

## ON CERTAIN $L$ -FUNCTIONS FOR DEFORMATIONS OF KNOT GROUP REPRESENTATIONS

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**ABSTRACT.** We study the twisted knot module for the universal deformation of an  $SL_2$ -representation of a knot group and introduce an associated  $L$ -function, which may be seen as an analogue of the algebraic  $p$ -adic  $L$ -function associated to the Selmer module for the universal deformation of a Galois representation. We then investigate two problems proposed by Mazur: Firstly we show the torsion property of the twisted knot module over the universal deformation ring under certain conditions. Secondly we compute the  $L$ -function by some concrete examples for 2-bridge knots.

### INTRODUCTION

It has been known that there are intriguing analogies between knot theory and number theory (cf. [Mo]). In particular, it may be noteworthy that there are close parallels between Alexander-Fox theory and Iwasawa theory ([Ma1], [Mo, Chs. 9–13]). From the viewpoint of deformations of group representations ([Ma2]), they are concerned with abelian deformations of representations of knot and Galois groups and the associated topological and arithmetic invariants such as the Alexander and Iwasawa polynomials, respectively. In [Ma3], Mazur proposed a number of problems in pursuing these analogies for non-abelian deformations of higher dimensional representations. To carry out Mazur's perspective, as a first step, we developed a deformation theory for  $SL_2$ -representations of knot groups in [MTTU]. In this paper, we continue our study and introduce a certain  $L$ -function associated to the twisted knot module for the universal deformation of a knot group representation, which may be seen as an analogue of the algebraic  $p$ -adic  $L$ -function associated to the Selmer module for the universal deformation of a Galois representation ([G]).

Let  $K$  be a knot in the 3-sphere  $S^3$  and let  $G_K := \pi_1(S^3 \setminus K)$  be the knot group. Fix a field  $k$  whose characteristic is not 2 and a complete discrete valuation ring  $\mathcal{O}$  whose residue field is  $k$ . Let  $\bar{\rho} : G_K \rightarrow SL_2(k)$  be a given absolutely irreducible representation. It was shown in [MTTU] that there exists the universal deformation  $\rho : G_K \rightarrow SL_2(\mathbf{R}_{\bar{\rho}})$  of  $\bar{\rho}$ , where  $\mathbf{R}_{\bar{\rho}}$  is a complete local  $\mathcal{O}$ -algebra whose residue field is  $k$ . Assume that  $\mathbf{R}_{\bar{\rho}}$  is a Noetherian factorial domain. In this paper, we study

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the twisted knot module  $H_1(\boldsymbol{\rho}) := H_1(S^3 \setminus K; \boldsymbol{\rho})$  with coefficients in the universal deformation  $\boldsymbol{\rho}$  and introduce the associated  $L$ -function  $L_K(\boldsymbol{\rho})$  defined on the universal deformation space  $\text{Spec}(\mathbf{R}_{\bar{p}})$  as  $\Delta_0(H_1(\boldsymbol{\rho}))$ , the greatest common divisor of generators of the initial Fitting ideal of  $H_1(\boldsymbol{\rho})$  over the universal deformation ring  $\mathbf{R}_{\bar{p}}$ . In terms of our  $H_1(\boldsymbol{\rho})$  and  $L_K(\boldsymbol{\rho})$ , we then formulate the problems proposed by Mazur (questions 1 and 2 of [Ma3, page 440]) as follows.

- (1) Is  $H_1(\boldsymbol{\rho})$  finitely generated and torsion as an  $\mathbf{R}_{\bar{p}}$ -module?
- (2) Investigate the order of the zeroes of  $L_K(\boldsymbol{\rho})$  at prime divisors of  $\text{Spec}(\mathbf{R}_{\bar{p}})$ .

The corresponding problems of (1) and (2) in the arithmetic counterpart, say  $(1)_{\text{arith}}$  and  $(2)_{\text{arith}}$  respectively, are important issues in number theory (cf. questions 1 and 2 of [Ma3, page 454]). In fact,  $(1)_{\text{arith}}$  is a part of the so-called main conjecture for  $p$ -adic deformations of a Galois representation. For the cyclotomic deformation of a Dirichlet character, the affirmative answer to  $(1)_{\text{arith}}$  is a basic result in Iwasawa theory ([I]), which asserts that the classical Iwasawa module is finitely generated and torsion over the Iwasawa algebra. For the Hida deformation (universal ordinary modular  $\text{GL}_2$ -deformation) ([H1], [H2]), the affirmative answer to  $(1)_{\text{arith}}$  has been shown by Kato and Ochiai ([Kt], [O1], [O2]), which asserts that the dual Selmer module of the Hida deformation is finitely generated and torsion over the universal ordinary modular deformation ring ( $p$ -adic Hecke algebra). The problem  $(2)_{\text{arith}}$  remains an interesting problem to be explored, and it is related to deep arithmetic issues (cf. question 3 of [Ma3, page 454], and Ribet's theorem on Herbrand's theorem for example [MW]).

So it may be interesting to study the above problems (1) and (2) in the knot theoretic situation. Our results concerning these are as follows. For (1), we give a criterion for  $H_1(\boldsymbol{\rho})$  to be finitely generated and torsion over  $\mathbf{R}_{\bar{p}}$  under certain conditions using a twisted Alexander invariant of  $K$  (cf. Theorem 3.2.4). For (2), we give some concrete examples for 2-bridge knots  $K$  such that  $L_K(\boldsymbol{\rho})$  has only one zero of order 0 or 2 (cf. Example 4.5).

Here are the contents of this paper. In Section 1, we recall the deformation theory for  $\text{SL}_2$ -representations of a group, which was developed in [MTTU]. In Section 2, we show the relation between the universal deformation ring and the character scheme over  $\mathbb{Z}$  of  $\text{SL}_2$ -representations. In Section 3, we study the twisted knot module with coefficients in the universal deformation of an  $\text{SL}_2$ -representation of a knot group and introduce an associated  $L$ -function. In Section 4, we discuss some examples for some 2-bridge knots, for which we study Mazur's problems.

*Notation.* For a local ring  $R$ , we denote by  $\mathfrak{m}_R$  the maximal ideal of  $R$ . For an integral domain  $A$ , we denote by  $\text{char}(A)$  the characteristic of  $A$  and by  $Q(A)$  the field of fractions of  $A$ . For  $a, b$  in a commutative ring  $A$ ,  $a \doteq b$  means  $a = bu$  for some unit  $u \in A^\times$ .

## 1. THE UNIVERSAL DEFORMATION

In this section, we present a summary of the deformation theory for  $\text{SL}_2$ -representations of a group, which was developed in [MTTU]. We also discuss the obstruction to the deformation problem for a group representation. Throughout this section, let  $G$  denote a group.

**1.1. Pseudo-representations and their deformations.** Let  $A$  be a commutative ring with identity. A map  $T : G \rightarrow A$  is called a *pseudo-SL<sub>2</sub>-representation* over  $A$  if the following four conditions are satisfied:

- (P1)  $T(e) = 2$  ( $e :=$  the identity element of  $G$ ),
- (P2)  $T(g_1g_2) = T(g_2g_1)$  for any  $g_1, g_2 \in G$ ,
- (P3)  $T(g_1)T(g_2)T(g_3) + T(g_1g_2g_3) + T(g_1g_3g_2) - T(g_1g_2)T(g_3) - T(g_2g_3)T(g_1) - T(g_1g_3)T(g_2) = 0$  for any  $g_1, g_2, g_3 \in G$ ,
- (P4)  $T(g)^2 - T(g^2) = 2$  for any  $g \in G$ .

Note that the conditions (P1)–(P3) are nothing but Taylor’s conditions for a pseudo-representation of degree 2 ([Ta]) and that (P4) is the condition for determinant 1. In the following, we say simply a *pseudo-representation* for a pseudo-SL<sub>2</sub>-representation. The trace  $\text{tr}(\rho)$  of a representation  $\rho : G \rightarrow \text{SL}_2(A)$  satisfies the conditions (P1)–(P4) ([Pr, Theorem 4.3]), and, conversely, a pseudo-SL<sub>2</sub>-representation is shown to be obtained as the trace of a representation under certain conditions (see Theorem 1.2.1 below).

Let  $k$  be a perfect field and let  $\mathcal{O}$  be a complete discrete valuation ring with the residue field  $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$ . There is a unique subgroup  $V$  of  $\mathcal{O}^\times$  such that  $k^\times \simeq V$  and  $\mathcal{O}^\times = V \times (1 + \mathfrak{m}_{\mathcal{O}})$ . The composition map  $\lambda : k^\times \simeq V \hookrightarrow \mathcal{O}^\times$  is called the *Teichmüller lift* which satisfies  $\lambda(\alpha) \bmod \mathfrak{m}_{\mathcal{O}} = \alpha$  for  $\alpha \in k$ . It is extended to  $\lambda : k \hookrightarrow \mathcal{O}$  by  $\lambda(0) := 0$ . Let  $\mathfrak{CL}_{\mathcal{O}}$  be the category of complete local  $\mathcal{O}$ -algebras with residue field  $k$ . A morphism in  $\mathfrak{CL}_{\mathcal{O}}$  is an  $\mathcal{O}$ -algebra homomorphism inducing the identity on residue fields.

Let  $\overline{T} : G \rightarrow k$  be a pseudo-representation over  $k$ . A pair  $(R, T)$  is called an *SL<sub>2</sub>-deformation* of  $\overline{T}$  if  $R \in \mathfrak{CL}_{\mathcal{O}}$  and  $T : G \rightarrow R$  is a pseudo-SL<sub>2</sub>-representation over  $R$  such that  $T \bmod \mathfrak{m}_R = \overline{T}$ . In the following, we say simply a *deformation* of  $\overline{T}$  for an SL<sub>2</sub>-deformation. A deformation  $(\mathbf{R}_{\overline{T}}, \mathbf{T})$  of  $\overline{T}$  is called a *universal deformation* if the following universal property is satisfied: “For any deformation  $(R, T)$  of  $\overline{T}$  there exists a unique morphism  $\psi : \mathbf{R}_{\overline{T}} \rightarrow R$  in  $\mathfrak{CL}_{\mathcal{O}}$  such that  $\psi \circ \mathbf{T} = T$ .” Namely, the correspondence  $\psi \mapsto \psi \circ \mathbf{T}$  gives the bijection

$$\text{Hom}_{\mathfrak{CL}_{\mathcal{O}}}(\mathbf{R}_{\overline{T}}, R) \simeq \{(R, T) \mid \text{deformation of } \overline{T}\}.$$

By the universal property, a universal deformation  $(\mathbf{R}_{\overline{T}}, \mathbf{T})$  of  $\overline{T}$  is unique (if it exists) up to isomorphism. The  $\mathcal{O}$ -algebra  $\mathbf{R}_{\overline{T}}$  is called the *universal deformation ring* of  $\overline{T}$ .

**Theorem 1.1.1** ([MTTU, Theorem 1.2.1]). *For a pseudo-representation  $\overline{T} : G \rightarrow k$ , there exists a universal deformation  $(\mathbf{R}_{\overline{T}}, \mathbf{T})$  of  $\overline{T}$ .*

We recall the construction of  $(\mathbf{R}_{\overline{T}}, \mathbf{T})$ . Let  $X_g$  denote a variable indexed by  $g \in G$ . Then the universal deformation ring  $\mathbf{R}_{\overline{T}}$  is given by

$$\mathbf{R}_{\overline{T}} = \mathcal{O}[[X_g \ (g \in G)]]/\mathcal{I},$$

where  $\mathcal{I}$  is the ideal of the formal power series ring  $\mathcal{O}[[X_g \ (g \in G)]]$  generated by the elements of the following type: Setting  $T_g := X_g + \lambda(\overline{T}(g))$ :

- (1)  $T_e - 2 = X_e + \lambda(\overline{T}(e)) - 2$ ,
- (2)  $T_{g_1g_2} - T_{g_2g_1} = X_{g_1g_2} - X_{g_2g_1}$ ,
- (3)  $T_{g_1}T_{g_2}T_{g_3} + T_{g_1g_2g_3} + T_{g_1g_3g_2} - T_{g_1g_2}T_{g_3} - T_{g_2g_3}T_{g_1} - T_{g_1g_3}T_{g_2}$ ,
- (4)  $T_g^2 - T_{g^2} - 2$ ,

for  $g, g_1, g_2, g_3 \in G$ . The universal deformation  $\mathbf{T} : G \rightarrow \mathbf{R}_{\overline{T}}$  is given by

$$\mathbf{T}(g) := T_g \text{ mod } \mathcal{I}.$$

Then, for any deformation  $(R, T)$  of  $\overline{T}$ , the morphism  $\psi : \mathbf{R}_{\overline{T}} \rightarrow R$  in  $\mathcal{CL}_{\mathcal{O}}$  defined by  $\psi(X_g) := T(g) - \lambda(\overline{T}(g))$  satisfies  $\psi \circ \mathbf{T} = T$ .

We note that  $\mathbf{R}_{\overline{T}}$  constructed above is a complete Noetherian local  $\mathcal{O}$ -algebra if  $G$  is a finitely generated group.

**1.2. Deformations of an  $\text{SL}_2$ -representation.** We keep the same notation as in Subsection 1.1. In this subsection we assume that  $\text{char}(k) \neq 2$ , so that 2 is invertible in  $\mathcal{O}$  and hence in any  $R \in \mathcal{CL}_{\mathcal{O}}$ . Let  $\overline{\rho} : G \rightarrow \text{SL}_2(k)$  be a given representation. We call a couple  $(R, \rho)$  an  $\text{SL}_2$ -deformation of  $\overline{\rho}$  if  $R \in \mathcal{CL}_{\mathcal{O}}$  and  $\rho : G \rightarrow \text{SL}_2(R)$  is a representation such that  $\rho \text{ mod } \mathfrak{m}_R = \overline{\rho}$ . In the following, we say simply a deformation of  $\overline{\rho}$  for an  $\text{SL}_2$ -deformation. A deformation  $(\mathbf{R}_{\overline{\rho}}, \boldsymbol{\rho})$  of  $\overline{\rho}$  is called a universal deformation of  $\overline{\rho}$  if the following universal property is satisfied: “For any deformation  $(R, \rho)$  of  $\overline{\rho}$  there exists a unique morphism  $\psi : \mathbf{R}_{\overline{\rho}} \rightarrow R$  in  $\mathcal{CL}_{\mathcal{O}}$  such that  $\psi \circ \boldsymbol{\rho} \approx \rho$ .” Here two representations  $\rho_1, \rho_2$  of degree 2 over a local ring  $A$  are said to be strictly equivalent, denoted by  $\rho_1 \approx \rho_2$ , if there is  $\gamma \in I_2 + \text{M}_2(\mathfrak{m}_A)$  such that  $\rho_2(g) = \gamma^{-1} \rho_1(g) \gamma$  for all  $g \in G$ . Namely, the correspondence  $\psi \mapsto \psi \circ \boldsymbol{\rho}$  gives the bijection

$$\text{Hom}_{\mathcal{CL}_{\mathcal{O}}}(\mathbf{R}_{\overline{\rho}}, R) \simeq \{(R, \rho) \mid \text{deformation of } \overline{\rho}\} / \approx.$$

By the universal property, a universal deformation  $(\mathbf{R}_{\overline{\rho}}, \boldsymbol{\rho})$  of  $\overline{\rho}$  is unique (if it exists) up to strict equivalence. The  $\mathcal{O}$ -algebra  $\mathbf{R}_{\overline{\rho}}$  is called the universal deformation ring of  $\overline{\rho}$ .

A deformation  $(R, \rho)$  of  $\overline{\rho}$  gives rise to a deformation  $(R, \text{tr}(\rho))$  of the pseudo-representation  $\text{tr}(\overline{\rho}) : G \rightarrow k$ . Assume that  $\overline{\rho}$  is absolutely irreducible, namely, the composite of  $\overline{\rho}$  with an inclusion  $\text{SL}_2(k) \hookrightarrow \text{SL}_2(\overline{k})$  is irreducible for an algebraic closure  $\overline{k}$  of  $k$ . Then, by using theorems of Caryaol [Ca, Theorem 1] and Nyssen [Ny, Theorem 1], it can be shown that this correspondence by the trace is indeed bijective. It is here that the condition  $\text{char}(k) \neq 2$  is used.

**Theorem 1.2.1** ([MTTU, Theorem 2.1.2]). *Let  $\overline{\rho} : G \rightarrow \text{SL}_2(k)$  be an absolutely irreducible representation and let  $R \in \mathcal{CL}_{\mathcal{O}}$ . Then the correspondence  $\rho \mapsto \text{tr}(\rho)$  gives the following bijection:*

$$\begin{aligned} & \{\rho : G \rightarrow \text{SL}_2(R) \mid \text{deformation of } \overline{\rho} \text{ over } R\} / \approx \\ & \longrightarrow \{T : G \rightarrow R \mid \text{deformation of } \text{tr}(\overline{\rho}) \text{ over } R\}. \end{aligned}$$

Now, by Theorem 1.1.1, there exists the universal deformation  $(\mathbf{R}_{\overline{T}}, \mathbf{T})$  of a pseudo-representation  $\overline{T} = \text{tr}(\overline{\rho})$ . By Theorem 1.2.1, we have a deformation  $\boldsymbol{\rho} : G \rightarrow \text{SL}_2(\mathbf{R}_{\overline{T}})$  of  $\overline{\rho}$  such that  $\text{tr}(\boldsymbol{\rho}) = \mathbf{T}$ . Then we can verify that  $(\mathbf{R}_{\overline{T}}, \boldsymbol{\rho})$  satisfies the desired property of the universal deformation of  $\overline{\rho}$ .

**Theorem 1.2.2** ([MTTU, Theorem 2.2.2]). *Let  $\overline{\rho} : G \rightarrow \text{SL}_2(k)$  be an absolutely irreducible representation. Then there exists the universal deformation  $(\mathbf{R}_{\overline{\rho}}, \boldsymbol{\rho})$  of  $\overline{\rho}$ , where  $\mathbf{R}_{\overline{\rho}}$  is given as  $\mathbf{R}_{\overline{T}}$  for  $\overline{T} := \text{tr}(\overline{\rho})$  in Theorem 1.1.1.*

**1.3. Obstructions.** We recall basic facts on a presentation of a complete local  $\mathcal{O}$ -algebra and the obstruction for the deformation problem. For  $R \in \mathcal{CL}_{\mathcal{O}}$ , we define the relative cotangent space  $\mathfrak{t}_{R/\mathcal{O}}^*$  of  $R$  by the  $k$ -vector space  $\mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_{\mathcal{O}}R)$  and the relative tangent space  $\mathfrak{t}_{R/\mathcal{O}}$  of  $R$  by the dual  $k$ -vector space of  $\mathfrak{t}_{R/\mathcal{O}}^*$ . We note

that they are the same as the cotangent and tangent spaces of  $R/\mathfrak{m}_{\mathcal{O}}R = R \otimes_{\mathcal{O}} k$ , respectively. The following lemma is a well-known fact which can be proved using Nakayama’s lemma (cf. [Ti, Lemma 5.1]).

**Lemma 1.3.1.** *Let  $d := \dim_k \mathfrak{t}_{R/\mathcal{O}}$  and assume  $d < \infty$ . Let  $x_1, \dots, x_d$  be elements of  $R$  whose images in  $R \otimes_{\mathcal{O}} k$  form a system of parameters of the local  $k$ -algebra  $R \otimes_{\mathcal{O}} k$ . Then there is a surjective  $\mathcal{O}$ -algebra homomorphism*

$$\eta : \mathcal{O}[[X_1, \dots, X_d]] \longrightarrow R$$

in  $\mathfrak{CL}_{\mathcal{O}}$  such that  $\eta(X_i) = x_i$  for  $1 \leq i \leq d$ .

Let  $\text{Ad}(\bar{\rho})$  be the  $k$ -vector space  $\mathfrak{sl}_2(k) := \{X \in \text{M}_2(k) \mid \text{tr}(X) = 0\}$  on which  $G$  acts by  $g.X := \bar{\rho}(g)X\bar{\rho}(g)^{-1}$  for  $g \in G$  and  $X \in \mathfrak{sl}_2(k)$ . It is well-known ([Ma2, §1.6]) that there is a canonical isomorphism between the relative cotangent space  $\mathfrak{t}_{\mathbf{R}_{\bar{\rho}}/\mathcal{O}}^*$  and the first group cohomology  $H^1(G, \text{Ad}(\bar{\rho}))$ . We say that the deformation problem for  $\bar{\rho}$  is *unobstructed* if the second cohomology  $H^2(G, \text{Ad}(\bar{\rho}))$  vanishes. The following proposition is also well-known.

**Proposition 1.3.2.** *Suppose that the deformation problem for  $\bar{\rho}$  is unobstructed and  $\dim_k H^1(G, \text{Ad}(\bar{\rho})) < \infty$ . Then the map  $\eta$  in Lemma 1.3.1 with  $R = \mathbf{R}_{\bar{\rho}}$  is isomorphic:*

$$\eta : \mathcal{O}[[X_1, \dots, X_d]] \xrightarrow{\sim} \mathbf{R}_{\bar{\rho}}.$$

In this paper, we are interested in the case that  $G$  is a knot group, namely, the fundamental group of the complement of a knot in the 3-sphere  $S^3$ . We note that the deformation problem is not unobstructed in general for a knot group representation  $\bar{\rho}$ , as shown in Subsection 2.3.

## 2. CHARACTER SCHEMES

In this section, we show the relation between the universal deformation ring in Section 1 and the character scheme of  $\text{SL}_2$ -representations.

In Subsection 2.1, we recall the constructions and some facts concerning the character scheme and the skein algebra over  $\mathbb{Z}$  and then describe their relation. For the details we consult [CS], [LM, Chapter 1], [Na], and [Sa]. In Subsection 2.2, via the skein algebra, we show that the universal deformation ring may be seen as an infinitesimal deformation of the character algebra. In Subsection 2.3, we show that the deformation problem is not unobstructed for a knot group in general, using Thurston’s result on the character variety.

**2.1. Character schemes and skein algebras over  $\mathbb{Z}$ .** Let  $G$  be a group. Let  $\mathcal{F}$  be the functor from the category  $\mathfrak{Com.Ring}$  of commutative rings with identity to the category of sets defined by

$$\mathcal{F}(A) := \{G \rightarrow \text{SL}_2(A) \mid \text{representation}\}$$

for  $A \in \mathfrak{Com.Ring}$ . The functor  $\mathcal{F}$  is represented by a pair  $(\mathcal{A}(G), \sigma_G)$ , where  $\mathcal{A}(G) \in \mathfrak{Com.Ring}$  and  $\sigma_G : G \rightarrow \text{SL}_2(\mathcal{A}(G))$  is a representation which satisfies the following universal property: “For any  $A \in \mathfrak{Com.Ring}$  and a representation  $\rho : G \rightarrow \text{SL}_2(A)$ , there is a unique morphism  $\psi : \mathcal{A}(G) \rightarrow A$  in  $\mathfrak{Com.Ring}$  such that  $\psi \circ \sigma_G = \rho$ .” Thus the correspondence  $\psi \mapsto \psi \circ \sigma_G$  gives the bijection

$$\text{Hom}_{\mathfrak{Com.Ring}}(\mathcal{A}(G), A) \simeq \{G \rightarrow \text{SL}_2(A) \mid \text{representation}\}.$$

By the universal property, the pair  $(\mathcal{A}(G), \sigma_G)$  is unique (if it exists) up to isomorphism. We call  $\mathcal{A}(G)$  the *universal representation algebra* over  $\mathbb{Z}$  and  $\sigma_G : G \rightarrow \mathrm{SL}_2(\mathcal{A}(G))$  the *universal representation*. The pair  $(\mathcal{A}(G), \sigma_G)$  is constructed as follows. Let  $X(g) = (X_{ij}(g))_{1 \leq i, j \leq 2}$  be a  $2 \times 2$  matrix whose entries  $X_{ij}(g)$ 's are variables indexed by  $1 \leq i, j \leq 2$  and  $g \in G$ . Then  $\mathcal{A}(G)$  is given as

$$\mathcal{A}(G) = \mathbb{Z}[X_{ij}(g) \ (1 \leq i, j \leq 2; g \in G)]/J,$$

where  $J$  is the ideal of the polynomial ring  $\mathbb{Z}[X_{ij}(g) \ (1 \leq i, j \leq 2; g \in G)]$  generated by

$$X_{ij}(e) - \delta_{ij}, \ X_{ij}(g_1 g_2) - \sum_{k=1}^2 X_{ik}(g_1) X_{kj}(g_2), \ \det(X(g)) - 1$$

for  $1 \leq i, j \leq 2$  and  $g \in G$ , and the representation  $\sigma_G : G \rightarrow \mathrm{SL}_2(\mathcal{A}(G))$  is given by

$$\sigma_G(g) := X(g) \bmod J \quad (g \in G).$$

We note that when  $G$  is presented by finitely many generators  $g_1, \dots, g_n$  subject to the relations  $r_l = 1 \ (l \in L)$ ,  $\mathcal{A}(G)$  is given by

$$\mathcal{A}(G) = \mathbb{Z}[X_{ij}(g_h) \ (1 \leq h \leq n, 1 \leq i, j \leq 2)]/J'$$

for the ideal  $J'$  generated by

$$r_l(X(g_1), \dots, X(g_n))_{ij} - \delta_{ij}, \ \det(X(g_h)) - 1,$$

where  $1 \leq i, j \leq 2, l \in L, 1 \leq h \leq n$  and  $r_l(X(g_1), \dots, X(g_n))_{ij}$  denotes the  $(i, j)$ -entry of  $r_l(X(g_1), \dots, X(g_n))$ . The universal representation  $\sigma_G$  is given by

$$\sigma_G(g_h) = X(g_h) \bmod J' \quad (1 \leq h \leq n).$$

So  $\mathcal{A}(G)$  is a finitely generated algebra over  $\mathbb{Z}$  if  $G$  is a finitely generated group. We denote by  $\mathcal{R}(G)$  the affine scheme  $\mathrm{Spec}(\mathcal{A}(G))$  and call it the *representation scheme* of  $G$  over  $\mathbb{Z}$ . So  $A$ -rational points of  $\mathcal{R}(G)$  correspond bijectively to representations  $G \rightarrow \mathrm{SL}_2(A)$  for any  $A \in \mathbf{Com.Ring}$ . For  $\mathfrak{p} \in \mathcal{R}(G)$ , we let  $\rho_{\mathfrak{p}} := \psi_{\mathfrak{p}} \circ \sigma_G : G \rightarrow \mathrm{SL}_2(\mathcal{A}(G)/\mathfrak{p})$  be the corresponding representation, where  $\psi_{\mathfrak{p}} : \mathcal{A}(G) \rightarrow \mathcal{A}(G)/\mathfrak{p}$  is the natural homomorphism.

We say that a representation  $\rho : G \rightarrow \mathrm{SL}_2(A)$  with  $A \in \mathbf{Com.Ring}$  is *absolutely irreducible* if the composite of  $\rho$  with the natural map  $\mathrm{SL}_2(A) \rightarrow \mathrm{SL}_2(k(\mathfrak{p}))$  is absolutely irreducible over the residue field  $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$  for any  $\mathfrak{p} \in \mathrm{Spec}(A)$ .

Let  $\mathrm{PGL}_2$  be the group scheme over  $\mathbb{Z}$  whose coordinate ring  $A(\mathrm{PGL}_2)$  is the subring of the graded ring  $\mathbb{Z}[Y_{ij} \ (1 \leq i, j \leq 2)]_{\det(Y)}$  consisting of homogeneous elements of degree 0, where the degree of  $Y_{ij}$  is 1. The adjoint action  $\mathrm{Ad} : \mathcal{R}(G) \times \mathrm{PGL}_2 \rightarrow \mathcal{R}(G)$  is given by the dual action

$$\mathrm{Ad}^* : \mathcal{A}(G) \longrightarrow \mathcal{A}(G) \otimes_{\mathbb{Z}} A(\mathrm{PGL}_2); \ X_{ij}(g) \mapsto (YX(g)Y^{-1})_{ij} \otimes Y_{kl},$$

where  $Y = (Y_{ij})_{1 \leq i, j \leq 2}$  and  $(YX(g)Y^{-1})_{ij}$  denotes the  $(i, j)$ -entry of  $YX(g)Y^{-1}$ . Let  $\mathcal{B}(G)$  be the invariant subalgebra of  $\mathcal{A}(G)$  under this action of  $\mathrm{PGL}_2$ :

$$\begin{aligned} \mathcal{B}(G) &:= \mathcal{A}(G)^{\mathrm{PGL}_2} \\ &:= \{x \in \mathcal{A}(G) \mid \mathrm{Ad}^*(x) = x \otimes 1\}. \end{aligned}$$

We call  $\mathcal{B}(G)$  the *character algebra* of  $G$  over  $\mathbb{Z}$ . We denote by  $\mathcal{X}(G)$  the affine scheme  $\mathrm{Spec}(\mathcal{B}(G))$  and call it the *character scheme* of  $G$  over  $\mathbb{Z}$ . The natural inclusion

$$\iota : \mathcal{B}(G) \hookrightarrow \mathcal{A}(G)$$

induces a morphism of schemes

$$\iota^\# : \mathcal{R}(G) \longrightarrow \mathcal{X}(G).$$

We denote the image of  $\mathfrak{p}(= \rho_{\mathfrak{p}}) \in \mathcal{R}(G)$  in  $\mathcal{X}(G)$  under  $\iota^\#$  by  $[\mathfrak{p}] (= [\rho_{\mathfrak{p}}])$ .

According to [PS, Definition 2.5] and [Sa, §3.1], we define the *skein algebra*  $\mathcal{C}(G)$  over  $\mathbb{Z}$  by

$$\mathcal{C}(G) := \mathbb{Z}[t_g \ (g \in G)]/I,$$

where  $I$  is the ideal of the polynomial ring  $\mathbb{Z}[t_g \ (g \in \Pi)]$  generated by the polynomials of the form

$$t_e - 2, \ t_{g_1}t_{g_2} - t_{g_1g_2} - t_{g_1^{-1}g_2} \quad (g_1, g_2 \in G).$$

We note that  $\mathcal{C}(G)$  is a finitely generated algebra over  $\mathbb{Z}$  if  $G$  is a finitely generated group ([Sa, §3.2]). We denote by  $\mathcal{S}(G)$  the affine scheme  $\text{Spec}(\mathcal{C}(G))$  and call it the *skein scheme* of  $G$  over  $\mathbb{Z}$ .

Since  $\text{tr}(\sigma_G(g)) \ (g \in G)$  is invariant under the adjoint action of  $\text{PGL}_2$  and we have the formula

$$\text{tr}(\sigma_G(g_1))\text{tr}(\sigma_G(g_2)) - \text{tr}(\sigma_G(g_1g_2)) - \text{tr}(\sigma_G(g_1^{-1}g_2)) = 0$$

for  $g_1, g_2 \in G$ , which is derived from the Cayley-Hamilton relation, we obtain a  $\mathbb{Z}$ -algebra homomorphism

$$\tau : \mathcal{C}(G) \longrightarrow \mathcal{B}(G)$$

defined by

$$\tau(t_g) := \text{tr}(\sigma_G(g)) \quad (g \in G).$$

It induces the morphism of schemes

$$\tau^\# : \mathcal{X}(G) \longrightarrow \mathcal{S}(G).$$

We set

$$\varphi := \iota \circ \tau : \mathcal{C}(G) \longrightarrow \mathcal{A}(G)$$

so that we have the morphism of schemes

$$\varphi^\# = \tau^\# \circ \iota^\# : \mathcal{R}(G) \longrightarrow \mathcal{S}(G).$$

Now we define the *discriminant ideal*  $\Delta(G)$  of  $\mathcal{C}(G)$  by the ideal generated by the images of the elements in  $\mathbb{Z}[t_g \ (g \in \pi)]$  of the form

$$\Delta(g_1, g_2) := t_{g_1g_2g_1^{-1}g_2^{-1}} - 2 = t_{g_1}^2 + t_{g_2}^2 + t_{g_1g_2}^2 - t_{g_1}t_{g_2}t_{g_1g_2} - 4 \quad (g_1, g_2 \in G),$$

and the *discriminant subscheme* by  $V(\Delta(G)) = \text{Spec}(\mathcal{C}(G)/\Delta(G))$ . We define the open subschemes  $\mathcal{S}(G)_{\text{a.i.}}$ ,  $\mathcal{X}(G)_{\text{a.i.}}$ , and  $\mathcal{R}(G)_{\text{a.i.}}$  of  $\mathcal{S}(G)$ ,  $\mathcal{X}(G)$ , and  $\mathcal{R}(G)$ , respectively, by

$$\begin{aligned} \mathcal{S}(G)_{\text{a.i.}} &:= \mathcal{S}(G) \setminus V(\Delta(G)), \\ \mathcal{X}(G)_{\text{a.i.}} &:= \mathcal{X}(G) \setminus (\tau^\#)^{-1}(V(\Delta(G))), \\ \mathcal{R}(G)_{\text{a.i.}} &:= \mathcal{R}(G) \setminus (\varphi^\#)^{-1}(V(\Delta(G))). \end{aligned}$$

The following theorem, due to Kyoji Saito, is fundamental for our purpose.

**Theorem 2.1.1** ([Sa, §§4.2, 4.3], [Na, Corollary 6.8]).

- (1) For  $\mathfrak{p} \in \mathcal{R}(G)$ ,  $\rho_{\mathfrak{p}}$  is absolutely irreducible if and only if  $\mathfrak{p} \in \mathcal{R}(G)_{\text{a.i.}}$ .
- (2) The restriction of  $\varphi^\#$  to  $\mathcal{R}(G)_{\text{a.i.}}$ ,

$$\varphi_{\text{a.i.}}^\# : \mathcal{R}(G)_{\text{a.i.}} \longrightarrow \mathcal{S}(G)_{\text{a.i.}}$$

is a principal  $\text{PGL}_2$ -bundle.

(3) *The restriction of  $\tau^\#$  to  $\mathcal{X}(G)_{\text{a.i}}$  is an isomorphism*

$$\tau_{\text{a.i}}^\# : \mathcal{X}(G)_{\text{a.i}} \xrightarrow{\sim} \mathcal{S}(G)_{\text{a.i}}.$$

By virtue of Theorem 2.1.1(1), we call  $\mathcal{S}(G)_{\text{a.i}}$ ,  $\mathcal{X}(G)_{\text{a.i}}$ , and  $\mathcal{R}(G)_{\text{a.i}}$  the *absolutely irreducible part* of  $\mathcal{S}(G)$ ,  $\mathcal{X}(G)$ , and  $\mathcal{R}(G)$ , respectively. We note that  $\mathcal{X}(G)_{\text{a.i}} (\simeq \mathcal{S}(G)_{\text{a.i}})$  represents the functor  $\overline{\mathcal{F}}$  from the category  $\mathfrak{Sch}$  of schemes to the category of sets, which associates to a scheme  $X$  the set of isomorphism classes of absolutely irreducible representations  $G \rightarrow \text{SL}_2(\Gamma(X, \mathcal{O}_X))$ :

$$\overline{\mathcal{F}}(X) := \{G \rightarrow \text{SL}_2(\Gamma(X, \mathcal{O}_X)) \mid \text{absolutely irreducible representation}\} / \sim.$$

Since  $\varphi_{\text{a.i}}^\#$  and  $\tau_{\text{a.i}}^\#$  are defined over  $\mathbb{Z}$ , they induce maps on  $A$ -rational points for  $A \in \mathbf{Com.Ring}$ :

$$\begin{aligned} \varphi_{\text{a.i}}^\#(A) : \mathcal{R}(G)_{\text{a.i}}(A) &\longrightarrow \mathcal{S}(G)_{\text{a.i}}(A), \\ \tau_{\text{a.i}}^\#(A) : \mathcal{X}(G)_{\text{a.i}}(A) &\xrightarrow{\sim} \mathcal{S}(G)_{\text{a.i}}(A). \end{aligned}$$

By Theorem 2.1.1(3), we have the following.

**Corollary 2.1.2.** *Let  $\bar{\rho} : G \rightarrow \text{SL}_2(k)$  be an absolutely irreducible representation over a field  $k$  so that  $\bar{\rho} \in \mathcal{R}(G)_{\text{a.i}}(k)$ . Let  $[\bar{\rho}] \in \mathcal{X}(G)_{\text{a.i}}(k)$  also denote the corresponding prime ideal of  $\mathcal{B}(G)$ . Then the morphism  $\tau$  induces an isomorphism of local rings:*

$$\mathcal{C}(G)_{\tau^\#([\bar{\rho}])} \simeq \mathcal{B}(G)_{[\bar{\rho}]}.$$

The following proposition can be proved by using the vanishing of the Galois cohomology  $H^1(k, \text{PGL}_2(\bar{k})) = 1$  for a field  $k$  whose Brauer group  $\text{Br}(k) = 0$  ([Se3, III, §2.2]) and by the Skolem-Noether theorem. For example, when  $k$  is a finite field or an algebraically closed field,  $\text{Br}(k) = 0$ .

**Proposition 2.1.3** ([Fu, Lemma 3.3.1], [Ha, Proposition 2.2.27]). *Let  $k$  be a field whose Brauer group  $\text{Br}(k) = 0$ . Then  $\varphi_{\text{a.i}}^\#$  induces the following bijection on  $k$ -rational points:*

$$\varphi_{\text{a.i}}^\#(k) : \mathcal{R}(G)_{\text{a.i}}(k)/\text{PGL}_2(k) \xrightarrow{\sim} \mathcal{S}(G)_{\text{a.i}}(k).$$

**2.2. The relation between the universal deformation ring and the character scheme.** Let  $k$  be a perfect field with  $\text{char}(k) \neq 2$  and let  $\mathcal{O}$  be a discrete valuation ring with residue field  $k$ . Let  $\bar{\rho} : G \rightarrow \text{SL}_2(k)$  be an absolutely irreducible representation and let  $\overline{T} : G \rightarrow k$  be a pseudo- $\text{SL}_2$ -representation over  $k$  given by the character  $\text{tr}(\bar{\rho})$ . Let  $\mathbf{R}_{\bar{\rho}} (= \mathbf{R}_{\overline{T}})$  be the universal deformation ring of  $\bar{\rho}$  (or  $\overline{T}$ ) as in Theorem 1.2.2. Recall that  $\mathbf{R}_{\overline{T}}$  is a complete local  $\mathcal{O}$ -algebra whose residue field is  $k$ . On the other hand, let  $\mathcal{B}(G)$  and  $\mathcal{S}(G)$  be the character algebra and skein algebra of  $G$  over  $\mathbb{Z}$ , respectively. We set

$$\begin{aligned} \mathcal{B}(G)_k &:= \mathcal{B}(G) \otimes_{\mathbb{Z}} k, & \mathcal{X}(G)_k &:= \text{Spec}(\mathcal{B}(G)_k) = \mathcal{X}(G) \otimes_{\mathbb{Z}} k, \\ \mathcal{C}(G)_k &:= \mathcal{C}(G) \otimes_{\mathbb{Z}} k, & \mathcal{S}(G)_k &:= \text{Spec}(\mathcal{C}(G)_k) = \mathcal{S}(G) \otimes_{\mathbb{Z}} k. \end{aligned}$$

We also denote by  $\mathcal{X}(G)_k^{\text{a.i}}$  and  $\mathcal{S}(G)_k^{\text{a.i}}$  the absolutely irreducible part of  $\mathcal{X}(G)_k$  and  $\mathcal{S}(G)_k$ , respectively. By Theorem 2.1.1(3), we have  $\mathcal{X}(G)_k^{\text{a.i}} \simeq \mathcal{S}(G)_k^{\text{a.i}}$ . The following theorem tells us that the universal deformation ring  $\mathbf{R}_{\bar{\rho}}$  may be seen as an infinitesimal deformation of the character  $k$ -algebra  $\mathcal{B}(G)_k$  at  $[\bar{\rho}]$ .



**Theorem 2.2.1.** *Let  $[\bar{\rho}]$  denote the maximal ideal of  $\mathcal{B}(G)_k$  corresponding to the representation  $\bar{\rho}$ . We then have an isomorphism of  $k$ -algebras,*

$$\mathbf{R}_{\bar{\rho}} \otimes_{\mathcal{O}} k \simeq (\mathcal{B}(G)_k)_{[\bar{\rho}]}^{\wedge},$$

where  $(\mathcal{B}(G)_k)_{[\bar{\rho}]}^{\wedge}$  denotes the  $[\bar{\rho}]$ -adic completion of  $\mathcal{B}(G)_k$ .

*Proof.* By the construction of  $\mathbf{R}_{\bar{T}}$  in Theorem 1.1.1, we have

$$\mathbf{R}_{\bar{\rho}} = \mathcal{O}[[X_g (g \in G)]]/\mathcal{I},$$

where  $\mathcal{I}$  is the ideal of the power series ring  $\mathcal{O}[[X_g (g \in G)]]$  generated by elements of the form: setting  $T_g := X_g + \lambda(\bar{T}(g))$  ( $\lambda$  : the Teichmüller lift),

- (1)  $T_e - 2,$
- (2)  $T_{g_1 g_2} - T_{g_2 g_1},$
- (3)  $T_{g_1} T_{g_2} T_{g_3} + T_{g_1 g_2 g_3} + T_{g_1 g_3 g_2} - T_{g_1 g_2} T_{g_3} - T_{g_2 g_3} T_{g_1} - T_{g_1 g_3} T_{g_2},$
- (4)  $T_g^2 - T_{g^2} - 2,$

where  $g, g_1, g_2, g_3 \in G$ .

On the other hand, let  $\psi : \mathcal{B}(G)_k \rightarrow k$  be the morphism in **Com.Ring** corresponding to  $[\bar{\rho}] \in \mathcal{X}(G)_{\text{a.i.}}(k)$ . Since  $\psi(\text{tr}(\sigma_G(g))) = \text{tr}(\bar{\rho}(g)) = \bar{T}(g)$  for  $g \in G$ , the maximal ideal  $[\bar{\rho}] = \text{Ker}(\psi)$  of  $\mathcal{B}(G)_k$  corresponds to the maximal ideal  $(t_g - \bar{T}(g) (g \in G))$  of  $\mathcal{C}(G)_k$ . Therefore Corollary 2.1.2 yields

$$(\mathcal{B}(G)_k)_{[\bar{\rho}]}^{\wedge} \simeq k[[x_g (g \in G)]]/I^{\wedge},$$

where  $x_g := t_g - \bar{T}(g) (g \in G)$  and  $I^{\wedge}$  is the ideal of the power series ring  $k[[x_g (g \in G)]]$  generated by elements of the form

$$t_e - 2, t_{g_1} t_{g_2} - t_{g_1 g_2} - t_{g_1^{-1} g_2} (g_1, g_2 \in G).$$

So, in order to show that the correspondence  $x_g \mapsto X_g \otimes 1$  gives the desired isomorphism  $(\mathcal{B}(G)_k)_{[\bar{\rho}]}^{\wedge} \simeq \mathbf{R}_{\bar{\rho}} \otimes_{\mathcal{O}} k$ , it suffices to show the following:

**Lemma 2.2.2.** *Let  $T$  be a function on  $G$  with values in an integral domain whose characteristic is not 2. Let (P) be the relations given by*

- (P1)  $T(1) = 2,$
- (P2)  $T(g_1 g_2) = T(g_2 g_1),$
- (P3)  $T(g_1)T(g_2)T(g_3) + T(g_1 g_2 g_3) + T(g_1 g_3 g_2) - T(g_1 g_2)T(g_3) - T(g_2 g_3)T(g_1) - T(g_1 g_3)T(g_2) = 0,$
- (P4)  $T(g)^2 - T(g^2) = 2,$

and let (C) be the relations given by

- (C1)  $T(1) = 2,$
- (C2)  $T(g_1)T(g_2) = T(g_1 g_2) + T(g_1^{-1} g_2),$

where  $g, g_1, g_2, g_3$  are any elements in  $G$ .

Then (P) and (C) are equivalent.

*Proof of Lemma 2.2.2.* (P)  $\Rightarrow$  (C): Letting  $g_2 = g_1$  in (P3), we have

$$T(g_1)^2 T(g_3) - T(g_1^2) T(g_3) + T(g_1^2 g_3) + T(g_1 g_3 g_1) - 2T(g_1 g_3) T(g_1) = 0.$$

Using (P2) and (P4), we have

$$2(T(g_3) + T(g_1^2 g_3) - T(g_1 g_3) T(g_1)) = 0.$$

Letting  $g_3$  be replaced by  $g_1^{-1} g_2$  in the above equation and noting that  $T$  has the value in an integral domain whose characteristic is not 2, we obtain (C2).

(C)  $\Rightarrow$  (P). Letting  $g_2 = 1$  in (C2) and using (C1), we have

$$T(g) = T(g^{-1}) \text{ for any } g \in G.$$

Exchanging  $g_1$  and  $g_2$  in (C2) for each other and using the above relation, we have

$$T(g_2)T(g_1) = T(g_2g_1) + T(g_2^{-1}g_1) = T(g_2g_1) + T(g_1^{-1}g_2)$$

and hence we obtain (P2). Next letting  $g_1$  be replaced by  $g_1g_3$  in (C2), we have

$$(2.2.2.1) \quad -T(g_1g_3)T(g_2) + T(g_1g_3g_2) + T(g_3^{-1}g_1^{-1}g_2) = 0,$$

and letting  $g_2$  be replaced by  $g_2g_3$  in (C2), we have

$$(2.2.2.2) \quad -T(g_1)T(g_2g_3) + T(g_1g_2g_3) + T(g_1^{-1}g_2g_3) = 0.$$

By (C2), we have

$$\begin{aligned} T(g_3^{-1}g_1^{-1}g_2) &= T(g_3)T(g_1^{-1}g_2) - T(g_3g_1^{-1}g_2) \\ &= T(g_3)T(g_1)T(g_2) - T(g_1g_2)T(g_3) - T(g_3g_1^{-1}g_2). \end{aligned}$$

Hence, using (P2) proved already, we have

$$(2.2.2.3) \quad \begin{aligned} T(g_3^{-1}g_1^{-1}g_2) + T(g_1^{-1}g_2g_3) &= T(g_1)T(g_2)T(g_3) - T(g_1g_2)T(g_3) \\ &\quad - T(g_3g_1^{-1}g_2) + T(g_1^{-1}g_2g_3) \\ &= T(g_1)T(g_2)T(g_3) - T(g_1g_2)T(g_3). \end{aligned}$$

Summing up (2.2.2.1) and (2.2.2.2) together with (2.2.2.3), we obtain (P3). Finally putting  $g_1 = g_2$  in (C2) and using (C1), we obtain (P4).  $\square$

By Lemma 1.3.1 and Theorem 2.2.1, we have the following:

**Corollary 2.2.3.** *Let  $[\bar{\rho}]$  be a regular  $k$ -rational point of  $\mathcal{X}(G)_k^{\text{a.i.}}$ . Let  $d$  be the dimension of the irreducible component of  $\mathcal{X}(G)_k^{\text{a.i.}}$  containing  $[\bar{\rho}]$  so that  $(\mathcal{B}(G)_k)_{[\bar{\rho}]}$  is a power series ring over  $k$  on a regular system of parameters  $z_1, \dots, z_d$ . Let  $x_1, \dots, x_d$  be elements of  $\mathbf{R}_{\bar{\rho}}$  such that the image of  $x_i$  in  $\mathbf{R}_{\bar{\rho}} \otimes_{\mathcal{O}} k \simeq (\mathcal{B}(G)_k)_{[\bar{\rho}]}$  is  $z_i$  for  $1 \leq i \leq d$ . Then there is a surjective  $\mathcal{O}$ -algebra homomorphism*

$$\eta : \mathcal{O}[[X_1, \dots, X_d]] \longrightarrow \mathbf{R}_{\bar{\rho}}$$

in  $\mathcal{CL}_{\mathcal{O}}$  such that  $\eta(X_i) = x_i$  for  $1 \leq i \leq d$ .

Corollary 2.2.3, we obtain the following criterion which determines the universal deformations for many examples. See Section 4.

**Theorem 2.2.4.** *Let notation and assumptions be as in Corollary 2.2.3. We suppose that there are  $g_1, \dots, g_d \in G$  such that  $z_i = t_i - \text{tr}(\bar{\rho}(g_i))$  for  $1 \leq i \leq d$ , where  $t_i$  denotes a variable corresponding to the regular function  $\text{tr}(\sigma_G(g_i))$ . Choose  $\alpha_i \in \mathcal{O}$  such that  $\alpha_i \text{ mod } \mathfrak{m}_{\mathcal{O}} = \text{tr}(\bar{\rho}(g_i))$  for  $1 \leq i \leq d$  and suppose that  $\rho : G \rightarrow \text{SL}_2(\mathcal{O}[[t_1 - \alpha_1, \dots, t_d - \alpha_d]])$  is a deformation of  $\bar{\rho}$  satisfying*

$$\text{tr}(\rho(g_i)) = t_i \quad (1 \leq i \leq d).$$

Then  $(\mathcal{O}[[t_1 - \alpha_1, \dots, t_d - \alpha_d]], \rho)$  is the universal deformation of  $\bar{\rho}$ .

*Proof.* By the universal property of  $(\mathbf{R}_{\bar{\rho}}, \rho)$ , there is a morphism

$$\psi : \mathbf{R}_{\bar{\rho}} \longrightarrow \mathcal{O}[[t_1 - \alpha_1, \dots, t_d - \alpha_d]]$$

in  $\mathcal{CL}_{\mathcal{O}}$  such that  $\psi \circ \rho \approx \rho$ . Hence we have

$$(2.2.4.1) \quad \psi(\text{tr}(\rho(g_i))) = \text{tr}(\rho(g_i)) = t_i, \quad 1 \leq i \leq d.$$

By Corollary 2.2.3, there is a surjective morphism

$$\eta : \mathcal{O}[[X_1, \dots, X_d]] \longrightarrow \mathbf{R}_{\bar{\rho}}$$

in  $\mathfrak{EL}_{\mathcal{O}}$  such that  $\eta(X_i) = \text{tr}(\rho(g_i)) - \alpha_i$  for  $1 \leq i \leq d$ . Since  $\psi \circ \eta : \mathcal{O}[[X_1, \dots, X_d]] \rightarrow \mathcal{O}[[t_1 - \alpha_1, \dots, t_d - \alpha_d]]$  is a morphism in  $\mathfrak{EL}_{\mathcal{O}}$  and satisfies, by (2.2.4.1),

$$\psi \circ \eta(X_i) = t_i - \alpha_i \quad (1 \leq i \leq d),$$

$\psi \circ \eta$  is an isomorphism in  $\mathfrak{EL}_{\mathcal{O}}$ . Since  $\eta$  is surjective,  $\eta$  must be isomorphic and so is  $\psi$ . □

**2.3. The case of a knot group.** Let  $K$  be a knot in the 3-sphere  $S^3$  and let  $E_K$  denote the knot complement  $S^3 \setminus K$ . Let  $G_K$  denote the knot group of  $K$ ,  $G_K := \pi_1(E_K)$ . It is well-known that  $G_K$  has the following presentation of deficiency one (for example, the Wirtinger presentation):

$$(2.3.1) \quad G_K = \langle g_1, \dots, g_n \mid r_1 = \dots = r_{n-1} = 1 \rangle.$$

Let  $k$  be a field with  $\text{char}(k) \neq 2$ . Let  $\bar{\rho} : G_K \rightarrow \text{SL}_2(k)$  be an absolutely irreducible representation and let  $\mathbf{R}_{\bar{\rho}}$  be the universal deformation ring as in Theorem 1.2.2. Since the character variety  $\mathcal{X}(G_K)_k$  of a knot group  $G_K$  over a field  $k$  has been extensively studied (see [CS], [Le], [Ha], etc.), we can determine  $\mathbf{R}_{\bar{\rho} \otimes_{\mathcal{O}} k}$  by Theorem 2.2.1 and even  $\mathbf{R}_{\bar{\rho}}$  by Theorem 2.2.4 for some knots  $K$ . In fact, in [MTTU], we determined  $\mathbf{R}_{\bar{\rho}}$  for a certain Riley-type representation  $\bar{\rho}$  for a 2-bridge knot  $K$ . See also Section 4 for other examples.

It is a delicate problem, however, to determine the universal deformation ring  $\mathbf{R}_{\bar{\rho}}$  for a knot group representation  $\bar{\rho}$  in general, since the deformation problem for  $\bar{\rho}$  is not unobstructed in general for a knot group  $G_K$ , as the following theorem shows.

**Theorem 2.3.2.** *We suppose that  $\rho : G_K \rightarrow \text{SL}_2(\mathbb{C})$  is an irreducible representation and that there is a subring  $A$  of a finite algebraic number field  $F$  and a finite prime  $\mathfrak{p}$  of  $F$  such that  $A$  is  $\mathfrak{p}$ -integral and the image of  $\rho$  is contained in  $\text{SL}_2(A)$ . Set  $k := A/\mathfrak{p}$  and  $\bar{\rho} := \rho \bmod \mathfrak{p} : G_K \rightarrow \text{SL}_2(k)$ . Then we have*

$$H^2(G_K, \text{Ad}(\bar{\rho})) \neq 0.$$

We note that the assumption in Theorem 2.3.2 is satisfied, for instance, when  $K$  is a hyperbolic knot and  $\rho$  is the holonomy representation attached to a hyperbolic structure on  $E_K$  such that the completion is a closed or a cone 3-manifold. For the proof of Theorem 2.3.2, we recall the following lemma, which is a special case of a more general result, due to Thurston, for 3-manifolds.

**Lemma 2.3.3** ([CS, Proposition 3.2.1]). *For an irreducible representation  $\rho : G_K \rightarrow \text{SL}_2(\mathbb{C})$ , the irreducible component of  $\mathcal{X}(G_K)_{\mathbb{C}}$  containing  $[\rho]$  has dimension greater than 0.*

*Proof of Theorem 2.3.2.* Let  $W$  be the CW complex attached to the presentation (2.3.1). We recall herewith the construction of  $W$ :

- We prepare 0-cell  $b^*$ , C1-cells  $g_1^*, \dots, g_n^*$ , where each  $g_i^*$  corresponds to the generator  $g_i$ , C2-cells  $r_1^*, \dots, r_{n-1}^*$ , where each  $r_j^*$  corresponds to the relator  $r_j$ .
- We attach each 1-cell  $g_i^*$  to the 0-cell  $b^*$  so that we obtain a bouquet.
- We attach the boundary of each 2-cell  $r_j^*$  to 1-cells of the bouquet, according to words in  $r_j$ .

We note that the knot complement  $E_K$  and the CW complex  $W$  are homotopically equivalent by Whitehead’s theorem, because they are both the Eilenberg-MacLane space  $K(G_K, 1)$ .

Let  $\text{Ad}(\rho)$  be the  $A$ -module  $\mathfrak{sl}_2(A)$  on which  $G_K$  acts by  $g.X := \rho(g)X\rho(g)^{-1}$  for  $g \in G_K$  and  $X \in \mathfrak{sl}_2(A)$ . We let  $\text{Ad}(\rho)_{\mathbb{C}} := \text{Ad}(\rho) \otimes_A \mathbb{C} = \mathfrak{sl}_2(\mathbb{C})$  on which  $G_K$  acts as  $g \otimes \text{id}_{\mathbb{C}}$  for  $g \in G_K$ . Since the Euler characteristic of  $W$  is zero, we have

$$\begin{aligned}
 \sum_{i=0}^2 (-1)^i \dim_{\mathbb{C}} H^i(G_K, \text{Ad}(\rho)_{\mathbb{C}}) &= \sum_{i=0}^2 (-1)^i \dim_{\mathbb{C}} C^i(W; \text{Ad}(\rho)_{\mathbb{C}}) \\
 (2.3.2.1) \qquad \qquad \qquad &= 3 \sum_{i=0}^2 (-1)^i \dim_{\mathbb{C}} C^i(W; \mathbb{C}) \\
 &= 0.
 \end{aligned}$$

Since  $\rho$  is irreducible, we have  $H^0(G_K, \text{Ad}(\rho)_{\mathbb{C}}) = 0$  by Schur’s lemma. So, by (2.3.2.1), we have

$$(2.3.2.2) \qquad \dim_{\mathbb{C}} H^2(G_K, \text{Ad}(\rho)_{\mathbb{C}}) = \dim_{\mathbb{C}} H^1(G_K, \text{Ad}(\rho)_{\mathbb{C}}).$$

Since  $H^1(G_K, \text{Ad}(\rho)_{\mathbb{C}})$  contains the tangent space of the character variety  $\mathcal{X}(G_K)_{\mathbb{C}}$  at  $[\rho]$  ([Po, Proposition 3.5]), Lemma 2.3.3 implies that  $H^1(G_K, \text{Ad}(\rho)_{\mathbb{C}}) \neq 0$ . So, by (2.3.2.2), we have  $H^2(G_K, \text{Ad}(\rho)_{\mathbb{C}}) \neq 0$ . Since  $H^2(G_K, \text{Ad}(\rho)_{\mathbb{C}}) = H^2(G_K, \text{Ad}(\rho)_A) \otimes_A \mathbb{C}$ , we have

$$(2.3.2.3) \qquad \qquad \qquad H^2(G_K, \text{Ad}(\rho)) \neq 0.$$

Let  $\text{Ad}(\bar{\rho}) := \text{Ad}(\rho) \otimes_A k = \mathfrak{sl}_2(k)$  on which  $G_K$  acts as  $g \otimes \text{id}_k$  for  $g \in G_K$ . Let us consider the differentials of cochains

$$\begin{aligned}
 d : C^1(W; \text{Ad}(\rho)) &\longrightarrow C^2(W; \text{Ad}(\rho)), \\
 \bar{d} := d \otimes (\text{mod } \mathfrak{p}) : C^1(W; \text{Ad}(\bar{\rho})) &\longrightarrow C^2(W; \text{Ad}(\bar{\rho})).
 \end{aligned}$$

By (2.3.2.3), all  $3n$ -minors of  $d$  are zero. Therefore all  $3n$ -minors of  $\bar{d}$  are zero and hence  $H^2(G_K, \text{Ad}(\bar{\rho})) \neq 0$ . □

### 3. $L$ -FUNCTIONS ASSOCIATED TO THE UNIVERSAL DEFORMATIONS

In this section, we study the twisted knot module  $H_1(\boldsymbol{\rho}) = H_1(E_K; \boldsymbol{\rho})$  with coefficients in the universal deformation  $\boldsymbol{\rho}$  of an  $\text{SL}_2$ -representation of a knot group  $G_K$  and introduce the associated  $L$ -function  $L_K(\boldsymbol{\rho})$ . We then formulate two problems proposed by Mazur ([Ma3]): the torsion property of  $H_1(\boldsymbol{\rho})$  over the universal deformation ring  $\mathbf{R}_{\bar{\rho}}$  (Problem 3.2.3) and the generic simplicity of the zeroes of  $L_K(\boldsymbol{\rho})$  (Problem 3.2.10). Our main theorem in this section (Theorem 3.2.4) gives a criterion for  $H_1(\boldsymbol{\rho})$  to be finitely generated and torsion over  $\mathbf{R}_{\bar{\rho}}$  using a twisted Alexander invariant of  $K$ .

**3.1. Fitting ideals and twisted Alexander invariants.** Let  $A$  be a Noetherian integrally closed domain. Let  $M, M'$  be finitely generated  $A$ -modules. We say that a homomorphism  $\varphi : M \rightarrow M'$  is a *pseudo-isomorphism* if the annihilators of  $\text{Ker}(\varphi)$  and  $\text{Coker}(\varphi)$  are not contained in height 1 prime ideals of  $A$ .

**Lemma 3.1.1** (cf. [Se1, Lemma 5]). *For any finitely generated torsion  $A$ -module  $M$ , there are positive integers  $e_1, \dots, e_s$ , height 1 prime ideals  $\mathfrak{p}_1, \dots, \mathfrak{p}_s$  of  $A$  for*

some  $s \geq 1$ , and a pseudo-isomorphism

$$\varphi : M \longrightarrow \bigoplus_{i=1}^s A/\mathfrak{p}_i^{e_i}.$$

Here the set  $\{(\mathfrak{p}_i, e_i)\}$  is uniquely determined by  $M$ . If  $A$  is a Noetherian factorial domain further, each prime ideal  $\mathfrak{p}_i$  of height 1 is a principal ideal  $\mathfrak{p}_i = (f_i)$  for a prime element  $f_i$  of  $A$ .

Note that a regular local ring is a Noetherian factorial local domain (Auslander-Buchsbaum). For example, the Iwasawa algebra  $\mathcal{O}[[X]]$  is a 2-dimensional regular local ring, where  $\mathcal{O}$  is a complete discrete valuation ring with  $\text{char}(\mathcal{O}) = 0$  and finite residue field. Then it is known in Iwasawa theory ([I]) that a height 1 prime ideal of  $\mathcal{O}[[X]]$  is  $(\varpi)$  for a prime element  $\varpi$  of  $\mathcal{O}$  or  $(f)$  for an irreducible distinguished polynomial  $f \in \mathcal{O}[X]$ , and a pseudo-isomorphism means a homomorphism with finite kernel and cokernel ([Ws, §13.2]).

Let  $A$  be a Noetherian factorial domain and let  $M$  be a finitely generated  $A$ -module. Let us take a finite presentation of  $M$  over  $A$ ,

$$A^m \xrightarrow{\partial} A^n \longrightarrow M \longrightarrow 0,$$

where  $\partial$  is an  $n \times m$  matrix over  $A$ . For a non-negative integer  $d$ , we define the  $d$ -th Fitting ideal (elementary ideal)  $E_d(M)$  of  $M$  to be the ideal generated by  $(n - d)$  minors of  $\partial$ . If  $d \geq n$ , we let  $E_d(M) := A$ , and if  $n - d > m$ , we let  $E_d(M) := 0$ . These ideals depend only on  $M$  and are independent of the choice of a presentation. The initial Fitting ideal  $E_0(M)$  is called the order ideal of  $M$ . Let  $\Delta_d(M)$  be the greatest common divisor of generators of  $E_d(M)$ , which is well defined up to multiplication by a unit of  $A$ . The rank of  $M$  over  $A$  is defined by the dimension of  $M \otimes_A Q(A)$  over  $Q(A)$ . The following facts are well-known ([Kw, 7.2], [Hi, Chapter 3]).

**Lemma 3.1.2.** *Let  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  be an exact sequence of finitely generated  $A$ -modules. Then we have the following:*

- (1)  $\Delta_0(M_2) \doteq \Delta_0(M_1)\Delta_0(M_3)$ .
- (2) *If the  $A$ -torsion subgroup of  $M_3$  is zero and  $r$  is the rank of  $M_3$  over  $A$ , then  $\Delta_d(M_2) \doteq \Delta_{d-r}(M_1)$ .*

For example, suppose  $A$  is a principal ideal domain and  $M$  is a finitely generated torsion  $A$ -module. Then we have  $M \simeq \bigoplus_{i=1}^s A/(a_i)$  with  $(a_1) \supset \cdots \supset (a_s)$ , and  $E_d(M) = (a_1 \cdots a_{s-d})$ ,  $\Delta_d(M) \doteq a_1 \cdots a_{s-d}$  for  $d < s$ . As another example, let  $A$  be the Iwasawa algebra  $\mathcal{O}[[X]]$  and let  $M$  be a finitely generated torsion  $A$ -module. Then there is a pseudo-isomorphism  $\varphi : M \rightarrow \bigoplus_{i=1}^s A/(f_i^{e_i})$ , where  $f_i$  is a prime element of  $\mathcal{O}$  or an irreducible distinguished polynomial in  $\mathcal{O}[X]$ . If  $\varphi$  is injective, in particular, if  $M$  has no non-trivial finite  $A$ -submodule, we have  $E_0(M) = (f)$ ,  $\Delta_0(M) \doteq f$ , where  $f$  is the Iwasawa polynomial  $\prod_{i=1}^s f_i^{e_i}$  ([MW, Appendix]). For higher Fitting ideals  $E_d(M)$  for  $d > 0$  in Iwasawa theory, we refer to [Ku].

Next, let  $C$  be a finite connected CW complex. Let  $G := \pi_1(C)$  be the fundamental group of  $C$  which is supposed to have the finite presentation

$$G = \langle g_1, \dots, g_n \mid r_1 = \cdots = r_m = 1 \rangle,$$

where relators  $r_1, \dots, r_m$  are words of the letters  $g_1, \dots, g_n$ . We suppose that there is a surjective homomorphism

$$\alpha : G \longrightarrow \langle t \rangle \simeq \mathbb{Z}.$$

Let  $A$  be a Noetherian factorial domain. We denote by the same  $\alpha$  the group  $A$ -algebra homomorphism  $A[G] \rightarrow A[t^{\pm 1}]$ , which is induced by  $\alpha$ . Let

$$\rho : G \longrightarrow \mathrm{GL}_N(A)$$

be a representation of  $G$  of degree  $N$  over  $A$  and let us denote by the same  $\rho$  the  $A$ -algebra homomorphism  $A[G] \rightarrow \mathrm{M}_N(A)$  induced by  $\rho$ . Then we have the tensor product representation

$$\rho \otimes \alpha : A[G] \longrightarrow \mathrm{M}_N(A[t^{\pm 1}]).$$

The twisted Alexander invariant  $\Delta(C, \rho; t) \in A[t^{\pm 1}]$  is defined as follows. Let  $F$  be the free group on  $g_1, \dots, g_n$  and let  $\pi : F \rightarrow G$  be the natural homomorphism. We denote by the same  $\pi$  the  $A$ -algebra homomorphism  $A[F] \rightarrow A[G]$  induced by  $\pi$ . Then we have the  $A$ -algebra homomorphism

$$\Phi := (\rho \otimes \alpha) \circ \pi : A[F] \longrightarrow \mathrm{M}_N(A[t^{\pm 1}]).$$

Let  $\frac{\partial}{\partial g_i} : A[F] \rightarrow A[F]$  be the Fox derivative over  $A$ , extended from  $\mathbb{Z}$  ([Fo]). Let us consider the (big)  $n \times m$  matrix  $P$ , called the *twisted Alexander matrix*, whose  $(i, j)$  component is defined by the  $N \times N$  matrix

$$\Phi \left( \frac{\partial r_j}{\partial g_i} \right).$$

For  $1 \leq i \leq n$ , let  $P_i$  denote the matrix obtained by deleting the  $i$ -th row from  $P$ , and we regard  $P_i$  as an  $(n - 1)N \times mN$  matrix over  $A[t^{\pm 1}]$ . We note that  $A[t^{\pm 1}]$  is also a Noetherian factorial domain. Let  $D_i$  be the greatest common divisor of all  $(n - 1)N$ -minors of  $P_i$ . Then it is known that there is  $i$  ( $1 \leq i \leq n$ ) such that  $\det(\Phi(g_i - 1)) \neq 0$  and that the ratio

$$(3.1.3) \quad \Delta(C, \rho; t) := \frac{D_i}{\det \Phi(g_i - 1)} \ (\in Q(A)(t))$$

is independent of such  $i$ 's and is called the *twisted Alexander invariant* of  $C$  associated to  $\rho$  ([Wd]).

**3.2.  $L$ -functions associated to the universal deformations.** Let  $K$  be a knot in the 3-sphere  $S^3$  and let  $E_K$  denote the knot complement  $S^3 \setminus K$ . Let  $G_K$  denote the knot group  $\pi_1(E_K)$  of  $K$ , which has the following presentation:

$$(3.2.1) \quad G_K = \langle g_1, \dots, g_n \mid r_1 = \dots = r_{n-1} = 1 \rangle.$$

Let  $F$  be the free group on the words  $g_1, \dots, g_n$  and let  $\pi : \mathbb{Z}[F] \rightarrow \mathbb{Z}[G]$  be the natural homomorphism of group rings. We write the same  $g_i$  for the image of  $g_i$  in  $G_K$ .

Let  $\bar{\rho} : G_K \rightarrow \mathrm{SL}_2(k)$  be an absolutely irreducible representation of  $G_K$  over a perfect field  $k$  with  $\mathrm{char}(k) \neq 2$ . Let  $\mathcal{O}$  be a complete discrete valuation ring with residue field  $k$  and let  $\mathfrak{C}\mathfrak{L}_{\mathcal{O}}$  be the category of complete local  $\mathcal{O}$ -algebras with residue field  $k$ . Let  $\boldsymbol{\rho} : G_K \rightarrow \mathrm{SL}_2(\mathbf{R}_{\bar{\rho}})$  be the universal deformation of  $\bar{\rho}$  (Theorem 1.2.2). We denote by the same  $\boldsymbol{\rho}$  the induced algebra homomorphism  $\mathbb{Z}[G_K] \rightarrow \mathrm{M}_2(\mathbf{R}_{\bar{\rho}})$ .

Let  $V_\rho$  be the representation space (of column vectors)  $(\mathbf{R}_{\bar{\rho}})^{\oplus 2}$  of  $\rho$  on which  $G_K$  acts from the left via  $\rho$ . We will compute the *twisted knot module*

$$H_*(\rho) := H_*(E_K; V_\rho)$$

with coefficients in  $V_\rho$  as the homology of the chain complex  $C_*(W; V_\rho)$  of the CW complex  $W$  attached to the presentation (3.2.1). The CW complex  $W$  was given in Subsection 2.3. Since  $H_1(W; \mathbb{Z}) = H_1(E_K; \mathbb{Z}) = \langle t \rangle \simeq \mathbb{Z}$ , we take  $\alpha : \pi_1(W) \rightarrow \langle t \rangle$  to be the abelianization map.

For a representation  $\rho : G_K \rightarrow \text{SL}_2(A)$ , where  $A$  is a Noetherian factorial domain, we define the *twisted Alexander invariant*  $\Delta_K(\rho; t)$  of  $K$  associated to  $\rho$  by

$$\Delta_K(\rho; t) := \Delta(W, \rho; t).$$

We note that  $\Delta_K(\rho; t)$  coincides with the Reidemeister torsion of  $E_K$  (or  $W$ ) associated to the representation  $\rho \otimes \alpha$  over  $Q(A)(t)$  (cf. [FV, Proposition 2.2]). It is known ([FV], [KL], [Ki]) that the relation between the twisted Alexander invariant  $\Delta_K(\rho; t)$  and the initial Fitting ideals of  $H_i(\rho \otimes \alpha) := H_i(E_K; \rho \otimes \alpha)$  ( $i = 0, 1$ ) is given by

$$(3.2.2) \quad \Delta_K(\rho; t) \doteq \frac{\Delta_0(H_1(\rho \otimes \alpha))}{\Delta_0(H_0(\rho \otimes \alpha))}.$$

Following Mazur’s question 1 of [Ma3, page 440], we may ask the following:

**Problem 3.2.3.** Is  $H_1(\rho)$  a finitely generated and torsion  $\mathbf{R}_{\bar{\rho}}$ -module?

Here is our main theorem, which gives an affirmative answer to Problem 3.2.3 under some conditions using a twisted Alexander invariant of  $K$ .

**Theorem 3.2.4.** *With notation being as above, suppose that the following two conditions are satisfied:*

- (1)  $\mathbf{R}_{\bar{\rho}}$  is a Noetherian integral domain.
- (2) There is a deformation  $\rho : G_K \rightarrow \text{SL}_2(R)$  of  $\bar{\rho}$ , where  $R \in \mathfrak{CL}_{\mathcal{O}}$  is a Noetherian factorial domain, and  $g \in G_K$  such that
  - (2-1)  $\det(\rho(g) - I) \neq 0$  and
  - (2-2)  $\Delta_K(\rho; 1) \neq 0$ .

Then  $H_1(\rho)$  is a finitely generated torsion  $\mathbf{R}_{\bar{\rho}}$ -module.

*Proof.* We may assume that  $g = g_n$  in the presentation (3.2.1) of  $G_K$ . We consider the following chain complex  $C_*(\rho) := C_*(W; V_\rho)$  ([Kw, §7.1]):

$$0 \longrightarrow C_2(\rho) \xrightarrow{\partial_2} C_1(\rho) \xrightarrow{\partial_1} C_0(\rho) \longrightarrow 0,$$

defined by

$$\begin{cases} C_0(\rho) := V_\rho, \\ C_1(\rho) := (V_\rho)^{\oplus n}, \\ C_2(\rho) := (V_\rho)^{\oplus(n-1)}, \end{cases} \quad \begin{cases} \partial_1 := (\rho(g_1) - I, \dots, \rho(g_n) - I), \\ \partial_2 := (\rho \circ \pi \left( \frac{\partial r_j}{\partial g_i} \right)), \end{cases}$$

where  $\frac{\partial}{\partial g_i} : \mathbb{Z}[F] \rightarrow \mathbb{Z}[F]$  denotes the Fox derivative ([Fo]), and  $\partial_2$  is regarded as a (big)  $n \times (n - 1)$  matrix whose  $(i, j)$ -entry is the  $2 \times 2$  matrix  $\rho \circ \pi \left( \frac{\partial r_j}{\partial g_i} \right)$ .

By condition (2), let  $\psi : \mathbf{R}_{\bar{\rho}} \rightarrow R$  be a morphism in  $\mathfrak{CL}_{\mathcal{O}}$  such that  $\psi \circ \rho \approx \rho$ . Since  $\psi(\det(\rho(g_n) - I)) = \det(\rho(g_n) - I) \neq 0$  by (2-1), we have  $\det(\rho(g_n) - I) \in Q(\mathbf{R}_{\bar{\rho}})^\times$  by the condition (1). Hence we have

$$(3.2.4.1) \quad H_0(\rho) \otimes_{\mathbf{R}_{\bar{\rho}}} Q(\mathbf{R}_{\bar{\rho}}) = 0.$$

Let  $C'_1(\boldsymbol{\rho})$  be the  $\mathbf{R}_{\bar{\rho}}$ -submodule of  $C_1(\boldsymbol{\rho})$  consisting of the first  $(n - 1)$  components so that  $C_1(\boldsymbol{\rho}) = C'_1(\boldsymbol{\rho}) \oplus V_{\boldsymbol{\rho}}$  and let  $\partial'_2$  be the  $(n - 1) \times (n - 1)$  matrix obtained by deleting the  $n$ -th row from  $\partial_2$ . Consider the  $\mathbf{R}_{\bar{\rho}}$ -homomorphism

$$\partial'_2 : C_2(\boldsymbol{\rho}) \longrightarrow C'_1(\boldsymbol{\rho}).$$

Then, by the definition (3.1.3) of the twisted Alexander invariant, we have

$$(3.2.4.2) \quad \Delta_K(\boldsymbol{\rho}; 1) = \frac{\psi(\det(\partial'_2))}{\psi(\det(\boldsymbol{\rho}(g_n) - I))}.$$

By the conditions (2-1), (2-2), and (3.2.4.2), we have  $\det(\partial'_2) \in Q(\mathbf{R}_{\bar{\rho}})^\times$ . Hence we have

$$(3.2.4.3) \quad H_2(\boldsymbol{\rho}) \otimes_{\mathbf{R}_{\bar{\rho}}} Q(\mathbf{R}_{\bar{\rho}}) = 0.$$

Since the Euler characteristic of  $W$  is zero, we have

$$(3.2.4.4) \quad \begin{aligned} \sum_{i=0}^3 (-1)^i \dim_{Q(\mathbf{R}_{\bar{\rho}})} H_i(\boldsymbol{\rho}) \otimes_{\mathbf{R}_{\bar{\rho}}} Q(\mathbf{R}_{\bar{\rho}}) &= \sum_{i=0}^3 (-1)^i \dim_{Q(\mathbf{R}_{\bar{\rho}})} C_i(\boldsymbol{\rho}) \otimes_{\mathbf{R}_{\bar{\rho}}} Q(\mathbf{R}_{\bar{\rho}}) \\ &= (\text{rank}_{\mathbf{R}_{\bar{\rho}}} V_{\boldsymbol{\rho}}) \sum_{i=0}^3 (-1)^i \text{rank}_{\mathbb{Z}} C_i(W) \\ &= 0. \end{aligned}$$

Therefore, by (3.2.4.1), (3.2.4.3), and (3.2.4.4), we have

$$\text{rank}_{\mathbf{R}_{\bar{\rho}}} H_1(\boldsymbol{\rho}) = \dim_{Q(\mathbf{R}_{\bar{\rho}})} H_1(\boldsymbol{\rho}) \otimes_{\mathbf{R}_{\bar{\rho}}} Q(\mathbf{R}_{\bar{\rho}}) = 0,$$

and hence  $H_1(\boldsymbol{\rho})$  is torsion over  $\mathbf{R}_{\bar{\rho}}$ . Since  $\mathbf{R}_{\bar{\rho}}$  is Noetherian and  $H_1(\boldsymbol{\rho})$  is a quotient of a submodule of  $(V_{\boldsymbol{\rho}})^{\oplus n} = (\mathbf{R}_{\bar{\rho}})^{\oplus 2n}$ ,  $H_1(\boldsymbol{\rho})$  is Noetherian, in particular, finitely generated over  $\mathbf{R}_{\bar{\rho}}$ .  $\square$

It may be interesting to note that the condition (2-2) in Theorem 3.2.4 on a twisted Alexander polynomial is reminiscent of Kato’s result in the number theoretic situation ([Kt]), which asserts that the non-vanishing of the  $L$ -function at 1 of a modular form implies the finiteness of the Selmer module of the associated  $p$ -adic Galois representation.

As a special case of Theorem 3.2.4, the above proof shows the following.

**Corollary 3.2.5.** *With notation being as above, suppose that the following two conditions are satisfied:*

- (1)  $\mathbf{R}_{\bar{\rho}}$  is a Noetherian integral domain.
- (2) There is  $g \in G_K$  such that  $\det(\bar{\rho}(g) - I) \neq 0$  and  $\Delta_K(\bar{\rho}; 1) \neq 0$ .

Then we have  $H_1(\boldsymbol{\rho}) = 0$ .

*Proof.* By the assumptions, we have  $\det(\boldsymbol{\rho}(g_n) - I), \det(\partial'_2) \in (\mathbf{R}_{\bar{\rho}})^\times$ , from which we easily see that  $\text{Ker}(\partial_1) = \text{Im}(\partial_2)$ , and hence  $H_1(\boldsymbol{\rho}) = 0$ .  $\square$

Assume that  $\mathbf{R}_{\bar{\rho}}$  is a Noetherian factorial domain and that condition (2) of Theorem 3.2.4 holds. When  $H_1(\boldsymbol{\rho})$  is a torsion  $\mathbf{R}_{\bar{\rho}}$ -module, we are interested in the invariant

$$(3.2.6) \quad L_K(\boldsymbol{\rho}) := \Delta_0(H_1(\boldsymbol{\rho})),$$

which we call the  $L$ -function of the knot  $K$  associated to  $\boldsymbol{\rho}$  (cf. Remark 3.2.8(3) below). We note that it is a computable invariant by the following.



**Proposition 3.2.7.** *With notation being as above, we have*

$$L_K(\rho) \doteq \Delta_2(\text{Coker}(\partial_2)).$$

*Proof.* This follows from the exact sequence of  $\mathbf{R}_{\bar{p}}$ -modules

$$0 \longrightarrow H_1(\rho) \longrightarrow \text{Coker}(\partial_2) \longrightarrow V_\rho = (\mathbf{R}_{\bar{p}})^{\oplus 2} \longrightarrow 0$$

and Lemma 3.1.2(2). □

*Remark 3.2.8.*

- (1) The  $L$ -function  $L_K(\rho)$  is determined up to multiplication by a unit of  $\mathbf{R}_{\bar{p}}$ .
- (2) When  $H_*(\rho) \otimes_{\mathbf{R}_{\bar{p}}} Q(\mathbf{R}_{\bar{p}}) = 0$ , we have the Reidemeister torsion  $\Delta_K(\rho; 1) \in Q(\mathbf{R}_{\bar{p}})$  of  $E_K$  associated to  $\rho$ , which is an invariant defined without indeterminacy. It may be non-trivial, even when  $H_*(\rho) = 0$ .
- (3) Our  $L$ -function  $L_K(\rho)$  may be seen as an analogue in knot theory of the algebraic  $p$ -adic  $L$ -function for the universal Galois deformation in number theory ([G]). In terms of [Ma3], the  $\mathbf{R}_{\bar{p}}$ -module  $H_1(\rho)$  gives a coherent torsion sheaf  $\mathcal{H}_1(\rho)$  on the universal deformation space  $\text{Spec}(\mathbf{R}_{\bar{p}})$ , and  $L_K(\rho)$  gives a non-zero section of  $\mathcal{H}_1(\rho)$ .

We find the following necessary condition for the  $L$ -function  $L_K(\rho)$  to be non-trivial under a mild condition.

**Proposition 3.2.9.** *Assume that  $\Delta_0(H_0(\rho)) \doteq 1$ . If  $L_K(\rho) \not\dot{=} 1$ , we have  $\Delta_K(\bar{p}; 1) = 0$ .*

*Proof.* By (3.2.2), (3.2.6), and our assumption, we have

$$(3.2.9.1) \quad \Delta_K(\rho; 1) \doteq L_K(\rho).$$

Suppose  $L_K(\rho) \not\dot{=} 1$ , which means  $L_K(\rho) \in \mathfrak{m}_{\mathbf{R}_{\bar{p}}}$ . Let  $\varphi : \mathbf{R}_{\bar{p}} \rightarrow k$  be the homomorphism taking mod  $\mathfrak{m}_{\mathbf{R}_{\bar{p}}}$ . Then, by the functorial property of the twisted Alexander invariant and (3.2.9.1), we have

$$\Delta_K(\bar{p}; 1) = \varphi(\Delta_K(\rho; 1)) = \varphi(L_K(\rho)) = 0.$$

□

Following Mazur’s question 2 of [Ma3, page 440], we may ask the following:

**Problem 3.2.10.** Investigate the order of the zeroes of  $L_K(\rho)$  on  $\text{Spec}(\mathbf{R}_{\bar{p}})$  at prime divisors.

In the next section, we verify Problem 3.2.10 affirmatively by some examples.

*Remark 3.2.11.* In [Ma3], Mazur works over a field  $k$  (in fact, the field of complex numbers), and so the  $L$ -function discussed there is, in our terms, given by

$$L_K(\rho_k) := \Delta_0(H_1(\rho_k)),$$

where  $\rho_k : G_K \rightarrow \text{SL}_2(\mathbf{R}_{\bar{p}} \otimes_{\mathcal{O}} k)$  is the representation obtained by taking mod  $\mathfrak{m}_{\mathcal{O}}$  of  $\rho$ . Therefore our  $L$ -function  $L_K(\rho)$  in (3.2.6) is a finer object than  $L_K(\rho_k)$ .

4. EXAMPLES

In this section, we discuss concrete examples of the universal deformations of some representations of 2-bridge knot groups over finite fields and the associated  $L$ -functions.

Let  $K$  be a 2-bridge knot in the 3-sphere  $S^3$ , given as the Schubert form  $B(m, n)$  where  $m$  and  $n$  are odd integers with  $m > 0, -m < n < m$ , and  $\text{g.c.d}(m, n) = 1$ . The knot group  $G_K$  is known to have a presentation of the form

$$G_K = \langle g_1, g_2 \mid wg_1 = g_2w \rangle,$$

where  $w$  is a word  $w(g_1, g_2)$  of  $g_1$  and  $g_2$  which has the following symmetric form:

$$w = w(g_1, g_2) = g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_1^{\epsilon_{m-2}} g_2^{\epsilon_{m-1}},$$

$$\epsilon_i = (-1)^{[in/m]} = \epsilon_{m-i} \quad ([\cdot] = \text{Gauss symbol}).$$

We write the same  $g_i$  for the image of the word  $g_i$  in  $G_K$ .

Let  $A$  be a commutative ring with identity. For  $a \in A^\times$  and  $b \in A$ , we consider two matrices  $C(a)$  and  $D(a, b)$  in  $\text{SL}_2(A)$  defined by

$$C(a) := \begin{pmatrix} a & 1 \\ 0 & a^{-1} \end{pmatrix}, \quad D(a, b) := \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}$$

and we set

$$W(a, b) := C(a)^{\epsilon_1} D(a, b)^{\epsilon_2} \cdots C(a)^{\epsilon_{m-2}} D(a, b)^{\epsilon_{m-1}}.$$

It is easy to see that there are (Laurent) polynomials  $w_{ij}(t, u) \in \mathbb{Z}[t^\pm, u]$  ( $1 \leq i, j \leq 2$ ) such that  $W(a, b) = (w_{ij}(a, b))$ . Let  $\varphi(t, u) := w_{11}(t, u) + (t^{-1} - t)w_{12}(t, u) \in \mathbb{Z}[t^\pm, u]$ . Then it is shown ([R]) that there is a unique polynomial  $\Phi(x, u) \in \mathbb{Z}[x, u]$  such that

$$\Phi(t + t^{-1}, u) = t^l \varphi(t, u)$$

for an integer  $l$ .

Let  $k$  be a field with  $\text{char}(k) \neq 2$  and let  $\mathcal{O}$  be a complete discrete valuation ring with residue field  $k$ . Let  $\mathcal{X}(G_K)_k$  denote the character variety of  $G_K$  over  $k$ . The proof of Proposition 1.4.1 of [CS] tells us that any  $\text{tr}(\sigma_{G_K}(g))$  ( $g \in G_K$ ) is given as a polynomial of  $\text{tr}(\sigma_{G_K}(g_1)) (= \text{tr}(\sigma_{G_K}(g_2)))$  and  $\text{tr}(\sigma_{G_K}(g_1g_2))$  with coefficients in  $\mathbb{Z}$ . In particular, the character algebra  $\mathcal{B}(G_K)_k$  is generated by  $\text{tr}(\sigma_{G_K}(g_1))$  and  $\text{tr}(\sigma_{G_K}(g_1g_2))$  over  $k$ . Let  $x$  and  $y$  denote the variables corresponding, respectively, to the coordinate functions  $\text{tr}(\sigma_{G_K}(g_1))$  and  $\text{tr}(\sigma_{G_K}(g_1g_2))$  on  $\mathcal{X}(G_K)$ . This variable  $x$  is consistent with the variable  $x$  of  $\Phi(x, u)$  (and so causes no confusion). Since  $\text{tr}(C(a)) = a + a^{-1}$  and  $\text{tr}(C(a)D(a, b)) = a^2 + a^{-2} + b$ , the coordinate variables  $x$  and  $y$  are related with  $t$  and  $u$  by

$$x = t + t^{-1}, \quad y = t^2 + t^{-2} + u = x^2 + u - 2.$$

The following theorem is due to Le.

**Theorem 4.1** ([Le, Theorem 3.3.1]). *We have*

$$\mathcal{X}(G_K)_k = \text{Spec}(k[x, y]/((y - x^2 + 2)\Phi(x, y - x^2 + 2))).$$

Here, for a  $k$ -algebra  $A$ , the  $A$ -rational points on  $\Phi(x, y - x^2 + 2) = 0$  correspond bijectively to isomorphism classes of absolutely irreducible representation  $G_K \rightarrow \text{SL}_2(A)$  except the finitely many intersection points with  $y - x^2 + 2 = 0$ .

**Example 4.2.**

- (1) When  $K$  is the trefoil knot  $B(3, 1)$ , we see that  $\Phi(x, y - x^2 + 2) = y - 1$ .
- (2) When  $K$  is the figure eight knot  $B(5, 3)$ , we have  $\Phi(x, y - x^2 + 2) = y^2 - (1 + x^2)y + 2x^2 - 1$ .
- (3) When  $K := B(7, 3)$ , the knot  $5_2$ , we have  $\Phi(x, y - x^2 + 2) = y^3 - (x^2 + 1)y^2 + (3x^2 - 2)y - 2x^2 + 1$ .

By Theorem 4.1, we have the following:

**Corollary 4.3.** *Let  $\bar{\rho} : G_K \rightarrow \text{SL}_2(k)$  be an absolutely irreducible representation so that  $[\bar{\rho}]$  is a regular  $k$ -rational point of  $\mathcal{X}(G_K)_k^{\text{a.i.}}$ . Then we have*

$$(\mathcal{B}(G_K)_k)_{[\bar{\rho}]}^\wedge \simeq k[[x - \text{tr}(\bar{\rho}(g_1))]].$$

So, by Theorem 2.2.4, we have

**Corollary 4.4.** *Let  $\bar{\rho}$  be as in Corollary 4.3. Suppose that*

$$\rho : G_K \rightarrow \text{SL}_2(\mathcal{O}[[x - \alpha]]),$$

where  $\alpha$  is an element of  $\mathcal{O}$  such that  $\alpha \bmod \mathfrak{m}_{\mathcal{O}} = \text{tr}(\bar{\rho}(g_1))$  is a deformation of  $\bar{\rho}$  satisfying

$$(4.4.1) \quad \text{tr}(\rho(g_1)) = x.$$

Then the pair  $(\mathcal{O}[[x - \alpha]], \rho)$  is the universal deformation of  $\bar{\rho}$ .

In the following, we discuss some concrete examples, where  $k$  will be a finite field  $\mathbb{F}_p$  for some odd prime number  $p$ .

*Convention.* Let  $R$  be a complete local ring with residue field  $R/\mathfrak{m}_R = \mathbb{F}_p$ . When the equation  $X^2 = a$  for  $a \in R$  has two simple roots in  $R$ , we denote by  $\sqrt{a}$  the “positive” solution, namely,  $(\sqrt{a})^2 = a$  and  $\sqrt{a} \bmod \mathfrak{m}_R \in \{1, \dots, \frac{p-1}{2}\}$ .

**Example 4.5.**

- (1) Let  $K := B(3, 1)$  be the trefoil knot whose group is given by

$$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle.$$

We have  $\mathcal{X}(G_K)_{\text{a.i.}}(k) = \{(x, y) \in k^2 \mid y = 1\}$ .

Let  $k = \mathbb{F}_3$  and  $\mathcal{O} = \mathbb{Z}_3$ , and consider the following absolutely irreducible representation whose  $\text{PGL}_2(\mathbb{F}_3)$ -conjugacy class corresponds to the regular  $\mathbb{F}_3$ -rational point  $(x, y) = (2, 1)$  of  $\mathcal{X}(G_K)_{\text{a.i.}}$  (Proposition 2.1.3):

$$\bar{\rho}_1 : G_K \rightarrow \text{SL}_2(\mathbb{F}_3); \quad \bar{\rho}_1(g_1) = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, \quad \bar{\rho}_1(g_2) = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}.$$

Let  $\rho_1 : G_K \rightarrow \text{SL}_2(\mathbb{Z}_3[[x - 2]])$  be the representation defined by

$$\rho_1(g_1) = \begin{pmatrix} \frac{x + \sqrt{x^2 - 3}}{2} & -1 \\ \frac{1}{4} & \frac{x - \sqrt{x^2 - 3}}{2} \end{pmatrix},$$

$$\rho_1(g_2) = \begin{pmatrix} \frac{x - \sqrt{x^2 - 3}}{2} & -1 \\ \frac{1}{4} & \frac{x + \sqrt{x^2 - 3}}{2} \end{pmatrix}.$$

We see by straightforward computation that  $\rho_1$  is indeed a representation of  $G_K$  and a deformation of  $\bar{\rho}_1$  (see our convention). Moreover, we have  $\text{tr}(\rho_1(g_1)) = x$ ; hence  $\rho_1$  satisfies the condition (4.4.1). Therefore  $(\mathbf{R}_{\bar{\rho}_1} = \mathbb{Z}_3[[x - 2]], \rho_1)$  is the universal deformation of  $\bar{\rho}_1$ .

We easily see that  $\Delta_0(H_0(\rho_1)) \doteq 1$  and  $\Delta_K(\bar{\rho}_1; t) = 1 + t^2$  hence  $\Delta_K(\bar{\rho}_1; 1) = 2 \neq 0$ . Therefore, by Proposition 3.2.9, we have

$$H_1(\rho_1) = 0, L_K(\rho_1) \doteq 1.$$

(2) Let  $K := B(5, 3)$  be the figure eight knot whose group is given by

$$G_K = \langle g_1, g_2 \mid g_1 g_2^{-1} g_1^{-1} g_2 g_1 = g_2 g_1 g_2^{-1} g_1^{-1} g_2 \rangle.$$

We have  $\mathcal{X}(G_K)_{\text{a.i.}}(k) = \{(x, y) \in k^2 \mid y^2 - (1 + x^2)y + 2x^2 - 1 = 0\} \setminus \{(\pm\sqrt{5}, 3)\}$ .

Let  $k = \mathbb{F}_7$  and  $\mathcal{O} = \mathbb{Z}_7$ , and consider the following absolutely irreducible representation whose  $\text{PGL}_2(\mathbb{F}_7)$ -conjugacy class corresponds to the regular  $\mathbb{F}_7$ -rational points  $(x, y) = (5, 5)$  of  $\mathcal{X}(G_K)_{\text{a.i.}}$ :

$$\bar{\rho}_2 : G_K \rightarrow \text{SL}_2(\mathbb{F}_7); \bar{\rho}_2(g_1) = \begin{pmatrix} 0 & 6 \\ 1 & 5 \end{pmatrix}, \bar{\rho}_2(g_2) = \begin{pmatrix} 5 & 6 \\ 1 & 0 \end{pmatrix}.$$

Let  $\rho_2 : G_K \rightarrow \text{SL}_2(\mathbb{Z}_7[[x + 2]])$  be the representation defined by

$$\rho_2(g_1) = \begin{pmatrix} \frac{x + \sqrt{\frac{x^2 - 5 + u(x)}{2}}}{2} & -1 \\ -\frac{x^2 - 3 - u(x)}{8} & \frac{x - \sqrt{\frac{x^2 - 5 + u(x)}{2}}}{2} \end{pmatrix},$$

$$\rho_2(g_2) = \begin{pmatrix} \frac{x - \sqrt{\frac{x^2 - 5 + u(x)}{2}}}{2} & -1 \\ -\frac{x^2 - 3 - u(x)}{8} & \frac{x + \sqrt{\frac{x^2 - 5 + u(x)}{2}}}{2} \end{pmatrix},$$

where  $u(x) := \sqrt{(x^2 - 1)(x^2 - 5)}$ . We see by straightforward computation that  $\rho_2$  is indeed a representation of  $G_K$  and a deformation of  $\bar{\rho}_2$ . Moreover, we have  $\text{tr}(\rho_2(g_1)) = x$ ; hence  $\rho_2$  satisfies the condition (4.4.1). Therefore  $(\mathbf{R}_{\bar{\rho}_2} = \mathbb{Z}_7[[x + 2]], \rho_2)$  is the universal deformation of  $\bar{\rho}_2$ .

We easily see that  $\det(\bar{\rho}_2(g_2) - I) = 4 \neq 0$  and that  $\Delta_K(\bar{\rho}_2; t) = t^{-2} + 4t^{-1} + 1$ , hence  $\Delta_K(\bar{\rho}_2; 1) = 6 \neq 0$ . Therefore, by Corollary 3.2.5, we have

$$H_1(\rho_2) = 0, L_K(\rho_2) \doteq 1.$$

(3) Let  $K := B(7, 3)$  be the knot  $5_2$  whose group is given by

$$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 g_1 = g_2 g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 \rangle.$$

We have  $\mathcal{X}(G_K)_{\text{a.i.}}(k) = \{(x, y) \in k^2 \mid y^3 - (x^2 + 1)y^2 + (3x^2 - 2)y - 2x^2 + 1 = 0\} \setminus \{(\pm\sqrt{\frac{7}{2}}, \frac{3}{2})\}$ .

Firstly, let  $k = \mathbb{F}_{11}$  and  $\mathcal{O} = \mathbb{Z}_{11}$ , and consider the following absolutely irreducible representation whose  $\text{PGL}_2(\mathbb{F}_{11})$ -conjugacy class corresponds to the regular  $\mathbb{F}_{11}$ -rational point  $(x, y) = (5, 5)$  of  $\mathcal{X}(G_K)_{\text{a.i.}}$ :

$$\bar{\rho}_3 : G_K \rightarrow \text{SL}_2(\mathbb{F}_{11}); \bar{\rho}_3(g_1) = \begin{pmatrix} 5 & 10 \\ 1 & 0 \end{pmatrix}, \bar{\rho}_3(g_2) = \begin{pmatrix} 5 & 1 \\ 10 & 0 \end{pmatrix}.$$

Let  $\alpha := \frac{3 - \sqrt{5}}{2}, \xi := \frac{4 - \sqrt{5}}{4} \in \mathbb{Z}_{11}$  so that  $\alpha \bmod 11 = 5, \xi \bmod 11 = 0 \in \mathbb{F}_{11}$ . Let  $s = s(x)$  be the unique solution in  $\mathbb{Z}_{11}[[x - \alpha]]$  satisfying the equation

$$(4.5.1) \quad 64s^3 - 16(2x^2 + 5)s^2 + 4(x^4 + 9x^2 + 6)s - 4x^4 - 6x^2 - 1 = 0$$

and

$$(4.5.2) \quad s(\alpha) = \xi.$$

Such an  $s(x)$  is proved, by Hensel’s lemma ([Se2, II, §4, Proposition 7]), to exist uniquely. Now, let  $\rho_3 : G_K \rightarrow \text{SL}_2(\mathbb{Z}_{11}[[x - \alpha]])$  be the representation defined by

$$\rho_3(g_1) = \begin{pmatrix} \frac{x + \sqrt{x^2 - 4s(x)}}{2} & -1 \\ -s(x) + 1 & \frac{x - \sqrt{x^2 - 4s(x)}}{2} \end{pmatrix},$$

$$\rho_3(g_2) = \begin{pmatrix} \frac{x + \sqrt{x^2 - 4s(x)}}{2} & 1 \\ s(x) - 1 & \frac{x - \sqrt{x^2 - 4s(x)}}{2} \end{pmatrix}.$$

We can verify by (4.5.1) that  $\rho_3$  is indeed a representation of  $G_K$  and by (4.5.2) that  $\rho_3$  is a deformation of  $\bar{\rho}_3$ . Moreover, we have  $\text{tr}(\rho_3(g_1)) = x$ ; hence  $\rho_3$  satisfies the condition (4.4.1). Therefore  $(\mathbf{R}_{\bar{\rho}_3} = \mathbb{Z}_{11}[[x - \alpha]], \rho_3)$  is the universal deformation of  $\bar{\rho}_3$ .

Consider the 11-adic lifting  $\rho_3 : G_K \rightarrow \text{SL}_2(\mathbb{Z}_{11})$  of  $\bar{\rho}_3$  defined by  $\rho_3|_{x=5}$ :

$$\rho_3(g_1) = \begin{pmatrix} \frac{5 + \sqrt{25 - 4\mu}}{2} & -1 \\ -\mu + 1 & \frac{5 - \sqrt{25 - 4\mu}}{2} \end{pmatrix},$$

$$\rho_3(g_2) = \begin{pmatrix} \frac{5 + \sqrt{25 - 4\mu}}{2} & 1 \\ \mu - 1 & \frac{5 - \sqrt{25 - 4\mu}}{2} \end{pmatrix},$$

where  $\mu$  is the unique solution in  $\mathbb{Z}_{11}$  satisfying (4.5.1) with  $x = 5$  and  $\mu \pmod{11} = 0$ . Then we easily see that  $\det(\rho_3(g_2) - I) = -3 \neq 0$  and that

$$\Delta_K(\rho_3; t) = -2\{-8\mu^2 + 58\mu - 52 + 5t + (-8\mu^2 + 58\mu - 52)t^2\},$$

hence  $\Delta_K(\rho_3; 1) = -2(-16\mu^2 + 116\mu - 99) \neq 0$ . Therefore, by Theorem 3.2.4,  $H_1(\rho_3)$  is a finitely generated torsion  $\mathbb{Z}_{11}[[x - \alpha]]$ -module.

We let  $r := g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 g_1 g_2^{-1} g_1^{-1} g_2 g_1 g_2^{-1} g_1^{-1} g_2^{-1}$  and set

$$\partial_2 = (\rho_3(\frac{\partial r}{\partial g_1}), \rho_3(\frac{\partial r}{\partial g_2})) = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4).$$

By computer calculation, we find that all 2-minors of  $\partial_2$  are given by

$$(4.5.3) \quad \begin{aligned} \det(\mathbf{a}_1, \mathbf{a}_2) &= 2(x - 2)\{4(s - 1)x^2 + x - 4(2s - 1)^2\}, \\ \det(\mathbf{a}_1, \mathbf{a}_3) &= -\frac{1}{2}\{4(s - 1)x^4 - 2(8s^2 - 2s - 5)x^2 + 4(s - 1)x \\ &\quad + (4s - 3)(12s - 5)\}(x - 2 - \sqrt{x^2 - 4s}), \\ \det(\mathbf{a}_1, \mathbf{a}_4) &= 4(s - 1)x^4 - 8(s - 1)x^3 - 4(4s^2 - 5s + 2)x^2 \\ &\quad + 4(8s^2 - 7s + 2)x - (4s - 1)^2, \\ \det(\mathbf{a}_2, \mathbf{a}_3) &= -\{4(s - 1)x^4 - 8(s - 1)x^3 - 4(4s^2 - 5s + 2)x^2 \\ &\quad + 4(8s^2 - 7s + 2)x - (4s - 1)^2\}, \\ \det(\mathbf{a}_2, \mathbf{a}_4) &= 2\{4(s - 1)x^2 + x - 4(2s - 1)^2\}(x - 2 + \sqrt{x^2 - 4s}), \\ \det(\mathbf{a}_3, \mathbf{a}_4) &= 2(x - 2)\{4(s - 1)x^2 + x - 4(2s - 1)^2\}. \end{aligned}$$

By (4.5.3) and computer calculation, we find that  $x = \alpha$  ( $s(\alpha) = \frac{4 - \sqrt{5}}{4}$ ) gives a common zero of all 2-minors of  $\partial_2$  and and their derivatives and is not a common zero of the third order derivatives of all 2-minors. Hence the greatest common divisor of all 2-minors is  $(x - \alpha)^2$ . Therefore, by Proposition 3.2.7, we have

$$H_1(\rho_3) \simeq \mathbb{Z}_{11} \oplus \mathbb{Z}_{11}, \quad L_K(\rho_3) \doteq (x - \alpha)^2.$$

Secondly, let  $k = \mathbb{F}_{19}$  and  $\mathcal{O} = \mathbb{Z}_{19}$ , and consider the following absolutely irreducible representation whose  $\mathrm{PGL}_2(\mathbb{F}_{19})$ -conjugacy class corresponds to the regular  $\mathbb{F}_{19}$ -rational point  $(x, y) = (6, 6)$  of  $\mathcal{X}(G_K)_{\mathrm{a.i}}$ :

$$\bar{\rho}_4 : G_K \rightarrow \mathrm{SL}_2(\mathbb{F}_{19}); \bar{\rho}_4(g_1) = \begin{pmatrix} 14 & 1 \\ 1 & 11 \end{pmatrix}, \bar{\rho}_4(g_2) = \begin{pmatrix} 11 & 1 \\ 1 & 14 \end{pmatrix}.$$

Let  $\beta := \frac{3+\sqrt{5}}{2}, \zeta := \frac{7+\sqrt{5}}{8} \in \mathbb{Z}_{19}$  so that  $\beta \bmod 19 = 6, \zeta \bmod 19 = 2 \in \mathbb{F}_{19}$ . Let  $v = v(x)$  be the unique solution in  $\mathbb{Z}_{19}[[x - \beta]]$  satisfying the equation

$$(4.5.4) \quad 64v^3 - 16(x^2 + 7)v^2 + 28(x^2 + 2)v - 12x^2 - 7 = 0$$

and

$$(4.5.5) \quad v(\beta) = \zeta.$$

Such a  $v(x)$  is proved, by Hensel’s lemma ([Se2, II, §4, Proposition 7]), to exist uniquely. Now, let  $\rho_4 : G_K \rightarrow \mathrm{SL}_2(\mathbb{Z}_{19}[[x - \beta]])$  be the representation defined by

$$\rho_4(g_1) = \begin{pmatrix} \frac{x+\sqrt{x^2-4v(x)}}{2} & 1 \\ v(x) - 1 & \frac{x-\sqrt{x^2-4v(x)}}{2} \end{pmatrix},$$

$$\rho_4(g_2) = \begin{pmatrix} \frac{x-\sqrt{x^2-4v(x)}}{2} & 1 \\ v(x) - 1 & \frac{x+\sqrt{x^2-4v(x)}}{2} \end{pmatrix}.$$

We can verify by (4.5.4) that  $\rho_4$  is indeed a representation of  $G_K$  and by (4.5.5) that  $\rho_4$  is a deformation of  $\bar{\rho}_4$ . Moreover, we have  $\mathrm{tr}(\rho_4(g_1)) = x$ ; hence  $\rho_4$  satisfies the condition (4.4.1). Therefore  $(\mathbf{R}_{\bar{\rho}_4} = \mathbb{Z}_{19}[[x - \beta]], \rho_4)$  is the universal deformation of  $\bar{\rho}_4$ .

Consider the 19-adic lifting  $\rho_4 : G_K \rightarrow \mathrm{SL}_2(\mathbb{Z}_{19})$  of  $\bar{\rho}_4$  defined by  $\rho_4|_{x=6}$ :

$$\rho_4(g_1) = \begin{pmatrix} \frac{6+\sqrt{36-4\nu}}{2} & 1 \\ \nu - 1 & \frac{6-\sqrt{36-4\nu}}{2} \end{pmatrix},$$

$$\rho_4(g_2) = \begin{pmatrix} \frac{6-\sqrt{36-4\nu}}{2} & 1 \\ \nu - 1 & \frac{6+\sqrt{36-4\nu}}{2} \end{pmatrix},$$

where  $\nu$  is the unique solution in  $\mathbb{Z}_{19}$  satisfying (4.5.4) with  $x = 6$  and  $\nu \bmod 19 = 2$ . Then we easily see that  $\det(\rho_4(g_2) - I) = -4 \neq 0$  and that

$$\Delta_K(\rho_4; t) = -2\{-8\nu^2 + 80\nu - 74 + 6t + (-8\nu^2 + 80\nu - 74)t^2\},$$

hence  $\Delta_K(\rho_4; 1) = -2(-16\nu^2 + 160\nu - 142) \neq 0$ . Therefore, by Theorem 3.2.4,  $H_1(\rho_4)$  is a finitely generated torsion  $\mathbb{Z}_{19}[[x - \beta]]$ -module.

We set

$$\partial_2 = (\rho_4(\frac{\partial r}{\partial g_1}), \rho_4(\frac{\partial r}{\partial g_2})) = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4).$$

By computer calculation, we find that all 2-minors of  $\partial_2$  are given by

$$\begin{aligned}
 \det(\mathbf{b}_1, \mathbf{b}_2) &= 2(x-2)\{4(v-1)x^2 + x - 4(2v-1)^2\}, \\
 \det(\mathbf{b}_1, \mathbf{b}_3) &= -\frac{1}{2}\{4(v-1)x^2 - 4(v-1)x - (4v-3)^2\}\sqrt{x^2-4v}, \\
 \det(\mathbf{b}_1, \mathbf{b}_4) &= 4(v-1)x^4 - (8v-9)x^3 - 2(8v^2-10v+5)x^2 \\
 &\quad + 4(8v^2-9v+3)x - (4v-3)^2 \\
 (4.5.6) \quad \det(\mathbf{b}_2, \mathbf{b}_3) &= -\{4(v-1)x^4 - (8v-9)x^3 - 2(8v^2-10v+5)x^2 \\
 &\quad + 4(8v^2-9v+3)x - (4v-3)^2\} \\
 &\quad - (x-2)\{4(v-1)x^2 + x - 4(2v-1)^2\}\sqrt{x^2-4v}, \\
 \det(\mathbf{b}_2, \mathbf{b}_4) &= 2\{4(v-1)x^2 + x - 4(2v-1)^2\}\sqrt{x^2-4v}, \\
 \det(\mathbf{b}_3, \mathbf{b}_4) &= 2(x-2)\{4(v-1)x^2 + x - 4(2v-1)^2\}.
 \end{aligned}$$

By (4.5.6) and computer calculation, we find that  $x = \beta$  ( $v(\beta) = \frac{7+\sqrt{5}}{8}$ ) is a common zero of all 2-minors of  $\partial_2$  and their derivatives and is not a common zero of the third order derivatives of all 2-minors. Hence the greatest common divisor of all 2-minors is  $(x - \beta)^2$ . Therefore, by Proposition 3.2.7, we have

$$H_1(\rho_4) \simeq \mathbb{Z}_{19} \oplus \mathbb{Z}_{19}, \quad L_K(\rho_4) \doteq (x - \beta)^2.$$

We see that all examples above answer Problem 3.2.3 affirmatively and answer Problem 3.2.10 concretely.

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