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ON THE PARAMETRIC BEHAVIOR OF A-HYPERGEOMETRIC SERIES

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ABSTRACT. We describe the parametric behavior of the series solutions of an A-hypergeometric system. More precisely, we construct explicit stratifications of the parameter space such that, on each stratum, the series solutions of the system are holomorphic.

1. Introduction

A classical object of study, the Gauss hypergeometric equation is

$$z(1-z)\frac{d^2F}{dz^2}+(c-z(a+b+1))\frac{dF}{dz}-abF=0\,; \qquad a,b,c\in\mathbb{C}. \tag{1.1}$$

The quantities a, b, c are considered parameters, and z is considered a variable. For $c \notin \mathbb{Z}$, the functions

(1.2)

$$F(z;a,b,c)=\sum_{n=0}^{\infty}\left[\prod_{\ell=0}^{n-1}\frac{(a+\ell)(b+\ell)}{(c+\ell)}\right]\cdot\frac{z^n}{n!}\,; \quad z^{1-c}F(z;a+1-c,b+1-c,2-c)$$

form a basis of the solution space of (1.1) in a neighborhood of the origin $0 \in \mathbb{C}$. It is not hard to see that as functions of a, b, c, the series (1.2) are meromorphic; the first has poles for $c \in \mathbb{Z}_{\leq 0}$, and the second one for $c \in \mathbb{Z}_{>1}$. In order to solve (1.1) for $c \in \mathbb{Z}$, one takes derivatives of the series (1.2) with respect to the parameters. For instance, when $c = 1$, both series (1.2) coincide, and

$$F(z;a,b,1)\log(z)+\sum_{n=0}^{\infty}\left[\frac{\partial}{\partial h}\prod_{\ell=0}^{n-1}\frac{(a+h+\ell)(b+h+\ell)}{(c+h+\ell)(1+h+\ell)}\right]_{h=0} z^n$$

is the second function needed to span the solution of (1.1) near $0 \in \mathbb{C}$. Since it is obtained as a derivative, the series above is an entire function of a, b, and holomorphic for z in any simply connected subset of the punctured disc $0 < |z| < 1$ in $\mathbb{C}$. In this way, it is possible to understand the solutions of (1.1) as functions of $(a, b, c) \in \mathbb{C}^3$.

The goal of this article is to perform a similar analysis for certain generalized hypergeometric systems in several variables: the *A-hypergeometric systems* of Gelfand, Graev, Kapranov, and Zelevinsky (Definition 1.1).

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We adopt the convention that $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. Throughout this article, let A be a $d \times n$ integer matrix of rank d whose columns, denoted by $a_1, \dots, a_n$, span $\mathbb{Z}^d$ as a lattice. We write $A = [a_{ij}]$ where $i = 1, \dots, d$ and $j = 1, \dots, n$. In Sections 2, 3, 4, and 5, we impose the additional assumption that A is *homogeneous*, i.e., that $(1, \dots, 1) \in \mathbb{Q}^n$ is in the $\mathbb{Q}$-row span of A. We return to the inhomogeneous case in Section 6.

The *Weyl algebra* D is the ring of differential operators on $\mathbb{C}^n$. In other words, D is the quotient of the free associative $\mathbb{C}$-algebra generated by the variables $x_1, \dots, x_n$, and $\partial_1, \dots, \partial_n$, by the two-sided ideal

$$\langle x_k x_\ell = x_\ell x_k,\ \partial_k \partial_\ell = \partial_\ell \partial_k,\ \partial_k x_\ell = x_\ell \partial_k + \delta_{k\ell} \mid k, \ell \in \{1, \dots, n\} \rangle ,$$

where $\delta_{k\ell}$ denotes the Kronecker δ-function. The *toric ideal* of the matrix A is defined by

$$I_A = \langle \partial^u - \partial^v \mid u, v \in \mathbb{N}^n,\ Au = Av \rangle \subseteq \mathbb{C}[\partial].$$

For each parameter $\beta \in \mathbb{C}^d$, the *Euler operators* associated to A and β are $E - \beta = \{E_i - \beta_i\}_{i=1}^d$, where

$$E_i = \sum_{j=1}^n a_{ij} x_j \partial_j, \qquad i = 1, \dots, d.$$

Definition 1.1. The *A-hypergeometric system* with parameter β is given by

$$H_A(\beta) := D \cdot (I_A + \langle E - \beta \rangle).$$

The D-module $D/H_A(\beta)$ is known as the *A-hypergeometric D-module with parameter β*.

A-hypergeometric systems [GGZ87, GZK88, GKZ89] were introduced by Gelfand, Graev, Kapranov, and Zelevinsky. We study here *A-hypergeometric functions*, the solutions of A-hypergeometric systems. Let x be a nonsingular point of the system $H_A(\beta)$. A *solution of $D/H_A(\beta)$ at x* is a germ φ of a holomorphic function such that $P \bullet \varphi = 0$ for each $P \in H_A(\beta)$. The solutions of $D/H_A(\beta)$ at x form a $\mathbb{C}$-vector space, which we denote by $\mathrm{Sol}_x(H_A(\beta))$. The *rank* of $D/H_A(\beta)$, denoted $\operatorname{rank} D/H_A(\beta)$, is the vector space dimension of $\mathrm{Sol}_x(H_A(\beta))$, which is independent of the choice of x.

Some properties of the Gauss hypergeometric equation (1.1) generalize to the A-hypergeometric setting. Of particular relevance is fact is that if β is sufficiently generic, the solutions of $D/H_A(\beta)$ have explicit combinatorial expressions, analogous to (1.2), which are easily shown to be holomorphic as functions of the parameters (see Section 3 for more details). We can apply the method of parametric derivatives, which produced the solutions of (1.1) at special parameters, to the A-hypergeometric situation. However, this method does not produce all A-hypergeometric functions unless the holonomic rank of $D/H_A(\beta)$ is constant as a function of β [BFM14, Section 3]. In general, however, the rank of $D/H_A(\beta)$ is not constant [MMW05, Ber11], and therefore the method of parametric derivatives will not serve the purpose of this article.

Our main tool to understand the parametric behavior of A-hypergeometric functions is the fact that these functions can be expanded as Nilsson series (convergent Puiseux series with logarithms) in an algorithmic way. When A is homogeneous, this follows from the fact that $D/H_A(\beta)$ is a regular holonomic D-module [Hot91], by the method of canonical series [SST00, Sections 2.5 and 2.6]. If A is not homogeneous, then $D/H_A(\beta)$ has irregular singularities [SW08] (these systems are also known as *confluent*), but if the domain of expansion is adequately chosen, one can still write A-hypergeometric functions as Nilsson series [DMM12].

When A-hypergeometric series have no logarithms, their parametric behavior is understood (Section 3). The core of our work is to understand the true logarithmic A-hypergeometric series as functions of the parameters. The key difficulty in this study is the fact that there are no general combinatorial expressions for those solutions. When such combinatorial expressions exist (for instance [AS14]), our arguments may be simplified.

To compute the solutions of $D/H_A(\beta)$ as power series, we choose special points (called *toric infinities*) around which to expand. This is equivalent to choosing a weight vector. A vector $w \in \mathbb{R}^n_{>0}$ is called a *weight vector* for A if it is sufficiently generic so that the initial ideal $\mathrm{in}_w(I_A)$ with respect to w is a monomial ideal. In this case, w induces a *coherent* (or *regular*) *triangulation* of $\mathrm{conv}(A)$, the convex hull of the columns of A, by projecting onto $\mathrm{conv}(A)$ the lower hull of the points (w_i, a_i), where $a_1, \dots, a_n$ are the columns of A. For Theorems 1.2 and 1.3, we also require w to be generic enough that the cones (2.1) and (2.2) are full dimensional.

Suppose that A is homogeneous. Let $w \in \mathbb{R}^n_{>0}$ be a weight vector and let Δ_w be the induced triangulation of $\mathrm{conv}(A)$. We define a stratification $\mathcal{S}$ of $\mathbb{C}^d$ associated with the triangulation Δ_w.

If $\sigma \subseteq \{a_1, \dots, a_n\}$ is a codimension 1 face of Δ_w, let H_σ be the hyperplane spanned (as a vector space) by the elements of σ. The *codimension* 0 *stratum of* $\mathcal{S}$ is defined as

$$\mathcal{S}_0 := \mathbb{C}^d \smallsetminus \Bigg(\bigcup_{\substack{\sigma \in \Delta_w \\ \mathrm{codim}(\sigma)=1}} \bigcup_{p \in \mathbb{Z}^d} (p + H_\sigma) \Bigg).$$

Inductively, $\overline{\mathcal{S}_i} \smallsetminus \mathcal{S}_i$ is an infinite, locally finite union of affine spaces of codimension $i+1$. (Here we use the closure in either the Euclidean or the Zariski topology of $\mathbb{C}^d$.) Then we define the *codimension* $i+1$ *stratum of* $\mathcal{S}$ as

$$\mathcal{S}_{i+1} := \bigcup_{H \in \overline{\mathcal{S}_i} \smallsetminus \mathcal{S}_i} H \smallsetminus \Bigg(\bigcup_{\substack{\sigma \in \Delta_w \\ \mathrm{codim}(\sigma)=1}} \bigcup_{\substack{p \in \mathbb{Z}^d \\ p + H_\sigma \not\supseteq H}} p + H_\sigma \Bigg),$$

where above, $H \in \overline{\mathcal{S}_i} \smallsetminus \mathcal{S}_i$ denotes an irreducible component of $\overline{\mathcal{S}_i} \smallsetminus \mathcal{S}_i$.

Note that the intersection of two irreducible components of $\overline{\mathcal{S}_i} \smallsetminus \mathcal{S}_i$ is not contained in $\mathcal{S}_{i+1}$ by construction. Consequently, if q is a point in $\mathcal{S}_{i+1}$, then it belongs to a unique irreducible component H of $\overline{\mathcal{S}_i} \smallsetminus \mathcal{S}_i$.

The stratification $\mathcal{S}$ has the property that if an integer translate of the span of a face of Δ_w intersects an irreducible component of $\overline{\mathcal{S}_i} \smallsetminus \mathcal{S}_i$, then it contains that whole irreducible component.

Theorem 1.2. *Assume that A is homogeneous and $w \in \mathbb{R}^n_{>0}$ is such that the cones* (2.1) *and* (2.2) *are full dimensional. Let Δ_w be a coherent triangulation of A arising from w, and consider the stratification $\mathcal{S}$ of $\mathbb{C}^d$ associated to Δ_w as above. Then there is an open set $U \subset \mathbb{C}^n$ (depending only on w and A) such that the solutions of $H_A(\beta)$ are holomorphic on $U \times \mathcal{S}_i$, for $i = 0, \dots, d-1$.*

If A is not homogeneous, then a slightly weaker version of Theorem 1.2 holds. For A a $d \times n$ integer matrix as above, we define $\rho(A)$ to be the $(d+1) \times (n+1)$

matrix obtained from A by appending $0 \in \mathbb{Z}^d$ on the left, and then appending $(1,\dots,1) \in \mathbb{Z}^{n+1}$ on top of the resulting $d\times(n+1)$ matrix. In this case, we consider coherent triangulations Δ_w of $\operatorname{conv}(\{0, a_1, \dots, a_n\})$ and construct the corresponding stratification $\mathcal{S}$ in the same way as before. Moreover, only special triangulations can be used in the following result.

Theorem 1.3. *Let $A \in \mathbb{Z}^{d\times n}$, not necessarily homogeneous, and let $w \in \mathbb{R}^n_{>0}$ be sufficiently generic that the cones* (2.1) *and* (2.2) *are full dimensional. Let $w \in \mathbb{R}^n_{>0}$ be a weight vector such that the closure of the cone* (2.1) *contains $(1,\dots,1) \in \mathbb{R}^n$. Let Δ_w be the coherent triangulation of $\operatorname{conv}(\{0, a_1, \dots, a_n\})$ arising from $(0, w)$, and let $\mathcal{S}$ be the corresponding stratification of $\mathbb{C}^d$. Then there exists an open set $U \subset \mathbb{C}^n$ (depending only on w and A) such that the solutions of $H_A(\beta)$ are holomorphic on $U \times \mathcal{S}_i$ for $i = 1,\dots,d-1$.*

We remark that A-hypergeometric functions do not vary according to isomorphism classes of A-hypergeometric systems, as computed in [Sai01].

1.1. **Relation to GG functions.** In [GG97], Gelfand and Graev introduced systems of differential and difference equations, much generalizing the hypergeometric systems studied here. In a special case, these were also rediscovered by Ohara and Takayama [OT09]. A key feature of the *GG systems* is that the parameters are now considered as variables, and solutions are required to be holomorphic in these new variables. However, the issues addressed in this article do not arise in the *GG* setting, because it no longer makes sense to restrict to special parameters. For instance, the *GG* system for the Gauss hypergeometric equation (1.1) is

$$\begin{aligned}
\frac{dF(z;a,b,c)}{dz} &= F(z;a+1,b+1,c+1),\\
F(z;a+1,b,c) &= aF(z;a,b,c) + zF(z;a+1,b+1,c+1),\\
F(z;a,b+1,c) &= bF(z;a,b,c) + zF(z;a+1,b+1,c+1),\\
F(z;a,b,c-1) &= cF(z;a,b,c) + zF(z;a+1,b+1,c+1),
\end{aligned}$$

whose solution space is spanned by

$$(1/\Gamma(c))F(z;a,b,c) \quad \text{and} \quad (z^{1-c}/\Gamma(2-c))F(z;a+1-c,b+1-c,2-c),$$

where Γ is the Euler gamma function and F is as in (1.2). These series are entire in a, b, c, and holomorphic for z in a punctured polydisc around $0 \in \mathbb{C}$. Thus, the Gauss *GG* system does not capture the solutions of the Gauss differential equation for $c \in \mathbb{Z}$. In general, the *GG* systems corresponding to the hypergeometric equations studied here give rise to the solutions of those systems for generic parameters, as is stated in [GG99, Theorem 4]. See also Section 3.

Outline. Sections 2-5 lead to a proof of Theorem 1.2. Section 2 contains background on canonical series solutions, mostly from [SST00, Chapter 2]. Section 3 introduces useful series of Horn type and shows that logarithm-free hypergeometric series are holomorphic in the parameters. Sections 4 and 5 extend this result to hypergeometric series involving logarithms, culminating in a proof of Theorem 1.2. Section 6 gives an upper bound for rank in the inhomogeneous case and provides the proof of Theorem 1.3.

2. Canonical series solutions of A-hypergeometric systems

In this section, we summarize the essential properties of the canonical series solutions from Chapter 2 of [SST00] that are needed in the sequel and prove Theorem 2.4.

A left D-module of the form D/I is said to be *holonomic* if $\operatorname{Ext}^i_D(D/I, D) = 0$ for all $i \neq n$. An ideal I is said to be holonomic if the quotient D/I is a holonomic D-module. In this case, by a result of Kashiwara (see [SST00, Theorem 1.4.19]) the *holonomic rank* of D/I, denoted by $\operatorname{rank}(D/I)$, which is by definition the dimension of the space of germs of holomorphic solutions of I near a generic nonsingular point of the system I, is finite. By [Ado94, Theorem 3.9], the quotient $D/H_A(\beta)$ is holonomic for all β, and thus has finite rank.

The algorithm to compute canonical series solutions presented in [SST00, Section 2.6] applies to all *regular holonomic* left D-ideals. For an A-hypergeometric system, regular holonomicity is equivalent to the matrix A being homogeneous [Hot91, SW08], an assumption which we impose for the remainder of this section.

Fix a weight vector $w \in \mathbb{R}^n$. The $(-w, w)$-*weight* of a monomial $x^u\partial^v \in D$ is by definition $-w \cdot u + w \cdot v$. Thus, the weight w induces a partial order on the set of monomials in D. For a nontrivial $f(x, \partial) = \sum_{u,v} c_{uv}x^u\partial^v \in D$, the *initial form* $\operatorname{in}_{(-w,w)}(f)$ of f with respect to $(-w, w)$ is the subsum of f consisting of its (nonzero) terms of maximal $(-w, w)$-weight. If I is a left D-ideal, its *initial ideal* with respect to $(-w, w)$ is

$$\operatorname{in}_{(-w,w)}(I) = \langle \operatorname{in}_{(-w,w)}(f) \mid f \in I, f \neq 0\rangle \subset D.$$

Similarly, if $J \subseteq \mathbb{C}[\partial]$ is an ideal, we define the initial ideal $\operatorname{in}_w(J)$ as the ideal generated by the initial terms of all nonzero polynomials in J.

Assume that w is sufficiently generic so that the rational polyhedral cones

$$C := \{w' \in \mathbb{R}^n \mid \operatorname{in}_{(-w',w')}(H_A(\bar\beta)) = \operatorname{in}_{(-w,w)}(H_A(\bar\beta)) \text{ for all } \bar\beta \in \mathbb{C}^d\} \quad \text{and} \tag{2.1}$$

$$C^* := \{\, v \in \mathbb{R}^n \mid v \cdot w' \geq 0 \text{ for all } w' \in C\} \tag{2.2}$$

are full dimensional. (See the proof of Theorem 2.4 for a proof that C, and hence also C^*, is a rational polyhedral cone.) Note that C is open, while C^* is closed (with nonempty interior). Identify $C^*_{\mathbb{Z}} = C^* \cap \mathbb{Z}^n$ with the set $\{x^v \mid v \in C^* \cap \mathbb{Z}^n\}$ (notice that this set depends on w), and set

$$N_w = \mathbb{C}[[C^*_{\mathbb{Z}}]][\log(x)],$$

where $\log(x) = (\log(x_1), \ldots, \log(x_n))$. We call an element of N_w of the form $\log(x)^\delta$ a *logarithmic monomial* and an element of the form $x^\gamma \log(x)^\delta$ a *mixed monomial.* A series $\varphi(x) = \sum_{\gamma,\delta} c_{\gamma\delta}\, x^\gamma \log(x)^\delta \in N_w$ is called a logarithmic series. By [SST00, Proposition 2.5.2], if φ is a logarithmic series which is a solution to $H_A(\beta)$, then, with notation as above, the set of real parts $\{\operatorname{Re}(\gamma\cdot w) \mid c_{\gamma\delta} \neq 0 \text{ for some } \delta\}$ achieves a (finite) minimum which is denoted $\mu(\varphi)$. Furthermore, the subseries of φ whose terms are $c_{\gamma\delta}x^\gamma \log(x)^\delta$ such that $c_{\gamma\delta} \neq 0$ and $\operatorname{Re}(\gamma\cdot w) = \mu(\varphi)$ is finitely supported. We call this finite sum the *initial series of φ with respect to w* and we denote it by $\operatorname{in}_w(\varphi)$.

Definition 2.1. A *canonical series solution of $H_A(\beta)$ with respect to w* is a series $\varphi \in N_w$ such that

(1) φ is a *formal solution* of $H_A(\beta)$,
(2) the initial series $\mathrm{in}_w(\varphi)$ is a mixed monomial,
(3) the mixed monomial $\mathrm{in}_w(\varphi)$ is the unique element of the set

$$\{\mathrm{in}_w(\varphi) \mid \varphi \in N_w \text{ is a (formal) solution of } H_A(\beta)\} \tag{2.3}$$

appearing in the logarithmic series φ with nonzero coefficient.

Let φ be a canonical series solution of $H_A(\beta)$ with respect to w, and write $\mathrm{in}_w(\varphi) = x^\gamma(\log(x))^\delta$. The vector γ is called the *exponent* of φ. The *exponents* of $H_A(\beta)$ with respect to w are the exponents of all its canonical series with respect to w.

Theorem 2.2 ([SST00, Theorem 2.5.16])**.** *Let A be homogeneous, let $\beta \in \mathbb{C}^d$, and let w be a weight vector. Then there exists a domain $W \subseteq \mathbb{C}^n$ (depending on w) such that, for any $x \in W$, the canonical series solutions of $H_A(\beta)$ form a basis of $\mathrm{Sol}_x(H_A(\beta))$.*

In particular, canonical series solutions exist. Note also that, with notation as in Theorem 2.2, for each $x \in U$ there are precisely $\mathrm{rank}\,(H_A(\beta))$-many canonical series solutions of $H_A(\beta)$ that converge in a neighborhood of x and are linearly independent over $\mathbb{C}$.

Recall that a weight vector also gives rise to a triangulation Δ_w of $\mathrm{conv}(A)$. The following fact, linking canonical series solutions and triangulations, is a consequence of [SST00, Lemma 4.1.3] and [SST00, equation 3.7].

Lemma 2.3. *Let γ be an exponent of $H_A(\beta)$ with respect to w. Then the set $\{i \in \{1,\dots,n\} \mid \gamma_i \notin \mathbb{N}\}$ are the vertices of a simplex of the triangulation Δ_w, and the columns of A indexed by this set are linearly independent.*

The following result provides a common domain of convergence in x for all canonical series solutions with respect to a weight vector w, regardless of the parameter.

Theorem 2.4. *Let $w \in \mathbb{R}^n_{>0}$ be a weight vector. Then there exists an open, nonempty subset $V \subseteq \mathbb{C}^n$ such that for all $\beta \in \mathbb{C}^d$, the canonical series solutions of $H_A(\beta)$ with respect to w are absolutely convergent for $x \in V$.*

Proof. Consider $D[\beta]$, the Weyl algebra with d commuting indeterminates $\beta_1,\dots,\beta_d$ adjoined. We claim that there exists a comprehensive Gröbner basis for $H_A(\beta)$ with respect to $(-w, w)$. This is a finite subset G of $D[\beta]$ whose specialization for any fixed $\bar\beta \in \mathbb{C}^d$ is a Gröbner basis of $H_A(\bar\beta)$ with respect to $(-w, w)$. This claim follows from [NOT16, Algorithm 2], and we also outline an argument as follows. Note first that [KW91, W92] provide the existence of comprehensive Gröbner bases with respect to a term order for rings of solvable type, one of which is the Weyl algebra. On the other hand, the weight vector $(-w, w)$ does not induce a term order in D, even after a tie-breaker is chosen, because of the existence of monomials of negative weight.

The tools needed to remedy this situation can be found in [SST00, Section 1.2]. Consider the free associative $\mathbb{C}$-algebra generated by $h, x_1, x_2, \dots, x_n, \partial_1, \partial_2, \dots, \partial_n$ modulo the two-sided ideal

$$\langle x_k x_\ell = x_\ell x_k,\ \partial_k\partial_\ell = \partial_\ell\partial_k,\ \partial_k x_\ell = x_\ell\partial_k + \delta_{k\ell}\cdot h^2,\ h x_k = x_k h,\\ h\partial_k = \partial_k h \mid k, \ell \in \{1,\dots,n\}\rangle.$$

We denote this homogenized Weyl algebra by $D^{(h)}$. When $t \in \mathbb{R}$ is chosen so that $t \leq 0$ (so that $h^2 \preceq x_k\partial_k$), the weight vector $(t, -w, w)$ on $D^{(h)}$ determines a multiplicative monomial order for which Buchberger's algorithm terminates [SST00, Proposition 1.2.2] whenever the input is homogeneous. Now, as in [SST00, Algorithm 1.2.6], by passing through $D^{(h)}$, it is possible to compute Gröbner bases in D with respect to the weight vector $(-w, w)$. This construction is compatible with the further addition of the indeterminates $\beta_1, \beta_2, \ldots, \beta_d$, by combining the results in [KW91, W92] applied to $D^{(h)}[\beta]$ with the (de)homogenization procedures in [SST00, Algorithm 1.2.6]. As such, the desired comprehensive Gröbner basis $G \subset D[\beta]$ for $H_A(\beta)$ exists.

Consider an element $P(\beta, x, \partial) = \sum q_{u,v}(\beta) x^u \partial^v$ of G. Given $\bar{\beta} \in \mathbb{C}^d$, the support of the initial form $\mathrm{in}_{(-w,w)}(P(\bar{\beta}, x, \partial))$ with respect to x and ∂ depends on (the vanishing of) the coefficients $q_{u,v}(\bar{\beta})$. For each possible support of a specialization of P at $\bar{\beta}$, the conditions on $w' \in \mathbb{R}^n$ so that $\mathrm{in}_{(-w',w')}(P) = \mathrm{in}_{(-w,w)}(P)$ are linear equations and inequalities, and therefore the set

$$\{w' \in \mathbb{R}^n \mid \mathrm{in}_{(-w',w')}(P(\bar{\beta}, x, \partial)) = \mathrm{in}_{(-w,w)}(P(\bar{\beta}, x, \partial)) \text{ for all } \bar{\beta} \in \mathbb{C}^d\}$$

is (in the Euclidean topology) a relatively open rational polyhedral cone. Applying the same argument to all elements of G, we conclude that the cone C from (2.1) is a relatively open rational polyhedral cone. If w is sufficiently generic, this cone is full dimensional, and therefore open in $\mathbb{C}^n$. Now [SST00, Theorem 2.5.16] implies that there exists $\bar{x} \in C$ such that every canonical series solution of $H_A(\bar{\beta})$ with respect to w converges absolutely for $(-\log|x_1|, \ldots, -\log|x_n|) \in \bar{x} + C$, and this holds for all $\bar{\beta} \in \mathbb{C}^d$. ☐

3. Logarithm-free hypergeometric series are holomorphic in the parameters

In this section, we assume that A is homogeneous and summarize known results about A-hypergeometric series without logarithms.

A series $\varphi \in N_w$ is *logarithm-free* if it contains no term $x^\gamma \log(x)^\delta$ with $\delta \neq 0$. A detailed study of such series and their exponents can be found in [SST00, Section 3.4].

The exponents of logarithm-free A-hypergeometric series have *minimal negative support*. This means that if $\bar{\alpha}$ is such an exponent, then $\{i \mid \bar{\alpha}_i \in \mathbb{Z}_{<0}\} \subseteq \{i \mid (\bar{\alpha} + u)_i \in \mathbb{Z}_{<0}\}$ for all $u \in \ker_{\mathbb{Z}}(A)$. Conversely, if $\bar{\alpha}$ is an exponent of $H_A(\bar{\beta})$ with respect to w that has minimal negative support, then it is the exponent of a (unique) logarithm-free canonical series solution of $H_A(\bar{\beta})$ (note that different canonical series solutions of $H_A(\bar{\beta})$ may have the same exponent). See [SST00, Theorem 3.4.14, Corollary 3.4.15].

More precisely, if $\bar{\alpha}$ is an exponent of $H_A(\bar{\beta})$ with minimal negative support and

$$S = \big\{u \in \ker_{\mathbb{Z}}(A) \mid \{i \mid \bar{\alpha}_i \in \mathbb{Z}_{<0}\} = \{i \mid (\bar{\alpha} + u)_i \in \mathbb{Z}_{<0}\}\big\}, \tag{3.1}$$

then

$$\sum_{u \in S} \frac{\prod_{u_i<0} \prod_{j=1}^{-u_i} (\bar{\alpha}_i - j + 1)}{\prod_{u_i>0} \prod_{j=1}^{u_i} (\bar{\alpha}_i + j)} x^{\bar{\alpha}+u}$$

is the unique logarithm-free canonical series solution of $H_A(\bar{\beta})$ with exponent $\bar{\alpha}$ [SST00, Proposition 3.4.13]. We remark that $S \subset C^* \cap \ker_{\mathbb{Z}}(A)$, where C^* is the cone from (2.2) (see the proof of [SST00, Theorem 3.4.14]).

Since the expression above is so explicit, we may use it to view logarithm-free A-hypergeometric series not only as a functions of x but also as functions of (some of the coordinates of) the corresponding exponent and, consequently, as functions of the parameters.

Let $\bar{\alpha}$ be an exponent of $H_A(\bar{\beta})$ with respect to a weight vector w, and assume that $\bar{\alpha}$ has minimal negative support. Let $\sigma = \{i \mid \gamma_i \notin \mathbb{Z}\}$. (In this case, σ is the set of vertices of a simplex in the triangulation Δ_w by Lemma 2.3.)

Now let $\alpha_j \in \mathbb{C}\smallsetminus\mathbb{Z}$ for $j \in \sigma$, and denote by $\alpha \in \mathbb{C}^n$ the vector whose coordinates indexed by σ are the α_i and whose remaining coordinates are the $\bar{\alpha}_i$. Then $\alpha \in (\mathbb{C}\smallsetminus\mathbb{Z})^\sigma \times \{(\bar{\alpha}_i)_{i\notin\sigma}\}$. In what follows, we abuse notation and write $\alpha \in (\mathbb{C}\smallsetminus\mathbb{Z})^\sigma$, as the other coordinates of α are fixed. It can be shown [SST00, Corollary 3.4.15] that α is an exponent of $H_A(A\cdot\alpha)$ with minimal negative support whose corresponding logarithm-free A-hypergeometric series is

$$\varphi(x;\alpha) = \sum_{u\in S} \frac{\prod_{u_i<0}\prod_{j=1}^{-u_i}(\alpha_i - j + 1)}{\prod_{u_i>0}\prod_{j=1}^{u_i}(\alpha_i + j)}\, x^{\alpha+u}, \tag{3.2}$$

where the sum is over the same set S as in (3.1). As this is a canonical series solution of a hypergeometric system, every time we fix $\alpha \in (\mathbb{C}\smallsetminus\mathbb{Z})^\sigma$, the series $\varphi(x;\alpha)$ converges on the open set V from Theorem 2.4.

Thus we see that we may consider $\varphi(x;\alpha)$ as a function of both x and α. The parameter $A\cdot\alpha$ is a function of α; on the other hand, we may also consider α as a function of the parameter $A\cdot\alpha$, because the columns of A indexed by σ are linearly independent by Lemma 2.3.

The following result gives precise information on the convergence of $\varphi(x;\alpha)$.

Theorem 3.1. *The series $\varphi(x;\alpha)$ is holomorphic for $(x,\alpha) \in V \times (\mathbb{C}\smallsetminus\mathbb{Z})^\sigma$, where $V \subset \mathbb{C}^n$ is the open set from Theorem* 2.4.

Proof. This is a well-known result (see for instance [OT09, Lemma 1] and [GG99, Theorem 4]). We sketch the proof here for the reader's convenience.

Let Γ denote the Euler gamma function. Then

$$\frac{1}{\prod_{i=1}^n \Gamma(\alpha_i+1)}\,\varphi(x;\alpha) = \sum_{u\in S}\frac{1}{\prod_{i=1}^n\Gamma(\alpha_i+u_i+1)}\,x^{\alpha+u}. \tag{3.3}$$

Multiply and divide on the right hand side to obtain

$$\sum_{u\in S}\prod_{j\in\sigma}\frac{\Gamma(\bar{\alpha}_j+u_j+1)}{\Gamma(\alpha_j+u_j+1)}\,\frac{1}{\prod_{i=1}^n\Gamma(\bar{\alpha}_i+u_i+1)}\,x^{\alpha+u},$$

and consider the factor $\prod_{j\in\sigma}\frac{\Gamma(\bar{\alpha}_j+u_j+1)}{\Gamma(\alpha_j+u_j+1)}$. Fix a compact set $K \subset (\mathbb{C}\smallsetminus\mathbb{Z})^\sigma$ and suppose $\alpha \in K$. Since the elements $u \in S$ lie in the pointed cone C^* from (2.2), the behavior of u_j for $j \in \sigma$ is constrained: either $|u_j|$ is bounded for all $u \in S$ or u_j is bounded below and $u_j \to \infty$ or u_j is bounded above and $u_j \to -\infty$. For $\alpha \in K$ and sufficiently large t, $\Gamma(\bar{\alpha}_j+t+1)/\Gamma(\alpha_j+t+1)$ is approximately one, for instance, by Stirling's approximation. Therefore, the series on the right hand side of (3.3) can be bounded term by term in absolute value by the series $\sum_{u\in S} x^{\alpha+u}/\prod_{i=1}^n\Gamma(\bar{\alpha}_i+u_i+1)$, and this has the same domain of convergence in x as the series $\sum_{u\in S} x^{\bar{\alpha}+u}/\prod_{i=1}^n\Gamma(\bar{\alpha}_i+u_i+1)$; recall that this domain V was found in Theorem 2.4.

Moreover, this argument shows that for each fixed $x \in V$, the partial sums of (3.3) converge absolutely and uniformly for α on compact subsets of $(\mathbb{C} \smallsetminus \mathbb{Z})^\sigma$. As each partial sum is holomorphic in α, a standard application of Morera's theorem yields that the limit is holomorphic in α as well. □

We remark that if $\beta \in \mathcal{S}_0$, where $\mathcal{S}$ is the stratification associated to the triangulation Δ_w constructed in Section 1, all solutions of $D/H_A(\beta)$ with respect to w are logarithm-free [SST00, Proposition 3.4.4 and Theorem 3.4.14]. Consequently, Theorem 3.1 proves Theorem 1.2 for the codimension zero stratum of $\mathcal{S}$.

3.1. **Series of Horn type.** In this subsection we consider logarithm-free series of Horn type, which are related to A-hypergeometric series. Horn series are a key ingredient in proving the results of Section 5.

Let $B_1, \dots, B_n \in \mathbb{Z}^m$ denote the rows of a rank m, $n \times m$ matrix $B = [b_{jk}]$ such that $A \cdot B = 0$, where $m = n - d$. For $k = 1, \dots, m$, let us define polynomials in the variables $\mu = (\mu_1, \dots, \mu_m)$ and parameters $(\alpha_1, \dots, \alpha_n)$ by

(3.4)
$$P_k(\mu;\alpha) = \prod_{b_{jk}>0} \prod_{\ell=0}^{b_{jk}-1} (B_j \cdot \mu + \alpha_j - \ell) \quad \text{and} \quad Q_k(\mu;\alpha) = \prod_{b_{jk}<0} \prod_{\ell=0}^{|b_{jk}|-1} (B_j \cdot \mu + \alpha_j - \ell).$$

For $\mu \in \mathbb{N}^n$, let

(3.5)
$$R_\mu(\alpha) = \prod_{\ell=1}^{m} \prod_{j_\ell=0}^{\mu_\ell - 1} \frac{P_\ell(\mu_1, \dots, \mu_{\ell-1}, j_\ell, 0, \dots, 0)}{Q_\ell(\mu_1, \dots, \mu_{\ell-1}, j_\ell + 1, 0, \dots, 0)} \ ; \quad \Phi_B(z;\alpha) = \sum_{\mu \in \mathbb{N}^n} R_\mu(\alpha) z^\mu.$$

Note that the $R_\mu(\alpha)$ are rational functions in α. For $\mu \in \mathbb{N}^n$, the poles of $R_\mu(\alpha)$ occur when some coordinates of α take specific integer values.

The proof of Theorem 3.1 applies to show the following.

Theorem 3.2. *Let $Y = \{\alpha \in \mathbb{C}^n \mid R_\mu(\alpha)$ has no poles $\forall \mu \in \mathbb{N}^n\} \supset (\mathbb{C} \smallsetminus \mathbb{Z})^n$. There exists a polydisc $W \subset \mathbb{C}^m$ centered at the origin such that $\Phi_B(z;\alpha)$ is holomorphic for $(z, \alpha) \in W \times Y$. The polydisc W depends only on the matrix B and not on the parameters α.*

4. Logarithmic A-hypergeometric series are continuous in the parameters

A is assumed to be homogeneous throughout this section. We already know from the previous section that a canonical logarithm-free A-hypergeometric series can be regarded as a function of its parameters. Since we lack such explicit expressions for logarithmic A-hypergeometric series in general ([AS14] provides combinatorial expressions for logarithmic A-hypergeometric series under special conditions), it is nontrivial to see that the coefficients in such a series depend continuously on the parameters. The goal of this section is to prove Theorem 4.1, which implies that for canonical logarithmic A-hypergeometric series this dependence is indeed continuous. We begin by describing the general form of such a series.

Let $w \in \mathbb{R}^n_{>0}$ be a weight vector, and consider the cones C and C^* from (2.1) and (2.2). Since w is generic, C^* is a convex, pointed rational polyhedral cone with nonempty interior.

Let $\bar{\beta} \in \mathbb{C}^d$, and let $\bar{\alpha}$ be an exponent of $H_A(\bar{\beta})$ with respect to the weight vector w. In particular, $A \cdot \bar{\alpha} = \bar{\beta}$. We can write the canonical series solution of $H_A(\bar{\beta})$ corresponding to $\bar{\alpha}$ as

$$\varphi(x;\bar{\alpha}) = x^{\bar{\alpha}} \sum_{u \in C^* \cap \ker_{\mathbb{Z}}(A)} p_u(\log(x);\bar{\alpha})\, x^u, \tag{4.1}$$

where $p_u(\log(x);\bar{\alpha})$ is a polynomial in the n variables $\log(x_1), \dots, \log(x_n)$ (whose coefficients depend on $\bar{\alpha}$). It is known that the degree in $\log(x)$ of the polynomial p_u is bounded by n times the rank of $D/H_A(\bar{\beta})$ (see [SST00, Theorem 2.5.14]). We remark that $\operatorname{rank}(D/H_A(\bar{\beta})) \leq 2^{2d}\operatorname{vol}(A)$ by [SST00, Corollary 4.1.2]; this gives the parameter-independent bound $n2^{2d}\operatorname{vol}(A)$ for the degree of the polynomials p_u.

Since C is open and $w \in C$, for any $T > 0$, the set $\{u \in C^* \mid u \cdot w \leq T\}$ is bounded. The set $\{u \in C^* \cap \ker_{\mathbb{Z}}(A) \mid p_u \neq 0\}$ is called the *support* of φ.

Assume that $\bar{\alpha}_1 \notin \mathbb{Z}$. In the following result, we use α to denote a vector all of whose coordinates coincide with those of $\bar{\alpha}$ except possibly the first one.

Theorem 4.1. *Fix a weight vector w and parameter $\bar{\beta} \in \mathbb{C}^d$. Let $\varphi(x;\bar{\alpha})$ be the logarithmic canonical series solution* (4.1)*. Suppose $\bar{\alpha}_1 \notin \mathbb{Z}$. Then there exist a neighborhood W of $\bar{\alpha}_1$ and $p_u(y_1,\dots,y_n;\alpha) \in \mathbb{C}(\alpha_1)[y_1,\dots,y_n]$ for $u \in C^* \cap \ker_{\mathbb{Z}}(A)$, such that for $\alpha_1 \in W$, the series $\varphi(x;\alpha) = x^{\alpha} \sum_{u \in C^* \cap \ker_{\mathbb{Z}}(A)} p_u(\log(x);\alpha) x^u$ is a canonical series solution of $H_A(A \cdot \alpha)$ with respect to w (and therefore converges for $x \in V$, where V is the open set from Theorem* 2.4*). Moreover when α is set to $\bar{\alpha}$, $\varphi(x;\alpha)$ specializes to* (4.1)*.*

To prepare for the proof of Theorem 4.1, we prove the following technical result. While it is easy to see that the derivative of an A-hypergeometric series is A-hypergeometric (with a shifted parameter), we provide a sufficient condition for a possibly logarithmic A-hypergeometric series to have an A-hypergeometric antiderivative.

Proposition 4.2. *Let $\varphi(x;\bar{\alpha})$ be a (logarithmic) canonical series solution of $H_A(\bar{\beta})$ with respect to a weight vector w, and let a_1 denote the first column of A. If $\bar{\alpha}_1 \notin \mathbb{Z}$, then there exists a canonical solution ψ of $H_A(\bar{\beta} + a_1)$ with respect to w such that $\partial_1 \psi = \varphi$.*

Lemma 4.3 ([DMM12, Lemma 3.12])**.** *Let p be a polynomial in n variables and let $\bar{\alpha} \in \mathbb{C}^n$ with $\bar{\alpha}_1 \neq -1$. Then, there exists a unique polynomial q in n variables such that* $\deg(q) = \deg(p)$ *and*

$$\partial_1 x_1 x^{\bar{\alpha}}\, q(\log(x)) = x^{\bar{\alpha}}\, p(\log(x)).$$

With the notation of Lemma 4.3, set

$$\partial_1^{-1}\big[x^{\bar{\alpha}}\, p(\log(x))\big] := x_1\, x^{\bar{\alpha}}\, q(\log(x)).$$

The next two results follow from the uniqueness in Lemma 4.3.

Lemma 4.4. *Let p be a polynomial in n variables and let $\bar{\alpha} \in \mathbb{C}^n$ with $\bar{\alpha}_1 \neq -1$. If $x^{\bar{\alpha}}\, p(\log(x))$ is a solution of $\langle E - \bar{\beta}\rangle$, then $\partial_1^{-1} x^{\bar{\alpha}}\, p(\log(x))$ is a solution of $\langle E - (\beta + a_1)\rangle$.*

Lemma 4.5. *Let p be a polynomial in n variables and let $\bar{\alpha} \in \mathbb{C}^n$ with $\bar{\alpha}_1 \neq -1$. Then for any $\mu \in \mathbb{N}^n$,*

$$\partial^{\mu}\big[\partial_1^{-1}\, x^{\bar{\alpha}}\, p(\log(x))\big] = \partial_1^{-1}\big[\partial^{\mu}\, x^{\bar{\alpha}}\, p(\log(x))\big].$$

Proof of Proposition 4.2. Since $\bar{\alpha}_1 \notin \mathbb{Z}$, the first coordinate of $u+\bar{\alpha}$ never equals -1. We may define a formal power series

$$\psi(x,\bar{\alpha}) \;=\; \sum_{u\in C^*\cap\ker_{\mathbb{Z}}(A)} \partial_1^{-1}\big[p_u(\log(x);\bar{\alpha})\, x^{u+\bar{\alpha}}\big].$$

By Lemmas 4.4 and 4.5, the series ψ is a solution of $H_A(\bar{\beta}+a_1)$ and $\partial_1\psi=\varphi$. From this equality, we also conclude that ψ is a canonical series solution of $H_A(\bar{\beta}+a_1)$. In particular, ψ is convergent. $\square$

Proof of Theorem 4.1. Our first task is to show that for α_1 in a neighborhood of $\bar{\alpha}_1$, α is an exponent of $H_A(A\cdot\alpha)$ with respect to w. Let G be a comprehensive Gröbner basis of $H_A(\beta)$ with respect to $(-w,w)$, as in the proof of Theorem 2.4. Restrict to parameters of the form $\beta = A\cdot\alpha$ with $\alpha_j=\bar{\alpha}_j$ for $j\neq 1$, so that the elements of G are polynomials whose monomials, in the variables x and ∂, have coefficients that are polynomial in α_1. Recall that the degrees of the logarithmic polynomial coefficients in an A-hypergeometric function are bounded by $n2^{2d}\mathrm{vol}(A)$. We may assume that there exists some $u\in C^*\cap\ker_{\mathbb{Z}}(A)$ such that the polynomial $p_u(\log(x);\bar{\alpha})$ has positive degree.

We have that $\partial_1^k\varphi$ is a solution of $H_A(\bar{\beta}-ka_1)$ for all $k\in\mathbb{N}$. Moreover, by repeated application of Proposition 4.2, for any $k\in\mathbb{N}$, there is a solution ψ_k of $H_A(\bar{\beta}+ka_1)$ such that $\partial_1^k\psi_k=\varphi$. All of these hypergeometric series have the same support as φ, and even the degrees of the logarithmic polynomial coefficients are preserved.

Since univariate polynomials have finitely many roots, the polynomials in α_1 that are the coefficients of the elements of G do not vanish for $\alpha_1=\bar{\alpha}_1+k$ when $k\in\mathbb{N}$ is sufficiently large. Fix such a k. Once the result is proven for $\bar{\alpha}+ka_1$, applying ∂_1^k yields the desired result. Thus, we may assume that all the polynomial coefficients in G are nonzero at $\bar{\alpha}_1$.

The conditions for $x^\alpha\, p_0(\log(x);\alpha)$ (where the degree of p_0 as a polynomial in $\log(x)$ does not depend on α) to be a solution of $\mathrm{in}_{(-w,w)}(H_A(A\alpha))$ for α in a neighborhood of $\bar{\alpha}$ is a linear system of equations whose augmented matrix has entries that are polynomial in α_1.

The solvability of a system of linear equations can be expressed as a rank condition on its augmented matrix. Thus, we have a solution for α in some algebraic subvariety $Y\subset\mathbb{C}$, defined by certain minors of the augmented matrix. By hypothesis, the required conditions are satisfied at $\bar{\alpha_1}$; hence Y is nonempty. Actually, there exists a solution for $\alpha_1=\bar{\alpha}_1\pm\ell$ if ℓ is a sufficiently large positive integer, implying that Y is of infinite cardinality. We conclude that $Y=\mathbb{C}$. Hence, there exists $\varepsilon>0$ such that for all α with $|\alpha_1-\bar{\alpha}_1|<\varepsilon$ (and $\alpha_j=\bar{\alpha}_j$ for $j\neq 1$) the system $\mathrm{in}_{(-w,w)}(H_A(A\cdot\alpha))$ has a solution of the form $x^\alpha\, p_0(\log(x);\alpha)$, where the degree of p_0 as a polynomial in $\log(x)$ does not depend on α. In particular, α is an exponent of $H_A(A\cdot\alpha)$ with respect to w by [SST00, Lemma 2.5.10, Corollary 2.5.11].

Now, for all α with $|\alpha_1-\bar{\alpha}_1|<\varepsilon$, we must construct a (canonical) series solution φ of $H_A(A\cdot\alpha)$ with respect to $(-w,w)$ of the form (4.1). We follow the algorithm to compute canonical series solutions that appear in [SST00, Section 2.5]. Given $u\in C^*\cap\ker_{\mathbb{Z}}(A)$, this algorithm allows us to set up a system of linear equations involving the coefficients of p_u and those of p_v for any

$$v\in C^*\cap\ker_{\mathbb{Z}}(A) \quad\text{such that}\quad w\cdot v\leq w\cdot u. \tag{4.2}$$

Solving this system we obtain the polynomial p_u. This system has finitely many equations, since there are only finitely many elements that satisfy (4.2). Moreover, the coefficients of the augmented matrix of the linear system depend polynomially on α_1. If ℓ is a sufficiently large positive integer, the polynomials whose nonvanishing is a condition for the solvability of the system under consideration have no roots when $|\alpha_1 - (\bar{\alpha}_1 + \ell)| < \varepsilon$. The same argument as above implies that the polynomials that are required to vanish for the system to be solvable are identically zero. Upon obtaining a solution $x^{u+\alpha} p_u(\log(x); \alpha)$ that is valid for $|\alpha_1 - (\bar{\alpha}_1 + \ell)| < \varepsilon$, the polynomial $\partial_1^\ell x^{u+\alpha} p_u(\log(x); \alpha)$ provides a solution that is valid for $|\alpha_1 - \bar{\alpha}_1| < \varepsilon$. To see this, two ingredients are necessary. First, that ∂_1^ℓ transforms solutions of $\langle E - A\alpha \rangle$ into solutions of $\langle E - A\alpha - \ell a_1 \rangle$, and second, that the toric operators have constant coefficients and therefore commute with ∂_1^k.

Finally, since there is a unique (up to a constant multiple) canonical solution of $H_A(A \cdot \alpha)$ corresponding to the starting term $x_0^\alpha(\log(x); \alpha)$, the fact that the coefficients of the polynomials $p_u(\log(x); \alpha)$ are determined by solving linear systems whose augmented matrices depend polynomially on α_1 implies that the coefficients of $p_u(\log(x); \alpha)$ are rational functions of α_1. □

In the proof of Theorem 4.1, we see that we can extend $\varphi(x; \bar{\alpha})$ not just to α_1 in a neighborhood of $\bar{\alpha}_1$, but to $\alpha_1 \in \mathbb{C} \smallsetminus \mathbb{Z}$, as all the quantities involved are rational functions of α_1, and α_1 only needs to avoid the poles of those rational functions. Moreover, although only one parameter was perturbed in Theorem 4.1, it is possible to perturb any noninteger coordinate of the exponent $\bar{\alpha}$ and obtain that the desired coefficients are rational functions of the perturbed coordinates.

Corollary 4.6. *Let $\bar{\alpha}$ be an exponent of $H_A(\bar{\beta})$ whose corresponding solution $\varphi(x; \bar{\alpha})$, as in* (4.1)*, is logarithmic. Let $\sigma = \{i \mid \bar{\alpha}_i \notin \mathbb{Z}\}$. Let α denote a vector whose coordinates indexed by σ are $\alpha_i \in \mathbb{C} \smallsetminus \mathbb{Z}$, and whose coordinates not indexed by σ coincide with those of $\bar{\alpha}$. Then there exist $p_u(y_1, \dots, y_n; \alpha) \in \mathbb{C}(\alpha_i \mid i \in \sigma)[y_1, \dots, y_n]$ for $u \in C^* \cap \ker_{\mathbb{Z}}(A)$, such that for $\alpha \in (\mathbb{C} \smallsetminus \mathbb{Z})^\sigma$, the series $\varphi(x; \alpha) = x^\alpha \sum_{u \in C^* \cap \ker_{\mathbb{Z}}(A)} p_u(\log(x); \alpha) x^u$ is a canonical series solution of $H_A(A \cdot \alpha)$ with respect to w (and therefore converges for $x \in V$, where V is the open set from Theorem* 2.4*). Moreover when α is set to $\bar{\alpha}$, $\varphi(x; \alpha)$ specializes to* (4.1)*.* □

5. Logarithmic A-hypergeometric series are holomorphic in the parameters

In this section, we again assume A is homogeneous. We show that logarithmic A-hypergeometric series are holomorphic functions of the parameters; then we prove Theorem 1.2.

Let $\bar{\alpha}$ be an exponent of $H_A(\bar{\beta})$ whose corresponding solution $\varphi(x, \bar{\alpha})$, as in (4.1), is logarithmic.

Theorem 5.1. *Resume the hypotheses and notation from Corollary* 4.6*. There exists an open set $U \subseteq V \subset \mathbb{C}^n$ (where V is from Theorem* 2.4*) such that $\varphi(x; \alpha)$ is holomorphic for $(x, \alpha) \in U \times (\mathbb{C} \smallsetminus \mathbb{Z})^\sigma$. The open set U does not depend on the logarithmic A-hypergeometric series* (4.1)*, on the set σ, or on the parameters α.*

We prove Theorem 5.1 through an inductive procedure, based on the following result. Recall that we use the convention that $0 \in \mathbb{N}$.

Lemma 5.2. *Rewrite a solution φ of $H_A(\bar{\beta})$ as in* (4.1) *as*

$$\varphi(x,\bar{\alpha}) = \sum_{\gamma\in\mathbb{N}^n} \varphi_\gamma(x,\bar{\alpha})\log(x)^\gamma,$$

where φ_γ are logarithm-free series. We refer to γ as the logarithmic exponent *of $\varphi_\gamma \log(x)^\gamma$. If δ is a componentwise maximal element of $\{\gamma \in \mathbb{N}^n \mid \varphi_\gamma \neq 0\}$, then φ_δ is a* (*logarithm-free*) *solution of $H_A(\bar{\beta})$.*

Proof. If P is a differential operator, then

$$P \bullet \left[\varphi_\gamma \log(x)^\gamma\right] = \log(x)^\gamma \left[P \bullet \varphi_\gamma\right] + \text{ terms with lower logarithmic exponent.}$$

Thus, if P annihilates φ, then P must annihilate φ_δ for any componentwise maximal δ. □

By Lemma 5.2, a logarithmic A-hypergeometric series has the property that the coefficients of the maximal logarithmic monomials are logarithm-free A-hypergeometric series of the same parameter. Such a series is holomorphic in the parameters by Theorem 3.1. Thus, in order to prove Theorem 5.1, it is enough to express the coefficients of logarithmic monomials of smaller exponents, in terms of coefficients of logarithmic monomials with larger exponents, in such a way that holomorphy is preserved. To achieve this goal, we perform a change of variables, which is aided by the following result.

Lemma 5.3. *If K is a pointed, full dimensional rational polyhedral cone in $\mathbb{R}^m$ and L is a full rank lattice in $\mathbb{R}^m$, then there exists a $\mathbb{Z}$-basis $\{\lambda_1,\ldots,\lambda_m\}$ of L such that $K \cap L$ is contained in the set $\{\nu_1\lambda_1 + \cdots + \nu_m\lambda_m \mid \nu_1,\ldots,\nu_m \in \mathbb{N}\}$.*

Proof. Since K is full dimensional and pointed, so is its polar cone K^*. Let L^* be the dual lattice of L, that is, $L^* = \{u \in \mathbb{R}^m \mid u\cdot\lambda \in \mathbb{Z} \text{ for all } \lambda \in L\}$. The group generated by the monoid $K^* \cap L^*$ is L^*, so the Hilbert basis of $K^*\cap L^*$ must contain a set of generators of L^*, say $\kappa_1,\ldots,\kappa_m$. This implies that K is contained in the cone given by $\nabla = \{z \mid \kappa_j \cdot z \geq 0 \text{ for } j = 1,\ldots,m\}$. Let $\lambda_1,\ldots,\lambda_m$ be the dual basis of $\kappa_1,\ldots,\kappa_m$; this is the desired basis of L because by construction, ∇ is the set of nonnegative linear combinations of $\lambda_1,\ldots,\lambda_m$. □

We state one more result required for the proof of Theorem 5.1.

Lemma 5.4. *Assume that*

$$\varphi_1(z,\alpha) = \sum_{\mu\in\mathbb{N}^m} c(\mu,\alpha)\, z^\mu \quad \text{and} \quad \varphi_2(z,\alpha) = \sum_{\mu\in\mathbb{N}^m} c'(\mu,\alpha)\, z^\mu \tag{5.1}$$

are holomorphic for (z,α) in a product of domains $U_1 \times U_2 \subset \mathbb{C}^m \times \mathbb{C}^k$, where U_1 contains the origin in $\mathbb{C}^m$. In particular, for each fixed $\alpha \in U_2$, the series (5.1) *are absolutely convergent on U_1. Then, there is a domain $U_3 \subset \mathbb{C}^m$ such that the Hadamard product*

$$(\varphi_1 * \varphi_2)(z,\alpha) := \sum_{\mu\in\mathbb{N}^m} c(\mu,\alpha)\, c'(\mu,\alpha)\, z^\mu$$

is holomorphic on $U_3 \times U_2$.

Proof. After possibly rescaling, we can assume that U_1 is the polydisc $\{z \mid |z_i| < 1, i = 1,\ldots,n\}$. In this case, if $(z,\alpha) \in U_1 \times U_2$, then so is $(\sqrt{|z|},\alpha)$, where $\sqrt{|z|} = (\sqrt{|z_1|},\ldots,\sqrt{|z_n|})$.

Let $\alpha_0 \in U_2$. Then, there are absolutely convergent power series expansions

$$\varphi_1(z,\alpha) = \sum_{\mu,\eta} c_{\mu,\eta}\,(\alpha-\alpha_0)^{\eta}\,z^{\mu} \quad \text{and} \quad \varphi_2(z,\alpha) = \sum_{\mu,\eta'} c'_{\mu,\eta}\,(\alpha-\alpha_0)^{\eta'}\,z^{\mu},$$

where $c(\mu,\alpha) = \sum_{\eta} c_{\mu,\eta}(\alpha-\alpha_0)^{\eta}$, and ditto for $c'(\mu,\alpha)$. We have that, formally,

$$\begin{aligned}(\varphi_1 * \varphi_2)(z,\alpha) &= \sum_{\mu,\eta,\eta'} c_{\mu,\eta}\,c'_{\mu,\eta'}\,(\alpha-\alpha_0)^{\eta}(\alpha-\alpha_0)^{\eta'}\,z^{\mu}\\ &= \sum_{\mu,\delta} C_{\mu,\delta}\,(\alpha-\alpha_0)^{\delta}\,z^{\mu},\end{aligned}$$

where $C_{\mu,\delta} = \sum_{\eta+\eta'=\delta} c_{\mu,\eta}\,c'_{\mu,\eta'}$. Thus, the statement follows from the elementary estimates

$$\begin{aligned}\sum_{\mu,\delta} |C_{\mu,\delta}|\;|\alpha-\alpha_0|^{\delta}\,|z|^{\mu} &\le \sum_{\mu,\eta,\eta'} |c_{\mu,\eta}|\,|c'_{\mu,\eta'}|\,|\alpha-\alpha_0|^{\eta}|\alpha-\alpha_0|^{\eta'}|z|^{\mu}\\ &\le \left(\sum_{\mu,\eta} |c_{\mu,\eta}|\,|\alpha-\alpha_0|^{\eta}\,|z|^{\frac{\mu}{2}}\right)\left(\sum_{\mu,\eta'} |c'_{\mu,\eta'}|\,|\alpha-\alpha_0|^{\eta'}\,|z|^{\frac{\mu}{2}}\right).\end{aligned}$$

□

Proof of Theorem 5.1. Consider a solution $\varphi(x,\bar{\alpha})$ of $H_A(\bar{\beta})$ as in (4.1). It is given that C^* is pointed, and we may assume that C^* is full dimensional. Note that $m = \operatorname{rank}(\ker_{\mathbb{Z}}(A)) = n-d$. Therefore, Lemma 5.3 applied to the cone $C^* \cap \ker_{\mathbb{R}}(A) \subset \ker_{\mathbb{R}}(A) \cong \mathbb{R}^m$ and the lattice $\ker_{\mathbb{Z}}(A)$ yields a basis $b_1,\dots,b_m$ of $\ker_{\mathbb{Z}}(A)$ such that

$$C^* \cap \ker_{\mathbb{Z}}(A) \subseteq \{\nu_1 b_1 + \cdots + \nu_m b_m \mid \nu_1,\dots,\nu_m \in \mathbb{N}\}.$$

Moreover, since $b_1,\dots,b_m$ is a lattice basis, any element $u \in C^* \cap \ker_{\mathbb{Z}}(A)$ may be expressed as $u = B\nu$ for some $\nu \in \mathbb{N}^m$, where B is the matrix whose columns are given by $b_1,\dots,b_m$. Therefore, we may write

$$\varphi(x;\bar{\alpha}) = x^{\bar{\alpha}} \sum_{\nu\in\mathbb{N}^m} p_{B\nu}(\log(x);\bar{\alpha})\,x^{B\nu},$$

where the p_u are polynomials in n variables whose coefficients depend on $\bar{\alpha}$. The support of the series above may be strictly contained in $\mathbb{N}^m$; however, it is more convenient to use this larger summation set. Furthermore, by [Sai02, Proposition 5.2], for each fixed $\bar{\alpha}$, the polynomial p_u (or $p_{B\nu}$) belongs to the symmetric algebra of the lattice $\ker_{\mathbb{Z}}(A)$. Thus, there are m-variate polynomials $q_\nu(y_1,\dots,y_m;\bar{\alpha}) \in \mathbb{C}[y_1,\dots,y_m]$ (whose coefficients depend on $\bar{\alpha}$) such that

$$p_{B\nu}(\log(x);\bar{\alpha}) = q_\nu(\log(x^{b_1}),\dots,\log(x^{b_m});\bar{\alpha}).$$

Consequently, the series

$$F(z;\bar{\alpha}) := \sum_{\nu\in\mathbb{N}^m} q_\nu(\log(z);\bar{\alpha})\,z^{\nu}$$

is such that

$$\varphi(x;\bar{\alpha}) = x^{\bar{\alpha}}\,F(x^{b_1},\dots,x^{b_m};\bar{\alpha}).$$

We refer to this passage from the n-variate series φ to the m-variate series F as *dehomogenizing the torus action*. This is helpful in our proof of Theorem 5.1 because it reduces the number of variables. In fact, while the argument could be carried out without this step, it greatly simplifies the necessary notation. A D-module theoretic study of this procedure and its implications for hypergeometric systems can be found in [BMW15]. Note that, by construction of the convergence domain in x of φ, the series $F(z)$ converges for z in a polydisc W around the origin in $\mathbb{C}^m$. Moreover, by Theorem 2.4 the same neighborhood W can be used for any parameter.

Using the notation from Corollary 4.6, let $\alpha \in (\mathbb{C} \smallsetminus \mathbb{Z})^\sigma$. For convenience, we assume that $1 \in \sigma$, and we fix the value of all parameters (the coordinates of α) except α_1. Then for $\alpha_1 \in \mathbb{C} \smallsetminus \mathbb{Z}$, $\varphi(x;\alpha)$ is a well-defined solution of $H_A(A\cdot\alpha)$; each coefficient $p_u(\log(x);\alpha)$ is a polynomial in $\log(x)$ with coefficients that are rational functions in α_1. Therefore, each q_ν is a polynomial in $\log(z)$ whose coefficients are rational functions of α_1. The degrees of the numerators and denominators of these rational functions need not be uniformly bounded.

Since x^α is entire as a function of α_1, the series φ is holomorphic for $\alpha_1 \in \mathbb{C} \smallsetminus \mathbb{Z}$ if and only if F is holomorphic for $\alpha_1 \in \mathbb{C} \smallsetminus \mathbb{Z}$.

We can write

$$F(z;\alpha) = \sum_{\gamma\in S} \log(z)^\gamma F_\gamma(z;\alpha), \tag{5.2}$$

where $S \subset \mathbb{N}^m$ is a finite set. Note that each series $F_\gamma(z;\alpha)$ is logarithm-free. The coordinatewise maximal elements of S correspond to coordinatewise maximal logarithmic monomials as in Lemma 5.2. This implies that the series F_δ corresponding to coordinatewise maximal elements δ of S are dehomogenizations of logarithm-free A-hypergeometric series. Note also that these logarithm-free A-hypergeometric series have exponents that depend on α_1 and are defined for $\alpha_1 \in \mathbb{C} \smallsetminus \mathbb{Z}$. Using the expression (3.2) of a logarithm-free A-hypergeometric series and Theorem 3.1, we conclude that these series are holomorphic for $\alpha_1 \in \mathbb{C} \smallsetminus \mathbb{Z}$ and $z \in U'$.

The above observation is the base case in an inductive argument to show that the series $F_\gamma(z;\alpha)$ is holomorphic in α. For the inductive step, we wish to express $F_\gamma(z,\alpha)$ in terms of series $F_\delta(z,\alpha)$ for which δ is larger than γ componentwise.

Let $B_1,\dots,B_n \in \mathbb{Z}^m$ denote the rows of the $n\times m$ matrix B obtained in the process of dehomogenizing the torus action. For $k = 1,\dots,m$, recall the polynomials in the variables $\mu = (\mu_1,\dots,\mu_m)$ from (3.4):

$$P_k(\mu;\alpha) = \prod_{b_{jk}>0} \prod_{\ell=0}^{b_{jk}-1} (B_j\cdot\mu+\alpha_j-\ell) \quad \text{and} \quad Q_k(\mu;\alpha) = \prod_{b_{jk}<0} \prod_{\ell=0}^{|b_{jk}|-1} (B_j\cdot\mu+\alpha_j-\ell).$$

We consider the *Horn operators* associated to B, defined as

$$H_k := Q_k(z_1\partial_{z_1},\dots,z_m\partial_{z_m};\alpha) - z_k P_k(z_1\partial_{z_1},\dots,z_m\partial_{z_m};\alpha), \quad k=1,\dots,m. \tag{5.3}$$

Given a vector $b \in \mathbb{Z}^n$, let $b_+, b_- \in \mathbb{N}^n$ be the unique vectors such that $b = b_+ - b_-$. Let $b_1,\dots,b_m$ denote the columns of B. For $k=1,\dots,m$, the operators $\partial^{b_{k+}} - \partial^{b_{k-}}$ annihilate φ. Using this fact, we can show that there exists $\omega \in \mathbb{C}^m$, depending linearly on α, such that $z^\omega F(z;\alpha)$ is a solution of the system defined by the Horn operators $H_1,\dots,H_m$ (see [BMW15] for details). In what follows, we assume that $\omega = 0$ to simplify notation and observe that all our arguments are directly applicable to the general case.

Let k be a fixed integer between 0 and m. As $F(z;\alpha)$ is a solution to the kth Horn operator, the coefficient of the logarithmic monomial $\log(z)^\gamma$ in the series obtained by applying the kth operator (5.3) to $F(z;\alpha)$ must vanish. This coefficient is the sum of the kth Horn operator applied to the series F_γ and a series arising from applying differential operators to coefficients of F_δ, for δ that are coordinatewise larger than γ. Thus,

$$H_k \bullet F_\gamma(z;\alpha) = G_k(z;\alpha), \tag{5.4}$$

where, by the inductive hypothesis, the series $G_k(z,\alpha)$ is holomorphic for α_1 in $\mathbb{C} \smallsetminus \mathbb{Z}$ and $z \in U_{k,\gamma}$ where $U_{k,\gamma}$ is a polydisc around 0 which depends only on k and γ, and not on α.

Let us expand the series $F_\gamma(z;\alpha)$ as

$$F_\gamma(z;\alpha) = \sum_{\mu\in\mathbb{N}^m} f_\mu(\alpha) z^\mu.$$

Our goal is to show that F_γ is holomorphic in α by performing a second induction over the dimension of the index set $\mathbb{N}^m$. Let $\eta \in \mathbb{N}^r$ for some $1 \le r \le m$. We identify η with its image under the natural injection $\mathbb{N}^r \to \mathbb{N}^r \times \mathbb{N}^{m-r}$. Furthermore, let $\hat\eta$ be the image of η under the projection $\mathbb{N}^r \to \mathbb{N}^{r-1}$ given by forgetting the last coordinate. We also use the notation $(\hat\eta, \ell) = (\eta_1, \ldots, \eta_{r-1}, \ell)$. In our second induction, we consider the r-variate partial sums

$$F_\gamma^r(z;\alpha) = \sum_{\eta\in\mathbb{N}^r} f_\eta(\alpha)\, z^\eta. \tag{5.5}$$

As a basis for the induction, we have that $f_0(\alpha)$ is a rational function of α_1 without poles in $\mathbb{C} \smallsetminus \mathbb{Z}$, that is, independent of z.

Expand the series $G_k(z;\alpha)$ as

$$G_k(z;\alpha) = \sum_{\mu\in\mathbb{N}^m} g_\mu^{(k)}(\alpha) z^\mu.$$

Identifying coefficients of (5.4), we find that

$$Q_k(\mu + e_k;\alpha) f_{\mu+e_k}(\alpha) - P_k(\mu;\alpha) f_\mu(\alpha) = g_{\mu+e_k}^{(k)}(\alpha).$$

For each k, this is a first order inhomogeneous recurrence which can be solved explicitly. We present here only the solution for $\eta \in \mathbb{N}^r$ in the case that $k = r$:

$$\begin{aligned} f_\eta(\alpha) = &\left[\prod_{\ell=0}^{\eta_r-1} \frac{P_r((\hat\eta,\ell);\alpha)}{Q_r((\hat\eta,\ell+1);\alpha)}\right] \\ &\times \left[f_{\hat\eta}(\alpha) + \sum_{j=0}^{\eta_r-1} \left(\frac{g^{(r)}_{(\hat\eta,j+1)}(\alpha)}{Q_r((\hat\eta,j+1);\alpha)} \Big/ \prod_{\ell=0}^{j} \frac{P_r((\hat\eta,\ell);\alpha)}{Q_r((\hat\eta,\ell+1);\alpha)} \right) \right]. \end{aligned} \tag{5.6}$$

Thus, the series with terms (5.5) can be expanded as a sum of two series. We consider each of them separately.

The first summand is the Hadamard product of the two series

$$\sum_{\eta\in\mathbb{N}^r} \prod_{\ell=0}^{\eta_r-1} \frac{P_r((\hat\eta,\ell);\alpha)}{Q_r((\hat\eta,\ell+1);\alpha)}\, z^\eta \tag{5.7}$$

and

$$\frac{1}{1-z_r} \sum_{\hat\eta\in\mathbb{N}^{r-1}} f_{\hat\eta}(\alpha)\, z^{\hat\eta} = \sum_{\eta\in\mathbb{N}^r} f_{\hat\eta}(\alpha)\, z^\eta.$$

The latter is holomorphic for $\alpha_1 \in \mathbb{C} \smallsetminus \mathbb{Z}$ and z in a polydisc around 0 that does not depend on α by the second induction hypothesis. To see that (5.7) is holomorphic for $\alpha_1 \in \mathbb{C} \smallsetminus \mathbb{Z}$ and z in a polydisc around 0 that does not depend on α, we bound this series term by term in absolute value using the series Φ_B from (3.5) and apply Theorem 3.2 and its proof. We conclude, by Lemma 5.4, that the first summand of (5.6) is holomorphic for $\alpha_1 \in \mathbb{C} \smallsetminus \mathbb{Z}$ and z in a polydisc around 0 that does not depend on α.

The series whose terms are given by the second summand in (5.6) can be rewritten as

(5.8)

$$\sum_{\eta \in \mathbb{N}^r} \left[\prod_{\ell=0}^{\eta_r - 1} \frac{P_r((\hat{\eta}, \ell), \alpha)}{Q_r((\hat{\eta}, \ell+1), \alpha)} \right] \left[\sum_{j=0}^{\eta_r - 1} \frac{g^{(r)}_{(\hat{\eta}, j+1)}(\alpha)}{Q_r((\hat{\eta}, j+1), \alpha)} \prod_{\ell=0}^{j} \frac{Q_r((\hat{\eta}, \ell+1), \alpha)}{P_r((\hat{\eta}, \ell), \alpha)} \right] z^\eta.$$

The series

$$\sum_{\eta \in \mathbb{N}^r} \frac{Q_r((\hat{\eta}, \ell+1), \alpha)}{P_r((\hat{\eta}, \ell), \alpha)} z^\eta \qquad \text{and} \qquad \sum_{\eta \in \mathbb{N}^r} \frac{z^\eta}{Q_r(\eta + e_r, \alpha)}$$

define holomorphic functions for $\alpha_1 \in \mathbb{C} \smallsetminus \mathbb{Z}$ and z in a polydisc around 0 independent of α, again using Theorem 3.2. The (first) inductive hypothesis gives that the series $\sum_{\eta \in \mathbb{N}^r} g^{(r)}_\eta(\alpha) z^\eta$ is holomorphic for $\alpha_1 \in \mathbb{C} \smallsetminus \mathbb{Z}$ and z in a polydisc around 0 that does not depend on α. Taking Hadamard products, Lemma 5.4 implies that the series

$$\sum_{\eta \in \mathbb{N}^r} \left[\frac{g^{(r)}_{\eta + e_r}(\alpha)}{Q_r(\eta + e_r, \alpha)} \prod_{\ell=0}^{\eta_r} \frac{Q_r((\hat{\eta}, \ell+1), \alpha)}{P_r((\hat{\eta}, \ell), \alpha)} \right] z^\eta \tag{5.9}$$

is holomorphic for $\alpha_1 \in \mathbb{C} \smallsetminus \mathbb{Z}$ and z in a polydisc around 0 that does not depend on α. Taking the product of (5.9) with the function $\sum_{\eta \in \mathbb{N}^r} z^\eta$ we see that the series whose coefficients consist of the second factor of (5.8) defines a holomorphic function for $\alpha_1 \in \mathbb{C} \smallsetminus \mathbb{Z}$ and z in a polydisc around 0 that does not depend on α. Finally, (5.8) is the Hadamard product of (5.7) and (5.9), so by Lemma 5.4, (5.8) is holomorphic for $\alpha_1 \in \mathbb{C} \smallsetminus \mathbb{Z}$ and z in a polydisc around 0 that does not depend on α.

Note that in the above process, the convergence domain in z of the series involved has to be shrunk several times as we build F_γ inductively. At each step, the domains in z that are used are independent of the parameters α. As a matter of fact, what these domains are depends only on the initial data of the matrix A, the weight vector w (when we use Theorem 2.4), the matrix B (when we use Theorem 3.2), and the indices of the inductions (which control the Hadamard products that need to be performed). Moreover, both inductions are finite: the second induction has m steps, while the first has at most $(2^{2d}\mathrm{vol}(A))^n$ steps. We conclude that there exists a polydisc W' around 0 in $\mathbb{C}^m$ such that $F(z;\alpha)$ is holomorphic for $(z, \alpha_1) \in W' \times (\mathbb{C} \smallsetminus \mathbb{Z})$. $\square$

Proof of Theorem 1.2. The stratification $\mathcal{S}$ introduced in Section 1 has the following properties. If $\beta \in \mathcal{S}_i$, then β belongs to a unique irreducible component S of the closure $\overline{\mathcal{S}}_i$, and this component is a codimension i affine subspace of $\mathbb{C}^d$. If in addition, β belongs to an integer translate L of the span of a face σ of the triangulation Δ_w, then $L \supseteq S$.

Consider the exponents of $H_A(\bar\beta)$ with respect to w, where $\bar\beta \in \mathcal{S}$. If $\bar\alpha$ is such an exponent, then $\sigma = \{i \mid \bar\alpha_i \notin \mathbb{Z}\}$ is a simplex in Δ_w. By the previous argument, the closure of $\{A\cdot\alpha \mid \alpha \in \mathbb{C}^n; \alpha_i = \bar\alpha_i,\ i \notin \sigma; \alpha_j \notin \mathbb{Z},\ j \in \sigma\}$ contains the irreducible component of $\bar\beta$ in $\overline{\mathcal{S}}_i$. Thus, by Theorem 5.1 the hypergeometric series corresponding to the exponent $\bar\alpha$ can be extended to a holomorphic function for (x,α) in $U \times Y$, where Y is a neighborhood of $\bar\beta$ in $\mathcal{S}_i$. □

6. The confluent case

In this section, we use ideas from [Ber11, DMM12] to prove Theorem 1.3. We assume in this section that A is not necessarily homogeneous. Let

$$\rho(A) = \left[\begin{array}{c|c} 1 & 1 \ \cdots \ 1 \\ \hline 0 & \\ \vdots & A \\ 0 & \end{array}\right]$$

be the *homogenization* of A. Observe that $\mathbb{Z}(\rho(A)) = \mathbb{Z}^{d+1}$ and $\rho(A)$ is homogeneous.

Theorem 6.1. *If $\beta \in \mathbb{C}^d$ and $\beta_0 \in \mathbb{C}$ are generic, then the modules corresponding to the A-hypergeometric systems $H_A(\beta)$ and $H_{\rho(A)}(\beta_0,\beta)$ have the same rank. In addition, for $x \in \mathbb{C}^{n+1}$ sufficiently generic, the following map is an isomorphism:*

$$\mathrm{Sol}_x(H_{\rho(A)}(\beta_0,\beta)) \to \mathrm{Sol}_{\overline{x}}(H_A(\beta)),$$

where $\overline{x}$ is obtained from x by setting x_0 equal to 1.

This result can be used to produce an upper bound for the rank of $H_A(\beta)$ in the confluent case.

Corollary 6.2. *If $A \in \mathbb{Z}^{d\times n}$ and $\beta \in \mathbb{C}^d$, then the rank of $H_A(\beta)$ is bounded above by $2^{2d+2}\cdot \mathrm{vol}(A)$.*

Proof. [SST00, Corollary 4.1.2] states that if A is homogeneous, then

$$\mathrm{rank}\,(D/H_A(\beta)) \le 2^{2d}\cdot \mathrm{vol}(A) \quad \text{for any } \beta.$$

The result then follows directly from Theorem 6.1, since $\mathrm{vol}(A) = \mathrm{vol}(\rho(A))$. □

To prove Theorem 6.1, we first recall a result from [Ber11] that computes the rank of $H_A(\beta)$ for any β. For a face $F \preceq A$, let $\mathrm{vol}(F)$ denote the normalized volume of F in $\mathbb{Z}F$, and consider the translates

$$\mathbb{E}_F^\beta := \left[\mathbb{Z}^d \cap (\beta + \mathbb{C}F)\right] \smallsetminus (\mathbb{N}A + \mathbb{Z}F) = \bigsqcup_{b\in R_F^\beta} (b + \mathbb{Z}F),$$

where R_F^β is a set of lattice translate representatives. Each lattice translate in an $\mathbb{E}_F^\beta$ is called a *ranking lattice* of β. Let

$$\mathbb{E}^\beta = \bigcup_{F\preceq A} \mathbb{E}_F^\beta = \bigcup_{\substack{F\preceq A\\ b\in R_F^\beta}} (b + \mathbb{Z}F) \tag{6.1}$$

denote the union of ranking lattices.

Theorem 6.3. *For any $A \in \mathbb{Z}^{d\times n}$ and $\beta \in \mathbb{C}^d$, the rank of $H_A(\beta)$ can be computed from the combinatorics of the ranking lattices in $\mathbb{E}^\beta$. The formula obtained uses only the lattice of intersections of the faces with maximal lattice translates appearing in $\mathbb{E}^\beta$ and, for each such face F, $\operatorname{vol}(F)$, $\operatorname{codim}(F)$, and $|R_F^\beta|$.*

In light of Theorem 6.3, the proof of Theorem 6.1 will proceed by showing that there is a bijection between the ranking lattices in $\mathbb{E}^\beta$ and the ranking lattices in $\mathbb{E}^{(\beta_0,\beta)}$ and, in addition, that the combinatorial data in the formula in Theorem 6.3 are preserved by homogenization.

We define the *homogenization map* $\rho\colon \mathbb{C}^d \to \mathbb{C}^{d+1}$ to be

$$\rho(b) = \left[\begin{smallmatrix} 1 \\ b \end{smallmatrix}\right]$$

for $b \in \mathbb{C}^d$. Note that the homogenization of A is $\rho(A) = [\rho(a_0),\ \rho(a_1)\ \cdots\ \rho(a_n)]$, where $a_0 = 0$ in $\mathbb{C}^d$. Given a face $F \preceq A$, setting

$$\rho(F) = \{\rho(0)\} \cup \{\rho(a_i) \mid a_i \in F\} \subseteq \rho(A)$$

induces a one-to-one correspondence between the faces of A and the faces of $\rho(A)$ containing $\rho(0)$.

Proposition 6.4. *For a face $F \preceq A$, the homogenization map ρ preserves its codimension, volume, and lattice index $[\mathbb{Z}A \cap \mathbb{R}F : \mathbb{Z}F]$.*

Proof. Homogenization preserves codimension because $\dim(\rho(F)) = \dim(F) + 1$. For volume, notice that the convex hull of $\rho(F)$ and the origin in $\mathbb{Z}(\rho(F)) \otimes_\mathbb{Z} \mathbb{R}$ is a cone over the convex hull of F and the origin in $\mathbb{Z}F \otimes_\mathbb{Z} \mathbb{R}$, under the obvious embedding, so this result follows from the definition of volume. Finally, if $b, c \in \mathbb{Z}^d$, then $b - c \in \mathbb{Z}F$ if and only if $\rho(b) - \rho(c) \in \mathbb{Z}(\rho(F))$, since $\rho(0) \in \rho(F)$. $\square$

Proposition 6.5. *If $\beta \in \mathbb{C}^d$ and $\beta_0 \in \mathbb{C}$ are generic, then there is a bijection between the ranking lattices in $\mathbb{E}^\beta$ and $\mathbb{E}^{(\beta_0,\beta)}$. In particular, if*

$$\mathcal{J}(\beta) = \left\{ (F, b) \;\middle|\; F \preceq A,\ b \in \mathbb{Z}^d,\ \text{and } (b + \mathbb{Z}F) \subseteq \mathbb{E}_F^\beta \right\},$$

then

$$\mathbb{E}^{(\beta_0,\beta)} = \bigcup_{(F,b)\in\mathcal{J}(\beta)} [\rho(b) + \mathbb{Z}(\rho(F))]\,.$$

Proof. Fix $F \preceq A$ and $b \in \mathbb{Z}^d$. Since $\rho(0) \in \rho(F)$, $(b + \mathbb{Z}F) \cap (\mathbb{N}A + \mathbb{Z}F) = \varnothing$ if and only if

$$[\rho(b) + \mathbb{Z}(\rho(F))] \cap [\mathbb{N}(\rho(A)) + \mathbb{Z}(\rho(F))] = \varnothing.$$

Again because $\rho(0) \in \rho(F)$, $b \in \beta + \mathbb{C}F$ is equivalent to $\rho(b) \in \rho(\beta) + \mathbb{C}(\rho(F)) = \beta' + \mathbb{C}(\rho(F))$. Therefore, if $b + \mathbb{Z}F \subseteq \mathbb{F}^\beta$, then $\rho(b) + \mathbb{Z}\rho(F) \subseteq \mathbb{E}^{(\beta_0,\beta)}$. Further, for the reverse containment, it is enough to show that a face $G \preceq \rho(A)$ with $\rho(0) \notin G$ does not contribute additional ranking lattices to $\mathbb{E}^{(\beta_0,\beta)}$.

Since $\{\rho(0) \mid 0 \in \mathbb{Z}^d\}$ is a face of $\rho(A)$, the union of ranking lattices $\bigcup_{\beta_0\in\mathbb{R}_{\geq 0}} \mathbb{E}^{(\beta_0,\beta)}$ necessarily involves finitely many lattice translates. Thus there is a Zariski open set from which to choose $\beta_0 \in \mathbb{C}$ so that any additional lattice translates from a $G \preceq \rho(A)$ with $\rho(0) \notin G$ can be avoided. In particular, for all $\beta \notin \mathbb{R}_{\geq 0}A$, only finitely many lattices which intersect $\mathbb{R}_{\geq 0}A$ need be avoided, and then any $\beta_0 \in \mathbb{R}_{\gg 0}$ will yield a proper choice. $\square$

Proof of Theorem 6.1. The first statement follows immediately from Theorem 6.3 and Proposition 6.4. Thus it is enough to show that the map

$$\mathrm{Sol}_x(H_{\rho(A)}(\beta_0,\beta)) \to \mathrm{Sol}_x(H_A(\beta))$$

given by setting x_0 equal to 1 is injective. But this is the natural map induced by

$$\mathbb{C}[\partial_1,\dots,\partial_n]/I_A \hookrightarrow \mathbb{C}[\partial_0,\partial_1,\dots,\partial_n]/I_{\rho(A)}$$

by [DMM12, Proposition 3.17], thanks to the proof of [DMM12, Lemma 3.12]. Since the resulting D-modules have the same rank, the proof is complete. □

Proof of Theorem 1.3. In light of the proof of Theorem 1.2 at the end of Section 5, it remains only to reduce the result when A is not homogeneous to the homogeneous case. We apply Theorem 1.2 to $\rho(A)$ using a weight vector which is a generic perturbation of $(0,1,1,\dots,1) \in \mathbb{R}^{n+1}$ and produce an open set $U' \subset \mathbb{C}^{n+1}$ and a stratification $\mathcal{S}'$ such that the solutions of $H_{\rho(A)}(\beta_0,\beta)$ are holomorphic on $U' \times S'$ for any stratum S' induced by $\mathcal{S}'$. By construction, the stratification $\mathcal{S}$ we are interested in is the projection of $\mathcal{S}'$ that forgets the zeroth coordinate. By the end of the proof of Proposition 6.5, it is possible to find real numbers $\alpha_0,\alpha_1,\dots,\alpha_n$ that are algebraically independent and irrational so that

$$Z = \mathrm{Var}(\alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n) \subset \mathbb{C}^{d+1}$$

satisfies $\pi(Z) = \mathbb{C}^d$ and $Z \cap [(\mathbb{Z}^{d+1} + \mathbb{Z}G) \smallsetminus \mathbb{N}A] = \varnothing$ for all $G \preceq \rho(A)$ with $\rho(0) \notin G$. As such, for each $\beta \in \mathbb{C}^d$, Z contains a unique lift (β_0,β) of β so that Theorem 6.1 holds, and the isomorphism therein is holomorphic on $U \times S$ for any stratum S induced by $\mathcal{S}$, where U is the projection of U' that forgets the zeroth coordinate. □

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