$. Then, substituting again in (*), we find $2X_- = Y \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_- \iff X_- \in \mathfrak{z}(\mathfrak{g})_-$.

We have proved conditions (1), (2), and (5) in Definition 3.1. The remaining ones follow directly from the fact that σ respects the Cartan involution induced by θ.

As for (3), we have that

$$\begin{aligned} &\mathrm{Ad}_G(g) \in \mathrm{Ad}_G(G)^\iota \iff \mathrm{Ad}_G(g) = \mathrm{Ad}_G(g^\iota) \\ &\iff g^{-1}g^\iota \in \mathrm{Ker}(\mathrm{Ad}_G) = Z(G) \iff g \in G_\iota. \end{aligned}$$

For point (4), we easily check that $Lie(N) =: \mathfrak{n} = \mathfrak{g}_\iota$, so conditions (1), (3), and (4) in Definition 3.1 follow from point (2) in this proposition. All that's left to check is polar decomposition, as it is clear that $N^\theta = N_H(G^\iota)$ is maximally compact. By Lemma 7.22 in [25] applied to the reductive group G (plus the fact that the proof of the quoted result does not use (SR) in Definition 3.1), since both N and N^θ normalize the θ-invariant Lie algebra $\mathfrak{g}^\iota$, it follows that $N_G(\mathfrak{g}^\iota) = N_U(\mathfrak{g}^\iota) \times e^{i\mathfrak{n}_{\mathfrak{h}}(\mathfrak{g}^\iota)}$. Now, $n \in N_G(\mathfrak{g}^\iota) \iff n^{-1}n^\iota \in Z_G(\mathfrak{g}^\iota)$. Likewise, $n \in N \iff n^{-1}n^\iota \in Z_G(G^\iota)$. Hence, we have (4).

Finally, (5) and (6) are easy to check from the definitions. In (6) note that $\mathrm{Ad}_{\mathfrak{g}}(N)$ is always reductive, as $\mathrm{Ad}_{\mathfrak{g}}(Z(G)) = \mathrm{Ad}_{\mathfrak{g}}(Z_G(\mathfrak{g}^\iota)) = 1$. □

Now, when ι defines a real form of a complex Lie group, Proposition 3.17 can be completed as follows.

Corollary 3.18. *Let $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B_{\mathbb{C}})$ be a real form defined by σ. Then:*

1. *The tuple (U_σ, θ, B) defines a reductive structure on $(G^{\mathbb{C}})_\sigma$.*
2. *We have $(G^{\mathbb{C}})_\sigma = N$ when $Z_{G^{\mathbb{C}}}(G) = Z(G^{\mathbb{C}})$. This is the case, for example, for semisimple groups.*
3. *The Lie algebra $\mathfrak{g}_\sigma \subset \mathfrak{g}^{\mathbb{C}}$ is a real form of $\mathfrak{g}^{\mathbb{C}} \oplus \mathfrak{z}(\mathfrak{g}^{\mathbb{C}})$.*

Proof. Point (1) follows from the equality $Z_U(G^{\mathbb{C}}) = Z(U)$, proved just as Lemma 3.5 was proved.

The first statement in (2) follows as in Proposition 3.17, while the second is a consequence of (1) in Lemma 3.15.

Point (3), is an easy remark, as from (2) in Proposition 3.17, we have $\mathfrak{g}_\sigma = \mathfrak{g} \oplus i\mathfrak{z}(\mathfrak{g})$. □

Note that strong reductivity need not be preserved.

Example 3.19. We easily see that $N_{\mathrm{SL}(2,\mathbb{C})}(\mathrm{SO}(2,\mathbb{C})) = \mathrm{SL}(2,\mathbb{C})_\theta$, which is the extension

$$0 \to \mathrm{SO}(2,\mathbb{C}) \to N \to \mathbb{Z}/2\mathbb{Z} \to 0$$

generated by the element $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$.

The following lemma points out an important relation between the groups $(G^{\mathbb{C}})_\sigma$ and $(G^{\mathbb{C}})_\theta$.

Lemma 3.20. *Let $G < G^{\mathbb{C}}$ be a real form of a semisimple Lie group whose defining involution we denote by σ. Then, if θ denotes the holomorphic involution corresponding to σ after a choice of a compatible maximal compact subgroup (see Remark 3.7), we have*

$$(G^{\mathbb{C}})_\sigma/G = (G^{\mathbb{C}})_\theta/H^{\mathbb{C}}.$$

Proof. We note that the above groups fit into the following exact sequences:

$$0 \to G \hookrightarrow (G^{\mathbb{C}})_\sigma \xrightarrow{f_1} Z(G^{\mathbb{C}}), \qquad f_1(g) = g^{-1}g^\sigma,$$

$$0 \to H^{\mathbb{C}} \hookrightarrow (G^{\mathbb{C}})_\theta \xrightarrow{f_2} Z(G^{\mathbb{C}}), \qquad f_2(g) = g^{-1}g^\theta.$$

Thus we just need to prove that $g^{-1}g^\sigma \in Z(G^{\mathbb{C}}) \iff g^{-1}g^\theta \in Z(G^{\mathbb{C}})$. By Lemma 3.5, $Z(G^{\mathbb{C}}) = Z(U)$. So let $g = ue^V$ be the polar decomposition of some element of $G^{\mathbb{C}}$. Then,

$$g^{-1}g^\sigma \in Z(U) \iff u^{-1}u^\sigma = u^{-1}u^\theta \in Z(U) \iff g^{-1}g^\theta \in Z(U).$$

□

Our interest in groups such as $(G^{\mathbb{C}})_\sigma$ is twofold. On the one hand, they produce examples of real Lie groups which are not real forms. On the other hand, we will see in Section 4 that the group $\mathrm{Ad}(G^{\mathbb{C}})_\theta = \mathrm{Ad}(G^{\mathbb{C}})^\theta$ is relevant in the study of the $H^{\mathbb{C}}$-module $\mathfrak{m}^{\mathbb{C}}$. Lemma 3.20 and Corollary 3.18 show that $\mathrm{Ad}(G^{\mathbb{C}})_\theta$ determines the real form $\mathrm{Ad}(G^{\mathbb{C}})^\sigma = \mathrm{Ad}((G^{\mathbb{C}})_\sigma) = \mathrm{Ad}(G^{\mathbb{C}})_\sigma$ and vice versa.

Proposition 3.21. *Let $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B)$ be a real form of a complex strongly reductive Lie group. Let*

$$A = e^{\mathfrak{a}}, \tag{6}$$

and consider

$$F = \{a \in A : a^2 \in Z(G)\}. \tag{7}$$

Then:

1. *We have that $G_\theta = F \cdot H$ and $\mathrm{Ad}(G_\theta) = \mathrm{Ad}(G)_\theta = Q \cdot \mathrm{Ad}(H)$, where $Q = \{a \in \mathrm{Ad}(\mathfrak{a}) \ : \ a^2 = 1\}$.*
2. *There are equalities*
$$\mathrm{Ad}(G^{\mathbb{C}})^\theta = \mathrm{Ad}(G^{\mathbb{C}})_\theta = \mathrm{Ad}((G^{\mathbb{C}})_\theta) = Q \cdot \mathrm{Ad}(H^{\mathbb{C}}) = \mathrm{Ad}(G_\theta)^{\mathbb{C}}.$$
3. *Let $\mathrm{Ad}_G : G \to \mathrm{Aut}(G)$ be the adjoint representation. Then G^θ is the preimage of $\mathrm{Ad}(G)^\theta$.*

Proof. To prove (1), consider the decomposition $G = HAH$ (see [25, VII.3], noting that the arguments leading to Theorem 7.39 do not require condition (SR) in Definition 3.1). Now, choose $g \in G_\theta$. By the above, it can be expressed as $g = h_1 a h_2$, where $h_1, h_2 \in H$, $a \in A$. Thus, $g(g^\theta)^{-1} = h_1 a^2 h_1^{-1} \in Z(G)$ if and only if $a^2 \in Z(G)$, whence the result.

As for (2), the first equality is a remark, whilst the second follows from

$$\mathrm{Ad}(g)^\theta = \mathrm{Ad}(g^\theta) = \mathrm{Ad}(g) \iff g^{-1}g^\theta \in Z(G^{\mathbb{C}}).$$

For the third equality, the same proof as in Proposition 1 in [29] can be used (note that the proposition itself can only be directly applied if $\mathrm{Ad}(G^{\mathbb{C}})$ is connected), yielding

$$\mathrm{Ad}(G^{\mathbb{C}})_\theta = Q \cdot \mathrm{Ad}(H^{\mathbb{C}}),$$

where $Q = \exp(i\mathrm{ad}(\mathfrak{a}))_{[2]}$. But then, $Q \subset \mathrm{Ad}(G)$, as $\sigma(g) = g^{-1} = g$ as Q is two torsion (see Proposition 2 in [29]). The proof of (3) follows from (3) in Proposition 3.17.

Finally, the last equality follows from (1), as $\mathrm{Ad}(F) = Q$. □

Remark 3.22. When $G^{\mathbb{C}}$ is the adjoint group of a complex reductive Lie algebra, we obtain that $(G^{\mathbb{C}})_\theta = (G^{\mathbb{C}})^\theta$, as the centre is trivial. This case is the one considered by Kostant and Rallis, who distinguish between two groups: K_θ, in our notation $(G^{\mathbb{C}})_\theta$, and K, the identity component of K_θ, in our notation $(H^{\mathbb{C}})^0$. This distinction is important for the orbit structure of $\mathfrak{m}^{\mathbb{C}}$ under the action of $(H^{\mathbb{C}})^0$ (see [29], Theorem 11). In the real case, if the centre of G is trivial, then $F \subset H$, as in this situation $a \in F$ if and only if $a^2 = 1$, so $a^{-1} = a^\theta = a$. Hence $G_\theta = H$.

3.3. **Maximal split subgroup.** Just as there is a maximal split subalgebra of a real reductive Lie algebra, we can define the maximal connected split subgroup of a reductive Lie group (G, H, θ, B). We introduce the following notions.

Definition 3.23. We say that a real reductive Lie group (G, H, θ, B) is split, quasi-split, etc., if $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ is split, quasi-split, etc., respectively.

Definition 3.24. Let G be a Lie group whose Lie algebra is reductive. The **maximal connected split subgroup** is defined to be the analytic subgroup $\widehat{G}_0 \leq G$ with Lie algebra $\widehat{\mathfrak{g}}$.

Consider the tuple $(\widehat{G}_0, \widehat{H}_0, \widehat{\theta}, \widehat{B})$ where $\widehat{H}_0 := \exp(\widehat{\mathfrak{h}}) \leq H$, and $\widehat{\theta}$ and $\widehat{B}$ are obtained by restriction.

Proposition 3.25. *If (G, H, θ, B) is a reductive Lie group, then the tuple $(\widehat{G}_0, \widehat{H}_0, \widehat{\theta}, \widehat{B})$ is a strongly reductive Lie group.*

Proof. By Proposition 2.7, conditions (1), (3), and (4) in Definition 3.1 hold. Since $\widehat{G}_0$ is connected, we may assume that G is connected, as $\widehat{G}_0 \subset G_0$. In this case, writing the polar decomposition of $g \in \widehat{G}_0$, we have, by connectedness of H, $g = e^X e^Y$, for some $X \in \mathfrak{h}$, $Y \in \mathfrak{m}$. By construction, $\widehat{\mathfrak{g}}^{\mathbb{C}}$ is self-normalizing within $\mathfrak{g}^{\mathbb{C}}$ (as it is the subalgebra generated by a principal normal TDS, $\mathfrak{a}^{\mathbb{C}}$, and the centre of $\mathfrak{g}^{\mathbb{C}}$), and the same holds for $\widehat{\mathfrak{g}}$. This implies that, modulo the kernel of the exponential, X and Y can be chosen in $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{m}}$. So we may work at the level of the universal cover G^u of G, to which there corresponds a maximal split subgroup $\widehat{G}_0^u$, and then induce the result for $\widehat{G}_0$.

This gives the polar decomposition, and maximality of $\widehat{H}_0$ follows from Proposition 7.19 in [25], just noticing that its proof does not use (SR) in Definition 3.1, and Remark 3.2. Strong reductivity follows from connectedness, as condition (5) in Definition 3.1 implies that $G = e^{\mathfrak{h}} \cdot e^{\mathfrak{m}}$, since H being compact and connected it must be $H = e^{\mathfrak{h}}$. A simple computation shows that in the case of matrix groups $\mathrm{Ad}_{e^X} \circ \mathrm{Ad}_{e^Y} \equiv \mathrm{Ad}_{e^{X+Y}} \in \mathrm{Aut}\ \mathfrak{g}$. Since $\mathrm{Ad}(G)$ is semisimple, it is a matrix group, and furthermore $\mathrm{Ad}\,(\mathrm{Ad}(G)) \cong \mathrm{Ad}(G)$, so condition (SR) in Definition 3.1 follows for connected groups. □

If $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B)$ is a real form of a complex reductive Lie group, there is an alternative natural candidate for a maximal split subgroup. Note that even in the situation when G has a complexification, $\widehat{G}_0$ need not be a real form of a complex Lie group. It is so just up to a finite extension.

Lemma 3.26. *Let $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B)$ be a real form of a complex reductive Lie group, and let σ be the corresponding antiholomorphic involution. Define $\widehat{G}^{\mathbb{C}} < G^{\mathbb{C}}$ to be the analytic subgroup corresponding to $\widehat{\mathfrak{g}}^{\mathbb{C}}$, where $\widehat{\mathfrak{g}}$ is defined as in Definition* 2.6. *Then:*

1. *The involution σ leaves $\widehat{G}^{\mathbb{C}}$ invariant.*
2. *Let $\widehat{G} = \left(\widehat{G}^{\mathbb{C}}\right)^{\sigma}$, and let $\widehat{H} \leq \widehat{G}$ be the maximal compact subgroup. Then $(\widehat{G}, \widehat{H}, \widehat{\theta}, \widehat{B})$, where $\widehat{\theta}$ and $\widehat{B}$ are as in Proposition* 3.25*, is a reductive Lie group and a real form of $(\widehat{G}^{\mathbb{C}}, U \cap \widehat{G}^{\mathbb{C}}, \tau|_{\widehat{\mathfrak{g}}^{\mathbb{C}}}, B|_{\widehat{\mathfrak{g}}^{\mathbb{C}}})$.*

Proof. We first note that $\widehat{G}^{\mathbb{C}} = (\widehat{G}_0)^{\mathbb{C}}$, as both are connected complex Lie subgroups of $G^{\mathbb{C}}$ with the same Lie algebra. Then the first statement follows from the following fact: by definition σ leaves G pointwise invariant, and it also leaves $\widehat{G}_0$. Thus, the complexification $(\widehat{G}_0)^{\mathbb{C}} = \widehat{G}^{\mathbb{C}}$ is σ-invariant. Indeed, $\widehat{G}_0 \subset \widehat{G}^{\mathbb{C}} \cap \sigma\left(\widehat{G}^{\mathbb{C}}\right)$; the intersection is a complex group, so that the complexification of $\widehat{G}_0$ is also contained in the intersection; namely, it is all of the intersection.

The second assertion follows from Proposition 3.25. □

Definition 3.27. Let $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B_{\mathbb{C}})$ be a real form of a complex reductive Lie group. Let $(\widehat{G}, \widehat{H}, \widehat{\theta}, \widehat{B})$ be as in Lemma 3.26. We call this group the **maximal split subgroup** of (G, H, θ, B).

Given a reductive Lie group, we would like to determine its maximal connected split subgroup. This is studied in work by Borel and Tits [8] in the case of real forms of complex semisimple algebraic groups. It is important to note that over $\mathbb{R}$, the category of semisimple algebraic groups differs from the category of semisimple Lie groups. For example, the semisimple algebraic group $\mathrm{Sp}(2n, \mathbb{R})$ has a finite cover of any given degree, all of which are semisimple Lie groups, but none of them is a matrix group. So although their results do not apply to real Lie groups in general, they do apply to real forms of complex semisimple Lie groups.

In an earlier work [7], in the context of reductive algebraic groups, Borel and Tits built a maximal connected split subgroup, unique up to the choice of a maximal split subtorus $\mathcal{A}$ and a choice of one unipotent generator of an $\mathcal{A}$-invariant three-dimensional subgroup corresponding to each root $\alpha \in \Delta$ such that $2\alpha \notin \Delta$.

Let $\mathcal{G}$ be a reductive algebraic group, and let $\widehat{\mathcal{G}}_0$ be the maximal connected split subgroup. In the case $\widehat{\mathcal{G}}$ has a complexification $\widehat{\mathcal{G}}^{\mathbb{C}}$, it is well known that the map taking a group to its complex points

$$\widehat{\mathcal{G}}^{\mathbb{C}} \mapsto \widehat{\mathcal{G}}^{\mathbb{C}}(\mathbb{C})$$

establishes an equivalence of categories between the categories $\mathcal{AG}$ of complex semisimple algebraic groups and $\mathcal{LG}$ of (holomorphic) complex semisimple Lie groups (also reductive, but on the holomorphic side we get a subcategory). This yields the following.

Proposition 3.28. *Let $G^{\mathbb{C}}$ be a complex semisimple Lie group, and let $\mathcal{G}^{\mathbb{C}}$ be the corresponding algebraic group, so that $G^{\mathbb{C}} = \mathcal{G}^{\mathbb{C}}(\mathbb{C})$. Let $G < G^{\mathbb{C}}$ be a real form. Then, there exists a real linear algebraic group $\mathcal{G}$ such that $\mathcal{G}(\mathbb{R}) = G$ and moreover $\widehat{\mathcal{G}}_0(\mathbb{R}) = \widehat{G}_0$.*

Proof. The equivalence between $\mathcal{AG}$ and $\mathcal{LG}$ implies that the holomorphic involution $\theta \curvearrowright G^{\mathbb{C}}$ corresponding to G via Corollary 3.8 is algebraic. Thus, both τ and σ are real algebraic, that is, defined by polynomial equations over the real numbers. This implies that they induce involutions (which we denote by the same letters) of $\mathcal{G}^{\mathbb{C}}$. Let $\mathcal{G} = \left(\mathcal{G}^{\mathbb{C}}\right)^{\sigma}$. Then, $\mathcal{G}(\mathbb{R}) = \left(\mathcal{G}^{\mathbb{C}}(\mathbb{C})\right)^{\sigma} = G$. By construction of $\widehat{G}_0$, the choices required for the uniqueness of Borel-Tits's maximal connected split subgroup are met. So there is a unique algebraic group $\widehat{\mathcal{G}}_0$ such that $\widehat{\mathcal{G}}_0(\mathbb{R}) = \widehat{G}_0$. □

The following lemma gives a necessary condition for a subgroup to be the maximal connected split subgroup.

Lemma 3.29. *Let $\mathcal{G}$ be a real semisimple algebraic group, and let $\widehat{\mathcal{G}}$ be a semisimple subgroup such that there exist maximal tori $\mathcal{T}, \widehat{\mathcal{T}}$ of $\mathcal{G}$ and $\widehat{\mathcal{G}}$, respectively, with $\widehat{\mathcal{T}} \subseteq \mathcal{T}$. Let Δ be a root system of $\mathcal{G}$ with respect to $\mathcal{T}$, and let $\widehat{\Delta}$ be the (non-zero) restriction of elements of Δ to $\widehat{\mathcal{T}}$. Assume $\widehat{\Delta}$ is a root system. If $\mathcal{G}$ is simply connected or $\widehat{\Delta}$ is a non-reduced root system, then $\widehat{\mathcal{G}}$ is simply connected.*

Remark 3.30. In the above lemma, simple connectedness is meant in the algebraic sense: namely, the lattice of inverse roots is maximal within the lattice of weights of the group. Note that the algebraic fundamental group for compact linear algebraic groups and the topological fundamental group of their corresponding groups of matrices of complex points are the same (see [13] for details). The polar decomposition implies the same for the class of reductive Lie groups. However, algebraic simple connectedness does not mean that the fundamental group is trivial.

Lemma 3.29 has the following consequence.

Corollary 3.31. *Let $G^{\mathbb{C}}$ be a complex semisimple Lie group, and let $G < G^{\mathbb{C}}$ be a real form that is either simply connected or of type BC. Then the analytic subgroup $\widehat{G}_0^{\mathbb{C}} \leq G^{\mathbb{C}}$ is (topologically) simply connected.*

Proof. By Proposition 3.28, we have algebraic groups $\mathcal{G}^{\mathbb{C}}$, $\widehat{\mathcal{G}}^{\mathbb{C}}$ and real forms $\mathcal{G}$, $\widehat{\mathcal{G}}$ to which the results of Borel and Tits may be applied. In particular $\widehat{\mathcal{G}}$ is simply connected. Assume $\widehat{\mathcal{G}}^{\mathbb{C}}$ was not. Then, it would have a finite cover $(\widehat{\mathcal{G}}^{\mathbb{C}})'$, which in turn would contain a real form $(\widehat{\mathcal{G}})'$ (defined by a lifting σ) that would be a finite cover of $\widehat{\mathcal{G}}$ and an algebraic group. □

Example 3.32. Take the real form $\mathrm{SU}(p,q) < \mathrm{SL}(p+q,\mathbb{C})$. Its fundamental group is

$$\pi_1(\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))) = \mathbb{Z}.$$

We know from [4] that the maximal split Lie subalgebra of $\mathfrak{su}(p,q)$, $p > q$, is $\mathfrak{so}(q+1,q)$, whereas the maximal split subalgebra of $\mathfrak{su}(p,p)$ is $\mathfrak{sp}(2p,\mathbb{R})$. In what follows, we analyse what the maximal split subgroup is in the various cases:

• $p > q$. Since the root system is non-reduced (see [25, VI.4]), Lemma 3.29 and Corollary 3.31 imply that the maximal split subgroup is the algebraic universal

cover of $\mathrm{SO}(q+1,q)_0$. We have the following table of fundamental groups of the connected component of $\mathrm{SO}(p+1,p)$:

$\pi_1(\mathrm{SO}(q+1,q)_0)$	$q=1$	$\mathbb{Z}$
	$q=2$	$\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
	$q\geq 3$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$

For $q=1$, we have the exact sequence

$$1\to\mathbb{Z}/2\mathbb{Z}\to\mathrm{Sp}(2,\mathbb{R})\to\mathrm{SO}(2,1)_0\to 1,$$

since $\mathrm{Sp}(2,\mathbb{R})$ is simply connected (for example, since no finite covering of it is a matrix group). In particular $\widehat{\mathrm{SU}(p,1)}=\mathrm{Spin}(2,1)_0\cong\mathrm{Sp}(2,\mathbb{R})$.

When $q=2$, the maximal split subgroup is again the algebraic universal cover of $\mathrm{SO}(3,2)_0$, which is a two cover considering the fundamental group. It is well known that $\mathfrak{so}(3,2)\cong\mathfrak{sp}(4,\mathbb{R})$, and that $\mathrm{Spin}(3,2)_0\cong\mathrm{Sp}(4,\mathbb{R})$ are connected. Since moreover $\mathrm{Spin}(3,2)_0$ is algebraically simply conected, we have $\widehat{SU}(2,3)\cong\mathrm{Spin}(3,2)_0$.

As for $q\geq 3$, the universal covering group of $\mathrm{SO}(q,q+1)_0$ is the connected component of $\mathrm{Spin}(q,q+1)$. This group is a 4-fold cover of $\mathrm{SO}(q,q+1)_0$, which is thus simply connected.

• $p=q$. Since $\mathrm{Sp}(2n,\mathbb{R})\subseteq\mathrm{SU}(n,n)$, the candidate to the maximal split subgroup is a finite cover of $\mathrm{Sp}(2n,\mathbb{R})$ embedding into $\mathrm{Sp}(2n,\mathbb{C})$ (which is simply connected). Thus $\widehat{\mathrm{SU}(n,n)}=\mathrm{Sp}(2n,\mathbb{R})$.

The group $\mathrm{SU}(p,q)$ is a group of Hermitian type, a class of groups which will become relevant in Section 5.

Definition 3.33. A reductive group (G,H,θ,B) is said to be of Hermitian type if the symmetric space associated to it admits a complex structure which is invariant by the group of isometries. If the group G is simple, this is equivalent to H having non-discrete centre.

The Lie algebras of simple such groups are $\mathfrak{sp}(2n,\mathbb{R})$, $\mathfrak{su}(p,q)$, $\mathfrak{so}^*(2,n)$, $\mathfrak{so}(2,n)$, $\mathfrak{e}_{6(-14)}$, and $\mathfrak{e}_{7(-25)}$.

4. The Kostant–Rallis section

Let (G,H,θ,B) be a reductive Lie group, and consider the decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ induced by θ. Let $\mathfrak{a}\subseteq\mathfrak{m}$ be a maximal anisotropic Cartan subalgebra, and let $H^{\mathbb{C}}$, $\mathfrak{g}^{\mathbb{C}}$, etc., denote the complexifications of the respective groups, algebras, etc. Note that we do not assume that $G^{\mathbb{C}}$ exists. In [29], Kostant and Rallis study the orbit structure of the $H^{\mathbb{C}}$ module $\mathfrak{m}^{\mathbb{C}}$ in the case when $G^{\mathbb{C}}$ is the adjoint group of a complex reductive Lie algebra $\mathfrak{g}^{\mathbb{C}}$, namely, $G^{\mathbb{C}}=\mathrm{Int}(\mathfrak{g}^{\mathbb{C}})=\mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})^0$. In this section, we study a generalization of their result to reductive Lie groups in the sense of Definition 3.1.

The first result we will be concerned about is the Chevalley restriction theorem, which is well known for Lie groups of adjoint type. Recall that given a complex reductive Lie algebra $\mathfrak{g}^{\mathbb{C}}$, its **adjoint group**, denoted by $\mathrm{Ad}(\mathfrak{g}^{\mathbb{C}})$, is the connected component of its automorphism group $\mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$. It coincides with the connected component of the image of the adjoint representation of any Lie group $G^{\mathbb{C}}$ such that $\mathrm{Lie}(G^{\mathbb{C}})=\mathfrak{g}^{\mathbb{C}}$. We need the following.

Definition 4.1. We define the **restricted Weyl group** of $\mathfrak{g}$ (resp. $\mathfrak{g}^{\mathbb{C}}$) associated to $\mathfrak{a}$ (resp. $\mathfrak{a}^{\mathbb{C}}$), $W(\mathfrak{a})$ (resp. $W(\mathfrak{a}^{\mathbb{C}})$) to be the group of automorphisms of $\mathfrak{a}$ (resp. $\mathfrak{a}^{\mathbb{C}}$) generated by reflections on the hyperplanes defined by the restricted roots $\lambda \in \Lambda(\mathfrak{a})$ (resp. $\Lambda(\mathfrak{a}^{\mathbb{C}})$).

The Chevalley restriction theorem asserts that, given a group G of adjoint type, the restriction $\mathbb{C}[\mathfrak{m}^{\mathbb{C}}] \to \mathbb{C}[\mathfrak{a}^{\mathbb{C}}]$ induces an isomorphism

$$\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}} \xrightarrow{\cong} \mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})}.$$

See for example [21].

The restricted Weyl group admits other useful characterisations in the case of strongly reductive Lie groups.

Lemma 4.2. *Let (G, H, θ, B) be a strongly reductive Lie group. We have:*

(1) $W(\mathfrak{a}) = N_H(\mathfrak{a})/C_H(\mathfrak{a})$, *where*

$$N_H(\mathfrak{a}) = \{h \in H \ : \ \mathrm{Ad}_h(x) \in \mathfrak{a} \ \text{ for all } \ x \in \mathfrak{a}\},$$

$$C_H(\mathfrak{a}) = \{h \in H \ : \ \mathrm{Ad}_h(x) = x \ \text{ for all } \ x \in \mathfrak{a}\}.$$

(2) $W(\mathfrak{a}^{\mathbb{C}}) = N_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})/C_H(\mathfrak{a}^{\mathbb{C}})$, *where $N_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$ and $C_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$ are defined as above.*

(3) *Moreover, $W(\mathfrak{a}^{\mathbb{C}}) = W(\mathfrak{a})$ as automorphism groups of $\mathfrak{a}^{\mathbb{C}}$, where the action of $W(\mathfrak{a})$ on $\mathfrak{a}^{\mathbb{C}}$ is defined by extension by complex linearity.*

Proof. The first statement follows from Proposition 7.24 in [25].

As for (3), it follows by definition of restricted roots.

To prove (2), it is therefore enough to prove that $W(\mathfrak{a}) = N_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})/C_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$ when acting on $\mathfrak{a}^{\mathbb{C}}$. Now, if (G, H, θ, B) is strongly reductive, then $(H^{\mathbb{C}}, H, \tau, B_{\mathfrak{h}})$ is also strongly reductive for τ, the involution defining $\mathfrak{h}$ inside its complexification, and a suitable choice of $B_{\mathfrak{h}}$. Hence, by Lemma 7.22 in [25], if $h = xe^{Y}$ is the polar decomposition of an element in $N_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$, we have, by τ-invariance of $\mathfrak{a}^{\mathbb{C}}$, that both x and Y normalise $\mathfrak{a}^{\mathbb{C}}$. This means that $x \in N_H(\mathfrak{a}^{\mathbb{C}}) = N_H(\mathfrak{a})$, and $Y \in \mathfrak{n}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$. Now, by Lemma 6.56 in [25], $\mathfrak{n}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}}) = \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$, so the statement is proved. □

We have the following.

Proposition 4.3. *Let (G, H, θ, B) be a strongly reductive Lie group and let $(\widehat{G}_0, \widehat{H}_0, \widehat{\theta}, \widehat{B})$ be the maximal connected split subgroup. Then, restriction induces an isomorphism*

$$\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}} \cong \mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})} \cong \mathbb{C}[\widehat{\mathfrak{m}}^{\mathbb{C}}]^{(\widehat{H}_0)^{\mathbb{C}}}.$$

If moreover $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B_{\mathbb{C}})$ is a real form, from Definition 3.27 one has the maximal split subgroup $(\widehat{G}, \widehat{H}, \widehat{\theta}, \widehat{B}) < (G, H, \theta, B)$, and

$$\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}} \cong \mathbb{C}[\widehat{\mathfrak{m}}^{\mathbb{C}}]^{\widehat{H}^{\mathbb{C}}}.$$

Proof. By Lemma 7.24 in [25],

$$\mathrm{Ad}(H) \subseteq \mathrm{Int}(\mathfrak{h} \oplus i\mathfrak{m}).$$

Then, given that $H^{\mathbb{C}} = He^{i\mathfrak{h}}$, $H^{\mathbb{C}}$ clearly acts on $\mathfrak{g}^{\mathbb{C}}$ by inner automorphisms of $\mathfrak{g}^{\mathbb{C}}$. So $\mathrm{Ad}(\mathfrak{h}^{\mathbb{C}}) = \mathrm{Ad}(H^{\mathbb{C}}) \subseteq (\mathrm{Ad}\ \mathfrak{g}^{\mathbb{C}})^{\theta}$, which implies that

$$\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{\mathrm{Ad}\ \mathfrak{h}^{\mathbb{C}}} = \mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{\mathrm{Ad}\ H^{\mathbb{C}}} \supseteq \mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{(\mathrm{Ad}\ \mathfrak{g}^{\mathbb{C}})^{\theta}}. \tag{8}$$

Now, Proposition 10 in [29] implies that

$$\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{(\mathrm{Ad}\ \mathfrak{g}^{\mathbb{C}})^{\theta}} = \mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{\mathrm{Ad}\ \mathfrak{h}^{\mathbb{C}}},$$

and so we obtain equalities in equation (8) above.

Since $W(\mathfrak{a}^{\mathbb{C}}) = N_{\mathrm{Ad}\ \mathfrak{h}}(\mathfrak{a})/C_{\mathrm{Ad}\ \mathfrak{h}}(\mathfrak{a})$, the isomorphism $\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}} \cong \mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})}$ follows from the adjoint group case and (8).

As for the split subgroup, by the adjoint case and Proposition 3.25, we have $\mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})} \cong \mathbb{C}[\widehat{\mathfrak{m}}^{\mathbb{C}}]^{\widehat{H}_0^{\mathbb{C}}}$. Also, by the definition of $\widehat{H}$ (see Definition 3.27), $\mathrm{Ad}(\widehat{H}_0) \subset \mathrm{Ad}(\widehat{H}) \subset \mathrm{Ad}(\widehat{G}_0)_\theta$, which by Proposition 3.21 and Proposition 10 in [29] implies that $\mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})} \cong \mathbb{C}[\widehat{\mathfrak{m}}^{\mathbb{C}}]^{\widehat{H}^{\mathbb{C}}}$. □

Proposition 4.4. *Let $a = \dim \mathfrak{a}^{\mathbb{C}}$. Then:*

1. *$\mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})}$ is generated by homogeneous polynomials of degrees $m_1, \dots, m_a$, canonically determined by (G, θ).*
2. *If $(\widehat{G}_0, \widehat{H}_0, \widehat{\theta}, \widehat{B}) < (G, H, \theta, B)$ is the maximal connected split subgroup, the exponents are the same for both groups.*

Proof. Statement (1) is well known and follows from Proposition 2.7(2).

(2) follows by Proposition 4.3 and the fact that the exponents of the group (G, H, B, θ) are $m_k - 1$, where m_k are the degrees of the generators of $\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}}$. □

Remark 4.5. Note that the ring of invariant polynomials depends on the choice of involution θ for non-semisimple groups, as the number of degree one generators of $\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}}$ is the dimension $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{m}$, which depends on θ if G is not semisimple. So two instances of Cartan data on the same Lie group will yield different rings of invariants.

We thus have an algebraic morphism

$$\chi : \mathfrak{m}^{\mathbb{C}} \twoheadrightarrow \mathfrak{m}^{\mathbb{C}} /\!\!/ H^{\mathbb{C}} \cong \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}}), \tag{9}$$

where the double quotient sign $/\!\!/$ stands for the affine GIT quotient.

We next build a section of the above surjective map. This is done by Kostant and Rallis in the case $G^{\mathbb{C}} = \mathrm{Ad}(\mathfrak{g}^{\mathbb{C}})$ for a complex reductive Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Let us start with some preliminary definitions.

Definition 4.6. An element $x \in \mathfrak{m}^{\mathbb{C}}$ is said to be **regular** if $\dim \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x) = \dim \mathfrak{a}^{\mathbb{C}}$. Here

$$\mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x) = \{y \in \mathfrak{m}^{\mathbb{C}} \ : \ [y, x] = 0\}. \tag{10}$$

Denote the subset of regular elements of $\mathfrak{m}^{\mathbb{C}}$ by $\mathfrak{m}^{\mathbb{C}}{}_{reg}$.

Regular elements are those whose $H^{\mathbb{C}}$-orbits are maximal dimensional, so this notion generalises the classical notion of regularity of an element of a complex reductive Lie algebra.

Remark 4.7. Note that the intersection $\mathfrak{m}^{\mathbb{C}} \cap \mathfrak{g}^{\mathbb{C}}_{reg}$ is either empty or the whole of $\mathfrak{m}_{reg}$. Here $\mathfrak{g}_{reg}$ denotes the elements of $\mathfrak{g}^{\mathbb{C}}$ with maximal dimensional $G^{\mathbb{C}}$-orbit.

The following definition follows naturally from the preceding remark.

Definition 4.8. A real form $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ is **quasi-split** if $\mathfrak{m} \cap \mathfrak{g}^{\mathbb{C}}_{reg} = \mathfrak{m}^{\mathbb{C}}_{reg}$. These include split real forms and the Lie algebras $\mathfrak{su}(p,p)$, $\mathfrak{su}(p,p+1)$, $\mathfrak{so}(p,p+2)$, and $\mathfrak{e}_{6(2)}$. Quasi-split real forms admit several equivalent characterizations: $\mathfrak{g}$ is quasi-split if and only if $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{a})$ is abelian, which holds if and only if $\mathfrak{g}^{\mathbb{C}}$ contains a θ-invariant Borel subalgebra and if and only if $\mathfrak{m} \cap \mathfrak{g}^{\mathbb{C}}_{reg} = \mathfrak{m}_{reg}$.

Theorem 4.9. *Let (G, H, θ, B) be a strongly reductive Lie group. Let $\mathfrak{s}^{\mathbb{C}} \subseteq \mathfrak{g}^{\mathbb{C}}$ be a principal normal TDS with normal basis $\{x, e, f\}$ (see Definition 2.5). Then*

1. *The affine subspace $f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$ is isomorphic to $\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$ as an affine variety.*
2. *$f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$ is contained in the open subset ${\mathfrak{m}^{\mathbb{C}}}_{reg}$, where $\mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$ is defined as in (10).*
3. *$f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$ intersects each $(\mathrm{Ad}(G_\theta))^{\mathbb{C}}$-orbit at exactly one point. Here G_θ is given in Definition 3.12.*
4. *$f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$ is a section of the Chevalley morphism (9).*
5. *Let $(\widehat{G}_0, \widehat{H}_0, \widehat{\theta}, \widehat{B}) < (G, H, \theta, B)$ be the maximal connected split subgroup. Then, $\mathfrak{s}^{\mathbb{C}}$ can be chosen so that $f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e) \subseteq \widehat{\mathfrak{m}}^{\mathbb{C}}$. If moreover G is a real form of $G^{\mathbb{C}}$, say, then $f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$ is the image of Kostant's section for $\widehat{G}^{\mathbb{C}}$ [28]. Here, $\widehat{\mathfrak{m}}^{\mathbb{C}}$ is defined as in Proposition 2.7 and $\widehat{G}^{\mathbb{C}}$ as in Lemma 3.26.*

Proof. We follow the proof due to Kostant and Rallis (see Theorems 11, 12, and 13 in [29]), adapting their arguments to our setting when necessary.

First note that Proposition 4.3 implies the existence of a surjective map

$$\mathfrak{m}^{\mathbb{C}} \to \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}}).$$

As in [29], consider the element

$$e_c = i \sum_j d_j y_j \in i\mathfrak{g}, \tag{11}$$

where $y_i \in \mathfrak{g}_{\lambda_i}$ are as in Section 2.2 and

$$d_j = \sqrt{\frac{-c_j}{b_j}}. \tag{12}$$

Here the elements c_j are defined so that

$$w = \sum_i c_i h_i \in \mathfrak{a} \tag{13}$$

is the only element in $\mathfrak{a}$ such that $\lambda(w) = 2$ for any $\lambda \in \Lambda(\mathfrak{a})$, and h_i is the dual of λ_i via the bilinear form B. Note that in order for e_c to belong to $i\mathfrak{g}$, we must prove that $c_i/b_i < 0$. Now, following the proof of Proposition 18 in [29], for any $y \in \mathfrak{g}$, we have $2B(y, \theta y) = B(y + \theta y, y + \theta y) < 0$ since $y + \theta y \in \mathfrak{h}$. Hence, if $b_i = B(y_i, \theta y_i)$ it must be a negative real number. Also the fact that $c_i > 0$ follows from general considerations on the representations of three-dimensional subalgebras (see Lemma 15 in [29]) and so does not depend on the choice of pairing B.

Once we have that, taking

$$f_c = \theta e_c,$$

it follows by the same arguments found in [29] that $\{e_c, f_c, w\}$ generate a principal normal TDS $\mathfrak{s}^{\mathbb{C}}$ stable by σ and θ (Proposition 22 in [29]). In particular, $\mathfrak{s}^{\mathbb{C}}$ has a

normal basis, say $\{e, f, x\}$. By construction, it is clear that $f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e) \subseteq \widehat{\mathfrak{m}}^{\mathbb{C}}_{reg}$. It is furthermore a section, which is proved as in [29], as groups act by inner automorphisms of the Lie algebra, together with Lemma 4.10 following this theorem. This proves (1), (2), and (4)

As for (3), it follows directly from Theorem 11 in [29], which asserts that the affine space $f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$ hits each $\mathrm{Ad}(G)^{\mathbb{C}}_{\theta}$ orbit exactly at one point, taking Remark 3.22 and (2) in Proposition 3.21 into account.

Statement (5) follows from the fact that $\widehat{G}_0$ is strongly reductive; hence the statement follows from Theorem 7 in [28], where a section for the Chevalley morphism for complex groups is defined, together with Remark 19 in [29] and its proof, where it is checked that $f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$ defines a section of the restriction of the Chevalley morphism to $\widehat{\mathfrak{m}}^{\mathbb{C}}_{reg} \hookrightarrow \mathfrak{m}^{\mathbb{C}}{}_{reg}$. □

Lemma 4.10. *The Lie algebra $\mathfrak{s}^{\mathbb{C}}$ is the image of a σ and θ-equivariant morphism $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}^{\mathbb{C}}$ where σ on $\mathfrak{sl}(2,\mathbb{C})$ is complex conjugation and θ on $\mathfrak{sl}(2,\mathbb{C})$ is defined by $X \mapsto -\mathrm{Ad}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)({}^tX)$.*

Proof. Consider the basis of $\mathfrak{sl}(2,\mathbb{R})$:

$$E = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \qquad F = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{14}$$

and note that $W \in \mathfrak{sym}_0(2,\mathbb{R}) =: \mathfrak{m}_{\mathfrak{sl}}$, $E = \theta F$, so that $E + F \in \mathfrak{so}(2,\mathbb{R})$.

Consider e_c, f_c, w as described in the proof of Theorem 4.9. Then the map defined by

$$\rho' : E \mapsto ie_c,\ F \mapsto if_c,\ W \mapsto -w \tag{15}$$

is the desired morphism. Indeed, it is σ-invariant by definition. Furthermore, $\mathfrak{so}(2,\mathbb{R}) \ni E + F \mapsto ie_c + if_c \in \mathfrak{h}$ by construction. Finally, $\mathfrak{m}_{\mathfrak{sl}}$ is generated by W and $E - F$, and so is $\mathfrak{s} \cap \mathfrak{m}$. Indeed, we must only prove that $ie_c - if_c$ is not a multiple of w. But this follows from simplicity of $\mathfrak{sl}(2,\mathbb{C})$, the fact that $\mathfrak{s}^{\mathbb{C}}$ is homomorphic to it, and $w \neq 0$, which forces S-triples to be independent. □

Remark 4.11. Theorem 4.9 implies that the GIT quotient $\mathfrak{m}^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$ does not parameterise $H^{\mathbb{C}}$ orbits or regular elements, but rather $\mathrm{Ad}(H^{\mathbb{C}})_{\theta}$ orbits, each of which contains finitely many $H^{\mathbb{C}}$-orbits. This is a consequence of the fact that not all normal principal TDS's are $H^{\mathbb{C}}$ conjugate, which yields different sections for different choices of a TDS. See [29] for more details.

By the above remark, we will need to keep track of conjugacy classes of principal normal TDS's.

Proposition 4.12. *Let $\mathfrak{s}^{\mathbb{C}} \subseteq \mathfrak{g}^{\mathbb{C}}$ be a normal TDS, and let (e, f, x) be a normal triple generating it. Then:*

1. *The triple is principal if and only if $e + f = \pm w$, where w is defined by* (13).
2. *There exist e', f' such that (e', f', w) is a TDS generating $\mathfrak{s}^{\mathbb{C}}$ and $e' = \theta f'$. Under these hypotheses, e' is uniquely defined up to sign.*

Proof. See Lemma 5 and Proposition 13 in [29]. □

In the classical setting of complex reductive Lie algebras, there is also a notion of principal TDS. These are defined to be Lie algebras homomorphic to $\mathfrak{sl}(2,\mathbb{C})$ generated by regular nilpotents, except that regularity is now taken in the sense of the whole Lie algebra $\mathfrak{g}^{\mathbb{C}}$, which need not coincide with the notion for a given real form $\mathfrak{g}$ (see Remark 4.7).

Let us recall some facts about three-dimensional subalgebras. Let $\mathfrak{s}^{\mathbb{C}}$ be a normal TDS (cf. Definition 2.5) generated by the normal S-triple $\{e,f,x\}$. Let $n = \dim \mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(e)$. The adjoint representation induces a splitting

$$\mathfrak{g}^{\mathbb{C}} \cong \bigoplus_{k=1}^{n} M_k \tag{16}$$

into irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules M_k, generated from the highest weight vector e_k by the action of f, possibly isomorphic to one another. Since the highest weight vectors are annihilated by the action of e, it follows that $\mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(e)$ is generated by the highest weight vectors. Note that $\mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(e)$ is θ-invariant. Given that $[x,\mathfrak{h}^{\mathbb{C}}] \subset \mathfrak{h}^{\mathbb{C}}$ and $[x,\mathfrak{m}^{\mathbb{C}}] \subset \mathfrak{m}^{\mathbb{C}}$, then $e_k \in \mathfrak{m}^{\mathbb{C}}$ or $e_k \in \mathfrak{h}^{\mathbb{C}}$.

Lemma 4.13. *Let $\mathfrak{g}$ be a real reductive Lie algebra, and let $\mathfrak{s} \subset \mathfrak{g}$ be such that $\mathfrak{s}^{\mathbb{C}}$ is a principal normal TDS of $\mathfrak{g}^{\mathbb{C}}$ with generating normal triple $\{e,f,x\}$. Let $\mathfrak{g}_\lambda \subset \mathfrak{g}^{\mathbb{C}}$ be the eigenspace of eigenvalue λ for the action of x. Let e_k, $k=1,\dots,n$, be highest weight vectors for the action of x with eigenvalues $m_k - 1 \geq 0$, $k=1,\dots,n$, and assume $m_k < m_{k+1}$, so that $m_0 \geq 1$. Then:*

1. *If $m_1 = 1$, then $\mathfrak{g}_0 = \mathfrak{c}_{\mathfrak{g}}^{\mathbb{C}}(\mathfrak{s}^{\mathbb{C}}) = \mathfrak{c}_{\mathfrak{h}}^{\mathbb{C}}(\mathfrak{s}^{\mathbb{C}}) \oplus \mathfrak{z}_{\mathfrak{m}^{\mathbb{C}}}(\mathfrak{g}^{\mathbb{C}}) = M_1^{\dim \mathfrak{c}_{\mathfrak{g}}^{\mathbb{C}}(\mathfrak{s}^{\mathbb{C}})}$.*
2. *Moreover, $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ is quasi-split if and only if $\mathfrak{g}_1 = \mathfrak{z}(\mathfrak{g}^{\mathbb{C}})$.*
3. *For all values of k,*

$$m_k - 1 := \frac{\dim M_k - 1}{2}. \tag{17}$$

Proof. To prove (1), note that it is clear that $\mathfrak{g}_1 = \mathfrak{c}_{\mathfrak{g}}^{\mathbb{C}}(\mathfrak{s}^{\mathbb{C}})$. We need to prove $\mathfrak{g}_1 \cap \mathfrak{m}^{\mathbb{C}}$ is central. Note that $\mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}) = \mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(w) \cap \mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(ie_c)$, where e_c and w are as in Proposition 4.3. By Theorem 3.6 in [27] $\mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(e)$ is fully composed of nilpotent elements; however, all elements in $\mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(w) = \mathfrak{a}^{\mathbb{C}}$ are semisimple, hence

$$\mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(w) \cap \mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(ie_c) = \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(w) \cap \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(ie_c).$$

For (2), by the proof of (1) above, $\mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}) = \mathfrak{z}(\mathfrak{g}^{\mathbb{C}})$ if and only if $\mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(w)$ is composed of semisimple elements, which happens if and only if $\Delta_i = 0$, for Δ_i as in (4). Namely, if and only if $\mathfrak{g}$ is quasi-split.

Finally, (3) follows from [27, 2.5(c) and (d)] (or simply, by the way the M_k's are generated). □

Remark 4.14. Note that m_k is an exponent of G whenever $e_k \in \mathfrak{m}^{\mathbb{C}}$.

Corollary 4.15. *Let $i : S \hookrightarrow G$ be a three-dimensional subgroup corresponding to a three-dimensional subalgebra $\mathfrak{s} \subset \mathfrak{g}$. Then i is irreducible into the component of the identity G_0 (namely, $Z_{G_0}(S) = Z(G_0)$) if and only if G is quasi-split.*

5. G-Higgs bundles

For this section, we follow [18].

5.1. **Basic theory.** Let X be a smooth complex projective curve, and let $L \to X$ be a holomorphic line bundle on X. Let (G, H, B, θ) be a real reductive Lie group as defined in Section 3, and consider $\mathfrak{h}$, $\mathfrak{m}$, etc., as defined in Section 2. Note that by condition (5) in Definition 3.1, we have a representation

$$\iota : H \to \mathrm{GL}(\mathfrak{m}), \tag{18}$$

which complexifies to $H^{\mathbb{C}} \curvearrowright \mathfrak{m}^{\mathbb{C}}$. We will refer to both as the isotropy representation.

Definition 5.1. An **L-twisted G-Higgs bundle** over X is a pair (E, φ), where E is a holomorphic principal $H^{\mathbb{C}}$-bundle on X and $\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes L)$. Here, $E(\mathfrak{m}^{\mathbb{C}})$ is the vector bundle associated to E via the isotropy representation. When $L = K$ is the canonical bundle of X, these pairs are referred to simply as **G-Higgs bundles**.

Remark 5.2. (1) When G is the real Lie group underlying a complex reductive Lie group, the above definition reduces to the classical definition for complex groups given by Hitchin [23]. Indeed, if $U < G$ is the maximal compact subgroup, then $G = (U^{\mathbb{C}})_{\mathbb{R}}$, so $\mathfrak{m}^{\mathbb{C}} = (i\mathfrak{u})^{\mathbb{C}} = \mathfrak{g}$ and the complexified isotropy representation is the adjoint representation.

(2) Note that the above definition uses all the ingredients of the Cartan data of G except the bilinear form B. Its role will become apparent in the definition of stability conditions, as well as the Hitchin equations for G-Higgs bundles.

Given $s \in i\mathfrak{h}$, we define:

$$\begin{array}{rcl} \mathfrak{p}_s & = & \{x \in \mathfrak{h}^{\mathbb{C}} \mid \mathrm{Ad}(e^{ts})(x) \text{ is bounded as } t \to \infty\}, \\ P_s & = & \{g \in H^{\mathbb{C}} \mid \mathrm{Ad}(e^{ts})(g) \text{ is bounded as } t \to \infty\}, \\ \mathfrak{l}_s & = & \{x \in \mathfrak{h}^{\mathbb{C}} \mid [x, s] = 0 = \mathfrak{c}_{\mathfrak{h}}(x)\}, \\ L_s & = & \{g \in H^{\mathbb{C}} \mid \mathrm{Ad}(e^{ts})(g) = g = C_{H^{\mathbb{C}}}(e^{\mathbb{R}s})\}, \\ \mathfrak{m}_s & = & \{x \in \mathfrak{m}^{\mathbb{C}} \ : \ \lim_{t \to 0} \iota(e^{ts})(x) \text{ exists}\}, \\ \mathfrak{m}_s^0 & = & \{x \in \mathfrak{m}^{\mathbb{C}} \ : \ \iota(e^{ts})(x) = x\}. \end{array} \tag{19}$$

We call P_s and $\mathfrak{p}_s$ (respectively L_s and $\mathfrak{l}_s$) the **parabolic** (respectively **Levi**) **subgroup** and **subalgebra** associated to s. For each $s \in i\mathfrak{h}$, we define χ_s to be the character of $\mathfrak{p}_s$ dual to s via the bilinear form B. We note it is a strictly antidominant character of $\mathfrak{p}_s$ (cf. [18]).

Consider an L-twisted G-Higgs bundle (E, φ). Given a parabolic subgroup $P_s \leq H^{\mathbb{C}}$ and $\sigma \in \Gamma(X, E(H^{\mathbb{C}}/P_s))$ a holomorphic reduction of the structure group to P_s, let E_σ denote the corresponding principal bundle. The isotropy representation restricts to actions $P_s \curvearrowright \mathfrak{m}_s$, $L_s \curvearrowright \mathfrak{m}_s^0$, so it makes sense to consider $E_\sigma(\mathfrak{m}_s)$. Similarly, given a choice of a holomorphic reduction of the structure group $\sigma_L \in \Gamma(X, P_s/L_s)$, one may consider $E_{\sigma_L}(\mathfrak{m}_s^0)$.

Let F_h be the curvature of the Chern connection of E with respect to a C^∞ reduction of the structure group $h \in \Omega^0(X, E(H^{\mathbb{C}}/H))$. Let $s \in i\mathfrak{h}$, and let $\sigma \in \Gamma(X, E(H^{\mathbb{C}}/P_s))$ be holomorphic. We define the **degree of E with respect to s and the reduction σ** as follows:

$$\deg E(s, \sigma) = \int_X \chi_s(F_h). \tag{20}$$

An alternative definition of the degree when the character χ_s lifts to a character $\delta_s : P_s \to \mathbb{C}^\times$ is given by

$$\deg E(s,\sigma) = \deg(E \times_{\delta_s} \mathbb{C}). \tag{21}$$

See [18] for the equivalence of both definitions.

We can now define the stability of a G-Higgs bundle. This notion naturally depends on an element in $i\mathfrak{z}$, which has special significance when G is a group of Hermitian type (cf. Definition 3.33).

Definition 5.3. Let $\alpha \in i\mathfrak{z}$. We say that the pair (E,φ) is:

(1) **α-semistable** if for any $s \in i\mathfrak{h}$ and any holomorphic reduction of the structure group $\sigma \in \Gamma(X, E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(X, E(\mathfrak{m}_s) \otimes L)$, then
$$\deg E(s,\sigma) - B(\alpha, s) \geq 0;$$

(2) **α-stable** if it is semistable and for any $s \in i\mathfrak{h} \setminus \operatorname{Ker}(d\iota)$, given any holomorphic reduction $\sigma \in \Gamma(X, E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(X, E(\mathfrak{m}_s) \otimes L)$, then
$$\deg E(s,\sigma) - B(\alpha, s) > 0;$$

(3) **α-polystable** if it is α-semistable and whenever
$$\deg E(s,\sigma) - B(\alpha, s) = 0$$
for some s and σ as above, there exists a reduction σ' to the corresponding Levi subgroup L_s such that φ takes values in $H^0(X, E_{\sigma'}(\mathfrak{m}_s^0) \otimes L)$.

The **moduli space of α-polystable L-twisted G-Higgs bundles** is defined as the set $\mathcal{M}_L^\alpha(G)$ of isomorphism classes of such objects. It coincides with the moduli space of S-equivalence classes of α-semistable Higgs bundles. For a more detailed account of these notions, as well as the geometry of $\mathcal{M}_L^\alpha(G)$, we refer the reader to [18].

Parameters appear naturally when studying the moduli problem from the gauge-theoretic point of view. This relation is established by the Hitchin–Kobayashi correspondence as follows (cf. [18]).

Theorem 5.4. *Let $\alpha \in i\mathfrak{z}$. Let $L \to X$ be a line bundle, and let h_L be a Hermitian metric on L. Fix ω a Kähler form on X. An L-twisted G-Higgs bundle (E,φ) is α-polystable if and only if there exists $h \in \Omega^0(X, E(H^{\mathbb{C}}/H))$ satisfying*

$$F_h - [\varphi, \tau_h(\varphi)]\omega = -i\alpha\omega, \tag{22}$$

where F_h is the curvature of the Chern connection on E corresponding to h, and $\tau_h : \Omega^0\left(E(\mathfrak{m}^{\mathbb{C}} \otimes L)\right) \to \Omega^0\left(E(\mathfrak{m}^{\mathbb{C}}) \otimes L\right)$ is the antilinear involution on $\Omega^0(E(\mathfrak{m}^{\mathbb{C}}) \otimes L)$ determined by h and h_L.

In the above theorem, we fix a G-Higgs bundle and look for a solution of equation (22). From a different perspective, we can construct the gauge moduli space associated to equation (22) as follows. Fix a C^∞ principal $H^{\mathbb{C}}$-bundle $\mathbb{E}$. Given a reduction $h \in \Omega^0(X, \mathbb{E}(H^{\mathbb{C}}/H))$, let $\mathbb{E}_h$ be the corresponding principal H-bundle. Consider pairs (A,φ) where A is a connection on $\mathbb{E}_h$ and $\varphi \in \Omega^0(X, \mathbb{E}_h \otimes L)$ is holomorphic with respect to the holomorphic structure defined by A and both satisfy (22). The gauge group $\mathcal{H} = \Omega^0(X, \operatorname{Ad} \mathbb{E}_h)$, where $\operatorname{Ad} \mathbb{E}_h := \mathbb{E}_h \times_{\operatorname{Ad}} H$ is the associated bundle of groups, acts on solutions of (22). Let $\mathcal{M}_{L,\mathbb{E}_h}^{gauge,\alpha}(G)$ be the gauge moduli space obtained by taking the quotient of the space of solutions to (22) by this

action. In a similar fashion, we can define the moduli space $\mathcal{M}^{\alpha}_{L,\mathbb{E}}(G) \subset \mathcal{M}^{\alpha}_{L}(G)$ of α-polystable G-Higgs bundles with underlying smooth $H^{\mathbb{C}}$-bundle $\mathbb{E}$. Theorem 5.4 defines a homeomorphism

$$\mathcal{M}^{\alpha}_{L,\mathbb{E}_h}(G) \cong \mathcal{M}^{gauge,\alpha}_{L,\mathbb{E}}(G). \tag{23}$$

In the case $L = K$, for $\alpha = 0$, there is a third moduli space that can be considered. Let $\mathcal{R}(G) = \mathrm{Hom}^{+}(\pi_1(X), G)/G$ be the quotient of the set of reductive homomorphisms $\rho : \pi_1(X) \to G$ by the conjugation action of G. Combining the homeomorphism (23) with Corlette–Donaldson's theorem [15, 16], from each $\rho \in \mathcal{R}(G)$ one obtains a polystable Higgs bundle $(E, \varphi) \in \mathcal{M}^{\alpha}_{L,\mathbb{E}}(G)$. This induces a homeomorphism

$$\mathcal{R}(G) \cong \mathcal{M}^{0}_{K}(G). \tag{24}$$

This correspondence is the basic content of **non-abelian Hodge theory**.

5.2. **Topological type of Higgs bundles.** Given a C^{∞} principal bundle $\mathbb{E}$, its isomorphism class is determined by a topological invariant, which in the case when G is connected is given by an element $d \in \pi_1(H)$. This goes as follows: consider the short exact sequence

$$1 \to \pi_1(H^{\mathbb{C}}) \to \widetilde{H^{\mathbb{C}}} \to H^{\mathbb{C}} \to 1.$$

Then, since $\dim_{\mathbb{R}}(X) = 2$ and the fundamental group of a Lie group is abelian (see Theorem 7.1 in [13]), one has that $H^2(X, \pi_1(H^{\mathbb{C}})) \cong \pi_1(H^{\mathbb{C}}) \cong \pi_1(H)$, where the last isomorphism follows from the fact that H is a deformation retract of $H^{\mathbb{C}}$. So through the associated long exact sequence in cohomology one associates to each class $[\mathbb{E}] \in H^1(X, H^{\mathbb{C}})$ an element $d(\mathbb{E}) \in \pi_1(H)$. In particular, given a G-Higgs bundle, (E, φ), one may consider the class corresponding to the differentiable principal bundle underlying E. Fixing the topological class $d \in \pi_1(H)$, we can consider the subspace $\mathcal{M}^{\alpha}_{L,d}(G) \subset \mathcal{M}^{\alpha}_{L}(G)$ consisting of isomorphism classes of α-polystable L-twisted G-Higgs bundles with class d.

In the case of groups of Hermitian type, there is an equivalent invariant that one can define called the **Toledo invariant**. The original definition of this invariant in the context of representations of the fundamental group is due to Toledo [34] when $G = \mathrm{PU}(n, 1)$, generalised by several authors for the various simple classical and exceptional groups of Hermitian type and extended to arbitrary groups of Hermitian type by Burger–Iozzi–Wienhard [14]. In the context of L-twisted G-Higgs bundles the Toledo invariant has been defined for arbitrary groups of Hermitian type in [5]. These two general definitions naturally coincide when $L = K$.

Let G be a simple Hermitian Lie group such that G/H is irreducible. In this situation the centre $\mathfrak{z}$ of $\mathfrak{h}$ is isomorphic to $\mathbb{R}$, and the adjoint action of an element $J \in \mathfrak{z}$ defines an almost complex structure on $\mathfrak{m} = T_o(G/H)$, where $o \in G/H$ corresponds to the coset H, making the symmetric space G/H into a Kähler manifold. The almost complex structure $\mathrm{ad}(J)$ gives a decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{+} + \mathfrak{m}^{-}$ in $\pm i$-eigenspaces, which is $H^{\mathbb{C}}$-invariant. An immediate consequence of this decomposition for an L-twisted G-Higgs bundle (E, φ) is that it gives a bundle decomposition $E(\mathfrak{m}^{\mathbb{C}}) = E(\mathfrak{m}^{+}) \oplus E(\mathfrak{m}^{-})$, and hence the Higgs field decomposes as $\varphi = (\beta, \gamma)$, where $\beta \in H^0(X, E(\mathfrak{m}^{+}) \otimes L)$ and $\gamma \in H^0(X, E(\mathfrak{m}^{-}) \otimes L)$.

There is a character of $\chi_T : \mathfrak{h}^{\mathbb{C}} \to \mathbb{C}$ called the Toledo character and a rational number q_T such that $q_T \chi_T$ lifts to a character $\tilde{\chi_T}$ of $H^{\mathbb{C}}$. We define the Toledo

invariant of an L-twisted G-Higgs bundle (E, φ) by

$$T(E) = \frac{1}{q_T} \deg(E \times_{\widetilde{\chi_T}} \mathbb{C}^\times). \tag{25}$$

One can define the ranks of β and γ (see [5]). These are integers bounded by the rank of the symmetric space G/H. The following can be found in [5] (see Theorem 3.18 and the discussion preceding Theorem 4.14 therein):

Proposition 5.5. *Let G be a simple group of Hermitian type with irreducible associated symmetric space, so that $\mathfrak{z}(\mathfrak{h}) = i\mathbb{R}$. Let $(E, (\beta, \gamma))$ be an L-twisted G-Higgs bundle, α-semistable for some $\alpha = i\lambda J$. Then:*

(1) *The Toledo invariant satisfies the Milnor–Wood inequality*

$$-\mathrm{rk}\beta \cdot d_L - \lambda\left(\frac{\dim \mathfrak{m}}{N} - \mathrm{rk}\beta\right) \leq T(E) \leq \mathrm{rk}\gamma \cdot d_L - \lambda\left(\frac{\dim \mathfrak{m}}{N} - \mathrm{rk}\gamma\right), \tag{26}$$

where N is the dual Coxeter number of $\mathfrak{g}^{\mathbb{C}}$ and d_L is the degree of L. Moreover, when G is of tube type (i.e., G/H is biholomorphic to a tube domain), T (resp. $-T$) is maximal if and only if $\gamma(x) \in \mathfrak{m}^{\mathbb{C}^+_{reg}}$ (resp. $\beta(x) \in \mathfrak{m}^{\mathbb{C}^-_{reg}}$) for all $x \in X$.

(2) *There exists a canonical $k > 0$ such that*

$$\overline{d}(E) = kT(E),$$

where $\overline{d}(E)$ denotes the projection of the topological class $d(E)$ to the torsion free part of $\pi_1(H)$.

Now, the curvature of a principal bundle E determines the torsion free part of its topological class $d(E)$ via the first Chern class. This information is partially determined by the parameter and vice versa. Let $\mathfrak{z}(\mathfrak{g})^\perp$ be the orthogonal complement of $\mathfrak{z}(\mathfrak{g})$ inside $\mathfrak{g}$.

Proposition 5.6. *Let (E, φ) be an α-polystable Higgs bundle. Let $\alpha = \alpha_0 + \alpha_1$, where $\alpha_0 \in i\mathfrak{z}(\mathfrak{h}) \cap i\mathfrak{z}(\mathfrak{g})$ and $\alpha_1 \in i\mathfrak{z}(\mathfrak{h}) \cap i\mathfrak{z}(\mathfrak{g})^\perp$ are the projections to $i\mathfrak{z}(\mathfrak{g})$ and $i\mathfrak{z}(\mathfrak{g})^\perp$. Then, α_0 determines $\overline{d}(E)$.*

Proof. In order to see this, we note that α_0 is determined by the image $\chi(\alpha)$ for all $\chi \in \mathrm{Char}(\mathfrak{h}^{\mathbb{C}}) \cap \mathrm{Char}(\mathfrak{g}^{\mathbb{C}})$. Now, $[H^{\mathbb{C}}, H^{\mathbb{C}}]$-invariance implies that it makes sense to evaluate $\chi(F_A - [\varphi, \varphi^*])$, and, moreover, the evaluation of all such characters determines $F_A - [\varphi, \varphi^*]$. Furthermore, for $\chi \in \mathrm{Char}(\mathfrak{h}^{\mathbb{C}}) \cap \mathrm{Char}(\mathfrak{g}^{\mathbb{C}})$, we have $\chi([\varphi, \varphi^*]) = 0$, as $[\varphi, \varphi^*]$ is a two form with values in $[\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}]$. This proves the statement. □

Remark 5.7 (Topological type and parameters). A non-zero parameter $\alpha \neq 0$ makes sense only when $\mathfrak{z}(\mathfrak{h}) \neq 0$. This includes the case of real groups underlying a complex non-semisimple reductive Lie group $(G^{\mathbb{C}})_{\mathbb{R}}$ (cf. Remark 5.2) or the case of simple groups of Hermitian type (cf. Definition 3.33).

Proposition 5.6 implies that when $G^{\mathbb{C}}$ has a positive dimensional centre, the topology of the bundle fully determines the parameter, and conversely, the torsion free piece of the topological type is also determined by the parameter. On the other hand, the same result implies that for Hermitian groups we are in the opposite situation, as these are characterised by having large $\mathfrak{z}(\mathfrak{h}) \cap \mathfrak{z}(\mathfrak{g})^\perp$.

5.3. **Morphisms induced by group homomorphisms.** Consider a morphism of reductive Lie groups $f : (G', H', \theta', B') \to (G, H, \theta, B)$.

Definition 5.8. Given a G'-Higgs bundle (E', φ'), we define the **extended G-Higgs bundle** (by the morphism f) to be the pair $(E'(H^{\mathbb{C}}), df(\varphi'))$, where $E'(H^{\mathbb{C}})$ is the principal $H^{\mathbb{C}}$-bundles associated to E' via f. Note that $df(\varphi')$ is well defined as df commutes with the adjoint action.

These pairs satisfy the following.

Proposition 5.9. *With notation as above, if the G'-Higgs bundle (E', φ') is α-polystable and $df(\alpha) \in i\mathfrak{z}(\mathfrak{h})$, then the corresponding extended G-Higgs bundle (E, φ) is $df(\alpha)$-polystable.*

Proof. By Theorem 5.4, polystability of (E', φ') is equivalent to the existence of a solution to the Hitchin equation (22). Let h' be the corresponding solution. Now, h' extends to a Hermitian metric on E, as f defines a map

$$\Omega^0(E((H')^{\mathbb{C}}/H')) \to \Omega^0(E(H^{\mathbb{C}}/H)).$$

Let $h \in \Omega^0(E(H^{\mathbb{C}}/H))$ be the image of h' via that map. Clearly $F_{h'}$ is a two form with values in $\mathfrak{h}$. But $F_h = df(F_{h'})$, where df is evaluated on the coefficients of the two-form $F_{h'}$, as the canonical connection ∇_h is defined by

$$dh = \langle \nabla_h \cdot, \cdot \rangle + \langle \cdot, \nabla_h \cdot \rangle.$$

Since $dh = df(dh')$, it follows that $\nabla_h = df(\nabla_{h'})$ solves the modified equations. By Theorem 5.4, this gives a polystable Higgs bundle, which by construction must be (E, φ). □

As a corollary we have the following.

Corollary 5.10. *With the above notation, if $\alpha \in i\mathfrak{z}'$ is such that $df(\alpha) \in i\mathfrak{z}$, then the map*

$$(E', \varphi') \mapsto \big(E'(H^{\mathbb{C}}), df(\varphi')\big)$$

induces a morphism

$$\mathcal{M}_d^{\alpha}(G') \to \mathcal{M}_{f_*d}^{df(\alpha)}(G),$$

where f_*d is the topological type of $E(H^{\mathbb{C}})$. When G is connected, this corresponds to the image via the map $f_* : \pi_1(H') \to \pi_1(H)$ induced by f.

Lemma 5.11. *Let $G' \subseteq G$ be two Lie groups. Let E, $\widetilde{E}$ be two principal G'-bundles over X, and suppose there exists a morphism*

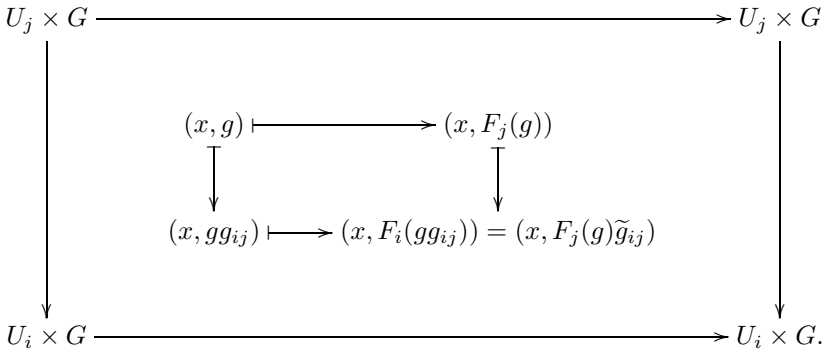
$$F : \ E(G) \to \widetilde{E}(G)$$

of principal G-bundles. Then there exists an isomorphism of principal $N_G(G')$-bundles $E(N_G(G')) \cong \widetilde{E}(N_G(G'))$.

Proof. By Theorem 10.3 in [33], F is an isomorphism. Denote $N_G(G')$ by N. Choose common trivialising neighbourhoods $U_i \to X$ such that

$$E|_{U_i} \cong U_i \times G', \qquad \widetilde{E}|_{U_i} \cong U_i \times G'.$$

Let $g_{ij}, \widetilde{g_{ij}}$ be the transition functions for E and $\widetilde{E}$ respectively and define $F_i := F|_{E(G)|_{U_i}}$. Then we have the following commutative diagram:



Now, since for any $n \in N,\ g \in G'$ we have that $ng \in N$, it follows that for all i, j, $F_i(N) = F_j(N)\widetilde{g_{ij}}$. Namely, the image bundle of $E(N)$ is isomorphic to $\widetilde{E}(N)$. □

5.4. **Deformation theory.** The deformation theory of Higgs bundles was studied by several authors, amongst which we cite [6] in the setting of arbitrary pairs and [18] and references therein for G-Higgs bundle when G is a real reductive Lie group. Let us recall the basics.

The deformation complex of a G-Higgs bundle $(E, \varphi) \to X$ is

$$C^\bullet : [d\varphi,\ \cdot\ ] : E(\mathfrak{h}^{\mathbb{C}}) \to E(\mathfrak{m}^{\mathbb{C}}) \otimes L, \tag{27}$$

whose hypercohomology sets fit into the exact sequence

$$\begin{aligned} 0 \to \mathbb{H}^0(C^\bullet) \to H^0(X, E(\mathfrak{h}^{\mathbb{C}})) \to H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes L) \\ \to \mathbb{H}^1(C^\bullet) \to H^1(E(\mathfrak{h}^{\mathbb{C}})) \to H^1(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes L) \to \mathbb{H}^2(C^\bullet) \to 0. \end{aligned} \tag{28}$$

In particular, we see that $\mathbb{H}^0(C^\bullet) = \mathfrak{aut}(E, \varphi)$, where $\mathfrak{aut}(E, \varphi)$ denotes the Lie algebra of the automorphism group of (E, φ).

On the other hand, the space of infinitesimal deformations of a pair (E, φ) is canonically isomorphic to $\mathbb{H}^1(C^\bullet)$ ([6, Theorem 2.3]). Hence, the expected dimension of the moduli space is the dimension of $\mathbb{H}^1(C^\bullet(E, \varphi))$ at a smooth point (E, φ).

Definition 5.12. A G-Higgs bundle (E, φ) is said to be **simple** if

$$\mathrm{Aut}(E, \varphi) = H^0(X, \mathrm{Ker}(\iota) \cap Z(H^{\mathbb{C}})).$$

(E, φ) is said to be **infinitesimally simple** if

$$\mathbb{H}^0(X, C^\bullet) \cong H^0(X, (\mathrm{Ker}(d\iota) \cap \mathfrak{z}(\mathfrak{h}^{\mathbb{C}}))).$$

Here ι is the isotropy representation of $H^{\mathbb{C}}$ in $\mathfrak{m}^C$.

These notions are deeply related to smoothness of the points of the moduli space, as the next result shows. For an alternative proof of the following proposition, see [11].

Proposition 5.13. *Let (E, φ) be a stable and simple G-Higgs bundle, where (G, H, θ, B) is a real strongly reductive Lie group. Let $\mathfrak{z}_\mathfrak{m} = \mathfrak{z}(\mathfrak{g}^{\mathbb{C}}) \cap \mathfrak{m}^{\mathbb{C}}$. Assume that $\mathbb{H}^2(C^\bullet) = H^1(X, \mathfrak{z}_\mathfrak{m} \otimes L)$. Then (E, φ) is a smooth point of the moduli space.*

Proof. This follows from Theorem 3.1 in [6] applied to the algebraic group $H^{\mathbb{C}}$ and the isotropy representation $\iota : H^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}})$.

Indeed, singularities of the moduli space can be of orbifold origin, which are discarded by the simplicity assumption, or caused by the existence of obstructions to deformations, measured by $\mathbb{H}^2(C^\bullet)$. Now, although Theorem 3.1 in [6] assumes the vanishing of the whole hypercohomology group, a simple argument shows that the centre plays no role in obstructing infinitesimal deformations.

To understand this, let $\mathfrak{m}_{ss} = [\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}] \cap \mathfrak{m}^{\mathbb{C}}$, $\mathfrak{h}_{ss}^{\mathbb{C}} = (\mathfrak{h} \cap \mathfrak{g}_{ss})^{\mathbb{C}}$, $\mathfrak{z}_{\mathfrak{m}} = \mathfrak{z}(\mathfrak{g}^{\mathbb{C}}) \cap \mathfrak{m}^{\mathbb{C}}$, and $\mathfrak{z}_{\mathfrak{h}} = \mathfrak{h}^{\mathbb{C}} \cap \mathfrak{z}(\mathfrak{g}^{\mathbb{C}})$. Observe that $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ factors through $G_{ss} := [G, G]$, which implies that

$$E(\mathfrak{h}^{\mathbb{C}}) \cong E_{ss}(\mathfrak{h}_{ss}^{\mathbb{C}}) \oplus X \times \mathfrak{z}_{\mathfrak{h}},\ E(\mathfrak{m}^{\mathbb{C}}) \cong E(\mathfrak{m}_{ss}^{\mathbb{C}}) \oplus X \times \mathfrak{z}_{\mathfrak{m}}.$$

Moreover, $[\varphi, E(\mathfrak{h}^{\mathbb{C}})] = [\varphi_{ss}, E_{ss}(\mathfrak{h}_{ss}^{\mathbb{C}})] \subset E_{ss}(\mathfrak{m}_{ss}^{\mathbb{C}})$, which implies that the complex $C^\bullet$ splits into a direct sum of complexes $C^\bullet = C_{ss}^\bullet \oplus Z(C^\bullet)$ where

$$C_{ss}^\bullet : E_{ss}(\mathfrak{h}_{ss}^{\mathbb{C}}) \to E_{ss}(\mathfrak{m}_{ss}^{\mathbb{C}}) \otimes L \tag{29}$$

and

$$Z(C^\bullet) := X \times \mathfrak{z}_{\mathfrak{h}} \xrightarrow{0} X \times \mathfrak{z}_{\mathfrak{m}} \otimes L. \tag{30}$$

Hence

$$\mathbb{H}^i(C^\bullet) = \mathbb{H}^i(C_{ss}^\bullet) \oplus \mathbb{H}^i(Z(C^\bullet)). \tag{31}$$

Now, following the proof of Theorem 3.1 in [6], we have complexes

$$\mathcal{G}_n^\bullet : p_n^* E(\mathfrak{h}^{\mathbb{C}}) \otimes \mathbb{C}[\epsilon]/\epsilon^n \to p_n^* E(\mathfrak{m}^{\mathbb{C}}) \otimes L \otimes \mathbb{C}[\epsilon]/\epsilon^n \to 0,$$

where $p_n : X \times \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^n) \to X$ is the projection on the first factor. With this we obtain a short exact sequence of complexes

$$0 \to C^\bullet \otimes \langle \epsilon^n \rangle \to \mathcal{G}_{n+1} \to \mathcal{G}_n \to 0, \tag{32}$$

which splits into the direct sum of

$$0 \to C_{ss}^\bullet \otimes \langle \epsilon^n \rangle \to \mathcal{G}_{n+1,ss} \to \mathcal{G}_{n,ss} \to 0$$

and

$$0 \to Z(C^\bullet) \otimes \langle \epsilon^n \rangle \to Z(\mathcal{G}_{n+1}) \to Z(\mathcal{G}_n) \to 0, \tag{33}$$

where $\mathcal{G}_{n,ss}^\bullet$, $Z(\mathcal{G}_n)$ are defined similarly to (29), (30). Hence, the long exact sequence in hypercohomology induced by (32) also splits. This, together with (31) and Theorem 3.1 in [6], implies that the only obstructions to deformation come from the long exact sequence induced by (33). We see that this long exact sequence splits into short exact sequences

$$0 \to \mathbb{H}^i(Z(C^\bullet)) \to \mathbb{H}^i(Z(\mathcal{G}_{n+1}^\bullet)) \to \mathbb{H}^i(Z(\mathcal{G}_n^\bullet)) \to 0,$$

and so we may conclude that no obstruction to deformation lies in $\mathbb{H}^2(Z(C^\bullet))$. □

The above has its counterpart in terms of the gauge moduli space. This is done in full detail in [18] in the case $\alpha = 0$, $L = K_X$. We extend it here to the deformation complex of an arbitrary pair. Coming back to the gauge moduli setup developed in Section 5.1, let (A, φ) be a pair of a connection on some differentiable

principal $H^{\mathbb{C}}$-bundle $\mathbb{E}$, and let $\varphi \in \Omega^0(\mathbb{E}(\mathfrak{m}^{\mathbb{C}}) \otimes L)$. Then, if h is the solution to (22) corresponding to (A, φ), we get a deformation complex,

$$\begin{aligned} C^\bullet(A,\varphi) : \Omega^0(X, E_h(\mathfrak{h})) &\xrightarrow{d_0} \Omega^1(X, E_h(\mathfrak{h})) \oplus \Omega^0(X, E_h(\mathfrak{m}^{\mathbb{C}}) \otimes L) \\ &\xrightarrow[d_1]{} \Omega^2(X, E_h(\mathfrak{h})) \oplus \Omega^{0,1}(X, E_h(\mathfrak{m}^{\mathbb{C}}) \otimes L), \end{aligned}$$

where E_h is the reduction of E to a principal H-bundle given by h, and the maps are defined by

$$d_0(\psi) = (d_A\psi, [\varphi, \psi]), \tag{34}$$

$$d_1(\dot{A}, \dot{\varphi}) = (d_A(\dot{A}) - [\dot{\varphi}, \tau\varphi]\omega - [\varphi, \tau\,\dot{\varphi}]\omega, \overline{\partial}_A\,\dot{\varphi} + [\dot{A}^{0,1}, \varphi]).$$

Definition 5.14. A pair (A, φ) is said to be **irreducible** if its group of automorphisms

$$\mathrm{Aut}(A,\varphi) := \{h \in \mathcal{H} \ : \ h^*A = A,\ \iota(h)(\varphi) = \varphi\} = Z(H) \cap \mathrm{Ker}(\iota). \tag{35}$$

It is said to be **infinitesimally irreducible** if

$$\mathfrak{aut}(A,\varphi) := \mathrm{Lie}(\mathrm{Aut}(A,\varphi)) = \mathfrak{z}(\mathfrak{h}) \cap \mathrm{Ker} d\iota.$$

The following two propositions are explained in full detail in [18] for moduli spaces of (0-polystable) Higgs bundles. For the general case, arguments are also standard and consist of resolving the hypercohomology complex $\mathbb{H}^1(C^\bullet(E,\varphi))$ and choosing harmonic representatives (see for example [26, VI.8]).

Proposition 5.15. *Let $(E, \varphi) \in \mathcal{M}^{\alpha}_{L,d}(G)$, and let $(A, \varphi) \in \mathcal{M}^{gauge}_{L,d}(G)$ be its corresponding gauge counterpart. Assume they are both smooth points of their respective moduli. Then*

$$\mathbb{H}^0(C^\bullet(E,\varphi)) \cong \mathbb{H}^0(C^\bullet(A,\varphi)).$$

Proposition 5.16. *Let $(E, \varphi) \in \mathcal{M}^{\alpha}_{L,d}(G)$, and let $(A, \varphi) \in \mathcal{M}^{gauge}_{L,d}(G)$ be its corresponding gauge counterpart. Then*

$$\mathbb{H}^1(C^\bullet(E,\varphi)) \cong \mathbb{H}^1(C^\bullet(A,\varphi)).$$

Proposition 5.17. *Under the correspondence established by Theorem* 5.4*, stable Higgs bundles correspond to infinitesimally irreducible solutions to* (22)*. On the other hand, simple and stable bundles correspond to irreducible solutions.*

6. The Hitchin map and the Hitchin–Kostant–Rallis section

Let (G, H, θ, B) be a reductive Lie group as in Definition 3.1, and let $\mathfrak{h}$, $\mathfrak{m}$, $\mathfrak{a}$, etc., be as in Sections 2 and 4.

Consider the Chevalley morphism defined in Section 4:

$$\chi : \mathfrak{m}^{\mathbb{C}} \to \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}}). \tag{36}$$

This map is $\mathbb{C}^\times$-equivariant. In particular, it induces a morphism

$$h_L : \mathfrak{m}^{\mathbb{C}} \otimes L \to \mathfrak{a}^{\mathbb{C}} \otimes L/W(\mathfrak{a}^{\mathbb{C}}).$$

The map χ is also $H^{\mathbb{C}}$-equivariant, thus defining a morphism

$$h_L : \mathcal{M}^{\alpha}_L(G) \to B_L(G) := H^0(X, \mathfrak{a}^{\mathbb{C}} \otimes L/W(\mathfrak{a}^{\mathbb{C}})). \tag{37}$$

Definition 6.1. The map h_L in (37) is called the **Hitchin map**, and the space $B_L(G)$ is called the **Hitchin base**.

Proposition 6.2. *Let $a = \dim \mathfrak{a}^{\mathbb{C}}$, and let $\mathfrak{z}_{\mathfrak{m}} = \mathfrak{z}(\mathfrak{g}^{\mathbb{C}}) \cap \mathfrak{m}^{\mathbb{C}}$. Let $\widehat{\mathfrak{g}}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ be the complexification of the maximal split subalgebra defined in Section 2.2. Assume $\deg L \geq g-1$. Then*

$$\dim B_L(G) = \frac{d_L}{2}(\dim \widehat{\mathfrak{g}}^{\mathbb{C}}) + a\left(\frac{d_L}{2} - g + 1\right) + h^1(L)\dim \mathfrak{z}_{\mathfrak{m}}. \tag{38}$$

Proof. By definition

$$\begin{aligned}\dim B_L(G) &= \sum_{e_k \in \mathfrak{m}^{\mathbb{C}}} h^0(L^{m_k}) = \sum_{e_k \in \mathfrak{m}^{\mathbb{C}}} (m_k d_L - g + 1) + \sum_{e_k \in \mathfrak{m}^{\mathbb{C}}} h^1(L^{m_k}) \\ &= \sum_{e_k \in \mathfrak{m}^{\mathbb{C}}} \left((2m_k - 1)\frac{d_L}{2} - g + 1 + \frac{d_L}{2}\right) + h^1(L)\dim \mathfrak{z}_{\mathfrak{m}} \\ &= \frac{d_L}{2}(\dim \widehat{\mathfrak{g}}^{\mathbb{C}}_{ss} + \dim \mathfrak{z}_{\mathfrak{m}}) + a\left(\frac{d_L}{2} - g + 1\right) + h^1(L)\dim \mathfrak{z}_{\mathfrak{m}},\end{aligned}$$

which yields (38) since by definition $\mathfrak{z}(\widehat{\mathfrak{g}}^{\mathbb{C}}) = \mathfrak{z}_{\mathfrak{m}}$. □

Corollary 6.3. *If $L = K$ and $G = (U^{\mathbb{C}})_{\mathbb{R}}$ is the real group underlying a complex reductive subgroup, then $\dim B_L(G) = (g-1)\dim \mathfrak{u}^{\mathbb{C}} + \dim \mathfrak{z}(\mathfrak{u}^{\mathbb{C}})$.*

Proof. We need only note that in this case $\mathfrak{m}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} = \mathfrak{u}^{\mathbb{C}}$, and $\widehat{\mathfrak{g}}$ is the split real form of $\mathfrak{u}^{\mathbb{C}}$ (cf. Remark 2.8). Hence, $\dim \widehat{\mathfrak{g}}^{\mathbb{C}} = \dim \mathfrak{u}^{\mathbb{C}}$, and $\dim \mathfrak{z}_{\mathfrak{h}} = \dim \mathfrak{z}_{\mathfrak{m}} = \dim \mathfrak{z}(\mathfrak{u}^{\mathbb{C}})$. □

Remark 6.4. We will see later that the dimension of $B_L(G)$ fails to be half the dimension of the moduli space unless $L = K$, the case considered by Hitchin [23].

In what follows, we proceed to the construction of a section of the Hitchin map (37). This generalises Hitchin's construction [24] in essentially two ways. First of all, Hitchin considers the case $L = K$, and he builds the section into $\mathcal{M}_K(G^{\mathbb{C}})$ for a complex Lie group $G^{\mathbb{C}}$ of adjoint type. A consequence of this is that $\alpha = 0$, as is the case for all semisimple groups (see Remark 5.7). Hitchin then checks that the monodromy of the corresponding representations takes values in G_{split}, the split real form of $G^{\mathbb{C}}$, so it is implicit in his construction that the section factors through $\mathcal{M}_K(G_{split})$. In what follows, we consider the existence of the section for arbitrary real reductive Lie groups, allowing arbitrary $\alpha \in i\mathfrak{z}(\mathfrak{h})$, and twisting by an arbitrary line bundle L; this requires the implementation of new techniques to prove stability and smoothness results. Moreover, our section is directly constructed into the moduli space of G-Higgs bundles, in particular, into $\mathcal{M}_K(G_{\mathrm{split}})$ when $G = G_{\mathrm{split}}$ is the split real form of a complex reductive Lie group $G^{\mathbb{C}}$ and $K = L$. In the latter case, this is precisely a factorisation of Hitchin's section through $\mathcal{M}_K(G_{\mathrm{split}})$. Recall (cf. Remark 5.7) that $\alpha \in i\mathfrak{z}(\mathfrak{h})$ decomposes as $\beta + \gamma \in \mathfrak{z}(\mathfrak{h}) \cap \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{z}(\mathfrak{h}) \cap \mathfrak{z}(\mathfrak{g})^{\perp}$. Then β is determined by the topology of the bundle and determines its torsion free part. As for γ, it is not of topological nature. Amongst groups with $\mathfrak{z}(\mathfrak{h}) \cap \mathfrak{z}(\mathfrak{g})^{\perp} \neq 0$ we find groups of Hermitian type (such as $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SU}(p, q)$, $\mathrm{SO}^*(2n)$ and $\mathrm{SO}(2, n)$) or any group containing one amongst its simple factors. On the other hand, $\mathfrak{z}(g)^{\perp} \cap \mathfrak{z}(\mathfrak{h}) = 0$ implies that the parameter is purely topological. This includes the case of complex reductive Lie groups. Indeed, $\mathfrak{z}(\mathfrak{g}^{\mathbb{C}}) = \mathfrak{z}(\mathfrak{u}) \oplus i\mathfrak{z}(\mathfrak{u})$, and so $\mathfrak{z}(\mathfrak{g}^{\mathbb{C}})^{\perp} \cap \mathfrak{z}(\mathfrak{u}) = 0$.

6.1. **Some representation theory.** The content of this section can be found in [27, 29].

Choose $\mathfrak{s}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$, a principal normal TDS (cf. Definition 2.5) defined by the homomorphism (15) of Lemma 4.10,

$$\rho' : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{s}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}, \tag{39}$$

which is σ- and θ-equivariant for the action of σ and θ on $\mathfrak{sl}(2,\mathbb{C})$ as defined in Lemma 4.10. Recall from (1) that the Cartan decomposition of $\mathfrak{sl}(2,\mathbb{R})$ under θ is

$$\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2) \oplus \mathfrak{shm}_0(2,\mathbb{R}), \tag{40}$$

which identifies $\mathfrak{so}(2)$ to trace zero diagonal matrices and $\mathfrak{shm}_0(2,\mathbb{R})$ to real antidiagonal matrices.

The image under ρ' of the standard basis

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto e, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto f, \quad \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto x \tag{41}$$

is a normal triple (e, f, x) (cf. Definition 2.5).

By θ-equivariance, $\rho' = \rho'_+ \oplus \rho'_-$, where

$$\rho'_+ : \mathfrak{so}(2,\mathbb{C}) \cong \mathbb{C} \to \mathfrak{h}^{\mathbb{C}}, \qquad \rho'_- : \mathfrak{shm}_0(2,\mathbb{C}) \cong \mathbb{C}^2 \to \mathfrak{m}^{\mathbb{C}}. \tag{42}$$

In particular, ρ'_+ fits into a commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\rho'_+} & \mathfrak{h}^{\mathbb{C}} \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota} \\ \mathfrak{gl}(\mathbb{C}^2) & \longrightarrow & \mathfrak{gl}(\mathfrak{m}^{\mathbb{C}}). \end{array} \tag{43}$$

We claim that the restriction of ρ' to $\mathfrak{sl}(2,\mathbb{R})$ lifts to a θ-equivariant group homomorphism

$$\rho : \mathrm{SL}(2,\mathbb{R}) \to G \tag{44}$$

taking $\mathrm{SO}(2)$ to H. Indeed, by connectedness of $\mathrm{SL}(2,\mathbb{R})$ and the polar decomposition, we can define $\rho(e^U e^V) = e^{\rho' X} e^{\rho' V}$ for given $U \in \mathfrak{so}(2,\mathbb{C})$, $V \in i\mathfrak{so}(2,\mathbb{C})$. We will abuse notation and use ρ_+ both for the restriction $\rho|_{\mathrm{SO}(2)}$ and its complexification. That is,

$$\rho_+ : \mathrm{SO}(2,\mathbb{C}) \to H^{\mathbb{C}}. \tag{45}$$

Now, by simple connectedness of $\mathrm{SL}(2,\mathbb{C})$, ρ' lifts to

$$\mathrm{Ad}(\rho) : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Ad}(G)^{\mathbb{C}}, \tag{46}$$

where $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$ is the adjoint representation and $\mathrm{Ad}(\rho)|_{\mathrm{SL}(2,\mathbb{R})} = \mathrm{Ad} \circ \rho$. Note that

$$\mathrm{Ker}(\mathrm{Ad}) = Z_G(\mathfrak{g}) \supseteq Z(G). \tag{47}$$

6.2. $\mathrm{SL}(2,\mathbb{R})$**-Higgs bundles.** Our basic case is $\mathrm{SL}(2,\mathbb{R})$, which is a group of Hermitian type, as $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ is the hyperbolic plane. Let us start by analysing $\mathcal{M}_d^\alpha(\mathrm{SL}(2,\mathbb{R}))_L$ for an arbitrary line bundle L of degree d_L.

An L-twisted $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle on a curve X is a line bundle $F \to X$ together with morphisms $\beta : F^* \to F \otimes L$ and $\gamma : F \to F^* \otimes L$.

Lemma 6.5. *The moduli space* $\mathcal{M}^\alpha_{L,d}(\mathrm{SL}(2,\mathbb{R}))$:

(1) *is empty if* $d > |d_L/2|$ *or* $d < \alpha$,
(2) *consists of all isomorphism classes of semistable* $\mathrm{SL}(2,\mathbb{R})$*-Higgs pairs if degree* $d > \alpha$, *and*
(3) *is isomorphic to* $\mathrm{Pic}^d(X)$ *if* $\alpha = d$.
(4) *Furthermore, if* $i\alpha' \leq i\alpha \in \mathfrak{h}$, *there is an inclusion*

$$\mathcal{M}^\alpha(\mathrm{SL}(2,\mathbb{R})) \subseteq \mathcal{M}^{\alpha'}(\mathrm{SL}(2,\mathbb{R})). \tag{48}$$

Proof. To prove (1), we first observe that the existence of sections $\beta \in H^0(X, F^2 \otimes L)$ and $\gamma \in H^0(X, F^{-2} \otimes L)$ implies that $|d_L/2| \geq |\deg F|$ with equality if and only if $F^{\pm 2} \cong L$. This accounts for the first condition.

For the second, since $H^{\mathbb{C}} \cong \mathbb{C}^\times$ is abelian, for all $s \in i\mathfrak{h}$, $P_s = H^{\mathbb{C}}$, and so the only reduction of the structure group is the identity; moreover, the only antidominant character is the identity (see [18, Section 2.2]) and $B(\alpha, id) = \alpha||id||_B$. Hence, a Higgs bundle is α-semistable if and only if

$$\deg F \geq \alpha||id||_B. \tag{49}$$

So after normalizing $||id||_B = 1$, we find that there will be no α-semistable bundles for $\alpha > d_L/2$, and for $\alpha \leq d_L/2$ we get bundles whose degree is at least $\lceil \alpha \rceil$ (where $\lceil \alpha \rceil$ is the lowest integer greater that real number α) and at most $[d_L/2]$.

Statements (2) and (3) follow from the above dicussion together with the fact that conditions for stability are limited to strictness of the inequality (49). Indeed, the Levi is again $H^{\mathbb{C}}$ itself. As for polystability, all stable bundles are polystable, so the only remaining case is when (49) is an equality. Then, $(F, (\beta, \gamma))$ is polystable if and only if $\beta = \gamma = 0$, as for $s \in \mathfrak{z} \setminus 0$, $\mathfrak{m}_s^0 = \{0\}$.

Assertion (4) follows from the definitions. □

Following [24], fix a holomorphic line bundle $L \to X$ of non-negative even degree, and consider

$$L^{1/2}, \qquad \varphi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in H^0(X, \mathrm{Hom}(L^{1/2}, L^{-1/2} \otimes L)). \tag{50}$$

By Lemma 6.5, the pair $(L^{1/2}, (0,1))$ is a stable L-twisted $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle whenever $i\alpha \leq d_L/2$. Furthermore, if $i\alpha \geq 0$, we can map

$$\mathcal{M}^\alpha_L(\mathrm{SL}(2,\mathbb{R})) \to \mathcal{M}^0_L(\mathrm{SL}(2,\mathbb{C})) \tag{51}$$

by (48), and the associated $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle is stable for $\deg L \neq 0$ (the case in which the pair is strictly polystable whenever $\beta = \gamma = 0$).

From now on we will assume that

$$d_L > 0, \qquad i\alpha \leq d_L/2, \qquad 2|d_L F. \tag{52}$$

We analyse the degree zero case in Remark 6.15.

Proposition 6.6. *Given $L \to X$ and $i\alpha \in \mathbb{R} = \mathfrak{z}(\mathfrak{so}(2))$ satisfying (52), we have two well-defined non-gauge equivalent sections to the Hitchin map*

$$h_L : \mathcal{M}_L^{\alpha}(\mathrm{SL}(2,\mathbb{R})) \to H^0(X, L^2)$$

given by

$$s_+ : \omega \mapsto (L^{1/2}, (1, \omega)) \tag{53}$$

and

$$s_- : \omega \mapsto (L^{-1/2}, (\omega, 1)). \tag{54}$$

Proof. Conditions (52) on α ensure polystability of the elements in the image of the section by Lemma 6.5. The same result ensures it is enough to consider the case $d_L = 2\alpha$.

Non-equivalence of (53) and (54) follows from the fact that both sections are conjugate via the complex gauge transformation $\mathrm{Ad}\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$ of $\mathcal{M}(\mathrm{SL}(2,\mathbb{C}))$ which is not in the image of $\mathrm{SO}(2,\mathbb{C})$ under $\mathrm{Ad} : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Aut}(\mathrm{SL}(2,\mathbb{C}))$. □

Remark 6.7. (1) By Remark 4.7, $\mathfrak{sym}_0(2,\mathbb{C})^{reg} \subset \mathfrak{sl}(2,\mathbb{C})^{reg}$, and since $\mathfrak{sl}(2,\mathbb{C})^{reg} = \mathfrak{sl}(2,\mathbb{C}) \setminus \{0\}$, the Higgs field of every element in the image of the section is trivially everywhere regular (cf. Definition 4.6).

(2) Note that for $i\alpha \geq 0$, the images of s_+ and s_- are identified in $\mathcal{M}_L^0(\mathrm{SL}(2,\mathbb{C}))$ under the morphism (51).

6.3. **The induced basic G-Higgs bundle.** We are interested in a section of (37) for arbitrary reductive groups (G, H, θ, B). It turns out that the $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle $(L^{1/2}, \varphi)$ defined in (50) induces a G-Higgs bundle as follows.

Let V be the principal bundle of frames of $L^{1/2}$. This has a structure group equal to $\mathbb{C}^\times$, which is isomorphic to $\mathrm{SO}(2,\mathbb{C})$. Let ρ_+ be as in (44), and consider the corresponding associated bundle

$$E = V(H^{\mathbb{C}}). \tag{55}$$

Letting ρ'_- be as in (42), we obtain a Higgs field

$$\Phi := \rho'_-(\varphi) \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes L), \tag{56}$$

where φ is as in (50) and $E(\mathfrak{m}^{\mathbb{C}})$ is the bundle associated to E via the isotropy representation.

Since E extends a principal $\mathbb{C}^\times$-bundle, the structure of $E(\mathfrak{m}^{\mathbb{C}}) \otimes L$ is determined by the action of $\mathrm{ad}(x)$, where x is defined in (41). Furthermore, Lemma 4.10 implies that e is a principal nilpotent element of $\mathfrak{m}^{\mathbb{C}}$.

Note that $V(\mathfrak{sym}_0(2,\mathbb{C})) \cong E(M_{\mathfrak{s}} \cap \mathfrak{m}^{\mathbb{C}})$ (where $M_{\mathfrak{s}}$ is the module as defined in (16) corresponding to the irreducible representation $\mathfrak{s}^{\mathbb{C}}$) is the bundle of symmetric endomorphisms of $L^{1/2} \oplus L^{-1/2}$, so we can identify it with $L \oplus L^{-1}$, as $L \cong \mathrm{Hom}(L^{-1/2}, L^{1/2})$. It follows that

$$E(M_k \cap \mathfrak{m}^{\mathbb{C}}) \cong \bigoplus_{i=0}^{\lfloor m_k - 1/2 \rfloor} L^{m_k - 1 - 2i} \quad \text{if } e_k \in \mathfrak{m}^{\mathbb{C}}. \tag{57}$$

In particular, Φ can be identified with the element $f \in \mathfrak{m}^{\mathbb{C}}$ considered as a section of $\mathfrak{m}^{\mathbb{C}}{}_{-1} \otimes L^{-1} \otimes L \subset E(\mathfrak{m}^{\mathbb{C}}) \otimes L$, where $\mathfrak{m}^{\mathbb{C}}_{\lambda}$ is the eigenspace of $\mathrm{ad}(x)$ with eigenvalue

λ. More generally

$$e_k \in \mathfrak{m}^{\mathbb{C}}_{m_k-1} \otimes L^{m_k-1} \otimes L^{-m_k+1} \subset E(M_k \cap \mathfrak{m}^{\mathbb{C}} \otimes L^{-m_k+1}), \tag{58}$$

since m_k is odd whenever $e_k \in \mathfrak{m}_{\mathbb{C}}$ by (17).

Definition 6.8. We call the pair (E, Φ) the **basic G-Higgs bundle**.

In what follows, we study stability and smoothness properties of the basic G-Higgs bundle.

Lemma 6.9. *Let (E, Φ) be defined by* (55) *and* (56). *Then $(E, \Phi) \in \mathcal{M}^0_L(G)$.*

Proof. By θ-equivariance of (44), we obtain a principal $H^{\mathbb{C}}$-bundle and a Higgs field taking values in $\mathfrak{m}^{\mathbb{C}}$. Corollary 5.10 gives the rest. □

Moreover, we have the following.

Proposition 6.10. *If G is quasi-split, the pair (E, Φ) defined by* (55) *and* (56) *is stable. Moreover, if G is strongly reductive and $Z(G) = Z_G(\mathfrak{g})$, then it is also simple.*

Before we prove Proposition 6.10, we need a lemma.

Lemma 6.11. *Let G be a strongly reductive quasi-split group* (*cf. Definition* 3.23). *Then, the map ρ* (*see* (44)) *satisfies that $Z_G(\mathrm{Im}(\rho)) = Z_G(\mathfrak{g})$.*

Proof. Let $S = \mathrm{Im}(\rho)$. Under the hypothesis on the group, by Lemma 4.13(2), we have that $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{s}) = 0$. Thus, by definition, $\mathrm{Ad}(\rho)$ (see (46)) is irreducible, so we have a three-dimensional subgroup $\mathrm{Ad}(S)^{\mathbb{C}} = \mathrm{Ad}(\rho)(\mathrm{SL}(2,\mathbb{C})) < \mathrm{Ad}(G)^{\mathbb{C}}$. In particular $Z_{\mathrm{Ad}(G)^{\mathbb{C}}}(\mathrm{Ad}(S)^{\mathbb{C}}) = 1$.

Now, let $g \in Z_G(S)$. Since $\mathrm{Ad}(G)^{\mathbb{C}}$ is a group of matrices, we have that $\mathrm{Ad}(S)^{\mathbb{C}} \subset \mathbb{C} \otimes \mathrm{Ad}(S) \subset \mathrm{End}(\mathfrak{g}^{\mathbb{C}})$, so $Z_{\mathrm{Ad}(G)^{\mathbb{C}}}(\mathrm{Ad}(S)) \subset Z_{\mathrm{Ad}(G)^{\mathbb{C}}}(\mathrm{Ad}(S)^{\mathbb{C}})$. This implies that $g \in \mathrm{Ker}(\mathrm{Ad}) = Z_G(\mathfrak{g})$. □

Proof of Proposition 6.10. Assume first G is connected. Note that (E, Φ) is obtained by extending the stable $\mathrm{SL}(2,\mathbb{R})$-Higgs pair (V, φ) via the morphism ρ defined in (44); by Proposition 5.9, (E, Φ) is polystable. By Theorem 5.4, there exists a solution $h \in \Omega^0(X, V(\mathrm{SO}(2,\mathbb{C})/\mathrm{SO}(2)))$ (resp. $h' \in \Omega^0(X, E(H^{\mathbb{C}}/H))$) to the Hitchin equations (22) for $\alpha = 0$ and group $\mathrm{SL}(2,\mathbb{R})$ (resp. G). Let A (resp. A') be the corresponding Chern connection for the given holomorphic structure of V (resp. E). From the proof of Proposition 5.9, we may assume that $A' = \rho'(A)$. Locally, write

$$A = d + M_A,$$

where $M_A \in \Omega^1(X, \mathfrak{so}(2))$. Then M_A is generically non-zero, as otherwise $L^{1/2}$ would be flat, which by assumption (52) is not the case. Now, an automorphism g of (A', Φ) satisfies that for each $x \in X$,

$$\mathrm{Ad}_{g_x} \rho'(M_{A,x}) = \rho'(M_{A,x})$$

and

$$\mathrm{Ad}_{g_x} \Phi_x = \Phi_x.$$

Since for generic x, $M_{A,x}$ and φ_x generate $\mathfrak{sl}(2,\mathbb{C})$, it follows that g_x must centralise $\rho'(\mathfrak{sl}(2,\mathbb{C})) = \mathfrak{s}^{\mathbb{C}}$. In particular, g_x centralises the subgroup $S = \rho(e^{\mathfrak{so}(2)} e^{\mathfrak{sym}_0(2,\mathbb{R})})$.

By Lemma 6.11, we have that $g_x \in H \cap Z_G(\mathfrak{g}) = Z_H(\mathfrak{g})$. Now, by closedness of $Z_H(\mathfrak{g})$ inside H, it follows that $g_x \in Z_H(\mathfrak{g}) \cap \mathrm{Ker}(\iota)$ for arbitrary $x \in X$. Thus

$$\mathfrak{aut}(A', \Phi) \subseteq \mathfrak{z}_{\mathfrak{h}}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{h}) \cap \mathrm{Ker}(d\iota) \subset \mathfrak{aut}(A', \Phi),$$

so (A', Φ) is infinitesimally irreducible, and by Proposition 5.17 (E, Φ) is stable.

When $Z(G) = Z_G(\mathfrak{g})$, then $g_x \in H \cap Z(G) = Z_H(G) = Z(H) \cap \mathrm{Ker}(\iota)$, and we also have $\mathrm{Aut}(A', \Phi) = Z(H) \cap \mathrm{Ker}(\iota)$. That is, (A', Φ) is irreducible, and so (E, Φ) is stable and simple by Proposition 5.17.

As for disconnected groups, we note that the basic G-Higgs bundle (E, Φ) reduces its structure group to G^0, the component of the identity in G. Let (E^0, Φ^0) be the G^0-Higgs bundle whose extension is (E, Φ). By the previous discussion, (E^0, Φ^0) is stable, and by Proposition 5.9, (E, Φ) is polystable. Assume $\sigma \in \Gamma(X, E(H^{\mathbb{C}}/P_s))$ is a reduction of the structure group to a parabolic subgroup $P_s \subset H$ violating the stability condition, namely, $\deg E(s, \sigma) > B(\alpha, s)$. We claim that σ induces a reduction $\sigma' \in \Gamma(X, E^0(H^{\mathbb{C}}/P_s \cap H_0^{\mathbb{C}}))$. Indeed, let $\sigma_\alpha(x) = (x, h_\alpha(x)P_s)$ be the expression of σ on a trivialising neighbourhood U_α. Then, on $U\alpha \cap U_\beta$, $\sigma_\beta(x) = (x, g_{\alpha\beta}(x)h_\alpha(x)P_s)$, where $g_{\alpha\beta} : U\alpha \cap U_\beta \to H_0^{\mathbb{C}}$ are the transition functions of $E = E^0(H^{\mathbb{C}})$. Then, we readily check that $\sigma'_\alpha(x) = (x, h_\alpha(x)P_s \cap H_0^{\mathbb{C}})$ is well defined, as $h_\alpha^{-1} g_{\alpha\beta} h_\alpha \in H_0^{\mathbb{C}}$. So we obtain a principal $P_s \cap H_0^{\mathbb{C}}$-bundle E_s such that $E_s(H^{\mathbb{C}}) = E$. Since $P_s \cap H_0^{\mathbb{C}} \subset H_0^{\mathbb{C}}$, also $E' = E_s(H_0^{\mathbb{C}})$. Let $\sigma^0 \in \Gamma(X, E^0(H_0^{\mathbb{C}}/P_s \cap H_0^{\mathbb{C}}))$ be the corresponding reduction of the structure group. We need to check that $\deg E(s, \sigma) = \deg E^0(s, \sigma^0)$, which is easily seen using the definition of the degree given in (21). This contradicts stability of (E, Φ).

Concerning simplicity, Lemma 6.11 applies just as in the connected case. □

Proposition 6.12. *If G is a strongly reductive Lie group and (E, φ) is the basic G-Higgs bundle as defined in* (55) *and* (56)*, then $\mathbb{H}^2(C^\bullet(E, \Phi)) = H^1(X, \mathfrak{z}_{\mathfrak{m}} \otimes L)$.*

Proof. First note that $S \hookrightarrow G$ factors through $S \hookrightarrow G_{ss}$. Let (E_{ss}, φ_{ss}) be the corresponding G_{ss} bundle. Then

$$E(\mathfrak{h}^{\mathbb{C}}) \cong E_{ss}(\mathfrak{h}_{ss}^{\mathbb{C}}) \oplus X \times \mathfrak{z}_{\mathfrak{h}},$$

where $\mathfrak{z}_{\mathfrak{h}} = \mathfrak{h}^{\mathbb{C}} \cap \mathfrak{z}(\mathfrak{g}^{\mathbb{C}})$ and $\mathfrak{h}_{ss}^{\mathbb{C}} = (\mathfrak{h} \cap \mathfrak{g}_{ss})^{\mathbb{C}}$. Likewise, $E(\mathfrak{m}^{\mathbb{C}}) \cong E_{ss}(\mathfrak{m}_{ss}^{\mathbb{C}}) \oplus X \times \mathfrak{z}_{\mathfrak{m}}$.

So the exact sequence (28) has the form:

$$\begin{aligned}\mathbb{H}^0(C^\bullet) &\hookrightarrow H^0(E_{ss}(\mathfrak{h}_{ss}^{\mathbb{C}})) \oplus H^0(\mathfrak{z}_{\mathfrak{h}}) \to H^0(E_{ss}(\mathfrak{m}_{ss}^{\mathbb{C}}) \otimes L) \oplus H^0(\mathfrak{z}_{\mathfrak{m}} \otimes L)\\ &\to \mathbb{H}^1(C^\bullet) \to H^1(E_{ss}(\mathfrak{h}_{ss}^{\mathbb{C}})) \oplus H^1(\mathfrak{z}_{\mathfrak{h}}) \to H^1(E_{ss}(\mathfrak{m}_{ss}^{\mathbb{C}}) \otimes L) \oplus H^1(\mathfrak{z}_{\mathfrak{m}} \otimes L)\\ &\to \mathbb{H}^2(C^\bullet) \to 0.\end{aligned}$$

Moreover,

$$[\varphi, E(\mathfrak{h}^{\mathbb{C}})] = [\varphi_{ss}, E_{ss}(\mathfrak{h}_{ss}^{\mathbb{C}})] \subset E_{ss}(\mathfrak{m}_{ss}^{\mathbb{C}}),$$

which implies that $H^{i-1}(\mathfrak{z}_{\mathfrak{m}} \otimes L) \hookrightarrow \mathbb{H}^i(C^\bullet)$, and thus

$$\mathbb{H}^2(C^\bullet) = \mathbb{H}^2(\mathrm{Ad}(C^\bullet)) \oplus H^1(X, \mathfrak{z}_{\mathfrak{m}} \otimes L).$$

With the notation of Proposition 5.13 we just need to prove that if $d_L \geq 2(g-1)$, then

$$[\varphi_{ss}, H^1(X, E_{ss}(\mathfrak{h}_{ss}^{\mathbb{C}}))] = H^1(X, E_{ss}(\mathfrak{m}_{ss}^{\mathbb{C}}) \otimes L).$$

By (57), we have

(59)
$$E_{ss}(\mathfrak{h}_{ss}^{\mathbb{C}} \cap M_k) = \begin{cases} L^{m_k-1} \oplus L^{m_k-3} \oplus \cdots \oplus L^{j_k} \oplus \cdots \oplus L^{-m_k+1} & \text{if } e_k \in \mathfrak{h}^{\mathbb{C}}, \\ L^{m_k-2} \oplus L^{m_k-4} \oplus \cdots \oplus L^{l_k} \oplus \cdots \oplus L^{-m_k+2} & \text{if } e_k \in \mathfrak{m}^{\mathbb{C}}, \end{cases}$$

where $j_k = 0$ if $m_k - 1 \equiv 0(2)$, $j_k = 1$ if $m_k - 1 \equiv 1(2)$, and $l_k = j_k + 1(2)$. In a similar way, we see that

(60)
$$E_{ss}(\mathfrak{m}_{ss}^{\mathbb{C}} \cap M_k) = \begin{cases} L^{m_k-2} \oplus L^{m_k-4} \oplus \cdots \oplus L^{j_k+1(2)} \oplus \cdots \oplus L^{-m_k+2} & \text{if } e_k \in \mathfrak{h}^{\mathbb{C}}, \\ L^{m_k-1} \oplus L^{m_k-3} \oplus \cdots \oplus L^{j_k} \oplus \cdots \oplus L^{-m_k+1} & \text{if } e_k \in \mathfrak{m}^{\mathbb{C}}. \end{cases}$$

Now, by definition

$$\Phi : L^j \mapsto \begin{cases} L^{j-1} \otimes L = L^j & \text{if } j > -m_k + 1,\ m_k \neq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{61}$$

Hence

$$[\Phi, H^1(E_{ss}(\mathfrak{h}_{ss}^{\mathbb{C}}) \cap M_k)] = \begin{cases} H^1(L^{m_k-1} \oplus \cdots \oplus L^{j_k} \oplus \cdots \oplus L^{-m_k+3}) & \text{if } e_k \in \mathfrak{h}^{\mathbb{C}}, \\ H^1(L^{m_k-2} \oplus L^{m_k-4} \oplus \cdots \oplus L^{-m_k+2}) & \text{if } e_k \in \mathfrak{m}^{\mathbb{C}}. \end{cases}$$

We thus have

$$\operatorname{Ker}(\operatorname{ad}(\Phi)) = \begin{cases} H^1(L^{-m_k+1}) & \text{if } e_k \in \mathfrak{h}^{\mathbb{C}}, \\ 0 & \text{if } e_k \in \mathfrak{m}^{\mathbb{C}}, \end{cases}$$

which implies that

$$\operatorname{Coker}(\operatorname{ad}(\Phi)) = \begin{cases} 0 & \text{if } e_k \in \mathfrak{h}^{\mathbb{C}}, \\ H^1(L^{m_k-1}) & \text{if } e_k \in \mathfrak{m}^{\mathbb{C}}. \end{cases}$$

If G is quasi-split, given that $m_k > 1$ (as we are only considering the semisimple part), we have $h^1(L^{m_k-1}) = h^0(L^{-m_k+1}K) = 0$, and thus $\mathbb{H}^2(\operatorname{Ad}(C^\bullet)) = 0$, which proves the statement.

If G is not quasi-split, the only thing that is different is the fact that the trivial representation $\mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}^{\mathbb{C}})$ has $m_1 = 1$ and positive multiplicity n_1 by Lemma 4.13. Therefore, $H^1(X, L \otimes \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(\mathfrak{s}^{\mathbb{C}})) \hookrightarrow \mathbb{H}^2(\operatorname{Ad}(C^\bullet))$. But $\mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(\mathfrak{s}^{\mathbb{C}}) = 0$ by (1) in Lemma 4.13. □

6.4. **Construction of the section.** We now have all the ingredients to yield to the Hitchin–Kostant–Rallis section. Let us recall some of the notation before stating the theorem. Let $\rho' : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}$ be the homomorphism defining the principal normal TDS $\mathfrak{s} \subset \mathfrak{g}$ (see (15)). Consider the group Q satisfying $(\operatorname{Ad}(G)^{\mathbb{C}})^\theta = Q\operatorname{Ad}(H^{\mathbb{C}})$ (see Proposition 3.21 for other characterizations). It is a finite group whose cardinality we denote by N.

Theorem 6.13. *Let (G, H, θ, B) be a strongly reductive Lie group, and let $(\widehat{G}_0, \widehat{H}_0, \widehat{\theta}, \widehat{B})$ be its maximal connected split subgroup. Let $L \to X$ be a line bundle with degree $d_L \geq 2g - 2$. Let $\alpha \in i\mathfrak{z}(\mathfrak{so}(2))$ be such that $\rho'(\alpha) \in \mathfrak{z}(\mathfrak{h})$. Then, the choice of a square root of L determines N inequivalent sections of the map*

$$h_L : \mathcal{M}_L^{\rho'(\alpha)}(G) \to B_L(G).$$

Each such section s_G satisfies the following:

1. *If G is quasi-split, $s_G(B_L(G))$ is contained in the stable locus of $\mathcal{M}_L^{\rho'(\alpha)}(G)$, and in the smooth locus if $Z(G) = Z_G(\mathfrak{g})$ and $d_L \geq 2g - 2$.*
2. *If G is not quasi-split, the image of the section is contained in the strictly polystable locus.*

(3) *For arbitrary groups, the Higgs field is everywhere regular.*
(4) *If* $\rho'(\alpha) \in i\mathfrak{z}\left(\widehat{\mathfrak{h}}\right)$, *the section factors through* $\mathcal{M}_L^{\rho'(\alpha)}(\widehat{G}_0)$. *This is in particular the case if* $\alpha = 0$.
(5) *If* $G_{\text{split}} < G^{\mathbb{C}}$ *is the split real form satisfying* $\mathfrak{z}(\mathfrak{g}^{\mathbb{C}}) \cap i\mathfrak{u} \subset \mathfrak{m}$, $K = L$, *and* $\alpha = 0$, *then* s_G *is the factorization of the Hitchin section through* $\mathcal{M}(G_{\text{split}})$.

Proof. The proof consists of three parts: firstly, we construct a section into $\mathcal{M}_L^0(G)$ for quasi-split real forms. This in particular includes the split group case. Secondly, using the maximal split subgroup, we are able to extend the section to $\mathcal{M}_L^0(G)$ for all groups. A third part deals with stability for other values of the parameter.

(1) Quasi-split groups. To start with, we note that the deformation argument used by Hitchin in [24] adapts to the case of quasi-split groups: for each $\overline{\gamma} \in \bigoplus_{i=1}^a H^0(X, L^{m_i})$, define the field

$$\Phi_{\overline{\gamma}} = f + \sum_{i=1}^{a} \gamma_i e_i,$$

where $e_i, i = 1, \dots, a$, generate $\mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$ and $e_1 = e$. Note that this is a well-defined section of $E(\mathfrak{m}^{\mathbb{C}} \otimes L)$ by (58).

Now, any family of Higgs bundles containing a stable point automatically contains a dense open set of stable points. In particular, by Proposition 6.10, (E, Φ) is 0-stable, so for sufficiently small γ_i's, we have that (E, Φ_γ) is 0-stable. Namely, the basic solution (E, Φ) can be deformed to a section from an open neighbourhood of $0 \in B_{G,L}$ into $\mathcal{M}_L^0(G)^{stable}$. Next, note that exponentiation of x produces an automorphism of E and $E(\mathfrak{m}^{\mathbb{C}}) \otimes L$ sending Φ_γ to

$$\Psi_\gamma = \mu^{-1} f_1 + \gamma_1 \mu^{m_1} e_1 + \cdots + \gamma_a \mu^{m_a} e_a.$$

That is, the automorphism transforms the family corresponding to (E, Φ) into the family corresponding to $(E, \mu^{-1}\Phi)$. The same arguments apply to the latter bundle, so that for sufficiently small $\mu^{m_i}\gamma_i$, Ψ_γ is stable. So every element of the family can be identified to one with small γ_i, as $m_i > 0$ by (17). Since gauge transformations preserve stability, we are done. Furthermore, by Propositions 6.12 and 5.13, if $Z(G) = Z_G(\mathfrak{g})$, the points in the image of the section are smooth.

For moduli spaces depending on an arbitrary parameter, we note that the hypotheses on the parameter and equation (48) imply that for $\alpha \neq 0$, $\mathcal{M}_L^0(\mathrm{SL}(2,\mathbb{R})) \subset \mathcal{M}_L^\alpha(\mathrm{SL}(2,\mathbb{R}))$, and stability is preserved. Since (E, Φ) is the extended G-Higgs bundle of (V, φ) via ρ (cf. Definition 5.8 and equation (44)), polystability is automatic for any $\rho'(\alpha)$ such that (V, φ) is α' stable, where $\rho' = d\rho$ is as in (15). Hence, we have $(E, \Phi) \in \mathcal{M}_L^{\rho'(\alpha)}(G)$ for all $i\alpha \leq 0$. Namely, for all $s \in i\mathfrak{h}$ and all $\sigma \in \Omega^0(X, E(H^{\mathbb{C}}/P_s))$ satisfying conditions in Definition 5.3, we have

$$\deg(E(s, \sigma)) \geq B(\rho'(\alpha), s).$$

Now, $B(\rho'(\alpha), s) = i\alpha B(\rho'(i), s)$, which, given that B is definite positive on $i\mathfrak{h}$, means that $i\alpha B(\rho'(i), s) \leq 0$. But 0-stability of (E, Φ) implies that

$$\deg(E(s, \sigma)) > 0 \geq B(\rho'(\alpha), s),$$

whence stability follows.

(2) Non-quasi-split groups. By (1), the elements in the image of the Hitchin–Kostant–Rallis section for the split subgroup are 0-stable, as split groups are quasi-split. So Corollary 5.10 and Theorem 4.9 imply the existence of a 0-polystable section for any group. Strict polystability follows from Proposition 2.14 in [18] and Corollary 4.15.

Points (3) and (4) follow by construction. For (5), we just note that from Definition 4.8, the principal normal TDS is in particular a TDS in the usual sense [28], so the construction matches Hitchin's as long as the rings of invariants $\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}}$ and $\mathbb{C}[\mathfrak{g}^{\mathbb{C}}]^{G^{\mathbb{C}}}$ match. This is guaranteed for a split subgroup satisfying the hypotheses of statement (5) above (see Remark 4.5). Such a split form always exists as the maximal connected split subgroup for a choice of Cartan data $(G^{\mathbb{C}}, U, \tau, B))$ satisfies the conditions.

Concerning the number of sections, the construction depends on a choice of principal normal TDS. By Theorem 6 in [29], all such are $(\mathrm{Ad}(H)_\theta)^{\mathbb{C}}$ conjugate, and by Proposition 3.21, the number of non-conjugate $H^{\mathbb{C}}$-orbits is determined by $\#Q$.

Finally, regularity follows from Theorem 4.9. □

Remark 6.14. The Hitchin–Kostant–Rallis section is a section in the sense that for a given choice of homogeneous generators $\{p_1, \dots, p_a\} \subset \mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}}$, the map

$$(p_1, \dots, p_a) \circ s_G : B_L(G) \to B_L(G)$$

is the identity. This follows from Theorem 7 in [28].

Remark 6.15 (Degree zero twisting). When $d_L = 0$, there are two cases to consider:

(1) Trivial bundle: if $L = \mathcal{O}_X$, the existence and construction of the section amounts to the results in [29]. Indeed, the Hitchin base

$$B_{G,\mathcal{O}} = H^0(X, \mathcal{O} \otimes \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})) \cong \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}}).$$

On the other hand, by (58), $e_i \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}))$. Thus everything follows from [29], modulo the choice of a square root of $\mathcal{O}$, i.e., an order two point of $Jac(X)$.

(2) Non-trivial bundle: this is a trivial case, as $B_{G,L} = 0$.

Propositions 5.13, 6.10, and 6.12 and Theorem 6.13 yield the following.

Corollary 6.16. *Let G be a strongly reductive quasi-split group. Assume that $Z(G) = Z_G(\mathfrak{g})$ and $d_L \geq 2g - 2$. Then the G-Higgs bundle (E, Φ) defined by* (55) *and* (56) *is a smooth point of $\mathcal{M}_L^0(G)$. In particular, for any quasi-split group, the associated* $\mathrm{Ad}(G)$*-Higgs bundle $(E([H]^{\mathbb{C}}), [\varphi])$ is a smooth point of $\mathcal{M}_L(\mathrm{Ad}(G))$, where $[H] = H/Z(G) \cap H$, $[\varphi] = \varphi/_{\mathfrak{z}_{\mathfrak{m}}}$.*

Proposition 6.17. *Let $\alpha \in i\mathfrak{z}(\mathfrak{h})$ be such that the basic G-Higgs bundle (E, Φ) is α-stable. Let $a = \dim \mathfrak{a}^{\mathbb{C}}$, $b = \dim \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$ and $c = \dim \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{s}^{\mathbb{C}})$, where $\mathfrak{s}^{\mathbb{C}}$ is a normal principal TDS. Let G be a strongly reductive Lie group. Then the expected dimension of the irreducible component of the moduli space $\mathcal{M}_L^\alpha(G)$ containing the image of the HKR section is*

$$\exp.\dim\left(\mathcal{M}_L^\alpha(G)\right) = c + h^1(\mathfrak{z}_{\mathfrak{m}} \otimes L) + \frac{d_L}{2}\dim \mathfrak{g}^{\mathbb{C}} + (a-b)\left(\frac{d_L}{2} - g + 1\right). \tag{62}$$

In particular, if G is quasi-split and $\deg L \geq 2g - 2$, the expected dimension is the actual dimension of the moduli space.

Proof. Letting $\mathbf{h}^i = \dim \mathbb{H}^i(X, C^\bullet)$, the expected dimension is $\mathbf{h}^1$. From the long exact sequence (28), we have

$$\mathbf{h}^1 = \chi(E(\mathfrak{m}^{\mathbb{C}}) \otimes L) - \chi(E(\mathfrak{h}^{\mathbb{C}})) + \mathbf{h}^0 + \mathbf{h}^2. \tag{63}$$

By (59)

$$\chi(E(\mathfrak{h}^{\mathbb{C}})) = \sum_{e_k \in \mathfrak{h}^{\mathbb{C}}} \sum_{0 \le j \le m_k - 2} \chi\left(L^{m_k - 2j - 1}\right) + \sum_{e_k \in \mathfrak{m}^{\mathbb{C}}} \sum_{1 \le j \le m_k - 1} \chi\left(L^{m_k - 2j}\right).$$

Similarly

$$\chi(E(\mathfrak{m}^{\mathbb{C}}) \otimes L) = \sum_{e_k \in \mathfrak{h}^{\mathbb{C}}} \sum_{0 \le i \le m_k - 2} \chi\left(L^{m_k - 2i - 1}\right) + \sum_{e_k \in \mathfrak{m}^{\mathbb{C}}} \sum_{0 \le i \le m_k - 1} \chi\left(L^{m_k - 2i}\right).$$

Also, by Proposition 6.12 $\mathbf{h}^2 = h^1(\mathfrak{z}_{\mathfrak{m}} \otimes L)$. On the other hand, we easily deduce from (61) that $\mathbf{h}^0 = \dim \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{s}^{\mathbb{C}})$. Substituting it all into (63) and applying Riemann–Roch yields

$$\mathbf{h}^1 = c + h^1(\mathfrak{z}_{\mathfrak{m}} \otimes L) + \sum_{e_k \in \mathfrak{m}^{\mathbb{C}}} (d_L m_k - g + 1) + \sum_{e_k \in \mathfrak{h}^{\mathbb{C}}} ((m_k - 1) d_L + g - 1).$$

Using (17), we obtain

$$\begin{aligned} \mathbf{h}^1 &= c + h^1(\mathfrak{z}_{\mathfrak{m}} \otimes L) + \sum_{e_k \in \mathfrak{m}^{\mathbb{C}}} \left(\frac{d_L}{2}(2m_k - 1) + \frac{d_L}{2} - g + 1 \right) \\ &+ \sum_{e_k \in \mathfrak{h}^{\mathbb{C}}} \left((2m_k - 1)\frac{d_L}{2} - \frac{d_L}{2} + g - 1 \right) \\ &= c + h^1(\mathfrak{z}_{\mathfrak{m}} \otimes L) + \frac{d_L}{2} \dim \mathfrak{g}^{\mathbb{C}} + (a - b)\left(\frac{d_L}{2} - g + 1 \right). \end{aligned}$$

This yields the result about the expected dimension.

The last assertion follows fom Corollary 6.16. □

Remark 6.18. We can give the expected dimension of the moduli space $\mathcal{M}_L(G)$. Indeed, let (E, φ) be a smooth point. This implies that $\mathbf{h}^0 = \dim \mathfrak{z}_{\mathfrak{h}}$ and $\mathbf{h}^2 = h^1(\mathfrak{z}_{\mathfrak{m}} \otimes L)$. On the other hand, by Hirzebruch–Riemann–Roch, we have

$$\chi(E(\mathfrak{h}^{\mathbb{C}})) = c_1(E(\mathfrak{h}^{\mathbb{C}})) + \dim(\mathfrak{h}^{\mathbb{C}})(1 - g).$$

Now, $c_1(\mathfrak{h}^{\mathbb{C}}) = 0$, as $\mathfrak{h}^{\mathbb{C}} \cong_{B_{\mathbb{C}}} (\mathfrak{h}^{\mathbb{C}})^\wedge$. Likewise, $\chi(E(\mathfrak{m}^{\mathbb{C}}) \otimes L) = c_1(E(\mathfrak{m}^{\mathbb{C}}) \otimes L) + \dim(\mathfrak{m}^{\mathbb{C}})(1 - g)$. Since $c_1(E(\mathfrak{h}^{\mathbb{C}})) + c_1(E(\mathfrak{m}^{\mathbb{C}})) = 0 = c_1(E(\mathfrak{g}^{\mathbb{C}}))$, then $\chi(E(\mathfrak{m}^{\mathbb{C}}) \otimes L) = \dim(\mathfrak{m}^{\mathbb{C}})(d_L + 1 - g)$. So altogether

$$\mathbf{h}^1 = \dim \mathfrak{z}_{\mathfrak{h}} + h^1(\mathfrak{z}_{\mathfrak{m}} \otimes L) + \dim(\mathfrak{m}^{\mathbb{C}})(d_L + 1 - g) - \dim(\mathfrak{h}^{\mathbb{C}})(1 - g).$$

Expressing $\dim(\mathfrak{m}^{\mathbb{C}})(d_L + 1 - g) = (\frac{d_L}{2} + \frac{d_L}{2} + 1 - g)$ and $\dim(\mathfrak{h}^{\mathbb{C}})(1 - g) = \dim(\mathfrak{h}^{\mathbb{C}})(-\frac{d_L}{2}\frac{d_L}{2} + 1 - g)$ we have

$$\mathbf{h}^1 = \dim \mathfrak{z}_{\mathfrak{h}} + h^1(\mathfrak{z}_{\mathfrak{m}} \otimes L) + \dim(\mathfrak{g}^{\mathbb{C}})\frac{d_L}{2} + (\dim \mathfrak{m}^{\mathbb{C}} - \dim \mathfrak{h}^{\mathbb{C}})(\frac{d_L}{2} + 1 - g).$$

By Proposition 5 in [29], $\dim \mathfrak{m}^{\mathbb{C}} - \dim \mathfrak{h}^{\mathbb{C}} = a - b$, where a and b are as in Proposition 6.17. So in the particular example of quasi-split groups, the expected dimension of the component containing the image of the HKR section is the expected dimension of the moduli space (as it should be by smoothness). On the other hand, we find that for non-quasi-split groups, the singularities of the image of the HKR section are of orbifold origin.

Corollary 6.19. *Assume $L = K$. Then:*

(1) *If G is the real group underlying a complex reductive Lie group $U^{\mathbb{C}}$, then*
$$\exp.\dim \mathcal{M}_K(U^{\mathbb{C}}) = \dim \mathcal{M}_K(U^{\mathbb{C}}) = 2\left(\dim(U^{\mathbb{C}})(g-1) + \dim Z(U^{\mathbb{C}})\right).$$

(2) *If $G < G^{\mathbb{C}}$ is a real form of a complex reductive Lie group, then*
$$\exp.\dim \mathcal{M}_K(G) = \frac{1}{2}\dim \mathcal{M}_K(U^{\mathbb{C}}) + \dim \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}_{ss}}(\mathfrak{s}^{\mathbb{C}}).$$
Therefore, it matches the expected dimension of the moduli space if and only if G is quasi-split.

Proof. To see (1) first remark that $G < G^{\mathbb{C}} \times G^{\mathbb{C}}$ is quasi-split, and by Proposition 6.17 the expected dimension at any element in the HKR section is the actual dimension. So under the given hypotheses, $c = \dim Z(U^{\mathbb{C}}) = \dim \mathfrak{z}_{\mathfrak{m}}$, where the first equality follows from Lemma 4.13.

For (2), we note that $c = \dim \mathfrak{z}_{\mathfrak{h}} + \dim \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}_{ss}}(\mathfrak{s}^{\mathbb{C}})$ and that $\mathfrak{z}(\mathfrak{g}^{\mathbb{C}}) = \mathfrak{z}_{\mathfrak{h}} \oplus \mathfrak{z}_{\mathfrak{m}}$. □

Proposition 6.20. *Let G be a quasi-split Lie group. Let $L \to X$ be a holomorphic line bundle such that $d_L := \deg L \geq 2g - 2$. Then the HKR section covers a connected component of the moduli space of L-twisted Higgs bundles if and only if G is split.*

Under the above possible hypothesis, our construction yields $N \cdot 2^{2g}$ Hitchin components, where N is defined as in Theorem 6.13.

Proof. Since G is quasi-split, by Theorem 6.13, the image of the section defines a closed subspace contained in the smooth locus of the moduli space. Moreover, by construction, the image of the section is open whenever

$$\dim B_L(G) \geq \dim \mathcal{M}_L(G), \tag{64}$$

as it is an affine subset of a manifold of the right dimension (cf. Theorem 3.4 in [18]), and it is open as it is a family of stable elements parameterised by the Hitchin base. So we apply Propositions 6.17 and 6.2, noting that by quasi-splitness $c = \dim \mathfrak{z}_{\mathfrak{h}}$ (cf. Lemma 4.13(3)). Comparing dimensions, we obtain that (64) holds if and only if

$$\frac{d_L}{2}\left(\dim \widehat{\mathfrak{g}}^{\mathbb{C}} - \dim \mathfrak{g}^{\mathbb{C}}\right) + b\left(\frac{d_L}{2} - g + 1\right) - \dim \mathfrak{z}_{\mathfrak{h}} \geq 0. \tag{65}$$

Note that $\dim \widehat{\mathfrak{g}}^{\mathbb{C}} - \dim \mathfrak{g}^{\mathbb{C}} = -b - 2 \cdot (\#\Delta - \#\widehat{\Lambda}(\mathfrak{a}))$, where Δ denotes the set of roots and $\widehat{\Lambda}(\mathfrak{a})$ the set of reduced restricted roots defined in (5). It follows that (65) is equivalent to

$$-d_L\left(\#\Delta - \#\widehat{\Lambda}(\mathfrak{a})\right) - b(g-1) - \dim \mathfrak{z}_{\mathfrak{h}} \geq 0,$$

which is possible if and only if each of the (negative) terms vanishes. But this implies in particular that $b = 0$, so that $\mathfrak{g}$ must be split, and consequently all other terms vanish.

As for the statement concerning the number of sections, the factor N is the one appearing in Theorem 6.13. The remaining choices correspond to taking a square root of L. We could have also chosen such for L^{-1}, but the sections obtained this way are identified with the ones resulting from using $L^{1/2}$ by the action of $\mathrm{Ad}(H_\theta)$. A way to see this is by considering the section into $\mathcal{M}(\mathrm{Ad}(G))$ and complexifying them. The conclusion follows from Remark 6.7, together with Proposition 3.21 and Lemma 5.11. The same reasoning implies inequivalence of the $N \cdot 2^{2g}$ sections. □

6.5. **Regularity.** Regularity of the Higgs field is directly related to smoothness of points in the Hitchin fiber. This essentially goes back to Kostant's [28], as it is proved by Biswas and Ramanan for complex Lie groups ([6], Theorem 5.9). Their proof applies to the real case, so we have the following.

Proposition 6.21. *Let $\omega \in B_L(G)$, and assume $(E,\varphi) \in \mathcal{M}(G) \cap h_G^{-1}(\omega)^{smooth}$ is a smooth point of $h_G^{-1}(\omega)$. Then $\varphi(x) \in \mathfrak{m}^{\mathbb{C}}{}_{reg}$ for all $x \in X$.*

Proof. Fixing $x \in X$, we have that $\mathrm{ev}_x \circ h(E,\varphi) = \chi\varphi_x$, where $\chi : \mathfrak{m}^{\mathbb{C}} \to \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a})$ is the Chevalley map. At a smooth point of the fiber, dh is surjective, and since ev_x is surjective too, it follows that $d(\chi \circ \mathrm{ev}_x)$ is itself surjective. Since $d\mathrm{ev}_x : H^0(X, E(\mathfrak{m}^{\mathbb{C}} \otimes K)) \to \mathfrak{m}^{\mathbb{C}} \otimes K_x$ is surjective and is itself an evaluation at x, this implies that $d_{\varphi_x}\chi$ is surjective. But Kostant–Rallis's work [29], citing Kostant [28], implies this happens if and only if φ_x is regular. □

7. Topological type of the elements in the image of the HKR section

Recall from Section 5.2 that to a Higgs bundle we can assign a topological invariant. We now come to the problem of determining the topological invariant of the component of the moduli space where the image of the HKR section falls in.

We remark that given a G-Higgs bundle (E,Φ), the topological type depends uniquely on E, so it is enough to compute the invariants for the principal bundle defined in (55). Moreover, by construction of the section, the type of E is independent of the value of $\alpha = 0$, as it is the principal bundle associated to some fixed $\mathrm{SO}(2,\mathbb{C})$ bundle.

Proposition 7.1. *Let $\mathcal{G}$ be a connected simply connected simple algebraic group over $\mathbb{R}$, let $\widehat{\mathcal{G}}$ be its maximal split subgroup, and let $G := \mathcal{G}(\mathbb{R})$, $\widehat{G} = \widehat{\mathcal{G}}(\mathbb{R})$ be the groups of their respective real points. Assume G is not of Hermitian type. Let E and $\widehat{E}$ be the principal bundles defined by (55) for the groups G and $\widehat{G}$ respectively. Then either $\pi_1(G) = 1$ (and then $d(E) = 0$) or $d(E) = d(\widehat{E}) \mod 2$.*

Proof. We observe that by simplicity, $\pi_1(G) = 1$, $\mathbb{Z}/2\mathbb{Z}$, or $\mathbb{Z}$, but the last option corresponds to Hermitian groups, so either $\pi_1(G) = 1$ or $\pi_1(G) = \mathbb{Z}/2\mathbb{Z}$. Likewise, $\pi_1(\widehat{G})$ is either $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$, as on the one hand $\widehat{G}$ is simple by construction, and on the other, split groups are never simply connected by Corollary 1.2 in [1]. We will prove that the map $i_* : \pi_1(\widehat{G}) \twoheadrightarrow \pi_1(G)$ (induced by the inclusion i) is surjective, which implies the statement. Indeed, the only homomorphisms $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ are constant or reduction modulo two, and similarly for $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$.

By Proposition 2.10 in [1],

$$\pi_1(G) = \mathbb{Z}\langle \Delta_{nc}^\vee(\mathfrak{g}^{\mathbb{C}}, \mathfrak{d}_c^{\mathbb{C}})\rangle / \Delta_{nc}^\vee(\mathfrak{g}^{\mathbb{C}}, \mathfrak{d}_c^{\mathbb{C}}) \cap \Delta^\vee(\mathfrak{h}^{\mathbb{C}}, \mathfrak{d}_c^{\mathbb{C}} \cap \mathfrak{h}^{\mathbb{C}}),$$

where $\mathfrak{d}_c \subset \mathfrak{g}$ is a maximally compact Cartan subalgebra. Let $\beta^\vee \in \Delta^\vee(\mathfrak{g}^{\mathbb{C}}, \mathfrak{d}_c^{\mathbb{C}})$ be a generator. Using the Cayley transform, we may identify it with a real coroot

$\alpha^\vee \in \Delta^\vee(\mathfrak{g}^{\mathbb{C}}, \mathfrak{d}^{\mathbb{C}})$, where $\mathfrak{d} \subset \mathfrak{g}$ is a maximally split Cartan subalgebra. Since $\alpha \in \Delta_r$, $(\alpha|_{\mathfrak{a}})^\vee = \alpha^\vee$, so $i_*(\alpha|_{\mathfrak{a}})^\vee = \alpha^\vee$. In particular, the image of $(\alpha|_{\mathfrak{a}})^\vee$ in $\pi_1(\widehat{G})$ is non-trivial, and so i_* is surjective. □

Proposition 7.2. *Let G be a connected simple real Lie group of Hermitian type. Then, the topological invariant $d(E)$ corresponding to the Hitchin–Kostant–Rallis section for the moduli space of Higgs bundles is maximal if G is of tube type and zero if it is of non-tube type.*

Proof. First of all, by Proposition 5.5(2), maximality or vanishing are equivalent whether we consider T or d, so we will use them indistinctly. As discussed above, it is enough to determine the degree of E.

Let G be of tube type. Then, by Theorem 6.13, the Higgs field is regular at every point, and thus Proposition 5.5(1) implies maximality of the Toledo invariant.

Now, if G is of non-tube type, $\widehat{G}_0$ is not of Hermitian type unless its split rank is one or two. Indeed, the simple Lie algebras of Hermitian non-tube type are $\mathfrak{su}(p,q)$ with $p \neq q$, $\mathfrak{so}^*(4p+2)$, and $e_{6(-14)}$. The maximal split subalgebra of all of them is $\mathfrak{so}(rk_{\mathbb{R}}(\mathfrak{g}), rk_{\mathbb{R}}(\mathfrak{g})+1)$, which is not of Hermitian type whenever the real rank is higher than two (see Table 1).

Now, the basic G-Higgs bundle E is associated to the basic $\widehat{G}_0$-Higgs bundle by extension of the structure group. By Corollary 5.10, if G has rank at least three, the topological type is zero, as it is the image of a torsion group inside $\pi_1(G) = \mathbb{Z}$.

As for ranks 1 and 2, for Lie groups with Lie algebra $\mathfrak{su}(n,1)$ with $n > 1$, $\mathfrak{su}(n,2)$ with $n > 2$, and $\mathfrak{e}_{6(-14)}$, as well as simply connected Lie groups, the result follows from Corollary 3.31.

The only remaining groups are $\mathfrak{so}^*(6)$ and $\mathfrak{so}^*(10)$, of ranks 1 and 2 respectively, which are covered by Lemma 7.3 below. □

Lemma 7.3. *Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{so}^*(6)$ or $\mathfrak{so}^*(10)$. Then, if $\{e, f, x\}$ is a normal triple generating a principal normal TDS, then the semisimple element x decomposes as*

$$x = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right), \quad \text{where } tr(A) = tr(B) = 0.$$

Proof. Following [25], we realise the Lie algebra $\mathfrak{so}^*(2n)$ as the subalgebra of $\mathfrak{sl}(2n, \mathbb{C})$, whose elements satisfy

$$-\mathrm{Ad}(I_{n,n})^t \overline{A} = A, \quad -\mathrm{Ad}(J_{n,n})^t A = A,$$

where

$$I_{n,n} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \quad J_{n,n} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

We also have

$$\mathfrak{h}^{\mathbb{C}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -^t A \end{pmatrix} \,:\, A \in \mathfrak{gl}(n, \mathbb{C}) \right\},$$

$$\mathfrak{m}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \,:\, B, C \in \mathfrak{gl}(n, \mathbb{C}), B +^t B = 0 = C +^t C \right\}.$$

In particular,

$$\theta \begin{pmatrix} A & B \\ C & -^t A \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & -^t A \end{pmatrix}. \tag{66}$$

Now, with the same notation of Theorem 4.9, we can easily compute generators e_c, f_c, w for a principal normal TDS. From these, a normal triple is given by $e = \frac{-e_c+f_c+w}{2}$, $e = \frac{e_c-f_c+w}{2}$, $x = e_c + f_c$. So to have x, it is enough to compute e_c, as $f_c = \theta e_c$.

We start with $\mathfrak{so}^*(6)$. In this case e_c is a multiple of an eigenvector $y \in \mathfrak{so}^*(6)$ for

$$w = \left(\begin{array}{c|c} 0 & A \\ \hline -A & 0 \end{array}\right), \qquad A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

By setting $[w, y] = y$, we obtain

$$y = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{array}\right).$$

Then, since both diagonal blocks of y have zero trace, so do the ones of $e_c = \lambda y$, and f_c by (66); hence the same holds for x.

As for $\mathfrak{so}^*(10)$, an element of the maximal anisotropic Cartan subalgebra has the form

$$w = \left(\begin{array}{c|c} 0 & A \\ \hline -A & 0 \end{array}\right) = ah_1 + bh_2, \qquad A = \begin{pmatrix} 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -b & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We compute y_i to be the eigenmatrix of y_i within $\mathfrak{so}^*(10)$. We see that

$$y_1 = \left(\begin{array}{c|c} C & D \\ \hline -D & C \end{array}\right), \qquad C = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 0 \end{pmatrix}.$$

As for y_2, $h_2 = Ah_1A^{-1}$ is obtained from h_1 by exchange of columns and rows $1 \leftrightarrow 2$, $4 \leftrightarrow 5$, $6 \leftrightarrow 7$, $9 \leftrightarrow 10$, so y_2 can be obtained from y_1 in the same way. We

readily check that

$$y_2 = \left(\begin{array}{c|c} E & F \\ \hline -F & E \end{array}\right), \qquad E = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

belong to $\mathfrak{so}^*(10)$, and so we are done, as $e_c = l_1 y_1 + l_2 y_2$ and the arguments used for the rank one case apply. □

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