

## SENSITIVITY, PROXIMAL EXTENSION AND HIGHER ORDER ALMOST AUTOMORPHY

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ABSTRACT. Let  $(X, T)$  be a topological dynamical system, and  $\mathcal{F}$  be a family of subsets of  $\mathbb{Z}_+$ .  $(X, T)$  is strongly  $\mathcal{F}$ -sensitive if there is  $\delta > 0$  such that for each non-empty open subset  $U$  there are  $x, y \in U$  with  $\{n \in \mathbb{Z}_+ : d(T^n x, T^n y) > \delta\} \in \mathcal{F}$ . Let  $\mathcal{F}_t$  (resp.  $\mathcal{F}_{ip}$ ,  $\mathcal{F}_{fip}$ ) consist of thick sets (resp. IP-sets, subsets containing arbitrarily long finite IP-sets).

The following Auslander-Yorke's type dichotomy theorems are obtained: (1) a minimal system is either strongly  $\mathcal{F}_{fip}$ -sensitive or an almost one-to-one extension of its  $\infty$ -step nilfactor; (2) a minimal system is either strongly  $\mathcal{F}_{ip}$ -sensitive or an almost one-to-one extension of its maximal distal factor; (3) a minimal system is either strongly  $\mathcal{F}_t$ -sensitive or a proximal extension of its maximal distal factor.

### 1. INTRODUCTION

Throughout this paper  $(X, T)$  denotes a *topological dynamical system* (t.d.s. for short), where  $X$  is a compact metric space, and  $T : X \rightarrow X$  is continuous and surjective. In this section, we first discuss the motivations of our research and then state the main results of the article.

The notion of sensitivity (sensitive dependence on initial conditions) was first used by Ruelle [31]. It is in the kernel of the definition of Devaney's chaos. According to Auslander and Yorke [5] a t.d.s.  $(X, T)$  is called *sensitive* if there exists  $\delta > 0$  such that for every  $x \in X$  and every neighborhood  $U_x$  of  $x$ , there exist  $y \in U_x$  and  $n \in \mathbb{N}$  with  $d(T^n x, T^n y) > \delta$ . For a t.d.s.  $(X, T)$ ,  $\delta > 0$  and an open (open and non-empty) subset  $U \subset X$ , put

$$N(\delta, U) = \{n \in \mathbb{N} : \exists x, y \in U \text{ with } d(T^n x, T^n y) > \delta\} = \{n \in \mathbb{N} : \text{diam}(T^n(U)) > \delta\}.$$

Then it is easy to see that  $(X, T)$  is sensitive if and only if there exists  $\delta > 0$  such that  $N(\delta, U) \neq \emptyset$  for each open subset  $U$ . A t.d.s.  $(X, T)$  is called *equicontinuous* if for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that whenever  $x, y \in X$  with  $d(x, y) < \delta$ , then  $d(T^n x, T^n y) < \varepsilon$  for  $n \in \mathbb{N}$ . Auslander and Yorke [5] proved the following dichotomy theorem: a minimal system is either equicontinuous or sensitive. A similar result obtained by Glasner and Weiss [17] states that: a transitive system is either almost equicontinuous or sensitive.

There have been several attempts to generalize the notion of sensitivity. Akin and Kolyada [1] introduced the notion of Li-Yorke sensitivity, combining the two well known notions (sensitivity and Li-Yorke chaos) together. The study of sensitivity related to families of non-negative integers was initiated by Moothathu

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in [30]. Let  $\mathcal{F}$  be a family. Recall that according to [30]  $(X, T)$  is  $\mathcal{F}$ -sensitive if there is  $\delta > 0$  such that for any open subset  $U$ ,  $N(\delta, U) \in \mathcal{F}$ .  $\mathcal{F}$ -sensitivity for some families was discussed in [7, 21, 25, 27, 28, 30]. It is known that for a minimal system  $\{\text{thick}\}$ -sensitivity is equivalent to  $\{\text{thickly syndetic}\}$ -sensitivity [28]. Very recently, a striking result obtained by Huang, Kolyada and Zhang [21, Theorem 3.1] states that: a minimal system is either  $\{\text{thick}\}$ -sensitive or an almost one-to-one extension of its maximal equicontinuous factor.

It is clear that when  $(X, T)$  is  $\mathcal{F}$ -sensitive, then for  $n \in N(\delta, U)$ , there are  $x_n, y_n \in U$  such that  $d(T^n x_n, T^n y_n) > \delta$ . If we require all  $x_n$  (resp.  $y_n$ ) are equal, then it leads the notion of *strong  $\mathcal{F}$ -sensitivity* which will be studied in detail in the paper. Recall that  $(X, T)$  is strongly  $\mathcal{F}$ -sensitive if there is  $\delta > 0$  such that for each non-empty open subset  $U$  there are  $x, y \in U$  with  $\{n \in \mathbb{Z}_+ : d(T^n x, T^n y) > \delta\} \in \mathcal{F}$ , where  $\mathcal{F}$  is a family of subsets of  $\mathbb{Z}_+$ . We remark that some notions of sensitivity similar to the strong sensitivity were studied in [7, 26], which appear naturally when studying mean equicontinuity. It was shown that a minimal system is either mean-sensitive, or mean equicontinuous.

When investigating strong sensitivity we find that for some families  $\mathcal{F}$  the requirement of all  $x_n$  or  $y_n$  being equal is too strong. So in this paper we also introduce a notion of sensitivity related to a family  $\mathcal{F}$ , called *block  $\mathcal{F}$ -sensitivity*. Roughly speaking, in this definition we require  $x_n$  (resp.  $y_n$ ) to be equal for a sequence of arbitrarily long finite segments from the family  $\mathcal{F}$ . For example, a t.d.s.  $(X, T)$  is called *block  $\{\text{thick}\}$ -sensitive* (resp. *block  $\{\text{IP}\}$ -sensitive*) if there is  $\delta > 0$  such that for each  $x \in X$ , every neighborhood  $U_x$  of  $x$  and  $l \in \mathbb{N}$  there are  $y_l \in U_x$  with  $\{n \in \mathbb{Z}_+ : d(T^n x, T^n y_l) > \delta\}$  containing  $\{m+1, \dots, m+l\}$  for some  $m = m(l) \in \mathbb{N}$  (resp. a finite IP-set of length at least  $l$ ). Thus

$$\text{strong } \mathcal{F}\text{-sensitivity} \subset \text{block } \mathcal{F}\text{-sensitivity} \subset \mathcal{F}\text{-sensitivity}.$$

In this paper first we investigate  $\mathcal{F}$ -sensitivity to warm up. Then we study block  $\mathcal{F}$ -sensitivity and some related strong  $\mathcal{F}$ -sensitivity notions for some families. Finally we will focus on strong  $\mathcal{F}$ -sensitivity. Note that for a minimal system we use  $X_{eq}$ ,  $X_\infty$  and  $X_D$  to denote the maximal equicontinuous factor, the maximal  $\infty$ -step nilfactor and the maximal distal factor of  $X$  respectively (for the definitions see Section 2). It is very interesting that for some well known families strong sensitivity for the family is closely related to other well known dynamical properties.

The main results of the paper are:

**Theorem A.** *Let  $(X, T)$  be a minimal system. Then the following conditions are equivalent:*

- (1)  $(X, T)$  is block  $\mathcal{F}_t$ -sensitive;
- (2)  $\pi: X \rightarrow X_{eq}$  is not proximal.

**Theorem B.** *Let  $(X, T)$  be an invertible minimal system. Then the following statements are equivalent:*

- (1)  $(X, T)$  is strongly  $\mathcal{F}_{ip}$ -sensitive;
- (2)  $(X, T)$  is block  $\mathcal{F}_{ip}$ -sensitive;
- (3)  $\pi: X \rightarrow X_\infty$  is not almost one-to-one.

**Theorem C.** *Let  $(X, T)$  be a minimal system. Then the following conditions are equivalent:*

- (1)  $(X, T)$  is strongly  $\mathcal{F}_{ip}$ -sensitive;
- (2)  $\pi: X \rightarrow X_D$  is not almost one-to-one.

**Theorem D.** *Let  $(X, T)$  be a minimal system. Then the following conditions are equivalent:*

- (1)  $(X, T)$  is strongly  $\mathcal{F}_t$ -sensitive;
- (2)  $\pi: X \rightarrow X_D$  is not proximal.

From Theorem B it is natural to ask if we can find some family  $\mathcal{F}$  such that strong  $\mathcal{F}$ -sensitivity is related to a  $d$ -step almost automorphy (see Section 5.2 for the definitions of the families appearing below),  $d \in \mathbb{N}$ . This leads us to study strong  $\mathcal{F}_{Poin_d}$ -sensitivity (where  $\mathcal{F}_{Poin_d}$  is the family of all  $d$ -step Poincaré sequences) for  $d \in \mathbb{N}$ . We show that if a minimal t.d.s.  $(X, T)$  is strongly  $\mathcal{F}_{Poin_d}$ -sensitive, then  $\pi: X \rightarrow X_d$  is not an almost one-to-one extension (Theorem 5.19), where  $X_d$  is the maximal  $d$ -step nilfactor of  $X$ . Examples show that the converse statement does not hold (see Example 5.22). It is an interesting open question to find a family  $\mathcal{F}$  such that for any minimal system  $(X, T)$ ,  $(X, T)$  is strongly  $\mathcal{F}$ -sensitive if and only if  $\pi: X \rightarrow X_\infty$  is not proximal.

For a minimal system, Table 1 gives the details of results obtained in the paper (the results related to sensitivity are essentially obtained in [21]).

TABLE 1. Relationships

|                     | not strongly sensitive                               | not block sensitive                        | not sensitive     |
|---------------------|--|--|-------------------|
| $\mathcal{F}_t$     | proximal extension of the maximal distal factor      | proximal extension of maximal equi. factor | almost automorphy |
| $\mathcal{F}_{ip}$  | almost one-to-one extension of maximal distal factor | $\infty$ -step almost automorphy           | almost automorphy |
| $\mathcal{F}_{fip}$ | $\infty$ -step almost automorphy                     | $\infty$ -step almost automorphy           | almost automorphy |

We remark that when defining strong sensitivity, except for the definition given before one may define strong  $\mathcal{F}$ -sensitivity as follows: if there is  $\delta > 0$  such that for each  $x \in X$  and each neighborhood  $U$  of  $x$ , there is  $y \in U$  with  $\{n \in \mathbb{Z}_+ : d(T^n x, T^n y) > \delta\} \in \mathcal{F}$ . It is easy to see that the two definitions coincide when  $\mathcal{F}$  has the Ramsey property. We also remark that since any sensitive minimal system is strongly  $\{syndetic\}$ -sensitive [30], we know that if a family  $\mathcal{F}$  contains the set of all syndetic subsets, then for a minimal system strong  $\mathcal{F}$ -sensitivity is equivalent to sensitivity. This fact restricts the families when we consider strong  $\mathcal{F}$ -sensitivity and try to obtain new results, and also explains the reason why we choose  $\mathcal{F}_t$ ,  $\mathcal{F}_{ip}$ , and  $\mathcal{F}_{fip}$  et al. to consider strong  $\mathcal{F}$ -sensitivity in this paper.

We also remark that for a transitive system, we may investigate the same problem. Because of the restriction on the length of the paper we leave this study to the readers.

The paper is organized as follows: In Section 2, we recall some definitions and some related theorems. In Section 3, we discuss sensitivity. In Section 4, we study

block sensitivity and some related notions of strong sensitivity, and prove Theorem A, Theorem B and Theorem C. In Section 5, we investigate strong sensitivity and show Theorem D.

## 2. PRELIMINARIES

In this section we will recall some basic notions and theorems we need in the following sections.

**2.1. Topological dynamical systems.** In the article, sets of integers, non-negative integers and natural numbers are denoted by  $\mathbb{Z}$ ,  $\mathbb{Z}_+$  and  $\mathbb{N}$  respectively. By a topological dynamical system we mean a pair  $(X, T)$ , where  $X$  is a compact metric space with a metric  $d$  and  $T : X \rightarrow X$  is continuous and surjective. A non-vacuous closed invariant subset  $Y \subseteq X$  defines naturally a subsystem  $(Y, T)$  of  $(X, T)$ . A system  $(X, T)$  is called *minimal* if it contains no proper subsystem. Each point belonging to some minimal subsystem of  $(X, T)$  is called a *minimal point*. The *orbit* of a point  $x \in X$  is the set  $Orb(x, T) = \{T^n x : n \in \mathbb{Z}_+\}$ .

For  $x \in X$  and  $U, V \subset X$ , put

$$N(x, U) = \{n \in \mathbb{Z}_+ : T^n x \in U\} \quad \text{and} \quad N(U, V) = \{n \in \mathbb{Z}_+ : U \cap T^{-n}V \neq \emptyset\}.$$

Recall that a dynamical system  $(X, T)$  is called *topologically transitive* (or just *transitive*) if for every two open subsets  $U, V$  of  $X$  the set  $N(U, V)$  is infinite. Any point with dense orbit is called a *transitive point*. Denote the set of all transitive points by  $Trans(X, T)$ . It is well known that for a transitive system,  $Trans(X, T)$  is a dense  $G_\delta$  subset of  $X$ .

Let  $M(X)$  be the set of all Borel probability measures on  $X$ . We are interested in those members of  $M(X)$  that are invariant measures for  $T$ , denoted by  $M(X, T)$ . This set consists of all  $\mu \in M(X)$  making  $T$  a measure-preserving transformation of  $(X, \mathcal{B}(X), \mu)$ , where  $\mathcal{B}(X)$  is the Borel  $\sigma$ -algebra of  $X$ . By the Krylov-Bogolyubov Theorem,  $M(X, T)$  is non-empty. The *support* of a measure  $\mu \in M(X)$ , denoted by  $\text{supp}(\mu)$ , is the smallest closed subset  $C$  of  $X$  such that  $\mu(C) = 1$ . We say that a measure has *full support* or is *fully supported* if  $\text{supp}(\mu) = X$ . If  $(X, T)$  is a minimal system, every  $T$ -invariant measure has full support.

**2.2. Distal, proximal, regionally proximal.** Let  $(X, T)$  and  $(Y, S)$  be two dynamical systems. If there is a continuous surjection  $\pi : X \rightarrow Y$  with  $\pi \circ T = S \circ \pi$ , then we say that  $\pi$  is a *factor map*, the system  $(Y, S)$  is a *factor* of  $(X, T)$  or  $(X, T)$  is an *extension* of  $(Y, S)$ . If  $\pi$  is a homeomorphism, then we say that  $\pi$  is a *conjugacy* and dynamical systems  $(X, T)$  and  $(Y, S)$  are *conjugate*. Conjugate dynamical systems can be considered the same from the dynamical point of view.

Let  $(X, T)$  be a dynamical system. A pair  $(x_1, x_2) \in X \times X$  is said to be *proximal* if for any  $\epsilon > 0$ , there exists a positive integer  $n$  such that  $d(T^n x_1, T^n x_2) < \epsilon$ . Let  $P(X, T)$  denote the collection of all proximal pairs in  $(X, T)$ .  $P$  is a reflexive symmetric  $T$  invariant relation, but is in general not transitive or closed. If  $(x, y)$  is not proximal, it is said to be a *distal pair*. A system  $(X, T)$  is called *distal* if any pair of distinct points in  $(X, T)$  is a distal pair.

Recall that the *regionally proximal relation*  $Q(X, T)$  is the set of all points  $(x_1, x_2) \in X \times X$  such that for each  $\epsilon > 0$  and each open neighborhood  $U_i$  of  $x_i$ ,  $i = 1, 2$ , there are  $x'_i \in U_i$ ,  $i = 1, 2$ , and  $n \in \mathbb{N}$  with  $d(T^n(x'_1), T^n(x'_2)) < \epsilon$ . Note that  $Q(X)$  is a reflexive symmetric  $T$  invariant closed relation, but is in general

not transitive. However for each minimal system  $(X, T)$ ,  $Q(X)$  is a closed invariant equivalence relation.

Every topological dynamical system  $(X, T)$  has a maximal distal factor  $(X_D, T)$  and a maximal equicontinuous factor  $(X_{eq}, T)$ . That is,  $(X_D, T)$  is distal and every distal factor of  $(X, T)$  is a factor of  $(X_D, T)$ .  $(X_{eq}, T)$  has the corresponding property for equicontinuous factors. Thus there are closed  $T$ -invariant equivalence relations  $S_D$  and  $S_{eq}$  such that  $X/S_D = X_D$  and  $X/S_{eq} = X_{eq}$ .  $S_D$  is the smallest closed  $T$ -invariant equivalence relation containing  $P(X)$ , and  $X_{eq}$  is the smallest closed  $T$ -invariant equivalence relation containing  $Q(X)$ .

An extension  $\phi : (X, T) \rightarrow (Y, S)$  is *proximal* if  $R_\phi \subset P(X, T)$  and is distal if  $R_\phi \cap P(X, T) = \Delta_X$ , where  $R_\phi = \{(x, y) \in X^2 : \phi(x) = \phi(y)\}$ . Observe that when  $Y$  is trivial (reduced to one point) the map  $\phi$  is distal if and only if  $(X, T)$  is distal. An extension  $\phi : (X, T) \rightarrow (Y, T)$  is almost one-to-one if the  $G_\delta$  set  $X_0 = \{x \in X : \phi^{-1}(\phi(x)) = x\}$  is dense.

**2.3. Nilmanifolds and nilsystems.** Let  $G$  be a group. For  $g, h \in G$ , we write  $[g, h] = ghg^{-1}h^{-1}$  for the commutator of  $g$  and  $h$  and we write  $[A, B]$  for the subgroup spanned by  $\{[a, b] : a \in A, b \in B\}$ . The commutator subgroups  $G_j, j \geq 1$ , are defined inductively by setting  $G_1 = G$  and  $G_{j+1} = [G_j, G]$ . Let  $k \geq 1$  be an integer. We say that  $G$  is *k-step nilpotent* if  $G_{k+1}$  is the trivial subgroup.

Let  $G$  be a  $k$ -step nilpotent Lie group and  $\Gamma$  a discrete cocompact subgroup of  $G$ . The compact manifold  $X = G/\Gamma$  is called a *k-step nilmanifold*. The group  $G$  acts on  $X$  by left translations and we write this action as  $(g, x) \mapsto gx$ . The Haar measure  $\mu$  of  $X$  is the unique probability measure on  $X$  invariant under this action. Let  $\tau \in G$  and  $T$  be the transformation  $x \mapsto \tau x$  of  $X$ . Then  $(X, T, \mu)$  is called a *basic k-step nilsystem*. When the measure is not needed for results, we omit it and write that  $(X, T)$  is a basic  $k$ -step nilsystem.

We also make use of inverse limits of nilsystems and so we recall the definition of an inverse limit of systems (restricting ourselves to the case of sequential inverse limits). If  $(X_i, T_i)_{i \in \mathbb{N}}$  are systems with  $diam(X_i) \leq M < \infty$  and  $\phi_i : X_{i+1} \rightarrow X_i$  are factor maps, the *inverse limit* of the systems is defined to be the compact subset of  $\prod_{i \in \mathbb{N}} X_i$  given by  $\{(x_i)_{i \in \mathbb{N}} : \phi_i(x_{i+1}) = x_i, i \in \mathbb{N}\}$ , which is denoted by  $\varprojlim \{X_i\}_{i \in \mathbb{N}}$ . It is a compact metric space endowed with the distance  $\rho(x, y) = \sum_{i \in \mathbb{N}} 1/2^i d_i(x_i, y_i)$ . We note that the maps  $\{T_i\}$  induce a transformation  $T$  on the inverse limit. Let  $(X_i, T_i) = (X, T)$  and  $\phi_i = T$ ; then the inverse limit of systems  $(\tilde{X}, \tilde{T})$  is called the natural extension of  $(X, T)$ .

If  $(X, T)$  is an inverse limit of basic  $(d - 1)$ -step minimal nilsystems,  $(X, T)$  is called a *(d - 1)-step nilsystem* or a *system of order (d - 1)*.

**2.4. Regionally proximal relation of order  $d$ ,  $\mathbf{RP}^{[d]}$ .** Let  $(X, T)$  be a t.d.s. and let  $d \geq 1$  be an integer. A pair  $(x, y) \in X \times X$  is said to be *regionally proximal of order  $d$*  if for any  $\delta > 0$ , there exist  $x', y' \in X$  and a vector  $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$  such that  $\rho(x, x') < \delta, \rho(y, y') < \delta$ , and

$$\rho(T^{\mathbf{n} \cdot \epsilon} x', T^{\mathbf{n} \cdot \epsilon} y') < \delta \text{ for any } \epsilon \in \{0, 1\}^d, \epsilon \neq (0, \dots, 0),$$

where  $\mathbf{n} \cdot \epsilon = \sum_{i=1}^d \epsilon_i n_i$ . The set of regionally proximal pairs of order  $d$  is denoted by  $\mathbf{RP}^{[d]}(X)$ , and is called *the regionally proximal relation of order  $d$* .

This notion was first introduced by Host-Kra-Maass in [19]. It is clear that

$$(2.1) \quad P(X) \subseteq \dots \subseteq \mathbf{RP}^{[d+1]} \subseteq \mathbf{RP}^{[d]} \subseteq \dots \subseteq \mathbf{RP}^{[2]} \subseteq \mathbf{RP}^{[1]} = Q(X).$$

It was shown [19,32] that for each minimal system  $(X, T)$ ,  $\mathbf{RP}^{[d]}(X)$  is a closed invariant equivalence relation for any  $d \in \mathbb{N}$ . When  $d = 1$ ,  $\mathbf{RP}^{[d]}(X)$  is nothing but the classical regionally proximal relation which determines the maximal equicontinuous factor for any minimal system. We remark that recently Glasner-Gutman-Ye [16] defined a new regionally proximal relation of order  $d$  for any group  $G$  (coinciding with the previous definition when  $G$  is abelian) and showed that it is an equivalence relation for any minimal system  $(X, G)$ .

Now we state a proposition from [19,32] which we need in what follows.

**Proposition 2.1.** *Let  $(X, T)$  be minimal systems and  $d \in \mathbb{N}$ . Then the following statements are equivalent:*

- (1)  $(X, T)$  is a  $d$ -step nilsystem;
- (2)  $\mathbf{RP}^{[d]}(X) = \Delta_X$ .

Let  $\mathbf{RP}^{[\infty]}(X) = \bigcap_{d=1}^{\infty} \mathbf{RP}^{[d]}(X)$ ; then  $\mathbf{RP}^{[\infty]}(X)$  is a closed invariant equivalence relation.

**Definition 2.2.** A minimal system  $(X, T)$  is an  $\infty$ -step nilsystem or a system of order  $\infty$ , if the equivalence relation  $\mathbf{RP}^{[\infty]}$  is trivial, i.e., coincides with the diagonal.

The following proposition was proved in [8].

**Proposition 2.3.** *A minimal system is an  $\infty$ -step nilsystem if and only if it is an inverse limit of minimal nilsystems.*

Let  $(X, T)$  be a t.d.s. and  $d \in \mathbb{N}$ . Put  $X_d = X/\mathbf{RP}^{[d]}(X)$  and  $X_\infty = X/\mathbf{RP}^{[\infty]}(X)$ .

**Definition 2.4.** Let  $(X, T)$  be a minimal system and  $d \in \mathbb{N} \cup \{\infty\}$ . A point  $x \in X$  is called a  $d$ -step almost automorphic point (or  $d$ -step AA point for short) if  $\mathbf{RP}^{[d]}(X)[x] = \{x\}$ .

A minimal system  $(X, T)$  is called  $d$ -step almost automorphic ( $d$ -step AA for short) if it has a  $d$ -step almost automorphic point.

$d$ -step almost automorphic systems were studied systematically in [23]; in particular we have

**Proposition 2.5** ([23, Theorem 8.13]). *Let  $(X, T)$  be a minimal system. Then  $(X, T)$  is a  $d$ -step almost automorphic system for some  $d \in \mathbb{N} \cup \{\infty\}$  if and only if it is an almost one-to-one extension of its maximal  $d$ -step nilfactor  $(X_d, T)$ .*

**2.5. Families.** Let  $\mathcal{P} = \mathcal{P}(\mathbb{Z}_+)$  be the collection of all subsets of  $\mathbb{Z}_+$ . A subset  $\mathcal{F}$  of  $\mathcal{P}$  is a *family* if it is hereditary upwards, i.e.,  $F_1 \subset F_2$  and  $F_1 \in \mathcal{F}$  imply  $F_2 \in \mathcal{F}$ . A family  $\mathcal{F}$  is *proper* if it is a proper subset of  $\mathcal{P}$ , i.e., neither empty nor all of  $\mathcal{P}$ . It is easy to see that  $\mathcal{F}$  is proper if and only if  $\mathbb{Z}_+ \in \mathcal{F}$  and  $\emptyset \notin \mathcal{F}$ . A family  $\mathcal{F}$  has the *Ramsey property* if  $F \in \mathcal{F}$  and  $F = F_1 \cup F_2$  imply that  $F_i \in \mathcal{F}$  for some  $i \in \{1, 2\}$ . Any subset  $\mathcal{A}$  of  $\mathcal{P}$  generates a family

$$[\mathcal{A}] = \{F \in \mathcal{P} : F \supset A \text{ for some } A \in \mathcal{A}\}.$$

If a proper family  $\mathcal{F}$  is closed under finite intersection, then  $\mathcal{F}$  is called a *filter*. For a family  $\mathcal{F}$ , the *dual family* is

$$\mathcal{F}^* = \{F \in \mathcal{P} : \mathbb{Z}_+ \setminus F \notin \mathcal{F}\} = \{F \in \mathcal{P} : F \cap F' \neq \emptyset \text{ for all } F' \in \mathcal{F}\}.$$

$\mathcal{F}^*$  is a family, proper if  $\mathcal{F}$  is. It is well known that a proper family has the Ramsey property if and only if its dual  $\mathcal{F}^*$  is a filter [12]. Clearly, for a family  $\mathcal{F}$

$$(\mathcal{F}^*)^* = \mathcal{F} \text{ and } \mathcal{F}_1 \subset \mathcal{F}_2 \Rightarrow \mathcal{F}_2^* \subset \mathcal{F}_1^*.$$

We say that a subset  $F$  of  $\mathbb{Z}_+$  is

- (1) *thick* if it contains arbitrarily long blocks of consecutive integers, that is, for every  $d \geq 1$  there is  $n \in \mathbb{N}$  such that  $\{n, n + 1, \dots, n + d\} \subset F$ ;
- (2) *syndetic* if it has bounded gaps, that is, for some  $N \in \mathbb{N}$  and every  $k \in \mathbb{N}$  we have  $\{k, k + 1, \dots, k + N\} \cap A \neq \emptyset$ ;
- (3) *piecewise syndetic* if it is the intersection of a syndetic set with a thick set;
- (4) *thickly syndetic* if it has non-empty intersection with every piecewise syndetic set

The collection of all syndetic (resp. thick) subsets is denoted by  $\mathcal{F}_s$  (resp.  $\mathcal{F}_t$ ). Note that  $\mathcal{F}_s^* = \mathcal{F}_t$  and  $\mathcal{F}_t^* = \mathcal{F}_s$ . The collection of all piecewise syndetic (resp. thickly syndetic) subsets is denoted by  $\mathcal{F}_{ps}$  (resp.  $\mathcal{F}_{ts}$ ).

Let  $\{b_i\}_{i \in I}$  be a finite or infinite sequence in  $\mathbb{Z}_+$ . One defines

$$FS(\{b_i\}_{i \in I}) = \left\{ \sum_{i \in \alpha} b_i : \alpha \text{ is a finite non-empty subset of } I \right\}.$$

$F$  is an IP-set if it contains some  $FS(\{p_i\}_{i=1}^\infty)$  where  $p_i \in \mathbb{N}$ . The collection of all IP-sets is denoted by  $\mathcal{F}_{ip}$ . A subset of  $\mathbb{Z}_+$  is called an *IP\*-set*, if it has non-empty intersection with any IP-set. IP-sets are important in the study of dynamical properties; see [6, 12].

If  $I$  is finite, then one says  $FS(\{p_i\}_{i \in I})$  is a *finite IP-set* of length  $|I|$ . The collection of all sets containing finite IP-sets with arbitrarily long lengths is denoted by  $\mathcal{F}_{fip}$ .

Let  $E$  be a finite or infinite set in  $\mathcal{P}(\mathbb{Z}_+)$ , One defines

$$\Delta(E) = \{a - b : a \geq b, a, b \in E\}.$$

A subset  $F$  of  $\mathbb{Z}_+$  is called a *difference set* if it contains some  $\Delta(E)$  with  $|E|$  infinite. The collection of all difference sets is denoted by  $\mathcal{F}_\Delta$ . A subset of  $\mathbb{Z}_+$  is called a  $\Delta^*$ -set, if it has non-empty intersection with any difference set.

If  $E$  is a finite set, then one says that  $\Delta(E)$  is a *finite difference set* of length  $|E|$ . The collection of all sets containing finite difference sets with arbitrarily long lengths is denoted by  $\mathcal{F}_{f\Delta}$ .

**2.6. Technical lemmas.** Note that a factor map is *semi-open* if it sends any opene set to a set containing an opene set. To end the section we state an easy lemma which follows from the continuity of  $\pi$ .

**Lemma 2.6.** *Let  $\pi : (X, T) \rightarrow (Y, S)$  be a semi-open factor map between two t.d.s. and let  $\mathcal{F}$  be a family. If  $(Y, S)$  is  $\mathcal{F}$ -sensitive (resp. block  $\mathcal{F}$ -sensitive, strongly  $\mathcal{F}$ -sensitive), so is  $(X, T)$ .*

The following proposition is easy to check.

**Proposition 2.7.** *Let  $(X, T)$  be a dynamical system, and  $(\tilde{X}, \tilde{T})$  be the natural extension of  $(X, T)$ . Then  $(X, T)$  is  $\mathcal{F}$ -sensitive (resp. block  $\mathcal{F}$ -sensitive, strongly  $\mathcal{F}$ -sensitive) if and only if  $(\tilde{X}, \tilde{T})$  is  $\mathcal{F}$ -sensitive (resp. block  $\mathcal{F}$ -sensitive, strongly  $\mathcal{F}$ -sensitive).*

The next lemma is from [21, Proposition 4.4] or [10, Lemma 2.4].

**Lemma 2.8.** *Let  $\pi : (X, T) \rightarrow (Y, S)$  be a factor map with  $(X, T)$  minimal and  $(Y, S)$  invertible. If  $\pi$  is not almost one-to-one, then  $l = \inf_{y \in Y} \text{diam}(\pi^{-1}(y)) > 0$ .*

### 3. SENSITIVITY FOR FAMILIES

To start our research we begin to study  $\mathcal{F}$ -sensitivity. The goal is to show the notion of  $\mathcal{F}$ -sensitivity is rough, meaning that for many families the notions are equivalent in the minimality setup.

Recall that the authors in [21] proved that: a minimal system is either  $\mathcal{F}_t$ -sensitive or an almost one-to-one extension of its maximal equicontinuous factor. Moreover, they showed in [20] that for minimal systems all of the following notions:  $\mathcal{F}_{ts}$ -sensitivity, multi-sensitivity (see [30] for a definition) and  $\mathcal{F}_t$ -sensitivity are equivalent. In this section, we prove that for minimal systems all of the following notions:  $\mathcal{F}_{ts}$ -sensitivity,  $\mathcal{F}_{ip}$ -sensitivity,  $\mathcal{F}_{fip}$ -sensitivity and  $\mathcal{F}_{f\Delta}$ -sensitivity are equivalent (the equivalence to  $\mathcal{F}_{\text{Poin}_d}$ -sensitivity will be given in Section 5).

First we need a proposition which is basically due to Furstenberg [12, Proposition 9.8]. Let  $(X, T)$  be a t.d.s. and let  $\mathcal{F}$  be a family. Note that we say that  $x \in X$  is  $\mathcal{F}$ -recurrent, if for each neighborhood  $U$  of  $x$ ,  $N(x, U) \in \mathcal{F}$ ; and  $(X, T)$  is  $\mathcal{F}$ -recurrent if each point of  $x \in X$  is  $\mathcal{F}$ -recurrent.

**Proposition 3.1.** *Let  $(X, T)$  be a minimal equicontinuous system. Then  $(X, T)$  is  $\mathcal{F}_{f\Delta}^*$ -recurrent.*

*Proof.* Since  $(X, T)$  is minimal and equicontinuous, we can assume that  $(X, T)$  is a Kronecker system. That is,  $X = G$ , an abelian compact group, and  $Tx = ax$  for a fixed  $a \in G$ . Let  $x_0$  be any point of  $X$  and let  $U$  be any open neighborhood of  $x_0$ . Let  $V$  be any neighborhood of  $x_0$  such that  $VV^{-1}x_0 \subseteq U$ . Since  $X$  is minimal, there are  $l_1, \dots, l_k \in \mathbb{N}$  such that  $\{a^{l_1}V, a^{l_2}V, \dots, a^{l_k}V\}$  is a cover of  $X$ .

Let  $\{S_n\}_{n=1}^m$  be any finite sequence with  $m > k$ ; then there are  $a^{S_u}, a^{S_v}$  contained in the same subset  $a^{l_i}V$ . Then  $a^{S_u - S_v}x_0 \in U$ , which implies that  $(X, T)$  is  $\mathcal{F}_{f\Delta}^*$ -recurrent.  $\square$

Using Proposition 3.1 and some theorem in [21], we have the following result.

**Theorem 3.2.** *Let  $(X, T)$  be minimal. Then the following statements are equivalent:*

- (1)  $(X, T)$  is  $\mathcal{F}_{ts}$ -sensitive.
- (2)  $(X, T)$  is  $\mathcal{F}_t$ -sensitive.
- (3)  $(X, T)$  is  $\mathcal{F}_{ip}$ -sensitive.
- (4)  $(X, T)$  is  $\mathcal{F}_{fip}$ -sensitive.
- (5)  $(X, T)$  is  $\mathcal{F}_{f\Delta}$ -sensitive.
- (6) There exists  $\delta > 0$  such that for every  $x \in X$  there is  $y \in X$  such that  $(x, y)$  is regional proximal and  $d(x, y) > \delta$ .
- (7)  $(X, T)$  is not an almost one-to-one extension of  $X_{\text{eq}}$ .



*Proof.* It is clear that  $\mathcal{F}_{ts} \subset \mathcal{F}_t \subset \mathcal{F}_{ip} \subset \mathcal{F}_{fip} \subset \mathcal{F}_{f\Delta}$ . By [21, Theorem 3.1], it remains to show (5)  $\Rightarrow$  (7) and (7)  $\Leftrightarrow$  (6)

(5)  $\Rightarrow$  (7) Assume that  $(X, T)$  is  $\mathcal{F}_{f\Delta}$ -sensitive with a sensitive constant  $\delta > 0$  and  $\pi : (X, T) \rightarrow (X_{eq}, T_{eq})$  is almost one-to-one. Since  $(X_{eq}, T_{eq})$  is a minimal equicontinuous system, there is a compatible metric  $d'$  such that  $d'(T_{eq}^i x, T_{eq}^i y) = d'(x, y)$ , for all  $x, y \in X_{eq}$ . Let  $y_0 \in X_{eq}$  with  $\pi^{-1}(y_0)$  singleton. We take an open set  $W \subset X$  containing  $\pi^{-1}(y_0)$  such that  $\text{diam}(W) < \delta$ , and then there is an open set  $V \subset X_{eq}$  containing  $y_0$  such that  $\pi^{-1}V \subset W$ .

Let  $B(y_0, \epsilon) \subset V$  for some  $\epsilon > 0$  and  $U = \pi^{-1}(V_1)$  with  $V_1 = B(y_0, \epsilon/2)$ . By Proposition 3.1,  $N(y_0, V_1) \in \mathcal{F}_{f\Delta}^*$ .

For  $n \in N(y_0, V_1)$ , we have  $d'(T_{eq}^n y_0, y_0) < \epsilon/2$ . Since  $d'(T_{eq}^m y_0, T_{eq}^m y_0) < \frac{\epsilon}{2}$  for all  $m \in \mathbb{N}$  and  $y \in V_1$ , we deduce that  $T_{eq}^n(V_1) \subset V$  for  $n \in N(y_0, V_1)$ . For  $U = \pi^{-1}(V_1)$  and  $n \in N(y_0, V_1)$  we get

$$T^n(U) = T^n \pi^{-1}(V_1) \subset \pi^{-1}(T_{eq}^n V_1) \subset \pi^{-1}(V) \subset W.$$

This means that  $N(U, \delta) \cap N(y_0, V_1) = \emptyset$ , which implies  $N(U, \delta) \notin \mathcal{F}_{f\Delta}$ .

(6)  $\Rightarrow$  (7) is obvious.

(7)  $\Rightarrow$  (6) follows from Lemma 2.8. □

#### 4. BLOCK SENSITIVITY AND STRONG $\mathcal{F}_{fip}$ , $\mathcal{F}_{ip}$ -SENSITIVITY

In this section we study block sensitivity and some related notions of strong sensitivity, and prove Theorems A, B and C. This will be done in the following three subsections.

**4.1. Block  $\mathcal{F}_t$ -sensitivity.** In this subsection, we discuss block  $\mathcal{F}_t$ -sensitivity and give a proof of Theorem A.

Recall that a t.d.s.  $(X, T)$  is called *block  $\mathcal{F}_t$ -sensitive* if there is  $\delta > 0$  such that for each  $x \in X$ , every neighborhood  $U_x$  of  $x$  and  $l \in \mathbb{N}$  there are  $y_l \in U_x$  with  $\{n \in \mathbb{Z}_+ : d(T^n x, T^n y_l) > \delta\}$  containing  $\{m + 1, \dots, m + l\}$  for some  $m \in \mathbb{N}$ . In fact we will show the following theorem which covers Theorem A.

**Theorem 4.1.** *Let  $(X, T)$  be a minimal dynamical system. Then the following conditions are equivalent:*

- (1)  $(X, T)$  is block  $\mathcal{F}_t$ -sensitive;
- (2) there exists  $\delta > 0$  such that for every  $x \in X$  there exists  $y \in X$  such that  $(x, y)$  is regional proximal and  $\inf_{n \in \mathbb{Z}_+} d(T^n x, T^n y) > \delta$ ;
- (3)  $\pi : X \rightarrow X_{eq}$  is not proximal.

We start with

**Proposition 4.2.** *Let  $(X, T)$  be a t.d.s. and  $\pi : (X, T) \rightarrow (X_{eq}, T_{eq})$  be the factor map. If  $(X, T)$  is block  $\mathcal{F}_t$ -sensitive, then  $\pi$  is not proximal.*

*Proof.* Let  $d, d'$  be the compatible metrics of  $X, X_{eq}$  respectively. Let  $\epsilon_k > 0$  with  $\epsilon_k \rightarrow 0$ . Then for each  $k \in \mathbb{N}$ , there is  $0 < \tau_k, \tau'_k < \epsilon_k$  such that if  $d'(w_1, w_2) < \tau_k$  with  $w_1, w_2 \in X_{eq}$ , then  $d'(T_{eq}^i w_1, T_{eq}^i w_2) < \epsilon_k$  for any  $i \in \mathbb{Z}_+$ ; and if  $w_1, w_2 \in X$  with  $d(w_1, w_2) < \tau'_k$ , then  $d'(\pi(w_1), \pi(w_2)) < \tau_k$ .

Pick  $x \in X$  and put  $U_k = B(\tau'_k, x)$ . By the assumption  $(X, T)$  is block  $\mathcal{F}_t$ -sensitive, thus for each  $j \in \mathbb{N}$ , there is  $y_k^j \in U_k$  such that  $F = \{n \in \mathbb{Z}_+ : d(T^n x, T^n y_k^j) > \delta\}$  containing  $\{a_k^j, a_k^j + 1, \dots, a_k^j + j\}$  (with  $\delta$  the sensitive constant).

Without loss of generality we assume that  $T^{a_k^j}x \rightarrow z_k^1$  and  $T^{a_k^j}y_k^j \rightarrow z_k^2$  when  $j \rightarrow \infty$ . It is clear that  $d(T^i z_k^1, T^i z_k^2) \geq \delta$  for each  $i \in \mathbb{Z}_+$ . Now let  $z_1 = \lim_{k \rightarrow \infty} z_k^1$  and  $z_2 = \lim_{k \rightarrow \infty} z_k^2$ . We have  $d(T^i z_1, T^i z_2) \geq \delta$  for each  $i \in \mathbb{Z}_+$ .

Now we show that  $\pi(z_1) = \pi(z_2)$ . Since  $y_k^j \in U_k$ , it is clear that  $d'(\pi(x), \pi(y_k^j)) < \tau_k$  and thus we have  $d'(T_{eq}^i \pi(x), T_{eq}^i \pi(y_k^j)) < \epsilon_k$  for each  $i \in \mathbb{Z}_+$ . Particularly,

$$d'(T_{eq}^{a_k^j} \pi(x), T_{eq}^{a_k^j} \pi(y_k^j)) < \epsilon_k$$

for each  $j \in \mathbb{N}$ . This implies that  $d'(\pi(z_k^1), \pi(z_k^2)) \leq \epsilon_k$ , and hence  $d'(\pi(z_1), \pi(z_2)) = 0$ . We have proved that  $\pi(z_1) = \pi(z_2)$ . This indicates that  $\pi$  is not proximal, finishing the proof. □

*Proof of Theorem 4.1.* (1) $\Rightarrow$ (3) follows from the above proposition.

(3) $\Rightarrow$ (2) There exists a regional proximal pair  $(z_1, z_2)$  which is not proximal. Let  $\delta = \frac{1}{2} \inf_{n \in \mathbb{Z}_+} d(T^n z_1, T^n z_2) > 0$ . Fix a point  $x \in X$ . As  $z_1$  is a minimal point of  $(X, T)$ , there exists a sequence of positive numbers  $\{n_i\}$  such that  $\lim_{i \rightarrow \infty} T^{n_i} z_1 \rightarrow x$ . By the compactness of  $X$ , without loss of generality, assume that  $\lim_{i \rightarrow \infty} T^{n_i} z_2 \rightarrow y$ . Then  $(x, y)$  is regional proximal, since  $Q(X, T)$  is closed and  $T \times T$ -invariant. We also have  $\inf_{n \in \mathbb{Z}_+} d(T^n x, T^n y) \geq \inf_{n \in \mathbb{Z}_+} d(T^n z_1, T^n z_2) > \delta$ .

(2) $\Rightarrow$ (1) Fix  $x \in X$  and a neighborhood  $U$  of  $x$  and  $l \in \mathbb{N}$ . There exists  $y \in X$  such that  $(x, y)$  is regional proximal and  $\inf_{n \in \mathbb{Z}_+} d(T^n x, T^n y) > \delta$ . Choose small enough neighborhood  $V \subset U$  of  $x$  and neighborhood  $W$  of  $y$  such that  $\min_{0 \leq i \leq l} d(T^i V, T^i W) > \frac{1}{2} \delta$

As  $(x, y)$  is regional proximal,  $N(x, W)$  is a  $\Delta$ -set [21, Proposition 4.7]. We also have that  $N(V, V)$  is a  $\Delta^*$ -set [12, Page 177]. Then  $N(x, W)$  intersects  $N(V, V)$ . Pick  $n \in N(x, W) \cap N(V, V)$  and  $x' \in V \cap T^{-n}W$ . Then  $T^n x \in W, T^n x' \in V$ . This implies that  $d(T^{n+i} x, T^{n+i} x') \geq \min_{0 \leq i \leq l} d(T^i V, T^i W) > \frac{1}{2} \delta$  for  $i = 0, 1, \dots, l$ . Therefore,  $(X, T)$  is block  $\mathcal{F}_t$ -sensitive. □

We have the following corollary.

**Corollary 4.3.** *There is a minimal system which is  $\mathcal{F}_t$ -sensitive and not strongly  $\mathcal{F}_t$ -sensitive.*

*Proof.* There is a minimal system such that  $\pi : X \rightarrow X_{eq}$  is a proximal extension and not an almost one-to-one extension [18]. Then  $(X, T)$  is  $\mathcal{F}_t$ -sensitive by [21, Theorem 3.1], and is not strongly  $\mathcal{F}_t$ -sensitive by Proposition 4.2. □

**4.2. Block  $\mathcal{F}_{ip}$ -sensitivity and strong  $\mathcal{F}_{fip}$ -sensitivity.** In this subsection, we investigate block  $\mathcal{F}_{ip}$ -sensitivity, strong  $\mathcal{F}_{fip}$ -sensitivity and show Theorem B. In this subsection we assume that  $T$  is a homeomorphism (since some results we use are stated for homeomorphisms and it will take some pages to show they are true for continuous and surjective maps).

Recall that a t.d.s.  $(X, T)$  is called *block  $\mathcal{F}_{ip}$ -sensitive* if there is  $\delta > 0$  such that for each  $x \in X$ , every neighborhood  $U_x$  of  $x$  and  $l \in \mathbb{N}$  there is  $y_l \in U$  such that  $\{n \in \mathbb{Z}_+ : d(T^n x, T^n y_l) > \delta\}$  contains a finite IP-set of length  $l$ . By the Ramsey property of  $\mathcal{F}_{fip}$ , an equivalent definition can be stated as follows: there is  $\delta > 0$  such that for any opene  $U$  of  $X$  and  $l \in \mathbb{N}$  there are  $y_l, z_l \in U$  such that  $\{n \in \mathbb{Z}_+ : d(T^n y_l, T^n z_l) > \delta\}$  contains a finite IP-set of length  $l$ . As before we will show the following theorem which covers Theorem B.

**Theorem 4.4.** *Let  $(X, T)$  be a minimal system. Then the following statements are equivalent:*

- (1)  $(X, T)$  is strongly  $\mathcal{F}_{f_{ip}}$ -sensitive;
- (2)  $(X, T)$  is block  $\mathcal{F}_{ip}$ -sensitive;
- (3) there exists  $\delta > 0$  such that for every  $x \in X$  there exists  $y \in X$  such that  $(x, y) \in \mathbf{RP}^{[\infty]}$  with  $d(x, y) \geq \delta$ ;
- (4)  $\phi : X \rightarrow X_\infty$  is not almost one-to-one.

To prove Theorem 4.4 we need some preparation. The following lemma is from [14].

**Lemma 4.5.** *Let  $(X, \mathcal{B}, \mu)$  be a probability space, and  $\{E_i\}_{i=1}^\infty$  be a sequence of measurable sets with  $\mu(E_i) \geq a > 0$  for some constant  $a$ . Then for any  $k \geq 1$  and  $\epsilon > 0$  there is  $N = N(a, k, \epsilon)$  such that for any tuple  $\{s_1 < s_2 < \dots < s_n\}$  with  $n \geq N$  there exist  $1 \leq t_1 < t_2 < \dots < t_k \leq n$  with*

$$(4.1) \quad \mu(E_{s_{t_1}} \cap E_{s_{t_2}} \cap \dots \cap E_{s_{t_k}}) \geq a^k - \epsilon.$$

We will use the next lemma derived from Lemma 4.5.

**Lemma 4.6.** *Let  $(X, T)$  be a t.d.s. with  $\mu \in M(X, T)$ . Let  $U \in \mathcal{B}_X$  with  $a = \mu(U) > 0$ . Then there is  $n = n(a)$  such that for any finite IP-set  $FS(\{p_i\}_{i=1}^n)$  there is  $q \in FS(\{p_i\}_{i=1}^n)$  such that  $\mu(U \cap T^{-q}U) \geq \frac{1}{2}a^2$ .*

*Proof.* Apply Lemma 4.5 to  $k = 2$ ,  $\epsilon = \frac{1}{2}a^2$  and consider the finite tuple

$$T^{-p_1}U, \dots, T^{-p_1 - \dots - p_n}U.$$

□

The notion of a *central set* was introduced in [12]. It is known that a central set contains an IP-set [12, Proposition 8.10].

**Proposition 4.7.** *Let  $(X, T)$  and  $(Y, S)$  be minimal. If  $\pi : X \rightarrow Y$  is proximal and not almost one-to-one, then  $(X, T)$  is strongly  $\mathcal{F}_{ip}$ -sensitive.*

*Proof.* By Lemma 2.8,  $l = \inf_{y \in Y} \text{diam}(\pi^{-1}(y)) > 0$ . For each  $y \in Y$ , choose  $x_1(y), x_2(y) \in \pi^{-1}(y)$  with  $d(x_1(y), x_2(y)) = l(y) \geq l$ .

For  $x \in X$ , let  $y = \pi(x)$ . Then we have  $d(x, x_1(y)) \geq \frac{l}{2}$  or  $d(x, x_2(y)) \geq \frac{l}{2}$ . Without loss of generality, we assume that  $d(x, x_1(y)) \geq \frac{l}{2}$ . Then  $(x, x_1(y))$  is proximal.

Let  $\delta = \frac{l}{8}$  and let  $U', V$  be open neighborhoods of  $x, x_1(y)$  with  $\text{diam}(U'), \text{diam}(V) < \frac{l}{8}$  respectively. Then  $d(U', V) > \delta$ . Choose a smaller  $U$  with the same properties and  $U' \supset \overline{U}$ . We know that  $N(x, V)$  is a central set and hence it contains an IP-set  $FS(\{p_i\}_{i=1}^\infty)$ . We are going to show that there is  $z \in U'$  such that  $d(T^l x, T^l z) > \delta$  for all  $l$  in a sub IP-set of  $FS(\{p_i\}_{i=1}^\infty)$ .

To do this let  $\mu \in M(X, T)$ ; then  $a = \mu(U) > 0$ . Applying Lemma 4.6 to  $U$ . There are  $n_1$  and  $q_1 \in FS(\{p_i\}_{i=1}^{n_1})$  such that  $\mu(U \cap T^{-q_1}U) \geq \frac{1}{2}a^2$ .

Let  $U_1 = U \cap T^{-q_1}U$  and apply Lemma 4.6 to  $U_1$ . There are  $n_2$  and  $q_2 \in FS(\{p_i\}_{i=n_1+1}^{n_2})$  such that  $\mu(U_1 \cap T^{-q_2}U_1) \geq \frac{1}{8}a^4$ . Note that we have  $U \cap T^{-q_1}U \cap T^{-q_2}U \cap T^{-q_1 - q_2}U \neq \emptyset$ .

Inductively for any  $k \in \mathbb{N}$  we obtain  $n_1, \dots, n_k, U_1, \dots, U_k$  and  $q_1, \dots, q_k$  such that  $q_{j+1} \in FS(\{p_i\}_{i=n_j+1}^{n_{j+1}})$ ,  $U_{j+1} = U_j \cap T^{-q_{j+1}}(U_j)$  with  $\mu(U_{j+1}) \geq C_j > 0$  for

$j = 0, \dots, k - 1$  (set  $U_0 = U$  and  $n_0 = 0$ ). This implies that

$$\mu(U \bigcap_{l \in FS(\{q_i\}_{i=1}^k)} T^{-l}U) > 0.$$

Thus, for each  $k \in \mathbb{N}$  there is  $z_k \in U$  such that  $T^l(z_k) \in U$  for all  $l \in FS(\{q_i\}_{i=1}^k)$ . Without loss of generality, assume that  $z = \lim_{k \rightarrow \infty} z_k$ ; then  $T^l(z) \in cl(U) \subset U'$  for  $l \in FS(\{q_j\}_{j=1}^\infty)$ . We know  $d(T^l x, T^l z) > \delta$  for  $l \in FS(\{q_j\}_{j=1}^\infty)$ . Since  $FS(\{q_j\}_{j=1}^\infty) \subseteq FS(\{p_j\}_{j=1}^\infty)$ , this implies that  $(X, T)$  is strongly  $\mathcal{F}_{ip}$ -sensitive.  $\square$

By [32, Theorem 3.2] we know that for any  $d \in \mathbb{N}$  and any minimal t.d.s.  $(X, T)$ ,  $(x, y) \in \mathbf{RP}^{[d]}(X)$  if and only if for any neighborhood  $V$  of  $y$ ,  $N(x, V)$  contains a finite IP-set of length  $d + 1$ . As  $\mathbf{RP}^{[\infty]}(X) = \bigcap_{d=1}^\infty \mathbf{RP}^{[d]}(X)$ , so we have

**Lemma 4.8.** *Let  $(X, T)$  be minimal and  $(x, y) \in X \times X$ . Then  $(x, y) \in \mathbf{RP}^{[\infty]}(X)$  if and only if for any neighborhood  $V$  of  $y$ ,  $N(x, V) \in \mathcal{F}_{ip}$ .*

With the help of the above lemma and Lemma 4.6 we are able to show

**Proposition 4.9.** *Let  $(X, T)$  be minimal and  $\pi : X \rightarrow X_\infty$  is not proximal. Then  $(X, T)$  is strongly  $\mathcal{F}_{ip}$ -sensitive.*

*Proof.* Since  $\pi$  is not proximal, there are  $(x_1, x_2) \in R_\pi$  which is a distal pair. It follows that  $(x_1, x_2) \in \mathbf{RP}^{[\infty]} = \bigcap_{d=1}^\infty \mathbf{RP}^{[d]}$  and  $d(T^n x_1, T^n x_2) \geq l$  for any  $n \in \mathbb{N}$ .

Let  $U, V$  be closed neighborhoods of  $x_1, x_2$  with  $\text{diam}(U), \text{diam}(V) < \frac{l}{4}$  respectively. Then  $d(U, V) > \frac{l}{2}$  and we let  $\delta = \frac{l}{2}$ . By Lemma 4.8,  $N(x_1, V) \in \mathcal{F}_{ip}$ . We are going to show that there is  $z \in U$  such that  $d(T^l x_1, T^l z) > \delta$  for all  $l \in F \in \mathcal{F}_{ip}$  with  $F \subset N(x_1, V)$ .

For  $k = 1$ , using the same argument as in Proposition 4.7, we get  $n_1^1 \in \mathbb{N}$  such that for any finite IP-set of length  $n_1^1$  with  $FS\{p_i^1\}_{i=1}^{n_1^1} \subset N(x_1, V)$ , there is  $q_1^1 \in FS\{p_i^1\}_{i=1}^{n_1^1}$  such that  $\mu(U \cap T^{-q_1^1}U) \geq \frac{1}{2}a^2$ . Set  $U^1 = U \cap T^{-q_1^1}U$ .

For  $k = 2$ , using the same argument as in Proposition 4.7 (with respect to  $U^1$ ), we get  $n_2^2 \in \mathbb{N}$  such that for any finite IP-set of length  $n_2^2$  with  $FS\{p_i^2\}_{i=1}^{n_2^2} \subset N(x_1, V)$ , there are  $q_1^2, q_2^2, q_1^2 + q_2^2 \in FS\{p_i^2\}_{i=1}^{n_2^2}$  such that if we set  $U^2 = U^1 \cap T^{-q_1^2}U^1 \cap T^{-q_2^2}U^1 \cap T^{-q_1^2 - q_2^2}U^1$ , then  $\mu(U^2) > 0$ . So we have

$$\mu(U \cap T^{-q_1^1} \cap T^{-q_1^2}U \cap T^{-q_2^2}U \cap T^{-q_1^2 - q_2^2}U) > 0.$$

Inductively, for any  $k \in \mathbb{N}$  we obtain  $n_1^j, \dots, n_j^j, U^1, \dots, U^j$  and  $q_1^j, \dots, q_j^j$  for  $1 \leq j \leq k$  such that

- for  $0 \leq m \leq j - 1$ ,  $q_{m+1}^j \in FS(\{p_i^j\}_{i=n_m^j+1}^{n_{m+1}^j}) \subset N(x_1, V)$  (set  $n_0^j = 0$ ).
- $U^{j+1} = U^j \cap \bigcap_{l \in FS\{q_i^j\}_{i=1}^j} T^{-l}U^j$  for  $0 \leq j \leq k - 1$  satisfy that  $\mu(U^{j+1}) > 0$  (set  $U^0 = U$ ). So we have  $\mu(A_k) > 0$ , where

$$A_k = U \cap \bigcap_{j=1}^k \bigcap_{l \in FS\{q_i^j\}_{i=1}^j} T^{-l}U.$$

Set  $F = \bigcup_{k=1}^{\infty} FS\{q_i^k\}_{i=1}^k$ . Then  $F \subset N(x_1, V)$  and  $F \in \mathcal{F}_{fip}$ . Take  $z \in \bigcap_{k=1}^{\infty} A_k$ ; then  $T^l z \in U$  for all  $l \in F$ . This implies that  $d(T^l x_1, T^l z) > \delta$  for all  $l \in F \in \mathcal{F}_{fip}$ .

For  $u \in X$  there is a sequence  $\{n_i\}$  such that  $T^{n_i} x_1 \rightarrow u$  and  $T^{n_i} x_2 \rightarrow v$ . Then  $(u, v) \in \mathbf{RP}^{[\infty]}$  and  $(u, v)$  is a distal pair with  $d(T^n u, T^n v) \geq l$ . Let  $W, W'$  be closed neighborhoods of  $u$  and  $v$  respectively with  $\text{diam}(W), \text{diam}(W') < \frac{l}{4}$ . By the proof above, we know that there is  $w \in W$  such that  $d(T^l u, T^l w) > \delta$  for all  $l \in F$ , where  $F \in \mathcal{F}_{fip}$  with  $F \subset N(u, W') \in \mathcal{F}_{fip}$ . So we have proved that  $(X, T)$  is strongly  $\mathcal{F}_{fip}$ -sensitive.  $\square$

The following lemma is well known.

**Lemma 4.10.** *Let  $F$  be a finite IP-set of length  $n$  and  $F = F_1 \cup F_2$ . Then there is  $i \in \mathbb{N}$  such that  $F_i$  is a finite IP-set of length  $l(n)$  with  $l(n) \rightarrow \infty$  when  $n \rightarrow \infty$ . This also implies that  $\mathcal{F}_{fip}$  has the Ramsey property.*

To end the proof we need another proposition.

**Proposition 4.11.** *Let  $(X, T)$  be a minimal block  $\mathcal{F}_{ip}$ -sensitive t.d.s. with the sensitive constant  $10\delta$ . Assume that  $x \in X$  and  $U$  is any neighborhood of  $x$ . Then there are  $z \in U$  and  $y \in X$  such that  $(y, z) \in \mathbf{RP}^{[\infty]}$  with  $d(z, y) \geq \delta$ .*

*Proof.* Since  $(X, T)$  is block  $\mathcal{F}_{ip}$ -sensitive, there is  $\delta > 0$  such that for any  $x_0 \in X$ , any neighborhood  $V$  of  $x_0$  and any  $n \in \mathbb{N}$  there is  $y_0, z_0 \in V$  such that  $\{m \in \mathbb{N} : d(T^m y_0, T^m z_0) > 10\delta\}$  contains a finite IP-set of length  $n$ .

Let  $U_0 = B(x, 4\delta)$  and  $U_1 = B(x, \delta)$ . Without loss of generality, we assume  $U \subseteq U_1$ . Then for  $n_1 \in \mathbb{N}$  large enough there are  $x_1^1, x_1^2 \in U$  such that  $F_1 = \{n \in \mathbb{N} : d(T^n x_1^1, T^n x_1^2) > 10\delta\}$  contains a finite IP-set of length  $n_1$ . By the method of Proposition 4.7, there is  $z_1 \in U$  satisfying  $T^n z_1 \in U$  for  $n \in F_1' \subseteq F_1$ , where  $F_1'$  is a finite IP-set of length  $k(n_1)$ . Then  $d(T^n x_1^1, T^n z_1) > 5\delta$  or  $d(T^n x_1^2, T^n z_1) > 5\delta$  for  $n \in F_1'$ . Without loss of generality, we assume that  $d(T^n x_1^1, T^n z_1) > 5\delta$  for  $n \in F_1'' \subseteq F_1'$ , where  $F_1''$  is a finite IP-set of length  $l(k(n_1))$  (Lemma 4.10). Then  $T^n x_1^1 \notin U_0$  for  $n \in F_1''$ . Let  $U_2 \subset U$  be an open neighborhood of  $x_1^1$  with diameter small enough such that  $T^n \overline{U_2} \cap U_0 = \emptyset$  for  $n \in F_1''$ .

Then for  $n_1 \ll n_2 \in \mathbb{N}$  large enough there are  $x_2^1, x_2^2 \in U_2$  such that  $F_2 = \{n \in \mathbb{N} : d(T^n x_2^1, T^n x_2^2) > 10\delta\}$  contains a finite IP-set of length  $n_2$ . By the method of Proposition 4.7 again, there is  $z_2 \in U_2$  satisfying  $T^n z_2 \in U_2$  for  $n \in F_2' \subseteq F_2$ , where  $F_2'$  is a finite IP-set of length  $k(n_2)$ . Then  $d(T^n x_2^1, T^n z_2) > 5\delta$  or  $d(T^n x_2^2, T^n z_2) > 5\delta$  for  $n \in F_2'$ . Without loss of generality, we assume  $d(T^n x_2^1, T^n z_2) > 5\delta$  for  $n \in F_2'' \subseteq F_2'$ , where  $F_2''$  is a finite IP-set of length  $l(k(n_2))$ . Then  $T^n x_2^1 \notin U_0$ . Let  $U_3 \subset U_2$  be an open neighborhood of  $x_2^1$  with diameter small enough such that  $T^n \overline{U_3} \cap U_0 = \emptyset$  for  $n \in F_2''$ .

To continue the process, we get  $F_k \supseteq F_k' \supseteq F_k''$ ,  $z_k$ ,  $n_k$  and  $U_k$  with  $\text{diam}(U_k) \rightarrow 0$  as  $k \rightarrow \infty$ . We have

- (1)  $d(T^n z_k, T^n x_1^k) \geq 5\delta$  for  $n \in F_k''$  with  $z_k \in U_k$ ,  $x_1^k \in U_{k+1} \subseteq U_k$ ;
- (2)  $T^n \overline{U_{k+1}} \cap U_0 \neq \emptyset$  for  $n \in F_k''$ .

Assume that  $\lim_{k \rightarrow \infty} z_k = z$ ; then  $\lim_{k \rightarrow \infty} x_1^k = z$ . Since  $z \in \bigcap_{k=1}^{\infty} \overline{U_k}$ , we have  $T^n z \notin U_0$  for  $n \in F_k''$ . Thus,  $N(z, U_0^c) \in \mathcal{F}_{fip}$ , for  $l(k(n)) \rightarrow \infty$  as  $n \rightarrow \infty$ .

Let  $W = B(x, 3\delta)$ . Since  $U_0^c$  is compact, we can cover  $U_0^c$  by finitely many closed balls  $\{V_1^1, V_2^1, \dots, V_{l_1}^1\}$  with diameter less than 1 and  $\bigcup_{k=1}^{l_1} V_k^1 \subset W^c$ . By the Ramsey property of  $\mathcal{F}_{fip}$ , there is  $1 \leq m_1 \leq l_1$  such that  $N(z, V_{m_1}^1) \in \mathcal{F}_{fip}$ . Since  $V_{m_1}^1$  is compact, we can cover  $V_{m_1}^1$  by finitely many closed balls  $\{V_1^2, V_2^2, \dots, V_{l_2}^2\}$  with diameter less than  $\frac{1}{2}$  and  $\bigcup_{k=1}^{l_2} V_k^2 \subset W^c$ . By the Ramsey property of  $\mathcal{F}_{fip}$  again, there is  $1 \leq m_2 \leq l_2$  such that  $N(z, V_{m_2}^2) \in \mathcal{F}_{fip}$ . To continue the process, we get  $V_{m_k}^k$  such that

$$N(z, V_{m_k}^k) \in \mathcal{F}_{fip}, \text{ diam}(V_{m_k}^k) \leq \frac{1}{k} \text{ and } V_{m_k}^k \subset W^c.$$

Let  $y \in \bigcap_{k=1}^{\infty} V_{m_k}^k$ . Then for any open neighborhood  $W'$  of  $y$ , we have  $N(z, W') \in \mathcal{F}_{fip}$  since  $W'$  contains  $V_{m_k}^k$  for some  $k \in \mathbb{N}$ . Lemma 4.8 implies that  $(y, z) \in \mathbf{RP}^{[\infty]}$ . Since  $y \in W^c$  and  $z \in \overline{U_1}$ , we conclude that  $d(z, y) \geq 2\delta > \delta$ . This ends the proof.  $\square$

*Proof of Theorem 4.4.* (1)  $\Rightarrow$  (2) is obvious.

(2)  $\Rightarrow$  (3) Assume that  $(X, T)$  is block  $\mathcal{F}_{ip}$ -sensitive. Fix  $x \in X$ . By Proposition 4.11 for every  $n \in \mathbb{N}$ , there exists  $x_n \in B(x, \frac{1}{n})$  and  $y_n \in X$  such that  $d(x_n, y_n) \geq \delta$  and  $(x_n, y_n) \in \mathbf{RP}^{[\infty]}(X)$ . Without loss of generality, assume that  $y_n \rightarrow y$ . Then  $d(x, y) \geq \delta$  and  $(x, y) \in \mathbf{RP}^{[\infty]}(X)$  as  $\mathbf{RP}^{[\infty]}(X)$  is closed.

(3)  $\Rightarrow$  (4) is obvious.

(4)  $\Rightarrow$  (1) Since  $\phi$  is not almost one-to-one,  $\phi$  is either not proximal, or proximal and not almost one-to-one. If  $\phi : X \rightarrow X_\infty$  is not proximal, then by Proposition 4.9 we get that  $(X, T)$  is strongly  $\mathcal{F}_{fip}$ -sensitive. If  $\phi : X \rightarrow X_\infty$  is proximal, not almost one-to-one, by Proposition 4.7 we get  $(X, T)$  is strongly  $\mathcal{F}_{fip}$ -sensitive.  $\square$

**4.3. Strong  $\mathcal{F}_{ip}$ -sensitive.** In this subsection, we study strong  $\mathcal{F}_{ip}$ -sensitivity and give the proof of Theorem C. Recall that we say a t.d.s.  $(X, T)$  is strongly  $\mathcal{F}_{ip}$ -sensitive if there is  $\delta > 0$  such that for each open subset  $U$  of  $X$ , there are  $x, y \in U$  with  $\{n \in \mathbb{Z}_+ : d(T^n x, T^n y) > \delta\} \in \mathcal{F}_{ip}$ . In fact we will show a stronger form of Theorem C.

**Theorem 4.12.** *Let  $(X, T)$  be a minimal system. Then the following conditions are equivalent:*

- (1)  $(X, T)$  is strongly  $\mathcal{F}_{ip}$ -sensitive;
- (2) there is  $\delta > 0$  such that for every non-empty open subset  $U$  of  $X$  there exists a proximal pair  $(x, y)$  with  $x \in U$  and  $d(x, y) > \delta$ ;
- (3)  $\pi : X \rightarrow X_D$  is not almost one-to-one, where  $(X_D, T)$  is the maximal distal factor of  $(X, T)$ .

We say that  $x$  is *strongly proximal* to  $y$  if  $(y, y) \in \omega((x, y), T \times T)$ , where  $\omega(x, y)$  is the  $\omega$ -limit set of  $(x, y)$ . Note that if  $(x, y)$  is proximal and  $y$  is a minimal point, then  $x$  is strongly proximal to  $y$ . We need two results from [24].

**Lemma 4.13** ([24, Lemma 4.8]). *Let  $(X, T)$  be a dynamical system and  $x, y \in X$ . Then  $x$  is strongly proximal to  $y$  if and only if for every neighborhood  $U$  of  $y$ ,  $N(x, U) \cap N(y, U)$  contains an IP-set.*

**Proposition 4.14** ([24, Proposition 5.9]). *Let  $(X, T)$  be a dynamical system,  $x \in X$  and  $Y \subset X$  be a closed subset of  $X$ . If  $N(x, Y)$  contains an IP-set, then there exists  $y \in Y$  such that  $x$  is strongly proximal to  $y$ .*

Now we show a proposition.

**Proposition 4.15.** *Let  $(X, T)$  be a minimal system. Then  $(X, T)$  is strongly  $\mathcal{F}_{ip}$ -sensitive if and only if there is  $\delta > 0$  such that for every non-empty open subset  $U$  of  $X$ , there is  $x \in U$  and  $y \in X$  with  $d(x, y) > \delta$  and  $x$  is strongly proximal to  $y$ .*

*Proof.* First assume that  $(X, T)$  is strongly  $\mathcal{F}_{ip}$ -sensitive with sensitive constant  $8\delta > 0$ . Fix a non-empty open subset  $U$  of  $X$ . Pick  $z \in U$  and let  $V = U \cap B(z, \delta)$ . There are  $x_1, x_2 \in V$  such that  $F = \{n \in \mathbb{Z}_+ : d(T^n x_1, T^n x_2) > 8\delta\}$  contains an IP-set. Let  $W = X \setminus B(z, 2\delta)$ . By the Ramsey property of  $\mathcal{F}_{ip}$ , either  $N(x_1, W)$  or  $N(x_2, W)$  contains an IP-set. By Proposition 4.14 there exists  $y \in W$  such that either  $x_1$  or  $x_2$  is strongly proximal to  $y$ . It is clear that  $d(x_1, y) > \delta$  and  $d(x_2, y) > \delta$ .

Now we show the sufficiency. Fix a non-empty open subset  $U$  of  $X$ . There is  $x \in U$  and  $y \in X$  with  $d(x, y) > \delta$  and  $x$  is strongly proximal to  $y$ . By Lemma 4.13,  $N(x, B(y, \delta/3))$  contains an IP-set  $FS(\{p_i\}_{i=1}^\infty)$ . By the method of Proposition 4.7, there exist  $z \in B(x, \delta/3)$  and an IP subset  $FS(\{q_j\}_{j=1}^\infty)$  such that  $FS(\{q_j\}_{j=1}^\infty) \subset N(z, B(x, \delta/3))$  and  $FS(\{q_j\}_{j=1}^\infty) \subset FS(\{p_i\}_{i=1}^\infty)$ . Then  $FS(\{q_j\}_{j=1}^\infty) \subset \{n \in \mathbb{Z}_+ : d(T^n x, T^n z) > \delta/3\}$ , which implies that  $(X, T)$  is strongly  $\mathcal{F}_{ip}$ -sensitive with the sensitive constant  $\delta/3$ . □

We are in the position to give:

*Proof of Theorem 4.12.* (1) $\Rightarrow$ (2) follows from Proposition 4.15.

(2) $\Rightarrow$ (3) For every point  $x \in X$ , there exists a sequence  $y_n$  and  $z_n$  such that  $\lim_{n \rightarrow \infty} y_n = x$  and  $(y_n, z_n)$  is proximal and  $d(y_n, z_n) > \delta$ . Without loss of generality, assume that  $\lim_{n \rightarrow \infty} z_n = z$ . Then  $d(x, z) \geq \delta$ . Note that  $(x, z) \in S_D$ , where  $S_D$  is the distal relation,  $X/S_D = X_D$ . Let  $\pi : X \rightarrow X_D$ . Then  $\{x, z\} \in \pi^{-1}(\pi(x))$ . So  $\pi$  is not almost one-to-one.

(3) $\Rightarrow$ (1) If  $\pi$  is proximal, then by Proposition 4.7,  $(X, T)$  is strongly  $\mathcal{F}_{ip}$ -sensitive. So we assume that  $\pi$  is not proximal. This implies that  $P(X)$  is not closed. So there is a distal pair  $(y, z)$  and proximal pairs  $(y_i, z_i)$  such that  $(y_i, z_i) \rightarrow (y, z)$ . Let  $\inf_{n \in \mathbb{Z}_+} d(T^n y, T^n z) = 4\delta$ .

Fix a non-empty open subset  $U$  of  $X$ . As  $y$  is a minimal point, there exists  $k \in \mathbb{N}$  such that  $T^k y \in U$ . There exists  $n \in \mathbb{N}$  such that  $T^k y_n \in U \cap B(T^k y, \delta)$  and  $d(T^k z_n, T^k z) < \delta$ . Let  $x_1 = T^k y_n$  and  $x_2 = T^k z_n$ . Then  $x_1 \in U$ ,  $d(x_1, x_2) > \delta$  and  $(x_1, x_2)$  is proximal. As  $x_2$  is a minimal point,  $x_1$  is strongly proximal to  $x_2$ . Then the result follows from Proposition 4.15. □

### 5. STRONG SENSITIVITY FOR OTHER FAMILIES

In this section we study strong sensitivity for other families and shall prove Theorem D. Namely, we will investigate the properties of strong  $\mathcal{F}_t$ - and strong  $\mathcal{F}_{Poin_d}$ -sensitivity.

**5.1. Strong  $\mathcal{F}_t$ -sensitivity.** In this subsection, we discuss strong  $\mathcal{F}_t$ -sensitivity, and prove Theorem D. Recall that for a t.d.s.  $(X, T)$ , we say  $(X, T)$  is strongly  $\mathcal{F}_t$ -sensitive if there is  $\delta > 0$  such that for each open subset  $U$  of  $X$ , there are  $x, y \in U$

with  $\{n \in \mathbb{Z}_+ : d(T^m x, T^n y) > \delta\} \in \mathcal{F}_t$ . So  $(X, T)$  is not strongly  $\mathcal{F}_t$ -sensitive if there are  $\delta_n \rightarrow 0$  and opene subsets  $U_n$  such that for any  $x_n, y_n \in U_n$ , there is a syndetic subset  $F$  of  $\mathbb{Z}_+$  with  $d(T^m x_n, T^m y_n) \leq \delta_n$  for all  $m \in F$ .

To prove Theorem D, we first show that strong  $\mathcal{F}_t$ -sensitivity passes through proximal extensions.

**Proposition 5.1.** *Let  $\pi : (X, T) \rightarrow (Y, S)$  be a proximal extension of minimal systems. If  $(Y, S)$  is not strongly  $\mathcal{F}_t$ -sensitive, then neither is  $(X, T)$ .*

*Proof.* Let  $d, d'$  be the compatible metrics of  $X, Y$  respectively. Since  $(Y, S)$  is not strongly  $\mathcal{F}_t$ -sensitive, there are  $\delta_k \rightarrow 0$  and opene subsets  $U_k$  of  $Y$  such that if  $x_k, y_k \in U_k$ , then there is a syndetic subset  $F$  (depends on  $x_k, y_k$ ) with  $d'(S^n x_k, S^n y_k) < \delta_k$  for every  $n \in F$ .

Assume the contrary that  $(X, T)$  is strongly  $\mathcal{F}_t$ -sensitive with a sensitive constant  $\delta > 0$ . Then for each opene subset  $U$ , there are  $x, y \in U$  such that  $\{n \in \mathbb{Z}_+ : d(T^n x, T^n y) > \delta\} \in \mathcal{F}_t$ . Thus, there are  $u_k, v_k \in \pi^{-1}(U_k)$  such that  $F_k := \{n \in \mathbb{Z}_+ : d(T^n u_k, T^n v_k) > \delta\} \in \mathcal{F}_t$ . Note that  $E_k := \{n \in \mathbb{Z}_+ : d'(S^n \pi(u_k), S^n \pi(v_k)) < \delta_k\}$  is a syndetic set. This implies that there exists  $b_k \in \mathbb{N}$  such that

$$d'(S^{b_k} \pi(u_k), S^{b_k} \pi(v_k)) < \delta_k \quad \text{and} \quad d(T^j(u_k), T^j(v_k)) > \delta$$

for  $j \in [b_k - k, b_k + k]$ . Without loss of generality, assume that  $T^{b_k} u_k \rightarrow u, T^{b_k} v_k \rightarrow v$ . Then  $d(T^n u, T^n v) \geq \delta, \forall n \in \mathbb{Z}_+$ . Since  $\pi(T^{b_k} u_k) \rightarrow \pi(u), \pi(T^{b_k} v_k) \rightarrow \pi(v)$  and  $d'(S^{b_k} \pi(u_k), S^{b_k} \pi(v_k)) < \delta_k$ , we conclude that  $\pi(u) = \pi(v)$ , a contradiction. This indicates that  $(X, T)$  is not strongly  $\mathcal{F}_t$ -sensitive, ending the proof. □

**Proposition 5.2.** *Let  $(X, T)$  be a minimal system. If  $\pi : X \rightarrow X_D$  is proximal, then  $(X, T)$  is not strongly  $\mathcal{F}_t$ -sensitive.*

*Proof.* By Proposition 5.1 and Theorem 4.12. □

To prove the converse of Theorem D, we need the structure theorem. So we assume that  $T$  is a homeomorphism first. When  $(X, T)$  is not invertible, we use natural extension to prove Theorem D.

Recall that an extension  $\pi : X \rightarrow Y$  of minimal systems is a *relatively incontractible (RIC) extension* if it is open and for every  $n \geq 1$  the minimal points are dense in the relation

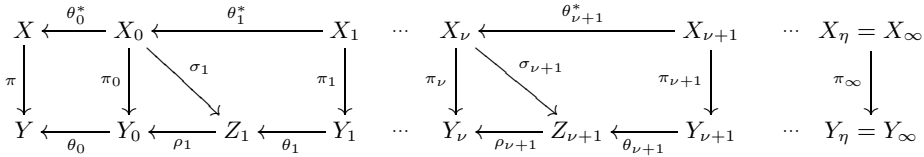
$$R_\pi^n = \{(x_1, \dots, x_n) \in X^n : \pi(x_i) = \pi(x_j) \forall 1 \leq i \leq j \leq n\}.$$

We say that a minimal system  $(X, T)$  is a *strictly PI system* if there is an ordinal  $\eta$  (which is countable when  $X$  is metrizable) and a family of systems  $\{(W_\iota, w_\iota)\}_{\iota \leq \eta}$  such that (i)  $W_0$  is the trivial system, (ii) for every  $\iota < \eta$  there exists a homomorphism  $\phi_\iota : W_{\iota+1} \rightarrow W_\iota$  which is either proximal or equicontinuous, (iii) for a limit ordinal  $\nu \leq \eta$  the system  $W_\nu$  is the inverse limit of the systems  $\{W_\iota\}_{\iota < \nu}$ , and (iv)  $W_\eta = X$ . We say that  $(X, T)$  is a *PI system* if there exists a strictly PI system  $\tilde{X}$  and a proximal homomorphism  $\theta : \tilde{X} \rightarrow X$ .

We have the following structure theorem for minimal systems.



**Lemma 5.3** (Structure theorem for minimal systems, [11]). *Given a homomorphism  $\pi : X \rightarrow Y$  of minimal dynamical system, there exists an ordinal  $\eta$  (countable when  $X$  is metrizable) and a canonically defined commutative diagram (the canonical PI-Tower)*



where for each  $\nu \leq \eta$ ,  $\pi_\nu$  is RIC,  $\rho_\nu$  is isometric,  $\theta_\nu, \theta_\nu^*$  are proximal and  $\pi_\infty$  is RIC and weakly mixing of all orders. For a limit ordinal  $\nu$ ,  $X_\nu, Y_\nu, \pi_\nu$ , etc., are the inverse limits (or joins) of  $X_\iota, Y_\iota, \pi_\iota$ , etc., for  $\iota < \nu$ .

Thus if  $Y$  is trivial, then  $X_\infty$  is a proximal extension of  $X$  and a RIC weakly mixing extension of the strictly PI system  $Y_\infty$ . The homomorphism  $\pi_\infty$  is an isomorphism (so that  $X_\infty = Y_\infty$ ) if and only if  $X$  is a PI system.

**Lemma 5.4** ([9, Lemma 7.16]). *Let  $\pi : X \rightarrow Y$  be a weakly mixing and RIC extension of minimal systems. Then there is a dense  $G_\delta$  subset  $Y_0$  of  $Y$  such that, for each  $y \in Y_0$  and each  $x \in \pi^{-1}(y)$ ,  $P_\pi[x]$  is dense in  $\pi^{-1}(y)$ , where  $P_\pi[x] = \{z \in \pi^{-1}(\pi(x)) : (x, z) \in P(X)\}$ .*

**Proposition 5.5.** *Let  $(X, T)$  be minimal. If  $(X, T)$  is not strongly  $\mathcal{F}_t$ -sensitive, then  $(X, T)$  is PI.*

*Proof.* First we claim: if  $(X, T)$  is minimal, and there is  $x \in X$  such that  $(x, y)$  is a distal pair, and  $y$  is proximal to  $z_i \in X$  with  $z_i \rightarrow x$ ,  $z_i \neq x$ ,  $i \in \mathbb{N}$ , then  $(X, T)$  is strongly  $\mathcal{F}_t$ -sensitive.

Let  $\delta = \frac{1}{3} \inf_{n \in \mathbb{N}} d(T^n x, T^n y)$  and fix an open set  $U$  of  $X$ . Then there is  $l \in \mathbb{N}$  with  $T^l x \in U$  by the minimality of  $X$ . This implies that  $(T^l x, T^l y)$  is a distal pair and  $T^l y$  is proximal to  $T^l z_i$  with  $T^l z_i \rightarrow T^l x$ ,  $T^l z_i \neq T^l x$ . There is  $i \in \mathbb{N}$  such that  $T^l z_i \in U$ . Since  $(T^l y, T^l z_i)$  is proximal, we get that  $\{n \in \mathbb{Z}_+ : d(T^{n+l} y, T^{n+l} z_i) < \delta\} \in \mathcal{F}_t$ . This implies that  $\{n \in \mathbb{N} : d(T^{n+l} x, T^{n+l} z_i) > \delta\} \in \mathcal{F}_t$ . We conclude that  $(X, T)$  is strongly  $\mathcal{F}_t$ -sensitive, finishing the proof of the claim.

Assume that  $(X, T)$  is not PI. By Lemma 5.3,  $\theta^* : X_\infty \rightarrow X$  is proximal,  $\pi_\infty : X_\infty \rightarrow Y_\infty$  is weakly mixing, RIC and not an isomorphism. By Lemma 5.4, there are  $s \in Y_\infty$  and  $u \in \pi_\infty^{-1}(s)$  such that  $P_{\pi_\infty}[u]$  is dense in the  $\pi_\infty^{-1}(s)$ . Since  $\pi_\infty$  is not proximal, there is  $v \in X_\infty$  such that  $(u, v)$  is distal and  $\pi_\infty(v) = \pi_\infty(u)$ . Since  $\theta^*$  is proximal, we know that  $(\theta^*(v), \theta^*(u))$  is distal. As  $P_{\pi_\infty}[u]$  is dense in the  $\pi_\infty^{-1}(s)$ , there are  $v_i \rightarrow v$  such that  $v_i \neq v$  and  $(v_i, u)$  is proximal. This implies that  $(\theta^*(v_i), \theta^*(u))$  is proximal. It is clear that  $\theta^*(v_i) \neq \theta^*(v)$  and  $\theta^*(v_i) \rightarrow \theta^*(v)$ . Applying the claim we just proved, we conclude that  $(X, T)$  is strongly  $\mathcal{F}_t$ -sensitive, a contradiction.  $\square$

Before proving the following key result for Theorem D we need two well known lemmas.

**Lemma 5.6.** *Let  $\pi : X \rightarrow Y$  be an open factor map between two t.d.s. Assume that  $y \in Y$  and  $y_i \rightarrow y$ . Then for any  $z \in \pi^{-1}(y)$  there are  $z_i \in \pi^{-1}(y_i)$  such that  $\lim z_i = z$ .*

Let  $E(X, T)$  be the enveloping semigroup of  $(X, T)$ .

**Lemma 5.7.** *Let  $\pi : X \rightarrow Y$  be a distal factor map between two minimal t.d.s. Then  $\pi$  is open and  $\pi^{-1}(py) = p\pi^{-1}(y)$  for any  $y \in Y$  and any  $p \in E(X)$ .*

**Theorem 5.8.** *Let  $(X_3, T)$  be minimal and  $X_1 \xrightarrow{\pi_1} X_2 \xrightarrow{\pi_2} X_3$ , where  $\pi_1$  is a non-trivial proximal extension,  $\pi_2$  is a non-trivial distal extension and  $X_1$  is distal. If  $P(X_3)$  is not closed, then  $X_3$  is strongly  $\mathcal{F}_t$ -sensitive.*

*Proof.* Since  $P(X_3)$  is not closed, there are a distal pair  $(x_1, x_2) \in X_3 \times X_3$  and proximal pairs  $(x_1(i), x_2(i)) \in X_3 \times X_3$  for all  $i \in \mathbb{N}$  such that  $(x_1(i), x_2(i)) \rightarrow (x_1, x_2)$ . Let  $\pi = \pi_1\pi_2$ . It is clear that  $\pi(x_1(i)) = \pi(x_2(i))$  since  $X_1$  is distal. This implies that  $\pi(x_1) = \pi(x_2)$ . Moreover, we may assume that  $(x_1, x_2)$  is a minimal point. As  $(\pi_2(x_1), \pi_2(x_2))$  is proximal and minimal we know that  $\pi_2(x_1) = \pi_2(x_2)$ . Put  $\delta = \inf_{n \in \mathbb{Z}_+} d(T^n x_1, T^n x_2)$  and let  $U_i$  be an open neighborhood of  $x_i$  with  $\text{diam}(U_i) < \delta/6, 1 \leq i \leq 2$ .

Set  $y = \pi_2(x_1)$  and  $y_i = \pi_2(x_1(i)), i \in \mathbb{N}$ . Then  $\lim_{i \rightarrow \infty} y_i = y$ . Let

$$M = \overline{\text{orb}((x_1, x_2), T \times T)} \quad \text{and} \quad K = \{x \in X_3 : (x_1, x) \in M\} \subset \pi_2^{-1}(y).$$

It is clear that  $x_2 \in K$ . Moreover,  $M$  is a minimal subsystem of  $X_3 \times X_3$  and for any  $(z_1, z_2) \in M$  we have that  $\pi_2(z_1) = \pi_2(z_2)$ . Let  $p : M \rightarrow X_3$  be the projection to the first coordinate. Then  $p^{-1}(x_1) = \{x_1\} \times K$  and  $p$  is a distal extension. Put  $p^{-1}(x_1(i)) = \{x_1(i)\} \times K_i, i \in \mathbb{N}$ .

Since  $M \cap (U_1 \times U_2)$  is an open neighborhood of  $(x_1, x_2)$  and  $p$  is open, by Lemma 5.6 there are  $x'_2(i) \in K_i$  such that  $(x_1(i), x'_2(i)) \in M \cap (U_1 \times U_2)$  since  $\lim_{i \rightarrow \infty} x_1(i) = x_1$ . Note that  $x'_2(i) \in K_i$  and thus  $\pi_2(x'_2(i)) = y_i$ .

We can choose a sequence  $\{n_k\}$  such that  $T^{n_k}(x_1) \rightarrow (x_1(i))$ . Then there is  $z \in K$  such that  $T^{n_k}(x_1, z) \rightarrow (x_1(i), x'_2(i)) \in M \cap (U_1 \times U_2)$  by Lemma 5.7 using the distality of  $p$ .

As  $(x_1(i), x_2(i))$  is proximal,  $(x_1(i), x'_2(i))$  (in the orbit closure of  $(x_1, x_2)$ ) is distal and  $x_2(i), x'_2(i) \in U_2$  we know that

$$\{n \in \mathbb{Z}_+ : d(T^n x_1(i), T^n x_2(i)) < \delta/6\} \in \mathcal{F}_t.$$

By the definition of  $\delta$  we get  $\inf_{k \in \mathbb{Z}_+} d(T^k x_1(i), T^k x'_2(i)) \geq \delta$  which implies that

$$\{n \in \mathbb{Z}_+ : d(T^n x_2(i), T^n x'_2(i)) > \delta/6\} \in \mathcal{F}_t.$$

Since this holds for each neighborhood of  $x_2$ , we conclude that  $X_3$  is strongly  $\mathcal{F}_t$ -sensitive. □

**Lemma 5.9.** *Let  $(Z_{n+1}, T)$  be minimal and consider the strictly PI-Tower  $Z_1 \xrightarrow{\theta_1} Y_1 \xrightarrow{\rho_2} Z_2 \xrightarrow{\theta_2} Y_2 \xrightarrow{\rho_3} Z_3 \xrightarrow{\theta_3} \dots \xrightarrow{\rho_n} Z_n \xrightarrow{\theta_n} Y_n \xrightarrow{\rho_{n+1}} Z_{n+1}$ , where  $\theta_i$  is a non-trivial proximal extension,  $\rho_i$  is a non-trivial distal extension and  $Z_1$  is distal. If  $(Z_{n+1}, T)$  is not strongly  $\mathcal{F}_t$ -sensitive, then  $P(Z_{n+1})$  is closed.*

*Proof.* We prove this lemma by induction on  $n$ . For  $n = 1$  it is just Theorem 5.8. Now we assume that the theorem holds for  $n \leq k - 1$ , and we prove it still holds for  $n = k$ . Let  $Z_D$  be the maximal distal factor of  $Z_2$  and let  $\pi_1 : Z_2 \rightarrow Z_D$  be the factor map. Since  $P(Z_2)$  is closed,  $\pi_1$  is proximal. So  $\pi_2 = \pi_1\theta_2 : Y_2 \rightarrow Z_D$  is proximal. Consider the new PI-Tower

$$Z_D \xrightarrow{\pi_2} Y_2 \xrightarrow{\rho_3} Z_3 \xrightarrow{\theta_3} \dots \xrightarrow{\rho_k} Z_n \xrightarrow{\theta_k} Y_k \xrightarrow{\rho_{k+1}} Z_{k+1}$$

and the theorem holds for  $n \leq k - 1$ . We know that  $P(Z_{k+1})$  is closed, i.e., the theorem holds for  $n = k$ . □

We also need the following two lemmas for the proof of Theorem D.

**Lemma 5.10.** *Let  $\pi : X \rightarrow Y$  be a factor map between minimal systems.*

- (1) *If  $P(X)$  is closed, then  $P(Y)$  is closed.*
- (2) *If  $\pi$  is proximal and  $P(Y)$  is closed, then  $P(X)$  is closed*

*Proof.* (1) follows from Lemma 2 in [4].

(2) Let  $(x_i, x'_i)$  be proximal pairs in  $P(X)$  such that  $(x_i, x'_i) \rightarrow (x, x')$ . Then  $(\pi(x_i), \pi(x'_i))$  are proximal pairs in  $P(Y)$  such that  $(\pi(x_i), \pi(x'_i)) \rightarrow (\pi(x), \pi(x'))$ . Since  $P(Y)$  is closed,  $(\pi(x), \pi(x')) \in P(Y)$ . So there exists  $p \in E(X)$  (where  $E(X)$  is the Ellis semigroup of  $X$ ) such that  $p\pi(x) = p\pi(x')$ , i.e.,  $\pi(px) = \pi(px')$ . Since  $\pi$  is proximal, there exists  $q \in E(X)$  such that  $qpx = qpx'$ , i.e.,  $(x, x') \in P(X)$ . So  $P(X)$  is closed. □

**Lemma 5.11.** *Let  $X$  be an inverse limit of minimal systems  $\{(X_i, T_i)\}_{i=1}^\infty$ . If  $P(X_i)$  is closed for each  $i \in \mathbb{N}$ , then  $P(X)$  is closed.*

*Proof.* If  $P(X_i)$  is closed, then by Theorem 2 in [2],  $P(\prod_{i=1}^\infty X_i)$  is closed. By the definition of inverse limit,  $P(X)$  is closed. □

With the above preparation we are ready to give the proof.

*Proof of Theorem D.* (2)  $\Rightarrow$  (1) follows from Proposition 5.2.

(1)  $\Rightarrow$  (2) We assume that  $(X, T)$  is invertible first.

By Theorem 5.5  $(X, T)$  is PI. Consider the strictly PI-Tower in the structure theorem,

$$Z_1 \xleftarrow{\varrho_1} Y_1 \xleftarrow{\varrho_2} Z_2 \xleftarrow{\varrho_2} Y_2 \xleftarrow{\varrho_3} Z_3 \xleftarrow{\varrho_3} \dots X_\infty.$$

By Proposition 5.1  $X_\infty$  is not strongly  $\mathcal{F}_t$ -sensitive. So each finite tower

$$Z_1 \xleftarrow{\varrho_1} Y_1 \xleftarrow{\varrho_2} Z_2 \xleftarrow{\varrho_2} Y_2 \xleftarrow{\varrho_3} Z_3 \xleftarrow{\varrho_3} \dots \xleftarrow{\varrho_n} Z_n \xleftarrow{\varrho_n} Y_n \xleftarrow{\varrho_{n+1}} Z_{n+1}$$

is not strongly  $\mathcal{F}_t$ -sensitive.

Then By Lemma 5.9,  $P(Z_{n+1})$  is closed. So  $P(Y_n)$  is closed by Lemma 5.10. By Lemma 5.11,  $P(X_\infty)$  is closed. By Lemma 5.10,  $P(X)$  is closed. So  $P(X)$  is an equivalence relation [29]; then  $\pi : X \rightarrow X_D$  is proximal.

When  $(X, T)$  is not invertible, let  $(\tilde{X}, \tilde{T})$  be the natural extension of  $(X, T)$ . If  $P(X, T)$  is not closed, then by Lemma 5.10  $P(\tilde{X}, \tilde{T})$  is not closed. Since  $(\tilde{X}, \tilde{T})$  is an invertible minimal system,  $(\tilde{X}, \tilde{T})$  is strong  $\mathcal{F}_t$ -sensitive. So by Proposition 2.7,  $(X, T)$  is strong  $\mathcal{F}_t$ -sensitive, a contradiction. So  $P(X, T)$  is closed; then  $\pi : X \rightarrow X_D$  is proximal. □

To get a better understanding of Theorem 5.8, we give a well known example which is strongly  $\mathcal{F}_t$ -sensitive.

To do so, first we give some other criteria of strongly  $\mathcal{F}_t$ -sensitivity.

**Proposition 5.12.** *Let  $(X, T)$  be minimal and invertible. If there are  $x \neq y$  such that  $x, y$  is proximal for  $T^{-1}$  and  $\inf_{n \in \mathbb{Z}_+} d(T^n x, T^n y) > 0$ , then  $(X, T)$  is strongly  $\mathcal{F}_t$ -sensitive.*

*Proof.* Let  $\inf_{n \in \mathbb{Z}_+} d(T^n x, T^n y) = 2\delta > 0$ , let  $U$  be any open set of  $X$  and  $l \in \mathbb{N}$  with  $T^l x \in U$ . Put  $x_1 = T^l x$  and  $y_1 = T^l y$ ; then  $(x_1, y_1)$  is proximal for  $T^{-1}$  and  $\inf_{n \in \mathbb{Z}_+} d(T^n x_1, T^n y_1) \geq 2\delta$ . Since  $U$  is a neighborhood of  $x_1$ , there is  $\epsilon > 0$  such that  $B_\epsilon(x_1) \subset U$  for  $\epsilon < \frac{\delta}{10}$ . Set  $V = B_{\frac{\delta}{2}}(x_1)$ . Since  $x_1, y_1$  is proximal for  $T^{-1}$ ,

$\{n < 0 : d(T^n x_1, T^n y_1) < \epsilon/2\}$  is thick in  $\mathbb{Z}_-$ . As  $(X, T)$  is minimal, we know that  $(X, T^{-1})$  is minimal. Thus,  $\{n < 0 : T^n x_1 \in V\}$  is syndetic in  $\mathbb{Z}_-$ . There is  $s < 0$  such that  $T^s x_1 \in V$  and  $d(T^s x_1, T^s y_1) < \epsilon/2$ . This implies that  $T^s x_1, T^s y_1 \in U$ . Set  $z_1 = T^s x_1$  and  $z_2 = T^s y_1$ . Then  $\{m \in \mathbb{Z}_+ : d(T^m z_1, T^m z_2) > \delta\} = [-s, \infty)$  is thick. So  $(X, T)$  is strongly  $\mathcal{F}_t$ -sensitive.  $\square$

We will give an application of Proposition 5.12, namely we shall show that the Morse minimal system is strongly  $\mathcal{F}_t$ -sensitive. The following results related to Morse system are basic and well known; see for example [15].

The Morse sequence  $\omega(n)$ :

$$0110100110010110 \dots$$

can be described by the following algorithms.

$\omega(0) = 0, \omega(2n) = \omega(n), \omega(2n + 1) = 1 - \omega(n) (n \in \mathbb{N})$ . Considering  $\omega$  as an element of  $\Omega = \{0, 1\}^{\mathbb{Z}}$  where  $\omega(-n) = \omega(n - 1)$ , let  $X \subset \Omega$  be its orbit closure under the shift  $\sigma$  with  $\sigma\xi(n) = \xi(n + 1)$ . Then  $(X, \sigma)$  is a minimal flow called the Morse minimal set.

The homeomorphism  $\varphi : \xi \rightarrow \bar{\xi}$  where  $\bar{\xi}(n) = \overline{\xi(n)}$  (and  $\bar{0} = 1, \bar{1} = 0$ ) preserves  $X$  and commutes with  $\sigma$ . The quotient space  $Y$ , of  $X$  modulo the group  $\{\varphi, \varphi^2 = id\}$  is a factor of  $(X, \sigma)$  in the sense that the natural projection  $\pi_1 : X \rightarrow Y$  satisfies  $\pi_1\sigma = \sigma\pi_1$ . For every  $\xi \in X$  there exists a sequence  $k_i$  such that  $\sigma^{k_i} \xi \rightarrow \xi$  and we can associate with  $\xi$  the dyadic sequence  $\{a_n\}, 0 \leq a_n \leq 2^n - 1$ , according to the rule  $a_n = \lim\{k_i \pmod{2^n}\}$ . It is easy to check that this limit exists and is independent of the particular choice of the sequence  $\{k_i\}$ . Clearly also the dyadic sequences corresponding to  $\xi$  and  $\bar{\xi}$  coincide, so that we can consider the map  $\pi_2 : Y \rightarrow G$  where  $G$  is the compact group of sequences  $\{\{a_n\} : 0 \leq a_n \leq 2^n - 1, a_{n+1} = a_n \pmod{2^n}\}$ . Moreover  $\pi_2\sigma y = (\pi_2 y) + 1$  where  $1 = (1, 0, 0, \dots) \in G$ . In fact it is not hard to describe  $\pi_2$  explicitly. If  $\eta \in \Omega$  is defined by  $\eta(n) = \omega(n)$  for  $n \geq 0$  and  $\eta(n) = \overline{\omega(n)}$  for  $n < 0$ , then  $\eta \in X$  and denoting  $y_1 = \pi_1(\omega), y_2 = \pi_1(\eta)$  we have for all  $n \in \mathbb{Z}, \pi_2^{-1}(n \cdot 1) = \{\sigma^n y_1, \sigma^n y_2\}$  while  $\pi_2^{-1}(g)$  is a singleton for every  $g \in G \setminus \{n \cdot 1; n \in \mathbb{Z}\}$ . The map  $\pi_2$  is therefore almost one-to-one hence proximal.

**Example 5.13.** The Morse minimal system is strongly  $\mathcal{F}_t$ -sensitive.

*Proof.* Let  $X$  be the Morse minimal system. Then  $\pi_1 : X \rightarrow Y$  is a group extension and  $\pi_2 : Y \rightarrow G$  is an almost one-to-one extension. It is easy to see that  $\inf_{n \in \mathbb{Z}_+} d(\sigma^n \bar{\omega}, \sigma^n \eta) > 0$  and  $(\bar{\omega}, \eta)$  is asymptotic for  $\sigma^{-1}$ . By Proposition 5.12, we conclude that the Morse minimal system is strongly  $\mathcal{F}_t$ -sensitive.  $\square$

**5.2. Strong  $\mathcal{F}_{Poin_d}$  and  $\mathcal{F}_{d,0}^*$ -sensitivity.** In this subsection, we discuss strong  $\mathcal{F}_{Poin_d}$  and  $\mathcal{F}_{d,0}^*$ -sensitivity. In this subsection we assume that  $T$  is a homeomorphism.

**Definition 5.14.** Let  $(X, T)$  be a t.d.s. We say  $(X, T)$  is strongly  $\mathcal{F}_{Poin_d}$  (resp.  $\mathcal{F}_{d,0}^*$ )-sensitive if there is  $\delta > 0$  such that for each opene subset  $U$  of  $X$ , there are  $x, y \in U$  with  $\{n \in \mathbb{Z} : d(T^n x, T^n y) > \delta\} \in \mathcal{F}_{Poin_d}$  (resp.  $\mathcal{F}_{d,0}^*$ ).

We state some basic notation, definitions and results related to  $\mathcal{F}_{Poin_d}$  (resp.  $\mathcal{F}_{d,0}^*$ ) first.

We say that  $S \subset \mathbb{Z}$  is a set of  $d$ -recurrence if for every measure preserving system  $(X, \chi, \mu, T)$  and for every  $A \in \chi$  with  $\mu(A) > 0$ , there exists  $n \in S \setminus \{0\}$  such that

$\mu(A \cap T^{-n}A \cap \dots \cap T^{-dn}A) > 0$ . Let  $\mathcal{F}_{Poin_d}$  be the family consisting of all sets of  $d$ -recurrence. By Furstenberg’s multiple ergodic theorem the definition is reasonable. A striking result due to Furstenberg and Katznelson [13, Theorem C] in our terms is that  $\mathcal{F}_{fip} \subset \mathcal{F}_{Poin_d}$ . So we have

**Proposition 5.15.** *If a minimal system  $(X, T)$  is not strongly  $\mathcal{F}_{Poin_d}$ -sensitive, then it is an almost one-to-one extension of its maximal  $\infty$ -step nilfactor.*

*Proof.* It follows from the fact  $\mathcal{F}_{fip} \subset \mathcal{F}_{Poin_d}$  and Theorem B. □

A subset  $A$  of  $\mathbb{Z}$  is a  $Nil_d$  Bohr<sub>0</sub>-set if there exist a  $d$ -step nilsystem  $(X, T)$ ,  $x_0 \in X$  and an open neighborhood  $U$  of  $x_0$  such that  $N(x_0, U) =: \{n \in \mathbb{Z} : T^n x_0 \in U\}$  is contained in  $A$ . Denote by  $\mathcal{F}_{d,0}$  the family consisting of all  $Nil_d$  Bohr<sub>0</sub>-sets. Let  $\mathcal{F}_{GP_d}$  be the family generated by the sets of forms

$$\bigcap_{i=1}^k \{n \in \mathbb{Z} : P_i(n) \pmod{\mathbb{Z}} \in (-\epsilon_i, \epsilon_i)\},$$

where  $k \in \mathbb{N}$ ,  $P_1, \dots, P_k$  are generalized polynomials of degree  $\leq d$ , and  $\epsilon_i > 0$ . For the definition of generalized polynomials, see [23, Page 21]. We have [23, Proposition 7.21, Proposition 7.24] for each  $d \in \mathbb{N}$ ,  $\mathcal{F}_{d,0}$  is a filter, and  $\mathcal{F}_{Poin_d}$  has the Ramsey property.

The following two lemmas will be used in the next theorem.

**Lemma 5.16** ([23, Theorem E]). *Let  $(X, T)$  be a minimal system and  $x, y \in X$ . Then the following statements are equivalent for  $d \in \mathbb{N} \cup \{\infty\}$ :*

- (1)  $(x, y) \in \mathbf{RP}^{[d]}$ .
- (2)  $N(x, U) \in \mathcal{F}_{d,0}^*$  for each neighborhood  $U$  of  $y$ .
- (3)  $N(x, U) \in \mathcal{F}_{Poin_d}$  for each neighborhood  $U$  of  $y$ .

**Lemma 5.17** ([23, Theorem F]). *Let  $(X, T)$  be a minimal system,  $x \in X$  and  $d \in \mathbb{N} \cup \{\infty\}$ . Then the following statements are equivalent:*

- (1)  $x$  is a  $d$ -step AA point.
- (2)  $N(x, V) \in \mathcal{F}_{d,0}$  for each neighborhood  $V$  of  $x$ .
- (3)  $N(x, V) \in \mathcal{F}_{Poin_d}^*$  for each neighborhood  $V$  of  $x$ .

Using Lemma 5.17 instead of using Proposition 3.1 we have the following result by the same proof of Theorem 3.2.

**Theorem 5.18.** *Let  $(X, T)$  be a minimal system and  $d \in \mathbb{N}$ . Then  $(X, T)$  is  $\mathcal{F}_{Poin_d}$ -sensitive if and only if  $\pi : X \rightarrow X_{eq}$  is not almost one-to-one.*

Using the idea of the proof of Proposition 4.7 and Proposition 4.11, we obtain the following result.

**Theorem 5.19.** *If  $(X, T)$  is a strongly  $\mathcal{F}_{Poin_d}$ -sensitive minimal system, then  $\pi : X \rightarrow X_d$  is not an almost one-to-one extension.*

*Proof.* Suppose that  $(X, T)$  is strongly  $\mathcal{F}_{Poin_d}$ -sensitive with the sensitive constant  $10\delta$  and  $\pi : X \rightarrow X_d$  is an almost one-to-one extension. Then there is  $x \in X$  such that  $\mathbf{RP}^{[d]}[x] = x$ .

Let  $\delta' < \delta$  and  $U = B(x, \delta')$ ; then there are  $y, z \in U$  such that

$$F = \{n \in \mathbb{Z} : d(T^n y, T^n z) > 10\delta\} \in \mathcal{F}_{Poin_d}.$$

By the Ramsey property of  $\mathcal{F}_{\mathcal{Poin}_d}$ ,  $F_1 = \{n \in \mathbb{Z} : d(T^n x, T^n u) > 5\delta\} \in \mathcal{F}_{\mathcal{Poin}_d}$ , where  $u = y$  or  $u = z$ . As  $\mathbf{RP}^{[d]}[x] = x$ , by Lemma 5.16 we have  $F_2 = N(x, U) \in \mathcal{F}_{\mathcal{Poin}_d}^*$ . So

$$F_3 = F_1 \cap F_2 \subset \{n \in \mathbb{Z} : d(x, T^n u) > 5\delta - \delta'\} \in \mathcal{F}_{\mathcal{Poin}_d}.$$

Then by the Ramsey property of  $\mathcal{F}_{\mathcal{Poin}_d}$  and using the same argument as in the proof of Proposition 4.11, we deduce that there are  $v \in X$  with  $d(u, v) \geq \delta$  and for each neighborhood  $V$  of  $v$ ,  $N(u, V) \in \mathcal{F}_{\mathcal{Poin}_d}$ . It is clear that  $(u, v) \in \mathbf{RP}^{[d]}(X)$  by Lemma 5.16. Moreover, we know that  $\pi(u) = \pi(v)$  since  $\mathbf{RP}^{[d]}(X_d) = \Delta$ . This contradicts the fact that  $\mathbf{RP}^{[d]}[x] = x$ , showing that  $\pi$  is not almost one-to-one.  $\square$

**Corollary 5.20.** *If  $(X, T)$  is a strongly  $\mathcal{F}_{d,0}^*$ -sensitive minimal system, then  $\pi : X \rightarrow X_d$  is not a one-to-one extension.*

*Proof.* The proof is similar to Theorem 5.19.  $\square$

It is unexpected that the converse of Theorem 5.19 fails. To give a counterexample we need the following lemma.

**Lemma 5.21** ([23, Theorem B, Corollary D]). *For  $d \in \mathbb{N}$ ,  $\mathcal{F}_{d,0} = \mathcal{F}_{GP_d}$  and  $\mathcal{F}_{\mathcal{Poin}_d} \subset \mathcal{F}_{d,0}^*$ .*

**Example 5.22.** There is a minimal system which is not an almost one-to-one extension of the maximal  $(d - 1)$ -step nilfactor and the system is not strongly  $\mathcal{F}_{\mathcal{Poin}_{d-1}}$ -sensitive.

*Proof.* For  $d \geq 2$  define  $T_{\alpha,d} : \mathbb{T}^d \rightarrow \mathbb{T}^d$  by

$$T_{\alpha,d}(\theta_1, \theta_2, \dots, \theta_d) = (\theta_1 + \alpha, \theta_2 + \theta_1, \dots, \theta_d + \theta_{d-1}),$$

where  $\alpha \in \mathbb{R}$ . When  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ ,  $(\mathbb{T}^d, T_{\alpha,d})$  is minimal. A simple computation yields that

$$T_{\alpha,d}^n(\theta_1, \theta_2, \dots, \theta_d) = (\theta_1 + n\alpha, \theta_2 + n\theta_1 + \frac{1}{2}n(n-1)\alpha, \dots, \sum_{i=0}^d \binom{n}{d-i} \theta_i),$$

where  $\theta_0 = \alpha, n \in \mathbb{Z}$  and  $\binom{n}{0} = 1, \binom{n}{i} = \frac{\prod_{j=0}^{i-1} (n-j)}{i!}$  for  $i = 1, 2, \dots, d$ .

$(\mathbb{T}^d, T_{\alpha,d})$  is a  $d$ -step nilsystem, so we have  $\mathbf{RP}^{[d]}(\mathbb{T}^d) = \Delta_{T^d}$ , and for  $s < d$

$$\mathbf{RP}^{[s]}(\mathbb{T}^d) = \{(\mathbf{x}, \mathbf{y}) : \text{the first } s \text{ coordinates of } \mathbf{x}, \mathbf{y} \text{ are the same}\}.$$

When  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ ,  $(\mathbb{T}^d, T_{\alpha,d})$  is minimal and not an almost one-to-one extension of its maximal  $(d - 1)$ -step nilfactor. We will prove that  $\mathbb{T}^d$  is not strongly  $\mathcal{F}_{\mathcal{Poin}_{d-1}}$ -sensitive.

Assume the contrary that it is strongly  $\mathcal{F}_{\mathcal{Poin}_{d-1}}$ -sensitive. That is, there is  $\delta > 0$  such that for any  $\mathbf{x} \in \mathbb{T}^d$  and  $\epsilon \in \mathbb{R}$ , there is  $\mathbf{y} \in \mathbb{T}^d$  such that  $\|\mathbf{x} - \mathbf{y}\| < \epsilon$  and  $\{n \in \mathbb{Z} : d(T_{\alpha,d}^n \mathbf{x}, T_{\alpha,d}^n \mathbf{y}) > 2\delta\} \in \mathcal{F}_{\mathcal{Poin}_{d-1}}$ . We can choose  $\mathbf{x} = \mathbf{0}$  and  $\epsilon = \delta$ ; then we have  $\mathbf{y} = (y_1, y_2, \dots, y_d)$  with

$$\{n \in \mathbb{Z} : d(T_{\alpha,d}^n \mathbf{0}, T_{\alpha,d}^n \mathbf{y}) > 2\delta\} \in \mathcal{F}_{\mathcal{Poin}_{d-1}}$$

and  $\|\mathbf{y}\| < \delta$ . A simple computation yields that

$$T_{\alpha,d}^n \mathbf{y} - T_{\alpha,d}^n \mathbf{0} = (y_1, 0, \dots, 0) + (0, T_{y_1, d-1}^n(y_2, y_3, \dots, y_d)).$$

So we have

$$F_1 = \{n \in \mathbb{Z} : \|T_{y_1, d-1}^n(y_2, y_3, \dots, y_d)\| \geq \delta\} \in \mathcal{F}_{\text{Poin}_{d-1}}$$

since  $F_1 \supset \{n \in \mathbb{Z} : d(T_{\alpha, d}^n \mathbf{0}, T_{\alpha, d}^n \mathbf{y}) > 2\delta\}$ .

Define

$$F_2 = \{n \in \mathbb{Z} : \text{the absolute value of each coordinate of } T_{y_1, d-1}^n(y_2, y_3, \dots, y_d) \text{ is less than } \frac{\delta}{(d-1)}\}.$$

We know that  $F_2 \in \mathcal{F}_{GP_{d-1}}$  by the definition of generalized polynomials. Moreover, we have

$$F_2 \subset F_3 = \{n \in \mathbb{Z} : \|T_{y_1, d-1}^n(y_2, y_3, \dots, y_d)\| < \delta\}.$$

Thus,  $F_1^c = F_3 \supset F_2$  which implies that  $F_1^c \in \mathcal{F}_{GP_{d-1}} = \mathcal{F}_{d-1,0}$ . So  $F_1 \notin \mathcal{F}_{d-1,0}^*$  which implies that  $F_1 \notin \mathcal{F}_{\text{Poin}_{d-1}}$  by Lemma 5.21, a contradiction.  $\square$

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