ON THE NUMBER OF ZEROS AND POLES
OF DIRICHLET SERIES

BAO QIN LI

ABSTRACT. This paper investigates lower bounds on the number of zeros and poles of a general Dirichlet series in a disk of radius r and gives, as a consequence, an affirmative answer to an open problem of Bombieri and Perelli on the bound. Applications will also be given to Picard type theorems, global estimates on the symmetric difference of zeros, and uniqueness problems for Dirichlet series.

1. INTRODUCTION AND THE MAIN RESULTS

A *general Dirichlet series* is an infinite series of the form

$$(1.1) \qquad \sum_{n=1}^{\infty} a_n e^{-\lambda_n s},$$

where the a_n's are complex numbers, $\{\lambda_n\}$ is a strictly increasing sequence of real numbers whose limit is infinity, and $s = \sigma + it$ is a complex variable. The special type $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$, also referred to as an *ordinary Dirichlet series*, is obtained by substituting $\lambda_n = \log n$. We refer to the monographs [1], [14], [23], etc., for general theory and applications of Dirichlet series. Dirichlet series appearing in many problems admit analytic continuation in the complex plane $\mathbf{C}$ as meromorphic functions, and information on their analytic properties are important in many areas from analysis to number theory to mathematical physics. This paper is concerned with the number of zeros and poles (as well as a-values) of these general Dirichlet series. Denote by $n(r, a; f)$ the number of solutions, counted with multiplicity, of the equation $f(s) = a$ in the disc $|s| \leq r$ for a function f meromorphic in $\mathbf{C}$ and a value a. Since we are dealing with general Dirichlet series without assumptions such as functional equation, this *counting function* (instead of the one in strips), or its integrated form (see below), is a natural measure for the zeros of $f(s) - a$. This measure, which is well known, is one of the most important quantities in the study of value distribution of meromorphic functions and their applications. The growth of $n(r, a; f)$ for a meromorphic function f can be at the same rate as, but cannot exceed, that of the function (see e.g. [15, p. 26]), which provides a general connection between the upper bound of the counting function and the growth of the function; there is, however, no such kind of connection in general between the lower bound of the counting function and the growth of the function. These bounds have long been studied for Dirichlet series, especially, of course, for the Riemann

zeta function $\sum_{n=1}^{\infty} \frac{1}{n^s}$, for which the well-known Riemann–von Mangoldt formula (see, e.g., [30, p. 214]) characterizes the counting function of its zeros, and also for other ordinary and general Dirichlet series under various analytic and arithmetic conditions (see, e.g., [20], [25], [26], [27], [28], and [29], to list a few). It appears in many known cases of Dirichlet series with "few" poles, besides various L-functions, that $n(r, 0; f)$ grows at least at the order r and that there are nontrivial Dirichlet series with growth order equal to r. It remained to be an open problem whether a lower bound of growth order equal to r can be given for the zeros and poles in a general setting, and it, if established, would be best possible. This is also closely tied to several other problems studied in complex analysis and number theory on Picard type theorems, global estimates on the symmetric difference of zeros, and uniqueness problems for Dirichlet series (see §2). In fact, Bombieri and Perelli [6] obtained a lower bound, which is of growth order r^γ for any $\gamma < 1$, on the number of zeros and poles for general Dirichlet series under mild analytic assumptions; more precisely, they proved the following theorem.

Theorem A. *Let $f(s) = \sum a_n e^{\lambda_n s}$, $\lambda_n \in \mathbb{R}$, be uniformly convergent in a half-plane $\Re(s) > b$ and admit an analytic continuation in $\mathbf{C}$ as a nonconstant meromorphic function of finite order. Suppose that $f(s)$ tends to a nonzero finite limit as $\Re(s) \to +\infty$. Then, for any fixed $\gamma < 1$, we have*

$$(1.2) \qquad \limsup_{r \to +\infty} \frac{N(r, 0; f) + N(r, \infty; f)}{r^\gamma} > 0.$$

In the above,

$$N(r, a; f) = \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t} dt + n(0, a; f) \log r,$$

which is the integrated counting function and plays a similar role as $n(r, a; f)$ for the solutions of the equation $f(s) = a$. (It will be seen from (2.1) in §2 that $N(r, a; f)$ can be replaced by the simpler un-integrated counting function $n(r, a; f)$.)

The lower bound given in (1.2) is not of growth order equal to r due to the assumption $\gamma < 1$. Bombieri and Perelli asked whether the conclusion of Theorem A holds with $\gamma = 1$, which would be best possible. This is stated in the following problem.

Problem A ([6, p. 71])**.** Under the same conditions of Theorem A, do we have

$$(1.3) \qquad \limsup_{r \to +\infty} \frac{N(r, 0; f) + N(r, \infty; f)}{r} > 0?$$

Bombieri and Perelli proved that (1.3) does hold under the following additional conditions [6].

Theorem B. *In addition to the hypotheses of Theorem A, suppose that $f(s)$ is uniformly almost periodic in some half-plane $\Re(s) < c$ and $f(s)$ tends to a nonzero finite limit as $\Re(s) \to -\infty$. Then, (1.3) holds.*

Here, the notion of a uniformly almost periodic function is in the sense of Bohr (see §2 for the definition).

In this paper, we will solve Problem A by an affirmative answer (and thus the additional conditions in Theorem B are superfluous); that is, we will prove the following theorem.

Theorem 1.1. *Under the same conditions of Theorem A, we have*

$$\limsup_{r \to +\infty} \frac{N(r,0;f) + N(r,\infty;f)}{r} > 0.$$

Remark. We show in this remark that Theorem 1.1 is best possible in the following senses:

(i) As mentioned above, the growth order r in the conclusion of Theorem 1.1 is the best possible, as seen from the Dirichlet series $f(s) = \sum_{n=0}^{\infty} \frac{a_n}{2^{ns}}$ with suitable a_n (for example, simply take $f(s) = 1 + \frac{1}{2^s}$).

(ii) The condition "$f(s)$ tends to a nonzero finite limit as $\Re(s) \to +\infty$" cannot be dropped, as seen from the function $f(s) = e^{-s}$ (resp., $f(s) = e^s$), which tends to zero as $\Re(s) \to +\infty$ (resp., tends to ∞ as $\Re(s) \to +\infty$). The function f satisfies all the other conditions of Theorem 1.1 but has no zeros or poles.

(iii) The condition "finite order" cannot be dropped in Theorem 1.1. To see this, consider the Dirichlet series $f(s) = \sum_{n=0}^{\infty} \frac{1}{n!} e^{-ns}$, which clearly satisfies all the conditions of Theorem 1.1 except that $f(s)$ is of infinite order, in view of the fact that $f(s) = e^{e^{-s}}$, which has no zeros or poles.

(iv) The sum $N(r,0;f) + N(r,\infty;f)$ in the conclusion of Theorem 1.1 cannot be strengthened as $N(r,0;f)$ or $N(r,\infty;f)$ alone, as seen from the Dirichlet series $f(s) = \frac{1}{1-e^{-s}} = \sum_{n=0}^{\infty} e^{-ns}$ ($\Re(s) > 0$), which has no zeros, and the Dirichlet series $f(s) = 1 - e^{-s}$, which has no poles.

Under the same conditions of Theorem 1.1 or Theorem A, we will actually establish the following result, which improves the original estimate (1.3) in Problem A; specifically we have the following theorem.

Theorem 1.2. *Let $f(s) = \sum a_n e^{\lambda_n s}$, $\lambda_n \in \mathbb{R}$, be uniformly convergent in a half-plane $\Re(s) > b$ and admit an analytic continuation in $\mathbf{C}$ as a nonconstant mero-morphic function of finite order. Suppose that $f(s)$ tends to a nonzero limit as $\Re(s) \to +\infty$. Then either*

$$(1.4) \qquad \liminf_{r \to +\infty} \frac{n(r,0;f) + n(r,\infty;f)}{r} > 0$$

or

$$(1.5) \qquad \int_{r_0}^{\infty} \frac{n(t,0;f) + n(t,\infty;f)}{t^3} dt = \infty$$

for any $r_0 > 0$.

2. Consequences of Theorem 1.2

Theorem 1.2 has various consequences. We include in this section its applications to Picard type theorems, global estimates on the symmetric difference of zeros, and uniqueness problems for Dirichlet series and, particularly, L-functions.

We first show that Theorem 1.1 is a direct consequence of Theorem 1.2, from which it is also clear (see (1.4) and (2.2) below) that *under the same conditions of Theorem* 1.1 *or Theorem* 1.2,

$$(2.1) \qquad \limsup_{r \to +\infty} \frac{n(r,0;f) + n(r,\infty;f)}{r} > 0.$$

In fact, each of (1.4) and (1.5), given in terms of the unintegrated counting function (often more convenient as a counting function), yields a stronger result than the conclusion of Theorem 1.1, i.e., (1.3).

If (1.4) holds, then there is a constant $A > 0$ such that

$$\frac{n(r, 0; f) + n(r, \infty; f)}{r} > A$$

for all large r. By the definition of the integrated counting function, we easily deduce, by integration, that

$$\liminf_{r \to +\infty} \frac{N(r, 0; f) + N(r, \infty; f)}{r} > 0,$$

which is a stronger statement than (1.3).

If (1.5) holds, then it is easy to see that for any small $\epsilon > 0$,

$$(2.2) \qquad \limsup_{r \to +\infty} \frac{n(r, 0; f) + n(r, \infty; f)}{r^{2-\epsilon}} = +\infty,$$

since otherwise there would be an $M > 0$ such that

$$n(r, 0; f) + n(r, \infty; f) < M r^{2-\epsilon}$$

for all large r, which clearly implies that

$$\int_{r_0}^{\infty} \frac{n(t, 0; f) + n(t, \infty; f)}{t^3} dt < +\infty$$

for any $r_0 > 0$, a contradiction to (1.5). In view of the fact that for $r \geq 1$,

$$n(r, 0; f) + n(r, \infty; f)$$
$$\leq \int_r^{er} \frac{n(t, 0; f) + n(t, \infty; f)}{t} dt$$
$$\leq N(er, 0; f) + N(er, \infty; f),$$

we deduce from (2.2) the following stronger result than (1.3):

$$\limsup_{r \to +\infty} \frac{N(r, 0; f) + N(r, \infty; f)}{r^{2-\epsilon}} = +\infty.$$

In Remark (ii) of Theorem 1.1, we pointed out that the condition "$f(s)$ tends to a nonzero finite limit as $\Re(s) \to +\infty$" cannot be dropped in Theorem 1.1 and thus in Theorem 1.2. However, this condition can be dropped for general Dirichlet series in both Theorem 1.1 and Theorem 1.2 (note that the series f in Theorem 1.1 and Theorem 1.2 is an infinite exponential sum, which is more general than general Dirichlet series), except the trivial case that $f(s) = ae^{-\lambda s}$ ($a \neq 0$) with only one term, which obviously has no zeros or poles. We particularly include the following corollary, in which (and in the sequel) f is called *nontrivial* if $f(s) \not\equiv ae^{-\lambda s}$ with a, λ being constant.

Corollary 2.1. *Let f be a nontrivial general Dirichlet series uniformly convergent in a half-plane $\Re(s) > b$ and admit an analytic continuation in $\boldsymbol{C}$ as a meromorphic function of finite order. Then,*

$$\limsup_{r \to +\infty} \frac{n(r, 0; f) + n(r, \infty; f)}{r} > 0.$$

In particular, if f has "few" poles in the sense that $n(r, \infty; f) = o(r)$, then

$$\limsup_{r \to +\infty} \frac{n(r, 0; f)}{r} > 0.$$

To show Corollary 2.1, suppose that $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ is a general Dirichlet series, i.e., $\{\lambda_n\}$ is a sequence of real increasing numbers tending to infinity (see (1.1)). We may then assume, without loss of generality, that $a_1 \neq 0$ (otherwise just use the first nonzero coefficient a_n). The Dirichlet series $e^{\lambda_1 s} f(s) = a_1 + \sum_{n=2}^{\infty} a_n e^{-(\lambda_n - \lambda_1)s}$ and $f(s)$ have the same zeros and poles. Thus, it suffices to consider $e^{\lambda_1 s} f(s)$ in Theorem 1.2, which is nonconstant since f is nontrivial, tends to a nonzero finite limit (equal to a_1) as $\Re(s) \to +\infty$, and thus satisfies all the conditions of Theorem 1.2. Corollary 2.1 then follows from Theorem 1.2 and (2.1).

The above results readily extend to those for a-values of f. In particular, for Corollary 2.1, all the conditions are still satisfied for $f - a$, provided that $f - a$ is nontrivial, i.e., $f - a$ is not of the form $ce^{-\lambda s}$ with c, λ being constant (cf. Corollary 2.1). Thus, we obtain from Corollary 2.1 the following quantitative Picard type theorem.

Corollary 2.2. *Let f be a general Dirichlet series uniformly convergent in a half-plane $\Re(s) > b$ and admit an analytic continuation in $\mathbf{C}$ as a meromorphic function of finite order. Then*

$$(2.3) \qquad \limsup_{r \to +\infty} \frac{n(r, a; f) + n(r, \infty; f)}{r} > 0$$

for any complex number a, provided that $f - a$ is nontrivial.

It is worth noting that in Corollary 2.2, if f is not of the form $a_0 + b_0 e^{-\lambda s}$ for some constants λ, a_0, b_0, then (2.3) holds for every complex number a without any exception (the situation when f is of the form $a_0 + b_0 e^{-\lambda s}$ is trivial). In the case that f is entire or has "few" poles in the sense stated above, i.e., $n(r, \infty; f) = o(r)$, the conclusion (2.3), giving a lower bound on how often f assumes a, implies particularly that f takes every complex number a infinitely often without any exception. This should be compared to the famous Picard theorem: A nonconstant entire (resp., meromorphic) function takes every complex number infinitely often with possibly one exception (resp., two exceptions), and it can be thought of as a quantitative Picard type theorem for general Dirichlet series. We refer to the monograph [23, Chapter III], for related but different results for Picard type theorems for Dirichlet series.

In many problems of significance in number theory, Dirichlet series admit an analytic continuation in $\mathbf{C}$ as meromorphic functions of order ≤ 1. Theorem 1.2 may further be applied to yield a global estimate, which is substantially stronger than the estimate from the upper limit result, on the counting functions of zeros, poles, and, more generally, a-values for these Dirichlet series f. In fact, when f is of order ≤ 1, it follows from Jensen's formula (see, e.g., [15, pp. 25–26]) that for any complex value a (including $a = \infty$), $n(r, a; f) = O(r^{1+\epsilon})$ for any $0 < \epsilon < 1$, which implies that

$$\int_{r_0}^{\infty} \frac{n(t, 0; f) + n(t, \infty; f)}{t^3} dt < +\infty$$

for any $r_0 > 0$. This last inequality actually holds for all f of order strictly less than 2, since then $n(r, a; f) = O(r^{2-\delta})$ for some small $\delta > 0$. Thus for these f,

(1.5) does not hold, and consequently, the global estimate (1.4) in Theorem 1.2 must hold. That is, we have the following (cf. Corollary 2.1).

Corollary 2.3. *Let f be a nontrivial general Dirichlet series uniformly convergent in a half-plane $\Re(s) > b$ and admit an analytic continuation in $\mathbf{C}$ as a meromorphic function of order < 2. Then*

$$(2.4) \qquad \liminf_{r \to +\infty} \frac{n(r, 0; f) + n(r, \infty; f)}{r} > 0.$$

As in Theorem 1.2, the nontrivial general Dirichlet series f in Corollary 2.3 can be replaced by a nonconstant infinite exponential sum $f(s) = \sum a_n e^{\lambda_n s}$, $\lambda_n \in \mathbb{R}$, with $f(s)$ tending to a nonzero limit as $\Re(s) \to +\infty$, and the result also extends to a-values of f as given in Corollary 2.2. We omit their statements here.

We next present another consequence of Theorem 1.2 (Corollary 2.3) as an application to L-functions, which are important objects and extensively studied in number theory, for lower bounds on the cardinality of the symmetric difference of their zeros. Let $F(s)$ and $G(s)$ be two general Dirichlet series, and let $m_F(\rho)$ (resp., $m_G(\rho)$) be the multiplicity of ρ as zero of F (resp., of G). The symmetric difference $D_{F,G}$ is defined as

$$(2.5) \qquad D_{F,G} = \sum_{|\rho| \leq T} |m_F(\rho) - m_G(\rho)|,$$

where $\rho = \sigma + it$ runs over the zeros of $F(s)G(s)$. As given in [6], the following holds on the cardinality $D_{F,G}$ of two distinct L-functions F, G satisfying the same functional equation:

$$(2.6) \qquad D_{F,G}(T) = \Omega(T)$$

(counting only the nontrivial zeros of $F(s)G(s)$ with $|t| \leq T$ in (2.5)), where the notation $f(x) = \Omega(g(x))$ denotes the negation of $f(x) = o(g(x))$ (see, e.g., [30, p. 184]), or $\limsup_{T \to +\infty} \frac{D_{F,G}(T)}{T} > 0$, under only the function-theoretic properties of F and G, disregarding their arithmetical aspects. Here and in the sequel, by abuse of language, an *L-function* means a Dirichlet series $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ with $a_1 = 1$, which is absolutely convergent for $\sigma > 1$, admits an analytic continuation in $\mathbf{C}$ as a meromorphic function of finite order with at most one pole at $s = 1$, and satisfies a standard functional equation (cf. [27]), without requiring the Ramanujan hypothesis and Euler product. Note that the same bound (2.6) was obtained in [24] for any two distinct L-functions F and G in the Selberg class, which consists of Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ satisfying the above-mentioned conditions and also the Ramanujan hypothesis and Euler product (see [27] for the details). Now, by the fact that a nonconstant L-function, as defined above, is actually of order equal to 1 (see, e.g., [27, p. 48]), which implies that for two L-functions F and G, the Dirichlet series $f = \frac{F}{G}$ (not an L-function in general) is of order at most 1 and thus from Corollary 2.3, the inequality (2.4) holds for $f = \frac{F}{G}$, which clearly improves the above mentioned result (2.6) of [6] to the following *global estimate for all large T:*

$$(2.7) \qquad D_{F,G}(T) > AT,$$

where $A > 0$ is a constant.

We mention that Bombieri and Perelli obtained in [5] that $D_{F,G}(T) \gg T \log T$ for a large class of L-functions, which is expected to hold for the Selberg class and would be optimal for this type of function.

Note from the above that the estimate (2.7) also holds for general Dirichlet series under much weaker conditions than those for L-functions. It also connects to another topic: uniqueness problems for L-functions, or, more generally, Dirichlet series, when F and G are identically equal in terms of the so-called shared values. This has recently been studied in various settings; see [7], [11], [13], [16], [18], [19], [22], [28], etc. From Corollary 2.3 and (2.7), we have the following.

Corollary 2.4. *Let F and G be two L-functions, or, more generally, let F, G be two general Dirichlet series so that F/G is uniformly convergent in a half-plane $\Re(s) > b$ and admits an analytic continuation in $\boldsymbol{C}$ as a meromorphic function of order < 2 with the limit equal to 1 as $\Re(s) \to +\infty$. Then $F \not\equiv G$ if and only if*

$$(2.8) \qquad D_{F,G}(T) > AT \quad or \quad n(T, 0; F/G) + n(T, \infty; F/G) > AT$$

for all large T, where $A > 0$ is a constant. Equivalently, $F \equiv G$ if and only if (2.8) does not hold.

The assumption in Corollary 2.4 that F/G tends to 1 as $\Re(s) \to +\infty$ together with $F \not\equiv G$ guarantees that F/G is nontrivial; otherwise, $F/G = ae^{-\lambda s}$ for some a, λ, from which it follows that $\lambda = 0$ and then $a = 1$ by letting $\Re(s) \to +\infty$ and thus $F \equiv G$, a contradiction. It is interesting to see that this limit condition is tied to the exceptional case in a more familiar form of uniqueness in terms of shared values (cf. Corollary 2.5 below).

If two L-functions F and G share the value 0, i.e., F and G have the same zeros with counting multiplicities, then $F \equiv G$ (see [28, p. 152], where the Ramanujan hypothesis is also required), which can now follow from Corollary 2.4 immediately since $D_{F,G}(T) = O(1)$ and thus (2.8) does not hold, and consequently $F \equiv G$. If two L-functions F and G share a complex number a, i.e., $F - a$ and $G - a$ have the same zeros with counting multiplicities, we may apply Corollary 2.4 to $F - a$ and $G - a$. Note that $\frac{F-a}{G-a}$ has the limit 1 as $\Re(s) \to +\infty$, except when $a = 1$, for which the limit might not be 1. We deduce from Corollary 2.4 the uniqueness that $F - a \equiv G - a$, i.e., $F \equiv G$, except the case $a = 1$, for which the result need not hold (compare the result in [28, p. 152]; see also [16]). In fact, the quantitative condition (2.8) can produce a quantitative condition for uniqueness, (see Corollary 2.5 below) in which we say that $F - a$ and $G - a$ have *"enough" common zeros (resp., poles)* if $F - a$ and $G - a$ have the same zeros (resp., poles) counted with multiplicity except an exceptional set E of their zeros (resp., poles) satisfying that $n(r, E) = o(r)$ as $r \to \infty$, where $n(r, E)$ is the counting function of E, i.e., the number of points in $E \cap \{|s| < r\}$ counted with multiplicity. From Corollary 2.4, we immediately have the following uniqueness theorem.

Corollary 2.5. *Let F and G be two general Dirichlet series and let $a \neq 1$ be a complex number so that $\frac{F-a}{G-a}$ is uniformly convergent in a half-plane $\Re(s) > b$ and admits an analytic continuation in $\boldsymbol{C}$ as a meromorphic function of order < 2 with the limit equal to 1 as $\Re(s) \to +\infty$. Then $F \equiv G$ if and only if $F - a$ and $G - a$ have "enough" common zeros and poles.*

In particular, if F and G are two L-functions and $a \neq 1$, then $F \equiv G$ if and only if $F - a$ and $G - a$ have "enough" common zeros.

The result need not hold when $a = 1$.

To see Corollary 2.5 need not hold when $a = 1$, consider the functions $F = 1 + \frac{2}{4^s}$ and $G = 1 + \frac{3}{9^s}$. Then F, G trivially satisfy all the sufficient conditions

of Corollary 2.5 with $a = 1$. Actually, F and G satisfy the functional equations $2^s L(s) = 2^{1-s}\overline{L(1-\bar{s})}$ and $3^s L(s) = 3^{1-s}\overline{L(1-\bar{s})}$, respectively (cf. [16]), and thus it is easy to see that F and G are also L-functions as defined above. But, $F \not\equiv G$.

We note that a connection (an equivalence) between the Riemann hypothesis and uniqueness of the Riemann zeta function $\zeta(s)$ and its analogue for L-functions has been given in [17].

3. Proof of Theorem 1.2

Throughout the proof, we will use ϵ to denote a positive constant which can be made arbitrarily small and C a positive constant, the actual values of which may vary from one occurrence to the next.

Recall the classic Poisson–Jensen formula (see, e.g., [30, p. 129], and [15, p. 1]): Let f be a meromorphic function in $|s| \leq R$. Then for each s in $|s| < R$ with $f(s) \neq 0, \infty$, we have

$$
(3.1) \qquad \log|f(s)| = \frac{1}{2\pi}\int_0^{2\pi}\Re\{\frac{Re^{i\phi}+s}{Re^{i\phi}-s}\}\log|f(Re^{i\phi})|d\phi
$$

$$
+ \sum_{k=1}^{M}\log|\frac{R(s-a_k)}{R^2-\bar{a}_k s}| - \sum_{k=1}^{N}\log|\frac{R(s-b_k)}{R^2-\bar{b}_k s}|,
$$

where $a_k(1 \leq k \leq M)$ and $b_k(1 \leq k \leq N)$ are the zeros and poles of f in $|s| < R$, respectively. In particular, if $s = 0$, we have the Jensen formula:

$$
(3.2) \quad \log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\phi})|d\phi + \sum_{k=1}^{M}\log|\frac{a_k}{R}| - \sum_{k=1}^{N}\log|\frac{b_k}{R}|.
$$

If $f(s) = 0$ or ∞, $\log|f(0)|$ is then replaced by $\log|a_m| + m\log R$, where $m = n(0,0;f) - n(0,\infty;f)$ and $a_m \neq 0$ is the coefficient in the Laurant or Taylor expansion $f = a_m s^m + \cdots$ of f at $s = 0$.

The underlying ideas of utilizing the Poisson–Jensen formula, which connects the modulus (and thus the growth) of a meromorphic function with its zeros and poles, can be traced to the relevant techniques in Nevanlinna theory (see [15, p. 36], [9], [8], etc.), while the present paper does not need familiarity with the theory. The Poisson–Jensen formula will be combined with several other tools in our proof.

We will make use of the following result (see [15, p .27], and [12, p. 56]), which can be proved directly by estimating the modulus of the infinite product below: Let $\{w_n\}$ be a sequence of nonzero complex numbers such that $\sum |w_n|^{-2}$ converges. Then the infinite product $\Pi(s) := \Pi_{n=1}^{\infty}(1 - \frac{s}{w_n})e^{\frac{s}{w_n}}$ is an entire function satisfying the following estimate:

$$
(3.3) \qquad \log|\Pi(s)|
$$

$$
\leq 4(2 + \log 2)\{|s|\int_0^{|s|}\frac{n(t,0,\Pi)}{t^2}dt + |s|^2\int_{|s|}^{+\infty}\frac{n(t,0,\Pi)}{t^3}dt\}.
$$

We will also employ the following result due to Cartan (see, e.g., [2, p. 360], and [21, p. 19]): Given any number $h > 0$ and complex numbers $a_1, a_2, \ldots, a_n$, there is a collection of disks in the complex plane with the sum of the radii equal to $2h$ such that for each point s lying outside these disks one has the inequality

$$
(3.4) \qquad \Pi_{k=1}^{n}|s - a_k| > (\frac{h}{e})^n.
$$

Recall also that an analytic function $f(s)$ is uniformly almost periodic (in the sense of Bohr) in a strip $\alpha < \Re(s) < \beta$ (α and β may be $\pm\infty$) if for every $\epsilon > 0$, the set of real numbers ω such that

$$|f(s + i\omega) - f(s)| < \epsilon, \quad \alpha < \Re(s) < \beta$$

is relatively dense, i.e., if for every $\epsilon > 0$ there is an $l > 0$ such that every interval of length l contains such a number ω (see, e.g., [3], [4]). If f is a nonconstant uniformly almost periodic function in a strip $\alpha < \Re(s) < \beta$ and if $f(s) = a$ has a solution in the strip, then we have that

$$(3.5) \qquad\qquad\qquad N(r, a; f) \geq cr$$

for some constant $c > 0$ and all large r. This follows from Rouché's theorem in a standard argument (see, e.g., [6], [10]). For completeness, we include its proof. In fact, suppose that s_0 is a solution of $f(s) = a$ in the strip and let ϵ be the minimum of $|f(s) - a|$ on the boundary of a disk containing s_0 and lying inside the strip with $f(s) \neq a$ on the boundary. Then by uniformly almost periodic property, there is a number $l > 0$ such that every interval of length l contains a τ such that $|f(s + i\tau) - f(s)| < \epsilon$ on the boundary. By Rouché's theorem, $f(s) - a$ and $f(s + i\tau) - a$ have the same number of zeros inside the disc, which clearly implies the inequality (3.5).

Let us now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. If the conclusion (1.5) holds, we have nothing to prove. Thus, it suffices to prove (1.4) when

$$(3.6) \qquad\qquad \int_{r_0}^{\infty} \frac{n(t, 0; f) + n(t, \infty; f)}{t^3} dt < \infty$$

for some $r_0 > 0$.

Suppose, to the contrary, that (1.4) does not hold. Then there exists an increasing sequence $\{r_n\}$ of positive numbers with $r_n \to +\infty$ such that for all n,

$$(3.7) \qquad\qquad \frac{n(r_n, 0; f) + n(r_n, \infty; f)}{r_n} < \frac{1}{n}.$$

We will arrive at a contradiction eventually.

By (3.6), for any given $\epsilon > 0$ we have that for large r,

$$\epsilon > \int_r^{2r} \frac{n(t, 0; f) + n(t, \infty; f)}{t^3} dt$$
$$\geq \frac{n(r, 0; f) + n(r, \infty; f)}{(2r)^3} r$$
$$= \frac{1}{8} \frac{n(r, 0; f) + n(r, \infty; f)}{r^2},$$

which implies that

$$(3.8) \qquad\qquad \frac{n(r, 0; f) + n(r, \infty; f)}{r^2} \to 0$$

as $r \to +\infty$.

Let a_k ($k = 1, 2, \dots$) be the nonzero zeros of f arranged with $|a_k| \leq |a_{k+1}|$ and let b_k ($k = 1, 2, \dots$) be the nonzero poles of f arranged with $|b_k| \leq |b_{k+1}|$. Then by

the Stieltjes integral and (3.8),

$$\sum_{k=1}^{\infty} |a_k|^{-2} = \int_0^{\infty} \frac{1}{t^2} d(n(t,0;f) - n(0,0;f))$$

$$= \lim_{r \to \infty} \left(\frac{n(r,0;f) - n(0,0;f)}{r^2} + 2 \int_0^r \frac{n(t,0;f) - n(0,0;f)}{t^3} dt \right)$$

$$= 2 \int_0^{\infty} \frac{n(t,0;f) - n(0,0;f)}{t^3} dt$$

converges by (3.6). One can then check (or see (3.3)) that the infinite product

$$f_1 = \Pi_{k=1}^{\infty} (1 - \frac{s}{a_k}) e^{\frac{s}{a_k}}$$

is an entire function having a_k's as its zeros. Similarly, the infinite product

$$f_2 = \Pi_{k=1}^{\infty} (1 - \frac{s}{b_k}) e^{\frac{s}{b_k}}$$

is an entire function having b_k's as its zeros. Therefore, we have the following factorization of f:

$$f(s) = s^m \frac{f_1(s)}{f_2(s)} e^{Q(s)},$$

where m is an integer, which is the order of zero or pole of f at $s = 0$, and Q is a polynomial of degree $\deg(Q) \le \rho(f)$, where $\rho(f)$ denotes the order of f.

Fix a small positive number τ (to be specified later). Define the operator Λ:

$$\Lambda f(s) = \frac{f(s+\tau)}{f(s)}, \quad \Lambda^k = \Lambda(\Lambda^{k-1}).$$

It is clear that $\Lambda e^{Q(s)} = e^{Q(z+\tau)-Q(z)}$ and $Q(z+\tau) - Q(z)$ is a polynomial of degree at most $\deg(Q) - 1$. Applying the operator Λ again, we see that $\Lambda^2 e^{Q(s)} = e^{Q_1(s)}$, where $Q_1(s)$ is a polynomial of degree at most $\deg(Q) - 2$. Applying Λ repeatedly for d_0 times, we obtain that $\Lambda^{d_0} e^{Q(s)} = c_0$, a nonzero constant, where $d_0 = \deg(Q)$. Set $d = d_0 + 1 \ge 1$. We thus obtain, in view of the above factorization of f, that

$$(3.9) \qquad F(s) := \Lambda^d f(s) = \frac{\Lambda^d f_1(s)}{\Lambda^d f_2(s)} R(s),$$

where $R(s) = \Lambda^d s^m$ is a nonzero rational function. (It is worth mentioning that $d \ge 1$, so we apply the operator Λ on f at least once even when the polynomial Q is a constant.)

By the given condition, $f(s)$ is nonconstant and tends to a nonzero finite limit, denoted by A_0, as $\Re(s) \to +\infty$. Thus, $\Lambda f(s)$ tends to 1 as $\Re(s) \to +\infty$. This implies that $\Lambda f(s)$ cannot be a constant function, since otherwise it must be identically equal to 1; that is, $f(s+\tau) \equiv f(s)$, a periodic function. But, f is bounded when $\Re(s) \to +\infty$. Hence, f is bounded in the complex plane (for any $\sigma_0 + it_0$, $|f(\sigma + it_0)| \le |A_0| + 1$ for large σ and thus for all σ, particularly σ_0, by the periodicity) and consequently must be a constant by Liouville's theorem, a contradiction. This same argument (cf. [6]) shows that $\Lambda^2 f(s), \ldots, \Lambda^d f(s)$ are all nonconstant. In particular, $F(s) = \Lambda^d f(s)$ is nonconstant. Note that an exponential series of the form $\sum_{n=1}^{\infty} a_n e^{\lambda_n s}$ uniformly convergent in a strip $\alpha < \Re(s) < \beta$ (α and β may be $\pm\infty$) must be uniformly almost periodic in the strip (see, e.g., [3, Theorem 6, Ch. III]). Thus, the Dirichlet series f is uniformly almost periodic in the half-plane

$\Re(s) > b$. It is then easy to verify directly that Λf, as the quotient of $f(s+\tau)$ and $f(s)$, which tend to the nonzero finite limit A_0 as $\Re(s) \to +\infty$, is also uniformly almost periodic in a right half-plane. In fact, we have that $\frac{|A_0|}{2} < |f(s)| < 2|A_0|$ and $\frac{|A_0|}{2} < |f(s+\tau)| < 2|A_0|$ for large $\Re(s)$. For any $\epsilon > 0$, there is an $l > 0$ such that every interval of length l contains a ω such that $|f(s+i\omega) - f(s)| < \epsilon$ for $\Re(s) > b$. Thus,

$$
\begin{aligned}
&|\Lambda f(s+i\omega) - \Lambda f(s)| \\
&= |\frac{f(s+\tau+i\omega)}{f(s+i\omega)} - \frac{f(s+\tau)}{f(s)}| \\
&= |\frac{\big(f(s+\tau+i\omega) - f(s+\tau)\big)f(s) + f(s+\tau)(f(s) - f(s+i\omega))}{f(s+i\omega)f(s)}| \\
&\leq \frac{4|A_0|\epsilon}{(\frac{|A_0|}{2})^2} = \frac{16\epsilon}{|A_0|}
\end{aligned}
$$

for large $\Re(s)$. This shows that $\Lambda f(s)$ is uniformly almost periodic in a right half-plane. The same argument shows that $\Lambda^2 f(s), \dots, \Lambda^d f(s)$ are all uniformly almost periodic in a right half-plane; in particular, $F(s) = \Lambda^d f(s)$ is uniformly almost periodic in a right half-plane. Take a point w_0 in this right half-plane and denote $a = F(w_0)$. Then by virtue of (3.5), there exists a constant $c > 0$ such that for large r,

$$
\text{(3.10)} \qquad\qquad N(r, a, F) \geq cr.
$$

We will establish a tight estimate on $N(r, a, F)$ in terms of the zeros of f_1 and f_2 and thus the zeros and poles of f. Fix any $r > e$. By the Jensen formula (3.2), we have that

$$
\begin{aligned}
&\log |F(0) - a| \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log |F(re^{i\theta}) - a| d\theta + \sum_{|\alpha_k| < r} \log \frac{|\alpha_k|}{r} - \sum_{|\beta_k| < r} \log \frac{|\beta_k|}{r},
\end{aligned}
$$

where α_k are the nonzero zeros of $F - a$ and β_k are the nonzero poles of $F - a$ (and thus F) in $|s| < r$. Here, if $F(0) = a$ or ∞, the term $\log |F(0) - a|$ needs to be replaced by $c_0 + c_1 \log r$ for some constants c_0 and c_1. Noting that

$$
\begin{aligned}
\sum_{|\alpha_k| < r} \log \frac{r}{|\alpha_k|} &= \int_0^r \log \frac{r}{t} d(n(t, a; F) - n(0, a; F)) \\
&= \int_0^r \frac{n(t, a; F) - n(0, a; F)}{t} dt \\
\text{(3.11)} \qquad &= N(r, a; F) - n(0, a; F) \log r
\end{aligned}
$$

and, similarly,

$$\sum_{|\beta_k|<r} \log \frac{r}{|\beta_k|} = N(r, \infty; F) - n(0, \infty; F) \log r,$$

we deduce that

$$N(r, a; F) \leq \frac{1}{2\pi} \int_0^{2\pi} \log |F(re^{i\theta}) - a| d\theta + N(r, \infty; F) + C \log r$$

$$= \frac{1}{2\pi} \int_{|F(re^{i\theta})| \leq |a|} \log |F(re^{i\theta}) - a| d\theta + \frac{1}{2\pi} \int_{|F(re^{i\theta})| > |a|} \log |F(re^{i\theta}) - a| d\theta$$
$$+ N(r, \infty; F) + C \log r$$

$$\leq \frac{1}{2\pi} \int_{|F(re^{i\theta})| \leq |a|} \log(2|a|) d\theta + \frac{1}{2\pi} \int_{|F(re^{i\theta})| > |a|} \log(2|F(re^{i\theta})|) d\theta$$
$$+ N(r, \infty; F) + C \log r$$

$$\leq \frac{1}{2\pi} \int_E \log |F(re^{i\theta})| d\theta + N(r, \infty; F) + C \log r$$

$$\leq \frac{1}{2\pi} \int_E \log |\frac{\Lambda^d f_1(re^{i\theta})}{\Lambda^d f_2(re^{i\theta})}| d\theta + N(r, \infty; F) + C \log r,$$

where E denotes the set of θ such that $|F(re^\theta)| > |a|$, in view of (3.9) and the fact that

$$\int_E \log |R(re^\theta)| d\theta = O(\log r)$$

since R is a rational function and thus $\log |R(re^\theta)| = O(\log r)$. By the definition of the operator Λ, it is easy to check that $\frac{\Lambda^d f_1(s)}{\Lambda^d f_2(s)}$ is a product of finitely many quotients of the form

$$\frac{f_1(s+\eta+\tau)}{f_1(s+\eta)}, \frac{f_1(s+\eta)}{f_1(s+\eta+\tau)}, \frac{f_2(s+\eta+\tau)}{f_2(s+\eta)}, \frac{f_2(s+\eta)}{f_2(s+\eta+\tau)},$$

where $\eta = k\tau$ and k is an integer with $0 \leq k \leq d-1$, i.e., there are finite sets I_j $(1 \leq j \leq 4)$, whose elements are of the form $k\tau$ with $0 \leq k \leq d-1$, such that

$$\frac{\Lambda^d f_1(s)}{\Lambda^d f_2(s)}$$
$$= \Pi_{\eta \in I_1} \frac{f_1(s+\eta+\tau)}{f_1(s+\eta)} \Pi_{\eta \in I_2} \frac{f_1(s+\eta)}{f_1(s+\eta+\tau)} \Pi_{\eta \in I_3} \frac{f_2(s+\eta+\tau)}{f_2(s+\eta)} \Pi_{\eta \in I_4} \frac{f_2(s+\eta)}{f_2(s+\eta+\tau)}.$$

(Actually, $I_4 = I_1$ and $I_3 = I_2$.) Thus, we obtain that with $s = re^{i\theta}$,

$$N(r, a, F) \leq \sum_{\eta \in I_1} \frac{1}{2\pi} \int_E \log |\frac{f_1(s+\eta+\tau)}{f_1(s+\eta)}| d\theta$$

$$+ \sum_{\eta \in I_2} \frac{1}{2\pi} \int_E \log |\frac{f_1(s+\eta)}{f_1(s+\eta+\tau)}| d\theta + \sum_{\eta \in I_3} \frac{1}{2\pi} \int_E \log |\frac{f_2(s+\eta+\tau)}{f_2(s+\eta)}| d\theta$$

$$(3.12) \quad + \sum_{\eta \in I_4} \frac{1}{2\pi} \int_E \log |\frac{f_2(s+\eta)}{f_2(s+\eta+\tau)}| d\theta + N(r, \infty; F) + C \log r.$$

We now estimate each term in (3.12). To estimate $N(r, \infty; F)$, we note that the poles of F come from the zeros of $f_1(s + k\tau)$ or $f_2(s + k\tau)$ or poles of $R(s)$, where

k is an integer with $0 \leq k \leq d$. When $|s| \leq r$, all these zeros lie within $|s| \leq r + d\tau$. Thus,

$$n(r, 0; f_1(s + k\tau)) \leq n(r + d\tau, 0; f_1(s)),$$
$$n(r, 0; f_2(s + k\tau)) \leq n(r + d\tau, 0; f_2(s)),$$

which clearly implies that

$$n(r, \infty, F) \leq C\big(n(r + d\tau, 0; f_1) + n(r + d\tau, 0; f_2) + 1\big)$$

for all $r > 0$, where $C > 0$ is a constant. (The last term $+1$ in the above inequality comes from the possible poles of $R(s)$.) We may take a small τ in the beginning so that f_1, f_2 do not vanish in $|s| \leq 2d\tau$ (such a τ clearly exists since $f_1(0)f_2(0) \neq 0$ by the definitions of f_1 and f_2). Then, for a fixed sufficiently small $\delta > 0$,

$$
\begin{aligned}
N(r, \infty, F) &= \int_0^r \frac{n(t, \infty; F) - n(0, \infty; F)}{t} dt + n(0, \infty; F) \log r \\
&= \int_\delta^r \frac{n(t, \infty; F) - n(0, \infty; F)}{t} dt + n(0, \infty; F) \log r \\
&\leq \int_\delta^r \frac{C(n(t + d\tau, 0; f_1) + n(t + d\tau, 0; f_2) + 1)}{t} dt + n(0, \infty; F) \log r \\
&\leq C\Big(\int_0^r \frac{n(t + d\tau, 0; f_1) + n(t + d\tau, 0; f_2)}{t} dt + \log r\Big) \\
&= C\Big(\int_{d\tau}^{r+d\tau} \frac{n(t, 0; f_1) + n(t, 0; f_2)}{t - d\tau} dt + \log r\Big) \\
&= C\Big(\int_{2d\tau}^{r+d\tau} \frac{n(t, 0; f_1) + n(t, 0; f_2)}{t}(1 + \frac{d\tau}{t - d\tau}) dt + \log r\Big) \\
&\leq C\big(N(r + d\tau, 0; f_1) + N(r + d\tau, 0; f_2) + \log r\big) \\
(3.13) \qquad &\leq C\big(N(r + d\tau, 0; f) + N(r + d\tau, \infty; f) + \log r\big).
\end{aligned}
$$

Next we estimate $\frac{1}{2\pi} \int_E \log |\frac{f_1(s+\eta+\tau)}{f_1(s+\eta)}| d\theta$ in (3.12); the same estimate for it given below applies to each of the integrals in (3.12). To this end, we use the Poisson–Jensen formula (3.1) for the entire function f_1 in $|s| < R$, where R is a large number to be determined later, but large enough at this moment so that the considered points $s = re^{i\theta}$ and $s + k\tau$ $(1 \leq k \leq d)$ all lie within $|s| < R$, i.e., $R > r + d\tau$. For all these $s = re^{i\theta}$, except those with $f_1(s + \eta) = 0$ or $f_1(s + \eta + \tau) = 0$, which constitute a finite set and do not affect the involved integrals below, we have that

$$
\begin{aligned}
&\log |f_1(s + \eta)| \\
&= \frac{1}{2\pi} \int_0^{2\pi} \Re\{\frac{Re^{i\phi} + s + \eta}{Re^{i\phi} - (s + \eta)}\} \log |f_1(Re^{i\phi})| d\phi \\
&\quad + \sum_{k=1}^M \log |\frac{R(s + \eta - a_k)}{R^2 - \bar{a_k}(s + \eta)}|
\end{aligned}
$$

and that

$$
\begin{aligned}
\log |f_1(s+\eta+\tau)| \\
= \frac{1}{2\pi} \int_0^{2\pi} \Re\{\frac{Re^{i\phi}+s+\eta+\tau}{Re^{i\phi}-(s+\eta+\tau)}\} \log |f_1(Re^{i\phi})| d\phi \\
+ \sum_{k=1}^{M} \log |\frac{R(s+\eta+\tau-a_k)}{R^2-\bar{a_k}(s+\eta+\tau)}|,
\end{aligned}
$$

where a_k $(k=1,2\ldots,M)$ are the zeros of f_1 in $|s| < R$. Observing that

$$
(3.14) \qquad |\frac{R^2-\bar{a_k}s}{R(s-a_k)}| \geq 1
$$

for any $|s| \leq R$, we deduce that

$$
\begin{aligned}
\log |\frac{f_1(s+\eta+\tau)}{f_1(s+\eta)}| \\
= \frac{1}{2\pi} \int_0^{2\pi} \Re\{\frac{Re^{i\phi}+s+\eta+\tau}{Re^{i\phi}-(s+\eta+\tau)} - \frac{Re^{i\phi}+s+\eta}{Re^{i\phi}-(s+\eta)}\} \log |f_1(Re^{i\phi})| d\phi \\
+ \sum_{k=1}^{M} \log |\frac{R(s+\eta+\tau-a_k)}{R^2-\bar{a_k}(s+\eta+\tau)}| - \sum_{k=1}^{M} \log |\frac{R(s+\eta-a_k)}{R^2-\bar{a_k}(s+\eta)}| \\
\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{2\tau R}{(R-r-d\tau)^2} |\log |f_1(Re^{i\phi})|| d\phi \\
+ \sum_{k=1}^{M} \log |\frac{R^2-\bar{a_k}(s+\eta)}{R(s+\eta-a_k)}|.
\end{aligned}
$$

$$(3.15)$$

It is easy to check that for any $x > 0$, $|\log x| = 2\log^+ x - \log x$, where $\log^+ x = \max\{\log x, 0\}$. Thus,

$$
\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} |\log |f_1(Re^{i\phi})|| d\phi \\
= \frac{1}{\pi} \int_0^{2\pi} \log^+ |f_1(Re^{i\phi})| d\phi - \frac{1}{2\pi} \int_0^{2\pi} \log |f_1(Re^{i\phi})| d\phi \\
\leq \frac{1}{\pi} \int_0^{2\pi} \log^+ |f_1(Re^{i\phi})| d\phi - \log |f_1(0)|
\end{aligned}
$$

$$(3.16)$$

by virtue of Jensen formula (3.2) for the last inequality. We further estimate the last integral in (3.16) using the tight estimate provided by (3.3). Applying this estimate to f_1 and in view of (3.6) and (3.8), we obtain that for any given $\epsilon > 0$,

$$
\begin{aligned}
\log^+ |f_1(s)| \leq 4(2+\log 2)\{|s| \int_0^{|s|} \frac{n(t,0,f_1)}{t^2} dt + |s|^2 \int_{|s|}^{+\infty} \frac{n(t,0,f_1)}{t^3} dt\} \\
\leq \epsilon |s|^2
\end{aligned}
$$

for large $|s|$. This together with (3.16) yields that

$$
\frac{1}{2\pi} \int_0^{2\pi} |\log |f_1(Re^{i\phi})|| d\phi \leq \epsilon R^2
$$

for large R. We then deduce from (3.15) that with $s = re^{i\theta}$,

(3.17)
$$\frac{1}{2\pi} \int_E \log \left| \frac{f_1(s+\eta+\tau)}{f_1(s+\eta)} \right| d\theta$$

$$\leq \frac{2\tau\epsilon R^3}{(R-r-d\tau)^2} + \sum_{k=1}^{M} \frac{1}{2\pi} \int_E \log \left| \frac{R^2 - \bar{a}_k(s+\eta)}{R(s+\eta-a_k)} \right| d\theta.$$

To estimate the summation in the above inequality, set $g(s) := \frac{R^2 - \bar{a}_k(s+\eta)}{R(s+\eta-a_k)}$. We notice by (3.14) that $|g(s)| \geq 1$ in $|s| \leq R$, and thus g has no zeros in $|s| \leq r < R$. Also, g has only one pole $a_k - \eta$ in $|s| \leq r$ if $|a_k - \eta| \leq r$ and has no poles in $|s| \leq r$ if $|a_k - \eta| > r$. Hence, applying Jensen formula (3.2) to $g(s)$ in $|s| \leq r$, we obtain that

$$\log |g(0)| = \log \left| \frac{R^2 - \bar{a}_k\eta}{R(\eta - a_k)} \right|$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{R^2 - \bar{a}_k(re^{i\theta} + \eta)}{R(re^{i\theta} + \eta - a_k)} \right| d\theta + \log^+ \frac{r}{|a_k - \eta|}$$

and thus, noting (3.14) again, that with $s = re^{i\theta}$,

$$\sum_{k=1}^{M} \frac{1}{2\pi} \int_E \log \left| \frac{R^2 - \bar{a}_k(s+\eta)}{R(s+\eta-a_k)} \right| d\theta$$

$$\leq \sum_{k=1}^{M} \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{R^2 - \bar{a}_k(s+\eta)}{R(s+\eta-a_k)} \right| d\theta$$

$$= \sum_{k=1}^{M} \log \left| \frac{R^2 - \bar{a}_k\eta}{R(\eta - a_k)} \right| - \sum_{k=1}^{M} \log^+ \frac{r}{|a_k - \eta|}$$

$$\leq \sum_{k=1}^{M} \log \frac{R + d\tau}{|\eta - a_k|}.$$

Since the zeros of $f_1(s)$ are a_k $(k = 1, 2, \ldots)$, the zeros of $f_1(s + \eta)$ are $a_k - \eta$ $(k = 1, 2, \ldots)$. Note also that when $|a_k| \leq R$, $|a_k - \eta| \leq R + |\eta| \leq R + d\tau$ by the definition of η. Thus, we deduce, in view of (3.11) and the fact that f_1 does not

vanish in $|s| < 2d\tau$, that

$$\sum_{k=1}^{M} \log \frac{R+d\tau}{|\eta - a_k|}$$

$$\leq \sum_{|a_k - \eta| \leq R+d\tau} \log \frac{R+d\tau}{|a_k - \eta|}$$

$$= \int_0^{R+d\tau} \frac{n(t,0;f_1(s+\eta)) - n(0,0;f_1(s+\eta))}{t} dt$$

$$\leq \int_0^{R+d\tau} \frac{n(t+d\tau,0;f_1(s))}{t} dt$$

$$= \int_{d\tau}^{R+2d\tau} \frac{n(t,0;f_1(s))}{t-d\tau} dt$$

$$= \int_{2d\tau}^{R+2d\tau} \frac{n(t,0;f_1(s))}{t}(1 + \frac{d\tau}{t-d\tau}) dt$$

$$\leq 2N(R+2d\tau,0;f_1).$$

This together with (3.17) yields that with $s = re^{i\theta}$,

$$\frac{1}{2\pi} \int_E \log |\frac{f_1(s+\eta+\tau)}{f_1(s+\eta)}| d\theta$$

$$\leq \frac{2\tau\epsilon R^3}{(R-r-d\tau)^2} + 2N(R+2d\tau,0;f_1).$$

This same estimate holds for all the integrals in (3.12) by the same argument. Thus, in view of (3.10), (3.12), and (3.13) we obtain that for a small number $\epsilon > 0$ and a constant $C > 0$,

$$(3.18) \quad cr \leq N(r,a,F)$$

$$\leq C\Big(\frac{\epsilon R^3}{(R-r-d\tau)^2} + N(R+2d\tau,0;f) + N(R+2d\tau,\infty;f) + \log r\Big)$$

for $r > e$ and large $R > r + d\tau$, where $c > 0$ is the constant in (3.10).

Next, we bound the right-hand side of estimate (3.18) in terms of $n(r,0;f) + n(r,\infty;f)$ so that we can make use of assumption (3.7) to arrive at the desired contradiction mentioned in the beginning of the proof. (It would be tempting to try to use the definition of the integrated counting function to bound $N(r,0;f) + N(r,\infty;f)$ using $n(r,0;f) + n(r,\infty;f)$ directly, but it would only yield the rough estimate that $N(r,0;f) + N(r,\infty;f) \leq \big(n(r,0;f) + n(r,\infty;f)\big)\log r + O(\log r)$, which does not serve our purpose.) To this end, fix a large positive number R_1. Let $a_1, a_2, \ldots, a_p$ be the zeros and $b_1, b_2, \ldots, b_q$ be the poles of f in $|s| \leq R_1$. Thus, $p = n(R_1,0;f)$ and $q = n(R_1,\infty,f)$. Applying the Cartan theorem (see (3.4)) to the zeros and poles of f, respectively, with $h = \frac{1}{32}R_1$, we obtain a collection $\mathcal{D}$ of the disks with the sum of the radii equal to $4h = \frac{1}{8}R_1$ such that for all $z \notin \mathcal{D}$ we have

$$(3.19) \quad \Pi_{k=1}^p |s - a_k| > (\frac{h}{e})^p$$

and

$$\Pi_{k=1}^q |s - b_k| > (\frac{h}{e})^q.$$

Since the sum of the radii of the disks in $\mathcal{D}$ is $\frac{1}{8}R_1$, it is easy to check that those disks in $\mathcal{D}$ cannot fill up the annulus $\mathcal{G}$: $\frac{1}{16}R_1 \leq |s| \leq \frac{1}{4}R_1$. Therefore, there must be a point $s_0 \in \mathcal{G}$ outside all the above disks, i.e., $s_0 \notin \mathcal{D}$ with $f(s_0) \neq 0, \infty$. Note that the estimate (3.18) was obtained for any f that satisfies the conditions of Theorem 1.2 and the assumption (3.6), which are satisfied by $f(s + s_0)$ as well. Therefore, applying the proved estimate (3.18) to $f(s+s_0)$, we obtain that for large R and r with $R > r + d\tau$,

$$cr \leq N(r, a, F(s + s_0))$$
$$\leq C\Big(\frac{\epsilon R^3}{(R - r - d\tau)^2} + N(R + 2d\tau, 0; f(s + s_0))$$
(3.20)
$$+N(R + 2d\tau, \infty; f(s + s_0)) + \log r\Big).$$

Note that $f(s)$ vanishes at a_k if and only if $f(s_0 + s)$ vanishes at $a_k - s_0$. Denote $\rho = \frac{3}{4}R_1$. We have, in view of (3.11), that

$$N(\rho, 0; f(s_0 + s))$$
$$= \sum_{|a_k - s_0| \leq \rho} \log \frac{\rho}{|a_k - s_0|}$$
$$= \log \frac{\rho^{n_1}}{\Pi_{|a_k - s_0| \leq \rho} |a_k - s_0|}$$
$$= \log \frac{\rho^{n_1} \Pi_{|a_k| \leq R_1, |a_k - s_0| > \rho} |a_k - s_0|}{\Pi_{|a_k| \leq R_1} |a_k - s_0|},$$

where n_1 is the number of the points a_k satisfying $|a_k - s_0| \leq \rho$; these a_k's satisfy that $|a_k| \leq \rho + |s_0| \leq \frac{3}{4}R_1 + \frac{1}{4}R_1 = R_1$. Let n_2 be the number of a_k's satisfying that $|a_k - s_0| > \rho$ and $|a_k| \leq R_1$. It is then clear that $n_1 + n_2 = p = n(R_1, 0; f)$. Thus, by virtue of (3.19) and the fact that $|a_k - s_0| \leq \frac{5}{4}R_1$ for any $k \leq p$, we deduce that

$$N(\rho, 0; f(s_0 + s))$$
$$\leq \log \frac{\rho^{n_1} (2R_1)^{n_2}}{(\frac{R_1}{32e})^p}$$
$$\leq \log \frac{(2R_1)^p}{(\frac{R_1}{32e})^p} = n(R_1, 0; f) \log(64e).$$

In the exactly same way, we also have that

$$N(\rho, \infty; f(s_0 + s)) \leq n(R_1, \infty; f) \log(64e).$$

Thus, we have that

$$N(\frac{3}{4}R_1, 0; f(s_0 + s)) + N(\frac{3}{4}R_1, \infty; f(s_0 + s))$$
(3.21)
$$\leq \big(n(R_1, 0; f) + n(R_1, \infty; f)\big) \log(64e).$$

Fix a large n. Take $R_1 = r_n$, where $\{r_n\}$, which tends to $+\infty$, is the sequence in (3.7). Then, take $r = \frac{1}{4}r_n$ and $R = \frac{3}{4}r_n - 2d\tau$ in (3.20). Combining (3.20), (3.21),

and (3.7) yields that

$$cr_n \leq C\big(\frac{\epsilon(\frac{3}{4}r_n - 2d\tau)^3}{(\frac{1}{2}r_n - 3d\tau)^2} + (n(r_n, 0; f) + n(r_n, \infty; f))\log(64e) + \log r_n\big)$$
$$\leq C(\epsilon + \frac{1}{n})r_n,$$

where $c, C > 0$ are fixed constants, $\epsilon > 0$ can be made arbitrarily small, and n can be made arbitrarily large, which is clearly impossible. This finally completes the proof of the theorem. $\qquad\square$

ACKNOWLEDGMENT

REFERENCES

[1] Tom M. Apostol, *Modular functions and Dirichlet series in number theory*, 2nd ed., Graduate Texts in Mathematics, vol. 41, Springer-Verlag, New York, 1990. MR1027834

[2] Carlos A. Berenstein and Roger Gay, *Complex variables*, An introduction, Graduate Texts in Mathematics, vol. 125, Springer-Verlag, New York, 1991. MR1107514

[3] A. S. Besicovitch, *Almost periodic functions*, Dover Publications, Inc., New York, 1955. MR0068029

[4] Harald Bohr, *Almost Periodic Functions*, Chelsea Publishing Company, New York, N.Y., 1947. MR0020163

[5] E. Bombieri and A. Perelli, *Distinct zeros of L-functions*, Acta Arith. **83** (1998), no. 3, 271–281. MR1611193

[6] Enrico Bombieri and Alberto Perelli, *Zeros and poles of Dirichlet series* (English, with English and Italian summaries), Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **12** (2001), 69–73 (2002). MR1898450

[7] Matthew Cardwell and Zhuan Ye, *A uniqueness theorem of L-functions with rational moving targets*, J. Math. Anal. **5** (2014), no. 1, 16–19. MR3199389

[8] Yik-Man Chiang and Shao-Ji Feng, *On the growth of logarithmic differences, difference quotients and logarithmic derivatives of meromorphic functions*, Trans. Amer. Math. Soc. **361** (2009), no. 7, 3767–3791, DOI 10.1090/S0002-9947-09-04663-7. MR2491899

[9] Chong Ji Dai, David Drasin, and Bao Qin Li, *On the growth of entire and meromorphic functions of infinite order*, J. Analyse Math. **55** (1990), 217–228, DOI 10.1007/BF02789202. MR1094716

[10] H. Davenport and H. Heilbronn, On the zeros of certain Dirichlet series, J. London Math. Soc., 11(1936), 181-185; Collected Works, vol. IV, Academic Press, 1977, 1774-1779.

[11] Rui Gao and Bao Qin Li, *An answer to a question on value distribution of the Riemann zeta-function*, Internat. J. Math. **23** (2012), no. 4, 1250044, 9, DOI 10.1142/S0129167X12500449. MR2903194

[12] Anatoly A. Goldberg and Iossif V. Ostrovskii, *Value distribution of meromorphic functions*, Translations of Mathematical Monographs, vol. 236, American Mathematical Society, Providence, RI, 2008. Translated from the 1970 Russian original by Mikhail Ostrovskii. With an appendix by Alexandre Eremenko and James K. Langley. MR2435270

[13] Steven M. Gonek, Jaeho Haan, and Haseo Ki, *A uniqueness theorem for functions in the extended Selberg class*, Math. Z. **278** (2014), no. 3-4, 995–1004, DOI 10.1007/s00209-014-1343-1. MR3278902

[14] G. H. Hardy and M. Riesz, *The general theory of Dirichlet's series*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 18, Stechert-Hafner, Inc., New York, 1964. MR0185094

[15] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964. MR0164038

[16] Pei-Chu Hu and Bao Qin Li, *A simple proof and strengthening of a uniqueness theorem for L-functions*, Canad. Math. Bull. **59** (2016), no. 1, 119–122, DOI 10.4153/CMB-2015-045-1. MR3451903

[17] Pei-Chu Hu and Bao Qin Li, A connection between the Riemann hypothesis and uniqueness of the Riemann zeta function, manuscript.

[18] Haseo Ki, *A remark on the uniqueness of the Dirichlet series with a Riemann-type function equation*, Adv. Math. **231** (2012), no. 5, 2484–2490, DOI 10.1016/j.aim.2012.07.027. MR2970456

[19] Haseo Ki and Bao Qin Li, *On uniqueness in the extended Selberg class of Dirichlet series*, Proc. Amer. Math. Soc. **141** (2013), no. 12, 4169–4173, DOI 10.1090/S0002-9939-2013-11749-1. MR3105859

[20] C. G. Lekkerkerker, *On the zeros of a class of Dirichlet series*, Van Gorcum & Comp. N.V., Assen, 1955. MR0069265

[21] B. Ja. Levin, *Distribution of zeros of entire functions*, Revised edition, Translations of Mathematical Monographs, vol. 5, American Mathematical Society, Providence, R.I., 1980. Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman. MR589888

[22] Bao Qin Li, *A uniqueness theorem for Dirichlet series satisfying a Riemann type functional equation*, Adv. Math. **226** (2011), no. 5, 4198–4211, DOI 10.1016/j.aim.2010.12.001. MR2770446

[23] S. Mandelbrojt, *Dirichlet series*, Principles and methods, D. Reidel Publishing Co., Dordrecht, 1972. MR0435370

[24] Maruti Ram Murty and Vijaya Kumar Murty, *Strong multiplicity one for Selberg's class* (English, with English and French summaries), C. R. Acad. Sci. Paris Sér. I Math. **319** (1994), no. 4, 315–320. MR1289304

[25] K. Ramachandra, *On the zeros of a class of generalized Dirichlet series*, J. Reine Angew. Math. **273** (1975), 31–40; addendum, ibid. 273 (1975), 60, DOI 10.1515/crll.1975.273.31. MR0369285

[26] K. Ramachandra, *On the zeros of a class of generalised Dirichlet series. VII*, Ann. Acad. Sci. Fenn. Ser. A I Math. **16** (1991), no. 2, 391–397, DOI 10.5186/aasfm.1991.1620. MR1139805

[27] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, In: E. Bombieri et al. (eds.), Proc. Amalfi Conf. Analytic Number Theory, Universit'a di Salerno 1992, 367-385; Collected Papers, vol. II, Springer-Verlag, 1991, 47-63.

[28] Jörn Steuding, *Value-distribution of L-functions*, Lecture Notes in Mathematics, vol. 1877, Springer, Berlin, 2007. MR2330696

[29] E. C. Titchmarsh, *The theory of functions*, Oxford University Press, Oxford, 1958. Reprint of the second (1939) edition. MR3155290

[30] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., Edited and with a preface by D. R. Heath-Brown, The Clarendon Press, Oxford University Press, New York, 1986. MR882550

Department of Mathematics and Statistics, Florida International University, Miami, Florida 33199

Email address: libaoqin@fiu.edu