TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 5, May 2018, Pages 3245–3264
http://dx.doi.org/10.1090/tran/7052
Article electronically published on November 17, 2017

TOTARO'S QUESTION FOR TORI OF LOW RANK

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ABSTRACT. Let G be a smooth connected linear algebraic group and let X be a G-torsor. Totaro asked: if X admits a zero-cycle of degree $d \geq 1$, then does X have a closed étale point of degree dividing d? This question is entirely unexplored in the literature for algebraic tori. We settle Totaro's question affirmatively for algebraic tori of rank ≤ 2.

1. INTRODUCTION

Let X be a smooth quasiprojective variety over a field k. Define its *index*, denoted $\operatorname{ind}(X)$, to be the minimal positive degree of a zero-cycle on X. This is nothing more than the greatest common divisor of degrees of field extensions L/k such that $X(L) \neq \varnothing$. If X has a rational point, then clearly $\operatorname{ind}(X) = 1$; but the converse is false in general. Striking counterexamples to the converse are found among conic bundles over $\mathbb{P}^1_{\mathbb{Q}_p}$ (due to Colliot-Thélène–Coray [CTC79]), affine homogeneous spaces under a smooth connected linear algebraic group over $\mathbb{Q}_p$ with finite stabilizers (due to Florence [Flo04]), and projective homogeneous spaces under a smooth connected linear algebraic group over $\mathbb{Q}_p((t))$ (due to Parimala [Par05]).

Serre asked if every index 1 *principal* homogeneous space (or torsor) under a smooth connected linear algebraic group G over a field k has a rational point [Ser95]. Such spaces are classified by the pointed Galois cohomology set $H^1(k, G)$; for any $X \in H^1(k, G)$ and any field extension L/k, $X(L) \neq \varnothing$ if and only if $X_L = 1 \in H^1(L, G_L)$. So the index of a G-torsor X over k is exactly the greatest common divisor of degrees of field extensions L/k such that $X_L = 1 \in H^1(L, G_L)$. Rephrased in the language of Galois cohomology,

Serre's question (1995). *Let G be a smooth connected linear algebraic group over a field k, and let $X \in H^1(k, G)$ be a G-torsor over k. If $\operatorname{ind}(X) = 1$, then is $X = 1 \in H^1(k, G)$?*

No counterexamples to Serre's question are known, and there are positive answers in some special cases: the case of PGL_n is known from the classical theory of central simple algebras; the case of SO_n is due to Springer [Spr52]; the case of unitary groups is a result of Bayer-Fluckiger–Lenstra [BFL90]; and Sansuc proved that Serre's question has a positive answer for any smooth connected linear algebraic group over a number field or a p-adic field [San81]. One should refer to Black [Bla11a, Bla11b] for further work on this question.

However, for abelian G, a positive answer to Serre's question is a trivial consequence of the fact that the order of X in the *abelian group* $H^1(k, G)$, called the

Received by the editors April 12, 2016, and, in revised form, July 25, 2016.
2010 *Mathematics Subject Classification.* Primary 11E72; Secondary 20G15, 14G05.

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period of X and denoted $\mathrm{per}(X)$, divides $\mathrm{ind}(X)$ (cf. Lemma 3.1). Totaro generalized Serre's question in a natural way that was non-obvious even for abelian G: he asked if the existence of a zero-cycle on X of degree $d \geq 1$ implies the existence of a closed étale point on X of degree dividing d [Tot04]. Reformulating in the language of Galois cohomology as before,

Totaro's question (2004)**.** Let G be a smooth connected linear algebraic group over a field k, and let $X \in H^1(k, G)$ be a G-torsor over k. Is there a separable field extension F/k of degree $\mathrm{ind}(X)$ such that $X_F = 1 \in H^1(F, G_F)$?

No counterexamples to Totaro's question are known, but affirmative proofs are scarcer than those for Serre's question: the case of PGL_n is again a classical theorem about central simple algebras; in the paper where he first asked the question, Totaro answered it positively for split simply connected groups of type G_2, F_4, or E_6 (with a partial result for E_7) [Tot04]; Garibaldi–Hoffmann improved upon this result to give a positive answer for groups of type G_2, reduced of type F_4, and simply connected of types ${}^1E^0_{6,6}$ or ${}^1E^{28}_{6,2}$ [GH06]; and Black–Parimala settled the question for simply connected semisimple groups of rank ≤ 2 over fields of characteristic $\neq 2$ [BP14]. Further exposition can be found in Black–Parimala [BP14].

Suffice it to say that Totaro's question has a rich history but is wide open. In particular, it is completely unexplored in the literature for tori. Our main result is (cf. Section 5)

Theorem 1.1. *Totaro's question has a positive answer for tori of rank* ≤ 2.

We remark that the theorem is true even if the ground field is not perfect. Define the *separable index* of a variety X over a field, denoted $\mathrm{ind_s}(X)$, to be the minimal positive degree of a zero-cycle of closed étale points on X. The question of equality between $\mathrm{ind}(X)$ and $\mathrm{ind_s}(X)$ was raised by Lang–Tate and answered affirmatively by recent work of Gabber–Liu–Lorenzini when X is a generically smooth and non-empty scheme of finite type over a field [GLL13]. Since torsors under tori over fields satisfy these hypotheses, we only need to consider *separable* field extensions in the proof of Theorem 1.1.

Now, if X is regular over a field and $U \subseteq X$ is open and dense, then $\mathrm{ind}(X) = \mathrm{ind}(U)$ by a general moving lemma for zero-cycles. So the index is a birational invariant among regular varieties over a given field. Together with Theorem 1.1, we obtain from this (cf. Section 6)

Corollary 1.2. *Let X be a regular variety over a field containing a principal homogeneous space of a smooth torus of rank ≤ 2 as a dense open subset. If X admits a zero-cycle of degree $d \geq 1$, then X has a closed étale point of degree dividing d.*

In particular, Manin proved that del Pezzo surfaces of degree 6 are toric varieties as in Corollary 1.2 [Man72]. So as a special case of the corollary, we have

Corollary 1.3. *Let X be a del Pezzo surface of degree* 6. *If X admits a zero-cycle of degree $d \geq 1$, then X has a closed étale point of degree dividing d.*

2. Preliminaries on tori

Let k be a field and k^{s} be its separable closure. For any étale algebra A/k, let $\mathbb{G}_{m,A}$ (or just $\mathbb{G}_m$ when the base is understood) be the abelian group scheme

$\operatorname{Spec} A[t, t^{-1}]$. A connected linear algebraic group T/k is called an *algebraic torus*, *k-torus*, or simply a *torus* if

$$T_{k^s} := T \times_k k^s \cong \mathbb{G}_{m,k^s}^r$$

for some $r \geq 1$, which is called the *rank* of the torus. If E/k is a field extension such that $T_E \cong \mathbb{G}_{m,E}^r$, then E is called a *splitting field* of (and is said to *split*) T.

For any finite étale algebra A/k, let $R_{A/k}$ denote the *Weil restriction* functor (also known as the *restriction of scalars* functor), which takes A-schemes to k-schemes and, in particular, takes A-tori to k-tori. In particular, for any finite separable field extension L/k and any L-torus T, $R_{L/k}T$ is a k-torus. A k-torus T is called *quasitrivial* if it is isomorphic to a finite product of tori of the form $R_{L_i/k}\,\mathbb{G}_m$ where each L_i/k is a finite separable field extension. For any finite separable field extension L/k, call

$$R_{L/k}^{(1)}\,\mathbb{G}_m := \ker[R_{L/k}\,\mathbb{G}_m \xrightarrow{N_{L/k}} \mathbb{G}_m]$$

the *norm torus* associated to that extension; $R_{L/k}^{(1)}\,\mathbb{G}_m$ evidently has rank $[L:k]-1$.

Now, let $\Gamma = \operatorname{Gal}(k^s/k)$. For any rank r k-torus T, define its *character module* to be

$$\mathbf{X}(T) := \operatorname{Hom}(T_{k^s}, \mathbb{G}_{m,k^s})\ [\,\cong \operatorname{Hom}(\mathbb{G}_{m,k^s}^r, \mathbb{G}_{m,k^s}) \cong \mathbb{Z}^r\,].$$

Then $\mathbf{X}(T)$ is a rank r Γ-module. The association $T \mapsto \mathbf{X}(T)$ is an antiequivalence between the categories of k-tori and finitely-generated Γ-modules; in fact, it is an antiequivalence between the categories of k-tori split by a finite Galois extension E/k and finitely-generated $\operatorname{Gal}(E/k)$-modules. The Γ-action on $\mathbf{X}(T)$ yields a continuous representation

$$\Gamma \to \operatorname{Aut}(\mathbf{X}(T)) \cong \operatorname{Aut}(\mathbb{Z}^r) \cong \operatorname{GL}_r(\mathbb{Z})$$

whose kernel $\mathfrak{h} \trianglelefteq \Gamma$ corresponds to the minimal splitting field of T, a finite Galois extension E/k. The group $\operatorname{GL}_r(\mathbb{Z})$ contains the image of this representation, a copy of $\Gamma/\mathfrak{h} \cong \operatorname{Gal}(E/k)$. Call this the *Galois group of* T. On the other hand, an embedding $\operatorname{Gal}(E/k) \to \operatorname{GL}_r(\mathbb{Z})$ lifts to a continuous representation $\Gamma \to \operatorname{GL}_r(\mathbb{Z})$, which determines a Γ-action on $\mathbf{X}(\mathbb{G}_m^r)$, identifying the rank r k-torus $\operatorname{Spec}\big(E[\mathbf{X}(\mathbb{G}_m^r)]^\Gamma\big)$ whose Galois group is $\operatorname{Gal}(E/k)$. Explicitly,

$$\begin{aligned}\{\text{rank } r\ k\text{-tori}\}/\cong\ &\leftrightarrow\ \{\text{rank } r\ \Gamma\text{-modules}\}/\cong\\ &\leftrightarrow\ H^1(k, \operatorname{Aut}(\mathbf{X}(\mathbb{G}_m^r)))\\ &\leftrightarrow\ H^1(k, \operatorname{Aut}(\mathbb{Z}^r))\\ &\leftrightarrow\ H^1(k, \operatorname{GL}_r(\mathbb{Z}))\\ &=\ \operatorname{Hom}(\Gamma, \operatorname{GL}_r(\mathbb{Z}))/\sim\end{aligned}$$

where $\rho \sim \rho'$ if and only if $\rho(\Gamma)$ and $\rho'(\Gamma)$ are conjugate in $\operatorname{GL}_r(\mathbb{Z})$.

To classify rank r tori, it is necessary to count the conjugacy classes of finite subgroups of $\operatorname{GL}_r(\mathbb{Z})$. There are 13 such classes in $\operatorname{GL}_2(\mathbb{Z})$; in [Vos65], however, Voskresenskiĭ gave explicit representations of *15* finite groups in terms of matrix generators along with their associated rank 2 tori. He later corrected this in a short geometric proof that rank 2 tori are rational [Vos98]; here, he noted that there are only two distinct maximal finite subgroups of $\operatorname{GL}_2(\mathbb{Z})$ up to conjugacy, D_4 and D_6, whereas he produced two faithful representations of each of these groups in $\operatorname{GL}_2(\mathbb{Z})$ in his earlier classification paper. For the convenience of the cross-referencing reader, the proof of Theorem 1.1 will follow his original classification.

3. Lemmata

In order to prove Theorem 1.1, a number of key lemmas will be cited repeatedly.

Lemma 3.1. *Totaro's question for* $\operatorname{ind}(X) = 1$ *has a positive answer for abelian algebraic groups (e.g., tori).*

Proof. Let G be an abelian algebraic group defined over a field k. By a well-known fact from Galois cohomology, the composition of the natural restriction and corestriction maps associated to any finite field extension L/k

$$H^1(k, G) \xrightarrow{\text{res}} H^1(L, G_L) \xrightarrow{\text{cores}} H^1(k, G)$$

is the multiplication-by-$[L : k]$ map. Now, fix $X \in H^1(k, G)$. If $X_L = 0 \in H^1(L, G_L)$ for some finite field extension L/k, then

$$[L : k]X = (\text{cores} \circ \text{res})(X) = \text{cores}(0) = 0 \in H^1(k, G),$$

and so $\operatorname{per}(X) \mid [L : k]$. Since L is arbitrary, $\operatorname{per}(X) \mid \operatorname{ind}(X)$. But $\operatorname{ind}(X) = 1$. Then $\operatorname{per}(X) = 1$, meaning that $X = 0 \in H^1(k, G)$. So it suffices to take $F = k$, as desired. □

Lemma 3.2. *Let* L/k *be a finite separable field extension and* $T = R^{(1)}_{L/k}\,\mathbb{G}_m$.

(a) $H^1(k, T) \cong k^\times/N_{L/k}(L^\times)$.
(b) *If* L/k *is cyclic, then* $H^1(k, T) \cong \operatorname{Br}(L/k)$.
(c) $H^1(L, T_L) = 0$. *In particular,* $\operatorname{ind}(X) \mid [L : k]$ *for all* $X \in H^1(k, T)$.

Proof. From the short exact sequence of k-tori

$$1 \to R^{(1)}_{L/k}\,\mathbb{G}_m \to R_{L/k}\,\mathbb{G}_m \xrightarrow{N_{L/k}} \mathbb{G}_m \to 1,$$

taking Galois cohomology yields (by Hilbert 90 and the universal property of Weil restriction) the long exact sequence of groups

$$L^\times \xrightarrow{N_{L/k}} k^\times \to H^1(k, T) \to 1,$$

from which (a) is clear. Now, for any finite cyclic field extension L/k with $\operatorname{Gal}(L/k) \cong \langle\sigma\rangle$, we have a canonical isomorphism (cf. Corollary 4.4.10 from Gille–Szamuely [GS06])

$$k^\times/N_{L/k}(L^\times) \cong \operatorname{Br}(L/k)$$

given by

$$\gamma \mapsto (L/k, \sigma, \gamma)$$

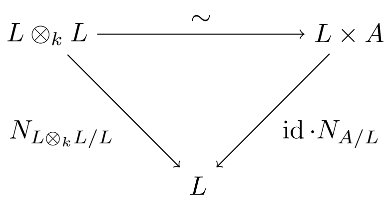
where $(L/k, \sigma, \gamma)$ is the cyclic algebra generated over L by u with relations $ux = \sigma(x)u$ for any $x \in L$ and $u^{[L:k]} = \gamma$. From this, (b) follows immediately. Finally, if $L \cong k[x]/\left(p(x)\right)$ and $a_1, \dots, a_m$ are the roots of $p(x)$ in L, then

$$p(x) = q(x)\prod_{i=1}^{m}(x - a_i)$$

for some $q(x) \in L[x]$. By the Chinese Remainder Theorem,

$$\begin{aligned} L \otimes_k L &\cong L \otimes_k k[x]/\left(p(x)\right) \\ &\cong L[x]/\left(q(x)\right) \times \prod_{i=1}^{m} L[x]/\left(x - a_i\right) \\ &\cong L \times A \end{aligned}$$

where A/L is a finite étale algebra. So the following diagram commutes.



In particular, $N_{L\otimes_k L/L}$ is surjective since

$$(\mathrm{id} \cdot N_{A/L})(\lambda, 1, \ldots, 1) = \lambda$$

for any $\lambda \in L$. Then

$$H^1(L, T_L) \cong L^\times / N_{L\otimes_k L/L}\left((L \otimes_k L)^\times\right) = 0,$$

hence (c). □

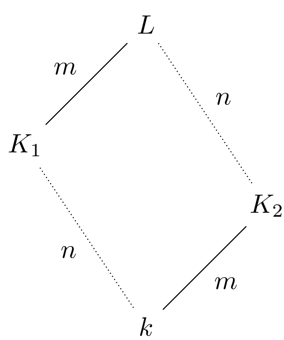
Lemma 3.3. *Let T be a k-torus with a (not necessarily minimal) splitting field E of finite degree over k, and let $X \in H^1(k, T)$.*

(a) $\mathrm{ind}(X) \mid [E : k]$.
(b) *If $[E : k]$ is prime, then Totaro's question has a positive answer for T.*

Proof. Since T_E is split, $H^1(E, T_E) = 0$ by Hilbert 90. Then $\mathrm{ind}(X) \mid [E : k]$. If $[E : k]$ is prime, then by (a), $\mathrm{ind}(X) = 1$ or $[E : k]$, hence either $F = k$ or E suffices, respectively. □

For any finite extension of étale algebras A/B, let $(A^\times)^{(1)}_B := \{a \in A^\times : N_{A/B}(a) = 1\}$.

Lemma 3.4. *Consider the following diagram of separable field extensions:*



for some $m, n > 1$, and let $T = R_{K_1/k}\left(R^{(1)}_{L/K_1}\mathbb{G}_m\right) \cap R_{K_2/k}\left(R^{(1)}_{L/K_2}\mathbb{G}_m\right)$.

(a) *The following sequences of k-tori are exact:*

$$1 \to T \to R_{K_1/k}(R^{(1)}_{L/K_1}\mathbb{G}_m) \xrightarrow{N_{L/K_2}} R^{(1)}_{K_2/k}\mathbb{G}_m \to 1,$$

$$1 \to T \to R_{K_2/k}(R^{(1)}_{L/K_2}\mathbb{G}_m) \xrightarrow{N_{L/K_1}} R^{(1)}_{K_1/k}\mathbb{G}_m \to 1.$$

(b) *The following sequences of abelian groups are exact:*

$$(L^\times)^{(1)}_{K_1} \xrightarrow{N_{L/K_2}} (K_2^\times)^{(1)}_k \to H^1(k,T) \xrightarrow{\delta_1} K_1^\times/N_{L/K_1}(L^\times),$$

$$(L^\times)^{(1)}_{K_2} \xrightarrow{N_{L/K_1}} (K_1^\times)^{(1)}_k \to H^1(k,T) \xrightarrow{\delta_2} K_2^\times/N_{L/K_2}(L^\times).$$

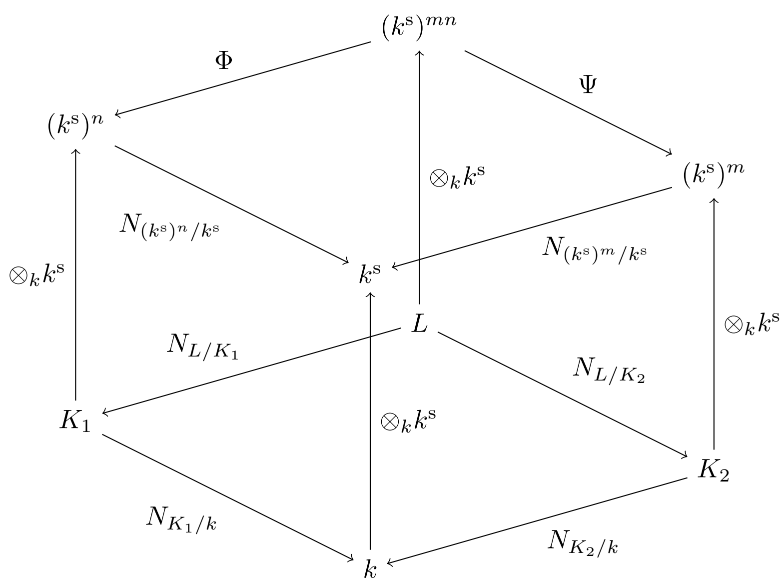
Proof. Left exactness of both sequences is clear from the construction of T, so proving (a) amounts to showing that N_{L/K_2} and N_{L/K_1} are surjective after extending scalars to k^{s}. If $\Phi : (k^{\mathrm{s}})^{mn} \to (k^{\mathrm{s}})^n$ and $\Psi : (k^{\mathrm{s}})^{mn} \to (k^{\mathrm{s}})^m$ are the maps defined by

$$\Phi(x_{ij} : 1 \le i \le m, 1 \le j \le n) = \left(\prod_{i=1}^m x_{i1}, \ldots, \prod_{i=1}^m x_{in}\right)$$

and

$$\Psi(x_{ij} : 1 \le i \le m, 1 \le j \le n) = \left(\prod_{j=1}^n x_{1j}, \ldots, \prod_{j=1}^n x_{mj}\right),$$

then the following diagram commutes.



Any $a \in (R^{(1)}_{K_2 \otimes_k k^{\mathrm{s}}/k^{\mathrm{s}}} \mathbb{G}_m)(k^{\mathrm{s}})$ then corresponds to an m-tuple $(a_1, \ldots, a_m) \in (k^{\mathrm{s}})^m$ such that $\prod_{i=1}^m a_i = 1$. But Ψ is surjective: if $x_{ij} = a_i$ when $j = 1$ and $x_{ij} = 1$ otherwise, then

$$\begin{aligned}\Psi(x_{ij} : 1 \le i \le m, 1 \le j \le n) &= \Psi(a_1, \underbrace{1, \ldots, 1}_{n-1 \text{ times}}, a_2, \underbrace{1, \ldots, 1}_{n-1 \text{ times}}, \ldots, a_m, \underbrace{1, \ldots, 1}_{n-1 \text{ times}}) \\ &= (a_1, \ldots, a_m),\end{aligned}$$

and in fact,

$$\Phi(x_{ij} : 1 \le i \le m, 1 \le j \le n) = \Phi(a_1, \underbrace{1, \dots, 1}_{n-1 \text{ times}}, a_2, \underbrace{1, \dots, 1}_{n-1 \text{ times}}, \dots, a_m, \underbrace{1, \dots, 1}_{n-1 \text{ times}}) = (\underbrace{1, \dots, 1}_{n \text{ times}}).$$

Therefore this mn-tuple yields a k^{s}-point of $R^{(1)}_{L\otimes_k k^{\mathrm{s}}/K_1\otimes_k k^{\mathrm{s}}}\mathbb{G}_m$ mapping to $a \in R^{(1)}_{K_2\otimes_k k^{\mathrm{s}}/k^{\mathrm{s}}}(k^{\mathrm{s}})$. Then N_{L/K_2} is surjective as a map of algebraic groups. By a symmetric argument, N_{L/K_1} is surjective too, proving (a). (b) follows by taking Galois cohomology of these short exact sequences of k-tori and applying Lemma 3.2. □

4. Technical results

Two technical propositions are needed for the proof of Theorem 1.1.

Proposition 4.1. *Let $L/K/k$ be a tower of separable quadratic extensions with no intermediate fields between k and L other than K, and let*

$$T = R_{K/k}(R^{(1)}_{L/K}\mathbb{G}_m).$$

Then Totaro's question has a positive answer for T.

Proof. Let M be the Galois closure of L/k in k^{s} and $G = \mathrm{Gal}(M/k)$. Either $M = L$, in which case $G \cong \mathbb{Z}/4\mathbb{Z}$, or $[M : L] = 2$, in which case $G \cong D_4$. Suppose that $M = L$. Then

$$K \otimes_k L \cong L \times L$$

as $K \subseteq L$, $[K : k] = 2$, and K/k is separable, and

$$L \otimes_k L \cong (L \times L) \times (L \times L)$$

as $[L : k] = 4$ and L/k is Galois. So the following diagram commutes.

$$\begin{array}{ccc} L \otimes_k L & \xrightarrow{\sim} & (L \times L) \times (L \times L) \\ \downarrow{\scriptstyle N_{L\otimes_k L/K\otimes_k L}} & & \downarrow{\scriptstyle N_{L\times L/L} \times N_{L\times L/L}} \\ K \otimes_k L & \xrightarrow{\sim} & L \times L \end{array}$$

Since $N_{L\times L/L} \times N_{L\times L/L}$ is surjective, so is $N_{L\otimes_k L/K\otimes_k L}$, and so by Lemma 3.2.(a),

$$H^1(L, T_L) \cong (K \otimes_k L)^\times / N_{L\otimes_k L/K\otimes_k L}\left((L \otimes_k L)^\times\right) = 0.$$

If $[M : L] = 2$, then since D_4 contains three distinct subgroups of order 2, there is another tower of separable extensions $M/L'/k$ such that $[M : L'] = 2$,

$$K \otimes_k L' \cong M,$$

and

$$L \otimes_k L' \cong M \times M.$$

So the following diagram commutes.

$$\begin{array}{ccc} L \otimes_k L' & \xrightarrow{\sim} & M \times M \\ \big\downarrow N_{L\otimes_k L'/K\otimes_k L'} & & \big\downarrow N_{M\times M/M} \\ K \otimes_k L' & \xrightarrow{\sim} & M \end{array}$$

Since $N_{M\times M/M}$ is surjective, so is $N_{L\otimes_k L'/K\otimes_k L'}$, and so by Lemma 3.2.(a),

$$H^1(L', T_{L'}) \cong (K \otimes_k L')^\times / N_{L\otimes_k L'/K\otimes_k L'}\left((L \otimes_k L')^\times\right) = 0.$$

So $\mathrm{ind}(X) \mid 4$ for any $X \in H^1(k, T)$, and if $\mathrm{ind}(X) = 4$, then either $F = L$ or L' will suffice.

Suppose now that $\mathrm{ind}(X) = 2$. Let $X = [\beta]$ with some $\beta \in K^\times$ that is not a norm from $L^\times$. Since $\mathrm{ind}(X) = 2$, it can be assumed by Theorem 9.2 from Gabber–Liu–Lorenzini [GLL13] using standard Galois theory reductions (cf. Lemma 1.5 from Garibaldi–Hoffmann [GH06]) that there is a tower of separable field extensions $E'/E/k$ such that $[E' : E] = 2$, $[E : k] = m$ for some odd m, and

$$\beta \in N_{L\otimes_k E'/K\otimes_k E'}\left((L \otimes_k E')^\times\right).$$

Write

$$E' \cong \begin{cases} E[x]/(x^2 + x + a) & \text{if } \mathrm{char}(k) = 2, \\ E[x]/(x^2 - a) & \text{if } \mathrm{char}(k) \neq 2, \end{cases}$$

for some $a \in E^\times$. In both cases, identify the class of x with $i \in E'$. Then there are $u_0, v_0 \in LE$ not both zero such that

$$\begin{aligned} \beta &= N_{L\otimes_k E'/K\otimes_k E'}(u_0 + v_0 i) \\ &= \left(N_{LE/KE}(u_0) + aN_{LE/KE}(v_0)\right) + T_E(u_0, v_0)i, \end{aligned}$$

where

$$T_K(u, v) = \begin{cases} \mathrm{tr}_{L/K}(u\overline{v}) + N_{L/K}(v) & \text{if } \mathrm{char}(k) = 2, \\ \mathrm{tr}_{L/K}(u\overline{v}) & \text{if } \mathrm{char}(k) \neq 2. \end{cases}$$

Since $\beta \in K^\times$, $T_E(u_0, v_0) = 0$, and so

$$\beta = N_{LE/KE}(u_0) + aN_{LE/KE}(v_0).$$

If $v_0 = 0$, then $\beta = N_{LE/KE}(u_0)$, in which case $\beta \in K^\times$ is represented by the K-quadratic form $N_{L/K}$ after extending scalars to KE. But $[KE : K] = [E : k] = m$ is odd. Then by Springer's Theorem [Spr52], $\beta \in N_{L/K}(L^\times)$, a contradiction. So $v_0 \neq 0$.

Now, write

$$K \cong \begin{cases} k[y]/(y^2 + y + b) & \text{if } \mathrm{char}(k) = 2, \\ k[y]/(y^2 - b) & \text{if } \mathrm{char}(k) \neq 2, \end{cases}$$

for some $b \in k^\times$. In both cases, identify the class of y with $j \in K$. Then there are $\beta_1, \beta_2 \in k$ not both zero such that

$$\beta = \beta_1 + \beta_2 j.$$

Let $N^1, N^2 : L \to k$ and $Q^1, Q^2 : L^2 \to k$ be the k-quadratic forms defined by

$$N_{L/K} = N^1 + N^2 j,$$

$$Q^1(u, v) = \beta_1 N^1(u) + b\beta_2 N^2(u) - N^1(v),$$

and

$$Q^2(u,v) = \begin{cases} (\beta_1+\beta_2)N^2(u) + \beta_2 N^1(u) + N^2(v) & \text{if } \mathrm{char}(k)=2, \\ \beta_1 N^2(u) + \beta_2 N^1(u) - N^2(v) & \text{if } \mathrm{char}(k)\neq 2. \end{cases}$$

Then setting $x_0 = v_0^{-1}$ and $y_0 = u_0 v_0^{-1}$,

$$\begin{aligned} a &= \beta N_{LE/KE}(x_0) - N_{LE/KE}(y_0) \\ &= (\beta_1 + \beta_2 j)(N^1_{LE} + N^2_{LE} j)(x_0) - (N^1_{LE} + N^2_{LE} j)(y_0) \\ &= Q^1_E(x_0,y_0) + Q^2_E(x_0,y_0)j. \end{aligned}$$

Since $a \in E^\times$, $Q^1_E(x_0,y_0) = a$ and $Q^2_E(x_0,y_0)=0$. Now, case by $\mathrm{char}(k)$.

First, suppose that $\mathrm{char}(k)\neq 2$. Since $\mathrm{tr}_{LE/KE}(y_0)=0$, the isotropic vector for Q^2_E comes from the subspace

$$LE \oplus (LE)^0 \cong (L \oplus L^0) \otimes_k E,$$

where $L^0 = \ker \mathrm{tr}_{L/K} \subseteq L$. But as $[E:k]=m$ is odd, Q^2 is isotropic by Springer's Theorem [Spr52]. So there is some $(x_1,y_1) \in L \oplus L^0$ such that

$$Q^2(x_1,y_1) = \beta_1 N^2(x_1) + \beta_2 N^1(x_1) - N^2(y_1) = 0.$$

If $x_1 = 0$, then y_1 is an isotropic vector for N^2. But isotropic quadratic forms are universal. So for any x, there is a y such that $N^2(y) = \beta_1 N^2(x) + \beta_2 N^1(x)$, i.e., $Q^2(x,y)=0$. Then we can assume that $x_1 \neq 0$. So

$$\begin{aligned} \alpha &= Q^1(x_1,y_1) \\ &= Q^1(x_1,y_1) + Q^2(x_1,y_1)j \\ &= \beta N_{L/K}(x_1) - N_{L/K}(y_1) \end{aligned}$$

means that

$$N_{L/K}(x_1^{-1})(N_{L/K}(y_1)+\alpha) = \beta.$$

With $F = k(\sqrt{\alpha})$, $[F:k]=2$, and since $y_1 \in L^0$,

$$N_{L\otimes_k F/K\otimes_k F}\left(\frac{y_1+\sqrt{\alpha}}{x_1}\right) = \beta.$$

Then $X_F = 0 \in H^1(F,T_F)$, as desired.

Now, suppose that $\mathrm{char}(k)=2$. Let $T^1, T^2 : L \to k$ be the k-linear maps defined by

$$\mathrm{tr}_{L/K} = T^1 + T^2 j.$$

Since

$$\begin{aligned} (T^1_E(y_0)+1) + T^2_E(y_0)j &= \mathrm{tr}_{LE/KE}(y_0) + 1 \\ &= \mathrm{tr}_{LE/KE}(u_0 v_0^{-1}) + 1 \\ &= N_{LE/KE}(v_0)\left(\mathrm{tr}_{LE/KE}(u_0\overline{v_0}) + N_{LE/KE}(v_0)\right) \\ &= 0, \end{aligned}$$

$T^2_E(y_0)=0$, and so the isotropic vector for Q^2_E comes from the subspace

$$LE \oplus (LE)^\# \cong (L \oplus L^\#) \otimes_k E,$$

where $L^\# = \ker T^2 \subseteq L$. But as $[E:k]=m$ is odd, Q^2 is isotropic by Springer's Theorem [Spr52]. So there is some $(x_1,y_1) \in L \oplus L^\#$ such that

$$Q^2(x_1,y_1) = (\beta_1+\beta_2)N^2(x_1) + \beta_2 N^1(x_1) + N^2(y_1) = 0.$$

If $x_1=0$, then y_1 is an isotropic vector for N^2. But the symmetric bilinear form

$$b_{N^2} : L^2 \to k$$

defined by

$$b_{N^2}(x,y) := N^2(x+y) - N^2(x) - N^2(y) = T^2(x\overline{y})$$

is non-degenerate. Then N^2 is regular and isotropic, hence universal [EKM08]. So as before, we can assume that $x_1 \neq 0$. Let $\gamma = T^1(y_1)$. If $\gamma = 0$, then $y_1 = 0$ as $y_1 \in L^{\#}$. Setting $\alpha = Q^1(x_1, 0)$ and $F = k[z]/(z^2 + z + \alpha)$ and identifying the class of z with $\lambda \in F$ yields that

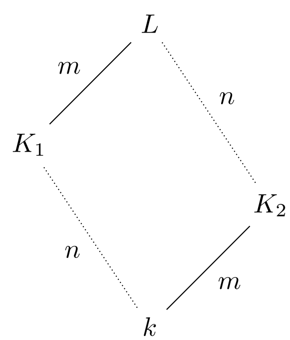
$$N_{L\otimes_k F/K\otimes_k F}\left(\frac{\lambda}{x_1}\right) = \beta.$$

If $\gamma \neq 0$, then

$$N_{L\otimes_k F/K\otimes_k F}\left(\frac{y_1 + \gamma\lambda}{\gamma x_1}\right) = \beta.$$

In both cases, $[F : k] = 2$ and $X_F = 0 \in H^1(F, T_F)$, as desired. □

Proposition 4.2. *Consider the following diagram of separable field extensions:*



for some coprime $m, n > 1$, and let

$$T = R_{K_1/k}\left(R^{(1)}_{L/K_1}\,\mathbb{G}_m\right) \cap R_{K_2/k}\left(R^{(1)}_{L/K_2}\,\mathbb{G}_m\right).$$

Then Totaro's question has a positive answer for $X \in H^1(k,T)$ of index m, n, and mn. Furthermore, if $(\mathrm{ind}(X), m) = 1$, *then* $\mathrm{ind}(X) \mid n$, *and if* $(\mathrm{ind}(X), n) = 1$, *then* $\mathrm{ind}(X) \mid m$.

Proof. By Lemma 3.4.(b), the following sequences of abelian groups are exact:

$$(L^\times)^{(1)}_{K_1} \xrightarrow{N_{L/K_2}} (K_2^\times)^{(1)}_k \to H^1(k,T) \xrightarrow{\delta_1} K_1^\times/N_{L/K_1}(L^\times),$$

$$(L^\times)^{(1)}_{K_2} \xrightarrow{N_{L/K_1}} (K_1^\times)^{(1)}_k \to H^1(k,T) \xrightarrow{\delta_2} K_2^\times/N_{L/K_2}(L^\times).$$

The proof will proceed according to the index.

First, suppose that $\mathrm{ind}(X) = m$. Since $[L : K_2] = n$, $K_2^\times/N_{L/K_2}(L^\times)$ is n-torsion. But $(m,n) = 1$, and $\mathrm{per}(X) \mid \mathrm{ind}(X)$. So $\delta_2(X) = 1$. Then X lifts to some $\beta \in (K_1^\times)^{(1)}_k$. Now,

$$K_2 \otimes_k K_2 \cong K_2 \times B,$$

where B/K_2 is an étale algebra as K_2/k is separable,

$$K_1 \otimes_k K_2 \cong L$$

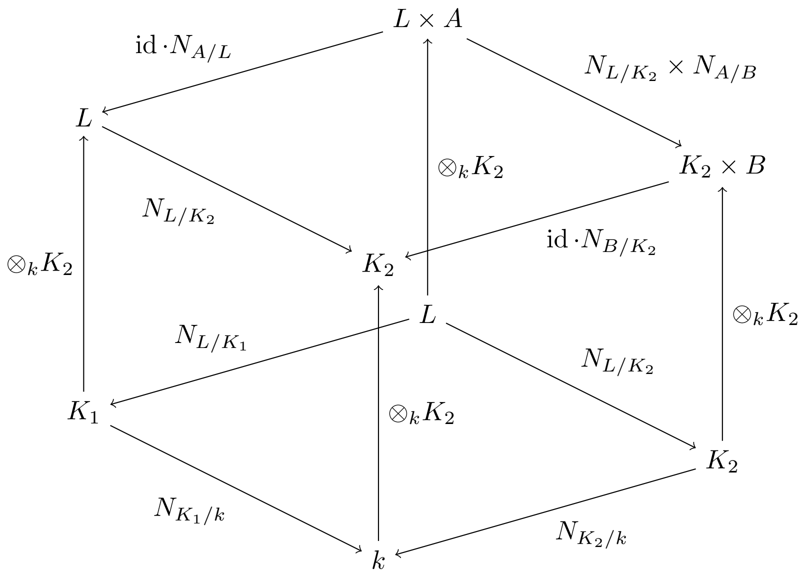
as $K_1, K_2 \subseteq L$ have coprime degrees and are therefore k-linearly disjoint such that

$$[K_1 : k][K_2 : k] = mn = [L : k],$$

and

$$L \otimes_k K_2 \cong L \times A,$$

where $A \cong B \otimes_{K_2} L/L$ is an étale algebra as $K_2 \subseteq L$ and K_2/k is separable. After identifying through the natural isomorphisms, the following diagram commutes.



Observe that

$$(\mathrm{id} \cdot N_{A/L})(\beta, 1) = \beta$$

and

$$(N_{L/K_2} \times N_{A/B})(\beta, 1) = \big(N_{K_1/k}(\beta), N_{A/B}(1)\big) = (1, 1),$$

meaning that $X_{K_2} = 0 \in H^1(K_2, T_{K_2})$. Since $\mathrm{ind}(X) = [K_2 : k] = m$, it suffices to take $F = K_2$. But only that $(\mathrm{ind}(X), n) = 1$ is needed to show that $X_{K_2} = 0$. So $(\mathrm{ind}(X), n) = 1$ implies that $\mathrm{ind}(X) \mid m$. By a symmetric argument, $F = K_1$ suffices when $\mathrm{ind}(X) = n$, and $(\mathrm{ind}(X), m) = 1$ implies that $\mathrm{ind}(X) \mid n$.

Now, suppose that $\mathrm{ind}(X) = mn$. Since the sequence of k-tori

$$1 \to T \to R_{K_1/k}(R^{(1)}_{L/K_1} \mathbb{G}_m) \xrightarrow{N_{L/K_2}} R^{(1)}_{K_2/k} \mathbb{G}_m \to 1$$

is short exact, so is the sequence of K_2-tori

$$1 \to T_{K_2} \to R_{L/K_2}(R^{(1)}_{L \times A/L} \mathbb{G}_m) \xrightarrow{N_{L \times A/K_2 \times B}} R^{(1)}_{K_2 \times B/K_2} \mathbb{G}_m \to 1.$$

Since K_2^{s}-points of $R^{(1)}_{K_2 \times B/K_2} \mathbb{G}_m$ take the form $(N_{B \otimes_{K_2} K_2^{\mathrm{s}}/K_2^{\mathrm{s}}}(\beta^{-1}), \beta)$ for $\beta \in (B \otimes_{K_2} K_2^{\mathrm{s}})^\times$,

$$R^{(1)}_{K_2 \times B/K_2} \mathbb{G}_m \cong \mathbb{G}_{m,B}\,.$$

By a similar argument,

$$R_{L/K_2}(R^{(1)}_{L \times A/L} \mathbb{G}_m) \cong R_{L/K_2} \mathbb{G}_{m,A}\,.$$

So

$$1 \to T_{K_2} \to R_{L/K_2} \mathbb{G}_{m,A} \xrightarrow{N_{A/B}} \mathbb{G}_{m,B} \to 1$$

is a short exact sequence of K_2-tori. Since A/L is an étale algebra, $H^1(L, \mathbb{G}_{m,A}) = 0$ by Hilbert 90. Taking Galois cohomology then yields the long exact sequence of abelian groups

$$A^\times \xrightarrow{N_{A/B}} B^\times \to H^1(K_2, T_{K_2}) \to 0.$$

So X_{K_2} lifts to some $\beta \in B^\times$. Let C/L be the étale algebra such that

$$L \otimes_{K_2} L \cong L \times C.$$

Then since

$$\begin{aligned} A \otimes_{K_2} L &\cong B \otimes_{K_2} L \otimes_{K_2} L \\ &\cong B \otimes_{K_2} (L \times C) \\ &\cong A \times (B \otimes_{K_2} C), \end{aligned}$$

the following diagram commutes.

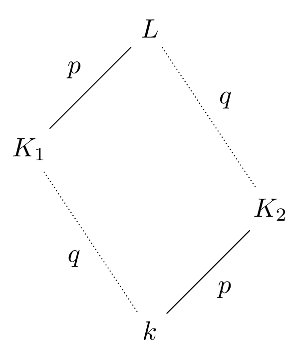
$$\begin{array}{ccc} A \otimes_{K_2} L & \xrightarrow{\sim} & A \times (B \otimes_{K_2} C) \\ \downarrow{\scriptstyle N_{A\otimes_{K_2} L/A}} & & \downarrow{\scriptstyle \mathrm{id} \cdot N_{B\otimes_{K_2} C/A}} \\ B \otimes_{K_2} L & \xrightarrow{\sim} & A \end{array}$$

But

$$(\mathrm{id} \cdot N_{B\otimes_{K_2} C/A})(\beta, 1) = \beta,$$

meaning that $X_L = (X_{K_2})_L = 0 \in H^1(L, T_L)$. Since $[L : k] = mn$, $F = L$ suffices. $\square$

Corollary 4.3. *Consider the following diagram of separable field extensions:*



for some distinct primes p and q, and let

$$T = R_{K_1/k}\left(R^{(1)}_{L/K_1}\,\mathbb{G}_m\right) \cap R_{K_2/k}\left(R^{(1)}_{L/K_2}\,\mathbb{G}_m\right).$$

Then Totaro's question has a positive answer for T.

Proof. The claim follows immediately from Proposition 4.2. $\square$

5. Proof of Theorem 1.1

The proof of Theorem 1.1 will proceed according to $\mathrm{Gal}(E/k)$ where E is the minimal splitting field of the torus. Recall that for a given group, there may be multiple isomorphism classes of tori associated to that group (over suitably general fields) depending on how many conjugacy classes represent its isomorphism class in $\mathrm{GL}_2(\mathbb{Z})$. Finally: by Lemma 3.1 and Lemma 3.3, one can reduce $\mathrm{ind}(X)$ to be a non-trivial proper divisor of $[E:k]$.

5.1. Rank 1 tori. There are only two (conjugacy classes of) finite subgroups of $\mathrm{GL}_1(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$: (1) and $\mathbb{Z}/2\mathbb{Z}$. These correspond to the two classes of rank 1 tori. For both types, a positive answer to Totaro's question is a trivial consequence of the previous reductions.

5.1.1 $\mathrm{Gal}(E/k) \cong (1)$ and $T \cong \mathbb{G}_m$.

Proof. T is quasitrivial, and so we are done by Hilbert 90. □

5.1.2 $\mathrm{Gal}(E/k) \cong \mathbb{Z}/2\mathbb{Z}$ and $T \cong R^{(1)}_{E/k}\,\mathbb{G}_m$.

Proof. $[E:k]$ is prime, and so we are done by Lemma 3.3.(b). □

5.2. Rank 2 tori. There are 9 isomorphism classes and 15 conjugacy classes of finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$.

5.2.1 $\mathrm{Gal}(E/k) \cong (1)$ and $T \cong \mathbb{G}_m \times \mathbb{G}_m$.

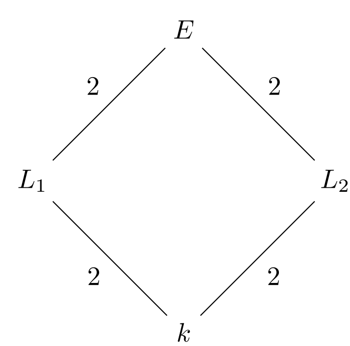
Proof. T is quasitrivial, and so we are done by Hilbert 90. □

5.2.2 $\mathrm{Gal}(E/k) \cong \mathbb{Z}/2\mathbb{Z}$.

(a) $T \cong R^{(1)}_{E/k} \times R^{(1)}_{E/k}\,\mathbb{G}_m$.
(b) $T \cong \mathbb{G}_m \times R^{(1)}_{E/k}\,\mathbb{G}_m$.
(c) $T \cong R_{E/k}\,\mathbb{G}_m$.

Proof. $[E:k]$ is prime, and so we are done by Lemma 3.3.(b). □

5.2.3 $\mathrm{Gal}(E/k) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.



(a) $T \cong R_{L_1/k}\left(R^{(1)}_{E/L_1}\,\mathbb{G}_m\right)$.

Proof. Since $[E : k] = 4$, we can assume that $\mathrm{ind}(X) = 2$. Then

$$H^1(k, T) \cong H^1(L_1, R^{(1)}_{E/L_1}\,\mathbb{G}_m) \cong \mathrm{Br}(E/L_1)$$

by Lemma 3.2.(b). Let $\delta : H^1(k, T) \to \mathrm{Br}(E/L_1)$ denote the composition. Since

$$\begin{aligned} \delta(X_{L_2}) &\cong \delta(X) \otimes_k L_2 \\ &\cong \delta(X) \otimes_{L_1} L_1 \otimes_k L_2 \\ &\cong \delta(X) \otimes_{L_1} E \end{aligned}$$

is split and $[L_2 : k] = 2$, it suffices to take $F = L_2$. □

(b) $T \cong R^{(1)}_{L_1/k}\,\mathbb{G}_m \times R^{(1)}_{L_2/k}\,\mathbb{G}_m$.

Proof. Since $[E : k] = 4$, we can assume that $\mathrm{ind}(X) = 2$. As

$$\begin{aligned} H^1(k, T) &\cong H^1(k, R^{(1)}_{L_1/k}\,\mathbb{G}_m \times R^{(1)}_{L_2/k}\,\mathbb{G}_m) \\ &\cong H^1(k, R^{(1)}_{L_1/k}\,\mathbb{G}_m) \times H^1(k, R^{(1)}_{L_2/k}\,\mathbb{G}_m) \\ &\cong \mathrm{Br}(L_1/k) \times \mathrm{Br}(L_2/k) \end{aligned}$$

by Lemma 3.2.(b), X can be identified with a pair of division algebras $D_1 \in \mathrm{Br}(L_1/k)$ and $D_2 \in \mathrm{Br}(L_2/k)$. Since D_1 and D_2 are both split over quadratic extensions L_1 and L_2, respectively, each is either a field or a quaternion division algebra. If either of D_1 or D_2 is a field, then it suffices to take either $F = L_2$ or L_1, respectively. So we can assume that both D_1 and D_2 are quaternion division algebras.

Let $D = D_1 \otimes_k D_2$. By Albert's Theorem [Alb72], either D is a division algebra or D_1 and D_2 have a common subfield F separable over k such that $[F : k] = 2$ that necessarily splits both algebras. Suppose that D is a division algebra. Then

$$\mathrm{ind}(D) = \deg(D) = \deg(D_1)\deg(D_2) = 4.$$

But since $\mathrm{ind}(X) = 2$, it can be assumed by Theorem 9.2 from Gabber–Liu–Lorenzini [GLL13] using standard Galois theory reductions (cf. Lemma 1.5 from Garibaldi–Hoffmann [GH06]) that there is a tower of separable field extensions $K'/K/k$ such that $[K' : K] = 2$, $[K : k]$ is odd, and ${D_1}_{K'}$ and ${D_2}_{K'}$ (hence $D_{K'}$) are split. Since $[K : k]$ is odd and $\mathrm{ind}(D) = 4$, D_K is a division algebra. But as $D_{K'}$ is split and $[K' : K] = 2$,

$$\mathrm{ind}(D) = \mathrm{ind}(D_K) = 2,$$

a contradiction. So D_1 and D_2 have a common subfield F separable over k such that $[F : k] = 2$ that necessarily splits both algebras, completing the proof. □

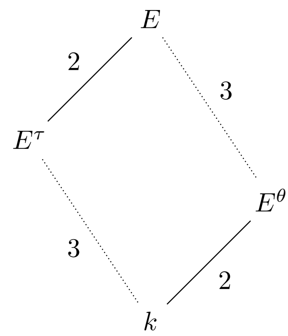
5.2.4 $\mathrm{Gal}(E/k) \cong \mathbb{Z}/3\,\mathbb{Z}$ and $T \cong R^{(1)}_{E/k}\,\mathbb{G}_m$.

Proof. $[E : k]$ is prime, and so we are done by Lemma 3.3.(b). □

5.2.5 $\mathrm{Gal}(E/k) \cong \mathbb{Z}/4\,\mathbb{Z} = \langle \phi \rangle$ and $T \cong R_{E^{\phi^2}/k}\left(R^{(1)}_{E/E^{\phi^2}}\,\mathbb{G}_m\right)$.

Proof. We are done by Proposition 4.1. □

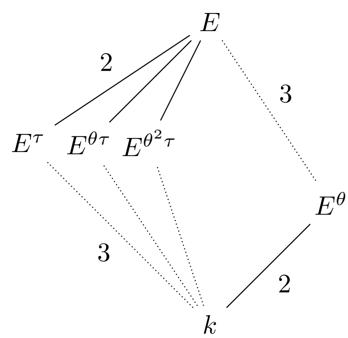
5.2.6 $\mathrm{Gal}(E/k) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle\theta\rangle \times \langle\tau\rangle$.



$$T = R_{E^\tau/k}\left(R^{(1)}_{E/E^\tau}\,\mathbb{G}_m\right) \cap R_{E^\theta/k}\left(R^{(1)}_{E/E^\theta}\,\mathbb{G}_m\right).$$

Proof. We are done by Corollary 4.3. □

5.2.7 $\mathrm{Gal}(E/k) \cong S_3 = \langle\theta\rangle \rtimes \langle\tau\rangle$.



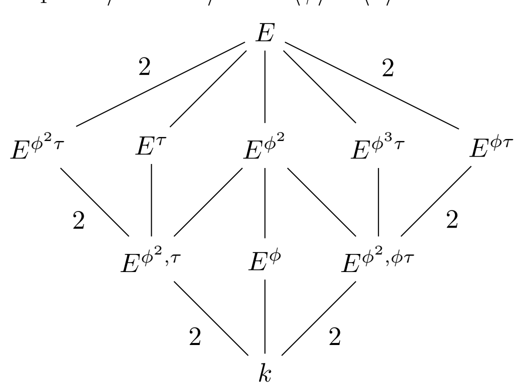
(a) $T \cong R^{(1)}_{E^\tau/k}\,\mathbb{G}_m$.

Proof. Since $[E:k] = 6$, the only cases to consider are $\mathrm{ind}(X) = 2$ and 3. But by Lemma 3.2.(c), only $\mathrm{ind}(X) = 3$ is possible, and $F = E^\tau$ suffices by Lemma 3.2.(c). □

(b) $T \cong R_{E^\tau/k}\left(R^{(1)}_{E/E^\tau}\,\mathbb{G}_m\right) \cap R_{E^\theta/k}\left(R^{(1)}_{E/E^\theta}\,\mathbb{G}_m\right)$.

Proof. We are done by Corollary 4.3. □

5.2.8 $\mathrm{Gal}(E/k) \cong D_4 \cong \mathbb{Z}/4\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} = \langle\phi\rangle \rtimes \langle\tau\rangle$.



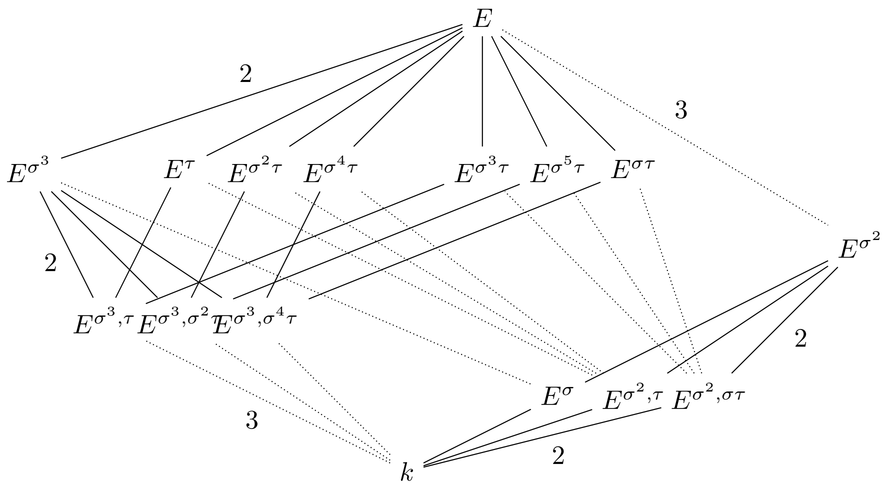
(a) $T \cong R_{E^{\phi^2,\tau}/k}\left(R^{(1)}_{E^{\tau}/E^{\phi^2,\tau}}\,\mathbb{G}_m\right)$.

Proof. We are done by Proposition 4.1. ☐

(b) $T \cong R_{E^{\phi^2,\phi\tau}/k}\left(R^{(1)}_{E^{\phi\tau}/E^{\phi^2,\phi\tau}}\,\mathbb{G}_m\right)$.

Proof. T is isomorphic to the torus from (a). ☐

5.2.9 $\mathrm{Gal}(E/k) \cong D_6 \cong \mathbb{Z}/6\,\mathbb{Z} \rtimes \mathbb{Z}/2\,\mathbb{Z} = \langle\sigma\rangle \rtimes \langle\tau\rangle$.



(a) $T \cong R_{E^{\sigma^2}/k}\left(R^{(1)}_{E/E^{\sigma^2}}\,\mathbb{G}_m\right) \cap R_{E^{\sigma^3}/k}\left(R^{(1)}_{E/E^{\sigma^3}}\,\mathbb{G}_m\right) \cap R_{E^{\tau}/k}\,\mathbb{G}_m$.

Proof. Observe that $t \in T(A)$ for a k-algebra A if and only if

$$\begin{aligned} t^{\sigma^2}t^{\sigma^4}t &= 1, \\ t^{\sigma^3}t &= 1, \\ t^{\tau} &= t, \end{aligned}$$

which means that

$$T \cong R_{E^{\sigma^2,\tau}/k}\left(R^{(1)}_{E^{\tau}/E^{\sigma^2,\tau}}\,\mathbb{G}_m\right) \cap R_{E^{\sigma^3,\tau}/k}\left(R^{(1)}_{E^{\tau}/E^{\sigma^3,\tau}}\,\mathbb{G}_m\right).$$

So we are done by Proposition 4.1. ☐

(b) $T \cong R_{E^{\sigma^2}/k}\left(R^{(1)}_{E/E^{\sigma^2}}\,\mathbb{G}_m\right) \cap R_{E^{\sigma^3}/k}\left(R^{(1)}_{E/E^{\sigma^3}}\,\mathbb{G}_m\right) \cap R_{E^{\tau}/k}\left(R^{(1)}_{E/E^{\tau}}\,\mathbb{G}_m\right)$.

Proof. T is isomorphic to the torus from (a). ☐

This exhausts Voskresenskiĭ's classification and thus completes the proof of Theorem 1.1.

6. del Pezzo surfaces

We now prove a general consequence of Theorem 1.1.

Corollary 6.1. *Let X be a regular variety over a field containing a principal homogeneous space of a smooth torus of rank ≤ 2 as a dense open subset. If X admits a zero-cycle of degree $d \geq 1$, then X has a closed étale point of degree dividing d.*

Proof. Write $X = \overline{Y}$ for some principal homogeneous space Y under a torus T of rank ≤ 2. By a general moving lemma for zero-cycles (cf. Theorem 6.8 from Gabber–Liu–Lorenzini [GLL13]), given a closed point on X of degree n, there is a zero-cycle on Y of degree n. So given a zero-cycle on X of degree d, there is a zero-cycle on Y of degree d. By Theorem 1.1, $Y \subseteq X$ has a closed étale point of degree dividing d. □

A *del Pezzo surface* is a smooth projective surface X over a field k whose anticanonical bundle ω_X^{-1} is ample. Its *degree* is the self-intersection number $D = (K_X, K_X)$ of its canonical divisor K_X and lies between 1 and 9. If $D = 8$, then X_{k^s} is isomorphic to either $\mathbb{P}^2_{k^s}$ blown up at a point or $\mathbb{P}^1_{k^s} \times \mathbb{P}^1_{k^s}$; otherwise, X_{k^s} is isomorphic to $\mathbb{P}^2_{k^s}$ blown up at $9 - D$ points in general position. Manin [Man86] is a standard reference for these results; in fact, it is a theorem of Manin that del Pezzo surfaces of degree 6 contain torsors of rank 2 tori as dense open subsets (cf. Teorema 8.6 from [Man72], Theorem 30.3.1 from [Man86]). This gives

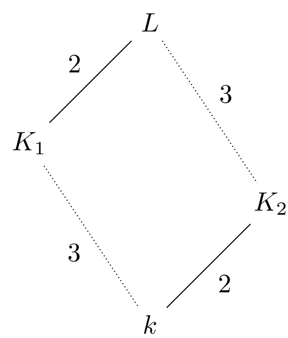
Corollary 6.2. *Let X be a del Pezzo surface of degree 6. If X admits a zero-cycle of degree $d \geq 1$, then X has a closed étale point of degree dividing d.*

Proof. This follows immediately from Corollary 6.1. □

Of independent interest are the particular rank 2 tori that arise from del Pezzo surfaces of degree 6 within Voskresenskiĭ's classification. By the explicit algebraic computations of Blunk [Blu10], over a non-separably-closed field k, each such torus takes the form

$$T = R_{K_2/k}\left(R^{(1)}_{L/K_2}\,\mathbb{G}_m\right)/R^{(1)}_{K_1/k}\,\mathbb{G}_m$$

for some diagram of separable field extensions



Lemma 6.3. $T \cong R_{K_1/k}\left(R^{(1)}_{L/K_1}\,\mathbb{G}_m\right) \cap R_{K_2/k}\left(R^{(1)}_{L/K_2}\,\mathbb{G}_m\right)$.

Proof. Let $\mathrm{Gal}(L/K_1) \cong \mathbb{Z}/2\,\mathbb{Z} = \langle\sigma\rangle$ and

$$S = R_{K_1/k}\left(R^{(1)}_{L/K_1}\,\mathbb{G}_m\right) \cap R_{K_2/k}\left(R^{(1)}_{L/K_2}\,\mathbb{G}_m\right).$$

It suffices to show that the sequence of k-tori

$$1 \to R^{(1)}_{K_1/k}\,\mathbb{G}_m \xrightarrow{\iota} R_{K_2/k}\left(R^{(1)}_{L/K_2}\,\mathbb{G}_m\right) \xrightarrow{\varphi} S \to 1,$$

where ι is the inclusion map and φ is defined functorially for any k-algebra A by

$$\begin{array}{ccc} R_{K_2/k}\left(R^{(1)}_{L/K_2}\,\mathbb{G}_m\right)(A) & \xrightarrow{\varphi(A)} & S(A), \\ a & \mapsto & \sigma(a)a^{-1}, \end{array}$$

is short exact. Left exactness is clear since $K_1 = L^\sigma$, so all that remains is to show that φ is surjective after passing to the separable closure k^{s}. Let $\beta \in S(k^{\mathrm{s}})$. Then

$$N_{L\otimes_k k^{\mathrm{s}}/K_1\otimes_k k^{\mathrm{s}}}(\beta) = 1 = N_{L\otimes_k k^{\mathrm{s}}/K_2\otimes_k k^{\mathrm{s}}}(\beta).$$

By Hilbert 90, $\beta = \sigma(\gamma)\gamma^{-1}$ for some $\gamma \in (L\otimes_k k^{\mathrm{s}})^\times$. Set $\lambda = N_{L\otimes_k k^{\mathrm{s}}/K_2\otimes_k k^{\mathrm{s}}}(\gamma)$. Then

$$\sigma(\lambda)\lambda^{-1} = N_{L\otimes_k k^{\mathrm{s}}/K_2\otimes_k k^{\mathrm{s}}}(\beta) = 1,$$

i.e., $\lambda \in ((K_2 \otimes_k k^{\mathrm{s}})^\sigma)^\times = (k^{\mathrm{s}})^\times$. Since K_1/k is separable and k^{s} is separably closed, $K_1 \otimes_k k^{\mathrm{s}} \cong (k^{\mathrm{s}})^3$. So there is some $\eta \in (K_1 \otimes_k k^{\mathrm{s}})^\times$ such that $\lambda = N_{K_1\otimes_k k^{\mathrm{s}}/k^{\mathrm{s}}}(\eta)$. Set $\alpha = \eta^{-1}\gamma$. Then

$$N_{L\otimes_k k^{\mathrm{s}}/K_2\otimes_k k^{\mathrm{s}}}(\alpha) = N_{L\otimes_k k^{\mathrm{s}}/K_2\otimes_k k^{\mathrm{s}}}\left(\eta^{-1}\gamma\right) = \lambda^{-1} N_{L\otimes_k k^{\mathrm{s}}/K_2\otimes_k k^{\mathrm{s}}}(\gamma) = 1,$$

i.e., $\alpha \in R_{K_2/k}\left(R^{(1)}_{L/K_2}\,\mathbb{G}_m\right)(k^{\mathrm{s}})$, and

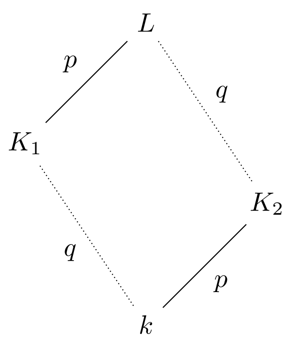
$$\varphi(\alpha) = \varphi\left(\eta^{-1}\gamma\right) = \sigma\left(\eta^{-1}\gamma\right)\left(\eta^{-1}\gamma\right)^{-1} = \sigma(\gamma)\gamma^{-1} = \beta,$$

completing the proof. □

7. Conclusions and an interesting open question

Theorem 7.1. *Totaro's question has a positive answer for:*

I. *quasitrivial tori,*
II. *norm tori of cyclic field extensions,*
III. *norm tori of prime degree field extensions,*
IV. *tori of rank $r \le 2$,*
V. *tori of the form $R_{K_1/k}\left(R^{(1)}_{L/K_1}\,\mathbb{G}_m\right) \cap R_{K_2/k}\left(R^{(1)}_{L/K_2}\,\mathbb{G}_m\right)$ where*



is a diagram of field extensions for distinct primes p and q.

Now, consider the following natural question about division algebras.

Open question: Let p be an odd prime and D and D' be non-split cyclic division algebras over k. If D_K and D'_K share a subfield of degree p over K for some finite separable field extension K/k such that $p \nmid [K : k]$, then do D and D' share a subfield of degree p over k?

A negative answer would yield the first known counterexample to Totaro's question.

Let k be a field, and let L and L' be cyclic field extensions of k of degree p such that $D \cong (L/k, \sigma, \gamma)$ and $D' \cong (L'/k, \sigma', \gamma')$. If $T = R^{(1)}_{L/k}\,\mathbb{G}_m \times R^{(1)}_{L'/k}\,\mathbb{G}_m$, then by Lemma 3.2.(b), $H^1(k,T) \cong \mathrm{Br}(L/k) \times \mathrm{Br}(L'/k)$. The pair (D, D') then identifies some $X \in H^1(k,T)$ that has a point over LL'. If $L = L'$, then this is the desired common subfield. Otherwise, $[LL' : k] = p^2$. The condition that D_K and D'_K have a common subfield, say E, of degree p over K means that D_E and D'_E are split, and so X has a point over E. But $[E : k] = p[K : k]$. So $\mathrm{ind}(X) = p$ since $\mathrm{ind}(X) \neq 1$ (because D and D' are non-split) and $\mathrm{ind}(X) \mid (p^2, p[K : k])$. Since a minimal splitting field of a division algebra is isomorphic to a maximal subfield of the algebra, the open question amounts to Totaro's question for T in the $\mathrm{ind}(X) = p$ case.

As a consequence of our much deeper understanding of quaternion algebras compared to cyclic algebras of odd prime degree, we know that the question has a positive answer when $p = 2$; this is just 5.2.3.(b) in the proof of Theorem 1.1. But unlike in our proof, even having an "Albert's Theorem" [Alb72] for odd primes would not be strong enough to immediately settle the question because $D \not\cong D^{\mathrm{op}}$, and so statements about the splitting fields of $D \otimes_k D'$ seem to be of limited utility. All this is to say that Totaro's question for tori thinly disguises many fundamental questions about division algebras whose answers, for now, remain elusive.

Acknowledgments

The author thanks R. Parimala, V. Suresh, N. Bhaskhar, and J.-L. Colliot-Thélène for their tremendous insights, advice, and support throughout the writing of this paper. The author was partially supported by the Robert W. Woodruff Foundation and the National Science Foundation under grants DMS-1401319 and FRG-1463882.

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