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THE DANCING METRIC, G_2-SYMMETRY AND PROJECTIVE ROLLING

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ABSTRACT. The "dancing metric" is a pseudo-Riemannian metric $\mathbf{g}$ of signature (2,2) on the space M^4 of non-incident point-line pairs in the real projective plane $\mathbb{RP}^2$. The null curves of $(M^4, \mathbf{g})$ are given by the "dancing condition": *at each moment, the point is moving towards or away from the point on the line about which the line is turning.* This is the standard homogeneous metric on the pseudo-Riemannian symmetric space $\mathrm{SL}_3(\mathbb{R})/\mathrm{GL}_2(\mathbb{R})$, also known as the "para-Kähler Fubini-Study metric", introduced by P. Libermann. We establish a dictionary between classical projective geometry (incidence, cross ratio, projective duality, projective invariants of plane curves, etc.) and pseudo-Riemannian 4-dimensional conformal geometry (null curves and geodesics, parallel transport, self-dual null 2-planes, the Weyl curvature, etc.). Then, applying a twistor construction to $(M^4, \mathbf{g})$, a G_2-symmetry is revealed, hidden deep in classical projective geometry. To uncover this symmetry, one needs to refine the "dancing condition" to a higher-order condition. The outcome is a correspondence between curves in the real projective plane and its dual, a projective geometric analog of the more familiar "rolling without slipping and twisting" for a pair of Riemannian surfaces.

CONTENTS

1. INTRODUCTION

Let us consider the following system of 4 ordinary differential equations for 6 unknown functions $p_1, p_2, p_3, q^1, q^2, q^3$ of the variable t,

$$p_i \frac{dq^i}{dt} = 0, \quad \frac{dp_i}{dt} = \epsilon_{ijk} q^j \frac{dq^k}{dt}, \quad i = 1, 2, 3.$$

(We are using the summation convention for repeated indices and the symbol ϵ_{ijk}, equal to 1 for an even permutation ijk of 123, -1 for an odd permutation, and 0 otherwise.)

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It is convenient to recast these equations in vector form by introducing the notation

$$\mathbf{q} = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix} \in \mathbb{R}^3, \quad \mathbf{p} = (p_1, p_2, p_3) \in (\mathbb{R}^3)^*.$$

Then, using the standard scalar and cross-product of vector calculus (and omitting the dot product symbol), the above system can be written more compactly as

$$\mathbf{p}\mathbf{q}' = 0, \quad \mathbf{p}' = \mathbf{q} \times \mathbf{q}'. \tag{1}$$

This simple-looking system of 4 ordinary differential equations for 6 unknown functions enjoys a number of remarkable properties and interpretations, forming new links between old subjects. The main themes are:

- generic (non-integrable) rank 2 distributions on 5-manifolds,
- 4-dimensional pseudo-Riemannian conformal geometry of split-signature,
- projective differential geometry of plane curves.

These themes are not new (the 1st and 3rd are over a century old), but the relations between them are new, which we believe is the main contribution of this article. We shall now give a brief review of these themes and the relations we establish between them in this paper.

1.1. **Summary of main results.**

1.1.1. *The Cartan-Engel* $(2,3,5)$*-distribution and its symmetries.* Geometrically, equations (1) define at each point $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^6$, away from a "small" subset, a 2-dimensional subspace $\mathcal{D}_{(\mathbf{q},\mathbf{p})} \subset T_{(\mathbf{q},\mathbf{p})}\mathbb{R}^6$. Put together, these subspaces define (generically) a rank 2 distribution $\mathcal{D} \subset T\mathbb{R}^6$, a field of tangent 2-planes, so that the solutions to our system of equations are precisely the *integral curves* of $\mathcal{D}$: the parametrized curves $(\mathbf{q}(t), \mathbf{p}(t))$ whose velocity vector $(\mathbf{q}'(t), \mathbf{p}'(t))$ lies in $\mathcal{D}_{(\mathbf{q}(t),\mathbf{p}(t))}$ at each moment t.

Furthermore, we see readily from (1) that the function $\mathbf{pq} = p^i q_i : \mathbb{R}^6 \to \mathbb{R}$ is a "conserved quantity" (constant along solutions), so $\mathcal{D}$ is tangent everywhere to the level surfaces of $\mathbf{pq}$. By a simple rescaling argument (Section 2.2), it suffices to consider one of its non-zero level surfaces, say $Q^5 := \{\mathbf{pq} = 1\}$. Restricted to Q^5, the equation $\mathbf{pq}' = 0$ is a consequence of $\mathbf{p}' = \mathbf{q} \times \mathbf{q}'$; hence our system of equations (1) reduces to

$$\mathbf{pq} = 1, \quad \mathbf{p}' = \mathbf{q} \times \mathbf{q}'. \tag{2}$$

The system $(Q^5, \mathcal{D})$ given by (2) does not have any more conserved quantities, since $\mathcal{D}$ *bracket-generates* TQ in two steps: $\mathcal{D}^{(2)} = [\mathcal{D}, \mathcal{D}]$ is a rank 3 distribution and $\mathcal{D}^{(3)} = [\mathcal{D}, \mathcal{D}^{(2)}] = TQ^5$. Such a distribution is called a $(2,3,5)$-distribution.

The study of $(2,3,5)$-distributions has a rich and fascinating history. Their local geometry was studied by Élie Cartan in his celebrated "5-variable paper" of 1910 [9], where he showed that the symmetry algebra of such a distribution (vector fields whose flow preserves $\mathcal{D}$) is at most 14-dimensional. The most symmetric case is realized, locally uniquely, on a certain compact homogeneous 5-manifold $\overline{Q}^5$ for the 14-dimensional simple exceptional non-compact Lie group G_2 equipped with a G_2-invariant $(2,3,5)$-distribution $\overline{\mathcal{D}}$ (Section 3). This maximally symmetric $(2,3,5)$-distribution $\overline{\mathcal{D}}$ is called the Cartan-Engel distribution and was in fact used by É. Cartan and F. Engel in 1893 [10,14] to *define* $\mathfrak{g}_2$ as the automorphism algebra of this distribution; the modern definition of G_2 as the automorphism group of the octonions did not appear until 1908 [11].

Using É. Cartan's theory of $(2,3,5)$-distributions—in particular, his *submaximality* result—we show (Theorem 2.6) that our distribution $(Q^5,\mathcal{D})$, as given by (2), is maximally-symmetric, i.e., admits a 14-dimensional symmetry algebra isomorphic to $\mathfrak{g}_2$ and hence is *locally* diffeomorphic to the Cartan-Engel distribution $(\overline{Q}^5,\overline{\mathcal{D}})$. Equations (2) thus provide an explicit model, apparently new, for the Cartan-Engel distribution.

Theorem 1.1. *The system $(Q^5,\mathcal{D})$ given by equations (2) is a $(2,3,5)$-distribution with a 14-dimensional symmetry algebra, isomorphic to $\mathfrak{g}_2$, the maximum possible for a $(2,3,5)$-distribution, and is thus locally diffeomorphic to the Cartan-Engel G_2-homogeneous distribution $(\overline{Q}^5,\overline{\mathcal{D}})$.*

Most of the symmetries of (1) implied by this theorem are not obvious at all ("hidden"). There is however an 8-dimensional subalgebra $\mathfrak{sl}_3(\mathbb{R})\subset\mathfrak{g}_2$ of "obvious" symmetries, generated by

$$(\mathbf{q},\mathbf{p})\mapsto(g\mathbf{q},\mathbf{p}g^{-1}),\quad g\in\mathrm{SL}_3(\mathbb{R}).$$

The cross-product in (1) can be defined via the standard volume form on $\mathbb{R}^3$, hence the occurrence of $\mathrm{SL}_3(\mathbb{R})$; see Section 2.4. The group $\mathrm{SL}_3(\mathbb{R})$ then acts transitively and effectively on Q^5, preserving $\mathcal{D}$, and will be our main tool for studying the system (1).

To explain the appearance of $\mathfrak{g}_2$ as the symmetry algebra of $(Q^5,\mathcal{D})$, we construct in Section 3 an embedding of $(Q^5,\mathcal{D})$ in the "standard model" $(\overline{Q}^5,\overline{\mathcal{D}})$ of the Cartan-Engel distribution, defined in terms of the split-octonions $\widetilde{\mathbb{O}}$. Using Zorn's "vector matrices" to represent split-octonions—usually it is done with pairs of "split-quaternions"—we get explicit formulas for the symmetry algebra of (1). In the next Theorem, $\Im(\widetilde{\mathbb{O}})$ stands for the imaginary split octonions.

Theorem 1.2. *There is an embedding $\mathrm{SL}_3(\mathbb{R})\hookrightarrow\mathrm{G}_2=\mathrm{Aut}(\widetilde{\mathbb{O}})$ and an $\mathrm{SL}_3(\mathbb{R})$-equivariant embedding $\mathbb{R}^6\hookrightarrow\mathbb{RP}^6=\mathbb{P}(\Im(\widetilde{\mathbb{O}}))$ (an affine chart), identifying Q^5 with the open dense orbit of $\mathrm{SL}_3(\mathbb{R})$ in $\overline{Q}^5=\mathrm{G}_2/P$ and mapping $\mathcal{D}$ over to $\overline{\mathcal{D}}$. The G_2-action on $\overline{Q}^5$ defines a Lie subalgebra of vector fields on Q^5 isomorphic to $\mathfrak{g}_2$ (a 14-dimensional simple Lie algebra), forming the symmetry algebra of $(Q^5,\mathcal{D})$.*

Corollary 1.3. *For each $A=(a^i_j)\in\mathfrak{sl}_3(\mathbb{R})$, $\mathbf{b}=(b^i)\in\mathbb{R}^3$ and $\mathbf{c}=(c_i)\in(\mathbb{R}^3)^*$ the vector field on $\mathbb{R}^6$,*

$$\begin{aligned}X_{A,\mathbf{b},\mathbf{c}} &= [2b^i+a^i_jq^j+\epsilon^{ijk}p_jc_k-(p_jb^j+c_jq^j)q^i]\partial_{q^i}\\ &\qquad +[2c_i-a^j_ip_j+\epsilon_{ijk}q^jb^k-(p_jb^j+c_jq^j)p_i]\partial_{p_i},\end{aligned}$$

is tangent to Q^5 and preserves $\mathcal{D}$. The collection of these vector fields is a 14-dimensional subalgebra of the Lie algebra of vector fields on Q^5, isomorphic to $\mathfrak{g}_2$, and forming the symmetry algebra of the system $(Q^5,\mathcal{D})$ defined by (1)

1.1.2. *4-dimensional conformal geometry in split signature.* Let $M^4\subset\mathbb{RP}^2\times\mathbb{RP}^{2*}$ be the (open dense) subset of *non-incident point-line pairs* (q,p). There is a principal fibration $\mathbb{R}^*\to Q^5\to M^4$, the "pseudo-Hopf-fibration", defined by regarding $(\mathbf{q},\mathbf{p})\in Q^5$ as homogeneous coordinates of the pair $(q,p)=([\mathbf{q}],[\mathbf{p}])\in M^4$. The pseudo-Riemannian metric on Q^5, of signature $(2,3)$, induced by the inclusion

$Q^5 \to \mathbb{R}^{3,3}$, descends to an $\mathrm{SL}_3(\mathbb{R})$-invariant pseudo-Riemannian metric $\mathbf{g}$ on M^4, of signature $(2,2)$ ("split-signature").

The pair $(M^4, \mathbf{g})$ is in fact isometric to the symmetric space $\mathrm{SL}_3(\mathbb{R})/\mathrm{GL}_2(\mathbb{R})$, equipped with the unique (up to scale) pseudo-Riemannian $\mathrm{SL}_3(\mathbb{R})$-invariant metric, first introduced by P. Libermann in [21, p. 89]. We rename $\mathbf{g}$ as the *dancing metric* to emphasize the projective geometric interpretation of its null curves, as described in the abstract of this paper. In Theorem 4.3 we summarize some of the (mostly known) remarkable properties of the dancing metric (self-dual, Einstein, irreducible, etc.). We prove some of these properties in Section 4.4.

We derive various explicit formulas for the dancing metric, some of which are (perhaps) new. The most elementary expression is the following: use the local coordinates (x, y, a, b) on M^4 where (x, y) are the Cartesian coordinates of a point $q \in \mathbb{RP}^2$ (in some affine chart) and (a, b) the coordinates of a line $p \in \mathbb{RP}^{2*}$ given by $y = ax + b$. Then

$$\mathbf{g} \sim da[(y-b)dx - xdy] + db[adx - dy],$$

where $\sim$ denotes conformal equivalence. See Section 5.2 for a quick derivation of this formula using the dancing condition. An explicit formula in homogeneous coordinates for the dancing *metric* $\mathbf{g}$ (not only its conformal class) is given in Section 4.1.1 (Propostition 4.2; this is essentially P. Libermann's formula of [21]). In Section 5.3 we give another formula for $\mathbf{g}$ in terms of the cross-ratio (this formula is probably new).

The main result of Section 4 is a correspondence between the geometries of $(Q^5, \mathcal{D})$ and $(M^4, \mathbf{g})$.

Theorem 1.4. *The above defined "pseudo-Hopf-fibration" $Q^5 \to M^4$ establishes a bijection between integral curves in $(Q^5, \mathcal{D})$ and non-degenerate null curves in $(M^4, \mathbf{g})$ with parallel tangent self-dual null 2-plane.*

The condition "parallel tangent self-dual null 2-plane" on a null curve in an oriented split-signature conformal 4-manifold can be regarded as "one-half" of the geodesic equations. Every null direction is the unique intersection of two null 2-planes, one self-dual and the other anti-self-dual. It follows that given a null curve in such a manifold there are two fields of tangent null 2-planes defined along it, one self-dual and the other anti-self-dual, intersecting in the tangent line. A null curve is a geodesic if and only if its tangent line is parallel, which is equivalent to the two tangent fields of null 2-planes being parallel. For our curves, only the self-dual field is required to be parallel, hence "half-geodesics".

Following the twistor construction for split-signature metrics as in [3], we show that $(Q^5, \mathcal{D})$ can be naturally identified with the *non-integrable locus* of the total space of the *self-dual twistor fibration* $\mathbb{RP}^1 \to \mathbb{T}^+(M^4) \to M^4$ associated with $(M^4, \mathbf{g})$, equipped with its twistor distribution $\mathcal{D}^+$. The non-integrability of $\mathcal{D}$ is then seen to be equivalent to the non-vanishing of the self-dual Weyl tensor of $\mathbf{g}$.

This explains also why we do not look at the "other-half" of the null geodesic equations on M^4. They correspond to integral curves of the twistor distribution $\mathcal{D}^-$ on the anti-self-dual twistor space $\mathbb{T}^-(M^4)$, which turns out to be integrable, due to the vanishing of the anti-self-dual Weyl tensor of $(M^4, \mathbf{g})$. The resulting "anti-self-dual-half-geodesics" can be easily described and are rather uninteresting from the point of view of this article (see Theorem 4.3, Section 4.1.3).

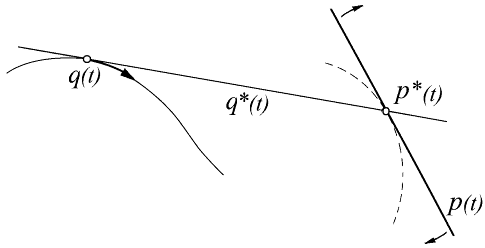

FIGURE 1. The dancing condition

1.1.3. *Projective geometry: Dancing pairs and projective rolling.* Every integral curve of $(Q^5, \mathcal{D})$ projects, via $Q^5 \to M^4 \subset \mathbb{RP}^2 \times \mathbb{RP}^{2*}$, to a pair of curves $q(t), p(t)$ in $\mathbb{RP}^2, \mathbb{RP}^{2*}$ (respectively). We offer two interrelated projective geometric interpretations of the class of pairs of curves thus obtained: "dancing pairs" and "projective rolling".

By "dancing" we refer to the interpretation of $q(t), p(t)$ as the coordinated motion of a non-incident point-line pair in $\mathbb{RP}^2$. We ask: what "rules of choreography" should the pair follow so as to define (1) a null curve in M^4 (2) with a parallel self-dual tangent plane? We call a pair of curves $q(t), p(t)$ satisfying these conditions a *dancing pair.*

The nullity condition on the pair turns out to have a rather simple "dancing" description. Consider a moving point tracing a curve $q(t)$ in $\mathbb{RP}^2$ with an associated tangent line along it, $q^*(t) \in \mathbb{RP}^{2*}$, the *dual curve* of $q(t)$. Likewise, a moving line in $\mathbb{RP}^2$ traces a curve $p(t)$ in $\mathbb{RP}^{2*}$, whose dual curve $p^*(t)$ is a curve in $\mathbb{RP}^2$, the *envelope* of the family of lines in $\mathbb{RP}^2$ represented by $p(t)$, or the curve in $\mathbb{RP}^2$ traced out by the "turning points" of the moving line $p(t)$.

Theorem 1.5. *A non-degenerate parametrized curve in* $(M^4, \mathbf{g})$ *is null if and only if the corresponding pair of curves* $(q(t), p(t))$ *satisfies the "dancing condition"* (*see Figure* 1): *at each moment* t*, the point* $q(t)$ *is moving towards or away from the turning point* $p^*(t)$ *of the line* $p(t)$.

In Section 5.6 we pose the following "dancing mate" problem: fix an arbitrary curve $q(t)$ in $\mathbb{RP}^2$ (with some mild non-degeneracy condition) and find its "dancing mates" $p(t)$, that is, curves $p(t)$ in $\mathbb{RP}^{2*}$ such that $(q(t), p(t))$ is a null curve in $(M^4, \mathbf{g})$ with parallel self-dual tangent plane. Abstractly, it is clear that there is a 3-parameter family of dancing mates for a given $q(t)$, corresponding to "horizontal lifts" of $q(t)$ to integral curves of $(Q^5, \mathcal{D})$ via $Q^5 \to M^4 \to \mathbb{RP}^2$, followed by the projection $Q^5 \to M^4 \to \mathbb{RP}^{2*}$.

We study the resulting correspondence of curves $q(t) \mapsto p(t)$ from the point of view of classical projective differential geometry. We find that this correspondence preserves the natural *projective structures* on the curves $q(t), p(t)$, but in general does not preserve the *projective arc length* or the *projective curvature.* These are the basic projective invariants of a plane curve; any two of the three invariants form a complete set of projective invariants for generic plane curves. We use the existence of a common projective parameter t on $q(t), p(t)$ and the dancing condition to derive the "dancing mate equation":

$$y^{(4)} + 2\frac{y'''y'}{y} + 3ry' + r'y = 0. \tag{3}$$

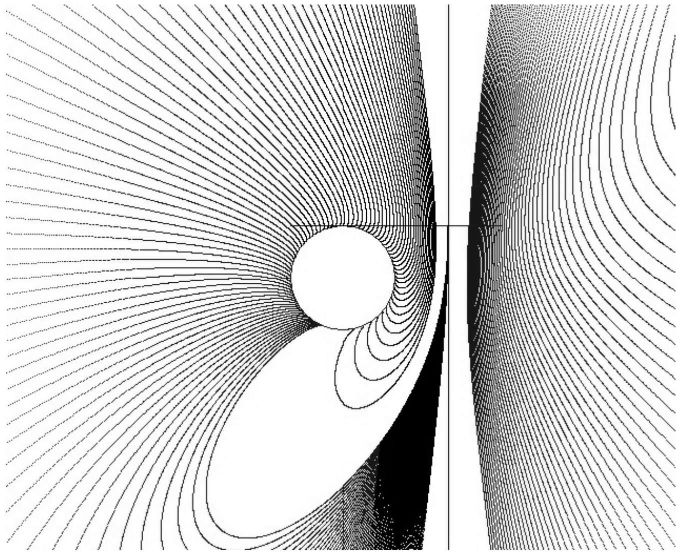

FIGURE 2. Dancing mates around the circle: the point-dancer moves along the central circle, starting at its "north pole", moving clockwise. The line-dancer starts in the vertical position ("y-axis"), keeping always tangent to one of the curves that spiral around the circle (the envelope of the line's motion). At all moments they comply with the dancing condition; the figure shows the tangent direction of the point-dancer at the moment it passes through the north pole (horizontal line segment) and its incidence with the "turning point" of the line-dancer at that moment.

Here, $q(t)$ is given in homogeneous coordinates by a "lift" $A(t) \in \mathbb{R}^3 \setminus 0$ satisfying $A''' + rA = 0$ for some function $r(t)$ (this is called the Laguerre-Forsyth form of the tautological equation for a plane curve), and the dual curve to $p(t)$ is given in homogeneous coordinates by $B = -y'A + yA'$.

We study the special case of *the dancing mates of the circle*. That is, we look for dancing pairs $(q(t), p(t))$ where $q(t)$ parametrizes a fixed circle $\mathcal{C} \subset \mathbb{RP}^2$ (or conic; projectively they are all equivalent). We show how the above dancing mate equation (3) reduces in this case to the 3rd order ODE $y'''y^2 = 1$. The dual dancing mates $p^*(t)$ form a 3-parameter family of curves in the exterior of the circle $\mathcal{C}$. We show in Figure 2 a computer generated image of a 1-parameter family of solutions; all other curves can be obtained from this family by the subgroup $\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{SL}_3(\mathbb{R})$ preserving $\mathcal{C}$.

As another illustration we give in Section 5.8 examples of dancing pairs with *constant projective curvature*: logarithmic spirals, "generalized parabolas", and exponential curves.

Finally, in Section 5.9 we turn to the "projective rolling" interpretation of (1): imagine the curves $q(t)$ and $p(t)$ as the contact points of the two projective planes $\mathbb{RP}^2, \mathbb{RP}^{2*}$ as they "roll" along each other. When rolling two surfaces along each other, one needs to pick at each moment, in addition to a pair of contact points (q, p) on the two surfaces, an identification of the tangent spaces $T_q\mathbb{RP}^2$, $T_p\mathbb{RP}^{2*}$ at these points. In the case of usual rolling of Riemannian surfaces, the identification is required to be an *isometry.* Here, we introduce the notion of "projective contact" between the corresponding tangent spaces: it is a linear isomorphism $\psi : T_q\mathbb{RP}^2 \to T_p\mathbb{RP}^{2*}$ which sends each line through q to its intersection point with the line p, thought of as a line in the tangent space to $\mathbb{RP}^{2*}$ at p.

Now a simple calculation shows that this "projective contact" condition is equivalent to the condition that the *graph of* ψ *is a self-dual null* 2*-plane* in $T_q\mathbb{RP}^2 \oplus T_q\mathbb{RP}^{2*} \simeq T_{(q,p)}M$. The configuration space for projective rolling is thus the space $\mathcal{PC}$ of such projective contact elements (q, p, ψ). Continuing the analogy with the rolling of Riemannian surfaces, we define *projective rolling without slipping* as a curve $(q(t), p(t), \psi(t))$ in $\mathcal{PC}$ satisfying $\psi(t)q'(t) = p'(t)$ for all t.

Theorem 1.6. *A curve* $(q(t), p(t), \psi(t))$ *in* $\mathcal{PC}$ *satisfies the no-slip condition* $\psi(t)q'(t) = p'(t)$ *if and only if* $(q(t), p(t))$ *is a null curve in* $(M^4, \mathbf{g})$*; equivalently, it satisfies the dancing condition of Theorem* 1.5.

Our next task is to translate the "half-geodesic" condition (parallel self-dual tangent plane) to rolling language. We use a notion of parallel transport of lines along (non-degenerate) curves in the projective plane, formulated in terms of Cartan's *development of the osculating conic along the curve* (the unique conic that touches a given point on the curve to 4th order; see Section 5.9.5). We then define the "no-twist" condition on a curve of projective contact elements $(q(t), p(t), \psi(t))$ as follows: if $\ell(t)$ is a parallel family of lines along $q(t)$, then $\psi(t)\ell(t)$ is a parallel family along $p(t)$.

Theorem 1.7. *A projective rolling curve* $(q(t), p(t), \psi(t))$ *satisfies the no-slip and no-twist condition if and only if* $(q(t), p(t))$ *is a null curve in* M^4 *with parallel self-dual tangent plane. Equivalently,* $(q(t), p(t))$ *is the projection via* $Q^5 \to M^4$ *of an integral curve of* $(Q^5, \mathcal{D})$*.*

The no-twist condition can be thought of as a "2nd dancing condition" for the dancing pair $(q(t), p(t))$; admittedly, it is a rather demanding one: the dancers should be aware of the 5th order derivative of their motion in order to comply with it. We thus end with the following:

Problem. Find a 2nd order projectively invariant condition for a curve of point-line pairs $(q(t), p(t))$ to satisfy the no-slip and no-twist condition of projective rolling (i.e., to define a null curve in the dancing space $(M^4, \mathbf{g})$ with parallel self-dual tangent 2-plane).

1.2. **Background.** Our original motivation for this article stems from the article of the third author with Daniel An [3], where the twistor construction for split-signature 4-dimensional conformal metrics was introduced, raising the following natural question: for which split-signature conformal 4-manifolds M^4 is the associated twistor distribution $\mathcal{D}^+$ on $\mathbb{T}^+M^4$ a *flat* $(2, 3, 5)$-distribution? That is, with $\mathfrak{g}_2$-symmetry, the maximum possible.

This is a hard problem, even when M^4 is a product of Riemannian surfaces (Σ_i, g_i), $i = 1, 2$, equipped with the difference metric $g = g_1 \ominus g_2$. In this case, the integral curves of the twistor distribution can be interpreted as modeling rolling without slipping or twisting of the two surfaces along each other. It was known for a while to R. Bryant and communicated in various places, such as [4, 29], that the only case of pairs of *constant curvature* surfaces that gives rise to a flat $(2,3,5)$-distribution is that of curvature ratio 9:1 (spheres of radius ratio 3:1, in the positive curvature case), but An-Nurowski found in [3] a new family of examples. It is still unknown if more examples exist.

These new examples of An-Nurowski motivated us to look for *irreducible* split signature 4-dimensional conformal metrics with flat twistor distribution. A natural place to start is with homogeneous 4-manifolds G/H, with $G \subset \mathrm{G}_2$. We know of a few such examples, but we found the case of $\mathrm{SL}_3(\mathbb{R})/\mathrm{GL}_2(\mathbb{R})$ studied in this article the most attractive due to its projective geometric flavor ("dancing" and "projective rolling" interpretations), so we decided to dedicate an article to this example alone.

2. The Cartan-Engel $(2,3,5)$-distribution and its symmetries

2.1. **First integral and reduction to the 5-manifold $Q^5 \subset \mathbb{R}^{3,3}$.** Let $\mathbb{R}^{3,3} := \mathbb{R}^3 \oplus (\mathbb{R}^3)^*$, equipped with the quadratic form $\mathbf{pq} = p^i q_i$. One can easily check that $\mathbf{pq}$ is a first integral of (1) (a conserved quantity). That is, for each $c \in \mathbb{R}$, a solution to (1) that starts on the level surface

$$Q_c = \{(\mathbf{q}, \mathbf{p}) | \mathbf{pq} = c\}$$

remains on Q_c for all times.

Furthermore, the map $(\mathbf{q}, \mathbf{p}) \mapsto (\lambda \mathbf{q}, \lambda^2 \mathbf{p})$, $\lambda \in \mathbb{R}$, maps solutions on Q_c to solutions on $Q_{\lambda^3 c}$. Hence it is enough to study solutions of the system restricted to one of the (non-zero) level surfaces, say

$$Q^5 := \{(\mathbf{q}, \mathbf{p}) | \mathbf{pq} = 1\},$$

a 5-dimensional affine quadric of signature $(3,3)$.

Remark. An *affine quadric* is a non-zero level set of a non-degenerate quadratic form on $\mathbb{R}^n$. Its *signature* is the signature of the defining quadratic form.

Remark. We leave out the less interesting case of the zero level surface Q_0.

Now restricted to Q^5, the equation $\mathbf{pq}' = 0$ is a consequence of $\mathbf{p}' = \mathbf{q} \times \mathbf{q}'$; hence we can replace equations (1) with the somewhat simpler system

$$\mathbf{pq} = 1, \quad \mathbf{p}' = \mathbf{q} \times \mathbf{q}'. \tag{4}$$

2.2. **A rank 2 distribution $\mathcal{D}$ on Q^5.** A geometric reformulation of (4) is the following: let us introduce the three 1-forms

$$\omega_i := dp_i - \epsilon_{ijk} q^j dq^k \in \Omega^1(Q^5), \quad i = 1, 2, 3,$$

or in vector notation,

$$\boldsymbol{\omega} = d\mathbf{p} - \mathbf{q} \times d\mathbf{q} \in \Omega^1(Q^5) \otimes (\mathbb{R}^3)^*.$$

Then the kernel of the 1-form $\boldsymbol{\omega}$ (the common kernel of its 3 components) defines at each point $(\mathbf{q}, \mathbf{p}) \in Q^5$ a 2-dimensional linear subspace $\mathcal{D}_{(\mathbf{q},\mathbf{p})} \subset T_{(\mathbf{q},\mathbf{p})} Q^5$. Put together, these subspaces define a rank 2 distribution $\mathcal{D} \subset TQ^5$ (a field of tangent 2-planes on Q^5), so that the solutions to our system of equations (4) are precisely the *integral curves* of $\mathcal{D}$: the parametrized curves $(\mathbf{q}(t), \mathbf{p}(t))$ whose velocity vector at each moment t belongs to $\mathcal{D}_{(\mathbf{q}(t),\mathbf{p}(t))}$.

Proposition 2.1. *The kernel of $\boldsymbol{\omega} = d\mathbf{p} - \mathbf{q} \times d\mathbf{q}$ defines on Q^5 a rank 2 distribution $\mathcal{D} \subset TQ^5$, whose integral curves are given by solutions to (4).*

Proof. One checks easily that the 3 components of $\boldsymbol{\omega}$ are linearly independent. $\square$

2.3. $\mathcal{D}$ **is a** $(2,3,5)$**-distribution.** We recall first some standard terminology from the general theory of distributions. A distribution $\mathcal{D}$ on a manifold is *integrable* if $[\mathcal{D},\mathcal{D}]\subset\mathcal{D}$. It is *bracket generating* if one can obtain any tangent vector on the manifold by successive Lie brackets of vector fields tangent to $\mathcal{D}$. Let $r_i = \mathrm{rank}(\mathcal{D}^{(i)})$, where $\mathcal{D}^{(i)} := [\mathcal{D},\mathcal{D}^{(i-1)}]$ and $\mathcal{D}^{(1)} := \mathcal{D}$. Then $(r_1, r_2, \ldots)$, is the *growth vector* of $\mathcal{D}$. In general, the growth vector of a distribution may vary from point to point of the manifold, although not in our case, since our distribution is homogeneous, as we shall soon see. A distribution with constant growth vector is *regular*. It can be shown that for a regular bracket-generating rank 2 distribution on a 5-manifold there are only two possible growth vectors: $(2,3,4,5)$, called *Goursat distributions*, or $(2,3,5)$, which is the generic case (see [5]).

Definition 2.2. A $(2,3,5)$-*distribution* is a bracket-generating rank 2 distribution $\mathcal{D}$ on a 5-manifold Q^5 with growth vector $(2,3,5)$ everywhere. That is, $\mathcal{D}^{(2)} = [\mathcal{D},\mathcal{D}]$ is a rank 3 distribution, and $\mathcal{D}^{(3)} = [\mathcal{D},\mathcal{D}^{(2)}] = TQ^5$.

Proposition 2.3. $\mathcal{D}{=}\mathrm{Ker}(\boldsymbol{\omega})\subset TQ^5$, *defined by* (4) *above, is a* (2,3,5)*-distribution.*

This is a calculation done most easily using the symmetries of the equations, so is postponed to the next subsection.

2.4. $\mathrm{SL}_3(\mathbb{R})$**-symmetry.** A *symmetry* of a distribution $\mathcal{D}$ on a manifold Q^5 is a diffeomorphism of Q^5 which preserves $\mathcal{D}$. An *infinitesimal symmetry* of $\mathcal{D}$ is a vector field on Q^5 whose flow preserves $\mathcal{D}$.

The use of the vector and scalar product on $\mathbb{R}^3$ in (4) may give the impression that $\mathcal{D}$ depends on the Euclidean structure on $\mathbb{R}^3$, so $(Q^5,\mathcal{D})$ only admits SO_3 as an obvious group of symmetries (a 3-dimensional group). In fact, it is quite easy to see, as we will show now, that $(Q^5,\mathcal{D})$ admits $\mathrm{SL}_3(\mathbb{R})$ as a symmetry group (8-dimensional). In the next section we will show the less obvious fact that the symmetry *algebra* of $(Q^5,\mathcal{D})$ is $\mathfrak{g}_2$ (14-dimensional).

Fix a volume form on $\mathbb{R}^3$, say

$$vol := dq^1\wedge dq^2\wedge dq^3,$$

and define the associated covector-valued "cross-product" $\mathbb{R}^3\times\mathbb{R}^3\to(\mathbb{R}^3)^*$ by

$$\mathbf{v}\times\mathbf{w} := vol(\mathbf{v},\mathbf{w},\cdot),$$

or in coordinates,

$$(\mathbf{v}\times\mathbf{w})_i = \epsilon_{ijk}v^jw^k,\quad i=1,2,3.$$

Let $\mathrm{SL}_3(\mathbb{R})$ be the group of 3×3 matrices with real entries and determinant 1, acting on $\mathbb{R}^{3,3}$ by

$$g\cdot(\mathbf{q},\mathbf{p}) = (g\mathbf{q},\mathbf{p}g^{-1}) \tag{5}$$

(recall that $\mathbf{q}$ is a column vector and $\mathbf{p}$ is a row vector). Clearly, this $\mathrm{SL}_3(\mathbb{R})$-action leaves the quadratic form $\mathbf{pq}$ invariant and thus leaves invariant also the quadric $Q^5\subset\mathbb{R}^{3,3}$.

Let e_1, e_2, e_3 (columns) be the standard basis of $\mathbb{R}^3$ and let e^1, e^2, e^3 (rows) be the dual basis of $(\mathbb{R}^3)^*$.

Proposition 2.4.

(a) $\mathrm{SL}_3(\mathbb{R})$ *acts on* Q^5 *transitively and effectively. The stabilizer of* (e_3, e^3) *is the subgroup*

$$H_0 = \left\{ \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \mid A \in \mathrm{SL}_2(\mathbb{R}) \right\}.$$

(b) $\mathrm{SL}_3(\mathbb{R})$ *acts on* Q^5 *by symmetries of* $\mathcal{D}$.

Proof. Part (a) is an easy calculation (omitted). For part (b), note that $\mathrm{SL}_3(\mathbb{R})$ leaves *vol* invariant; hence the vector product $\mathbb{R}^3 \times \mathbb{R}^3 \to (\mathbb{R}^3)^*$ is $\mathrm{SL}_3(\mathbb{R})$-equivariant: $(g\mathbf{v}) \times (g\mathbf{w}) = (\mathbf{v} \times \mathbf{w})g^{-1}$. It follows that $\boldsymbol{\omega} = d\mathbf{p} - \mathbf{q} \times d\mathbf{q}$ is also $\mathrm{SL}_3(\mathbb{R})$-equivariant, $g^*\boldsymbol{\omega} = \boldsymbol{\omega} g^{-1}$; hence $\mathcal{D} = \mathrm{Ker}(\boldsymbol{\omega})$ is $\mathrm{SL}_3(\mathbb{R})$-invariant. $\square$

Proof of Proposition 2.3. Let $\mathfrak{h}_0 \subset \mathfrak{sl}_3(\mathbb{R})$ be the Lie algebra of the stabilizer at $(e_3, e^3) \in Q^5$. Pick two elements $Y_1, Y_2 \in \mathfrak{sl}_3(\mathbb{R})$ whose infinitesimal action at (e_3, e^3) generates $\mathcal{D}$. Then we need to show that

$$Y_1, Y_2, [Y_1, Y_2], [Y_1, [Y_1, Y_2]], [Y_2, [Y_1, Y_2]]$$

span $\mathfrak{sl}_3(\mathbb{R})$ mod $\mathfrak{h}_0$. (This will show that $\mathcal{D}$ is $(2,3,5)$ at (e_3, e^3), so by homogeneity everywhere.) Now $\mathfrak{h}_0$ consists of matrices of the form

$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ \hline 0 & 0 & 0 \end{array} \right), \quad A \in \mathfrak{sl}_2(\mathbb{R}).$$

Furthermore, $Y \in \mathfrak{sl}_3(\mathbb{R})$ satisfies $Y \cdot (e_3, e^3) \in \mathcal{D}$ if and only if

$$Y = \begin{pmatrix} A & \mathbf{v} \\ \mathbf{v}^* & 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \mathbf{v}^* = (v_2, -v_1), \quad A \in \mathfrak{sl}_2(\mathbb{R}).$$

We can thus take

$$Y_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix};$$

then $[Y_1, Y_2] = Y_3$, $[Y_1, Y_3] = Y_4$ and $[Y_2, Y_3] = Y_5$, where

$$Y_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad Y_4 = \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & -3 & 0 \end{pmatrix}, \quad Y_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -3 \\ 3 & 0 & 0 \end{pmatrix},$$

which together with Y_1, Y_2 span $\mathfrak{sl}_3(\mathbb{R})/\mathfrak{h}_0 \simeq T_{(e_3,e^3)}Q^5$. $\square$

2.5. **$\mathfrak{g}_2$-symmetry via Cartan's submaximality.** Here we show that the symmetry *algebra* of our distribution $(Q^5, \mathcal{D})$, given by (4), is isomorphic to $\mathfrak{g}_2$, a 14-dimensional simple Lie algebra, the maximum possible for a $(2,3,5)$-distribution. We show this as an immediate consequence of a general theorem of Cartan (1910) on $(2,3,5)$-distributions. In the next section this "hidden symmetry" is explained and written explicitly by defining an embedding of $(Q^5, \mathcal{D})$ in the standard G_2-homogeneous model $(\overline{Q}^5, \overline{\mathcal{D}})$ using split-octonions.

Remark. Of course, there is a third way, by "brute force", using computer algebra. We do not find it too illuminating, but it does produce quickly a list of 14 vector fields on $\mathbb{R}^{3,3}$, generating the infinitesimal $\mathfrak{g}_2$-action, as given in Corollary 1.3 or Corollary 3.8 below.

In a well-known paper of 1910 (the "5-variable paper"), Cartan proved the following.

Theorem 2.5 (Cartan [9])**.**

(1) *The symmetry algebra of a* $(2,3,5)$*-distribution on a connected* 5*-manifold has dimension at most* 14*, in which case it is isomorphic to the real split-form of the simple Lie algebra of type* $\mathfrak{g}_2$*.*

(2) *All* $(2,3,5)$*-distributions with* 14*-dimensional symmetry algebra are locally diffeomorphic.*

(3) *If the symmetry algebra of a* $(2,3,5)$*-distribution has dimension* < 14*, then it has dimension at most* 7*.*

The last statement is sometimes referred to as Cartan's "submaximality" result for $(2,3,5)$-distribution. A $(2,3,5)$-distribution with the maximal symmetry algebra $\mathfrak{g}_2$ is called *flat*.

Using Proposition 2.4 and the fact that $\mathrm{SL}_3(\mathbb{R})$ is 8-dimensional, we immediately conclude the following from Cartan's submaximality result for $(2,3,5)$-distributions.

Theorem 2.6. *The symmetry algebra of the* $(2,3,5)$*-distribution defined by* (4) *is* 14*-dimensional, isomorphic to the Lie algebra* $\mathfrak{g}_2$*, containing the Lie subalgebra isomorphic to* $\mathfrak{sl}_3(\mathbb{R})$ *generated by the linear* $\mathrm{SL}_3(\mathbb{R})$*-action given by equation* (5)*.*

Remark. In fact, Cartan [10] and Engel [14] *defined* in 1893 the Lie algebra $\mathfrak{g}_2$ as the symmetry algebra of a certain $(2,3,5)$-distribution on an open set in $\mathbb{R}^5$, using formulas similar to our (4). For example, Engel considers in [14] the $(2,3,5)$-distribution obtained by restricting $d\mathbf{p}-\mathbf{q}\times d\mathbf{q}$ to the linear subspace in $\mathbb{R}^{3,3}$ given by $q_3 = p^3$.

3. G_2-symmetry via split-octonions

In this section we describe a relation between the algebra of split-octonions $\widetilde{\mathbb{O}}$ and our equations (2), thus explaining the appearance of the "hidden $\mathfrak{g}_2$-symmetry" in Theorem 2.6 of the previous section. We first review some well-known facts concerning the algebra of split-octonions $\widetilde{\mathbb{O}}$ and its automorphism group G_2. We then define the "standard model" for the flat $(2,3,5)$-distribution, a compact hypersurface $\overline{Q}^5 \subset \mathbb{RP}^6$, the projectivized null cone of imaginary split-octonions, equipped with a $(2,3,5)$-distribution $\overline{\mathcal{D}} \subset T\overline{Q}^5$. The group $G_2 = \mathrm{Aut}(\widetilde{\mathbb{O}})$ acts naturally on all objects defined in terms of the split-octonions, such as $\overline{Q}^5$ and $\overline{\mathcal{D}}$.

The relation of $(\overline{Q}^5, \overline{\mathcal{D}})$ with our system $(Q^5, \mathcal{D})$ is seen by finding an embedding of groups $\mathrm{SL}_3(\mathbb{R}) \hookrightarrow G_2$ and an $\mathrm{SL}_3(\mathbb{R})$-equivariant embedding $(Q, \mathcal{D}) \hookrightarrow (\overline{Q}^5, \overline{\mathcal{D}})$. In this way we obtain an explicit realization of $\mathfrak{g}_2$ as the 14-dimensional symmetry algebra of $(Q, \mathcal{D})$, containing the 8-dimensional subalgebra of "obvious" $\mathfrak{sl}_3(\mathbb{R})$-symmetries, as defined in (5) of Section 2.4. This construction explains also why the infinitesimal $\mathfrak{g}_2$-symmetry of $(Q^5, \mathcal{D})$ does not extend to a global G_2-symmetry.

3.1. **Split-octonions via Zorn's vector matrices.** We begin with a brief review of the algebra of split-octonions, using a somewhat unfamiliar notation due to Max Zorn (of Zorn's Lemma fame in set theory), which we found quite useful in our context. See [26] for a similar presentation.

The split-octonions $\widetilde{\mathbb{O}}$ is an 8-dimensional non-commutative and non-associative real algebra, whose elements can be written as "vector matrices"

$$\zeta = \begin{pmatrix} x & \mathbf{q} \\ \mathbf{p} & y \end{pmatrix}, \quad x, y \in \mathbb{R}, \quad \mathbf{q} \in \mathbb{R}^3, \quad \mathbf{p} \in (\mathbb{R}^3)^*,$$

with the "vector-matrix-multiplication", denoted here by $*$,

$$\zeta * \zeta' = \begin{pmatrix} x & \mathbf{q} \\ \mathbf{p} & y \end{pmatrix} * \begin{pmatrix} x' & \mathbf{q}' \\ \mathbf{p}' & y' \end{pmatrix} := \begin{pmatrix} xx' - \mathbf{p}'\mathbf{q} & x\mathbf{q}' + y'\mathbf{q} + \mathbf{p} \times \mathbf{p}' \\ x'\mathbf{p} + y\mathbf{p}' + \mathbf{q} \times \mathbf{q}' & yy' - \mathbf{p}\mathbf{q}' \end{pmatrix},$$

where, as before, we use the vector products $\mathbb{R}^3 \times \mathbb{R}^3 \to (\mathbb{R}^3)^*$ and $(\mathbb{R}^3)^* \times (\mathbb{R}^3)^* \to \mathbb{R}^3$, given by

$$\mathbf{q} \times \mathbf{q}' := vol(\mathbf{q}, \mathbf{q}', \cdot), \quad \mathbf{p} \times \mathbf{p}' := vol^*(\mathbf{p}, \mathbf{p}', \cdot),$$

vol is the standard volume form on $\mathbb{R}^3$,

$$vol = dq^1 \wedge dq^2 \wedge dq^3,$$

and vol^* is the dual volume form on $(\mathbb{R}^3)^*$,

$$vol^* = dp_1 \wedge dp_2 \wedge dp_3.$$

In coordinates,

$$(\mathbf{q} \times \mathbf{q}')_i = \epsilon_{ijk} q^j q'^k, \quad (\mathbf{p} \times \mathbf{p}')^i = \epsilon^{ijk} p_j p'_k.$$

Remark. These "vector matrices" were introduced by Max Zorn in [30] (p. 144). There are some minor variations in the literature in the signs in the multiplication formula, but they are all equivalent to ours by some simple change of variables. We are using Zorn's original formulas. For example, the formula in Wikipedia's article "Split-octonion" is obtained from ours by the change of variable $\mathbf{p} \mapsto -\mathbf{p}$. A better-known formula for octonion multiplication uses pairs of quaternions, but we found the above formulas of Zorn more suitable; they also fit nicely with the original Cartan and Engel 1894 formulas.

Conjugation in $\widetilde{\mathbb{O}}$ is given by

$$\zeta = \begin{pmatrix} x & \mathbf{q} \\ \mathbf{p} & y \end{pmatrix} \mapsto \overline{\zeta} = \begin{pmatrix} y & -\mathbf{q} \\ -\mathbf{p} & x \end{pmatrix},$$

satisfying

$$\overline{\overline{\zeta}} = \zeta, \quad \overline{\zeta * \zeta'} = \overline{\zeta'} * \overline{\zeta}, \quad \zeta * \overline{\zeta} = \langle \zeta, \zeta \rangle \mathrm{I},$$

where $\mathrm{I} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and

$$\langle \zeta, \zeta \rangle = xy + \mathbf{p}\mathbf{q}$$

is a quadratic form of signature $(4, 4)$ on $\widetilde{\mathbb{O}}$.

Define as usual

$$\Re(\zeta) = (\zeta + \overline{\zeta})/2, \quad \Im(\zeta) = (\zeta - \overline{\zeta})/2,$$

so that

$$\widetilde{\mathbb{O}} = \Re(\widetilde{\mathbb{O}}) \oplus \Im(\widetilde{\mathbb{O}}),$$

where $\Re(\widetilde{\mathbb{O}}) = \mathbb{R}\mathrm{I}$ and $\Im(\widetilde{\mathbb{O}})$ are vector matrices of the form $\zeta = \left(\begin{smallmatrix} x & \mathbf{q} \\ \mathbf{p} & -x \end{smallmatrix}\right)$.

3.2. **About** G_2**.**

Definition 3.1. G_2 is the subgroup of $\mathrm{GL}(\widetilde{\mathbb{O}}) \simeq \mathrm{GL}_8(\mathbb{R})$ satisfying $g(\zeta * \zeta') = g(\zeta) * g(\zeta')$ for all $\zeta, \zeta' \in \widetilde{\mathbb{O}}$.

Remark. There are in fact three essentially distinct groups denoted by G_2 in the literature: the complex Lie group $\mathrm{G}_2^{\mathbb{C}}$ and its two real forms: the compact form and the non-compact form, "our" G_2. See, for example, Theorem 6.105 on p. 421 of [18].

Proposition 3.2. *Every* $g \in \mathrm{G}_2$ *preserves the splitting* $\widetilde{\mathbb{O}} = \Re(\widetilde{\mathbb{O}}) \oplus \Im(\widetilde{\mathbb{O}})$. *The action of* G_2 *on* $\Re(\widetilde{\mathbb{O}})$ *is trivial. Thus* G_2 *embeds naturally in* $\mathrm{GL}(\Im(\widetilde{\mathbb{O}})) \simeq \mathrm{GL}_7(\mathbb{R})$.

Proof. Let $g \in \mathrm{G}_2$. Since I is invertible so is $g(\mathrm{I})$. Now $g(\mathrm{I}) = g(\mathrm{I} * \mathrm{I}) = g(\mathrm{I}) * g(\mathrm{I})$, hence $g(\mathrm{I}) = \mathrm{I}$. It follows that g acts trivially on $\Re(\widetilde{\mathbb{O}}) = \mathbb{R}\mathrm{I}$.

Next, to show that $\Im(\widetilde{\mathbb{O}})$ is g-invariant, define $S := \{\zeta \in \widetilde{\mathbb{O}} | \zeta * \zeta = -\mathrm{I}\}$. Then it is enough to show that (1) S is g-invariant, (2) $S \subset \Im(\widetilde{\mathbb{O}})$, (3) S spans $\Im(\widetilde{\mathbb{O}})$.

(1) is immediate from $g(-\mathrm{I}) = -\mathrm{I}$. For (2), let $\zeta = \left(\begin{smallmatrix} x & \mathbf{q} \\ \mathbf{p} & -y \end{smallmatrix}\right) \in S$. Then $\zeta * \zeta = -\mathrm{I} \Longrightarrow x^2 - \mathbf{pq} = y^2 - \mathbf{pq} = -1, (x+y)\mathbf{q} = (x+y)\mathbf{p} = 0 \Longrightarrow x+y = 0 \Longrightarrow \zeta \in \Im(\widetilde{\mathbb{O}})$. For (3), it is easy to find a basis of $\Im(\widetilde{\mathbb{O}})$ in S. $\square$

The Lie algebra of G_2 is the sub-algebra $\mathfrak{g}_2 \subset \mathrm{End}(\widetilde{\mathbb{O}})$ of *derivations* of $\widetilde{\mathbb{O}}$: the elements $X \in \mathrm{End}(\widetilde{\mathbb{O}})$ such that $X(\zeta * \zeta') = (X\zeta) * \zeta' + \zeta * (X\zeta')$ for all $\zeta, \zeta' \in \widetilde{\mathbb{O}}$. It follows from the last proposition that $\mathfrak{g}_2$ embeds as a subalgebra of $\mathrm{End}(\Im(\widetilde{\mathbb{O}}))$. In his 1894 thesis É. Cartan gave explicit formulas for the image of this embedding, as follows.

For each $(A, \mathbf{b}, \mathbf{c}) \in \mathfrak{sl}_3 \oplus \mathbb{R}^3 \oplus (\mathbb{R}^3)^*$ define $\rho(A, \mathbf{b}, \mathbf{c}) \in \mathrm{End}(\Im(\widetilde{\mathbb{O}}))$, written as a block matrix, corresponding to the decomposition $\Im(\widetilde{\mathbb{O}}) \simeq \mathbb{R}^3 \oplus (\mathbb{R}^3)^* \oplus \mathbb{R}$, $\left(\begin{smallmatrix} x & \mathbf{q} \\ \mathbf{p} & -x \end{smallmatrix}\right) \mapsto (\mathbf{q}, \mathbf{p}, x)$, by

$$\rho(A, \mathbf{b}, \mathbf{c}) = \begin{pmatrix} A & R_{\mathbf{c}} & 2\mathbf{b} \\ L_{\mathbf{b}} & -A^t & 2\mathbf{c} \\ \mathbf{c}^t & \mathbf{b}^t & 0 \end{pmatrix},$$

where $L_{\mathbf{b}} : \mathbb{R}^3 \to (\mathbb{R}^3)^*$ is given by $\mathbf{q} \mapsto \mathbf{b} \times \mathbf{q}$ and $R_{\mathbf{c}} : (\mathbb{R}^3)^* \to \mathbb{R}^3$ is given by $\mathbf{p} \mapsto \mathbf{p} \times \mathbf{c}$.

Now define $\tilde{\rho} : \mathfrak{sl}_3 \oplus \mathbb{R}^3 \oplus (\mathbb{R}^3)^* \to \mathrm{End}(\widetilde{\mathbb{O}})$ by

$$\tilde{\rho}(A, \mathbf{b}, \mathbf{c})\zeta = \rho(A, \mathbf{b}, \mathbf{c})\Im(\zeta).$$

Explicitly, we find

$$\tilde{\rho}(A, \mathbf{b}, \mathbf{c}) \begin{pmatrix} x & \mathbf{q} \\ \mathbf{p} & y \end{pmatrix} = \begin{pmatrix} \mathbf{pb} + \mathbf{cq} & A\mathbf{q} + (x-y)\mathbf{b} + \mathbf{p} \times \mathbf{c} \\ -\mathbf{p}A + \mathbf{b} \times \mathbf{q} + (x-y)\mathbf{c} & -\mathbf{pb} - \mathbf{cq} \end{pmatrix}.$$

Proposition 3.3. *The image of* $\tilde{\rho}$ *in* $\mathrm{End}(\widetilde{\mathbb{O}})$ *is* $\mathfrak{g}_2$. *That is, for all* $(A, \mathbf{b}, \mathbf{c}) \in \mathfrak{sl}_3 \oplus \mathbb{R}^3 \oplus (\mathbb{R}^3)^*$, $\tilde{\rho}(A, \mathbf{b}, \mathbf{c})$ *is a derivation of* $\widetilde{\mathbb{O}}$, *and all derivations of* $\widetilde{\mathbb{O}}$ *arise in this way. Thus* G_2 *is a* 14*-dimensional Lie group. It is a simple Lie group of type* $\mathfrak{g}_2$ (*the non-compact real form*).

Proof (*This is a sketch; for more details see for example* [26]). One shows first that $\tilde{\rho}(A, \mathbf{a}, \mathbf{b})$ is a derivation by direct calculation. In the other direction, if $X \in \mathfrak{g}_2$, i.e., is a derivation, then its restriction to $\Im(\widetilde{\mathbb{O}})$ is antisymmetric with respect to the quadratic form $\mathrm{J} = x^2 - \mathbf{pq}$, i.e., is in the 21-dimensional Lie algebra $\mathfrak{so}_{4,3}$ of the orthogonal group corresponding to J. One then needs to show the vanishing of the projection of X to $\mathfrak{so}_{4,3}/\mathrm{Im}(\rho)$ (a 21−14=7 dimensional space). The latter decomposes under $\mathrm{SL}_3(\mathbb{R})$ as $\mathbb{R}^3 \oplus (\mathbb{R}^3)^* \oplus \mathbb{R}$, so by the Schur Lemma it is enough to check the claim for one X in each of the three irreducible summands.

Now one can pick a Cartan subalgebra and root vectors showing that this algebra is of type $\mathfrak{g}_2$ (see Cartan's thesis [7], p. 146). □

Remark. Cartan gave the above representation of $\mathfrak{g}_2$ in his 1894 thesis [7] with no reference to octonions; the relation with octonions was published by him later in 1908 [11]. He presented $\mathfrak{g}_2$ as the symmetry algebra of a rank 3 distribution on the null cone in $\Im(\widetilde{\mathbb{O}})$.

3.3. **The distribution** $(\overline{Q}^5, \overline{\mathcal{D}})$**.** Imaginary split-octonions $\Im(\widetilde{\mathbb{O}})$ satisfy $\zeta = -\overline{\zeta}$ and are given by vector-matrices of the form

$$\zeta = \begin{pmatrix} x & \mathbf{q} \\ \mathbf{p} & -x \end{pmatrix},$$

where $(\mathbf{q}, \mathbf{p}, x) \in \mathbb{R}^3 \oplus (\mathbb{R}^3)^* \oplus \mathbb{R}$.

Definition 3.4. Let $\mathbf{\Omega} := \zeta * d\zeta$ (an $\widetilde{\mathbb{O}}$-valued 1-form on $\Im(\widetilde{\mathbb{O}})$). Explicitly,

$$\mathbf{\Omega} := \begin{pmatrix} x\,dx - \mathbf{q}\,d\mathbf{p} & x\,d\mathbf{q} - \mathbf{q}\,dx + \mathbf{p} \times d\mathbf{p} \\ \mathbf{p}\,dx - x\,d\mathbf{p} + \mathbf{q} \times d\mathbf{q} & x\,dx - \mathbf{p}\,d\mathbf{q} \end{pmatrix}.$$

Proposition 3.5. *Let* $\mathrm{Ker}(\mathbf{\Omega})$ *be the distribution* (*with variable rank*) *on* $\Im(\widetilde{\mathbb{O}})$ *annihilated by* $\mathbf{\Omega}$ *and let* $C \subset \Im(\widetilde{\mathbb{O}})$ *be the null cone,* $C = \{\zeta \in \Im(\widetilde{\mathbb{O}}) | x^2 - \mathbf{pq} = 0\}$. *Then* $\mathrm{Ker}(\mathbf{\Omega})$ *is:*

(1) G_2*-invariant,*
(2) $\mathbb{R}^*$*-invariant, under* $\zeta \mapsto \lambda\zeta$, $\lambda \in \mathbb{R}^*$,
(3) *tangent to* $C \setminus 0$,
(4) *a rank* 3 *distribution when restricted to* $C \setminus 0$.
(5) *The* $\mathbb{R}^*$*-orbits on* C *are tangent to* $\mathrm{Ker}(\mathbf{\Omega})$.

Proof.

(1) $\mathbf{\Omega}$ is G_2-equivariant, i.e., $g^*\mathbf{\Omega} = g\mathbf{\Omega}$ for all $g \in \mathrm{G}_2$, hence $\mathrm{Ker}(\mathbf{\Omega})$ is g-invariant. Details: $g^*(\zeta * d\zeta) = (g\zeta) * d(g\zeta) = (g\zeta) * [g(d\zeta)] = g(\zeta * d\zeta) = g\mathbf{\Omega}$.
(2) $\lambda^*\mathbf{\Omega} = \lambda^2\mathbf{\Omega} \Longrightarrow \mathrm{Ker}(\lambda^*\mathbf{\Omega}) = \mathrm{Ker}(\lambda^2\mathbf{\Omega}) = \mathrm{Ker}(\mathbf{\Omega})$.
(3) C is the 0 level set of $f(\zeta) = \zeta * \overline{\zeta} = -\zeta * \zeta$; hence the tangent bundle to $C \setminus 0$ is the kernel of $df = -(d\zeta) * \zeta - \zeta * d\zeta = -\mathbf{\Omega} - \overline{\mathbf{\Omega}} = -2\Re(\mathbf{\Omega})$, hence $\mathrm{Ker}(\mathbf{\Omega}) \subset \mathrm{Ker}(df)$.
(4) Use the fact that $\mathrm{G}_2 \times \mathbb{R}^*$ acts transitively on $C \setminus 0$, so it is enough to check at say $\mathbf{q} = e_1$, $\mathbf{p} = 0$, $x = 0$. Then $\mathrm{Ker}(\mathbf{\Omega})$ at this point is given by $dp_1 = dq^2 = dq^3 = dx = 0$, which defines a 3-dimensional subspace of $\Im(\widetilde{\mathbb{O}})$).
(5) The $\mathbb{R}^*$-action is generated by the Euler vector field

$$E = p_i \frac{\partial}{\partial_{p_i}} + q^i \frac{\partial}{\partial_{q^i}} + x \frac{\partial}{\partial_x},$$

hence $\mathbf{\Omega}(E) = \zeta * d\zeta(E) = \zeta * \zeta = 0$, for $\zeta \in C$.

□

Corollary 3.6. $\mathrm{Ker}(\mathbf{\Omega})$ *descends to a* G_2*-invariant rank* 2 *distribution* $\overline{\mathcal{D}}$ *on the projectivized null cone* $\overline{Q}^5 = (C \setminus 0)/\mathbb{R}^* \subset \mathbb{P}(\Im(\widetilde{\mathbb{O}})) \cong \mathbb{RP}^6$.

Now define an embedding $\iota : \mathbb{R}^{3,3} \to \Im(\widetilde{\mathbb{O}})$ by $(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{q}, \mathbf{p}, 1)$. The pull-back of $\mathbf{\Omega}$ by this map is easily seen to be

$$\iota^*\mathbf{\Omega} = \begin{pmatrix} -\mathbf{q}\, d\mathbf{p} & d\mathbf{q} + \mathbf{p} \times d\mathbf{p} \\ -d\mathbf{p} + \mathbf{q} \times d\mathbf{q} & -\mathbf{p}\, d\mathbf{q} \end{pmatrix}. \tag{6}$$

Let $\mathrm{SL}_3(\mathbb{R})$ act on $\widetilde{\mathbb{O}}$ by

$$\begin{pmatrix} x & \mathbf{q} \\ \mathbf{p} & y \end{pmatrix} \mapsto \begin{pmatrix} x & g\mathbf{q} \\ \mathbf{p}g^{-1} & y \end{pmatrix}, \quad g \in \mathrm{SL}_3(\mathbb{R}).$$

This defines an embedding $\mathrm{SL}_3(\mathbb{R}) \hookrightarrow \mathrm{Aut}(\widetilde{\mathbb{O}})$.

Theorem 3.7. *Let* $Q = \{\mathbf{pq} = 1\} \subset \mathbb{R}^{3,3}$. *Then*

(a) *the composition*

$$Q \xrightarrow{\iota} C \setminus 0 \xrightarrow{\mathbb{R}^*} \overline{Q}^5, \quad (\mathbf{q}, \mathbf{p}) \mapsto [(\mathbf{q}, \mathbf{p}, 1)] \in \overline{Q}^5 \subset \mathbb{P}(\Im(\widetilde{\mathbb{O}})) \cong \mathbb{RP}^6$$

is an $\mathrm{SL}_3(\mathbb{R})$*-equivariant embedding of* $(Q, \mathcal{D})$ *in* $(\overline{Q}^5, \overline{\mathcal{D}})$.

(b) *The image of* $Q^5 \to \overline{Q}^5$ *is the open-dense orbit of the* $\mathrm{SL}_3(\mathbb{R})$*-action on the projectivized null cone* $\overline{Q}^5 \subset \mathbb{P}(\Im(\widetilde{\mathbb{O}})) \cong \mathbb{RP}^6$*; its complement is a closed* 4*-dimensional submanifold.*

Proof. (a) Under $\mathrm{SL}_3(\mathbb{R})$, $\Im(\widetilde{\mathbb{O}})$ decomposes as $\mathbb{R}^{3,3} \oplus \mathbb{R}$; hence $\mathbb{R}^{3,3} \to \Im(\widetilde{\mathbb{O}})$, $(\mathbf{q}, \mathbf{p}) \mapsto [\mathbf{q}, \mathbf{p}, 1]$ is an $\mathrm{SL}_3(\mathbb{R})$-equivariant embedding. Formula (6) for $i^*\mathbf{\Omega}$ shows that $\mathcal{D}$ is mapped to $\overline{\mathcal{D}}$.

(b) From the previous item, the image of Q^5 in $\overline{Q}^5$ is a single $\mathrm{SL}_3(\mathbb{R})$-orbit, 5-dimensional, hence open. It is dense, since the complement is a 4-dimensional submanifold in $\overline{Q}^5$, given in homogeneous coordinates by the intersection of the hyperplane $x = 0$ with the quadric $\mathbf{pq} - x^2 = 0$. Restricted to $x = 0$, a 5-dimensional projective subspace in $\mathbb{RP}^6$, the equation $\mathbf{pq} = 0$ defines a smooth 4-dimensional hypersurface, a projective quadric of signature $(3, 3)$. $\square$

Now if we consider the projectivized $\mathfrak{g}_2$-action on $[\Im(\widetilde{\mathbb{O}}) \setminus 0]/\mathbb{R}^*$ and pull it back to $\mathbb{R}^{3,3}$ via $(\mathbf{q}, \mathbf{p}) \mapsto [(\mathbf{q}, \mathbf{p}, 1)]$, we obtain a realization of $\mathfrak{g}_2$ as a Lie algebra of vector fields on $\mathbb{R}^{3,3}$ tangent to Q, whose restriction to Q forms the symmetry algebra of $(Q, \mathcal{D})$.

Corollary 3.8. *For each* $(A, \mathbf{b}, \mathbf{c}) \in \mathfrak{sl}_3 \oplus \mathbb{R}^3 \oplus (\mathbb{R}^3)^*$ *the vector field on* $\mathbb{R}^{3,3}$,

$$\begin{aligned} X_{A,\mathbf{b},\mathbf{c}} &= [2\mathbf{b} + A\mathbf{q} + \mathbf{p} \times \mathbf{c} - (\mathbf{pb} + \mathbf{cq})\mathbf{q}]\partial_{\mathbf{q}} \\ &\quad + [2\mathbf{c} - \mathbf{p}A + \mathbf{q} \times \mathbf{b} - (\mathbf{pb} + \mathbf{cq})\mathbf{p}]\partial_{\mathbf{p}}, \end{aligned}$$

is tangent to $Q \subset \mathbb{R}^{3,3}$. *The resulting* 14*-dimensional vector space of vector fields on* Q *forms the symmetry algebra of* $(Q, \mathcal{D})$.

Explicitly, if $A = (a^i_j)$, $\mathbf{b} = (b^i)$, $\mathbf{c} = (c_i)$, then

$$\begin{aligned} X_{A,\mathbf{b},\mathbf{c}} &= [2b^i + a^i_j q^j + \epsilon^{ijk} p_j c_k - (p_j b^j + c_j q^j) q^i]\partial_{q^i} \\ &\quad + [2c_i - a^j_i p_j + \epsilon_{ijk} q^j b^k - (p_j b^j + c_j q^j) p_i]\partial_{p_i}. \end{aligned}$$

Proof. Let $\mathbf{u} = (\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{3,3}$; then $\iota(\mathbf{u}) = (\mathbf{u}, 1) \in \Im(\widetilde{\mathbb{O}})$. Any linear vector field X on $\Im(\widetilde{\mathbb{O}})$ can be block decomposed as

$$(X_{11}\mathbf{u} + X_{12}x)\partial_{\mathbf{u}} + (X_{21}\mathbf{u} + X_{22}x)\partial_x,$$

with

$$X_{11} \in \mathrm{End}(\mathbb{R}^{3,3}), \quad X_{12} \in \mathbb{R}^3, \quad X_{21} \in (\mathbb{R}^{3,3})^*, \quad X_{22} \in \mathbb{R}.$$

The induced vector field on $\mathbb{R}^{3,3}$, obtained by projectivization and pulling back via ι, is the quadratic vector field

$$[X_{12} + (X_{11} - X_{22}x)\mathbf{u} - (X_{21}\mathbf{u})\mathbf{u}]\partial_{\mathbf{u}}.$$

Now plug in the formula for X from the last corollary. □

4. Pseudo-Riemannian geometry in signature $(2, 2)$

In this section we relate the geometry of the (2,3,5)-distribution $(Q^5, \mathcal{D})$ given by equations (2) to 4-dimensional conformal geometry by giving Q^5 the structure of a principal $\mathbb{R}^*$-bundle $Q^5 \to M^4$, the "pseudo-Hopf-fibration", inducing on M^4 a split-signature pseudo-Riemannian metric $\mathbf{g}$, which we call the "dancing metric".

We then show in Theorem 4.8 (Section 4.3), using the Maurer-Cartan structure equations of $\mathrm{SL}_3(\mathbb{R})$, that the projection $Q^5 \to M^4$ establishes a bijection between integral curves in $(Q^5, \mathcal{D})$ and (non-degenerate) null curves in $(M^4, \mathbf{g})$ with *parallel self-dual tangent null* 2*-plane.*

A more conceptual explanation to Theorem 4.8 is given in Theorem 4.9, where we show that $(Q^5, \mathcal{D})$ can be naturally embedded in the total space of the *self-dual twistor fibration* $\mathbb{RP}^1 \to \mathbb{T}^+(M^4) \to M^4$ associated with $(M^4, \mathbf{g})$, equipped with its canonical *twistor distribution* $\mathcal{D}^+$, as introduced in [3]. The non-integrability of $\mathcal{D}$ is then seen to be due to the non-vanishing of the self-dual Weyl tensor of $\mathbf{g}$.

4.1. **The pseudo-Hopf-fibration and the dancing metric.**

4.1.1. *First definition of the dancing metric.* Recall from Section 2.1 that $Q^5 = \{(\mathbf{q}, \mathbf{p}) | \mathbf{pq} = 1\} \subset \mathbb{R}^{3,3}$ (the "unit pseudo-sphere"). To each pair $(\mathbf{q}, \mathbf{p}) \in Q^5$ we assign the pair $\Pi(\mathbf{q}, \mathbf{p}) = ([\mathbf{q}], [\mathbf{p}]) = (q, p) \in \mathbb{RP}^2 \times \mathbb{RP}^{2*}$, where $q \in \mathbb{RP}^2$, $p \in \mathbb{RP}^{2*}$ are the points with homogeneous coordinates $\mathbf{q}, \mathbf{p}$ (respectively). Let $\mathbb{I}^3 \subset \mathbb{RP}^2 \times \mathbb{RP}^{2*}$ be the subset of pairs (q, p) given in homogeneous coordinates by the equation $\mathbf{pq} = 0$, also called *incident pairs* (the name comes from the geometric interpretation of such a pair as a (*point, line*) pair, such that the *line* passes through the *point*; more on this in Section 5). It is easy to see from the equation $\mathbf{pq} = 0$ that $\mathbb{I}^3$ is a 3-dimensional closed submanifold of $\mathbb{RP}^2 \times \mathbb{RP}^{2*}$. Its complement

$$M^4 := (\mathbb{RP}^2 \times \mathbb{RP}^{2*}) \setminus \mathbb{I}^3$$

is the set of *non-incident point-line pairs*, a connected open dense subset of $\mathbb{RP}^2 \times \mathbb{RP}^{2*}$. Clearly, if $\mathbf{pq} = 1$, then $([\mathbf{q}], [\mathbf{p}]) \notin \mathbb{I}^3$; thus $\Pi : Q^5 \to M^4$ is well-defined and clearly surjective.

Define an $\mathbb{R}^*$-action on Q^5, where $\lambda \in \mathbb{R}^*$ acts by

$$(\mathbf{q}, \mathbf{p}) \mapsto (\lambda\mathbf{q}, \lambda^{-1}\mathbf{p}), \quad \lambda \in \mathbb{R}^*. \tag{7}$$

This is a free $\mathbb{R}^*$-action whose orbits are precisely the fibers of

$$\Pi : Q^5 \to M^4, \quad (\mathbf{q}, \mathbf{p}) \mapsto ([\mathbf{q}], [\mathbf{p}]).$$

That is, Π is a principal $\mathbb{R}^*$-fibration. Now the quadratic form $\mathbf{pq}$ defines on $\mathbb{R}^{3,3}$ a flat split-signature metric, whose restriction to $Q^5 \subset \mathbb{R}^{3,3}$ is a $(2,3)$-signature metric. Furthermore, the principal $\mathbb{R}^*$-action on Q^5 is by isometries, generated by a negative definite vector field. Combining these, we get:

Proposition 4.1 (**Definition of the dancing metric).** *Restrict the flat split-signature metric $-2d\mathbf{p}\,d\mathbf{q}$ on $\mathbb{R}^{3,3}$ to Q^5. Then there is a unique pseudo-Riemannian metric $\mathbf{g}$ on M^4, of signature $(2,2)$, rendering $\Pi : Q^5 \to M^4$ a pseudo-Riemannian submersion. We call $\mathbf{g}$ the* dancing metric.

Remark. The factor -2 in the above definition is not essential and is introduced merely for simplifying later explicit formulas for $\mathbf{g}$.

Remark. This definition is analogous to the definition of the Fubini-Study metric on $\mathbb{CP}^2$ via the (usual) Hopf fibration $S^1 \to S^5 \to \mathbb{CP}^2$. In fact, M^4 is referred to by some authors as the "para-complex projective plane" and $\mathbf{g}$ as the "para-Fubini-Study metric" [2, 12].

Using the $\mathrm{SL}_3(\mathbb{R})$-invariance of $\mathbf{g}$ it is not difficult to come up with an explicit formula for $\mathbf{g}$ in homogenous coordinates $\mathbf{q}, \mathbf{p}$ on $\mathbb{RP}^2, \mathbb{RP}^{2*}$ (respectively).

Proposition 4.2. *Let $\widetilde{\Pi} : \mathbb{R}^{3,3} \setminus \{\mathbf{pq} = 0\} \to M^4$, $(\mathbf{q}, \mathbf{p}) \mapsto ([\mathbf{q}], [\mathbf{p}])$. Then*

$$\widetilde{\Pi}^*\mathbf{g} = -2\frac{(\mathbf{q} \times d\mathbf{q})(\mathbf{p} \times d\mathbf{p})}{(\mathbf{pq})^2}. \tag{8}$$

Proof. The expression on the right hand side of (8) is a quadratic 2-form, $\mathbb{R}^* \times \mathbb{R}^*$-invariant, $\widetilde{\Pi}$-horizontal (vanishes on $\widetilde{\Pi}$-vertical vectors), and $\mathrm{SL}_3(\mathbb{R})$-invariant. It thus descends to an $\mathrm{SL}_3(\mathbb{R})$-invariant quadratic 2-form on M. By examining the isotropy representation of the stabilizer of a point in M (equation (23) below) we see that M admits a unique $\mathrm{SL}_3(\mathbb{R})$ quadratic 2-form, up to a constant multiple. It is thus sufficient to verify the formula on a single non-null vector, say $e_1 - e^1 \in T_{(e_3, e^3)}Q$. We omit this easy verification. $\square$

Remark. Using standard vector identities, formula (8) can be rewritten also as

$$\widetilde{\Pi}^*\mathbf{g} = -2\frac{(\mathbf{pq})(d\mathbf{p}\,d\mathbf{q}) - (\mathbf{p}\,d\mathbf{q})(d\mathbf{p}\,\mathbf{q})}{(\mathbf{pq})^2}. \tag{9}$$

An advantage of this formula is that it makes sense in higher dimensions, defining the "para-Fubini-Study" metric on $\left[\mathbb{R}^{n+1,n+1} \setminus \{\mathbf{pq} = 0\}\right] / (\mathbb{R}^* \times \mathbb{R}^*)$. It also compares nicely with the formula for the standard Fubini-Study metric $\mathbf{g}_{\mathbf{FS}}$ on $\mathbb{C}P^n = \left[\mathbb{C}^{n+1} \setminus \{0\}\right] / \mathbb{C}^*$, given in complex homogenous coordinates $\mathbf{z} = (z_0, \ldots, z_n)^t \in \mathbb{C}^{n+1}$, $\mathbf{z}^* := \bar{\mathbf{z}}^t$, by

$$\widetilde{\Pi}^*\mathbf{g}_{FS} = \frac{(\mathbf{z}^*\mathbf{z})(d\mathbf{z}^*\,d\mathbf{z}) - (\mathbf{z}^*\,d\mathbf{z})(d\mathbf{z}^*\,\mathbf{z})}{(\mathbf{z}^*\mathbf{z})^2}.$$

We give later three more explicit formulas for $\mathbf{g}$: in Proposition 4.10 (item (a)) $\mathbf{g}$ is expressed in terms of the Maurer-Cartan form of $\mathrm{SL}_3(\mathbb{R})$, analogous to a formula for the Fubini-Study metric on $\mathbb{C}P^n$ in terms of the Maurer-Cartan form of SU_{n+1}. In Proposition 5.5 we give a "cross-ratio" formula for $\mathbf{g}$. Lastly, in Section 5.2 we derive a simple formula in local coordinates for the *conformal class* $[\mathbf{g}]$, using the "dancing condition".

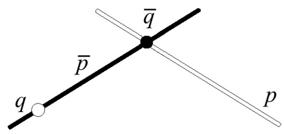

FIGURE 3. The definition of $\Sigma_{\bar{q},\bar{p}}$

4.1.2. *Orientation.* We define an orientation on M^4 via its *para-complex structure.* Namely, using the decomposition $T_{(q,p)}M^4 = T_q\mathbb{RP}^2 \oplus T_p\mathbb{RP}^{2*}$, define $K : TM \to TM$ by $K(q',p') = (q',-p')$. A *para-complex basis* for $T_{(q,p)}M^4$ is then an ordered basis of the form (v_1, v_2, Kv_1, Kv_2). One can check easily that any two such bases are related by a matrix with positive determinant; hence these bases give a well-defined orientation on M^4. See Proposition 4.10(c) below for an alternative definition in terms of the Maurer-Cartan form of $\mathrm{SL}_3(\mathbb{R})$.

4.1.3. *Some properties of the dancing metric.* The dancing metric has remarkable properties. We group in the next theorem some of them.

Theorem 4.3.

(1) $(M^4, \mathbf{g})$ *is the homogeneous symmetric space* $\mathrm{SL}_3(\mathbb{R})/H$, *where* $H \simeq \mathrm{GL}_2(\mathbb{R})$ (*the precise subgroup* H *is described below in Section* 4.4). *The* $\mathrm{SL}_3(\mathbb{R})$-*action on* M^4 *is induced from the standard action on* $\mathbb{R}^{3,3}$, $([\mathbf{q}],[\mathbf{p}]) \mapsto ([g\mathbf{q}],[\mathbf{p}g^{-1}])$. *The* $\mathrm{GL}_2(\mathbb{R})$-*structure endows* M^4 *with a structure of a* para-Kähler *manifold.*

(2) $(M^4, \mathbf{g})$ *is a complete, Einstein, irreducible, pseudo-Riemannian* 4-*manifold of signature* $(2,2)$. *It is self-dual* (*with respect to the above orientation*), *i.e., its anti-self-dual Weyl tensor* $\mathcal{W}^- \equiv 0$, *but is not conformally flat. Its self-dual Weyl curvature tensor* $\mathcal{W}^+$ *is nowhere vanishing, of Petrov type D.*

(3) *The splitting* $T_{(q,p)}M^4 = T_q\mathbb{RP}^2 \oplus T_p\mathbb{RP}^{2*}$ *equips* M^4 *with a pair of complementary null, self-dual, parallel, integrable, rank* 2 *distributions. Their integral leaves generate a pair of foliations of* M^4 *by totally geodesic self-dual null surfaces, the fibers of the double fibration*

$$\begin{tikzcd} & M^4 \arrow[dl, "\pi"'] \arrow[dr, "\bar{\pi}"] & \\ \mathbb{RP}^2 & & \mathbb{RP}^{2*} \end{tikzcd}$$

(4) M^4 *admits a* 3-*parameter family of anti-self-dual totally geodesic null surfaces, naturally parametrized by the incidence variety* $\mathbb{I}^3 := \{(\bar{q},\bar{p}) | \bar{q} \in \bar{p}\} \subset \mathbb{RP}^2 \times \mathbb{RP}^{2*}$. *For each incident pair* $(\bar{q},\bar{p}) \in \mathbb{I}^3$, *the corresponding surface is the set* $\Sigma_{\bar{q},\bar{p}}$ *of non-incident pairs* (q,p) *such that* $q \in \bar{p}$ *and* $\bar{q} \in p$.

Remark. The last point (4) can be reformulated as follows: let $N^5 \subset M^4 \times \mathbb{I}^3$ be defined via the incidence diagram of Figure 3, i.e.,

$$N^5 = \{(q,p,\bar{q},\bar{p}) | q \notin p, \bar{q} \in \bar{p}, q \in \bar{p}, \bar{q} \in p\} \subset \mathbb{RP}^2 \times \mathbb{RP}^{2*} \times \mathbb{RP}^2 \times \mathbb{RP}^{2*}.$$

Then N^5 is a 5-dimensional submanifold of $M^4 \times \mathbb{I}^3$, equipped with the double fibration

$$\begin{tikzcd} & N^5 \arrow[dl, "\pi_{12}"'] \arrow[dr, "\pi_{34}"] & \\ M^4 & & \mathbb{I}^3 \end{tikzcd} \tag{10}$$

The right hand fibration $\pi_{34} : N^5 \to \mathbb{I}^3$ foliates N^5 by 2-dimensional surfaces, each of which projects via $\pi_{12} : N^5 \to M^4$ to one of the surfaces $\Sigma_{\bar{q},\bar{p}}$. That is, $\Sigma_{\bar{q},\bar{p}} = \pi_{12}\left(\pi_{34}^{-1}(\bar{q},\bar{p})\right)$. The left hand fibration $\pi_{12} : N^5 \to M^4$ foliates N^5 by projective lines and can be naturally identified with the *anti-self-dual twistor fibration* $\mathbb{T}^- M^4 \to M^4$ associated with $(M^4, \mathbf{g})$ (see Section 4.2.4). The fibers of π_{34} then correspond to the integral leaves of the anti-self-dual twistor distribution $\mathcal{D}^-$, which is integrable in our case, due to the vanishing of $\mathcal{W}^-$ (see Corollary 4.11, Section 4.4 below).

Most claims of this theorem can be found in various sources in the literature (see e.g. [2] and the many references within). Using the Maurer-Cartan equations of $SL_3(\mathbb{R})$ (Section 4.4), it is quite straightforward to prove these results. Alternatively, one can write explicitly the dancing metric in local coordinates (Section 5.2) and let a computer calculate curvature, symmetries, etc.

4.2. **Rudiments of 4-dimensional geometry in split-signature.**

4.2.1. *Linear algebra.* Let V be an oriented 4-dimensional real vector space equipped with a quadratic form $\langle\, , \rangle$ of signature $(+ + - -)$. It is convenient to introduce *null bases* in such a V. This is a basis $\{e_1, e_2, e^1, e^2\} \subset V$ such that

$$\langle e_a, e_b \rangle = \langle e^a, e^b \rangle = 0,\ \langle e^a, e_b \rangle = \delta^a_b, \qquad a, b = 1, 2.$$

Note that if $\{x^1, x^2, x_1, x_2\} \subset V^*$ is the *dual basis*, i.e., $x^a(e^b) = x_a(e_b) = 0, x^a(e_b) = x_b(e^a) = \delta^a_b$, then

$$\langle\, , \rangle = 2(x^1 x_1 + x^2 x_2). \tag{11}$$

Remark. Our convention is that the symmetric tensor product $xy \in S^2 V^*$ of two elements $x, y \in V^*$ is the symmetric bilinear form

$$(xy)(v, w) := [x(v)y(w) + y(v)x(w)]/2, \quad v, w \in V. \tag{12}$$

Now let $vol := x^1 \wedge x^2 \wedge x_1 \wedge x_2 \in \Lambda^4 V^*$ and let $* : \Lambda^2 V^* \to \Lambda^2 V^*$ be the corresponding Hodge dual, satisfying $\alpha \wedge *\beta = \langle \alpha, \beta \rangle vol$, $\alpha, \beta \in \Lambda^2(V^*)$. Then $*^2 = 1$ and one has the splitting

$$\Lambda^2 V^* = \Lambda^2_+ V^* \oplus \Lambda^2_- V^*, \tag{13}$$

where $\Lambda^2_\pm V^*$ are the ± 1 eigenspaces of $*$, called the SD (self-dual) and ASD (anti-self-dual) 2-forms (respectively).

Let $SO_{2,2} \subset GL(V)$ be the corresponding orientation-preserving orthogonal group and let $\mathfrak{so}_{2,2} \subset \operatorname{End} V$ be its Lie algebra. With respect to a null basis, the matrices of elements in $\mathfrak{so}_{2,2}$ are of the form

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \quad A, B, C \in Mat_{2\times 2}(\mathbb{R}),\ B^t = -B,\ C^t = -C. \tag{14}$$

There are a natural isomorphism (equivalence of $\mathrm{SO}_{2,2}$-representations)

$$\mathfrak{so}_{2,2} \xrightarrow{\sim} \Lambda^2 V^*, \quad T \mapsto \frac{1}{2}\langle\,\cdot\,, T\,\cdot\,\rangle \tag{15}$$

and a Lie algebra decomposition $\mathfrak{so}_{2,2} = \mathfrak{sl}_2^+(\mathbb{R}) \oplus \mathfrak{sl}_2^-(\mathbb{R})$, given by

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ 0 & -A_0^t \end{pmatrix} + \begin{pmatrix} \frac{\mathrm{tr}A}{2}\mathrm{I} & B \\ C & -\frac{\mathrm{tr}A}{2}\mathrm{I} \end{pmatrix}, \quad A_0 = A - \frac{\mathrm{tr}A}{2}\mathrm{I} \in \mathfrak{sl}_2(\mathbb{R}),$$

matching the decomposition of (13), i.e., $\mathfrak{sl}_2^\pm(\mathbb{R}) \xrightarrow{\sim} \Lambda^2_\pm V^*$.

Given a 2-plane $W \subset V$ pick a basis θ^1, θ^2 of the annihilator $W^0 \subset V^*$ and let $\beta = \theta^1 \wedge \theta^2$. If we pick another basis of W^0, then β is multiplied by a non-zero constant (the determinant of the matrix of change of basis); hence $\mathbb{R}\beta \subset \Lambda^2(V^*)$ is well-defined in terms of W alone. This defines the *Plücker embedding* of the grassmannian of 2-planes $Gr(2,V) \hookrightarrow \mathbb{P}(\Lambda^2 V^*) \simeq \mathbb{RP}^5$. Its image is given in homogeneous coordinates by the quadratic equation $\beta \wedge \beta = 0$. We say that a 2-plane W is SD (self-dual) if $\mathbb{R}\beta \subset \Lambda^2_+ V^*$ and ASD (anti-self-dual) if $\mathbb{R}\beta \subset \Lambda^2_- V^*$. We denote by

$$\mathbb{T}^+ V := \{W \subset V \mid W \text{ is an SD 2-plane}\}.$$

Using the Plücker embedding, $\mathbb{T}^+V$ is naturally identified with the conic in $\mathbb{P}(\Lambda^2_+ V^*) \simeq \mathbb{RP}^2$ given by the equations $\beta \wedge \beta = 0$, $*\beta = \beta$. Similarly for the ASD 2-planes $\mathbb{T}^- V$.

A *null subspace* is a subspace of V on which the quadratic form $\langle\,,\rangle$ vanishes. The maximum dimension of a null subspace is 2, in which case we call it a *null 2-plane*. It turns out that the null 2-planes are precisely the SD and ASD 2-planes.

Proposition 4.4. *Let V be an oriented 4-dimensional vector space equipped with a quadratic form of signature $(2,2)$. Then*

(1) *A 2-plane $W \subset V$ is null if and only if it is SD or ASD. Thus the space $Gr_0(2,V)$ of null 2-planes in V is naturally identified with*

$$Gr_0(2,V) = (\mathbb{T}^+V) \sqcup (\mathbb{T}^-V), \quad \mathbb{T}^\pm V \simeq \mathbb{RP}^1.$$

(2) *Every 1-dimensional null subspace $N \subset V$ is the intersection of precisely two null 2-planes, one SD and one ASD, $N = W^+ \cap W^-$.*

The proof is elementary (omitted). Let us describe briefly the picture that emerges from the last assertion (see Figure 4). The set of 1-dimensional null subspaces $N \subset V$ forms the *projectivized null cone* $\mathbb{P}C$, a 2-dimensional quadric surface in $\mathbb{P}V \simeq \mathbb{RP}^3$, given in homogeneous coordinates, with respect to a null basis in V, by the equation $x^a x_a = 0$. The statement then is that the SD and ASD null 2-planes in V define a *double ruling* of $\mathbb{P}C$. That is, the surface $\mathbb{P}C \subset \mathbb{P}V$, although not flat, contains many lines, forming a pair of foliations, so that through each point $e \in \mathbb{P}C$ pass exactly two lines, one from each foliation. The two lines through e can also be found by intersecting $\mathbb{P}C$ with the tangent plane to $\mathbb{P}C$ at e. In some affine chart, if $\mathbb{P}C$ is given by $z = xy$ and $e = (x_0, y_0, x_0y_0)$, then the two null lines through e are given by $z = x_0 y$, $z = x y_0$.

4.2.2. *The Levi-Civita connection and its curvature.* Now let M be an oriented smooth 4-manifold equipped with a pseudo-Riemannian metric $\mathbf{g}$ of signature $(2,2)$. Denote by $\boldsymbol{\Lambda}^k := \Lambda^k(T^*M)$ the bundle of differential k-forms on M and by $\Gamma(\boldsymbol{\Lambda}^k)$ its space of smooth sections. In a (local) null coframe $\eta = (\eta^1, \eta^2, \eta_1, \eta_2)^t \in \Gamma(\boldsymbol{\Lambda}^1 \otimes \mathbb{R}^4)$

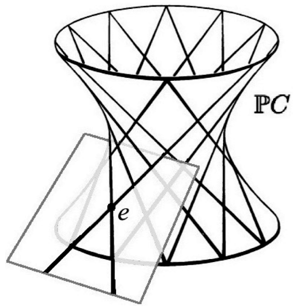

FIGURE 4. The double-ruling of the projectivized null cone $\mathbb{P}C \subset \mathbb{P}V$.

the metric is given by $\mathbf{g} = 2\eta_a\eta^a$, and the Levi-Civita connection is given by the unique $\mathfrak{so}_{2,2}$-valued 1-form Θ satisfying $d\eta + \Theta \wedge \eta = 0$; i.e., the connection is *torsion-free.* The associated covariant derivative is $\nabla\eta = -\Theta \otimes \eta$, and the curvature is the $\mathfrak{so}_{2,2}$-valued 2-form $\Phi = d\Theta + \Theta \wedge \Theta$. The curvature form Φ defines via the isomorphism $\mathfrak{so}_{2,2} \simeq \Lambda^2(T^*_m M)$ of (15) the curvature *operator* $\mathcal{R} \in \Gamma(\mathrm{End}(\mathbf{\Lambda}^2))$, which is self-adjoint with respect to $\mathbf{g}$, i.e., $\mathcal{R}^* = \mathcal{R}$. Now we use the decomposition $\mathbf{\Lambda}^2 = \mathbf{\Lambda}^2_+ \oplus \mathbf{\Lambda}^2_-$ to block decompose

$$\mathcal{R} = \begin{pmatrix} \mathcal{A}^+ & \mathcal{B} \\ \mathcal{B}^* & \mathcal{A}^- \end{pmatrix}, \tag{16}$$

where $\mathcal{B} \in \mathrm{Hom}(\mathbf{\Lambda}^2_+, \mathbf{\Lambda}^2_-)$ and $\mathcal{A}^\pm \in \mathrm{End}\,\mathbf{\Lambda}^2_\pm$ are self-adjoint. This can be further refined into an irreducible decomposition

$$\mathcal{R} \sim (\mathrm{tr}\mathcal{A}^\pm,\, \mathcal{B},\, \mathcal{A}^+ - \tfrac{1}{3}\mathrm{tr}\mathcal{A}_+,\, \mathcal{A}^- - \tfrac{1}{3}\mathrm{tr}\mathcal{A}^-),$$

where $\mathrm{tr}\mathcal{A}^+ = \mathrm{tr}\mathcal{A}^- = \frac{1}{4}$ scalar curvature, $\mathcal{B}$ is the traceless Ricci tensor, and the last two components are traceless self-adjoint endomorphisms $\mathcal{W}^\pm \in \Gamma(\mathrm{End}_0(\mathbf{\Lambda}^2_\pm))$, defining the conformally invariant *Weyl tensor*, $\mathcal{W} := \mathcal{W}^+ \oplus \mathcal{W}^-$ [27]. Thus the metric is *Einstein* if and only if $\mathcal{B} = 0$, *conformally flat* if and only if $\mathcal{W} = 0$, *self-dual* if and only if $\mathcal{W} = \mathcal{W}^+$ (i.e., $\mathcal{W}^- = 0$), and *anti-self-dual* if and only if $\mathcal{W} = \mathcal{W}^-$ (i.e., $\mathcal{W}^+ = 0$).

4.2.3. *Principal null 2-planes.* Associated with the Weyl tensor $\mathcal{W}$ are its *principal null 2-planes*, as follows. Recall from Section 4.2.1 (just before Proposition 4.4) that a 2-plane $W \subset T_m M$ corresponds to a unique 1-dimensional space $\mathbb{R}\beta \subset \Lambda^2(T^*_m M)$ satisfying $\beta \wedge \beta = 0$. Also, W is SD if and only if $\beta \in \mathbf{\Lambda}^2_+$, ASD if and only if $\beta \in \mathbf{\Lambda}^2_-$.

Definition 4.5. A null 2-plane $W \subset T_m M$ is *principal* if the associated non-zero elements $\beta \in \Lambda^2(T^*_m M)$ satisfy $\beta \wedge \mathcal{W}\beta = 0$.

If $\mathcal{W}^+_m = 0$, then all SD null 2-planes in $T_m M$ are principal (by definition). Otherwise, the quadratic equation $\beta \wedge \mathcal{W}^+\beta = 0$ defines a conic (possibly degenerate) in $\mathbb{P}\Lambda^2_+(T^*_m M) \simeq \mathbb{RP}^2$, intersecting the conic $\mathbb{T}^+(T_m M)$ given by $\beta \wedge \beta = 0$ in at most 4 points, corresponding precisely to the principal SD 2-planes. The possible patterns of intersection of these two conics give rise to an algebraic classification of the SD Weyl tensor $\mathcal{W}^+$, called the *Petrov classification.* A similar classification holds for $\mathcal{W}^-$.

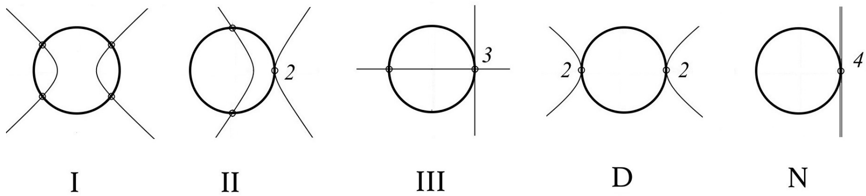

FIGURE 5. The Petrov classification

Remark. The diagram in Figure 5 depicts the classification over $\mathbb{C}$. In the real case (such as ours) there are more subcases, as some of the intersection points might be non-real. See for example [16] for the complete classification.

4.2.4. *The twistor fibration and distribution.* (We shall state the results for the SD twistor fibration, but they apply verbatim to the ASD case as well.) Let M be an oriented 4-manifold with a split-signature pseudo-Riemannian metric, as in the previous subsection. The *SD* (*self-dual*) *twistor fibration* is the fiber bundle

$$\mathbb{RP}^1 \to \mathbb{T}^+M \to M$$

whose fiber at a point $m \in M$ is the set $\mathbb{T}^+(T_mM)$ of SD null 2-planes in T_mM (see Proposition 4.4 of Section 4.2.1 above). The total space $\mathbb{T}^+M$ is a 5-manifold equipped with a natural rank 2 distribution $\mathcal{D}^+ \subset T(\mathbb{T}^+M)$, the *SD twistor distribution*, defined by the Levi-Civita connection, as follows: a point $\tilde{m} \in \mathbb{T}^+_mM$ corresponds to an SD 2-plane $W \subset T_mM$; the 2-plane $\mathcal{D}^+_{\tilde{m}} \subset T_{\tilde{m}}(\mathbb{T}^+M)$ is the horizontal lift of W via the Levi-Civita connection. One can check that $\mathcal{D}^+$ depends only on the conformal class $[\mathbf{g}]$ of the metric on M. By construction, the integral curves of $\mathcal{D}^+$ project to null curves in M with parallel self-dual tangent 2-plane. Conversely, each null curve in M with parallel SD null 2-plane lifts uniquely to an integral curve of $(\mathbb{T}^+M, \mathcal{D}^+)$.

This is the split-signature version of the famous twistor construction of Roger Penrose [25]. A standard feature of the twistor construction is the relation between the integrability properties of $\mathcal{D}^+$ and the vanishing of the SD Weyl tensor $\mathcal{W}^+$. Namely, $\mathcal{D}^+$ is integrable if and only if $\mathcal{W}^+ \equiv 0$ (i.e., M is ASD). Less standard is the case of non-vanishing $\mathcal{W}^+$, treated by An-Nurowski in [3].

Theorem 4.6 ([3]). *Let $(\mathbb{T}^+M, \mathcal{D}^+)$ be the SD twistor space and distribution of a split-signature oriented pseudo-Riemannian conformal 4-manifold $(M, [\mathbf{g}])$ with a nowhere-vanishing SD Weyl tensor $\mathcal{W}^+$. Then $\mathcal{D}^+$ is $(2,3,5)$ away from the principal locus of $\mathbb{T}^+M$. That is, $\mathcal{D}^+$ is $(2,3,5)$ when restricted to the open subset $\mathbb{T}^+_*M \subset \mathbb{T}^+M$ obtained by removing the set of points corresponding to the principal SD 2-planes (at most 4 points on each fiber of $\mathbb{T}^+M \to M$; see Definition 4.5 above).*

See the theorem in [3], right before Corollary 1.

4.3. **The tangent SD 2-plane along a null curve in the dancing space.** Now we return to our case of $M^4 \subset \mathbb{RP}^2 \times \mathbb{RP}^{2*}$ equipped with the dancing metric $\mathbf{g}$, as defined in Proposition 4.1.

Definition 4.7. Let Γ be a parametrized curve in M^4, $\Gamma(t) = (q(t), p(t))$. Then Γ is *non-degenerate* if $q(t), p(t)$ are regular curves in $\mathbb{RP}^2, \mathbb{RP}^{2*}$ (respectively); i.e., $q'(t) \neq 0$ and $p'(t) \neq 0$ for all t.

Note that the non-degeneracy condition is reparametrization independent; hence it applies to unparametrized curves $\Gamma \subset M$ (1-dimensonal submanifolds). It means that Γ is nowhere tangent to the leaves of the double fibration $\mathbb{RP}^2 \leftarrow M^4 \rightarrow \mathbb{RP}^{2*}$. Equivalently, the projections of Γ to $\mathbb{RP}^2$ and $\mathbb{RP}^{2*}$ are non-singular.

Now let Γ be a null curve in $(M^4, \mathbf{g})$. Then, by Proposition 4.4, there are two tangent null 2-plane fields defined along Γ, one SD and the other ASD, whose intersection is the tangent line field along Γ.

Theorem 4.8. *Every integral curve $\tilde{\Gamma}$ of $(Q^5, \mathcal{D})$ projects to a non-degenerate null curve Γ in M^4 with a parallel SD tangent 2-plane. Conversely, every non-degenerate null curve in $(M^4, \mathbf{g})$ with parallel SD tangent 2-plane lifts uniquely to an integral curve of $(Q^5, \mathcal{D})$.*

Theorem 4.9. *For each $(\mathbf{q}, \mathbf{p}) \in Q^5$, the 2-plane*

$$\Pi_* \mathcal{D}_{(\mathbf{q},\mathbf{p})} \subset T_{(q,p)} M^4,$$

where $(q,p) = \Pi((\mathbf{q}, \mathbf{p}))$ is a non-principal self-dual 2-plane. The resulting map

$$Q^5 \to \mathbb{T}^+ M^4, \quad (\mathbf{q}, \mathbf{p}) \mapsto \Pi_* \mathcal{D}_{(\mathbf{q},\mathbf{p})},$$

is an $\mathrm{SL}_3(\mathbb{R})$-equivariant embedding, identifying Q^5 with the non-principal locus of $\mathcal{D}^+$ in $\mathbb{T}^+ M^4$, and mapping $\mathcal{D}$ over to $\mathcal{D}^+$.

The proofs of these two theorems will be carried out in the next subsection, using the Maurer-Cartan structure equations of $\mathrm{SL}_3(\mathbb{R})$.

4.4. **Proofs of Theorems 4.8 and 4.9.** Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$ and let $\{e^1, e^2, e^3\}$ be the dual basis of $(\mathbb{R}^3)^*$. Recall (Section 2.4) that $G = \mathrm{SL}_3(\mathbb{R})$ acts transitively on $(Q^5, \mathcal{D})$ and $(M^4, \mathbf{g})$ by $g \cdot (\mathbf{q}, \mathbf{p}) = (g\mathbf{q}, \mathbf{p}g^{-1})$, $g \cdot ([\mathbf{q}], [\mathbf{p}]) = ([g\mathbf{q}], [\mathbf{p}g^{-1}])$, preserving $\mathcal{D}$ and $\mathbf{g}$ (respectively). Fix $\tilde{m}_0 = (e_3, e^3) \in Q^5$ and $m_0 = \Pi(\tilde{m}_0) = ([e_3], [e^3]) \in M^4$. Define

$$\begin{tikzcd} G \arrow[r, "\tilde{j}"] \arrow[dr, "j"'] & Q^5 \arrow[d, "\Pi"] \\ & M^4 \end{tikzcd} \tag{17}$$

by $\tilde{j}(g) = g \cdot \tilde{m}_0 = (ge_3, e^3 g^{-1})$, $j(g) = g \cdot m_0 = ([ge_3], [e^3 g^{-1}]) = (\Pi \circ \tilde{j})(g)$. Then j is a principal H-fibration and $\tilde{j}$ is a principal H_0-fibration, where

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & a^{-1} \end{pmatrix} \mid A \in \mathrm{GL}_2(\mathbb{R}),\ a = \det(A) \right\} \simeq \mathrm{GL}_2(\mathbb{R})$$

is the stabilizer subgroup of m_0, with Lie algebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} X & 0 \\ 0 & -x \end{pmatrix} \mid X \in \mathfrak{gl}_2(\mathbb{R}),\ x = \mathrm{tr}(X) \right\} \simeq \mathfrak{gl}_2(\mathbb{R}), \tag{18}$$

and

$$H_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \mid A \in \mathrm{SL}_2(\mathbb{R}) \right\} \simeq \mathrm{SL}_2(\mathbb{R})$$

is the stabilizer subgroup of $\tilde{m}_0$, with Lie algebra

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \mid X \in \mathfrak{sl}_2(\mathbb{R}) \right\} \simeq \mathfrak{sl}_2(\mathbb{R}).$$

The left-invariant MC (Maurer-Cartan) form on $G = \mathrm{SL}_3(\mathbb{R})$ is the $\mathfrak{g}$-valued 1-form $\omega = (\omega^i{}_j) := g^{-1}dg$, i.e., $\mathrm{tr}(\omega) = \omega^i{}_i = 0$, $i, j \in \{1, 2, 3\}$ (using, as always, the summation convention on repeated indices). The components of ω provide a global coframing on G, whose basic properties (immediate from its definition) are

$$
\begin{array}{llll}
 & \text{(a)} & \omega_e = id_{\mathfrak{g}}, & \\
(19) & \text{(b)} & (L_g)^*\omega = \omega & \text{(left invariance)}, \\
 & \text{(c)} & (R_g)^*\omega = g^{-1}\omega g & \text{(right Ad-equivariance)}, \\
 & \text{(d)} & d\omega = -\omega \wedge \omega & \text{(the MC structure equation)}.
\end{array}
$$

Now let us rename the components of ω:

$$
(20) \qquad \eta^a := \omega^a{}_3,\ \eta_b := \omega^3{}_b,\ \phi := \omega^a{}_a = -\omega^3{}_3,\ \theta^a_{\ b} := \omega^a{}_b + \delta^a_b\phi, \quad a, b \in \{1, 2\}.
$$

Furthermore, introduce the matrix notation

$$
(21) \quad \eta := \begin{pmatrix} \eta^1 \\ \eta^2 \\ \eta_1 \\ \eta_2 \end{pmatrix}, \Theta := \begin{pmatrix} \theta & 0 \\ 0 & -\theta^t \end{pmatrix}, \theta := (\theta^a_{\ b}) = \begin{pmatrix} 2\omega^1{}_1 + \omega^2{}_2 & \omega^1{}_2 \\ \omega^2{}_1 & \omega^1{}_1 + 2\omega^2{}_2 \end{pmatrix}.
$$

With this notation,(19c) now reads

$$
(22) \qquad (R_h)^*\eta = \rho_h^{-1}\eta, \quad (R_h)^*\Theta = \rho_h^{-1}\Theta\rho_h, \quad h \in H,
$$

where $\rho : H \to \mathrm{SO}_{2,2}$ is the *isotropy representation*

$$
(23) \quad h = \begin{pmatrix} A & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto \rho_h = \begin{pmatrix} aA & 0 \\ 0 & (aA^t)^{-1} \end{pmatrix}, \quad A \in \mathrm{GL}_2(\mathbb{R}), \quad a = \det A.
$$

The MC structure equation (19d) also breaks into two equations:

$$
(24) \qquad d\eta + \Theta \wedge \eta = 0, \qquad d\Theta + \Theta \wedge \Theta = \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^t \end{pmatrix},
$$

where

$$
(25) \qquad \varphi := d\theta + \theta \wedge \theta = \begin{pmatrix} 2\eta_1 \wedge \eta^1 + \eta_2 \wedge \eta^2 & \eta_2 \wedge \eta^1 \\ \eta_1 \wedge \eta^2 & \eta_1 \wedge \eta^1 + 2\eta_2 \wedge \eta^2 \end{pmatrix}.
$$

From formula (18) for $\mathfrak{h}$, we see that the four 1-forms $\eta^a, \eta_b \in \Omega^1(G)$ are pointwise linearly independent and j-horizontal, i.e., vanish on the fibers of $j : G \to M$, hence span $j^*(T^*M) \subset T^*(G)$. Similarly, η^a, η_b, ϕ span $\tilde{j}^*(T^*Q^5)$.

Proposition 4.10. *Consider the principal fibrations $j, \tilde{j}$ of* (17) *and the left-invariant* 1*-forms $\eta, \phi, \theta, \Theta, \varphi$ on G, as defined above in* (20)-(25)*. Then*

(a) *$j^*\mathbf{g} = 2\eta_a\eta^a$, where $\mathbf{g}$ is the dancing metric on M^4, as defined in Proposition* 4.1.

(b) *Let ∇ be the covariant derivative on T^*M associated with the Levi-Civita connection of $\mathbf{g}$ and let $\widetilde{\nabla} = j^*(\nabla)$ be its pull-back to $j^*(T^*M)$. Then $\widetilde{\nabla}\eta^a = -\theta^a_{\ b} \otimes \eta^b$, $\widetilde{\nabla}\eta_b = \theta^a_{\ b} \otimes \eta_a$, or in matrix form, $\widetilde{\nabla}\eta = -\Theta \otimes \eta$. The associated curvature* 2*-form is $\Phi := d\Theta + \Theta \wedge \Theta$, given in terms of η by* (24)-(25) *above.*

(c) *Let $vol \in \Omega^4(M)$ be the positively oriented unit volume form on M^4 (see Section* 4.1.2*). Then $j^*(vol) = \eta^1 \wedge \eta^2 \wedge \eta_1 \wedge \eta_2$.*

(d) *Let $\mathcal{D} \subset TQ^5$ be the rank* 2 *distribution given by $d\mathbf{p} = \mathbf{q} \times d\mathbf{q}$ and let $\mathcal{D}^0 \subset T^*Q^5$ be its annihilator. Then $\tilde{j}^*(\mathcal{D}^0) = \mathrm{Span}\{\eta^2 - \eta_1,\ \eta^1 + \eta_2,\ \phi\}$.*

Remark. We can rephrase the above in terms of coframes on M and Q, as follows: let σ be a local section of $j : G \to M$. Then (a) $\hat\eta = \sigma^*\eta$ is a null coframe on M, so that $\mathbf{g} = 2\hat\eta_a\hat\eta^a$, (b) $\hat\Theta = \sigma^*(\Theta)$ is the connection 1-form of the Levi-Civita connection of $\mathbf{g}$ with respect to the coframe $\hat\eta$, (c) $vol = \hat\eta^1 \wedge \hat\eta^2 \wedge \hat\eta_1 \wedge \hat\eta_2$, and (d) $\mathcal{D} = \mathrm{Ker}\{\tilde\eta^2 - \tilde\eta_1,\ \tilde\eta^1 + \tilde\eta_2,\ \tilde\phi\}$, where $\tilde\eta = \tilde\sigma^*\eta$, $\tilde\phi = \tilde\sigma^*\phi$ and $\tilde\sigma = \sigma \circ \Pi$ (a local section of $\tilde j : G \to Q$).

Proof. (a) First, the formula $(R_h)^*\eta = \rho_h^{-1}\eta$ of equation (22) implies that $\eta_a\eta^a$, a G-left-invariant j-horizontal symmetric 2-form on G, is H-right-invariant, hence descends to a well-defined G-invariant symmetric 2-form on M. Next, by examining the isotropy representation of H (equation (23)), one sees that $T_{m_0}M$ admits a unique H-invariant quadratic form, up to a constant multiple; hence M admits a unique G-invariant symmetric 2-form, up to a constant multiple. It follows that it is enough to verify the equation $j^*\mathbf{g} = 2\eta_a\eta^a$ on a single non-null element $Y \in \mathfrak{g} = T_eG$; for example, $Y = Y_1$ from the proof of Proposition 2.3. We omit this (easy) verification.

(b) The relations $d\eta + \Theta \wedge \eta = 0$, $(R_h)^*\Theta = \rho_h^{-1}\Theta\rho_h$, and the formula for Θ ((21)-(24)) show that Θ is an $\mathfrak{so}_{2,2}$-valued 1-form on G, defining a torsion-free $\mathrm{SO}_{2,2}$-connection on T^*M, hence is in fact the Levi-Civita connection of $\mathbf{g}$.

(c) First, one verifies that $\eta^1\wedge\eta^2\wedge\eta_1\wedge\eta_2$ is a volume form of norm 1 with respect to $2\eta_a\eta^a$. Then, to compare to the orientation definition of Section 4.1.2, we check that $K^*\eta^a = \eta^a$, $K^*\eta_b = -\eta_b$; hence $\eta^1 + \eta_1, \eta^2 + \eta_2, \eta^1 - \eta_1, \eta^2 - \eta_2$ is a para-complex coframe. Now one calculates $(\eta^1+\eta_1)\wedge(\eta^2+\eta_2)\wedge(\eta^1-\eta_1)\wedge(\eta^2-\eta_2) = 4\eta^1\wedge\eta^2\wedge\eta_1\wedge\eta_2$; hence $\eta^1, \eta^2, \eta_1, \eta_2$ is a positively oriented coframe.

(d) Let $E_j : G \to \mathbb{R}^3$ be the function that assigns to an element $g \in G$ its j-th column, $j = 1,2,3$. Then $\omega = g^{-1}dg$ is equivalent to $dE_j = E_i\omega^i{}_j$. Next, let $E^i : G \to (\mathbb{R}^3)^*$ be the function assigning to $g \in G$ the i-th row of g^{-1}. Then clearly $E^iE_j = \delta^i_j$ (matrix multiplication of a row by column vector), and by taking the exterior derivative of the last equation we obtain $dE^i = -\omega^i{}_jE^j$. Also, $\det(g) = 1$ implies that $E_i \times E_j = \epsilon_{ijk}E^k$, $E^i \times E^j = \epsilon^{ijk}E_k$. Next, by definition of $\tilde j$, $E_3 = \mathbf{q}\circ\tilde j$, $E^3 = \mathbf{p}\circ\tilde j$. Now we calculate $\tilde j^*(d\mathbf{p} - \mathbf{q}\times d\mathbf{q}) = dE^3 - E_3 \times dE_3 = -\omega^3{}_jE^j - (E_3\times E_i)\omega^i{}_3 = (\eta^2 - \eta_1)E^1 - (\eta^1+\eta_2)E^2 + \phi E^3$. □

Corollary 4.11 (Proofs of Theorems 4.8 and 4.9)**.**

(a) $(M^4, \mathbf{g})$ *is Einstein but not Ricci-flat, SD (i.e.,* $\mathcal{W}^- \equiv 0$*), and* $\mathcal{W}^+$ *is nowhere vanishing, of Petrov type D (see Figure 5). More precisely, at each* $(q,p) \in M$ *there are exactly two principal SD null 2-planes, each of multiplicity 2, given by* $T_q\mathbb{RP}^2 \oplus \{0\}$ *and* $\{0\} \oplus T_p\mathbb{RP}^{2*}$.

(b) *Every integral curve* $\tilde\Gamma$ *of* $(Q, \mathcal{D})$ *projects to a non-degenerate null curve* $\Gamma := \Pi\circ\tilde\Gamma$ *in* $(M^4, \mathbf{g})$ *with parallel SD tangent 2-plane.*

(c) *Every non-degenerate null curve* Γ *in* $(M^4, \mathbf{g})$ *with parallel SD tangent 2-plane lifts uniquely to an integral curve* $\tilde\Gamma$ *of* $(Q^5, \mathcal{D})$.

(d) *For every* $\tilde m \in Q^5$, $\Pi_*\mathcal{D}_{\tilde m} \subset T_{\Pi(\tilde m)}M^4$ *is a non-principal SD null 2-plane.*

(e) *Let* $\mathbb{T}^+_*M \subset \mathbb{T}^+M$ *be the non-principal locus (the complement of the principal points). Then the map* $\nu : Q^5 \to \mathbb{T}^+_*M$, $\tilde m \mapsto \Pi_*\mathcal{D}_{\tilde m}$ *is an* $\mathrm{SL}_3(\mathbb{R})$*-equivariant diffeomorphism, mapping* $\mathcal{D}$ *onto* $\mathcal{D}^+$.

Proof. (a) By Proposition 4.10(a) and 4.10(c), the coframe $\eta^1, \eta^2, \eta_1, \eta_2$ is null and positively oriented. It follows from the definition of the Hodge dual that

$$j^*(\Lambda^2_+ M) = \mathrm{Span}\{\eta_1 \wedge \eta^1 + \eta_2 \wedge \eta^2,\ \eta^1 \wedge \eta^2,\ \eta_1 \wedge \eta_2\}, \tag{26a}$$

$$j^*(\Lambda^2_- M) = \mathrm{Span}\{\eta_1 \wedge \eta^1 - \eta_2 \wedge \eta^2,\ \eta_1 \wedge \eta^2,\ \eta_2 \wedge \eta^1\}. \tag{26b}$$

Then using the formula for the curvature form Φ (equations (24)-(25)) and the definition of the curvature operator $\mathcal{R}$ (Section 4.2.2), one finds that $j^*(\mathcal{R})$ is diagonal in the above bases, with matrix

$$j^*(\mathcal{R}) = \left(\begin{array}{ccc|ccc} -3 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ \hline & & & -1 & & \\ & & & & -1 & \\ & & & & & -1 \end{array}\right).$$

Comparing this expression with the decomposition of $\mathcal{R}$ of (16), we see that the dancing metric is Einstein ($\mathcal{B} \equiv 0$), the scalar curvature is -12, $\mathcal{W}^- \equiv 0$, and

$$j^*(\mathcal{W}^+) = \begin{pmatrix} -2 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Now let a, b, c be the coordinates dual to the basis of $j^*(\Lambda^2_+ M)$ of equation (26a). Then $\beta \wedge \beta = 0$ is given by $a^2 - bc = 0$ and $\beta \wedge \mathcal{W}^+ \beta = 0$ by $2a^2 + bc = 0$. This system of two homogeneous equations has two non-zero solutions (up to a non-zero multiple), $a = b = 0$ and $a = c = 0$, each with multiplicity 2 (the pair of conics defined in each fiber of $\mathbb{P}\Lambda^2_+ M$ by these equations is tangent at its two intersection points). The corresponding SD 2-forms are $\eta_1 \wedge \eta_2, \eta^1 \wedge \eta^2$, corresponding to the principal SD null 2-planes $T_q\mathbb{RP}^2 \oplus \{0\}$ and $\{0\} \oplus T_p\mathbb{RP}^{2*}$ (respectively), as claimed.

(b) Let $\tilde\Gamma(t) = (\mathbf{q}(t), \mathbf{p}(t))$ be a regular parametrization of an integral curve of $(Q^5, \mathcal{D})$; i.e., $\tilde\Gamma' = (\mathbf{q}', \mathbf{p}')$ is nowhere vanishing and $\mathbf{p}' = \mathbf{q} \times \mathbf{q}'$. We first show that $\Gamma = \Pi \circ \tilde\Gamma$ is non-degenerate. Let $\Gamma(t) = \Pi(\tilde\Gamma(t)) = (q(t), p(t))$, where $q(t) = [\mathbf{q}(t)]$, $p(t) = [\mathbf{p}(t)]$. We need to show that q', p' are nowhere vanishing.

Lemma 4.12. *The distribution $\mathcal{D} \subset TQ^5$, given by $d\mathbf{p} = \mathbf{q} \times d\mathbf{q}$, is also given by $d\mathbf{q} = -\mathbf{p} \times d\mathbf{p}$.*

Proof. Let $\mathcal{D}' = \mathrm{Ker}(d\mathbf{q} + \mathbf{p} \times d\mathbf{p}) \subset TQ^5$. Then both $\mathcal{D}, \mathcal{D}'$ are $\mathrm{SL}_3(\mathbb{R})$-invariant; hence it is enough to compare them at, say, $(e_3, e^3) \in Q^5$. At this point $\mathcal{D}$ is given by $dp_1 + dq^2 = dp_2 - dq^1 = dp_3 = dp^3 + dq_3 = 0$, and $\mathcal{D}'$ by $dq^1 - dp_2 = dq^2 + dp_1 = dq^3 = dp^3 + dq_3 = 0$. These obviously have the same 2-dimensional space of solutions. □

Now $q' = \mathbf{q}' \pmod{\mathbf{q}}$; hence $q' = 0 \Longrightarrow \mathbf{q} \times \mathbf{q}' = 0 \Longrightarrow \mathbf{p}' = 0$, so by Lemma 4.12, $\mathbf{q}' = -\mathbf{p} \times \mathbf{p}' = 0$. Similarly, $p' = 0 \Longrightarrow \mathbf{q}' = \mathbf{p}' = 0$; hence Γ is non-degenerate.

Next we show that Γ is null. Let σ be a lift of $\tilde\Gamma$ (hence of Γ) to $G = \mathrm{SL}_3(\mathbb{R})$. Let $\sigma^*\eta^a = s^a dt$, $\sigma^*\eta_b = s_b dt$, $a, b = 1, 2$, for some real-valued functions (of t) s^1, s^2, s_1, s_2. Then, by Propositions 4.10(a) and 4.10(d),

$$\mathbf{g}(\Gamma', \Gamma') = 2 s_a s^a = 2(s_1(-s_2) + s_2 s_1) = 0;$$

hence Γ is a null curve.

Next we show that the SD null 2-plane along Γ is parallel. Let $\hat\eta = \eta \circ \sigma$ be the coframing of $\Gamma^*(TM)$ determined by the lift σ of Γ (a "moving coframe" along Γ).

Remark. $\hat\eta$ should not be confused with $\sigma^*\eta = (s^1, s^2, s_1, s_2)^t dt$, the restriction of $\hat\eta$ to $T\Gamma$.

Let W be the 2-plane field along Γ defined by $\hat\eta^1 + \hat\eta_2 = \hat\eta^2 - \hat\eta_1 = 0$. By Proposition 4.10(d), $\sigma^*(\eta^1 + \eta_2) = \sigma^*(\eta^2 - \eta_1) = 0$; hence W is tangent to Γ. The 2-form corresponding to W is $\beta = (\hat\eta^1 + \hat\eta_2)\wedge(\hat\eta^2 - \hat\eta_1) = \hat\eta^1\wedge\hat\eta^2 + \hat\eta_1\wedge\hat\eta_2 + \hat\eta_1\wedge\hat\eta^1 + \hat\eta_2\wedge\hat\eta^2$, which is SD by formula (26a); hence W is the SD tangent 2-plane field along Γ. Now a short calculation, using Proposition 4.10(b), shows that

$$\nabla\beta = 3(\sigma^*\phi)\otimes(\hat\eta_1\wedge\hat\eta_2 - \hat\eta^1\wedge\hat\eta^2).$$

By Proposition 4.10(d), $\sigma^*\phi = 0$, hence $\nabla\beta = 0$, so W is parallel.

(c) Let σ be a lift of Γ to G, with $\sigma^*\eta^a = s^a dt$, $\sigma^*\eta_b = s_b dt$.

Lemma 4.13. *Given a non-degenerate parametrized null curve $\Gamma : \mathbb{R} \to M$, there exists a lift σ of Γ to G such that $s^1 = s_2 = 0$, $s_1 = s^2 = 1$. In other words,*

$$\sigma^*\omega = \begin{pmatrix} * & * & 0 \\ * & * & 1 \\ 1 & 0 & * \end{pmatrix} dt.$$

Remark. We call such a lift σ *adapted* to Γ.

Proof. Starting with an arbitrary lift σ, any other lift is of the form $\bar\sigma = \sigma h$, where $h : \mathbb{R} \to H$ is an arbitrary H-valued smooth function, i.e.,

$$h = \begin{pmatrix} A & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad A : \mathbb{R} \to \mathrm{GL}_2(\mathbb{R}),\ a = \det(A).$$

Now a short calculation shows that

$$\bar\sigma^*\omega = h^{-1}(\sigma^*\omega)h + h^{-1}dh = \begin{pmatrix} * & * & 0 \\ * & * & 1 \\ 1 & 0 & * \end{pmatrix} dt$$

provided

$$aA\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} s^1 \\ s^2 \end{pmatrix}, \quad (1,\ 0) = (s_1,\ s_2)aA. \tag{27}$$

Now one checks that the last system of equations can be solved for A if and only if $(s^1, s^2) \neq 0$, $(s_1, s_2) \neq 0$, and $s_a s^a = 0$. These are precisely the non-degeneracy and nullity conditions on Γ. From A we obtain h and the desired $\bar\sigma$. $\square$

Once we have an adapted lift σ of Γ, with associated moving coframe $\hat\eta := \eta\circ\sigma$, we define a 2-plane field W along Γ by $\hat\eta^1 + \hat\eta_2 = \hat\eta^2 - \hat\eta_1 = 0$. Then $\sigma^*(\eta^1 + \eta_2) = (s^1 + s_2)dt = 0$, $\sigma^*(\eta^2 - \eta_1) = (s^2 - s_1)dt = 0$; hence W is tangent to Γ. Let $\beta := (\hat\eta^1 + \hat\eta_2)\wedge(\hat\eta^2 - \hat\eta_1)$. Then β is SD, so W is the SD tangent 2-plane along Γ. Now W is parallel $\iff$ $\nabla\beta = 3(\sigma^*\phi)\otimes(\hat\eta_1\wedge\hat\eta_2 - \hat\eta^1\wedge\hat\eta^2) \equiv 0 \pmod{\beta}$ $\iff$ $\sigma^*\phi = 0$, since $\hat\eta_1\wedge\hat\eta_2 - \hat\eta^1\wedge\hat\eta^2$ is a non-zero ASD form, hence $\not\equiv 0 \pmod{\beta}$. It follows that σ satisfies $\sigma^*(\eta^1 + \eta_2) = \sigma^*(\eta^2 - \eta_1) = \sigma^*(\phi) = 0$; hence, by Proposition 4.10(d), $\tilde\Gamma := \tilde j\circ\sigma$ is a lift of Γ to an integral curve of $(Q, \mathcal{D})$.

To show uniqueness, if $\tilde\Gamma(t) = (\mathbf{q}(t), \mathbf{p}(t))$, then any other lift of Γ to Q^5 is of the form $(\lambda(t)\mathbf{q}(t), \mathbf{p}(t)/\lambda(t))$ for some non-vanishing real function $\lambda(t)$. If this other lift is also an integral curve of $(Q^5, \mathcal{D})$, then $(\mathbf{p}/\lambda)' - (\lambda\mathbf{q})\times(\lambda\mathbf{q})' = -(\lambda'/\lambda^2)\mathbf{p} + (1/\lambda - \lambda^2)\mathbf{p}' = 0$. Multiplying the last equation by $\mathbf{q}$ and using $\mathbf{pq} = 1, \mathbf{p}'\mathbf{q} = 0$, we get $\lambda' = 0 \Longrightarrow (1-\lambda^3)\mathbf{p}' = 0$. Now $\mathbf{p}' \neq 0$ since Γ is non-degenerate $\Longrightarrow \lambda^3 = 1 \Longrightarrow \lambda = 1$.

(d) Let $m = \Pi(\tilde{m})$, $W = \Pi_*(\mathcal{D}_{\tilde{m}}) \subset T_mM$, $g \in G$ such that $\tilde{j}(g) = \tilde{m}$, and $\hat{\eta} = \eta(g)$ the corresponding coframing of T_mM. Then, by Proposition 4.10(d), $W = \mathrm{Ker}\{\hat{\eta}^1 + \hat{\eta}_2, \hat{\eta}^2 - \hat{\eta}_1\}$. As before (item (b)), one checks that $\beta := (\hat{\eta}^1 + \hat{\eta}_2) \wedge (\hat{\eta}^2 - \hat{\eta}_1) = \hat{\eta}^1 \wedge \hat{\eta}^2 + \hat{\eta}_1 \wedge \hat{\eta}_2 + \hat{\eta}_1 \wedge \hat{\eta}^1 + \hat{\eta}_2 \wedge \hat{\eta}^2$ is SD $\Longrightarrow W$ is SD, but not principal (the SD 2-planes are given by $\hat{\eta}^1 \wedge \hat{\eta}^2$ and $\hat{\eta}_1 \wedge \hat{\eta}_2$; see the proof of item (a) above).

(e) One checks that ν is $\mathrm{SL}_3(\mathbb{R})$-equivariant, Q^5 and $\mathbb{T}^+_* M$ are $\mathrm{SL}_3(\mathbb{R})$-homogeneous manifolds, with the same stabilizer at $\tilde{m}_0 = (e_3, e^3)$ and $\nu(\tilde{m}_0)$; hence ν is a diffeomorphism. It remains to show that $\nu_*\mathcal{D} = \mathcal{D}^+$. This is just a reformulation of items (b) and (c) above. $\square$

5. Projective geometry: Dancing pairs and projective rolling

We give here two related projective geometric interpretations of the Cartan-Engel distribution $(Q^5, \mathcal{D})$: "dancing pairs" and "projective rolling". We start in Section 5.1 with the *dancing condition*, characterizing null curves in $(M^4, \mathbf{g})$. In Section 5.2 we use this characterization for an elementary derivation of an explicit coordinate formula for the conformal class of $\mathbf{g}$. In Section 5.3 we give yet another formula for the dancing metric $\mathbf{g}$, this time in terms of the *cross-ratio*, a classical projective invariant of 4 colinear points. This is followed in Section 5.4 by a study of the relation between the *projective structures* of the members of a dancing pair (the structures happily match up), which we use in Section 5.6 for deriving the "dancing mate equation". To illustrate all these concepts we study two examples: the "dancing mates of the circle" (Section 5.7) and "dancing pairs with constant projective curvature" (Section 5.8).

We mention also in Sect. 5.5 a curious geometric interpretation for (1) (hence of the Cartan-Engel distribution) that we found during the proof of Proposition 5.9: curves in $\mathbb{R}^3$ with constant "centro-affine torsion".

The rest of the section (Section 5.9) is dedicated to *projective rolling*. Our motivation comes from the intrinsic geometric formulation of ordinary (Riemannian) rolling, as appears in [5]. After making the appropriate definitions, the nullity condition of the dancing metric $(M^4, \mathbf{g})$ becomes the "no-slip" condition for the projective rolling of $\mathbb{RP}^2$ along $\mathbb{RP}^{2*}$, self-dual null 2-planes become "projective contact elements" of the two surfaces, and the condition of "parallel self-dual tangent 2-plane" is the "no-twist" condition of projective rolling, expressed in terms of the osculating conic of a plane curve and its developments, as appear in É. Cartan's book [6].

5.1. Projective duality and the dancing condition.

Let $\mathbb{RP}^2 := \mathbb{P}(\mathbb{R}^3)$ be the *real projective plane*, i.e., the space of 1-dimensional linear subspaces in $\mathbb{R}^3$, with

$$\pi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}^2, \quad \mathbf{q} \mapsto \mathbb{R}\mathbf{q},$$

the canonical projection. If $q = \pi(\mathbf{q}) \in \mathbb{RP}^2$, where $\mathbf{q} = (q^1, q^2, q^3)^t \in \mathbb{R}^3 \setminus \{0\}$, we write $q = [\mathbf{q}]$ and say that q^1, q^2, q^3 are the *homogeneous coordinates* of q. Similarly, $\mathbb{RP}^{2*} := \mathbb{P}((\mathbb{R}^3)^*)$ is the *dual projective plane*, with

$$\bar{\pi} : (\mathbb{R}^3)^* \setminus 0 \to \mathbb{RP}^{2*}, \quad \mathbf{p} \mapsto \mathbb{R}\mathbf{p},$$

the canonical projection, $\bar{\pi}(\mathbf{p}) = [\mathbf{p}]$.

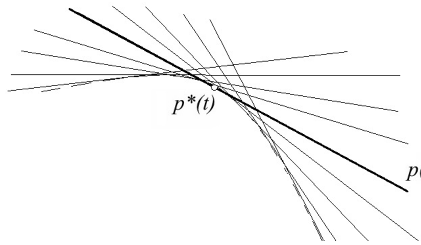

FIGURE 6. The envelope of a family of lines

A *projective line* in $\mathbb{RP}^2$ is the projectivization of a 2-dimensional linear subspace in $\mathbb{R}^3$, i.e., the set of 1-dimensional subspaces of $\mathbb{R}^3$ contained in a fixed 2-dimensional linear subspace of $\mathbb{R}^3$. The space of projective lines in $\mathbb{RP}^2$ is naturally identified with $\mathbb{RP}^{2*}$; to each $p = [\mathbf{p}] \in \mathbb{RP}^{2*}$ corresponds a projective line in $\mathbb{RP}^2$, the projectivization of $\mathbf{p}^0 = \{\mathbf{q} \in \mathbb{R}^3 | \mathbf{pq} = 0\}$, and each projective line in $\mathbb{RP}^2$ is of this form. Similarly, $\mathbb{RP}^2$ is naturally identified with the space of projective lines in $\mathbb{RP}^{2*}$; to each point $q = [\mathbf{q}] \in \mathbb{RP}^2$ corresponds a projective line in $\mathbb{RP}^{2*}$, the projectivization of the 2-dimensional subspace $\mathbf{q}^0 = \{\mathbf{p} \in (\mathbb{R}^3)^* | \mathbf{pq} = 0\}$.

We say that $(q, p) \in \mathbb{RP}^2 \times \mathbb{RP}^{2*}$ are *incident* if q belongs to the projective line in $\mathbb{RP}^2$ corresponding to p (same as p belongs to the projective line in $\mathbb{RP}^{2*}$ corresponding to q). We also write this condition as $q \in p$. In homogeneous coordinates this is simply $\mathbf{pq} = 0$.

Given a smooth curve $\gamma \subset \mathbb{RP}^2$ (a 1-dimensional submanifold), the *duality map* $* : \gamma \to \mathbb{RP}^{2*}$ assigns to each point $q \in \gamma$ its tangent line $q^* \in \mathbb{RP}^{2*}$. The image of γ under the duality map is the *dual curve* $\gamma^* \subset \mathbb{RP}^{2*}$. In homogeneous coordinates, if γ is parametrized by $q(t) = [\mathbf{q}(t)]$, where $\mathbf{q}(t) \in \mathbb{R}^3 \setminus \{0\}$, then γ^* is parametrized by $q^*(t) = [\mathbf{q}^*(t)]$, where $\mathbf{q}^*(t) := \mathbf{q}(t) \times \mathbf{q}'(t) \in (\mathbb{R}^3)^* \setminus \{0\}$. If γ is a smooth curve without inflection points (points where $q'' \equiv 0 \bmod q'$; see Definition 5.22 below), then γ^* is smooth as well. More generally, inflection points of γ map to singular (or "cusp") points of γ^*, where $(q^*)' = 0$.

Similarly, given a curve $\bar{\gamma} \subset \mathbb{RP}^{2*}$, the duality map $\bar{\gamma} \to \mathbb{RP}^2$ assigns to each line $p \in \bar{\gamma}$ its *turning point* p^*. In homogeneous coordinates, if $\bar{\gamma}$ is parametrized by $p(t) = [\mathbf{p}(t)]$, then its dual $\bar{\gamma}^* \subset \mathbb{RP}^2$ is parametrized by $p^*(t) = [\mathbf{p}^*(t)]$, where $\mathbf{p}^*(t) = \mathbf{p}(t) \times \mathbf{p}'(t)$.

Geometrically, $\bar{\gamma}$ is a 1-parameter family of lines in $\mathbb{RP}^2$, and its dual $\bar{\gamma}^*$ is the *envelope* of the family (see Figure 6). Using the above formulas for the duality map, it is easy to verify that, away from inflection points, $(\gamma^*)^* = \gamma$; that is, $p(t)$ is the tangent line to $\bar{\gamma}^*$ at $p^*(t)$.

Definition 5.1. (See Figure 1 of Section 1.1.3.) A pair of parametrized curves $q(t), p(t)$ in $\mathbb{RP}^2, \mathbb{RP}^{2*}$ (respectively) satisfies the *dancing condition* if for each t:

(1) $(q(t), p(t))$ is non-incident;
(2) if $q'(t) \neq 0$ and $p'(t) \neq 0$, then the tangent line $q^*(t)$ at $q(t)$ is incident to the turning point $p^*(t)$ of $p(t)$.

Remark. In condition (2), if either q' or p' vanish, then q^* or p^* is not well-defined, in which case, by definition, the curves satisfy the dancing condition.

Proposition 5.2. *The following conditions on a parametrized curve* $\Gamma(t) = (q(t), p(t))$ *in* M^4 *are equivalent:*

(1) $\Gamma(t)$ *is a null curve in* $(M^4, \mathbf{g})$*;*
(2) *the pair of curves* $q(t), p(t)$ *satisfies the dancing condition.*

Proof. We use the notation of the proof of Proposition 4.10(d). Let σ be a lift of Γ to $G = \mathrm{SL}_3(\mathbb{R})$ (see diagram (17) of Section 4.4). Then $(\mathbf{q}(t), \mathbf{p}(t)) := (E_3(\sigma(t)), E^3(\sigma(t)))$, $\sigma^*\eta^a = s^a dt$, $\sigma^*\eta_b = s_b dt$. If either q' or p' vanish, then either $(s^1, s^2) = (0,0)$ or $(s_1, s_2) = (0,0)$. Hence, by Proposition 4.10(a), $\mathbf{g}(\Gamma', \Gamma') = 2s_a s^a = 0$, so (1) and (2) are both satisfied. If neither q' nor p' vanishes, then $q^* = [\mathbf{q}^*]$, $p^* = [\mathbf{p}^*]$, where

$$\begin{aligned}\mathbf{q}^* &= \mathbf{q} \times \mathbf{q}' = E_3 \times E_3' = E_3 \times E_a s^a = s^1 E^2 - s^2 E^1,\\ \mathbf{p}^* &= \mathbf{p} \times \mathbf{p}' = E^3 \times (E^3)' = -E^3 \times E^b s_b = -s_1 E_2 + s_2 E_1.\end{aligned}$$

The dancing condition is then $\mathbf{q}^*\mathbf{p}^* = 0$, i.e., $(s^1E^2 - s^2E^1)(-s_1E_2 + s_2E_1) = -s_a s^a = 0$, which is the nullity condition on Γ. ☐

Remark. It is clear that both the dancing metric and the dancing condition are $\mathrm{SL}_3(\mathbb{R})$-invariant and homogeneous in the velocity Γ' of a parametrized curve Γ in M^4, thus defining each a field of tangent cones on M^4. It is also clear from the formula of the isotropy representation (23) that M^4 admits a unique $\mathrm{SL}_3(\mathbb{R})$-invariant conformal metric (of whatever signature). The main point of the last proposition, perhaps less obvious, is then that the dancing condition is *quadratic* in the velocities Γ', thus defining *some* conformal metric on M^4. This point can be proved in an elementary fashion, as we now proceed to show, and thus gives an alternative proof of the last proposition.

5.2. **A coordinate formula for the conformal class of the dancing metric.** Let us use Cartesian coordinates (x, y) for a point $q \in \mathbb{RP}^2$ (in some affine chart) and the coordinates (a, b) for a line $y = ax + b$ (a point $p \in \mathbb{RP}^{2*}$). If $q(t)$ is given by $(x(t), y(t))$, then its tangent line $y = Ax + B$ at time t satisfies

$$y(t) = Ax(t) + B, \quad y'(t) = Ax'(t). \tag{28}$$

Likewise, if $p(t)$ is a curve in $\mathbb{RP}^{2*}$ given by $y = a(t)x + b(t)$, then its "turning point" (X, Y) at time t satisfies

$$Y = a(t)X + b(t), \quad 0 = a'(t)X + b'(t). \tag{29}$$

The dancing condition ("the turning point of $p(t)$ lies on the tangent line to $q(t)$") is then

$$Y = AX + B.$$

Expressing A, B, X, Y in the last equation in terms of x, y, a, b and their derivatives via (28)-(29), we obtain $a'[(y - b)x' - xy'] + b'[ax' - y'] = 0$. We have shown the following:

Proposition 5.3. *The dancing metric* $\mathbf{g}$ *on* M *is given in the above local coordinates* x, y, a, b *by*

$$\mathbf{g} \sim da[(y - b)dx - xdy] + db[adx - dy],$$

where $\sim$ *denotes conformal equivalence (equality up to multiplication by some non-vanishing function on* M*).*

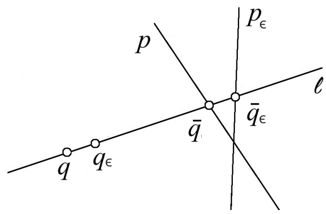

FIGURE 7. The cross-ratio definition of the dancing metric

Remark. In fact, although somewhat less elementary, it is not hard to show that the missing conformal factor on the right hand side of the above formula is a constant multiple of $(y - ax - b)^{-2}$.

5.3. **A cross-ratio formula for the dancing metric.**

Definition 5.4. The cross-ratio of 4 distinct points a_1, a_2, a_3, a_4 on a line $\ell \subset \mathbb{RP}^2$ is
$$[a_1, a_2, a_3, a_4] := \frac{x_1 - x_3}{x_1 - x_4} \cdot \frac{x_4 - x_2}{x_3 - x_2},$$
where x_i is the coordinate of a_i with respect to some affine coordinate x on ℓ.

It is well-known, and not hard to verify, that this definition is independent of the affine coordinate chosen on ℓ and that it is $\mathrm{SL}_3(\mathbb{R})$-invariant.

Now consider a non-degenerate parametrized curve Γ in M^4 and two points on it, $\Gamma(t) = (q(t), p(t))$ and $\Gamma(t+\epsilon) = (q(t+\epsilon), p(t+\epsilon))$. These determine 4 colinear points $q, q_\epsilon, \bar{q}, \bar{q}_\epsilon$, where $q := q(t)$, $q_\epsilon = q(t+\epsilon)$, and $\bar{q}, \bar{q}_\epsilon$ are the intersection points of the two lines $p := p(t), p_\epsilon := p(t+\epsilon)$ with the line ℓ through q, q_ϵ (respectively), as shown in Figure 7. (The line ℓ is well-defined, for small enough ϵ, by the non-degeneracy assumption on Γ.)

Let us expand the cross-ratio of $q, q_\epsilon, \bar{q}, \bar{q}_\epsilon$ in powers of ϵ.

Proposition 5.5. *Let $\Gamma(t)$ be a non-degenerate parametrized curve in M^4, let $q, q_\epsilon, \bar{q}, \bar{q}_\epsilon$ be as defined above, and let $v = \Gamma'(t)$. Then*
$$[q, q_\epsilon, \bar{q}, \bar{q}_\epsilon] = 1 - \frac{1}{2}\epsilon^2 \mathbf{g}(v, v) + O(\epsilon^3),$$
where $\mathbf{g}$ is the dancing metric on M, as defined in Proposition 4.1.

Proof. Lift $\Gamma(t)$ to a curve $\tilde{\Gamma}(t) = (\mathbf{q}(t), \mathbf{p}(t))$ in Q. Then $\ell = [\mathbf{q} \times \mathbf{q}_\epsilon]$, $\bar{q} = [\bar{\mathbf{q}}]$, and $\bar{q}_\epsilon = [\bar{\mathbf{q}}_\epsilon]$, where
$$\bar{\mathbf{q}} = (\mathbf{q} \times \mathbf{q}_\epsilon) \times \mathbf{p} = \mathbf{q}_\epsilon - (\mathbf{p}\mathbf{q}_\epsilon)\mathbf{q},$$
$$\bar{\mathbf{q}}_\epsilon = (\mathbf{q} \times \mathbf{q}_\epsilon) \times \mathbf{p}_\epsilon = (\mathbf{p}_\epsilon \mathbf{q})\mathbf{q}_\epsilon - \mathbf{q}.$$

Now it is easy to show that if 4 colinear points $a_1, \ldots, a_4 \in \mathbb{RP}^2$ are given by homogeneous coordinates $\mathbf{a}_i \in \mathbb{R}^3 \setminus 0$, such that $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$, $\mathbf{a}_4 = k\mathbf{a}_1 + \mathbf{a}_2$, then $[a_1, a_2, a_3, a_4] = k$ (see for example [17]). Using this formula and the above expressions for $\bar{\mathbf{q}}, \bar{\mathbf{q}}_\epsilon$, we obtain, after some manipulations,
$$[q, q_\epsilon, \bar{q}, \bar{q}_\epsilon] = \frac{1}{(\mathbf{p}\mathbf{q}_\epsilon)(\mathbf{p}_\epsilon \mathbf{q})} = 1 + \epsilon^2 (\mathbf{q} \times \mathbf{q}')(\mathbf{p} \times \mathbf{p}') + O(\epsilon^3).$$
Now we use the expression for $\mathbf{g}$ of Proposition 4.2. $\square$

5.4. **Dancing pairs and their projective structure.**

Definition 5.6. A *dancing pair* is a pair of parametrized curves $q(t), p(t)$ in $\mathbb{RP}^2$, $\mathbb{RP}^{2*}$ (respectively) obtained from the projections $q(t) = [\mathbf{q}(t)], p(t) = [\mathbf{p}(t)]$ of an integral curve $(\mathbf{q}(t), \mathbf{p}(t))$ of $(Q^5, \mathcal{D})$. If $q(t), p(t)$ is a dancing pair we say that $p(t)$ is a *dancing mate* of $q(t)$.

Equivalently, by Theorem 4.8, this is the pair of curves one obtains from a non-degenerate null curve in M^4 with parallel self-dual tangent plane when projecting it to $\mathbb{RP}^2$ and $\mathbb{RP}^{2*}$.

We already know that dancing pairs satisfy the dancing condition. We now want to study further the projective geometry of such pairs of curves, using the classical notions of projective differential geometry, such as the projective structure of a plane curve, projective curvature, and projective arc length. We will derive a 4th order ODE whose solutions give the *dancing mates* $p(t)$ of a given (locally convex) curve $q(t)$. We will give several examples of dancing pairs, including the surprisingly non-trivial case of the dancing mates associated with a point moving on a circle.

Differential projective geometry is not so well-known nowadays, so we begin with a brief review of the pertinent notions. Our favorite references are É. Cartan's book [6] and the more modern references of Ovsienko-Tabachnikov [24] and Konovenko-Lychagin [19].

The projective structure and arc length. There are three basic projective invariants of an embedded plane curve $\gamma \subset \mathbb{RP}^2$: its projective structure, the projective arc length, and the projective curvature. They form a complete set of projective invariants: a diffeomeorphism of plane curves that preserves the invariants is a restriction of a projective transformation in $\mathrm{SL}_3(\mathbb{R})$.

Definition 5.7. A *projective structure* on a curve γ (a 1-dimensional manifold) is a maximal atlas of charts (U_α, t_α), where $\{U_\alpha\}$ is an open cover of γ and the $t_\alpha : U_\alpha \to \mathbb{RP}^1$, called projective coordinates, are embeddings whose transition functions $t_\alpha \circ t_\beta^{-1}$ are given by (restrictions of) Möbius transformation in $\mathrm{PGL}_2(\mathbb{R})$.

For example, stereographic projection from any point q on a conic $\mathcal{C} \subset \mathbb{RP}^2$ to some line ℓ (non-incident to q) gives $\mathcal{C}$ a projective structure, independent of the point q and line ℓ chosen (a theorem attributed to Steiner; see [24, p. 7]).

An embedded curve $\gamma \subset \mathbb{RP}^2$ is *locally convex* if it has no inflection points (points where the tangent line has a 2nd order contact with the curve). Every locally convex curve $\gamma \subset \mathbb{RP}^2$ inherits a canonical projective structure. There are various equivalent ways to define this projective structure, but we will give the most classical one, using the *tautological ODE* associated with a plane curve (we follow here closely Cartan's book [6]).

Let $q(t)$ be a regular parametrization of an embedded curve $\gamma \subset \mathbb{RP}^2$, i.e., $q' \neq 0$, and $\mathbf{q}(t)$ a lift of $q(t)$ to $\mathbb{R}^3 \setminus \{0\}$, i.e., $q(t) = [\mathbf{q}(t)]$. Then local-convexity (absence of inflection points) is equivalent to $\det(\mathbf{q}(t), \mathbf{q}'(t), \mathbf{q}''(t)) \neq 0$, so there are unique a_0, a_1, a_2 (functions of t) such that

$$\mathbf{q}''' + a_2\mathbf{q}'' + a_1\mathbf{q}' + a_0\mathbf{q} = 0. \tag{30}$$

The last equation is called the *tautological ODE* associated with γ (or rather its parametrized lift $\mathbf{q}(t)$). Solving for the unknowns a_0, a_1, a_2 (by Kramer's rule), we get

$$a_0 = -\frac{J}{I}, \quad a_1 = \frac{K}{I}, \quad a_2 = -\frac{I'}{I},$$

where

$$I := \det(\mathbf{q}, \mathbf{q}', \mathbf{q}''), \quad J = \det(\mathbf{q}', \mathbf{q}'', \mathbf{q}'''), \quad K := \det(\mathbf{q}, \mathbf{q}'', \mathbf{q}''').$$

The tautological ODE equation (30) depends on the choice of parametrized lift $\mathbf{q}(t)$. One can modify $\mathbf{q}(t)$ in two ways:

- *Rescaling*: $\mathbf{q}(t) \mapsto \bar{\mathbf{q}}(t) = \lambda(t)\mathbf{q}(t)$, where $\lambda(t) \in \mathbb{R}^*$. This changes $I \mapsto \lambda^3 I$, so if $I \neq 0$ (no inflection points) one can rescale (uniquely) to $I = 1$, then obtain $a_2 = 0$.
- *Reparametrization*: $t \mapsto \bar{t} = f(t)$, for some diffeomorphism f. This changes $I \mapsto (f')^3 I$, so again, if $I \neq 0$, then one can reparametrize (uniquely up to an additive constant) to $I = 1$ so as to obtain $a_2 = 0$.

So one can achieve a tautological ODE for γ with $a_2 = 0$ by either rescaling or reparametrization. Can we combine reparametrization and rescaling so as to reduce the tautological ODE to $\mathbf{q}''' + a_0\mathbf{q} = 0$?

The answer is "yes" and the resulting ODE is called the *Laguerre-Forsyth form* (LF) of the tautological ODE for γ. A straightforward calculation ([6], pp. 48-50) shows that

Proposition 5.8. *Given a locally convex curve $\gamma \subset \mathbb{RP}^2$ with a parametrized lift $\mathbf{q}(t)$ satisfying $\mathbf{q}''' + a_1\mathbf{q}' + a_0\mathbf{q} = 0$,*

(1) *one can achieve the LF form by modifying $\mathbf{q}(t)$ to $\bar{\mathbf{q}}(\bar{t}) = f'(t)\mathbf{q}(t)$, where $\bar{t} = f(t)$ solves*

$$S(f) = \frac{a_1}{4}$$

and where

$$S(f) = \frac{1}{2}\frac{f'''}{f'} - \frac{3}{4}\left(\frac{f''}{f'}\right)^2$$

is the Schwarzian derivative *of f.*

(2) *The LF form is unique up to the change $\mathbf{q}(t) \mapsto \bar{\mathbf{q}}(\bar{t}) = f'(t)\mathbf{q}(t)$, where $\bar{t} = f(t)$ is a Möbius transformation. Hence the LF form defines a projective structure on γ.*

(3) *Given an LF form $\mathbf{q}''' + a_0\mathbf{q} = 0$ for γ, the one form $d\sigma := (a_0)^{1/3}dt$ is a well-defined 1-form on γ (independent of the particular LF form chosen), called the* projective arc length [6, p. 50].

Remark. It is possible to extend the definition of the projective structure to all embedded curves in $\mathbb{RP}^2$, not necessarily locally convex (see [13]).

The points on γ where the projective arc length vanishes, $d\sigma = 0$, are called *sextactic points* and are characterized geometrically as the points where the *osculating conic* to the curve (the conic which "best approximates" the curve around a given point) has higher order of contact with the curve than expected (5th or higher).

The projective curvature. The projective curvature κ of a curve $\gamma \subset \mathbb{RP}^2$ is a function defined along γ, away from *sextactic* points, where $d\sigma = 0$. Away from such points one can use σ as a natural parameter on γ and compare it to a projective parameter t, given by the projective structure (see Proposition 5.8). Namely, σ defines a local diffeomorphism $\mathbb{RP}^1 \to \gamma$, whose Schwarzian derivative is the quadratic form $\kappa(\sigma)(d\sigma)^2$. The pair $(d\sigma, \kappa)$ forms a complete set of projective invariants for curves in $\mathbb{RP}^2$ (for which they are defined), analogous to the square of the arc length element and curvature for regular curves in the Euclidean plane. For curves with constant projective curvature, the constant κ by itself is a complete invariant (same as in the Euclidean case). Along a conic $d\sigma \equiv 0$, and so κ is not defined.

Remark. In the book of Ovsienko-Tabachnikov [24] (a beautiful book; we highly recommend it) the term "projective curvature" is used to denote what we call here the projective *structure*, and it is stated that "the projective curvature is, by no means, a function on the curve" ([24, p. 14], the online version). This can be somewhat confusing if one does not realize the difference in usage of terminology. We adhere to the classical terminology, as in Cartan's book [6].

Example. A classical application of the last proposition is to show that *the duality map $\gamma \to \gamma^*$ preserves the projective structure and curvature but reverses the projective arc length* (provided both γ and γ^* are locally convex): parametrize γ by $[\mathbf{q}(t)]$ in the LF form, i.e., $\mathbf{q}''' + a_0\mathbf{q} = 0$; then γ^* is parametrized by $[\mathbf{p}(t)]$, where $\mathbf{p}(t) = \mathbf{q}(t) \times \mathbf{q}'(t)$. Then one can calculate easily that $\mathbf{p}(t)$ satisfies $\mathbf{p}''' - a_0\mathbf{p} = 0$, which is also in the LF form. Hence t is a common projective parameter on γ, γ^*, so $[\mathbf{q}(t)] \mapsto [\mathbf{p}(t)]$ preserves the projective structure but reverses the projective arc length.

Proposition 5.9. *Let $\gamma, \bar\gamma$ be a pair of non-degenerate curves in $\mathbb{RP}^2, \mathbb{RP}^{2*}$ (respectively), parametrized by a dancing pair $q(t), p(t)$ (i.e., $q(t) = [\mathbf{q}(t)], p(t) = [\mathbf{p}(t)]$, where $\mathbf{p}' = \mathbf{q} \times \mathbf{q}', \mathbf{pq} = 1$). Then the map $\gamma \to \bar\gamma$, $q(t) \mapsto p(t)$ is projective, i.e., preserves the natural projective structures on $\gamma, \bar\gamma$ induced by their embedding in $\mathbb{RP}^2, \mathbb{RP}^{2*}$ (respectively).*

Proof. According to the last proposition, it is enough to show that $(\mathbf{q}(t), \mathbf{p}(t))$ can be reparametrized in such a way that $\mathbf{q}(t), \mathbf{p}(t)$ each satisfy a tautological ODE with $a_2 = 0$ and the same a_1.

Lemma 5.10. *Let $(\mathbf{q}(t), \mathbf{p}(t)) \in \mathbb{R}^{3,3}$ be a solution of $\mathbf{p}' = \mathbf{q} \times \mathbf{q}'$, $\mathbf{pq} = 1$, with $I(\mathbf{q}) = \det(\mathbf{q}, \mathbf{q}', \mathbf{q}'') \neq 0$ and $I(\mathbf{p}) = \det(\mathbf{p}, \mathbf{p}', \mathbf{p}'') \neq 0$. Let $I = I(\mathbf{q}), \bar I = I(\mathbf{p}), J = J(\mathbf{q}), \bar J = J(\mathbf{p})$, etc. Then*

(1) $\mathbf{p}'\mathbf{q} = \mathbf{p}\mathbf{q}' = \mathbf{p}'\mathbf{q}' = \mathbf{p}\mathbf{q}'' = \mathbf{p}''\mathbf{q} = 0$.
(2) $I\mathbf{p} = \mathbf{q}' \times \mathbf{q}''$, $\bar I\mathbf{q} = \mathbf{p}' \times \mathbf{p}''$.
(3) $\mathbf{q}' = -\mathbf{p} \times \mathbf{p}'$.
(4) $I^2 + J = \bar I^2 - \bar J = 0$.
(5) $\bar I = I$, $\bar J = -J$, $\bar K = K$.
(6) $\bar a_2 = a_2$, $\bar a_1 = a_1$, $\bar a_0 = -a_0$.

Proof. (1) From $\mathbf{p}' = \mathbf{q} \times \mathbf{q}' \Longrightarrow \mathbf{p}'\mathbf{q} = \mathbf{p}'\mathbf{q}' = 0$. From $\mathbf{pq} = 1 \Longrightarrow \mathbf{p}'\mathbf{q} + \mathbf{p}\mathbf{q}' = 0 \Longrightarrow \mathbf{p}\mathbf{q}' = 0 \Longrightarrow 0 = (\mathbf{p}\mathbf{q}')' = \mathbf{p}'\mathbf{q}' + \mathbf{p}\mathbf{q}'' = \mathbf{p}\mathbf{q}''$. Similarly, $0 = (\mathbf{p}'\mathbf{q})' = \mathbf{p}''\mathbf{q} + \mathbf{p}'\mathbf{q}' = \mathbf{p}''\mathbf{q}$.

(2) From (1), $\mathbf{p}\mathbf{q}' = \mathbf{p}\mathbf{q}'' = 0 \Longrightarrow c\mathbf{p} = \mathbf{q}' \times \mathbf{q}''$ for some function c (we assume $I \neq 0$, hence $\mathbf{q}' \times \mathbf{q}'' \neq 0$). Taking the dot product of the last equation with $\mathbf{q}$ and using $\mathbf{p}\mathbf{q} = 1$ we get $c = I \Longrightarrow I\mathbf{p} = \mathbf{q}' \times \mathbf{q}''$.

Next, from (1), $\mathbf{p}'\mathbf{q} = \mathbf{p}''\mathbf{q} = 0 \Longrightarrow \bar{c}\mathbf{q} = \mathbf{p}' \times \mathbf{p}''$ for some function $\bar{c}$ (here we assume $\bar{I} \neq 0$). Take the dot product with $\mathbf{p}$ and get $\bar{c} = \bar{I} \Longrightarrow \bar{I}\mathbf{q} = \mathbf{p}' \times \mathbf{p}''$.

(3) (This was already shown in Lemma 4.12, but we give another proof here.) From (1), $\mathbf{p}\mathbf{q}' = \mathbf{p}'\mathbf{q}' = 0 \Longrightarrow \mathbf{q}' = f\mathbf{p} \times \mathbf{p}'$ for some function f. Cross-product with $\mathbf{q}$, use the vector identity

$$(\mathbf{p}_1 \times \mathbf{p}_2) \times \mathbf{q} = (\mathbf{p}_1\mathbf{q})\mathbf{p}_2 - (\mathbf{p}_2\mathbf{q})\mathbf{p}_1,$$

and get $-\mathbf{p}' = \mathbf{q}' \times \mathbf{q} = f(\mathbf{p} \times \mathbf{p}') \times \mathbf{q} = f[(\mathbf{p}\mathbf{q})\mathbf{p}' - (\mathbf{p}'\mathbf{q})\mathbf{p}] = f\mathbf{p}' \Longrightarrow f = -1 \Longrightarrow \mathbf{q}' = -\mathbf{p} \times \mathbf{p}'$.

(4) $I\mathbf{p} = \mathbf{q}' \times \mathbf{q}'', \mathbf{p}' = \mathbf{q} \times \mathbf{q}' \Longrightarrow I'\mathbf{p} + I(\mathbf{q} \times \mathbf{q}') = \mathbf{q}' \times \mathbf{q}'''$. Now dot product with $\mathbf{q}''$, use $\mathbf{p}\mathbf{q}'' = 0$, and get $I^2 + J = 0$. Very similarly, get $(\bar{I})^2 - \bar{J} = 0$.

(5) Use the vector identity

$$\det(\mathbf{q}_1 \times \mathbf{q}_2, \mathbf{q}_2 \times \mathbf{q}_3, \mathbf{q}_3 \times \mathbf{q}_1) = [\det(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)]^2$$

to get $I\bar{I} = \det(I\mathbf{p}, \mathbf{p}', \mathbf{p}'') = \det(\mathbf{q}' \times \mathbf{q}'', \mathbf{q} \times \mathbf{q}', \mathbf{q} \times \mathbf{q}'') = I^2$, hence $I = \bar{I}$.

From (4), $\bar{J} = \bar{I}^2 = I^2 = -J$.

From $\mathbf{p}' = \mathbf{q} \times \mathbf{q} \Longrightarrow \mathbf{p}'' = \mathbf{q} \times \mathbf{q}'' \Longrightarrow \mathbf{p}''\mathbf{q}'' = 0 \Longrightarrow \mathbf{p}'''\mathbf{q}'' + \mathbf{p}''\mathbf{q}''' = 0$. Now $K = \det(\mathbf{q}, \mathbf{q}'', \mathbf{q}''') = (\mathbf{q} \times \mathbf{q}'')\mathbf{q}''' = \mathbf{p}''\mathbf{q}'''$, $\bar{K} = \det(\mathbf{p}, \mathbf{p}'', \mathbf{p}''') = -\mathbf{p}'''\mathbf{q}''$, hence $K - \bar{K} = \mathbf{p}''\mathbf{q}''' + \mathbf{p}'''\mathbf{q}'' = (\mathbf{p}''\mathbf{q}'')' = 0$.

(6) Immediate from item (5) and the definition of a_0, a_1, a_2. $\square$

Now γ is locally convex, so we can reparametrize $\mathbf{q}(t)$ to achieve $I(\mathbf{q}) = 1$. The equation $\mathbf{p}' = \mathbf{q} \times \mathbf{q}'$ is reparametrization invariant, so it still holds. It follows from item (5) of the lemma that $I(\mathbf{p}) = 1$ as well, hence both $a_2 = \bar{a}_2 = 0$. From item (6) of the lemma we have that $a_1 = \bar{a}_1$. Hence the equation for projective parameter $S(f) = a_1/4$ is the same equation for both curves $\mathbf{q}(t)$ and $\mathbf{p}(t)$. $\square$

5.5. **An aside: Space curves with constant "centro-affine torsion".** We mention here in passing a curious geometric interpretation of a formula that appeared during the proof of Proposition 5.9 (see part (4) of Lemma 5.10):

$$J(\mathbf{q}) + I^2(\mathbf{q}) = 0, \tag{31}$$

where

$$I(\mathbf{q}) = \det(\mathbf{q}, \mathbf{q}', \mathbf{q}''), \quad J(\mathbf{q}) = \det(\mathbf{q}, \mathbf{q}'', \mathbf{q}''').$$

Effectively, this formula means that it is possible to eliminate the $\mathbf{p}$ variable from our system of equations (2), reducing them to a *single* 3rd order ODE for a space curve $\mathbf{q}(t)$.

In fact, it is not hard to show that (31) is *equivalent* to (2); given a non-degenerate ($I(\mathbf{q}) \neq 0$) solution $\mathbf{q}(t)$ to (31), use the "moving frame" $\mathbf{q}(t), \mathbf{q}'(t), \mathbf{q}''(t)$ to define $\mathbf{p}(t)$ by

$$\mathbf{p}(t)\mathbf{q}(t) = 1,\ \mathbf{p}(t)\mathbf{q}'(t) = 0,\ \mathbf{p}(t)\mathbf{q}''(t) = 0. \tag{32}$$

Then check that (31) implies that $(\mathbf{q}(t), \mathbf{p}(t))$ is a solution to (2).

The curve $\mathbf{p}(t)$ associated to a non-degenerate curve $\mathbf{q}(t)$ via (32) represents the *osculating plane* H_t along $\mathbf{q}(t)$, via the formula $H_t = \{\mathbf{q} | \mathbf{p}(t)\mathbf{q} = 1\}$.

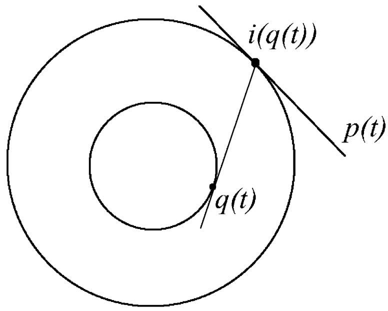

Figure 8. A projective involute which is not a dancing pair

For any space curve (with $I \neq 0$) the quantity $\mathcal{J} = J/I^2$ is parametrization-independent and $\mathrm{SL}_3(\mathbb{R})$-invariant, called by some authors the (*unimodular*) *centro-affine torsion* [23]. Hence (2) can also be interpreted as the equations for space curves with $\mathcal{J} = -1$.

5.6. **Projective involutes and the dancing mate equation.** The reader may suspect now that the necessary condition of Proposition 5.9 is also sufficient for a pair of curves to be a dancing pair. This is not so, as the following example shows.

Example (See Figure 8)**.** Let $\gamma, \bar\gamma$ be the pair consisting of a circle $q(t) = [\cos(t), \sin(t), 1]$ and the dual of the concentric circle $p^*(t) = [\sqrt{2}\cos(t + \pi/4), \sqrt{2}\sin(t + \pi/4), 1]$. One can check easily that $(q(t), p(t))$ satisfies the dancing condition (i.e., defines a null curve in M^4) and that the map $q(t) \mapsto p(t)$ is projective (as the restriction to γ of an element in $\mathrm{SL}_3(\mathbb{R})$: a dilation followed by a rotation). Nevertheless, the pair of curves $q(t), p(t)$ is not a dancing pair (there is no way to lift $(q(t), p(t))$ to a solution $(\mathbf{q}(t), \mathbf{p}(t))$ of $\mathbf{pq} = 1, \mathbf{p}' = \mathbf{q} \times \mathbf{q}'$).

We are going to study carefully the situation now and find an extra condition that the map $q(t) \mapsto p(t)$ should satisfy for the pair of curves $q(t), p(t)$ to be a dancing pair.

Definition 5.11. Let $\gamma \subset \mathbb{RP}^2$ be a locally convex curve. A *projective involute* of γ is a smooth map $i : \gamma \to \mathbb{RP}^2$ such that

- for all $q \in \gamma$, $i(q) \in q^*$ (the tangent line to γ at q),
- i is a projective immersion.

The last phrase means that i is an immersion and the resulting local diffeomorphism between γ and its image is projective with respect to the natural projective structures on γ and $i(\gamma)$, defined by their embedding in $\mathbb{RP}^2$.

Proposition 5.12. *Near a non-inflection point of a curve $\gamma \subset \mathbb{RP}^2$ there is a 4-parameter family of projective involutes, given by the solutions of the following 4th order ODE: if γ is given by a tautological ODE in the LF form $A''' + rA = 0$, then its projective involutes are given by $[A(t)] \mapsto [B(t)]$, where $B(t) = (C - y'(t))A(t) + y(t)A'(t)$, C is an arbitrary constant, and $y(t)$ is a solution of the ODE*

$$y^{(4)} + 2\frac{y'''(y' - C)}{y} + 3ry' + r'y = 0.$$

Proof. Calculate, using $A''' + rA = 0$:

$$\begin{aligned}
B &= xA + yA', \\
B' &= x'A + (x + y')A' + yA'', \\
B'' &= (x'' - ry)A + (2x' + y'')A' + (x + 2y')A'', \\
B''' &= (x''' - r'y - r(x + 3y'))A + (3x'' - ry + y''')A' + 3(x' + y'')A''.
\end{aligned}$$

Hence $B \times B''' = 0 \Longrightarrow y(x'+y'') = x(x'+y'') = x(3x''+y''') - y(x'''-3ry'-r'y) = 0$. Then $y \neq 0 \Longrightarrow x' + y'' = 0 \Longrightarrow x + y' = C$, for some constant C, hence

$$y^{(4)} = 2\frac{y'''(C - y')}{y} - 3ry' - r'y.$$

This gives a 5-parameter family of solutions. Then imposing, say, $\det(B, B', B'') = 1$ reduces it to a 4-parameter family (removing the scaling ambiguity on B). $\square$

Now given a non-degenerate $\gamma \subset \mathbb{RP}^2$, parametrized by $q(t)$, we know, by Proposition 5.9 and the preceding example, that each of its dancing mates $p(t)$ gives rise to a projective involute $q(t) \mapsto p(t) \mapsto p^*(t)$. The dancing mates of γ form a 3-parameter sub-family of the projective involutes, as they are obtained by lifting γ horizontally via $Q^5 \to \mathbb{RP}^2$, followed by the projection $Q^5 \to \mathbb{RP}^{2*}$. We are thus looking for a single equation characterizing projective involutes of γ that correspond to dancing mates.

Proposition 5.13. *Let $\gamma \subset \mathbb{RP}^2$ be a non-degenerate curve with a tautological ODE in LF form $A''' + rA = 0$. Let $i : \gamma \to \mathbb{RP}^2$ be a projective involute given in homogeneous coordinates by $B = xA + yA'$. Then $b = B \times B'$ is a dancing mate of A if and only if $x + y' = 0$. That is, $C = 0$ in the previous proposition so y satisfies the ODE*

$$y^{(4)} + 2\frac{y'''y'}{y} + 3ry' + r'y = 0.$$

Proof. Let $(\mathbf{q}(t), \mathbf{p}(t))$ be an integral curve of $(Q^5, \mathcal{D})$. Then $\mathbf{p}^* = \mathbf{p} \times \mathbf{p}' = -\mathbf{q}'$. Then, to bring both $\mathbf{q}, \mathbf{p}^*$ to LF form we need the same projective parameter $\bar{t} = f(t)$, so that $A(\bar{t}) = f'(t)\mathbf{q}(t)$, $B(\bar{t}) = f'(t)\mathbf{p}^*(t) = -f'(t)\mathbf{q}'(t)$. Taking the derivative of $A(\bar{t}) = f'(t)\mathbf{q}(t)$ with respect to t, we get $f'(dA/d\bar{t}) = f''\mathbf{q} + f'\mathbf{q}' = (f''/f')A - B$, hence $B = xA + y(dA/d\bar{t})$, with $x = f''/f'$, $y = -f' \Longrightarrow x + dy/d\bar{t} = f''/f' - f''/f' = 0$. $\square$

Remark. The geometric meaning of the condition $x + y' = 0$ is the following. Since $B = xA + yA'$ then $B' = x'A + (x + y')A' + yA''$. Hence the condition $x + y' = 0$ means that B' is the intersection point of the line b and the line $a' = A \times A''$ (the line connecting A and A''). In Section 5.9 below, we will further interpret this condition in terms of the osculating conic and Cartan's developments.

5.7. **Example: Dancing around a circle.** Take γ to be a conic, e.g. a circle, $A = (1 + t^2, 2t, 1 - t^2)$. Then $A''' = 0$, so $A(t)$ is in the LF form with $r = 0$. Then the dancing mate equation in this case is

$$y^{(4)} + 2\frac{y'''y'}{y} = 0.$$

Any quadratic polynomial solves this (since $y'''=0$), and the corresponding involute $B=-y'A+yA$ is a straight line. To show this, take $y=at^2+bt+c$. Then

$$B=(bt^2+2(c-a)t-b,-2at^2+2c,-bt^2-2(a+c)t-b),$$

which is contained in the 2-plane

$$(a+c)x+by+(c-a)z=0,$$

so projects into a straight line in $\mathbb{RP}^2$.

If y is not a quadratic polynomial, then in a neighborhood of t where $y'''(t)\neq 0$,

$$0=\frac{y^{(4)}}{y'''}+2\frac{y'}{y}=[\log(y'''y^2)]'=0\Longrightarrow y'''y^2=\text{const.}$$

Now we can assume, without loss of generality, that const. $=1$ (multiplying y by a constant does not affect $[B(t)]$), so we end up with the ODE

$$y'''y^2=1.$$

We do not know how to solve this equation explicitly, so we do it numerically. The result is Figure 2 of Section 1.1.3 above.

A few words about this drawing: we make the drawing in the XY plane, where the circle is $X^2+Y^2=1$ and dancing curves around it are obtained from solutions of $y'''y^2=1$ via the formulas

$$\begin{aligned}B&=-y'A+yA'-y'\left(1+t^2-2tz,2(t-z),1-t^2-2tz\right),\\(X,Y)&=\left(\frac{B_2}{B_1},\frac{B_3}{B_1}\right)=\frac{(2(t-z),1-t^2-2tz)}{1-t^2-2tz},\quad z=y/y'.\end{aligned}$$

The projective coordinate t on a conic $\mathcal{C}$ misses a point (the point at "infinity"), so when integrating this equation numerically, one needs a second coordinate, $\bar{t}=f(t)=1/t$, and the transformation formulas $\bar{y}=f'y$, etc.

5.8. **Example: Dancing pairs of constant projective curvature.**
The idea: fix a point $(\mathbf{q}_0,\mathbf{p}_0)\in Q$ and an element $Y\in\mathfrak{sl}_3(\mathbb{R})$ such that $Y\cdot(\mathbf{q}_0,\mathbf{p}_0)$ is $\mathcal{D}$-horizontal. That is, $\mathbf{p}_0d\mathbf{q}$ and $d\mathbf{p}-\mathbf{q}_0\times d\mathbf{q}$ both vanish on $Y\cdot(\mathbf{q}_0,\mathbf{p}_0)=(Y\mathbf{q}_0,-\mathbf{p}_0Y)$. The subspace of such Y has codimension 3 in $\mathfrak{sl}_3(\mathbb{R})$, i.e. is 5-dimensional, since $\mathrm{SL}_3(\mathbb{R})$ acts transitively on Q and $\mathcal{D}$ has corank 3.

Then the orbit of $(\mathbf{q}_0,\mathbf{p}_0)$ under the flow of Y,

$$(\mathbf{q}(t),\mathbf{p}(t))=\exp(tY)\cdot(\mathbf{q}_0,\mathbf{p}_0)=(\exp(tY)\mathbf{q}_0,\mathbf{p}_0\exp(-tY))$$

is an integral curve of $\mathcal{D}$ (this follows from the $\mathrm{SL}_3(\mathbb{R})$-invariance of $\mathcal{D}$). The projected curves $q(t)=[\mathbf{q}(t)],p(t)=[\mathbf{p}(t)]$ are then a dancing pair. Each of the curves is an orbit of the 1-parameter subgroup $\exp(tY)$ of $\mathrm{SL}_3(\mathbb{R})$. Such curves are called W-curves or "pathcurves". They are very interesting curves, studied by Klein and Lie in 1871 [20]. They are: straight lines and conics, exponential curves, logarithmic spirals, and "generalized parabolas" (see below for explicit formulas). This class of curves (except lines and conics, considered degenerate) coincides with the class of curves with *constant projective curvature* κ.

The classification of curves with constant projective curvature κ, up to projective equivalence, is as follows. There are two generic cases, divided (strangely enough) by the borderline value $\kappa_0 = -3(32)^{-1/3} \approx -0.94$:

- $\kappa > \kappa_0$: logarithmic spirals, $r = e^{a\theta}$, $a > 0$ (in polar coordinates);
- $\kappa = \kappa_0$: the exponential curve $y = e^x$;
- $\kappa < \kappa_0$: generalized parabolas, $y = x^m$, $m > 0$, $m \neq 2, 1, 1/2$.

The curves. Take $\mathbf{q}_0 = (0,0,1)^t$, $\mathbf{p}_0 = (1,0,0)$. Then $Y \cdot (\mathbf{q}_0, \mathbf{p}_0) \in \mathcal{D}_{(\mathbf{q}_0,\mathbf{p}_0)}$, $Y \in \mathfrak{sl}_3(\mathbb{R})$, implies that

$$Y = \left(\begin{array}{c|c} A & \mathbf{v} \\ \hline \mathbf{v}^* & 0 \end{array}\right), \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \mathbf{v}^* = (v_2, -v_1), \quad A \in \mathfrak{sl}_2(\mathbb{R}).$$

Let $H_0 \cong \mathrm{SL}_2(\mathbb{R})$ be the stabilizer subgroup of $(\mathbf{q}_0, \mathbf{p}_0)$. It acts on Y by the adjoint representation,

$$(A, \mathbf{v}) \mapsto (hAh^{-1}, h\mathbf{v}), \quad h \in \mathrm{SL}_2(\mathbb{R}).$$

Then, reducing by this $\mathrm{SL}_2(\mathbb{R})$-action as well as by rescaling, $Y \mapsto \lambda Y$, $\lambda \in \mathbb{R}^*$ (this just reparametrizes the orbit), and removing orbits which are fixed points and straight lines, we are left with a list of "normal forms" of Y (two one-parameter families and one isolated case):

$$Y_1 := \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & a \\ a & -1 & 0 \end{pmatrix}, \quad a > 0, \tag{33a}$$

$$Y_2 := \begin{pmatrix} 0 & 1 & b \\ -1 & 0 & 0 \\ 0 & -b & 0 \end{pmatrix}, \quad b > 0, \tag{33b}$$

$$Y_3 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{33c}$$

Proposition 5.14. *The pair of curves* $[\mathbf{q}(t)], [\mathbf{p}(t)]$ *in* $\mathbb{RP}^2, \mathbb{RP}^{2*}$ *(respectively), where* $\mathbf{q}(t) = \exp(tY)\mathbf{q}_0$, $\mathbf{p}(t) = \mathbf{p}_0 \exp(-tY)$, $\mathbf{q}_0 = (0,0,1)^t$, $\mathbf{p}_0 = (0,0,1)$, *and* Y *is any of the matrices in equations* (33) *above, is a dancing pair of curves with constant projective curvature* κ *(same value of* κ *for each member of the pair). All values of* $\kappa \in \mathbb{R}$ *can be obtained in such a way.*

Proof. A matrix Y with $\mathrm{tr}(Y) = 0$ has characteristic polynomial of the form $\det(\lambda I - Y) = \lambda^3 + a_1\lambda + a_0$. Then, using $Y^3 + a_1 Y + a_0 I = 0$ (Cayley-Hamilton), we have that $\mathbf{q}(t) := \exp(tY)\mathbf{q}_0$ satisfies the tautological ODE

$$\mathbf{q}''' + a_1\mathbf{q}' + a_0\mathbf{q} = 0.$$

From Cartan's formulas ([6], pp. 69 and 71), we then find easily

$$\kappa = a_1 a_0^{-2/3}/2.$$

Now in our case, the characteristic polynomials are

$$\text{(a)} \quad \lambda^3 - \lambda - 2a, \qquad \text{(b)} \quad \lambda^3 + \lambda - b^2, \qquad \text{(c)} \quad \lambda^3 - 1;$$

hence we get projective curvatures

$$\text{(a)} \quad \kappa = -(32a^2)^{-1/3}, \qquad \text{(b)} \quad \kappa = b^{-4/3}/2, \qquad \text{(c)} \quad \kappa = 0.$$

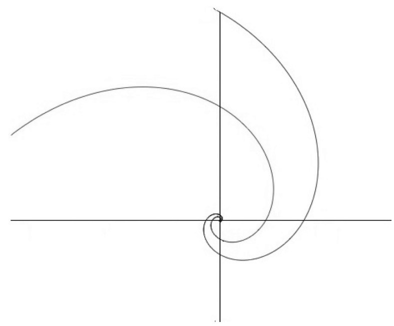

FIGURE 9. A dancing pair of logarithmic spirals with $\kappa = 0$

We thus get all possible values of κ. $\square$

To visualize such a pair, we draw in Figure 9 the pair $(q(t), p^*(t))$, where $p^*(t) = [\mathbf{p}^*(t)]$ is the curve dual to $p(t)$, given by $\mathbf{p}^*(t) = \mathbf{p}(t) \times \mathbf{p}'(t) = -Y\mathbf{q}(t)$.

5.9. **Projective rolling without slipping and twisting.**

5.9.1. *About Riemannian rolling.* Let us first describe ordinary (Riemannian) rolling, following [5, p. 456]. Let $(\Sigma_i, \mathbf{g_i})$, $i = 1, 2$, be two Riemannian surfaces. The configuration space for rolling the two surfaces along each other is the space $\mathcal{RC}$ of *Riemannian contact elements* (u_1, u_2, ψ), where $u_i \in \Sigma_i$ and

$$\psi : T_{u_1}\Sigma_1 \to T_{u_2}\Sigma_2$$

is an isometry. $\mathcal{RC}$ is a 5-manifold, and if Σ_i are oriented, then $\mathcal{RC}$ is the disjoint union $\mathcal{RC} = \mathcal{RC}^+ \sqcup \mathcal{RC}^-$, where each $\mathcal{RC}^\pm$ is a circle bundle over $\Sigma_1 \times \Sigma_2$ in an obvious way, so that $\mathcal{RC}^+$ consists of the orientation preserving Riemannian contact elements and $\mathcal{RC}^-$ are the orientation reversing.

A parametrized curve $(u_1(t), u_2(t), \psi(t))$ in $\mathcal{RC}$ satisfies the *non-slip condition* if

$$u_2'(t) = \psi(t)u_1'(t)$$

for all t. It satisfies also the *no-twist condition* if for every parallel vector field $v_1(t)$ along $u_1(t)$,

$$v_2(t) = \psi(t)v_1(t)$$

is parallel along $u_2(t)$, where "parallel" is taken with respect to the Levi-Civita connection of the corresponding metric.

It is easy to show that these two conditions define together a rank 2 distribution $\mathcal{D} \subset T\mathcal{RC}$ which is $(2,3,5)$ unless the surfaces are isometric ([5], p. 458). For some special pairs of surfaces (e.g., a pair of round spheres of radius ratio $3:1$) $\mathcal{D}$ is maximally symmetric, i.e., admits $\mathfrak{g}_2$-symmetry (maximum possible). Recently [3], some new pairs of surfaces were found where the corresponding *rolling distribution* $(\mathcal{RC}, \mathcal{D})$ admits $\mathfrak{g}_2$-symmetry, but the general case is not settled yet.

Now in [3] it was noticed that Riemannian rolling can be reformulated as follows. Let $M^4 := \Sigma_1 \times \Sigma_2$, equipped with the difference metric $\mathbf{g} = \mathbf{g_1} \ominus \mathbf{g_2}$. This is a pseudo-Riemannian metric of signature $(2,2)$. Then one can check easily that $\psi : T_{u_1}\Sigma_1 \to T_{u_2}\Sigma_2$ is an isometry if and only if its graph

$$W_\psi = \{(v, \psi v) | v \in T_{u_1}\Sigma_1\} \subset T_{u_1}\Sigma_1 \oplus T_{u_2}\Sigma_2 \simeq T_{(u_1,u_2)}M^4$$

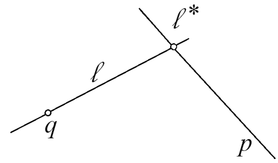

FIGURE 10. The natural isomorphism $\Psi_{q,p} : \ell \mapsto \ell^*$

is a *non-principal null 2-plane*; i.e., a null 2-plane not of the form $T_{u_1}\Sigma_1 \oplus \{0\}$ or $\{0\} \oplus T_{u_2}\Sigma_2$ (compare with Corollary 4.11a). This defines an embedding of $\mathcal{RC}$ in the total space of the twistor fibration $\mathbb{T}M^4 \to M^4$ of $(M^4, \mathbf{g})$ (see Section 4.2.4). Furthermore, if Σ_i are oriented, then one can orient M^4 so that $\mathcal{RC}^+$ (orientation preserving ψ's) is mapped to the self-dual twistor space $\mathbb{T}^+M^4$ and $\mathcal{RC}^-$ to the anti-self-dual twistor space $\mathbb{T}^-M^4$. Finally, it is shown in [3] that under the embedding $\mathcal{RC} \hookrightarrow \mathbb{T}M^4$, the *rolling distribution* $\mathcal{D}$ on $\mathcal{RC}$ goes over to the *twistor distribution* associated with Levi-Civita connection of $(M^4, \mathbf{g})$.

In what follows, we give a similar "rolling interpretation" of the self-dual twistor space of the dancing space $(M^4, \mathbf{g})$, and thus, via the identification $Q^5 \hookrightarrow \mathbb{T}^+M^4$ of Theorem 4.9, a "rolling interpretation" of our equations (2). The novelty here is that the dancing metric $(M^4, \mathbf{g})$ is irreducible, i.e., not a difference metric as in the case of Riemannian rolling. And yet, it can be given a rolling interpretation of some sort and in addition admits $\mathfrak{g}_2$-symmetry.

We try to keep our terminology as close as possible to the above terminology of Riemannian rolling in order to make the analogy transparent.

5.9.2. *A natural isomorphism of projective spaces.* A projective isomorphism of two projective spaces $\mathbb{P}(V)$, $\mathbb{P}(W)$ is the projectivization $[T]$ of a linear isomorphism $T : V \xrightarrow{\sim} W$ of the underlying vector spaces, $[T] : [v] \mapsto [Tv]$. Two linear isomorphisms $T, T' : V \to W$ induce the same projective isomorphism if and only if $T' = \lambda T$ for some $\lambda \in \mathbb{R}^*$.

For each non-incident pair $(q, p) \in M^4$ we define a projective isomorphism

$$\Psi_{q,p} : \mathbb{P}(T_q\mathbb{RP}^2) \xrightarrow{\sim} \mathbb{P}(T_p\mathbb{RP}^{2*}) \tag{34}$$

by first identifying $\mathbb{P}(T_q\mathbb{RP}^2)$ with the pencil of lines through q and $\mathbb{P}(T_p\mathbb{RP}^{2*})$ with the points on the line p. We then send a line ℓ through q to its intersection point ℓ^* with p. One can verify easily that $\Psi_{q,p}$ is a projective isomorphism. See Figure 10.

5.9.3. *Projective contact.*

Definition 5.15. A *projective contact element* between $\mathbb{RP}^2$ and $\mathbb{RP}^{2*}$ is a triple (q, p, ψ) where $(q, p) \in M^4$ and

$$\psi : T_q\mathbb{RP}^2 \to T_p\mathbb{RP}^{2*}$$

is a linear isomorphism covering the natural projective isomorphism $\Psi_{q,p}$ of (34); that is, $[\psi] = \Psi_{q,p}$. The set of projective contact elements forms a principal $\mathbb{R}^*$-fibration $\mathcal{PC} \to M^4$, $(q, p, \psi) \mapsto (q, p)$.

Remark. We only allow projective contacts between $\mathbb{RP}^2$ and $\mathbb{RP}^{2*}$ at a *non-incident* pair $(q, p) \in M^4$.

Let us look at the projective contact condition on ψ. We take a non-zero $v \in T_q\mathbb{RP}^2$ and let $w = \psi(v)$. To v corresponds a line ℓ through q, tangent to v at q. Likewise, to w corresponds a point $\ell^* \in p$, whose dual line in $\mathbb{RP}^{2*}$ (the pencil of lines through ℓ^*) is tangent to w at p. The projective contact condition on ψ is then the incidence relation $\ell^* \in \ell$. But this is precisely the *dancing condition*; i.e., $(v, w) \in T_{(q,p)}M^4$ is a null vector. In other words, the graph of ψ,

$$W_\psi = \{(v, \psi(v)) \,|\, v \in T_q\mathbb{RP}^2\} \subset T_q\mathbb{RP}^2 \oplus T_p\mathbb{RP}^{2*} = T_{(q,p)}M^4,$$

is a null 2-plane. More precisely,

Proposition 5.16. *A linear isomorphism $\psi : T_q\mathbb{RP}^2 \to T_p\mathbb{RP}^{2*}$ is a projective contact if and only if its graph $W_\psi \subset T_{(q,p)}M^4$ is a non-principal self-dual null 2-plane* (*see Corollary* 4.11).

Proof. We recall from Section 4.4: a local section σ of $j : \mathrm{SL}_3(\mathbb{R}) \to M$ around $(q,p) \in M$ provides a null coframing $\hat\eta := \sigma^*\eta = (\hat\eta^1, \hat\eta^2, \hat\eta_1, \hat\eta_2)^t$ such that $T_q\mathbb{RP}^2 \oplus \{0\} = \{\hat\eta_1 = \hat\eta_2 = 0\}$, $\{0\} \oplus T_p\mathbb{RP}^{2*} = \{\hat\eta^1 = \hat\eta^2 = 0\}$ and $\mathbf{g} = 2\hat\eta_a\,\hat\eta^a$. Let $f = (f_1, f_2, f^1, f^2)$ be the dual framing, $\psi(f_a) = \psi_{ab}f^b$. Now the projective contact condition is $\psi(v)(v) = 0 \Longrightarrow \psi_{ab} = -\psi_{ba}$. Say $\psi(f_1) = \lambda f^2$, $\psi(f_2) = -\lambda f^1$ for some $\lambda \in \mathbb{R}^* \Longrightarrow W_\psi = \mathrm{Span}\{f_1 + \lambda f^2, f_2 - \lambda f^1\} = \mathrm{Ker}\{\lambda\hat\eta^1 - \hat\eta_2, \lambda\hat\eta^2 - \hat\eta_1\}$. The 2-form corresponding to W_ψ is thus $\beta = (\lambda\hat\eta^1 - \hat\eta_2) \wedge (\lambda\hat\eta^2 + \hat\eta_1)$. Using formula (26a), this is easily seen to be the general form of an SD non-principal null 2-plane. ☐

Corollary 5.17. *The map $\psi \mapsto W_\psi$ defines an $\mathrm{SL}_3(\mathbb{R})$-equivariant embedding*

$$\mathcal{PC} \hookrightarrow \mathbb{T}^+M^4$$

whose image is the set $\mathbb{T}^+_*M^4$ of non-principal SD 2-planes in TM^4 (*the non-integrable locus of the twistor distribution $\mathcal{D}^+$*).

Now combining this last corollary with Theorem 4.9, we obtain the identifications

$$Q^5 \simeq \mathbb{T}^+_*M^4 \simeq \mathcal{PC}.$$

Tracing through our definitions, we find

Proposition 5.18. *There is an isomorphism of principal $\mathbb{R}^*$-bundles over M^4,*

$$Q^5 \xrightarrow{\sim} \mathcal{PC},$$

sending $(\mathbf{q}, \mathbf{p}) \in Q^5$ to the projective contact element (q, p, ψ), where $q = [\mathbf{q}], p = [\mathbf{p}]$, and $\psi : T_q\mathbb{RP}^2 \to T_p\mathbb{RP}^{2*}$ is given in homogeneous coordinates by

$$\psi([\mathbf{v}]) = [\mathbf{q} \times \mathbf{v}].$$

That is, if $v = d\pi_{\mathbf{q}}(\mathbf{v})$, then $\psi(v) = d\bar\pi_{\mathbf{p}}(\mathbf{q} \times \mathbf{v})$.

Definition 5.19. A parametrized curve $(q(t), p(t), \psi(t))$ in $\mathcal{PC}$ satisfies the *no-slip condition* if

$$\psi(t)q'(t) = p'(t)$$

for all t.

Proposition 5.20. *The projection $\mathcal{PC} \to M^4$ defines a bijection between curves in $\mathcal{PC}$ satisfying the no-slip condition and non-degenerate null curves in M^4.*

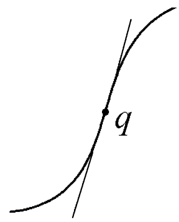

FIGURE 11. An inflection point

Proof. If $(q(t), p(t), \psi(t))$ satisfies the no-slip condition, then $(q'(t), p'(t)) \in W_{\psi(t)}$, which is a null plane; hence $(q'(t), p'(t))$ is a null vector. Conversely, if $(q(t), p(t))$ is null and non-degenerate, then for all t there is a unique non-principal SD null 2-plane W_t containing the null vector $(q'(t), p'(t))$. By the previous proposition, there is a unique $\psi(t)$ such that $W_t = W_{\psi(t)}$; hence $\psi(t)q'(t) = p'(t)$ and so $(q(t), p(t), \psi(t))$ satisfies the no-slip condition. $\square$

5.9.4. *The normal acceleration.*

Definition 5.21. Given a parametrized regular curve $q(t)$ in $\mathbb{RP}^2$, i.e., $q'(t) \neq 0$, its *normal acceleration*, denoted by q'', is a section of the normal line bundle of the curve, defined as follows: lift the curve to $\mathbf{q}(t)$ in $\mathbb{R}^3 \setminus 0$, then let

$$q'' := d\pi_{\mathbf{q}}(\mathbf{q}'') \ (\mathrm{mod}\ q'),$$

where $\pi : \mathbb{R}^3 \setminus 0 \to \mathbb{RP}^2$ is the canonical projection $\mathbf{q} \mapsto [\mathbf{q}]$.

Claim. This definition is independent of the lift $\mathbf{q}(t)$ chosen.

Proof. Note first that $\mathbb{R}\mathbf{q} = \mathrm{Ker}(d\pi_{\mathbf{q}})$ and that $d\pi_{\mathbf{q}}\mathbf{q}' = q'$. Now if we modify the lift by $\mathbf{q} \mapsto \lambda\mathbf{q}$, where λ is some non-vanishing real function of t, then $\mathbf{q}'' \mapsto (\lambda\mathbf{q})'' \equiv \lambda\mathbf{q}'' \ (\mathrm{mod}\ \mathbf{q}, \mathbf{q}') \Longrightarrow d\pi_{\lambda\mathbf{q}}(\lambda\mathbf{q})'' \equiv d\pi_{\lambda\mathbf{q}}(\lambda\mathbf{q}'') = d\pi_{\lambda\mathbf{q}}(d\lambda_{\mathbf{q}}(\mathbf{q}'')) = d(\pi \circ \lambda)_{\mathbf{q}}(\mathbf{q}'') = d\pi_{\mathbf{q}}(\mathbf{q}'') \ (\mathrm{mod}\ q')$. $\square$

Remark. If we write $q(t)$ in some affine coordinate chart, $q(t) = (x(t), y(t))$, then the above definition implies that $q'' = (x'', y'') \bmod (x', y')$. The disadvantage of this simple formula is that it is not so easy to show directly that this definition is independent of the affine coordinates chosen (the reader is invited to try).

Definition 5.22. An inflection point of a regular curve in $\mathbb{RP}^2$ is a point where the normal acceleration vanishes. (See Figure 11.)

Remark. It is easy to check that the definition is parametrization independent. In fact, it is equivalent to the following, perhaps better-known, definition: an inflection point is a point where the tangent line has a higher order of contact with the curve than expected (second order or higher).

Now given a curve $(q(t), p(t), \psi(t))$ in $\mathcal{PC}$, if it satisfies the no-slip condition $\psi(t)q'(t) = p'(t)$, then $\psi(t)$ induces a bundle map, denoted also by $\psi(t)$, between the normal line bundles along $q(t)$ and $p(t)$.

Proposition 5.23. *For any curve $(q(t), p(t), \psi(t))$ in $\mathcal{PC}$ satisfying the no-slip condition $\psi(t)q'(t) = p'(t)$,*

$$\psi(t)q''(t) = p''(t).$$

Proof. First note that for the normal accelerations q'', p'' to be well-defined, both q', p' must be non-vanishing; i.e., $\Gamma(t) := (q(t), p(t))$ is a non-degenerate null curve in M^4 (see Definition 4.7). It follows (see Lemma 4.13) that we can choose an *adapted* lift σ of Γ to $\mathrm{SL}_3(\mathbb{R})$ with associated coframing $\hat{\eta} = \sigma^*\eta$ and dual framing $\{f_1, f_2, f^1, f^2\}$ such that $\Gamma' = f_2 + f^1$. Let $\tilde{\Gamma} = \tilde{j} \circ \sigma$, with $\tilde{\Gamma}(t) = (\mathbf{q}(t), \mathbf{p}(t))$, $s^i{}_j = \omega^i{}_j(\tilde{\Gamma}')$, $E_i(t) = E_i(\sigma(t))$. Then $\mathbf{q}' = E_3' = E_2 + s^3{}_3 E_3 \Longrightarrow \mathbf{q}'' \equiv E_2' \equiv s^1{}_2 E_1 \pmod{\mathbf{q}, \mathbf{q}'} \Longrightarrow q'' \equiv s^1{}_2 f_1 \pmod{q'}$, and similarly $p'' \equiv s^1{}_2 f^2 \pmod{p'}$. Now

$$W^+ = \mathrm{Span}\{f_2 + f^1, f_1 - f^2\} \Longrightarrow \psi f_2 = f^1 \Longrightarrow \psi q'' = \psi(s^2{}_1 f_2) = s^2{}_1 f^1 = p'' \pmod{p'}.$$

□

Corollary 5.24. *For a pair of regular curves* $(q(t), p(t))$ *satisfying the dancing condition* (*equivalently,* $\Gamma(t) = (q(t), p(t))$ *is a non-degenerate null curve in* M^4), $q(t)$ *is an inflection point if and only if* $p(t)$ *is an inflection point.*

5.9.5. *Osculating conics and Cartan's developments.* To complete the "projective rolling" interpretation of $(Q^5, \mathcal{D})$ we introduce a projective connection associated with a plane curve $\gamma \subset \mathbb{RP}^2$, defined on its fibration of osculating conics $\mathcal{C}_\gamma$; the associated horizontal curves of this connection project to plane curves which are "Cartan's developments" of γ. The "no twist" condition for projective rolling is then expressed in terms of this connection, in analogy with the rolling of Riemannian surfaces.

Let $\gamma \subset \mathbb{RP}^2$ be a smooth locally convex curve (i.e., without inflection points). For each $q \in \gamma$ there is a unique conic $\mathcal{C}_q \subset \mathbb{RP}^2$ which is tangent to γ to order 4 at q. This is the projective analog of the osculating circle to a curve in Euclidean differential geometry. Define

$$\mathcal{C}_\gamma = \{(q, x) \,|\, q \in \gamma, x \in \mathcal{C}_q\} \subset \gamma \times \mathbb{RP}^2.$$

We get a fibration

$$\mathcal{C}_\gamma \to \gamma, \quad (q, x) \to q.$$

Remark. The fibration $\mathcal{C}_\gamma \to \gamma$ has some remarkable properties, see Figure 12. We refer the reader to the beautiful article [15] for further details..

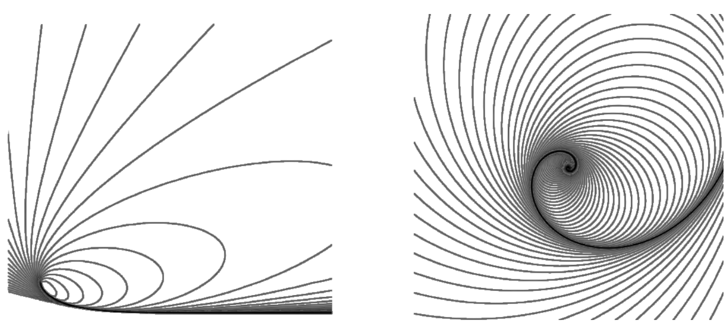

FIGURE 12. The osculating conics of a tractrix and a logarithmic spiral

There is a *projective connection* defined on $\mathcal{C}_\gamma \to \gamma$, i.e., a line field on $\mathcal{C}_\gamma$, transverse to the fibers, whose associated parallel transport identifies the fibers

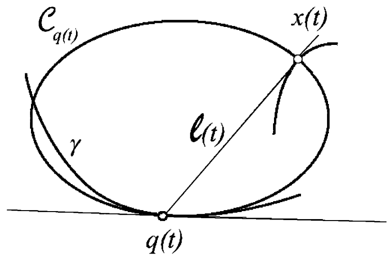

FIGURE 13. Cartan's development $x(t)$ of γ

of $\mathcal{C}_\gamma$ projectively. Its integral curves (the horizontal lifts of γ to $\mathcal{C}_\gamma$) are defined as follows ([6, p. 58]): if we parametrize γ by $q(t)$, then its horizontal lifts are parametrized curves $(q(t), x(t)) \in \mathcal{C}_\gamma$ such that the projected curve $x(t)$ is tangent to the line $\ell(t)$ passing through $x(t)$ and $q(t)$. The projections $x(t)$ of such horizontal curves on $\mathbb{RP}^2$ are *Cartan's developments* of γ. See Figure 13.

Next we consider another fibration of projective lines along γ,

$$\mathcal{L}_\gamma := \mathbb{P}(T\mathbb{RP}^2)|_\gamma \to \gamma.$$

The fiber over $q \in \gamma$ is the projectivized tangent space $\mathbb{P}(T_q\mathbb{RP}^2)$, which we can also identify with the pencil of lines through q.

We identify the fiber bundles $\mathcal{C}_\gamma \simeq \mathcal{L}_\gamma$ using the usual "stereographic projection": a point $x \in \mathcal{C}_q$, $x \neq q$, is mapped to the line ℓ joining x with q, while q itself is mapped to the tangent line to γ at q. Thus the projective connection on $\mathcal{C}_\gamma$ defines, via the identification $\mathcal{C}_\gamma \simeq \mathcal{L}_\gamma$, a projective connection on $\mathcal{L}_\gamma$.

5.9.6. *Examples of developments.* These are important examples and will be used later.

(1) Parametrize a locally convex curve $\gamma \subset \mathbb{RP}^2$ by $A(t)$ in LF form, i.e., $A''' + rA = 0$ (see Proposition 5.9). Using homogeneous coordinate (x, y, z) on $\mathbb{RP}^2$ with respect to the frame $A(t), A'(t), A''(t)$, the osculating conic $\mathcal{C}_t$ at $[A(t)]$ is given by $y^2 = 2xz$ (see [6], p. 55).

    In particular, taking $x = y = 0$, we get that $[A''(t)]$ is on the osculating conic at $[A(t)]$. In fact, $x(t) := [A''(t)]$ *is a development of* γ.

    Proof. $(A'')' = -rA$, so the tangent line to $A''(t)$ passes through $A(t)$. □

    The associated parallel line $\ell(t)$ along γ is given by $a' = A \times A''$ where $a = A \times A'$ is the dual curve.

(2) In fact, the development $[A''(t)]$ of the previous item is not so special. It is easy to see that for any point $x \in \mathcal{C}_q$ (other than $x = q$), one can pick a parametrization $A(t)$ of γ in LF form such that $x = [A''(0)]$.

    Proof (*Sketch*). Start with any $A(t)$ such that $q = [A(0)]$, then find a Möbius transformation $\bar{t} = f(t)$ such that $f(0) = 0$ and $\bar{A}(\bar{t}) = f'(t)A(t)$ satisfies $x = [\bar{A}''(0)]$. □

(3) Another way to get all developments of γ, using the notation of the first example, is to parametrize $\mathcal{C}_t$ by $P(u) = A(t) + uA'(t) + (u^2/2)A''(t)$. Then the developments are given by $x(t) = [P(u(t))]$, where $u(t)$ satisfies $u' + 1 = 0$; i.e., $P_c(t) = A(t) + (c - t)A'(t) + [(c - t)^2/2]A''(t)$ is a development of γ for

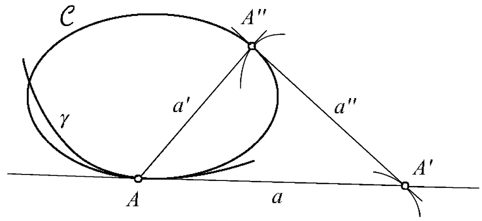

FIGURE 14. Development of the 2nd kind (development of the dual curve)

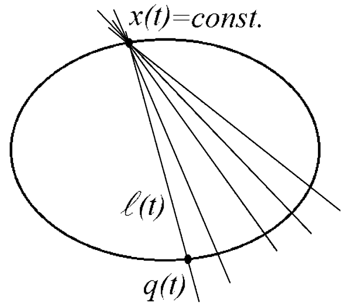

FIGURE 15. Parallel transport of a line along a conic

every constant c. (Note that these developments miss exactly the first example $x(t) = [A''(t)]$ above.)

Using this formula, Cartan shows that every development curve $P_c(t)$ is tangent to γ as $t \to c$, with a cusp at $t = c$.

(4) Consider the curve $\gamma^* \subset \mathbb{RP}^{2*}$ dual to a curve $\gamma \subset \mathbb{RP}^2$ with a parametrization $A(t)$ in LF form. Parametrize γ^* by $a = A \times A'$. One can check easily that $a(t)$ satisfies $a''' - ra = 0$, so is also in LF form. It follows, as in the last example, that $a''(t)$ is a development of γ^*. The associated "parallel line" along $a(t)$ (a point on $a(t)$) is $A' = a \times a''$.

Remark. Cartan calls the curve $A'(t)$ a *development of the 2nd kind* of γ (see Figure 14). It can also be characterized as the envelope (or dual) of the family of tangents to osculating conics along the development $A''(t)$.

(5) When γ is itself a conic $\mathcal{C}$ the osculating conic is obviously $\mathcal{C}$ itself for all $q \in \mathcal{C}$; hence the development curves $(q(t), x(t))$ satisfy $x(t) = \text{const}$. It follows that if we parallel transport a line along a conic, we get a family of *concurrent* lines $\ell(t)$. See Figure 15.

5.9.7. *The no-twist condition.*

Definition 5.25. A *projective rolling without slipping or twisting* of $\mathbb{RP}^2$ along $\mathbb{RP}^{2*}$ is a parametrized curve $(q(t), p(t), \psi(t))$ in $\mathcal{PC}$, satisfying for all t:

- the no-slip condition: $\psi(t)q'(t) = p'(t)$;
- the no-twist condition: if $u(t)$ is a parallel section of $\mathbb{P}(T\mathbb{RP}^2)$ along $q(t)$, then $\psi(t)u(t)$ is a parallel section of $\mathbb{P}(T\mathbb{RP}^{2*})$ along $p(t)$.

Theorem 5.26. *Under the identification $Q^5 \simeq \mathcal{PC}$, integral curves of the Cartan-Engel distribution $(Q^5, \mathcal{D})$ correspond to projective rolling curves in $\mathcal{PC}$ satisfying the no-slip and no-twist conditions.*

Proof. Let $(\mathbf{q}(t), \mathbf{p}(t))$ be an integral curve of $(Q^5, \mathcal{D})$ and let $(q(t), p(t), \psi(t))$ be the corresponding projective rolling curve in $\mathcal{PC}$. Then $(q(t), p(t))$ is a null curve in M^4; hence $(q(t), p(t), \psi(t))$ satisfies the no-slip condition. Let $\ell(t)$ be a parallel line along $q(t)$. We need to show that $\ell^*(t) := \psi(t)\ell(t) = \ell(t) \cap p(t)$ is parallel along $p(t)$. Pick a projective parameter t for $q(t)$ and a lift $A(t)$ of $q(t)$ to $\mathbb{R}^3 \setminus 0$ such that (1) $A''' + rA = 0$ (the LF form) and (2) $\ell(t)$ is the line $a' = A \times A''$ connecting $A(t)$ and $A''(t)$ (see example (2) in Section 5.9.6). Now $p(t)$ is a dancing mate of $q(t)$; hence its dual $p^*(t)$ is given by $B = xA + yA'$, where $B''' \times B = 0, x + y' = 0$. It follows that $\ell^*(t) = [B'(t)]$ (see the remark following Proposition 5.13), which is parallel along $p(t) = [b(t)]$, by example (4) in Section 5.9.6. $\square$

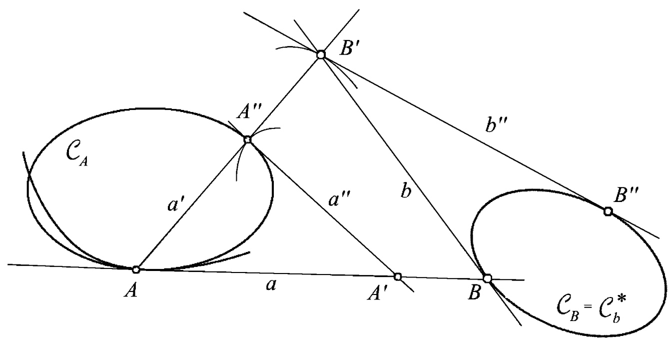

FIGURE 16. The proof of Theorem 5.26

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