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# QUANTITATIVE VOLUME SPACE FORM RIGIDITY UNDER LOWER RICCI CURVATURE BOUND II

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ABSTRACT. This is the second paper of two in a series under the same title; both study the quantitative volume space form rigidity conjecture: a closed $n$-manifold of Ricci curvature at least $(n-1)H$, $H = \pm 1$ or 0 is diffeomorphic to an $H$-space form if for every ball of definite size on $M$, the lifting ball on the Riemannian universal covering space of the ball achieves an almost maximal volume, provided the diameter of $M$ is bounded for $H \neq 1$.

In the first paper, we verified the conjecture for the case that the Riemannian universal covering space $\tilde{M}$ is not collapsed. In the present paper, we will verify this conjecture for the case that Ricci curvature is also bounded above, while the above non-collapsing condition on $\tilde{M}$ is not required.

## INTRODUCTION

This is the second paper of two in a series under the same title, concerning the quantitative version of the following volume space form rigidity.

Let $M$ be a compact $n$-manifold of Ricci curvature bounded below by $(n-1)H$, a constant. For $p \in M$ and $r > 0$, the volume of the $r$-ball at $p$, $\text{vol}(B_r(p)) \leq \text{vol}(\underline{B}_r^H)$, and "=" if and only if the open ball $B_r(p)$ is isometric to $\underline{B}_r^H$ (Bishop volume comparison), which denotes the $r$-ball in the $n$-dimensional simply connected $H$-space form.

The following statement is a consequence of the Bishop volume comparison.

**Theorem 0.1** (Volume space form rigidity). *Let $\rho > 0$. If a compact $n$-manifold $M$ satisfies*

$$\text{Ric}_M \geq (n-1)H, \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^H)} = 1 \quad \forall\, x \in M,$$

*then $M$ is isometric to a space form of constant curvature $H$, where $\pi^* : (\widetilde{B_\rho(x)}, x^*) \to (B_\rho(x), x)$ is the (incomplete) Riemannian universal covering space.*

All $H$-space forms satisfy the local volume condition in Theorem 0.1. On the other hand, given any $\rho, \epsilon > 0$ and $H = \pm 1$ or 0, there is an $H$-space form which contains a point $x$ such that $\text{vol}(B_\rho(x)) < \epsilon$, i.e., $B_\rho(x)$ is collapsed.

In [CRX], we proposed the following quantitative version of Theorem 0.1.

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**Conjecture 0.2** (Quantitative volume space form rigidity)**.** *Given $n, \rho, d > 0$ and $H = \pm 1$ or $0$, there exists a constant $\epsilon(n, \rho, d) > 0$ such that for any $0 < \epsilon < \epsilon(n, \rho, d)$, if a compact $n$-manifold $M$ satisfies*

$$\mathrm{Ric}_M \geq (n-1)H, \quad d \geq \mathrm{diam}(M), \quad \frac{\mathrm{vol}(B_\rho(x^*))}{\mathrm{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon \quad \forall\, x \in M,$$

*then $M$ is diffeomorphic and $\Psi(\epsilon|n, \rho, d)$-close in the Gromov-Hausdorff topology to a space form of constant curvature $H$, where $d = \pi$ or $1$ when $H = 1$ or $0$ respectively, where $\Psi(\epsilon|n, \rho, d)$ denotes a non-negative function such that $\Psi(\epsilon|n, \rho, d) \to 0$ as $\epsilon \to 0$ while $n, \rho$ and $d$ are fixed.*

Note that Conjecture 0.2 for $H \neq 1$ does not hold if one removes a bound on diameter (see [CRX]). Furthermore, for $H \neq -1$, $M$ in Conjecture 0.2 may have arbitrarily small volume, i.e., $M$ is collapsed.

Conjecture 0.2 implies the Riemannian universal cover is not collapsed; for $H = -1$, because any compact hyperbolic $n$-manifold has a definite positive volume (cf. [He]), by the volume convergence ([Col97]), Conjecture 0.2 implies that $M$ is not collapsed. Observe that a collapsed ball on an $H$-space form ($H \neq -1$) has a torus bundle structure (cf. [CFG92]), again by the volume convergence one sees that Conjecture 0.2 implies the following:

**Conjecture 0.3** (Non-collapsing on Riemannian universal cover)**.** *Given $n, \rho, d > 0$, $H = \pm 1$ or $0$, there exist constants, $\epsilon(n, \rho, d), v(n, \rho, d) > 0$, such that if a compact $n$-manifold $M$ satisfies*

$$\mathrm{Ric}_M \geq (n-1)H, \quad d \geq \mathrm{diam}(M), \quad \frac{\mathrm{vol}(B_\rho(x^*))}{\mathrm{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon(n, \rho, d), \quad \forall\, x \in M,$$

*then for some $\tilde{q}$ in the Riemannian universal cover $\tilde{M}$, $\mathrm{vol}(B_1(\tilde{q})) \geq v(n, \rho, d) > 0$.*

In [CRX], among other things we proved that Conjecture 0.3 implies Conjecture 0.2. Precisely, the following theorem is a combination of Theorem A, B and C in [CRX] (corresponding to $H = 1, -1$ and $0$).

**Theorem 0.4.** *Given $n, \rho, d, v > 0$ and $H = \pm 1$ or $0$, there exists a constant $\epsilon(n, \rho, d, v) > 0$ such that for any $0 < \epsilon < \epsilon(n, \rho, d, v)$, if a compact $n$-manifold $M$ satisfies (for some $\tilde{q} \in \tilde{M}$),*

$$\mathrm{Ric}_M \geq (n-1)H,\ d \geq \mathrm{diam}(M),\ \mathrm{vol}(B_1(\tilde{q})) \geq v,\ \frac{\mathrm{vol}(B_\rho(x^*))}{\mathrm{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon \quad \forall\, x \in M,$$

*then $M$ is diffeomorphic and $\Psi(\epsilon|n, \rho, d, v)$-close to a space form of constant curvature $H$.*

For $H = 1$, Theorem 0.4 generalizes the differential sphere theorem in [CC97] (cf. [Pe], [Col96]; see Remark 0.7 in [CRX]), and for $H = -1$, Theorem 0.4 is equivalent to a quantitative version of the maximal volume entropy rigidity in [LW] (see Theorem D, Corollary 0.6 in [CRX]).

In the present paper, we will verify Conjecture 0.2 under an additional assumption: Ricci curvature is also bounded above (Theorem D). This regularity condition allows us to find a nearby metric of almost constant sectional curvature (Theorem B) by the smoothing method ([DWY]) via renormalized Ricci flows in the sense of [TW]. As an application we verify Conjecture 0.3 in this case (Theorem C).

We now begin to state the main results in this paper.

The first result says that under bounded Ricci curvature, the almost maximality of volume on local coverings measures how far the metric is from being an $H$-Einstein metric (compare to Remark 0.5).

**Theorem A.** *Given* $n, \rho, \Lambda > 0$, $H = \pm 1$ *and* $0$, *there exists a constant,* $\epsilon(n, \rho, \Lambda) > 0$, *such that for* $0 < \epsilon < \epsilon(n, \rho, \Lambda)$, *if a compact Riemannian n-manifold* $(M, g)$ *satisfies*

$$\Lambda \geq \operatorname{Ric}(g) \geq (n-1)H, \quad \frac{\operatorname{vol}(B_\rho(x^*))}{\operatorname{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon \quad \forall\, x \in M,$$

*then* $g$ *is almost Einstein in the* $L^p$*-sense for any* $p \geq 1$, *i.e.,*

$$\fint_M |\operatorname{Ric}(g) - (n-1)Hg|^p < \Psi(\epsilon | n, \rho, \Lambda, p).$$

The additional upper bound on Ricci curvature implies a uniform $C^{1,\alpha}$-harmonic radius on $B_{\frac{\rho}{2}}(x^*)$ (Lemma 1.3), and a local version of Theorem A on $B_{\frac{\rho}{2}}(x^*)$ (Lemma 1.4). By a packing argument via relative volume comparison, we obtain Theorem A.

Consider the Ricci flow on $(M, g)$; following [DWY] we see that bounded Ricci curvature and a uniform $C^{1,\alpha}$-harmonic radius on $B_{\frac{\rho}{2}}(x^*)$ (independent of $x$) imply that the Ricci flow on $M$ exists for a definite time (Theorem 1.5), and that the renormalized Ricci flow ([TW]) preserves the almost Einstein property in the $L^p$-sense (Lemma 1.7). Using the two properties, we will prove the following strong smoothing result.

**Theorem B** (Smoothing to almost constant curvature)**.** *Given* $n, \rho, \Lambda, \delta > 0$ *and* $H = \pm 1$ *or* $0$, *there exists a constant,* $\epsilon(n, \rho, \Lambda, \delta) > 0$, *such that for any* $0 < \epsilon < \epsilon(n, \rho, \Lambda, \delta)$, *if a compact n-manifold* $(M, g)$ *satisfies*

$$\Lambda \geq \operatorname{Ric}(g) \geq (n-1)H, \quad \frac{\operatorname{vol}(B_\rho(x^*))}{\operatorname{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon \quad \forall\, x \in M,$$

*then* $M$ *admits a metric* $g'$ *such that* $|g' - g| < \delta$ *and for any* $0 \leq k < \infty$,

$$|\operatorname{Rm}(g')|_{C^k, M} \leq C(n, \rho, \Lambda, \delta, k), \quad |\sec_{g'} - H| < \Psi(\delta, \epsilon | n, \rho, \Lambda).$$

Using the existence of a nearby metric of almost constant sectional curvature, we are able to verify Conjecture 0.3 for the case of bounded Ricci curvature.

**Theorem C.** *Given* $n, \rho, d, \Lambda > 0$ *and* $H = \pm 1$ *or* $0$, *there exist positive constants,* $\epsilon(n, \rho, d, \Lambda), v(n, \rho, d, \Lambda) > 0$, *such that if a compact n-manifold* $M$ *satisfies*

$$\Lambda \geq \operatorname{Ric}_M \geq (n-1)H,\, d \geq \operatorname{diam}(M),\, \frac{\operatorname{vol}(B_\rho(x^*))}{\operatorname{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon(n, \rho, d, \Lambda) \quad \forall\, x \in M,$$

*then* $\tilde{M}$ *is not collapsed, i.e.,* $\operatorname{vol}(B_1(\tilde{p})) \geq v(n, \rho, d, \Lambda)$ *for any* $\tilde{p} \in \tilde{M}$, *where* $d = \pi$ *or* $\infty$ *when* $H = 1$ *or* $-1$.

By Theorem C, we can apply Theorem 0.4 to conclude Conjecture 0.2 in this case.

**Theorem D.** *Given $n, \rho, d, \Lambda > 0$ and $H = \pm 1$ or $0$, there exists a constant $\epsilon(n, \rho, d, \Lambda) > 0$ such that for any $0 < \epsilon < \epsilon(n, \rho, d, \Lambda)$, if a compact $n$-manifold $M$ satisfies*

$$\Lambda \geq \mathrm{Ric}_M \geq (n-1)H, \quad d \geq \mathrm{diam}(M), \quad \frac{\mathrm{vol}(B_\rho(x^*))}{\mathrm{vol}(\underline{B}_\rho^H)} \geq 1-\epsilon \quad \forall x \in M,$$

*then $M$ is diffeomorphic and $\Psi(\epsilon|n, \rho, d, \Lambda)$-close to a space form of constant curvature $H$, where $d = \pi$ or $1$ when $H = 1$ or $0$ respectively.*

As mentioned in the above, there is a uniform lower bound on $C^{1,\alpha}$-harmonic radius on $B_{\frac{\rho}{2}}(x^*)$ (see Lemma 1.3). Together with the above theorem and Theorem 2.1 in [CRX], we obtain the following $C^{1,\alpha}$-compactness result.

**Theorem E.** *Given $n, \rho, d, \Lambda, v > 0$, there exist $\epsilon = \epsilon(n, \rho, d, \Lambda, v)$ such that the collection of compact $n$-manifolds satisfying*

$$\Lambda \geq \mathrm{Ric}_M \geq (n-1)H, \, d \geq \mathrm{diam}(M), \, \mathrm{vol}(M) \geq v, \, \frac{\mathrm{vol}(B_\rho(x^*))}{\mathrm{vol}(\underline{B}_\rho^H)} \geq 1-\epsilon \quad \forall x \in M$$

*is compact in the $C^{1,\alpha}$-topology, where the condition, "$\mathrm{vol}(M) \geq v$" can be removed when $H = -1$.*

A few remarks are in order:

*Remark* 0.5. In the proof of Theorem A, we actually proved that $g$ is almost Einstein on any $B_{\frac{\rho}{8}}(x)$, $x \in M$ (see (2.1.2)); compare to Problem 2.4. Roughly, one may interpret this as under bounded Ricci curvature, a ball with almost maximal 'rewinding volume' is an almost 'Einstein ball'.

*Remark* 0.6. The existence of a nearby metric of almost constant curvature in Theorem B is crucial to our proof of Theorem C (and Theorem D). Indeed, we do not know, even assuming a higher regularity on the original metric, how to prove Theorem C without using a nearby metric of almost constant sectional curvature.

*Remark* 0.7. In Theorem C, there is no restriction on diameter for $H = -1$. For $H = 0$, the condition on bounded diameter cannot be removed. Here is a counterexample: for each $i$, let $S_i^3$ denote a round 3-sphere of radius $i$, and let $g_i$ be a collapsed Berger's metric such that $\mathrm{vol}(B_1(p, g_i)) < i^{-1}$ and $i^{-5} < \sec(g_i) < 4i^{-2}$ (p. 81, [Pet]). It is easy to see that $\frac{\mathrm{vol}(B_1(p^*, g_i^*))}{\mathrm{vol}(\underline{B}_1^0)} \to 1$, as $i \to \infty$.

*Remark* 0.8. Note that Theorem E and the $C^{1,\alpha}$-compactness theorem in [And90] may have only a 'small' overlap. This is because the local volume condition in Theorem E and the injectivity radius condition in [And90] are somewhat 'parallel': a lower bound on injectivity radius may not imply the volume condition in Theorem E, and vice versa the volume conditions may not imply a lower bound on injectivity radius. (Note that for $H = -1$, in Theorem E a priori $M$ could be collapsed.)

The rest of the paper is organized as follows:

In Section 1, we will supply notions and basic properties that will be used throughout the rest of the paper.

In Section 2, we will prove Theorems A-E. At the end, we will ask a few questions relating to the approach in this paper.

## 1. Preliminaries

The purpose of this section is to supply notions and basic properties that will be used throughout the rest of the paper; we refer readers to [And90], [CC96] and [DWY] for details.

**a. Almost maximal volume ball is an almost space form ball.** Let $N$ be a Riemannian $(n-1)$-manifold, let $k:(a,b)\to\mathbb{R}$ be a smooth positive function and let $(a,b)\times_k N$ be the $k$-warped product whose Riemannian tensor is

$$g = dr^2 + k^2(r) g_N.$$

The Riemannian distance $|(r_1,x_1)(r_2,x_2)|$ $(x_1\neq x_2)$ equals the infimum of the length

$$\int_0^l \sqrt{(c_1'(t))^2 + k^2(c_1(t))}dt$$

for any smooth curve $c(t)=(c_1(t),c_2(t))$ such that $c(0)=(r_1,x_1)$, $c(l)=(r_2,x_2)$ and $|c_2'|\equiv 1$, and $|(r_1,x)(r_2,x)|=|r_2-r_1|$. Thus given $a,b,k$, there is a function (e.g., the law of cosine on space forms)

$$\rho_{a,b,k}(r_1,r_2,|x_1x_2|) = |(r_1,x_2)(r_2,x_2)|.$$

Using the same formula for $|(r_1,x_2)(r_2,x_2)|$, one can extend the $k$-warped product $(a,b)\times_k Y$ to any metric space $Y$ (not necessarily a length space); see [CC96].

The following theorem in [CC96] asserts that an almost volume annulus (see (1.1.1) below) is an almost metric annulus (see (1.1.2)).

**Theorem 1.1** ([CC96]). *Let $M$ be a Riemannian manifold, let $r$ be a distance function to a compact subset in $M$, let $A_{a,b}=r^{-1}((a,b))$, let*

$$\mathcal{V}(u) = \inf\left\{ \frac{\mathrm{vol}(B_u(q))}{\mathrm{vol}(A_{a,b})} \middle| \text{ for all } q\in A_{a,b} \text{ with } B_u(q)\subset A_{a,b} \right\},$$

*and let $0<\alpha'<\alpha, \alpha-\alpha'>\xi>0$. If*

$$\mathrm{Ric}_M \geq -(n-1)\frac{k''(a)}{k(a)} \qquad (\text{on } r^{-1}(a)),$$

$$\Delta r \leq (n-1)\frac{k'(a)}{k(a)} \qquad (\text{on } r^{-1}(a)),$$

$$\frac{\mathrm{vol}(A_{a,b})}{\mathrm{vol}(r^{-1}(a))} \geq (1-\epsilon)\frac{\int_a^b k^{n-1}(r)dr}{k^{n-1}(a)}. \tag{1.1.1}$$

*Then there exists a length metric space $Y$, with at most $\#(a,b,k,\mathcal{V})$ components $Y_i$, satisfying*

$$\mathrm{diam}(Y_i)\leq D(a,b,k,\mathcal{V}),$$

*such that the Gromov-Hausdorff distance,*

$$d_{GH}(A_{a+\alpha,b-\alpha},(a+\alpha,b-\alpha)\times_k Y)\leq \Psi(\epsilon|n,k,a,b,\alpha',\xi,\mathcal{V}) \tag{1.1.2}$$

*with respect to the two metrics $d^{\alpha',\alpha}$ and $\underline{d}^{\alpha',\alpha}$, where $d^{\alpha',\alpha}$ (resp. $\underline{d}^{\alpha',\alpha}$) denotes the restriction of the intrinsic metric of $A_{a+\alpha',b-\alpha'}$ on $A_{a+\alpha,b-\alpha}$ (resp. $(a+\alpha',b-\alpha')\times_k Y$) on $(a+\alpha,b-\alpha)\times_k Y$).*

Let

$$\mathrm{sn}_H(r) = \begin{cases} \frac{\sin \sqrt{H} r}{\sqrt{H}}, & H > 0, \\ r, & H = 0, \\ \frac{\sinh \sqrt{-H} r}{\sqrt{-H}}, & H < 0. \end{cases}$$

Applying Theorem 1.1 to $k = \mathrm{sn}_H(r)$ with $r(x) = d(p, x) : M \to \mathbb{R}$, we conclude the following result that is used in the proof of Theorems A-E.

**Theorem 1.2.** *For $n, \rho, \epsilon > 0$, if a complete $n$-manifold $M$ containing a point $p$ satisfies*

$$\mathrm{Ric}_M \geq (n-1)H, \quad \frac{\mathrm{vol}(B_\rho(p))}{\mathrm{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon,$$

*then $d_{GH}(B_{\frac{\rho}{2}}(p), \underline{B}^H_{\frac{\rho}{2}}) < \Psi(\epsilon | n, \rho, H)$.*

Note that $\frac{\mathrm{vol}(B_\rho(p))}{\mathrm{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon$ implies (1.1.1), as $a \to 0$. Since the almost maximal volume condition holds at all points near $p$ (which contains regular points), by a simple blow up argument one concludes that $Y$ is isometric to $S_1^{n-1}$.

**b. Almost maximal volume and $C^{1,\alpha}$-harmonic radius estimate.** In this and the next subsections, we will always assume bounded Ricci curvature: $\Lambda \geq \mathrm{Ric} \geq (n-1)H$, $H = \pm 1$ or 0.

Let $M$ be a complete $n$-manifold. For $p \in M$, $k \geq 0$, $0 < \alpha < 1$ and $Q \geq 1$, the $C^{k,\alpha}$-harmonic radius at $p$ with respect to $Q$ is the largest radius $r_h(p)$ of the ball at $p$ such that there are harmonic coordinates on $B_{r_h}(p)$ and $r^{k+\alpha} |g_{ij}|_{C^{k,\alpha}, B_{r_h}(p)} \leq Q$. The harmonic radius of a subset is the infimum of the harmonic radii of points in the subset.

**Lemma 1.3.** *For $n, \rho, \Lambda > 0$, $Q > 1$ and $0 < \alpha < 1$, there are constants, $\epsilon(n, \rho, \Lambda), r_h(n, \rho, \Lambda, Q, \alpha) > 0$, such that if a compact Riemannian $n$-manifold $(M, g)$ satisfies*

$$\Lambda \geq \mathrm{Ric}(g) \geq (n-1)H, \quad \frac{\mathrm{vol}(B_\rho(x^*))}{\mathrm{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon(n, \rho, \Lambda) \quad \forall x \in M,$$

*the $C^{1,\alpha}$-harmonic radius on $B_{\frac{\rho}{2}}(x^*)$ with respect to $Q$ is at least $r_h(n, \rho, \Lambda, Q, \alpha)$.*

*Proof.* We argue by contradiction, and the proof follows the same argument as in the proof of Main Lemma 2.2 in [And90]; where the almost maximal volume condition is replaced with a lower bound on injectivity radius which guarantees that any blow up limit is $\mathbb{R}^n$. We claim that a contradicting sequence, $\frac{\mathrm{vol}(B_\rho(x_i^*))}{\mathrm{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon_i \to 1$, also satisfies that any blow up limit is $\mathbb{R}^n$. Hence, the same proof in [And90] goes through here to derive a contradiction.

To see the claim, for any $y_i^* \in B_{\frac{\rho}{2}}(x_i^*)$, $R > 0$ and $r_i \to \infty$, by Bishop-Gromov relative volume comparison and the volume convergence in [Col97] we derive

$$\frac{\mathrm{vol}(B_R(y_i^*, r_i^2 g_i^*))}{\mathrm{vol}(\underline{B}_R^{r_i^{-2}H})} = \frac{\mathrm{vol}(B_{r_i^{-1}R}(y_i^*))}{\mathrm{vol}(\underline{B}^H_{r_i^{-1}R})} \geq \frac{\mathrm{vol}(B_{\frac{\rho}{8}}(y_i^*))}{\mathrm{vol}(\underline{B}^H_{\frac{\rho}{8}})} \geq 1 - \Psi(\epsilon_i | n, \rho, \Lambda).$$

Since $r_i^{-2} H \to 0$, by Theorem 1.2 we conclude that

$$d_{GH}(B_R(y_i^*, r_i^2 g_i^*), \underline{B}_R^0) \to 0.$$

Since $R$ is arbitrarily chosen, the desired claim follows. □

As an application of Lemma 1.3, we will prove a non-collapsed local version of Theorem A.

**Lemma 1.4.** *Given $n, \rho, \Lambda > 0$ and $H = \pm 1$ and $0$, there is $\epsilon(n, \rho, \Lambda) > 0$ such that for $0 < \epsilon < \epsilon(n, \rho, \Lambda)$, if a complete $n$-manifold $(M, g, x)$ satisfies*

$$\Lambda \geq \operatorname{Ric}(g) \geq (n-1)H, \quad \frac{\operatorname{vol}(B_\rho(x))}{\operatorname{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon,$$

*then for all $p \geq 1$,*

$$\fint_{B_{\frac{\rho}{2}}(x)} |\operatorname{Ric}(g) - (n-1)Hg|^p \leq \Psi(\epsilon | n, \rho, \Lambda, p).$$

*Proof.* Arguing by contradiction, assume a contradicting sequence, $(M_i, g_i, x_i)$, satisfying

$$\Lambda \geq \operatorname{Ric}_{M_i} \geq (n-1)H, \quad \frac{\operatorname{vol}(B_\rho(x_i))}{\operatorname{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon_i \to 1,$$

but $\fint_{B_{\frac{\rho}{2}}(x_i)} |\operatorname{Ric}(g_i) - (n-1)Hg_i|^{p_0} \geq \delta_0 > 0$, for some $p_0 \geq 1$.

By Theorem 1.2, we may assume that $B_{\frac{\rho}{2}}(x_i) \xrightarrow{GH} \underline{B}^H_{\frac{\rho}{2}}$. By [CC97], we may assume that for $i$ large, $B_{\frac{\rho}{2}}(x_i)$ are diffeomorphic to $\underline{B}^H_{\frac{\rho}{2}}$. From the expression of Ricci curvature in a harmonic coordinate, a bound on Ricci curvature implies that $g_i \to \underline{g}_H$ in $L^{2,p}$-norm for all $p \geq 1$. Consequently, $h_i = \operatorname{Ric}(g_i) - (n-1)Hg_i \to h = \operatorname{Ric}(g) - (n-1)Hg \equiv 0$ on $\underline{B}^H_{\frac{\rho}{2}}$ in $L^p$-norm, a contradiction. □

**c. Almost maximal volume and Ricci flows.** The main reference for this subsection is [DWY].

Let $(M, g)$ be a compact Riemannian manifold. The Ricci flow was introduced by Hamilton as the solution of the following degenerate parabolic PDE:

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}(g(t)), \qquad g(0) = g.$$

The solution always exists for a short time $t > 0$, and if the maximal flow time $T_{\max} < \infty$, then $\max |\operatorname{Rm}(g(t))| \to +\infty$ as $t \to T_{\max}$ ([Ha]).

A basic property of Ricci flow is that it improves the regularity of the initial metric ([Sh1, Sh2]). However, the regularity depends on the flow time. For our purpose, a uniform definite flow time is important. We have

**Theorem 1.5.** *For $n, \rho, \Lambda > 0$ and $H = \pm 1$ or $0$, there are positive constants, $\epsilon(n, \rho, \Lambda), T(n, \rho, \Lambda)$, such that if a compact Riemannian $n$-manifold $(M, g)$ satisfies*

$$\Lambda \geq \operatorname{Ric}(g) \geq (n-1)H, \quad \frac{\operatorname{vol}(B_\rho(x^*))}{\operatorname{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon(n, \rho, \Lambda) \quad \forall\, x \in M,$$

*then the Ricci flow,*

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}(g(t)), \quad g(0) = g,$$

*exists for $t \in [0, T(n, \rho, \Lambda)]$ and*

$$|g(t) - g| < 4t, \quad |\operatorname{Rm}(g(t))|_{C^k} \leq C, \quad \Lambda + ct^{\frac{1}{2}} \geq \operatorname{Ric}(g(t)) \geq (n-1)H - ct^{\frac{1}{2}},$$

*where $C = C(n, \rho, \Lambda, k, t)$ and $c = c(n, \rho, \Lambda)$.*

Note that Theorem 1.5 is similar to Theorem 1.1 in [DWY], where the volume condition on local covering is replaced by a positive lower bound on conjugate radius. Note that the condition on conjugate radius is solely used to show an $L^{2,p}$-harmonic radius lower bound on a local covering space for all $p \geq 1$ (see Remark 1 in [DWY]), which is required to apply Moser's weak maximum principle (Theorem 2.1 in [DWY]). Because a lower bound on the $L^{2,p}$-harmonic radius follows from Lemma 1.3 and bounded Ricci curvature condition, the same proof in [DWY] will give a proof of Theorem 1.5 with the obvious modification (cf. [Sh1, Sh2]).

Let $(M, g)$ be as in Theorem 1.5. Inspired by [DWY] we will show that if $g$ is almost $H$-Einstein in the $L^p$-sense, then the renormalized Ricci flow solution $g(t)$ in (1.6.1) below is again almost $H$-Einstein in the $L^p$-sense (Lemma 1.7).

Consider the renormalized Ricci flow in the sense of [TW]:

$$\frac{\partial}{\partial t} g = -\operatorname{Ric}(g) + (n-1)Hg, \tag{1.6.1}$$

and let

$$\bar{g}(s) = \begin{cases} (1 - 2(n-1)Hs) \cdot g\left(\frac{\ln(1-2(n-1)Hs)}{-(n-1)H}\right), & H = \pm 1, \\ g(2s), & H = 0. \end{cases}$$

Then $\bar{g}(s)$ satisfies $\bar{g}(0) = g(0)$ and

$$\frac{\partial}{\partial s} \bar{g} = -2\operatorname{Ric}(\bar{g}(s)). \tag{1.6.2}$$

Let $g^*(t)$ (resp. $\bar{g}^*(s)$) be the lifting of $g(t)$ (resp. $\bar{g}(s)$) on $B_\rho(x^*)$. Then

$$\bar{R}^*_{ijkl}(s) = \begin{cases} (1 - 2(n-1)Hs) \cdot R^*_{ijkl}\left(\frac{\ln(1-2(n-1)Hs)}{-(n-1)H}\right), & H = \pm 1, \\ R^*_{ijkl}(2s), & H = 0. \end{cases}$$

Let

$$h^*_{ij} = R^*_{ij} - (n-1)Hg^*_{ij}. \tag{1.6.3}$$

Then

$$\frac{\partial}{\partial t} h^*_{ij} = \frac{1}{2}\Delta h^*_{ij} + R^*_{pijq} h^*_{pq} - h^*_{ip} h^*_{pj}.$$

To get rid of the $\frac{1}{2}$-factor, we make a change of variable $t = 2t'$ (for simple notation, switch to $t' = t$). Then the above implies

$$\frac{\partial}{\partial t} |h^*| \leq \Delta |h^*| + 2|\mathrm{Rm}^*||h^*|. \tag{1.6.4}$$

By applying Moser's weak maximum principle, we conclude the following:

**Lemma 1.7.** *Let the assumptions be as in Theorem* 1.5, *let $h^*_{ij}(t)$ be defined in the above, and*

$$\bar{T}(n, \rho, \Lambda) = \begin{cases} \frac{\ln(1-2(n-1)H \cdot T(n,\rho,\Lambda))}{-(n-1)H}, & H = \pm 1, \\ T(n, \rho, \Lambda), & H = 0. \end{cases}$$

*Then for $t \in (0, \bar{T}(n, \rho, \Lambda)]$,*

$$\max_{x \in M} |h^*(t)|_{p, B_{\frac{\rho}{4}}(x^*, g)} \leq \max_{x \in M} |h^*(0)|_{p, B_{\frac{\rho}{2}}(x^*, g)} \cdot \frac{1}{1 - c(n, \rho, \Lambda)t}.$$

*Proof.* Given Lemma 1.3, by (1.6.4) the rest of the proof is an imitation of the proof of Lemma 3.3 in [DWY]. □

## 2. Proof of Theorems A-E

*Proof of Theorem* A. Because Ricci curvature is bounded in absolute value, it suffices to prove Theorem A for $p = 1$. For any $x \in M$, by Lemma 1.4 we have

$$\fint_{B_{\frac{\rho}{2}}(x^*)} |\operatorname{Ric}(g^*) - (n-1)Hg^*| \leq \Psi_1(\epsilon|n, \rho, \Lambda). \tag{2.1.1}$$

We claim that for

$$\Psi_2(\epsilon|n, \rho, \Lambda) = \Psi_1(\epsilon|n, \rho, \Lambda) \cdot \frac{\operatorname{vol}(\underline{B}^H_{\frac{\rho}{2}})}{\operatorname{vol}(\underline{B}^H_{\frac{\rho}{8}})},$$

the following holds:

$$\fint_{B_{\frac{\rho}{8}}(x)} |\operatorname{Ric}(g) - (n-1)Hg| \leq \Psi_2(\epsilon|n, \rho, \Lambda). \tag{2.1.2}$$

Let $A = \{x_i\}$ denote an $\frac{\rho}{8}$-net on $M$. Then $B_{\frac{\rho}{16}}(x_i) \cap B_{\frac{\rho}{16}}(x_j) = \emptyset$ $(i \neq j)$ and $M \subseteq \bigcup_{x_i \in A} B_{\frac{\rho}{8}}(x_i)$. Assuming (2.1.2), we derive

$$\fint_M |\operatorname{Ric}(g) - (n-1)Hg| \leq \frac{1}{\operatorname{vol}(M)} \sum_{x_i \in A} \int_{B_{\frac{\rho}{8}}(x_i)} |\operatorname{Ric}(g) - (n-1)Hg| \tag{1}$$

$$= \frac{1}{\operatorname{vol}(M)} \sum_{x_i} \operatorname{vol}(B_{\frac{\rho}{8}}(x_i)) \fint_{B_{\frac{\rho}{8}}(x_i)} |\operatorname{Ric}(g) - (n-1)Hg| \tag{2}$$

$$\leq \frac{1}{\operatorname{vol}(M)} \sum_{x_i \in A} \operatorname{vol}(B_{\frac{\rho}{8}}(x_i)) \cdot \Psi_2(\epsilon|n, \rho, \Lambda) \tag{3}$$

$$\leq \frac{1}{\operatorname{vol}(M)} \sum_{x_i \in A} \operatorname{vol}(B_{\frac{\rho}{16}}(x_i)) \cdot \frac{\operatorname{vol}(\underline{B}^H_{\frac{\rho}{8}})}{\operatorname{vol}(\underline{B}^H_{\frac{\rho}{16}})} \cdot \Psi_2(\epsilon|n, \rho, \Lambda) \tag{4}$$

$$\leq \Psi(\epsilon|n, \rho, \Lambda). \tag{5}$$

We now verify (2.1.2). Let $D$ denote the Dirichlet fundamental domain at $x^* \in \widetilde{B_\rho(x)}$, and let $\Gamma(\frac{\rho}{4}) = \{\gamma \in \pi_1(B_\rho(x)),\ |x^*\gamma(x^*)| \leq \frac{\rho}{4}\}$. Then

$$B_{\frac{\rho}{8}}(x^*) \subset \bigcup_{\gamma \in \Gamma(\frac{\rho}{4})} \gamma(B_{\frac{\rho}{8}}(x^*) \cap D) \subset B_{\frac{\rho}{2}}(x^*).$$

We claim that there is a $\gamma \in \Gamma(\frac{\rho}{4})$ such that

$$\fint_{\gamma(B_{\frac{\rho}{8}}(x^*) \cap D)} |\operatorname{Ric}(g^*) - (n-1)Hg^*| \leq \Psi_2(\epsilon|n, \rho, \Lambda),$$

i.e.,

$$\fint_{B_{\frac{\rho}{8}}(x)} |\operatorname{Ric}(g) - (n-1)Hg| \leq \Psi_2(\epsilon|n, \rho, \Lambda).$$

If the claim fails, i.e., for all $\gamma \in \Gamma(\frac{\rho}{4})$,

$$\fint_{\gamma(B_{\frac{\rho}{8}}(x^*) \cap D)} |\operatorname{Ric}(g^*) - (n-1)Hg^*| > \Psi_2(\epsilon|n, \rho, \Lambda),$$

then

$$(6)\quad \fint_{B_{\frac{\rho}{2}}(x^*)} |\operatorname{Ric}(g^*) - (n-1)Hg^*|$$

$$(7)\quad \geq \sum_{\gamma\in\Gamma(\frac{\rho}{4})} \frac{\operatorname{vol}(\gamma(B_{\frac{\rho}{8}}(x^*)\cap D))}{\operatorname{vol}(B_{\frac{\rho}{2}}(x^*))} \fint_{\gamma(B_{\frac{\rho}{8}}(x^*)\cap D)} |\operatorname{Ric}(g^*) - (n-1)Hg^*|$$

$$(8)\quad > \frac{\Psi_2(\epsilon|n,\rho,\Lambda)}{\operatorname{vol}(B_{\frac{\rho}{2}}(x^*))} \sum_{\gamma\in\Gamma(\frac{\rho}{4})} \operatorname{vol}(\gamma(B_{\frac{\rho}{8}}(x^*)\cap D))$$

$$(9)\quad \geq \frac{\Psi_2(\epsilon|n,\rho,\Lambda)\operatorname{vol}(B_{\frac{\rho}{8}}(x^*))}{\operatorname{vol}(B_{\frac{\rho}{2}}(x^*))}$$

$$(10)\quad \geq \Psi_1(\epsilon|n,\rho,\Lambda),$$

a contradiction to (2.1.1). □

*Proof of Theorem* B. Arguing by contradiction, assume a contradicting sequence, $(M_i, g_i) \xrightarrow{GH} X$, such that

$$\Lambda \geq \operatorname{Ric}(g_i) \geq (n-1)H, \quad \frac{\operatorname{vol}(B_\rho(x_i^*))}{\operatorname{vol}(\underline{B}_\rho^H)} \geq 1-\epsilon_i \to 1 \quad \forall\, x_i \in M_i,$$

and $M_i$ admits no nearby metric to $g_i$ with almost constant sectional curvature $H$.

Fixing a small $\delta \in (0, T(n,\rho,\Lambda)]$ (Theorem 1.5), let $g_i(\delta)$ denote the renormalized Ricci flow in (1.6.1). By Theorem 1.5, for any $x_i \in M_i$, passing to a subsequence we may assume that the lifting metric $g_i^*(\delta)$ on $B_\rho(x_i^*)$ satisfies

$$B_{\frac{\rho}{2}}(x_i^*, g_i^*(\delta)) \xrightarrow{C^k} B_{\frac{\rho}{2}}(x_\delta^*, g_\infty^*(\delta)), \quad h_i(g_i^*(\delta)) \xrightarrow{C^k} h(g_\infty^*(\delta)),$$

where $h_i$ is defined in (1.6.3), and the $C^k$-convergence can be seen from the Cheeger-Gromov convergence theorem. Consequently, $g_\infty^*(\delta)$ is a smooth metric and $h(g_\infty^*(\delta))$ is a smooth tensor on $B_{\frac{\rho}{2}}(x_\delta^*, g_\infty^*(\delta))$. By Lemma 1.4 and Lemma 1.7, for any $x_i \in M_i$,

$$|h_i(g_i^*(\delta))|_{p,B_{\frac{\rho}{4}}(x_i^*)} \to 0.$$

Consequently, $h(g_\infty^*(\delta))|_{B_{\frac{\rho}{4}}(x_\delta^*, g_\infty^*(\delta))} \equiv 0$, i.e., $g_\infty^*(\delta)|_{B_{\frac{\rho}{4}}(x_\delta^*, g_\infty^*(\delta))}$ is $H$-Einstein.

Clearly, $B_{\frac{\rho}{4}}(x_\delta^*, g_\infty^*(\delta)) \xrightarrow{GH} \underline{B}_{\frac{\rho}{4}}^H$ as $\delta \to 0$. Since $g_\infty^*(\delta)$ is $H$-Einstein for all $\delta$, $B_{\frac{\rho}{4}}(x_\delta^*, g_\infty^*(\delta)) \xrightarrow{C^k} \underline{B}_{\frac{\rho}{4}}^H$, for any $k \geq 1$ ([CC97]). Consequently, for $\delta_0$ sufficiently small, $g_\infty^*(\delta_0)$ has almost constant sectional curvature $H$. Since $B_{\frac{\rho}{4}}(x_i^*, g_i^*(\delta_0)) \xrightarrow{C^k} B_{\frac{\rho}{4}}(x_{\delta_0}^*, g_\infty^*(\delta_0))$, for $i$ large, $g_i^*(\delta_0)$ has almost constant curvature $H$. Since $x_i$ is arbitrarily chosen, we conclude that $g_i(\delta_0)$ has almost constant sectional curvature $H$, a contradiction. □

*Proof of Theorem* C. Fixing a small $\delta > 0$, by Theorem B we may assume a nearby metric $g(\delta)$ such that

$$|g - g(\delta)| < \delta, \quad H - \delta \leq \sec_{g(\delta)} \leq H + \delta.$$

*Case* 1. Assume $H = -1$. For any $\tilde p \in \tilde M$, the exponential map, $\exp_{\tilde p}^{\tilde g(\delta)} : T_{\tilde p}\tilde M \to \tilde M$, is a diffeomorphism such that its differential has a bounded norm on $B_1(0)$

depending on $n, \delta$. Consequently, $\mathrm{vol}(B_1(\tilde p, \tilde g(\delta)))$ has a positive lower bound depending only on $n$ and $\delta$. Since $|\tilde g - \tilde g(\delta)| < \delta$, we conclude the desired result.

*Case* 2. Assume $H = 0$. By the splitting theorem of Cheeger-Gromoll, $\tilde M = \mathbb{R}^k \times N$, where $N$ is a simply connected $(n-k)$-manifold of non-negative Ricci curvature. We claim that $N$ is a point. Note that $\mathrm{diam}(N) \le c(n,d)$ (see the proof of Theorem C in [CRX] where we normalize $d = 1$). We may assume $\delta^{-\frac{1}{2}} > 4\,\mathrm{diam}(N)$. Note that since $\sec_{g(\delta)} < \delta$, $\exp_{\tilde p}^{\tilde g(\delta)} : B_{\frac{1}{\sqrt{\delta}}}(0) \to B_{\frac{1}{\sqrt{\delta}}}(\tilde p, \tilde g(\delta))$ is a local diffeomorphism. Note that $B_{\frac{1}{2\sqrt{\delta}}}(\tilde p)$ can be deformed to $0 \times N$ ($\tilde p = (0, x)$) and thus $B_{\frac{1}{2\sqrt{\delta}}}(\tilde p)$ is simply connected. Consequently, the lifting of $B_{\frac{1}{2\sqrt{\delta}}}(\tilde p)$ via $\exp_{\tilde p}^{\tilde g(\delta)}$ is contained in the segment domain (i.e. each $\tilde x \in B_{\frac{1}{2\sqrt{\delta}}}(\tilde p)$ is connecting to $\tilde p$ by a unique minimal geodesic; if $c_1$ and $c_2$ are two distinct minimal geodesics, then $c_1 * c_2^{-1}$ is a loop at $\tilde p$, and so is the lifting of $c_1 * c_2^{-1}$ a loop at 0. Note that with respect to the pullback metric on $T_{\tilde p}\tilde M$, we obtain two geodesics from 0 to some $v$; a contradiction). Therefore, $B_{\frac{1}{2\sqrt{\delta}}}(\tilde p)$ is contractible in $\tilde M$, a contradiction.

*Case* 3. Assume $H = 1$. The classical 1/4-pinched injectivity radius estimate implies that the pullback metric $\tilde g(\delta)$ on $\tilde M$ has injectivity radius $> \frac{\pi}{2}$, and thus $\mathrm{vol}(B_1(\tilde p))$ has a positive lower bound depending on $n$. By now the desired result follows. □

As seen in the introduction, Theorem D follows from Theorem C and Theorem 0.4. Note that the proof of Theorem 0.4 in [CRX] for $H = 1$ is quite involved due to the lack of regularity. Using the regularity of the Ricci flow solution in the proof of Theorem B, we are able to give a simple direct proof.

In the rest of the paper, we will freely use properties of equivariant Gromov-Hausdorff convergence; see b. of Section 1 in [CRX] for details.

**Lemma 2.2.** *Let $M_i$ be a sequence of compact $n$-manifolds satisfying*

$$\mathrm{Ric}_{M_i} \ge (n-1), \quad |\mathrm{Rm}|_{C^1, M_i} \le C, \quad \mathrm{vol}(\tilde M_i) \ge v > 0,$$

*and the commutative diagram,*

$$\begin{array}{ccc} (\tilde M_i, \Gamma_i) & \xrightarrow{GH} & (\tilde M_\infty, G) \\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi} \\ M_i & \xrightarrow{GH} & X, \end{array}$$

*Then for i large,*
*(2.2.1) There is an injective homomorphism and $\epsilon_i$-GHA ($\epsilon_i \to 0$), $\phi_i : \Gamma_i \to G$, such that $\phi_i(\Gamma_i)$ acts freely on $\tilde M_\infty$.*
*(2.2.2) There is a $\Gamma_i$-conjugate diffeomorphism, $\tilde f_i : (\tilde M_i, \Gamma_i) \to (\tilde M_\infty, \phi_i(\Gamma_i))$, which is also an $\epsilon_i$-GHA.*

*Proof.* Lemma 2.2 is essentially Theorem 3.5 in [CRX] where condition

$$|\mathrm{Rm}|_{C^1, M_i} \le C$$

is replaced with

$$\frac{\mathrm{vol}(B_\rho(\tilde x_i))}{\mathrm{vol}(\underline{B}_\rho^1)} \ge 1 - \epsilon_i \to 1,$$

and the proof of Theorem 3.5 will go through with the following minor modifications:

(i) There is a uniform lower bound on the injectivity radius of $\tilde M_i$, and thus $\tilde M_\infty$ is a Riemannian manifold, and for any $r_i \to \infty$, passing to a subsequence $(\tilde M_i, \tilde p_i, r_i^2 \tilde g_i)$ converges to $\mathbb{R}^n$; which guarantees (2.2.1).

(ii) In the proof of Theorem 3.5, the method of center of mass was applied to modify an $\epsilon_i$-equivariant GHA to a conjugate map $\tilde f_i$, while the main work was to show that $\tilde f_i$ is a diffeomorphism. Here, by the $C^1$-regularity we automatically obtain that $\tilde f_i$ is a diffeomorphism ([GK73]). □

*Proof of Theorem* D *for* $H = 1$. Arguing by contradiction, assume a contradicting sequence, $(M_i, g_i)$, such that $g_i$ satisfies the conditions of Theorem B for $\epsilon_i \to 0$ but none of $M_i$ is diffeomorphic to a spherical space form.

For each $i$, let $g_i(\delta)$ be as in Theorem B, such that for all $1 \le k < \infty$,

$$|\operatorname{Rm}(g_i(\delta))|_{C^k} \le C(n, \rho, \Lambda, \delta, k), \quad |\sec_{g_i(\delta)} - 1| < \delta.$$

Passing to a subsequence we may assume the following commutative diagram:

$$\begin{array}{ccc} (\tilde M_i, \tilde g_i(\delta), \Gamma_i) & \xrightarrow{GH} & (\tilde M_\infty(\delta), \tilde g_\infty(\delta), G(\delta)) \\ \Big\downarrow \pi_i & & \Big\downarrow \pi \\ (M_i, g_i(\delta)) & \xrightarrow{GH} & (X, d_\infty(\delta)), \end{array}$$

where $\Gamma_i$ denotes the deck transformations. Since $\tilde M_i$ is not collapsed (Theorem C), by Lemma 2.2 there is a $\Gamma_i$-conjugate diffeomorphism, $\tilde f_i(\delta) : (\tilde M_i, \tilde g_i(\delta), \Gamma_i) \to (\tilde M_\infty(\delta), \tilde g_\infty(\delta), \phi_i(\delta)(\Gamma_i))$. From the proof of Theorem B, we see that $(\tilde M_\infty(\delta), \tilde g_\infty(\delta))$ is 1-Einstein. It is clear that $(\tilde M_\infty(\delta), \tilde g_\infty(\delta), G(\delta)) \xrightarrow{GH} (S^n_1, \underline{g}_1, G)$, as $\delta \to 0$. Consequently, for all $k < \infty$, $(\tilde M_\infty(\delta), \tilde g_\infty(\delta), G(\delta)) \xrightarrow{C^k} (S^n_1, \underline{g}_1, G)$ ([CC97]).

For each $\delta$, we may choose $i$ large such that $d_{GH}(\phi_i(\delta)(\Gamma_i), G(\delta)) < \delta_i \to 0$, i.e., $(M_\infty(\delta), \tilde g_\infty(\delta), \phi_i(\delta)(\Gamma_i)) \xrightarrow{GH} (S^n_1, \underline{g}_1, G)$. We then apply Lemma 2.2 again to conclude that for a fixed small $\delta$, there is $\phi_i(\delta)(\Gamma_i)$-conjugate diffeomorphism, $\tilde f_\infty(\delta)$ : $(\tilde M_\infty(\delta), \phi_i(\delta)(\Gamma_i)) \to (S^n_1, \psi_i(\delta) \circ \phi_i(\delta)(\Gamma_i))$. Then $\tilde f_\infty(\delta) \circ \tilde f_i(\delta) : (\tilde M_i, \Gamma_i) \to (S^n_1, \psi_i(\delta) \circ \phi_i(\delta)(\Gamma_i))$ is $\Gamma_i$-conjugate diffeomorphism, and thus $M_i$ is diffeomorphic to a spherical space form, $S^n_1/(\psi_i(\delta) \circ \phi_i(\delta)(\Gamma_i))$, a contradiction. □

*Remark* 2.3. Given Theorem B, the conclusion of Theorem D for $H = 0$ and $H = 1$ can also be seen from the work [Gr] and [BS09] respectively.

*Proof of Theorem* E. It suffices to show that for any $Q \ge 1$ and $0 < \alpha < 1$, there is a constant $r_h = r_h(n, \rho, d, \Lambda, v, \alpha, Q) > 0$ such that $M$ has $C^{1,\alpha}$-harmonic radius with respect to $Q$ bounded below by $r_h$; because $\Lambda \ge \operatorname{Ric}_M \ge (n-1)H$.

Arguing by contradiction, assume for some $Q_0 \ge 1$ and $0 < \alpha_0 < 1$, there is a contradicting sequence, $M_i$, satisfying

$$\Lambda \ge \operatorname{Ric}_{M_i} \ge (n-1)H, \quad d \ge \operatorname{diam}(M_i), \quad \frac{\operatorname{vol}(B_\rho(x_i^*))}{\operatorname{vol}(\underline{B}^H_\rho)} \ge 1 - \epsilon_i \to 1 \quad \forall x_i \in M_i,$$

and $p_i \in M_i$ such that the $C^{1,\alpha_0}$-harmonic radius $r_h(p_i) \to 0$. Passing to a subsequence, we may assume the following commutative diagram:

$$\begin{array}{ccc} (\widetilde{B_\rho(p_i)}, p_i^*, K_i) & \xrightarrow{GH} & (X^*, p^*, K) \\ \Big\downarrow \pi_i & & \Big\downarrow \pi \\ (B_\rho(p_i), p_i) & \xrightarrow{GH} & (B_\rho(p), p), \end{array}$$

where $K_i$ denotes the fundamental group of $B_\rho(p_i)$. Since

$$\frac{\operatorname{vol}(B_\rho(p_i^*))}{\operatorname{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon_i \to 1,$$

by Theorem 1.2 we see that $B_\rho(p^*)$ is local isometric to an $H$-space form. If $H \neq -1$, $K_i$ is discrete because $\operatorname{vol}(M_i) \geq v$. We claim that $K$ is discrete when $H = -1$. Hence, in any case we are able to apply Theorem 2.1 in [CRX] to conclude that $K$ acts freely on $X^*$. We may assume that any element in $K_i$ moves any $x_i^*$ in $B_{\frac{\rho}{2}}(p_i^*)$ at least $\delta$-distance, where $\delta$ depends on $(X^*, K)$. By Lemma 1.3, we may assume that $r_h(p_i^*) \geq r_h(n, \rho, \Lambda, \alpha_0, Q_0) > 0$, and thus $2r_h(p_i) \geq \min\{\delta, r_h(p_i^*)\} > 0$, a contradiction.

To see that $K$ is discrete, note that by Theorem D we conclude that $M_i$ is $\Psi(\epsilon|n, \rho, d, \Lambda)$ close to a hyperbolic manifold $\mathbb{H}^n/\Gamma_i$. By the Margulis-Heintze lemma ([He]), $\mathbb{H}^n/\Gamma_i$ is not collapsed, and by the volume convergence in [Col97] we then conclude that $M_i$ is not collapsed (so $B_\rho(x_i)$ is not collapsed), and thus $K$ is discrete. □

We will conclude this paper with the following questions related to the present approach to Conjecture 0.3:

**Problem 2.4.** Does Theorem A hold without an upper bound on Ricci curvature? Indeed, it seems that it is even not known whether the scalar curvature is almost constant in the $L^p$-sense.

**Problem 2.5** (Ricci flow time)**.** For $n, \rho > 0$, and $H = \pm 1$ or 0, are there constants, $\epsilon(n, \rho) > 0, T(n, \rho) > 0$, such that for any $0 < \epsilon < \epsilon(n, \rho)$, if a compact $n$-manifold $(M, g)$ satisfies

$$\operatorname{Ric}(g) \geq (n-1)H, \quad \frac{\operatorname{vol}(B_\rho(x^*))}{\operatorname{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon \quad \forall\, x \in M,$$

then the Ricci flow from $g$ exists for $t \in [0, T(n, \rho)]$?

In a forth coming paper [HKRX], we will give a positive answer to Problem 2.5.

**Problem 2.6** (Ricci flows preserves almost Einstein)**.** Let $(M, g)$ be a compact $n$-manifold of $\operatorname{Ric}_M \geq (n-1)H$ and

$$\fint_{B_{\frac{\rho}{2}}(p^*)} |\operatorname{Ric}(g^*) - (n-1)Hg^*| < \epsilon.$$

Let $g(t)$ be a renormalized Ricci flow of $g$ (see (1.6.1)). Is

$$\fint_{B_{\frac{\rho}{2}}(p^*, g^*(t))} |\operatorname{Ric}(g^*(t)) - (n-1)Hg^*(t)| < \Psi(\epsilon|n, \rho, t)?$$

Note that if there are affirmative answers to Problems 2.4-2.6, then the approach in the present paper can be extended toward a proof of Conjecture 0.2.

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## References

[And90] Michael T. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math. **102** (1990), no. 2, 429–445, DOI 10.1007/BF01233434. MR1074481

[BS09] Simon Brendle and Richard Schoen, *Manifolds with 1/4-pinched curvature are space forms*, J. Amer. Math. Soc. **22** (2009), no. 1, 287–307, DOI 10.1090/S0894-0347-08-00613-9. MR2449060

[CC96] Jeff Cheeger and Tobias H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) **144** (1996), no. 1, 189–237, DOI 10.2307/2118589. MR1405949

[CC97] Jeff Cheeger and Tobias H. Colding, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. **46** (1997), no. 3, 406–480. MR1484888

[CFG92] Jeff Cheeger, Kenji Fukaya, and Mikhael Gromov, *Nilpotent structures and invariant metrics on collapsed manifolds*, J. Amer. Math. Soc. **5** (1992), no. 2, 327–372, DOI 10.2307/2152771. MR1126118

[Col96] Tobias H. Colding, *Large manifolds with positive Ricci curvature*, Invent. Math. **124** (1996), no. 1-3, 193–214, DOI 10.1007/s002220050050. MR1369415

[Col97] Tobias H. Colding, *Ricci curvature and volume convergence*, Ann. of Math. (2) **145** (1997), no. 3, 477–501, DOI 10.2307/2951841. MR1454700

[CRX] L. Chen, X. Rong, and S. Xu, *Quantitive volume rigidity of space form under lower Ricci curvature bound I*, to appear in J. Differential Geom.

[DWY] Xianzhe Dai, Guofang Wei, and Rugang Ye, *Smoothing Riemannian metrics with Ricci curvature bounds*, Manuscripta Math. **90** (1996), no. 1, 49–61, DOI 10.1007/BF02568293. MR1387754

[GK73] Karsten Grove and Hermann Karcher, *How to conjugate $C^1$-close group actions*, Math. Z. **132** (1973), 11–20. MR0356104

[Ha] Richard S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306. MR664497

[Gr] M. Gromov, *Almost flat manifolds*, J. Differential Geom. **13** (1978), no. 2, 231–241. MR540942

[He] Ernst Heintze, *Mannigfaltigkeiten negativer Krümmung* (German), Bonner Mathematische Schriften [Bonn Mathematical Publications], vol. 350, Universität Bonn, Mathematisches Institut, Bonn, 2002. MR1940403

[HKRX] H. Huang, L. Kong, X. Rong, and S. Xu, Collapsed manifolds with local Ricci bounded covering geometry. In preparation.

[LW] François Ledrappier and Xiaodong Wang, *An integral formula for the volume entropy with applications to rigidity*, J. Differential Geom. **85** (2010), no. 3, 461–477. MR2739810

[Pe] G. Perelman, *Manifolds of positive Ricci curvature with almost maximal volume*, J. Amer. Math. Soc. **7** (1994), no. 2, 299–305, DOI 10.2307/2152760. MR1231690

[Pet] Peter Petersen, *Riemannian geometry*, 2nd ed., Graduate Texts in Mathematics, vol. 171, Springer, New York, 2006. MR2243772

[Sh1] Wan-Xiong Shi, *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. **30** (1989), no. 1, 223–301. MR1001277

[Sh2] Wan-Xiong Shi, *Ricci deformation of the metric on complete noncompact Riemannian manifolds*, J. Differential Geom. **30** (1989), no. 2, 303–394. MR1010165

[TW] Gang Tian and Bing Wang, *On the structure of almost Einstein manifolds*, J. Amer. Math. Soc. **28** (2015), no. 4, 1169–1209, DOI 10.1090/jams/834. MR3369910

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