TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 8, August 2018, Pages 5425–5489
http://dx.doi.org/10.1090/tran/7132
Article electronically published on April 17, 2018

$\mathscr{B}$-FREE SETS AND DYNAMICS

AURELIA DYMEK, STANISŁAW KASJAN, JOANNA KUŁAGA-PRZYMUS,
AND MARIUSZ LEMAŃCZYK

Abstract. Given $\mathscr{B} \subset \mathbb{N}$, let $\eta = \eta_{\mathscr{B}} \in \{0,1\}^{\mathbb{Z}}$ be the characteristic function of the set $\mathcal{F}_{\mathscr{B}} := \mathbb{Z} \setminus \bigcup_{b\in\mathscr{B}} b\mathbb{Z}$ of $\mathscr{B}$-free numbers. The $\mathscr{B}$-free shift (X_η, S), its hereditary closure $(\widetilde{X}_\eta, S)$, and (still larger) the $\mathscr{B}$-admissible shift $(X_{\mathscr{B}}, S)$ are examined. Originated by Sarnak in 2010 for $\mathscr{B}$ being the set of square-free numbers, the dynamics of $\mathscr{B}$-free shifts was discussed by several authors for $\mathscr{B}$ being Erdös; i.e., when $\mathscr{B}$ is infinite, its elements are pairwise coprime, and $\sum_{b\in\mathscr{B}} 1/b < \infty$: in the Erdös case, we have $X_\eta = \widetilde{X}_\eta = X_{\mathscr{B}}$.

It is proved that X_η has a unique minimal subset, which turns out to be a Toeplitz dynamical system. Furthermore, a $\mathscr{B}$-free shift is proximal if and only if $\mathscr{B}$ contains an infinite coprime subset. It is also shown that for $\mathscr{B}$ with light tails, i.e., $\overline{d}(\sum_{b>K} b\mathbb{Z}) \to 0$ as $K \to \infty$, proximality is the same as heredity.

For each $\mathscr{B}$, it is shown that η is a quasi-generic point for some natural S-invariant measure ν_η on X_η. A special role is played by subshifts given by $\mathscr{B}$ which are taut, i.e., when $\boldsymbol{\delta}(\mathcal{F}_{\mathscr{B}}) < \boldsymbol{\delta}(\mathcal{F}_{\mathscr{B}\setminus\{b\}})$ for each $b \in \mathscr{B}$ ($\boldsymbol{\delta}$ stands for the logarithmic density). The taut class contains the light tail case; hence all Erdös sets and a characterization of taut sets $\mathscr{B}$ in terms of the support of ν_η are given. Moreover, for any $\mathscr{B}$ there exists a taut $\mathscr{B}'$ with $\nu_{\eta_{\mathscr{B}}} = \nu_{\eta_{\mathscr{B}'}}$. For taut sets $\mathscr{B}, \mathscr{B}'$, it holds that $X_{\mathscr{B}} = X_{\mathscr{B}'}$ if and only if $\mathscr{B} = \mathscr{B}'$.

For each $\mathscr{B}$, it is proved that there exists a taut $\mathscr{B}'$ such that $(\widetilde{X}_{\eta_{\mathscr{B}'}}, S)$ is a subsystem of $(\widetilde{X}_{\eta_{\mathscr{B}}}, S)$ and $\widetilde{X}_{\eta_{\mathscr{B}'}}$ is a quasi-attractor. In particular, all invariant measures for $(\widetilde{X}_{\eta_{\mathscr{B}}}, S)$ are supported by $\widetilde{X}_{\eta_{\mathscr{B}'}}$. Moreover, the system $(\widetilde{X}_\eta, S)$ is shown to be intrinsically ergodic for an arbitrary $\mathscr{B}$. A description of all probability invariant measures for $(\widetilde{X}_\eta, S)$ is given. The topological entropies of $(\widetilde{X}_\eta, S)$ and $(X_{\mathscr{B}}, S)$ are shown to be the same and equal to $\overline{d}(\mathcal{F}_{\mathscr{B}})$ ($\overline{d}$ stands for the upper density).

Finally, some applications in number theory on gaps between consecutive $\mathscr{B}$-free numbers are given, and some of these results are applied to the set of abundant numbers.

Contents

1. Introduction 5426
2. Preliminaries 5432
3. Topological dynamics 5442
4. Tautness 5451
5. Heredity (proofs of Theorems D and G) 5464
6. Entropy 5471
7. Tautness and support of ν_η (proof of Theorem H) 5475

Received by the editors November 6, 2015, and, in revised form, June 3, 2016 and November 10, 2016.

2010 *Mathematics Subject Classification.* Primary 37A35, 37A45, 37B05, 37B10, 37B40; Secondary 11N25.

This research was supported by Narodowe Centrum Nauki grant UMO-2014/15/B/ST1/03736.

8. Invariant measures (proof of Theorem I) 5476
9. Tautness revisited 5483
10. Intrinsic ergodicity (proof of Theorem J) 5484
11. Remarks on abundant numbers 5485
Acknowledgment 5487
References 5487

1. Introduction

1.1. Motivation.

Sets of multiples. For a subset $\mathscr{B}\subset\mathbb{N}:=\{1,2,\dots\}$, we consider its *set of multiples* $\mathcal{M}_{\mathscr{B}}:=\bigcup_{b\in\mathscr{B}}b\mathbb{Z}$ and the associated set of *$\mathscr{B}$-free numbers* $\mathcal{F}_{\mathscr{B}}:=\mathbb{Z}\setminus\mathcal{M}_{\mathscr{B}}$. The interest in sets of multiples was initiated in the 1930s by the study of the set of *abundant numbers*, i.e., of $n\in\mathbb{Z}$ for which $|n|$ is less than or equal to the sum of its (positive) proper divisors. In [7], Bessel-Hagen asked whether the set of abundant numbers has asymptotic density, and the positive answer was given independently by Davenport [12], Chowla [10] and Erdös [19]. Nowadays, abundant numbers are still of certain interest in number theory (see, e.g., [31, 32, 35]).

The works of Davenport, Chowla, and Erdös led to various problems on general sets of multiples. In particular, the natural question whether all sets of multiples have asymptotic density was answered negatively by Besicovitch [6]. On the other hand, Davenport and Erdös [13, 14] showed that $\mathcal{M}_{\mathscr{B}}$ (equivalently, $\mathcal{F}_{\mathscr{B}}$) always has logarithmic density equal to the lower density (respectively, upper density). In many cases, $\mathcal{M}_{\mathscr{B}}$ has even density: it is the case for $\mathscr{B}$ Erdös [22], that is, when $\mathscr{B}$ is infinite, its elements are pairwise coprime and $\sum_{b\in\mathscr{B}}1/b<\infty$; see, e.g., [27]. Following [28], all sets $\mathscr{B}\subset\mathbb{N}$ for which $\mathcal{M}_{\mathscr{B}}$ has density are called *Besicovitch*.

An important example of an Erdös set, hence Besicovitch, is the set $\mathscr{S}=\{p^2: p\in\mathscr{P}\}$ of squares of primes. Then, $\mathcal{F}_{\mathscr{S}}$ is called the set of *square-free* integers and its density equals $6/\pi^2$; see, e.g., [29]. The characteristic function of $\mathcal{F}_{\mathscr{S}}$ is the square $\boldsymbol{\mu}^2$ of the Möbius function $\boldsymbol{\mu}$ extended to $\mathbb{Z}$ in the natural way: $\boldsymbol{\mu}(-n)=\boldsymbol{\mu}(n),\boldsymbol{\mu}(0)=0$. (Recall that $\boldsymbol{\mu}(n)=(-1)^k$ when n is a product of $k\geq 1$ distinct primes, $\boldsymbol{\mu}(1)=1$, and $\boldsymbol{\mu}(n)=0$ if $n\in\mathbb{N}$ is not square-free.)

With each set $\mathcal{F}_{\mathscr{B}}$ of $\mathscr{B}$-free numbers, we associate three subshifts $X_\eta\subset\widetilde{X}_\eta\subset X_{\mathscr{B}}$ (by a *subshift*, we mean a dynamical system (X,S), where $X\subset\{0,1\}^{\mathbb{Z}}$ is closed, S-invariant and S stands for the left shift):

- *$\mathscr{B}$-free subshift* (X_η,S), where X_η is the closure of the orbit $\mathcal{O}_S(\eta):=\{S^m\eta:m\in\mathbb{Z}\}$ of $\eta=\eta_{\mathscr{B}}=\mathbb{1}_{\mathcal{F}_{\mathscr{B}}}\in\{0,1\}^{\mathbb{Z}}$;
- the subshift $(\widetilde{X}_\eta,S)$, where $\widetilde{X}_\eta$ is defined to be the smallest hereditary subshift containing X_η (a subshift (X,S) is *hereditary* whenever $x\in X$ and $y\leq x$ coordinatewise, then $y\in X$);
- *$\mathscr{B}$-admissible subshift* $(X_{\mathscr{B}},S)$, where $X_{\mathscr{B}}$ is the set of *$\mathscr{B}$-admissible* sequences, i.e., of $x\in\{0,1\}^{\mathbb{Z}}$, such that, for each $b\in\mathscr{B}$, the support $\operatorname{supp}x:=\{n\in\mathbb{Z}:x(n)=1\}$ of x taken modulo b is a proper subset of $\mathbb{Z}/b\mathbb{Z}$.

Similarly to the notion of admissible sequences, we can define admissible blocks and admissible subsets of integers. Notice that the set $X_{\mathscr{B}}$ is closed as the $\mathscr{B}$-admissibility of x is equivalent to the $\mathscr{B}$-admissibility of all finite subsets of $\operatorname{supp} x$. Clearly, η is $\mathscr{B}$-admissible. Moreover, $(X_{\mathscr{B}}, S)$ is hereditary. As above, if no confusion arises, we will write η instead of $\eta_{\mathscr{B}}$. We adopt this convention also for $\mathscr{B}'$ by writing η' instead of $\eta_{\mathscr{B}'}$.

Relations with number theory. Consider two more examples. Let

$$\mathscr{B} := \{pq : p, q \in \mathscr{P}\} \quad \text{and} \quad \mathscr{B}' := \mathscr{P}. \tag{1}$$

Then $\mathcal{F}_{\mathscr{B}} = \mathscr{P} \cup (-\mathscr{P}) \cup \{-1, 1\}$ (cf. [42], p. 173) and $\mathcal{F}_{\mathscr{B}'} = \{-1, 1\}$. Let $\eta := \mathbb{1}_{\mathcal{F}_{\mathscr{B}}}$, $\eta' := \mathbb{1}_{\mathcal{F}_{\mathscr{B}'}}$. Clearly, $X_{\eta'} \subsetneq \widetilde{X}_{\eta'} \subsetneq X_{\mathscr{P}}$, and it is easy to see that $X_{\mathscr{P}}$ is uncountable. Recall the following famous number-theoretical conjectures:

Prime k-Tuples Conjecture. *For each $k \geq 1$ and each $\mathscr{P}$-admissible subset $\{a_1, \dots, a_k\} \subset \mathbb{N} \cup \{0\}$, there exist infinitely many $n \in \mathbb{N}$ such that $\{a_1 + n, \dots, a_k + n\} \subset \mathscr{P}$.*

Note that the set $\{0, 2\}$ is $\mathscr{P}$-admissible and the Prime k-Tuples Conjecture in this case is the Twin Prime Conjecture. Note also that if, for some $p \in \mathscr{P}$, we have $\{a_i \bmod p : 1 \leq i \leq k\} = \mathbb{Z}/p\mathbb{Z}$ and $\{a_1 + n, \dots, a_k + n\} \subset \mathscr{P}$, then $n = p - a_i$ for some $1 \leq i \leq k$, whence the set of $n \in \mathbb{N}$ such that $\{a_1 + n, \dots, a_k + n\} \subset \mathscr{P}$ is finite.

Remark 1.1. It is not hard to see that the Prime k-Tuples Conjecture is equivalent to $X_{\mathscr{P}} \subset \widetilde{X}_\eta$. Indeed, for the necessity, we need to show that if a block $B \in \{0,1\}^s$ is $\mathscr{P}$-admissible, then there is a block $B' \in \{0,1\}^s$ appearing in η such that $B \leq B'$. The existence of such a B' follows directly from the Prime k-Tuples Conjecture. Conversely, let $F = \{a_1, \dots, a_k\}$ be $\mathscr{P}$-admissible. Take $i_0 \geq 1$ large enough, so that $2|F| < p_{i_0+1}$, where p_i stands for the i-th prime number. Then the sets $F \cup (F + kp_1 \dots p_{i_0})$, $k \geq 1$, are also $\mathscr{P}$-admissible. These sets, for each $k \geq 1$, correspond to some blocks C_k appearing in $X_{\mathscr{P}}$. By assumption, this implies the existence of C_k' in η with $C_k \leq C_k'$, $k \geq 1$. It follows that we have $n, m \in \mathbb{Z}$ such that $F + n, F + m \subset \mathscr{P}$ with $|n - m|$ arbitrarily large, and the Prime k-Tuples Conjecture follows.

Dickson's conjecture [15]. *Let $a_i \in \mathbb{Z}$, $b_i \in \mathbb{N}$ for $1 \leq i \leq k$. If for each $p \in \mathscr{P}$ there exists $n \in \mathbb{N}$ such that $p \nmid \prod_{1 \leq i \leq k}(b_i n + a_i)$, then there are infinitely many $n \in \mathbb{N}$ such that $b_i n + a_i \in \mathscr{P}$ for $1 \leq i \leq k$.*

Note that if $b_i = 1$ for $1 \leq i \leq k$, the condition that for each $p \in \mathscr{P}$ there exists $n \in \mathbb{N}$ such that $p \nmid \prod_{1 \leq i \leq k}(b_i n + a_i)$ is equivalent to the $\mathscr{P}$-admissibility of $\{a_1, \dots, a_k\}$.

Remark 1.2. The following consequence of Dickson's conjecture (more specifically, of its special case when $b_i = 1$ for $1 \leq i \leq k$) was pointed out to us by Professor A. Schinzel; see C_{13} in [50]:

> If $\{a_1, \dots, a_k\} \in [-n, n] \cap \mathbb{Z}$ is $\mathscr{P}$-admissible, then, for infinitely many $x \in \mathbb{N}$, we have $[x - n, x + n] \cap \mathscr{P} = \{x + a_i : i = 1, \dots, k\}$.

This can be rephrased as $X_{\mathscr{P}} \subset X_\eta$.

Dynamical approach. The above suggests that the sets of multiples and the associated subshifts are difficult to study in full generality. Sarnak, in his seminal paper [49], suggested studying dynamical properties of the *square-free subshift* $(X_{\boldsymbol{\mu}^2}, S)$. In [49], he also announced the following results:

(i) $\boldsymbol{\mu}^2$ is *generic* for an ergodic S-invariant measure $\nu_{\boldsymbol{\mu}^2}$ on $\{0,1\}^{\mathbb{Z}}$ such that the measure-theoretical dynamical system $(X_{\boldsymbol{\mu}^2}, \nu_{\boldsymbol{\mu}^2}, S)$ has zero Kolmogorov entropy,
(ii) the topological entropy of $(X_{\boldsymbol{\mu}^2}, S)$ is equal to $6/\pi^2$,
(iii) $X_{\boldsymbol{\mu}^2} = X_{\mathscr{S}}$,
(iv) $(X_{\boldsymbol{\mu}^2}, S)$ is *proximal*,
(v) $(X_{\boldsymbol{\mu}^2}, S)$ has a non-trivial *topological joining* with a rotation on a compact Abelian group

(we will explain the notions appearing in (i)-(v) later). Today, proofs of these facts are available; (i)-(v) have also been studied for some natural generalizations of $(X_{\boldsymbol{\mu}^2}, S)$, see [1, 4, 8, 9, 30, 46, 48] (cf. also [5, 33, 41] for the harmonic analysis viewpoint). In particular, in [1], Abdalaoui, Lemańczyk, and de la Rue cover the counterparts of (i)-(iii) from Sarnak's list for each $\mathscr{B}$ which is Erdös. In this case, by (iii), we have $X_\eta = \widetilde{X}_\eta = X_{\mathscr{B}}$. During the conference Ergodic Theory and Dynamical Systems in Toruń, Poland, 2014, M. Boshernitzan asked if one can relax the assumption on $\mathscr{B}$ being Erdös and tackle similar problems to (i)-(v) for a general $\mathscr{B}$. Note that $X_{\mathscr{B}'} \subset X_{\mathscr{B}}$ whenever $\mathscr{B} \subset \mathscr{B}' \subset \mathbb{N}$. In other words, any $(X_{\mathscr{B}}, S)$ has subsystems of the form $(X_{\mathscr{B}'}, S)$ for certain sets $\mathscr{B}' \subset \mathbb{N}$ whose elements are no longer pairwise coprime. (Another way to obtain a natural subsystem of $(X_{\mathscr{B}}, S)$ is to choose $b' \mid b$ for each $b \in \mathscr{B}$ and then note that $X_{\mathscr{B}'} \subset X_{\mathscr{B}}$, where $\mathscr{B}' = \{b' : b \in \mathscr{B}\}$.) In particular, the square-free subshift contains $X_{\mathscr{B}}$ whenever $\mathscr{S} \subset \mathscr{B} \subset \{pq : p, q \in \mathscr{P}\}$.

Recall also that in [36] a description of all invariant measures for $(X_{\mathscr{B}}, S)$ was given for $\mathscr{B}$ Erdös. Moreover, under the same assumptions, $(X_{\mathscr{B}}, S)$ was proved to be *intrinsically ergodic*; that is, the system has only one invariant measure ν such that the Kolmogorov entropy of $(X_{\mathscr{B}}, \nu, S)$ is equal to the topological entropy of $(X_{\mathscr{B}}, S)$ (the intrinsic ergodicity of $(X_{\boldsymbol{\mu}^2}, S)$ was proved in [46]).

The present paper seems to be the first attempt to consider Sarnak's list (i)-(v) and the problem of invariant measures for a general $\mathscr{B} \subset \mathbb{N}$. Sometimes, we put certain restrictions on $\mathscr{B}$. In particular, we deal with $\mathscr{B}$ that:

- are *thin*, i.e., $\sum_{b\in\mathscr{B}} 1/b < \infty$,
- have *light tails*, i.e., $\overline{d}(\sum_{b>K} b\mathbb{Z}) \to 0$ when $K \to \infty$.

Each thin $\mathscr{B}$ has light tails, and if $\mathscr{B}$ is pairwise coprime, these two notions coincide. Moreover, light tail sets are Besicovitch. A more subtle notion, which turns out to be crucial in our studies, is that of *tautness* [28]:

- $\mathscr{B}$ is *taut* when $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}\setminus\{b\}}) < \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}})$ for each $b \in \mathscr{B}$, where $\boldsymbol{\delta}$ stands for the logarithmic density.

Any *primitive* set $\mathscr{B}$ (i.e., such that, for $b, b' \in \mathscr{B}$, we have $b \nmid b'$) with light tails is taut.

The main difference between the general situation and the Erdös case is that X_η no longer has a characterization in terms of admissible sequences; i.e., it may happen that the $\mathscr{B}$-admissible subshift $(X_{\mathscr{B}}, S)$ is strictly larger than the $\mathscr{B}$-free subshift (X_η, S). What is more, while $X_{\mathscr{B}}$ is always hereditary, X_η need not be so, and as we have already seen by inspecting the case $\mathscr{B} = \mathscr{P}$, we may even have $X_\eta \subsetneq \widetilde{X}_\eta \subsetneq X_{\mathscr{B}}$. On the other hand, there are many similarities or analogies between the Erdös case and the general case.

1.2. **Main results.** Our main results can be divided into three groups: structural results, results on invariant measures and entropy, and number theoretical results.

1.2.1. *Structural results.* This group of results contains both topological and measure-theoretical results. Namely, we have:

Theorem A. *For any $\mathscr{B} \subset \mathbb{N}$, the subshift (X_η, S) has a unique minimal subset. Moreover, this subset is the orbit closure of a Toeplitz sequence.*

As a consequence of Theorem A, we obtain the following results:

Corollary 1.3. *For any $\mathscr{B} \subset \mathbb{N}$, each point $x \in X_\eta$ is proximal to a point in the orbit closure of a Toeplitz sequence.*

Corollary 1.4. *Let $\mathscr{B} \subset \mathbb{N}$. Then (X_η, S) is minimal if and only if (X_η, S) is a Toeplitz system.*

We also give a simple characterization of those $\mathscr{B} \subset \mathbb{N}$ for which the unique minimal subset of (X_η, S) is a singleton:

Theorem B. *Let $\mathscr{B} \subset \mathbb{N}$. The following conditions are equivalent:*

(i) $\{\dots 0.00 \dots\}$ *is the unique minimal subset of* (X_η, S),
(ii) (X_η, S) *is proximal,*
(iii) $\mathscr{B}$ *contains an infinite pairwise coprime subset.*

Given a topological dynamical system (X, T), by $\mathcal{P}(X, T)$ we denote the set of all probability Borel T-invariant measures on X. It turns out that measure-theoretic properties of the subshift $(\widetilde{X}_\eta, S)$ strongly depend on the notion of tautness. We have:

Theorem C. *For any $\mathscr{B} \subset \mathbb{N}$, there exists a unique taut set $\mathscr{B}' \subset \mathbb{N}$ such that $\mathcal{F}_{\mathscr{B}'} \subset \mathcal{F}_{\mathscr{B}}$, $\widetilde{X}_{\eta'} \subset \widetilde{X}_\eta$, and $\mathcal{P}(\widetilde{X}_\eta, S) = \mathcal{P}(\widetilde{X}_{\eta'}, S)$.*

Corollary 1.5. *For any $\mathscr{B} \subset \mathbb{N}$, there exists a unique taut set $\mathscr{B}' \subset \mathbb{N}$ such that $\mathcal{F}_{\mathscr{B}'} \subset \mathcal{F}_{\mathscr{B}}$ and any point $x \in \widetilde{X}_\eta$ is attracted to $\widetilde{X}_{\eta'}$ along a sequence of integers of density 1; i.e., there exists $E_x \subset \mathbb{N}$ of zero density such that*

$$\lim_{n\to\infty, n\notin E_x} d(S^n x, \widetilde{X}_{\eta'}) = 0.$$

A key ingredient in the proof of Theorem C is the description of all invariant measures on $\widetilde{X}_\eta$. Indeed, it follows from Theorem I below that in order to prove Theorem C, it suffices to construct a taut set $\mathscr{B}'$ such that $\nu_{\eta'} = \nu_\eta$.

If $\mathscr{B}$ is Erdös, then, as shown in [1], we have $X_\eta = \widetilde{X}_\eta = X_{\mathscr{B}}$. In general, this need not be the case.

Theorem D. *Let $\mathscr{B} \subset \mathbb{N}$ have light tails and contain an infinite, pairwise coprime subset. Then $X_\eta = \widetilde{X}_\eta$.*

In other words, for primitive $\mathscr{B}$ with light tails, (X_η, S) is proximal if and only if it is hereditary. Since every $\mathscr{B}$ that is primitive and has light tails is taut, a natural question arises whether the assertion of Theorem D remains true for all taut $\mathscr{B} \subset \mathbb{N}$. We conjecture that the answer is positive.

1.2.2. *Results on invariant measures and entropy.*

Proposition E. *For any $\mathscr{B}\subset\mathbb{N}$, $\eta=\mathbb{1}_{\mathcal{F}_\mathscr{B}}$ is a quasi-generic point for a natural ergodic S-invariant measure ν_η on $\{0,1\}^{\mathbb{Z}}$. In particular, $\nu_\eta(X_\eta)=1$. Moreover, $\mathscr{B}$ is Besicovitch if and only if η is generic for ν_η.*

Remark 1.6. Recall that η is quasi-generic for ν_η if, for some (N_k), we have the weak convergence $\frac{1}{N_k}\sum_{n\leq N_k}\delta_{S^n\eta}\to\nu_\eta$. Recall also that in the Erdös case, this convergence holds along (N_k) with $N_k=k$ (see [1]), (i.e., η is *generic* in this case); hence $\mathscr{B}$ is Besicovitch.

We call ν_η the Mirsky measure (in the square-free case the frequencies of blocks in η were first studied by Mirsky [43, 44]).

Theorem F. *Suppose that $\mathscr{B}\subset\mathbb{N}$ is taut. Then (X_η,ν_η,S) is isomorphic to $(G,\mathbb{P},T)$, where G is the closure of $\{(n \bmod b_k)_{k\geq 1}\in\prod_{k\geq1}\mathbb{Z}/b_k\mathbb{Z}:n\in\mathbb{Z}\}$ in $\prod_{k\geq1}\mathbb{Z}/b_k\mathbb{Z}$ and $Tg=g+(1,1,\dots)$. In particular, (X_η,ν_η,S) has zero entropy.*

Theorem G. *If $\mathscr{B}\subset\mathbb{N}$ has light tails, then X_η is the topological support of ν_η.*

Theorem H. *Let $Y:=\{x\in\{0,1\}^{\mathbb{Z}}:|supp\ y \bmod b|=b-1$ for each $b\in\mathscr{B}\}$. For $\mathscr{B}\subset\mathbb{N}$ infinite (and primitive), the following conditions are equivalent:*

(a) *$\mathscr{B}$ is taut,*
(b) *$\mathcal{P}(Y\cap\widetilde{X}_\eta,S)\neq\emptyset$,*
(c) *$\nu_\eta(Y\cap X_\eta)=1$.*

Theorem I. *For any $\mathscr{B}\subset\mathbb{N}$ and any $\nu\in\mathcal{P}(\widetilde{X}_\eta,S)$, there exists*

$$\rho\in\mathcal{P}(X_\eta\times\{0,1\}^{\mathbb{Z}},S\times S)$$

whose projection onto the first coordinate equals ν_η and such that $M_*(\rho)=\nu$, where $M\colon X_\eta\times\{0,1\}^{\mathbb{Z}}\to\widetilde{X}_\eta$ stands for the coordinatewise multiplication.

Theorem J. *For any $\mathscr{B}\subset\mathbb{N}$, the subshift $(\widetilde{X}_\eta,S)$ is intrinsically ergodic.*

An important tool here, which can also be of independent interest, is the following result:

Proposition K. *For any $\mathscr{B}\subset\mathbb{N}$, we have $h_{top}(\widetilde{X}_\eta,S)=h_{top}(X_\mathscr{B},S)=\boldsymbol{\delta}(\mathcal{F}_\mathscr{B})$.*

The last entropy result we would like to highlight here is the following immediate consequence of Theorem C and the variational principle:

Corollary 1.7. *For any $\mathscr{B}\subset\mathbb{N}$, there exists a taut set $\mathscr{B}'\subset\mathbb{N}$ such that $\mathcal{F}_{\mathscr{B}'}\subset\mathcal{F}_\mathscr{B}$ and $h_{top}(\widetilde{X}_\eta,S)=h_{top}(\widetilde{X}_{\eta'},S)$.*

1.2.3. *Number theoretical results.*

General consequences: Our first result in this section shows, in particular, that a taut set $\mathscr{B}$ is determined by the family of $\mathscr{B}$-admissible subsets.

Theorem L. *Suppose that $\mathscr{B},\mathscr{B}'\subset\mathbb{N}$ are taut. Then the following conditions are equivalent:*

(a) *$\mathscr{B}=\mathscr{B}'$,*
(b) *$\mathcal{M}_\mathscr{B}=\mathcal{M}_{\mathscr{B}'}$,*
(c) *$X_\mathscr{B}=X_{\mathscr{B}'}$,*
(d) *$\widetilde{X}_\eta=\widetilde{X}_{\eta'}$,*

(e) $X_\eta = X_{\eta'}$,
(f) $\nu_\eta = \nu_{\eta'}$,
(g) $\mathcal{P}(\widetilde{X}_\eta, S) = \mathcal{P}(\widetilde{X}_{\eta'}, S)$.

Remark 1.8. Theorem L extends an analogous result from [36], where it was shown that $X_{\mathscr{B}} = X_{\mathscr{B}'}$ is equivalent to $\mathscr{B} = \mathscr{B}'$ for $\mathscr{B}, \mathscr{B}' \subset \mathbb{N}$ Erdös.

As an immediate consequence of Proposition E and Theorem G, we obtain:

Corollary 1.9. *If $\mathscr{B} \subset \mathbb{N}$ has light tails and $F, M \subset \mathbb{N}$ are finite sets such that $F \subset \mathcal{F}_{\mathscr{B}}$, $M \subset \mathcal{M}_{\mathscr{B}}$, then the density of the set of $n \in \mathbb{N}$ such that $F + n \subset \mathcal{F}_{\mathscr{B}}$, $M + n \subset \mathcal{M}_{\mathscr{B}}$ is positive.*

Consecutive gaps between $\mathscr{B}$-free numbers: Fix $\mathscr{B} \subset \mathbb{N}$ and denote by $(n_j)_{j\geq 1}$ the sequence of consecutive natural $\mathscr{B}$-free numbers. In [3], the following was shown when $\mathscr{B} \subset \mathbb{N}$ is Erdös:

(2) Let $\delta, \sigma > 0$ be such that $20\sigma > 9 + 3606\delta$. Then, for N large enough there exists $j = j(N) \geq 1$ such that $n_j \in [N, N + N^\sigma]$ and $\min(n_{j+1} - n_j, n_j - n_{j-1}) > \Phi(N)$, where $\Phi(N)$ is the largest positive integer such that $\prod_{j=1}^{3\Phi(N)} b_j \leq N^\delta$.

In particular,

$$\limsup_{j\to\infty} \min(n_{j+2} - n_{j+1}, n_{j+1} - n_j) = \infty. \tag{3}$$

Proposition M. *Suppose that $\mathscr{B} \subset \mathbb{N}$ has light tails and contains an infinite coprime subset. Denote by (n_j) the sequence of consecutive $\mathscr{B}$-free numbers. Then*

$$\limsup_{j\to\infty} \min_{0\leq k\leq K} (n_{j+k+1} - n_{j+k}) = \infty \quad \textit{for any } K \geq 1.$$

Proof. It follows from Theorem D that $X_\eta = \widetilde{X}_\eta$. Moreover, by Theorem G, X_η is the topological support of ν_η. Since, by Proposition E η is quasi-generic for ν_η, the result follows. $\square$

Even though, contrary to (2), the result included in Proposition M is not quantitative, it seems new and it strengthens (3).

Consequences for abundant numbers: Denote by $\mathbf{A} \subset \mathbb{N}$ the set of *abundant numbers*, i.e., the set of $n \in \mathbb{N}$ for which $\sum_{d|n} d \geq 2n$. In Section 11, we will show that $\mathbf{A}$ is the set of multiples of a primitive set $\mathscr{B}_{\mathbf{A}} \subset \mathbb{N}$ which is thin and contains an infinite coprime set. Denote $\overline{\eta} = \eta_{\mathscr{B}_{\mathbf{A}}}$.

Corollary 1.10. *The subshift $(X_{\overline{\eta}}, S)$ is hereditary and proximal. Moreover, $(X_{\overline{\eta}}, S)$ is intrinsically ergodic, and we have $h_{top}(X_{\overline{\eta}}, S) = 1 - d(\mathbf{A})$.*

Proof. It follows from Theorem D that $X_{\overline{\eta}} = \widetilde{X}_{\overline{\eta}}$. In particular, by Theorem B, $(X_{\overline{\eta}}, S)$ is proximal. The intrinsic ergodicity of $(X_{\overline{\eta}}, S)$ follows from its heredity and Theorem J. Finally, the intrinsic ergodicity of $(X_{\overline{\eta}}, S)$ and Proposition K yield $h_{top}(X_{\overline{\eta}}, S) = 1 - d(\mathbf{A})$. $\square$

Moreover, in Section 11, an analog of Corollary 1.9 for $\mathbf{A}$ will be obtained.

1.3. ‘Map’ of the paper.

Result	Proof	Main tools
Theorem A	Section 3.1	Corollary 2.17, Lemma 2.18
Corollary 1.3	Section 1.2.1	Theorem A, Proposition 2.10
Corollary 1.4	Section 1.2.1	Theorem A, Corollary 1.3
Theorem B	Section 3.2.2	Chinese Remainder Theorem
Theorem C	Section 9.2	Proposition 2.28 and Proposition 2.30, Theorem 4.5, Theorem I, and Theorem L
Corollary 1.5	Section 9.2	Theorem C, Lemma 9.3
Theorem D	Remark 5.4	Proposition 5.10
Proposition E	Section 4.1	Theorem 2.23
Theorem F	Section 8.3	Lemma 8.7, Theorem 8.14
Theorem G	Section 5	Proposition 5.10, Proposition 5.11
Theorem H	Section 7	Theorem C, Proposition E
Theorem I	Section 8.2	Theorem 8.2, Theorem 8.4
Theorem J	Section 10	Theorem 10.1, Theorem C, and the variational principle
Proposition K	Section 6.1	Lemma 6.1
Corollary 1.7	Section 1.2.2	Theorem C and the variational principle
Theorem L	Section 9.1	Theorem 4.29, Proposition 4.31 Theorem I, Proposition K
Corollary 1.9	Section 1.2.3	Proposition E, Theorem G
Proposition M	Section 1.2.3	Theorem D, Proposition E, Theorem G
Corollary 11.3	Section 11	Lemma 11.1 and Corollary 1.9
Corollary 11.4	Section 11	Corollary 11.3
Corollary 11.8	Section 11	Proposition M, Lemma 11.1, Lemma 11.6
Corollary 1.10	Section 1.2.3	Lemma 11.1, Lemma 11.6, Theorem B, Theorem D, Theorem J, Proposition K

2. Preliminaries

2.1. Topological dynamics: Basic notions. A *topological dynamical system* is a pair (X,T), where X is a compact space endowed with a metric d and T is a homeomorphism of X. A point $x\in X$ is called *recurrent* if, for any open set $U\ni x$, there exists $n\neq 0$ such that $T^nx\in U$. Denote by $\mathcal{O}_T(x)$ the *orbit* of $x\in X$ under T, i.e., $\mathcal{O}_T(x)=\{T^nx : n\in\mathbb{Z}\}$. We say that (X,T) is *transitive* if it has a dense orbit, and each point $x\in X$ whose orbit is dense in X is called *transitive.*

Remark 2.1. Recall that (X,T) is transitive if and only if, for any non-empty open sets $U,V\subset X$, there exists $n\in\mathbb{Z}$ such that $T^{-n}U\cap V\neq\emptyset$.

A dynamical system (X,T) is called *topologically weakly mixing* if the product system $(X\times X,T\times T)$ is transitive. A *minimal set* $M\subset X$ is a non-empty, closed, T-invariant set that is minimal with respect to these properties. Equivalently, $M\subset X$ is minimal if for any $x\in M$, we have $\overline{\mathcal{O}_T(x)}=M$. If $M=X$, then T is called *minimal.* A point $x\in X$ is called *minimal* if $(\overline{\mathcal{O}_T(x)},T)$ is minimal.

Let (X,T), (Z,R) be topological dynamical systems. Then (Z,R) is a *factor* of (X,T) if there is a surjective $\pi: X\to Z$ which is continuous and $\pi\circ T=R\circ\pi$.

A subset $C \subset X$ is called *wandering* whenever the sets T^nC, $n \in \mathbb{Z}$, are pairwise disjoint. By $\mathcal{P}(X,T)$ we denote the set of all Borel probability T-invariant measures on X and by $\mathcal{P}^e(X,T)$ the subset of $\mathcal{P}(X,T)$ of ergodic measures (cf. Definition 2.7). We say that (X,T) is *uniquely ergodic* if $\mathcal{P}(X,T)$ is a singleton. A point $x \in X$ is called *generic* for $\mu \in \mathcal{P}(X,T)$ if the ergodic theorem holds for T at x for any continuous function $f \in C(X)$: $\frac{1}{N}\sum_{n\leq N} f(T^nx) \to \int f\, d\mu$.

Remark 2.2. In any uniquely ergodic systems all points are generic for the unique invariant measure and the above convergence is uniform.

A topological dynamical system (X,T) is called *equicontinuous* (see, e.g., [25]) if the family of maps $\{T^n : n \in \mathbb{Z}\}$ is equicontinuous. Every topological dynamical system has, up to isomorphism, the largest equicontinuous factor which is called the *maximal equicontinuous factor*.

Example 2.3. Consider (G,T), where G is a compact Abelian group and $Tg = g + g_0$ for some $g_0 \in G$. If (G,T) is minimal, then it is uniquely ergodic and Haar measure $\mathbb{P}$ is the unique member of $\mathcal{P}(G,T)$. In particular, all points $g \in G$ are generic for $\mathbb{P}$. All compact Abelian group rotations (G,T) are equicontinuous.

Example 2.4. Let A be a finite set and let $S\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be the left shift, i.e., $S((x_n)_{n\in\mathbb{Z}}) = (y_n)_{n\in\mathbb{Z}}$, where $y_n = x_{n+1}$ for each $n \in \mathbb{Z}$. Let $X \subset A^{\mathbb{Z}}$ be closed and S-invariant. Then we say that (X,S) is a *subshift*.

Definition 2.5. We say that $x \in \{0,1\}^{\mathbb{Z}}$ is a *Toeplitz sequence* whenever for any $n \in \mathbb{Z}$ there exists $d_n \in \mathbb{N}$ such that $x(n + k\cdot d_n) = x(n)$ for any $k \in \mathbb{Z}$. A subshift (Z,S), $Z \subset \{0,1\}^{\mathbb{Z}}$ is said to be *Toeplitz* if $Z = \overline{\mathcal{O}_S(y)}$ for some Toeplitz sequence $y \in \{0,1\}^{\mathbb{Z}}$.

Remark 2.6. Usually, one requires from a Toeplitz sequence that it not be periodic. For convenience, periodic sequences are included in Definition 2.5. We refer the reader, e.g., to [16] for more information on Toeplitz sequences.

2.2. **Measure-theoretic dynamics: Basic notions.** A *measure-theoretic dynamical system* is a 4-tuple $(X,\mathcal{B},\mu,T)$, where $(X,\mathcal{B},\mu)$ is a standard Borel probability space and T is an automorphism of $(X,\mathcal{B},\mu)$. The set of all automorphisms of $(X,\mathcal{B},\mu)$ will be denoted by $\mathrm{Aut}(X,\mathcal{B},\mu)$.

Definition 2.7. We say that $T \in \mathrm{Aut}(X,\mathcal{B},\mu)$ is *ergodic* if, for $A \in \mathcal{B}$, $A = T^{-1}A$ (μ-a.e.) implies $\mu(A) \in \{0,1\}$.

For $T\in\mathrm{Aut}(X,\mathcal{B},\mu)$, we define the associated *Koopman operator* $U_T\colon L^2(X,\mathcal{B},\mu) \to L^2(X,\mathcal{B},\mu)$ by setting $U_Tf = f\circ T$. We say that $\lambda \in \mathbb{S}^1$ is in the *discrete spectrum* of T if it is an eigenvalue of U_T; i.e., for some (eigenfunction) $0 \neq f \in L^2(X,\mathcal{B},\mu)$, we have $U_Tf = \lambda f$. We say that T has *purely discrete spectrum* if the eigenfunctions of U_T are linearly dense in $L^2(X,\mathcal{B},\mu)$. Following [26], we say that $T \in Aut(X,\mathcal{B},\mu)$ is *coalescent* if each endomorphism of $(X,\mathcal{B},\mu)$ commuting with T is invertible.

Remark 2.8. All ergodic automorphisms with purely discrete spectrum are coalescent.

Let $T \in \mathrm{Aut}(X,\mathcal{B},\mu)$, $S \in \mathrm{Aut}(Y,\mathcal{C},\nu)$, and let ρ be a $T\times S$-invariant measure on $X \times Y$. We say that ρ is a *joining* of T and S if $\rho|_X = \mu$ and $\rho|_Y = \nu$. In a similar way, joinings of at most countable families are defined. Following [24], T and S are called *disjoint* if product measure is the only joining of T and S.

Let $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ and let $C \in \mathcal{B}$ be such that $\mu(C) > 0$. Then the function n_C given by $n_C(x) = \min\{n \geq 1 : T^n x \in C\}$ is well-defined for μ-a.e. $x \in C$. The map $T_C\colon C \to C$ given by $T_C x = T^{n_C(x)}x$ is called the *induced transformation.* Then $T_C \in \mathrm{Aut}(C, \mathcal{B}_C, \mu_C)$, where $\mathcal{B}_C = \mathcal{B}|_C$ and $\mu_C(A) = \frac{\mu(A)}{\mu(C)}$ for any $A \in \mathcal{B}_C$. Moreover, T_C is ergodic whenever T is also.

2.3. **Entropy: Basic notions.** There are two basic notions of entropy: *topological entropy* $h_{top}(X, T)$ of (X, T) and *measure-theoretic entropy* (or Kolmogorov entropy) $h(X, \mu, T)$ of $(X, \mathcal{B}, \mu, T)$. We skip the definitions and refer the reader, e.g., to [17].

For any topological dynamical system (X, T) the variational principle holds, that is, $h_{top}(X, T) = \sup_{\mu \in \mathcal{P}(X,T)} h(X, \mu, T)$. We say that $\mu \in \mathcal{P}(X, T)$ is a *measure of maximal entropy* of (X, T) if $h_{top}(X, T) = h(X, \mu, T)$. A measure of maximal entropy may not exist; however, subshifts always have at least one measure of maximal entropy. Following [52], a topological system (X, T) is *intrinsically ergodic* if it has exactly one measure of maximal entropy.

2.4. **Topological dynamics: More on minimal subsets.** Let (X, T) be a topological dynamical system. A set $S \subset \mathbb{Z}$ is called *syndetic* if there exists a finite set K such that $K + S = \mathbb{Z}$.

We will be particularly interested in the situation when (X, T) has a unique minimal subset. We first recall well-known results related to the proximal case.

A pair $(x, y) \in X \times X$ is called *proximal* if $\liminf_{n\to\infty} d(T^n x, T^n y) = 0$. We denote the set of proximal pairs (x, y) by $\mathrm{Prox}(T)$. We say that T is *proximal* if $\mathrm{Prox}(T) = X \times X$. A pair $(x, y) \in X \times X$ is called *syndetically proximal* if $\{n \in \mathbb{Z} : d(T^n x, T^n y) < \varepsilon\}$ is syndetic for any $\varepsilon > 0$. We denote the set of all syndetically proximal pairs (x, y) by $\mathrm{SyProx}(T)$. We say that T is *syndetically proximal* if $\mathrm{SyProx}(T) = X \times X$.

Remark 2.9. Note that if $(x, Tx) \in \mathrm{Prox}(T)$, then clearly T has a fixed point. Moreover, (X, T) is proximal if and only if it has a fixed point that is the unique minimal subset of X.

Proposition 2.10 (Auslander - Ellis; see, e.g., [2])**.** *Let (X, T) be a topological dynamical system. Then for any $x \in X$ there exists a minimal point $y \in X$ such that x and y are proximal.*

Remark 2.11. Clearly, a subsystem of a (syndetically) proximal system remains (syndetically) proximal. Both relations, Prox and SyProx, are reflexive and symmetric. Moreover, SyProx is always an equivalence relation, whereas Prox need not be.

Remark 2.12. It is easy to see that if T is syndetically proximal, then $T^{\times n}$ is syndetically proximal for each $n \geq 1$.

Proposition 2.13 ([11, 54]; see also Theorem 19 in [45])**.** *The following are equivalent:*

- *$Prox(T)$ is an equivalence relation,*
- *$Prox(T) = SyProx(T)$,*
- *the orbit closure of any point $(x, y) \in X \times X$ in the dynamical system $(X \times X, T \times T)$ contains exactly one minimal subset.*

As an immediate consequence of Remark 2.11 and Remark 2.9, we obtain:

Corollary 2.14. *Suppose that $Tx_0 = x_0$ and SyProx(T) $\cap (\{x_0\} \times X) = \{x_0\} \times X$. Then Prox(T) $\supset$ SyProx(T) $= X \times X$; i.e., T is syndetically proximal and $\{x_0\}$ is the unique minimal subset of X.*

The following result is related to Lemma 1 in [18].

Proposition 2.15. *Let (X, T) be a topological dynamical system with a transitive point $\eta \in X$. The following are equivalent:*

(a) *(X, T) has a unique minimal subset M.*

(b) *There exists a closed, T-invariant subset $M' \subset X$ such that for any $x \in M'$ and $y \in X$, there exists $(m_n)_{n\geq 1} \subset \mathbb{Z}$ such that $T^{m_n}y \to x$.*

(c) *There exists $x_M \in X$ such that for any $y \in X$ there exists $(m_n)_{n\geq 1} \subset \mathbb{Z}$ such that $T^{m_n}y \to x_M$.*

(d) *There exists a closed, T-invariant subset $M'' \subset X$, such that $\{k \in \mathbb{Z} : T^k\eta \in U\}$ is syndetic for any open set U intersecting M''.*

(e) *There exists a sequence of open sets $(U_n)_{n\geq 1} \subset X$ such that:*
   - *diam$(U_n) \to 0$ as $n \to \infty$,*
   - *$\{k \in \mathbb{Z} : T^k\eta \in U_n\}$ is syndetic for each $n \in \mathbb{N}$.*

Furthermore, if any of the above hold, then $M = M' = M''$ and $x_M \in M$.

Proof. (a)$\Rightarrow$(b) Let $M' = M$, where M is as in (a). Let $y \in X$. The set $\omega(y) = \{z \in X : \exists m_n \to \infty \text{ with } T^{m_n}y \to z\}$ is closed, non-empty, and T-invariant, so it contains M', which yields (b) and shows that any M' as in (b) satisfies $M' = M$.

Clearly, (b) implies (c).

(c)$\Rightarrow$(d) Let x_M be as in (c). Denote by M'' the orbit closure of x_M. Let U be an open set and let $U \cap M'' \neq \emptyset$. Then there exists $N \in \mathbb{N}$ such that $x_M \in V := T^{-N}(U)$. Thus $N + \{k \in \mathbb{Z} : T^k\eta \in V\} \subset \{k \in \mathbb{Z} : T^k\eta \in U\}$, and the latter set is syndetic if the former is. If $\{k \in \mathbb{Z} : T^k\eta \in V\}$ is not syndetic, then for any $n \in \mathbb{N}$ there exists $k_n \in \mathbb{Z}$ for which $T^{k_n-n}\eta, \dots, T^{k_n+n}\eta \in X \setminus V$. Then the orbit closure of any limit point of $(T^{k_n}\eta)_{n\geq 1}$ lies in $X \setminus V$ and cannot contain x_M, which contradicts (c).

Clearly, (d) implies (e).

(e)$\Rightarrow$(*a*) Assume that (e) holds and $M_1, M_2 \subset X$ are minimal with $\min\{d(x_1, x_2) : x_i \in M_i, i = 1, 2\} =: \varepsilon > 0$. Let U_n be as in (e) with $\operatorname{diam} U_n < \varepsilon/2$. We may assume that $U_n \cap M_2 = \emptyset$. Let $V \supset M_2$ be open such that $V \cap U_n = \emptyset$. Let N be the maximal gap in $\{k \in \mathbb{Z} : T^k\eta \in U_n\}$. Since M_2 is T-invariant, there is an open set W with $M_2 \subset W$ such that $T^jW \subset V$ for $j = 0, 1, \dots, N$. But the orbit of η is dense in X; therefore there is $k \in \mathbb{Z}$ with $T^k\eta \in W$, contradicting the choice of N.

The same reasoning shows that M'' as in (d) must be unique and hence $M = M''$. To finish the proof note that we also must have $x_M \in M = M' = M''$ since x_M must be a minimal point. □

Remark 2.16. There is a well-known characterization of minimality of an orbit closure. Let $x \in X$. Then $(\overline{\mathcal{O}_T(x)}, T)$ is minimal if and only if, for any open set $U \ni x$, the set $\{n \in \mathbb{Z} : T^n x \in U\}$ is syndetic. In particular, if x is transitive, then (X, T) is minimal if and only if, for any open set $U \subset X$, the set $\{n \in \mathbb{Z} : T^n x \in U\}$ is syndetic.

Notice that Proposition 2.15 includes this characterization of minimal orbit closures as a special case. Indeed, if (X, T) is minimal, then any open set U intersects

$M = X$, whence $\{n \in \mathbb{Z} : T^n x \in U\}$ is syndetic by (d). On the other hand, if $\{n \in \mathbb{Z} : T^n x \in U\}$ is syndetic for any open set U, it follows that $M' := X$ satisfies (d). Therefore the only minimal subset M is also equal to X; i.e., (X, T) is minimal.

Corollary 2.17. *Let (X, T) be a transitive subshift. Then (X, T) has a unique minimal subset M if and only if there exists an infinite family of pairwise distinct blocks that appear in η with bounded gaps.*

Proof. It follows from the equivalence of (a) and (e) in Proposition 2.15. □

If (X, T) is a subshift, sometimes more can be said about the unique minimal subset.

Lemma 2.18. *Let $\eta \in \{0,1\}^{\mathbb{Z}}$. Suppose that there exist $B_n \in \{0,1\}^{[\ell_n, r_n]}$ for $n \geq 1$, with $\ell_n \searrow -\infty, r_n \nearrow \infty$, $(m_n)_{n\geq 1} \subset \mathbb{Z}$, and $(d_n)_{n\geq 1} \subset \mathbb{N}$, satisfying, for each $n \geq 1$:*

(a) $d_n \mid d_{n+1}$,
(b) $d_n \mid m_{n+1} - m_n$,
(c) $\eta[m_n + kd_n + \ell_n, m_n + kd_n + r_n] = B_n$ *for each* $k \in \mathbb{Z}$. (*Note that conditions* (a), (b), *and* (c) *imply that* $B_{n+1}[\ell_n, r_n] = B_n$, $n \geq 1$.)

Then η has a Toeplitz sequence x in its orbit closure X_η. Moreover, (X_η, S) has a unique minimal subset M which is the orbit closure of x.

Proof. Fix $n_0 \in \mathbb{N}$ and let $n \geq n_0$. Then, by (a) and (b), we have $d_{n_0} \mid m_n - m_{n_0}$. Therefore, in view of (c), for any $k \in \mathbb{Z}$, we have

$$S^{m_n}\eta[\ell_{n_0} + kd_{n_0}, r_{n_0} + kd_{n_0}]$$
$$= \eta[m_{n_0} + (kd_{n_0} + m_n - m_{n_0}) + \ell_{n_0}, m_{n_0} + (kd_{n_0} + m_n - m_{n_0}) + r_{n_0}] = B_{n_0}.$$

It follows that $x := \lim_{n\to\infty} S^{m_n}\eta$ is well-defined and Toeplitz. The last assertion follows by Corollary 2.17. □

2.5. **Asymptotic densities.** For $A \subset \mathbb{Z}$, we recall several notions of asymptotic density (in fact, these are densities of the positive part of the set A, i.e., of $A \cap \mathbb{N}$). We have:

- $\underline{d}(A) := \liminf_{N\to\infty} \frac{1}{N}|A \cap [1, N]|$ (*lower density* of A),
- $\overline{d}(A) := \limsup_{N\to\infty} \frac{1}{N}|A \cap [1, N]|$ (*upper density* of A).

If the lower and the upper densities of A coincide, their common value $d(A) := \underline{d}(A) = \overline{d}(A)$ is called the *density* of A. We also have:

- $\underline{\boldsymbol{\delta}}(A) := \liminf_{N\to\infty} \frac{1}{\log N} \sum_{a\in A, 1\leq a\leq N} \frac{1}{a}$ (*lower logarithmic density* of A),
- $\overline{\boldsymbol{\delta}}(A) := \limsup_{N\to\infty} \frac{1}{\log N} \sum_{a\in A, 1\leq a\leq N} \frac{1}{a}$(*upper logarithmic density* of A).

If the lower and the upper logarithmic densities of A coincide, we set $\boldsymbol{\delta}(A) := \underline{\boldsymbol{\delta}}(A) = \overline{\boldsymbol{\delta}}(A)$ and call it the *logarithmic density* of A. It is easy to see that

$$\underline{d}(A) \leq \underline{\boldsymbol{\delta}}(A) \leq \overline{\boldsymbol{\delta}}(A) \leq \overline{d}(A). \tag{4}$$

2.6. **Sets of multiples, $\mathscr{B}$-free numbers, and their density.** For $\mathscr{B} \subset \mathbb{N}$, let $\mathcal{M}_{\mathscr{B}} := \bigcup_{b\in\mathscr{B}} b\mathbb{Z}$ and $\mathcal{F}_{\mathscr{B}} := \mathbb{Z} \setminus \mathcal{M}_{\mathscr{B}}$.

Definition 2.19. We say that:

- $\mathscr{B}$ is *coprime* if $\gcd(b, b') = 1$ for $b \neq b'$ in $\mathscr{B}$,
- $\mathscr{B}$ is *thin* if $\sum_{b\in\mathscr{B}} 1/b < +\infty$,

- $\mathscr{B}$ has *light tails* if $\lim_{K\to\infty}\overline{d}\left(\bigcup_{b>K}b\mathbb{Z}\right)=0$,
- $\mathscr{B}$ is *taut* [28] if for any $b\in\mathscr{B}$, we have $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}})>\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}\setminus\{b\}})$.

Following [28], we say that $\mathscr{B}$ is *Besicovitch* if $d(\mathcal{M}_{\mathscr{B}})$ exists (clearly, this is equivalent to the existence of $d(\mathcal{F}_{\mathscr{B}})$). A set $\mathscr{B}\subset\mathbb{N}\setminus\{1\}$ is called *Behrend* if $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}})=1$.

Remark 2.20 (see Chapter 0 in [28]). Let $P(\mathscr{B})$ be the intersection of all sets $\mathscr{B}'\subset\mathbb{N}$ such that $\mathcal{M}_{\mathscr{B}}=\mathcal{M}_{\mathscr{B}'}$. Then $\mathcal{M}_{P(\mathscr{B})}=\mathcal{M}_{\mathscr{B}}$. Moreover, $P(\mathscr{B})$ is *primitive* (i.e., no element of $P(\mathscr{B})$ divides any other). Therefore, throughout the paper, whenever $\mathscr{B}$ is arbitrary, we will tacitly assume that it is primitive.

Since $\overline{d}\left(\bigcup_{b>K}b\mathbb{Z}\right)\leq\sum_{b>K}{}^{1}/{}_{b}$,

$$\text{if } \mathscr{B} \text{ is thin, then } \mathscr{B} \text{ has light tails.} \tag{5}$$

Recall that $d(\mathcal{M}_{\mathscr{B}})$ may not exist; the first counterexample was provided by Besicovitch [6]. Recall also the result by Erdös:

Theorem 2.21 ([21])**.** *A set $\mathscr{B}=\{b_k:k\geq 1\}$ is Besicovitch if and only if*

$$\lim_{0<\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\sum_{n^{1-\varepsilon}<b_k\leq n}|[0,n]\cap b_k\mathbb{Z}\cap\mathcal{F}_{\{b_1,\dots,b_{k-1}\}}|=0.$$

Corollary 2.22 (Theorem 1.6 in [28])**.** *If $\mathscr{A}$ and $\mathscr{B}$ are Besicovitch, then $\mathscr{A}\cup\mathscr{B}$ is also Besicovitch.*

On the other hand, we have the following result of Davenport and Erdös:

Theorem 2.23 ([13, 14])**.** *For any $\mathscr{B}$, the logarithmic density $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}})$ of $\mathcal{M}_{\mathscr{B}}$ exists. Moreover,*

$$\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}})=\underline{d}(\mathcal{M}_{\mathscr{B}})=\lim_{K\to\infty}d(\mathcal{M}_{\{b\in\mathscr{B}:b\leq K\}}). \tag{6}$$

Remark 2.24. Formula (6) follows from the proof of Theorem 2.23 in [14] (see also [28]). Notice that (6) implies that $\mathscr{B}$ is Besicovitch if and only if

$$\lim_{K\to\infty}\overline{d}(\mathcal{M}_{\{b\in\mathscr{B}:b>K\}}\setminus\mathcal{M}_{\{b\in\mathscr{B}:b\leq K\}})=0.$$

In particular, if $\mathscr{B}$ has light tails, then $\mathscr{B}$ is Besicovitch (this follows also from Theorem 2.21).

Corollary 2.25. *Let $\mathscr{A}=\mathscr{A}_1\cup\mathscr{A}_2\cup\dots$. Then*

$$\boldsymbol{\delta}(\mathcal{M}_{\mathscr{A}})=\underline{d}(\mathcal{M}_{\mathscr{A}})=\lim_{K\to\infty}\boldsymbol{\delta}(\mathcal{M}_{\mathscr{A}_1\cup\mathscr{A}_2\cup\ldots\cup\mathscr{A}_K}).$$

Proof. Let $\Delta(\mathscr{A}):=\lim_{K\to\infty}\boldsymbol{\delta}(\mathcal{M}_{\mathscr{A}_1\cup\ldots\cup\mathscr{A}_K})$. Clearly, $\Delta(\mathscr{A})\leq\boldsymbol{\delta}(\mathcal{M}_{\mathscr{A}})$. We will now show that $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{A}})\leq\Delta(\mathscr{A})$. For $K\geq 1$, let N_K be such that $\{a\in\mathscr{A}:a\leq K\}\subset\mathscr{A}_1\cup\ldots\cup\mathscr{A}_{N_K}$. Using Theorem 2.23, we obtain

$$\boldsymbol{\delta}(\mathcal{M}_{\mathscr{A}})=\lim_{K\to\infty}\boldsymbol{\delta}(\mathcal{M}_{\{a\in\mathscr{A}:a\leq K\}})\leq\lim_{K\to\infty}\boldsymbol{\delta}(\mathcal{M}_{\mathscr{A}_1\cup\ldots\cup\mathscr{A}_{N_K}})=\Delta(\mathscr{A}).$$

This completes the proof. □

Remark 2.26 (Cf. Remark 2.24). Let $\mathscr{A}=\mathscr{A}_1\cup\mathscr{A}_2\cup\dots$ and suppose additionally that the density of $\mathscr{A}_1\cup\ldots\cup\mathscr{A}_K$ exists, for each $K\geq 1$. As a consequence of Corollary 2.25, we obtain that $\mathscr{A}$ is Besicovitch if and only if

$$\lim_{K\to\infty}\overline{d}(\mathcal{M}_{\mathscr{A}}\setminus\mathcal{M}_{\mathscr{A}_1\cup\ldots\cup\mathscr{A}_K})=0.$$

Remark 2.27. Clearly, any superset of a Behrend set that does not contain 1 remains Behrend. Moreover, if $\mathscr{B}$ is Behrend, then $\mathscr{B}$ is Besicovitch. Note also that by Theorem 2.23, $\mathscr{B} \subset \mathbb{N} \setminus \{1\}$ is Behrend if and only if $d(\mathcal{M}_{\mathscr{B}}) = 1$.

Proposition 2.28 ([28], Corollary 0.14)**.** *The set $\mathscr{A} \cup \mathscr{B}$ is Behrend if and only if at least one of $\mathscr{A}$ and $\mathscr{B}$ is Behrend.*

For $\mathscr{B}$, $a \in \mathbb{N} \setminus \{1\}$ let

$$\mathscr{B}'(a) := \left\{ \frac{b}{\gcd(b,a)} : b \in \mathscr{B} \right\}.$$

Proposition 2.29 ([28], Theorem 0.8)**.** *Let $a \notin \mathcal{M}_{\mathscr{B}}$. Then $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}\cup\{a\}}) > \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}})$ if and only if $\mathscr{B}'(a)$ is not Behrend.*

Proposition 2.30 ([28], Corollary 0.19)**.** *The set $\mathscr{B}$ is taut if and only if it is primitive and does not contain $c\mathscr{A}$ with $c \in \mathbb{N}$ and $\mathscr{A} \subset \mathbb{N} \setminus \{1\}$ that is Behrend.*

Corollary 2.31. *Suppose that $\mathscr{B}$ is taut. If $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}\cup\{a\}}) = \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}})$, then $a \in \mathcal{M}_{\mathscr{B}}$.*

Proof. Suppose that $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}\cup\{a\}}) = \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}})$ and $a \notin \mathcal{M}_{\mathscr{B}}$. By Proposition 2.29, $\mathscr{B}'(a)$ is Behrend. Since a has finitely many divisors, it follows from Proposition 2.28 that at least one of the sets

$$\mathscr{B}'_d(a) := \left\{ \frac{b}{d} : b \in \mathscr{B} \quad \text{and} \quad \gcd(b,a) = d \right\},$$

where $d \mid a$ is Behrend. Moreover, $d \cdot \mathscr{B}'_d(a) \subset \mathscr{B}$. Notice that $1 \notin \mathscr{B}'_d(a)$. Indeed, if $1 \in \mathscr{B}'_d(a)$, then $d = \gcd(d,a) \in \mathscr{B}$. In particular, $d \mid a$, i.e., $a \in \mathcal{M}_{\mathscr{B}}$, which is not possible by the choice of a. It follows from Proposition 2.30 that $\mathscr{B}$ cannot be taut. $\square$

Furthermore, notice that

$$\text{if } \mathscr{B} \text{ has light tails (and is primitive), then } \mathscr{B} \text{ is taut.} \tag{7}$$

Indeed, if $\mathscr{B}$ is not taut, by Proposition 2.30, we have that $\mathscr{B} \supset c\mathscr{A}$ with $\mathscr{A}$ Behrend. Moreover, given $K \geq 1$, there exists $L = L(K,c)$ such that

$$c \cdot \{a \in \mathscr{A} : a > L\} \subset \bigcup_{b>K} b\mathbb{Z}.$$

But, in view of Proposition 2.28, $\{a \in \mathscr{A} : a > L\}$ is Behrend. It follows that $\boldsymbol{\delta}(\bigcup_{b>K} b\mathbb{Z}) \geq 1/c$ for all $K \geq 1$, which means that $\mathscr{B}$ cannot have light tails. In particular, we obtain that if $\mathscr{B}$ is finite, then $\mathscr{B}$ is taut.

2.7. **Canonical odometer associated with $\mathscr{B}$.** Denote the elements of $\mathscr{B}$ by b_k, $k \geq 1$, and consider the compact Abelian group $G_{\mathscr{B}} := \prod_{k\geq 1} \mathbb{Z}/b_k\mathbb{Z}$, with the coordinatewise addition. The product topology on $G_{\mathscr{B}}$ is metrizable with a metric d given by

$$d(g,g') = \sum_{k\geq 1} \frac{1}{2^k} \frac{|g_k - g'_k|}{1 + |g_k - g'_k|}. \tag{8}$$

In this metric, closeness of two sequences implies that they agree on a long initial segment of coordinates. Let $\mathbb{P}_{G_{\mathscr{B}}}$ be Haar measure of $G_{\mathscr{B}}$, i.e., $\mathbb{P}_{G_{\mathscr{B}}} = \bigotimes m_{\mathbb{Z}/b_k\mathbb{Z}}$, where, for $c \in \mathbb{N}$, $m_{\mathbb{Z}/c\mathbb{Z}}$ stands for the normalized counting measure on $\mathbb{Z}/c\mathbb{Z}$. For $n \in \mathbb{Z}$, let

$$\underline{n}_{\mathscr{B}} := (n \bmod b_1, n \bmod b_2, \ldots) \in G_{\mathscr{B}}. \tag{9}$$

Denote by G the smallest closed subgroup of $G_{\mathscr{B}}$ that contains $\underline{1}_{\mathscr{B}}$, i.e.,

$$G := \overline{\{\underline{n}_{\mathscr{B}} : n \in \mathbb{Z}\}} \subset G_{\mathscr{B}}. \tag{10}$$

Remark 2.32. By its definition, $G \subset G_{\mathscr{B}}$ contains a dense cyclic subgroup; i.e., G is monothetic and the homeomorphism $Tg = g + \underline{1}_{\mathscr{B}}$ yields a uniquely ergodic dynamical system (G, T) (with Haar measure $\mathbb{P}$ as the only invariant measure).

We will now provide another model of (G, T). First, given $1 \le k < \ell$, denote by

$$\pi_{k,\ell} \colon \mathbb{Z}/\operatorname{lcm}(b_1, \ldots, b_k, \ldots, b_\ell)\mathbb{Z} \to \mathbb{Z}/\operatorname{lcm}(b_1, \ldots, b_k)\mathbb{Z}$$

the natural homomorphism given, for each $r \in \mathbb{Z}/\operatorname{lcm}(b_1, \ldots, b_k, \ldots, b_\ell)\mathbb{Z}$, by

$$\pi_{k,\ell}(r) = r \bmod \operatorname{lcm}(b_1, \ldots, b_k).$$

Note that whenever $1 \le k < \ell < m$, we have $\pi_{k,\ell} \circ \pi_{\ell,m} = \pi_{k,m}$. Also, for each $k \ge 1$, we set $\pi_k := \pi_{k,k+1}$. This yields an inverse limit

$$\mathbb{Z}/b_1\mathbb{Z} \overset{\pi_1}{\leftarrow} \mathbb{Z}/\operatorname{lcm}(b_1, b_2)\mathbb{Z} \overset{\pi_2}{\leftarrow} \ldots \overset{\pi_{k-1}}{\leftarrow} \mathbb{Z}/\operatorname{lcm}(b_1, \ldots, b_k)\mathbb{Z} \overset{\pi_k}{\leftarrow} \ldots,$$

and we define

$$\begin{aligned} G' &:= \varprojlim \mathbb{Z}/\operatorname{lcm}(b_1, \ldots, b_k)\mathbb{Z} \\ &= \left\{ g \in \prod_{k \ge 1} \mathbb{Z}/\operatorname{lcm}(b_1, \ldots, b_k)\mathbb{Z} : \pi_k(g_{k+1}) = g_k \quad \text{for each } k \ge 1 \right\}, \end{aligned} \tag{11}$$

where $g = (g_1, g_2, \ldots)$. Then G' is closed and invariant under the coordinatewise addition. Hence, G' is Abelian, compact, and metrizable; cf. (8). We denote by $\mathbb{P}'$ Haar measure on G'. By the above, for each $n \ge 1$, we have

$$\underline{n} := (n \bmod b_1, n \bmod \operatorname{lcm}(b_1, b_2), \ldots) \in G', \tag{12}$$

in particular, $\underline{1} \in G'$. On G', we also define a homeomorphism $T'g = g + \underline{1}$.

Remark 2.33. Notice that if $(g_1, g_2, \ldots) \in G'$, then, since $g_k = g_j \bmod \operatorname{lcm}(b_1, \ldots, b_j)$ for $j = 1, \ldots, k$, we have

$$\underline{g_k} = (g_k \bmod b_1, g_k \bmod \operatorname{lcm}(b_1, b_2), \ldots) \to (g_1, g_2, \ldots) \quad \text{when } k \to \infty.$$

It follows that $\{\underline{n} : n \in \mathbb{Z}\}$ is dense in G' (and hence G' is monothetic).

Lemma 2.34. *The map $W \colon \{\underline{n}_{\mathscr{B}} : n \in \mathbb{Z}\} \to G'$ given by $W(\underline{n}_{\mathscr{B}}) = \underline{n}$ extends continuously to G in a unique way. Moreover, it yields a topological isomorphism of the dynamical systems (G, T) and (G', T').*

Proof. Notice first that W is uniformly continuous (and equivariant). Indeed, for any $K \ge 1$, if $d(\underline{n}_{\mathscr{B}}, \underline{m}_{\mathscr{B}})$ is sufficiently small, then $n = m \bmod b_k$ for $1 \le k \le K$. It follows that $n = m \bmod \operatorname{lcm}(b_1, \ldots, b_k)$ for $1 \le k \le K$; i.e., $d(\underline{n}, \underline{m})$ is small, provided that K is large. Therefore, W extends to a continuous map from G to G'. Moreover, by Remark 2.33, $W \colon G \to G'$ is surjective.

It remains to show that W is injective. For this, it suffices to show that the map $\underline{n} \mapsto \underline{n}_{\mathscr{B}}$ is also uniformly continuous. Fix $K \ge 1$. If $d(\underline{n}, \underline{m})$ is sufficiently small, then $n = m \bmod \operatorname{lcm}(b_1, \ldots, b_k)$ for $1 \le k \le K$. It follows that, for $1 \le k \le K$, we have $n = m \bmod b_k$; i.e., $d(\underline{n}_{\mathscr{B}}, \underline{m}_{\mathscr{B}})$ is arbitrarily small, provided that K is large. This completes the proof. □

Definition 2.35. We say that $(G, \mathbb{P}, T)$ is the *canonical odometer* associated to $\mathscr{B}$.

Remark 2.36. It follows from the proof of the above lemma that for $g \in G$, we have

$$W(g) = (g_1 \bmod b_1, g_2 \bmod b_2, \dots). \tag{13}$$

Example 2.37. When $\mathscr{B}$ is coprime, then $\mathbb{Z}/\operatorname{lcm}(b_1,\dots,b_k)\mathbb{Z} = \mathbb{Z}/(b_1\cdot\ldots\cdot b_k\mathbb{Z})$ is, by the Chinese Remainder Theorem, canonically isomorphic to $\mathbb{Z}/b_1\mathbb{Z}\times\ldots\times\mathbb{Z}/b_k\mathbb{Z}$ via

$$j \mapsto (j \bmod b_1, \dots, j \bmod b_k),$$

so π_k corresponds to

$$\mathrm{proj}_k \colon \mathbb{Z}/b_1\mathbb{Z}\times\ldots\times\mathbb{Z}/b_k\mathbb{Z}\times\mathbb{Z}/b_{k+1}\mathbb{Z} \to \mathbb{Z}/b_1\mathbb{Z}\times\ldots\times\mathbb{Z}/b_k\mathbb{Z},$$

i.e., the projection on the k first coordinates. The inverse limit G' given by the system $\{\mathrm{proj}_k : k \geq 1\}$ is naturally identified with the direct product $G_{\mathscr{B}}$. Moreover, $\underline{1} \in G'$ corresponds to $\underline{1}_{\mathscr{B}} \in G_{\mathscr{B}}$. It follows that $G = G_{\mathscr{B}}$, and thus the canonical odometer associated to $\mathscr{B}$ is the same as in [1] whenever $\mathscr{B}$ is coprime.

We will now show that the canonical odometer "outputs" $\mathcal{F}_{\mathscr{B}}$. Consider the following sets:

$$C := \{(g_1, g_2, \dots) \in G : \text{ for all } k \geq 1,\ g_k \not\equiv 0 \bmod b_k\}, \tag{14}$$

$$C' := \{(g_1, g_2, \dots) \in G' : \text{ for all } k \geq 1,\ g_k \not\equiv 0 \bmod b_k\}. \tag{15}$$

Remark 2.38. By Remark 2.36, we have $W(C) = C'$. In particular, for each $n \in \mathbb{Z}$, we have $\underline{n}_{\mathscr{B}} \in C \iff \underline{n} \in C' \iff n \in \mathcal{F}_{\mathscr{B}}$.

Let $\eta \in \{0,1\}^{\mathbb{Z}}$ be the sequence corresponding to $\mathbb{1}_{\mathcal{F}_{\mathscr{B}}}$. Recall that (X_η, S) denotes the $\mathscr{B}$-free subshift, i.e., $X_\eta := \{x \in \{0,1\}^{\mathbb{Z}}$: each block appearing in x appears in $\eta\}$.

Define $\varphi\colon G \to \{0,1\}^{\mathbb{Z}}$ by setting $\varphi(g)(n) := \mathbb{1}_C(T^n g)$ and notice that

$$\varphi(g)(n) = 1 \iff n \not\equiv -g_k \bmod b_k \quad \text{for all } k \geq 1. \tag{16}$$

Finally, notice that $\varphi\circ T = S\circ\varphi$ and $\eta = \varphi(0,0,\dots)$.

2.8. Admissibility.

Recall that $(X_{\mathscr{B}}, S)$ denotes the $\mathscr{B}$-admissible subshift consisting of $x \in \{0,1\}^{\mathbb{Z}}$ such that, for each $b \in \mathscr{B}$, we have $|\{n \in \mathbb{Z} : x(n) = 1\} \bmod b| < b$; see [1,49].

Remark 2.39. Consider $\varphi_{\mathscr{B}}\colon G_{\mathscr{B}} \to \{0,1\}^{\mathbb{Z}}$ given, for $g \in G_{\mathscr{B}}$, by the same formula as in (16). Arguing as in [1], we easily obtain $\varphi_{\mathscr{B}}(G_{\mathscr{B}}) \subset X_{\mathscr{B}}$. In particular, since $\eta = \varphi_{\mathscr{B}}(0,0,\dots)$, we have $\eta \in X_{\mathscr{B}}$, so $X_\eta \subset X_{\mathscr{B}}$.

Definition 2.40 (Cf. [34,38])**.** We say that $X \subset \{0,1\}^{\mathbb{Z}}$ is *hereditary* if for $x \in X$ and $y \in \{0,1\}^{\mathbb{Z}}$ with $y \leq x$ (coordinatewise), we have $y \in X$.

It follows directly from the definition of admissibility that $X_{\mathscr{B}}$ is hereditary. Denote by $\widetilde{X}_\eta$ the smallest hereditary subshift containing X_η. It is clear that $X_\eta \subset \widetilde{X}_\eta \subset X_{\mathscr{B}}$.

Remark 2.41. Note that $X_{\mathscr{B}}$ is always uncountable. Indeed, for $\mathscr{B}$ infinite, it suffices to notice that the characteristic function $\mathbb{1}_A$ of $A := \{b_1 \cdot \ldots b_k : k \geq 1\}$ has infinite support; thus the set $\{x \in \{0,1\}^{\mathbb{Z}} : x \leq \mathbb{1}_A\} \subset X_{\mathscr{B}}$ is uncountable. If $\mathscr{B} = \{b_1, \ldots, b_k\}$ is finite, then consider $A := \{(b_1 \cdot \ldots \cdot b_k)^\ell : \ell \geq 1\}$.

For $\mathscr{B}$ Erdös, we have $X_\eta = X_{\mathscr{B}}$; see [1]. It is also easy to see that $X_\eta \subsetneq \widetilde{X}_\eta = X_{\mathscr{B}}$ for $\mathscr{B} \neq \emptyset$ finite and coprime. This need not always be the case:

Example 2.42 ($X_\eta \subsetneq \widetilde{X}_\eta \subsetneq X_{\mathscr{B}}$)**.** Let $\mathscr{B} := \mathscr{P}$. Then $\mathcal{F}_{\mathscr{B}} = \{\pm 1\}$ and $\mathscr{B}$ is Behrend. It follows that

$$X_\eta = \{S^n\eta : n \in \mathbb{Z}\} \cup \{\ldots 0.00 \ldots\},$$
$$\widetilde{X}_\eta = \{S^n\eta : n \in \mathbb{Z}\} \cup \{S^n(\ldots 0.10 \ldots) : n \in \mathbb{Z}\} \cup \{\ldots 0.00 \ldots\}.$$

Hence, $X_\eta \subsetneq \widetilde{X}_\eta$ and $\widetilde{X}_\eta$ is countable; thus $\widetilde{X}_\eta \subsetneq X_{\mathscr{B}}$ by Remark 2.41. Note also that $(\widetilde{X}_\eta, S)$ fails to be transitive.

Example 2.43 ($\widetilde{X}_\eta \subsetneq X_{\mathscr{B}}$)**.** Suppose that $4, 6 \in \mathscr{B}$ and $b > 12$ for $b \in \mathscr{B} \setminus \{4, 6\}$. Let $y \in \{0,1\}^{\mathbb{Z}}$ be such that $y[1, 12] = 110011100110$ and $y(n) = 0$ for all $n \in \mathbb{Z} \setminus \{1, 2, \ldots, 12\}$. It follows that $y \in X_{\mathscr{B}}$. We claim that $y \notin \widetilde{X}_\eta$. Suppose that

$$y[1, 12] \leq \eta[k, k+11] \text{ for some } k \in \mathbb{Z}. \tag{17}$$

Recall that $4 \in \mathscr{B}$. Since $y[1] = \eta[k] = y[2] = \eta[k+1] = 1$, it follows that $4 \mid k+2$ or $4 \mid k+3$. Since $y[7] = \eta[k+6] = 1$, we cannot have $4 \mid k+2$. Hence $4 \mid k+3$. On the other hand, we have $6 \in \mathscr{B}$. Since $y[i+1] = \eta[k+i] = 1$ for $i \in \{0, 1, 4, 5, 6\}$ and $k+2$ is odd, we have $6 \mid k+3$. It follows that $6 \mid k+9$, whence $\eta[k+9] = 0$. This, however, contradicts (17).

One can modify $\mathscr{B}$ so that $d(\mathcal{F}_{\mathscr{B}})$ exists and is positive. Furthermore, one can obtain both $X_\eta = \widetilde{X}_\eta \subsetneq X_{\mathscr{B}}$ and $X_\eta \subsetneq \widetilde{X}_\eta \subsetneq X_{\mathscr{B}}$; see Example 5.6.

The subshift $(\widetilde{X}_\eta, S)$ has some natural S-invariant subsets we will be interested in. Given a sequence $(s_k)_{k \geq 1}$ with $0 \leq s_k \leq b_k$ for $k \geq 1$, let

$$Y_{s_1, s_2, \ldots} := \{x \in \{0,1\}^{\mathbb{Z}} : |\text{supp } x \bmod b_k| = b_k - s_k \text{ for each } k \geq 1\},$$
$$Y_{\geq s_1, \geq s_2, \ldots} := \{x \in \{0,1\}^{\mathbb{Z}} : |\text{supp } x \bmod b_k| \leq b_k - s_k \text{ for each } k \geq 1\}.$$

Remark 2.44. For $0 \leq s_k \leq b_k$, $k \geq 1$, define auxiliary subsets

$$Y^k_{s_k} := \{x \in \{0,1\}^{\mathbb{Z}} : |\text{supp } x \bmod b_k| = b_k - s_k\},$$
$$Y^k_{\geq s_k} := \{x \in \{0,1\}^{\mathbb{Z}} : |\text{supp } x \bmod b_k| \leq b_k - s_k\}.$$

Then $Y^k_{s_k} = Y^k_{\geq s_k} \setminus Y^k_{\geq s_k+1}$ and $Y^k_{\geq s_k}, Y^k_{\geq s_k+1}$ are closed. Moreover

$$Y_{s_1, s_2, \ldots} = \bigcap_{k \geq 1} Y^k_{s_k}, \; Y_{\geq s_1, \geq s_2, \ldots} = \bigcap_{k \geq 1} Y^k_{\geq s_k}.$$

In particular, $Y_{s_1, s_2, \ldots}$ is Borel and $Y_{\geq s_1, \geq s_2, \ldots}$ is closed, for any choice of $0 \leq s_k \leq b_k$, $k \geq 1$. Additionally, sets $Y_{s_1, s_2, \ldots}$ are pairwise disjoint for different choices of $(s_1, s_2, \ldots)$ and

$$\{0,1\}^{\mathbb{Z}} = \bigcup_{0 \leq s_k \leq b_k, k \geq 1} Y_{s_1, s_2, \ldots}.$$

We will write Y for $Y_{1,1,\ldots}$. Notice also that $Y_{\geq s_1, \geq s_2, \ldots}$ is the smallest hereditary subshift containing $Y_{s_1, s_2, \ldots}$.

Following [46], we define a map $\theta\colon Y \cap \widetilde{X}_\eta \to G_{\mathscr{B}}$ by

$$\theta(y) = g \iff (\mathrm{supp}\ y) \cap (b_k\mathbb{Z} - g_k) = \emptyset \quad \text{for each } k \geq 1. \tag{18}$$

Notice that given $y \in Y$ and $k_0 \geq 1$, there exists $N \geq 1$ such that

$$|(\mathrm{supp}\ y) \cap [-N, N] \bmod b_k| = b_k - 1 \text{ for } 1 \leq k \leq k_0. \tag{19}$$

Remark 2.45. Notice that

$$\theta(Y \cap \widetilde{X}_\eta) \subset G. \tag{20}$$

Indeed, take $y \in Y \cap \widetilde{X}_\eta$. Given $k_0 \geq 1$, let $N \geq 1$ be such that (19) holds and let $M \in \mathbb{Z}$ be such that $y[-N, N] \leq \eta[-N + M, N + M]$. It follows that $\theta(y) = (g_1, g_2, \dots)$, where $g_k \equiv -M \bmod b_k$ for $1 \leq k \leq k_0$. This yields (20).

Note also that θ is continuous. Indeed, given $y \in Y$ and $k_0 \geq 1$, let N be such that (19) holds. If $y' \in Y$ is sufficiently close to y, then (19) holds for y' as well. Therefore, if $y_n \to y$ in Y, then $\theta(y_n) \to \theta(y)$.

Remark 2.46. Note that:

- $T \circ \theta = \theta \circ S$,
- for each $y \in Y \cap \widetilde{X}_\eta$, $y \leq \varphi(\theta(y))$,
- for any $\nu \in \mathcal{P}(Y \cap \widetilde{X}_\eta, S)$, $\theta_*(\nu) = \mathbb{P}$

(the first two properties follow by a direct calculation; the third one is a consequence of the unique ergodicity of T).

2.9. **Mirsky measure ν_η.**

Definition 2.47. The image $\nu_\eta := \varphi_*(\mathbb{P})$ of $\mathbb{P}$ via φ is called the *Mirsky measure* of $\mathscr{B}$.

Remark 2.48 (Cf. Example 2.37). In the previous works [1,36], the Mirsky measure was defined in a different way. In the new notation, the "old Mirsky measure" was given by $\nu_{\mathscr{B}} := (\varphi_{\mathscr{B}})_*(\mathbb{P}_{\Omega_{\mathscr{B}}})$. We have

$$\nu_{\mathscr{B}}(\{x \in \{0,1\}^{\mathbb{Z}} : x(0) = 1\}) = \prod_{b \in \mathscr{B}} \left(1 - \frac{1}{b}\right)$$

(we follow word for word the proof of this formula from [1]). This implies that $\nu_{\mathscr{B}} \neq \delta_{\dots 0.00\dots}$ if and only if $\mathscr{B}$ is thin. An advantage of ν_η is that $\nu_\eta \neq \delta_{\dots 0.00\dots}$ whenever $\mathscr{B} \subset \mathbb{N}$ is not Behrend (see Remark 4.2). Moreover, we will see that ν_η plays a similar role and has similar properties as the "old Mirsky measure". This is why we call ν_η the Mirsky measure, not $\nu_{\mathscr{B}}$. Notice that if $\mathscr{B}$ is Erdös, we have $\nu_\eta = \nu_{\mathscr{B}}$.

3. Topological dynamics

3.1. **Unique minimal subset (proof of Theorem A).**

In the square-free case, i.e., when $\mathscr{B} = \{p^2 : p \in \mathscr{P}\}$, the subshift (X_η, S) is proximal [49]. In particular, by Remark 2.9, it has a fixed point that yields the only minimal subset of X_η (this fixed point is the sequence $\dots 0.00 \dots$). It turns out that in general there are $\mathscr{B}$-free subshifts (X_η, S) that are not proximal. Indeed, this happens, e.g., when $\mathscr{B}$ is finite (cf. Section 4.3.1), and more examples will be seen later (we give necessary and sufficient conditions for proximality in Section 3.2.2).

Proof of Theorem A. We begin the proof by showing the validity of the second assertion. Suppose first that η contains arbitrarily long blocks of zeros. Then the Toeplitz sequence $\dots 0.00 \dots$ is in X_η.

Suppose now that the length of blocks of zeros that appear in η is bounded. We will use Lemma 2.18 and sequences $(B_n)_{n\in\mathbb{N}}, (m_n)_{n\in\mathbb{N}}, (d_n)_{n\in\mathbb{N}}$ will be constructed inductively. First, we will choose the longest block of zeros that appears in η. Then we will extend it to the right and to the left by the shortest possible blocks of ones such that the extended block appears in η. Next, the obtained block will be extended to the right and then to the left by the longest possible blocks of zeros, so that the block we obtain still appears in η. This procedure will be repeated to obtain longer and longer blocks.

Let B_1 be the longest block of zeros that appears in η. For convenience, we will treat B_1 as an element of $\{0,1\}^{[0,|B_1|-1]}$ (i.e., we set $\ell_1 := 0$, $r_1 := |B_1|-1$). Then, since $\eta = \mathbb{1}_{\mathcal{F}_\mathscr{B}}$, there exists $d_1 \in \mathbb{N}$ such that B_1 appears in η periodically, with period d_1; i.e., for some $m_1 \in \mathbb{Z}$, we have

$$\eta[m_1 + kd_1 + \ell_1, m_1 + kd_1 + r_1] = B_1 \quad \text{for each } k \in \mathbb{Z}.$$

Suppose now that $B_n \in \{0,1\}^{[\ell_n, r_n]}$, $m_n \in \mathbb{Z}$, and $d_n \in \mathbb{N}$ for $1 \le n \le 4n_0+1$ are chosen so that (a) and (b) from Lemma 2.18 hold for $1 \le n \le 4n_0$ and (c) from Lemma 2.18 holds for $1 \le n \le 4n_0+1$. We will now define $B_n \in \{0,1\}^{[\ell_n, r_n]}$, $m_n \in \mathbb{Z}$, $d_n \in \mathbb{N}$ for $4n_0+2 \le n \le 4n_0+5$.

Let $B_{4n_0+2} \in \{0,1\}^{[\ell_{4n_0+2}, r_{4n_0+2}]}$, where $\ell_{4n_0+2} = \ell_{4n_0+1}$ (and $r_{4n_0+2} = \ell_{4n_0+2} + |B_{4n_0+2}| - 1$), be the **shortest** block of the form $B_{4n_0+1}1\dots1$ such that the block $B_{4n_0+1}1\dots10$ appears in η and begins at position $m_{4n_0+1} + \ell_{4n_0+1} + k_0 d_{4n_0+1}$ for some $k_0 \in \mathbb{Z}$, i.e.,

$$\eta[m_{4n_0+2} + \ell_{4n_0+2}, m_{4n_0+2} + r_{4n_0+2}] = B_{4n_0+2},$$

where $m_{4n_0+2} = m_{4n_0+1} + k_0 d_{4n_0+1}$. Then, clearly, $d_{4n_0+1} \mid m_{4n_0+2} - m_{4n_0+1}$. Moreover, by the definition of B_{4n_0+2}, we have

$$\eta[m_{4n_0+2} + \ell_{4n_0+2} + kd_{4n_0+1}, m_{4n_0+2} + r_{4n_0+2} + kd_{4n_0+1}] = B_{4n_0+2}$$

for each $k \in \mathbb{Z}$; i.e., we may set $d_{4n_0+2} := d_{4n_0+1}$. This way, we have extended our block B_{4n_0+1} to the right by a block of ones.

The block B_{4n_0+3} is defined in a similar way as B_{4n_0+2}, but now we extend B_{4n_0+2} to the left. Let $B_{4n_0+3} \in \{0,1\}^{[\ell_{4n_0+3}, r_{4n_0+3}]}$, where $r_{4n_0+3} = r_{4n_0+2}$ (and $\ell_{4n_0+3} = r_{4n_0+3} - |B_{4n_0+3}| + 1$), be the **shortest** block of the form $1\dots1B_{4n_0+2}$ such that the block $01\dots1B_{4n_0+2}$ appears in η and ends at position $m_{4n_0+2} + r_{4n_0+2} + k_0 d_{4n_0+2}$ for some $k_0 \in \mathbb{Z}$, i.e.,

$$\eta[m_{4n_0+3} + \ell_{4n_0+3}, m_{4n_0+3} + r_{4n_0+3}] = B_{4n_0+3},$$

where $m_{4n_0+3} = m_{4n_0+2} + k_0 d_{4n_0+2}$. Then, clearly, $d_{4n_0+2} \mid m_{4n_0+3} - m_{4n_0+2}$. Moreover, by the definition of B_{4n_0+3}, we have

$$\eta[m_{4n_0+3} + \ell_{4n_0+3} + kd_{4n_0+3}, m_{4n_0+3} + r_{4n_0+3} + kd_{4n_0+3}] = B_{4n_0+3}$$

for each $k \in \mathbb{Z}$; i.e., we may set $d_{4n_0+3} := d_{4n_0+2}$. This way, we have extended our block B_{4n_0+2} to the left by a block of ones.

Let $B_{4n_0+4} \in \{0,1\}^{[\ell_{4n_0+4}, r_{4n_0+4}]}$, where $\ell_{4n_0+4} = \ell_{4n_0+3}$ (and $r_{4n_0+4} = \ell_{4n_0+4} + |B_{4n_0+4}| - 1$), be the **longest** block of the form $B_{4n_0+3}0\dots0$ that appears in η and begins at position $m_{4n_0+3}+\ell_{4n_0+3}+k_0d_{4n_0+3}$ for some $k_0\in\mathbb{Z}$, i.e.,

$$\eta[m_{4n_0+4}+\ell_{4n_0+4},m_{4n_0+4}+r_{4n_0+4}]=B_{4n_0+4},$$

where $m_{4n_0+4}=m_{4n_0+3}+k_0d_{4n_0+3}$. Then, clearly, $d_{4n_0+3}\mid m_{4n_0+4}-m_{4n_0+3}$. Moreover, since each zero in η appears with some period, there exists d'_{4n_0+4} such that the pattern of zeros from B_{4n_0+4} repeats in η periodically, with period d'_{4n_0+4}. Thus, by taking $d_{4n_0+4}:=\mathrm{lcm}(d'_{4n_0+4},d_{4n_0+3})$, we obtain

$$\eta[m_{4n_0+4}+\ell_{4n_0+4}+kd_{4n_0+4},m_{4n_0+4}+r_{4n_0+4}+kd_{4n_0+4}]=B_{4n_0+4}$$

for each $k\in\mathbb{Z}$.

Finally, let $B_{4n_0+5}\in\{0,1\}^{[\ell_{4n_0+5},r_{4n_0+5}]}$, where $r_{4n_0+5}=r_{4n_0+4}$ (and $\ell_{4n_0+5}=r_{4n_0+5}-|B_{4n_0+5}|+1$), be the **longest** block of the form $0\dots0B_{4n_0+4}$ that appears in η and ends at position $m_{4n_0+4}+r_{4n_0+4}+k_0d_{4n_0+4}$ for some $k_0\in\mathbb{Z}$, i.e.,

$$\eta[m_{4n_0+5}+\ell_{4n_0+5},m_{4n_0+5}+r_{4n_0+5}]=B_{4n_0+5},$$

where $m_{4n_0+5}=m_{4n_0+4}+k_0d_{4n_0+4}$. Then, clearly $d_{4n_0+4}\mid m_{4n_0+5}-m_{4n_0+4}$. Moreover, since each zero in η appears with some period, there exists d'_{4n_0+5} such that the pattern of zeros from B_{4n_0+5} repeats in η periodically, with period d'_{4n_0+5}. Thus, by taking $d_{4n_0+5}:=\mathrm{lcm}(d'_{4n_0+5},d_{4n_0+4})$, we obtain

$$\eta[m_{4n_0+5}+\ell_{4n_0+5}+kd_{4n_0+5},m_{4n_0+5}+r_{4n_0+5}+kd_{4n_0+5}]=B_{4n_0+5}$$

for each $k\in\mathbb{Z}$.

The first assertion follows from the first part of the proof and Lemma 2.18. The proof of Theorem A is complete. □

By Corollary 1.4, (X_η,S) is minimal if and only if it is Toeplitz. In fact, η may even happen to be a Toeplitz sequence:

Example 3.1. Let $\mathscr{B}:=\{b_i2^i:i\geq1\}$, where $b_i\geq2$ for $i\geq1$. We will show that η is a Toeplitz sequence. Indeed, for each $n\in\mathbb{Z}$ such that $\eta(n)=0$, there is $k_n\geq1$ such that $\eta(n+jk_n)=0$ for all $j\in\mathbb{Z}$. Now let $n\in\mathbb{Z}$ be such that $\eta(n)=1$, i.e.,

$$n\not\equiv0 \bmod b_i2^i\quad\text{for each } i\geq1. \tag{21}$$

Let m be odd such that $n=m2^a$. We claim that

$$\eta(n+jb_1\dots b_a2^{a+1})=1\quad\text{for all } j\in\mathbb{Z}. \tag{22}$$

Suppose not, so that for some i_0, we have

$$n+j_0b_1\dots b_a2^{a+1}=K_0b_{i_0}2^{i_0}\quad\text{for some } j_0,K_0\in\mathbb{Z}. \tag{23}$$

Then $i_0\leq a$; if not, by (23), $2^{a+1}\mid n$, which is impossible. But now, again by (23), $b_{i_0}2^{i_0}\mid n$, which contradicts (21).

Remark 3.2. It follows by Proposition 4.25 that η in Example 3.1 is a Toeplitz sequence that is not periodic. Consider $d_n:=b_1\cdot\ldots\cdot b_n2^{n+1}$. Two cases appear:

- If $s\in\mathcal{M}_{\mathscr{B}}$, then $b_i2^i\mid s$ for some $i\geq1$. If $i\leq n$, then $s+d_n\mathbb{Z}\subset\mathcal{M}_{\mathscr{B}}$. Otherwise, we have $2^{n+1}\mid s$.
- If $s\in\mathcal{F}_{\mathscr{B}}$, then we let m be odd such that $s=m\cdot2^a$. Then, by (22), $s+b_1\cdot\ldots\cdot b_a2^{a+1}\mathbb{Z}\subset\mathcal{F}_{\mathscr{B}}$. If $a\leq n$, then clearly $s+d_n\mathbb{Z}\subset\mathcal{F}_{\mathscr{B}}$. Otherwise, we have $2^{n+1}\mid s$.

It follows that if $s \in \mathbb{Z}$ satisfies $(s+d_n\mathbb{Z})\cap\mathcal{M}_{\mathscr{B}}\neq\emptyset$ and $(s+d_n\mathbb{Z})\cap\mathcal{F}_{\mathscr{B}}\neq\emptyset$, then $2^{n+1} \mid s$. It follows that, in each integer interval of length d_n, the proportion of s for which the sequence $(\eta(s+jd_n))_{j\in\mathbb{Z}}$ is not constant equals $2^{-(n+1)}$, hence tends to zero as $n\to\infty$. Toeplitz sequences satisfying such a property are called *regular*; see, e.g., [16]. In particular, (X_η, S) is minimal and uniquely ergodic.

3.2. **Proximality of** (X_η, S)**.** We will first show that for $\mathscr{B}$ pairwise coprime and infinite, $(\widetilde{X}_\eta, S)$ is proximal. This implies, by Remark 2.11, that (X_η, S) is proximal and if $\widetilde{X}_{\eta'}\subset\widetilde{X}_\eta$, then $(\widetilde{X}_{\eta'}, S)$ and $(X_{\eta'}, S)$ are both proximal. Our aim (see Theorem 3.7) is to show that the converse is also true: if $(X_{\eta'}, S)$ and $(\widetilde{X}_{\eta'}, S)$ are proximal, then there exists η associated with coprime and infinite $\mathscr{B}$ such that $\widetilde{X}_{\eta'}\subset\widetilde{X}_\eta$.

3.2.1. *Coprime case.*

Proposition 3.3. *If $\mathscr{B}\subset\mathbb{N}$ is infinite and coprime, then $(\widetilde{X}_\eta, S)$ is syndetically proximal. In particular, (X_η, S) is syndetically proximal.*

Proof. By Corollary 2.14, it suffices to show that for any $x\in\widetilde{X}_\eta$ and $\varepsilon>0$ the set

$$\{n\in\mathbb{Z}\colon d(S^n x, \dots 0.00\dots)<\varepsilon\} \text{ is syndetic.} \tag{24}$$

Fix $x\in\widetilde{X}_\eta$. For $n\in\mathbb{N}$ and $k\geq 1$ there exists $m=m_{n,k}\in\mathbb{Z}$ such that

$$x[n,\dots,n+b_1\cdot\ldots\cdot b_k+k-1]\leq\eta[m,\dots,m+b_1\cdot\ldots\cdot b_k+k-1].$$

By the Chinese Remainder Theorem, there exists a unique $0\leq i_0\leq b_1\cdot\ldots\cdot b_k-1$ ($i_0=i_0(m,n)$) such that

$$m+i_0+j\equiv 0 \bmod b_{j+1}\quad\text{for } 0\leq j\leq k-1,$$

i.e., $x(n+i_0+j)\leq\eta(m_{n,k}+i_0+j)=0$ for $0\leq j\leq k-1$. This yields (24) and completes the proof. □

As an immediate consequence of Proposition 3.3 and Remark 2.12, we obtain the following:

Corollary 3.4. *For $\mathscr{B}\subset\mathbb{N}$ infinite and coprime, the maximal equicontinuous factor of $(X_\eta^{\times N}, S^{\times N})$ is trivial for each $N\geq 1$.*

3.2.2. *General case* (*proof of* Theorem B).

Definition 3.5. We say that $\mathscr{B}\subset\mathbb{N}$ satisfies condition (Au) whenever there exists infinite pairwise coprime $\mathscr{B}'\subset\mathscr{B}$. We say that $\mathscr{B}\subset\mathbb{N}$ satisfies condition ($\mathrm{T}_{\mathrm{prox}}$) whenever for any $k\in\mathbb{N}$ there exist $b_1^{(k)},\dots,b_k^{(k)}\in\mathscr{B}$ such that $\gcd(b_i^{(k)},b_j^{(k)})\mid(j-i)$ for all $1\leq i<j\leq k$.

Remark 3.6. Clearly, if (Au) holds, then $\eta\leq\eta'$, whence $X_\eta\subset\widetilde{X}_{\eta'}$.

Theorem 3.7. *Let $\mathscr{B}\subset\mathbb{N}$. The following conditions are equivalent:*

(a) *$(X_{\mathscr{B}}, S)$ is proximal,*
(b) *$(\widetilde{X}_\eta, S)$ is proximal,*
(c) *(X_η, S) is proximal,*
(d) *$\dots 0.00\dots\in X_\eta$,*
(e) *$\mathscr{B}$ satisfies (T_{prox}),*
(f) *for any choice of $q_1,\dots,q_m>1$, $m\geq 1$, we have $\mathscr{B}\not\subset\bigcup_{i=1}^m\mathbb{Z}q_i$,*

(g) $\mathscr{B}$ *satisfies* (Au),

(h) $\mathcal{F}_{\mathscr{B}}$ *does not contain an infinite arithmetic progression.*

Proof. Since $X_\eta \subset \widetilde{X}_\eta \subset X_{\mathscr{B}}$, by Remark 2.11, we have (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (d). If (X_η, S) is proximal, then, by Remark 2.9, it has a fixed point, i.e. either $\dots 0.00 \dots \in X_\eta$ or $\dots 1.11 \dots \in X_\eta$. The latter of the two is impossible, since each zero on η appears in η with bounded gaps, and the claim follows.

(d) $\Rightarrow$ (e). If $\dots 0.00 \dots \in X_\eta$, then there are arbitrarily long blocks of consecutive zeros on η. In other words, given $k \geq 1$, there are $s_1, \dots, s_k$ such that we can solve the system of congruences:

$$i_0 + i - 1 \equiv 0 \bmod b_{s_i},\ 1 \leq i \leq k.$$

Suppose that $d \mid \gcd(b_{s_i}, b_{s_j})$. Then $d \mid i_0+i-1$ and $d \mid i_0+j-1$, whence $d \mid (j-i)$.

(e) $\Rightarrow$ (f). Suppose that (e) holds but (f) does not hold and let $q_1, \dots, q_m$, $k \geq 1$, be such that $\mathscr{B} \subset \bigcup_{i=1}^m \mathbb{Z}q_i$. Without loss of generality, we can assume that $\{q_1, \dots, q_m\}$ is coprime (indeed, we can always find a coprime set $\{q_1', \dots, q_n'\}$ such that $\bigcup_{i=1}^m q_i\mathbb{Z} \subset \bigcup_{i=1}^n q_i'\mathbb{Z}$). Let $k \geq q_1 \dots q_m$ and choose $b_1^{(k)}, \dots, b_k^{(k)} \in \mathscr{B}$ satisfying condition ($\mathrm{T_{prox}}$). For $i = 1, \dots, m$, let $M_i := \{1 \leq \ell \leq k : b_\ell^{(k)} \in q_i\mathbb{Z}\}$. Then, by ($\mathrm{T_{prox}}$), $q_i \mid (\ell - \ell')$ for any $\ell, \ell' \in M_i$, whence

$$M_i \subset q_i\mathbb{Z} + r_i \quad \text{for some } 0 \leq r_i < q_i. \tag{25}$$

For $i = 1, \dots, m$, choose a natural number r_i' such that $q_i \nmid (r_i - r_i')$. By the Chinese Remainder Theorem there exists a natural number $j \leq q_1, \dots, q_m \leq k$ such that $j \equiv r_i' \bmod q_i$ for $i = 1, \dots, m$. It follows from (25) that $j \notin M_i$ for any $i = 1, \dots, m$. Thus $b_j^{(k)} \notin q_1\mathbb{Z} \cup \dots \cup q_m\mathbb{Z}$, a contradiction.

(f) $\Rightarrow$ (g). We will proceed inductively. Fix $c_1 \in \mathscr{B}$. Suppose that for $k \geq 1$ we have found pairwise coprime subset $\{c_1, \dots, c_k\} \subset \mathscr{B}$. Let $\{q_1, \dots, q_m\}$ be the set of all prime divisors of $c_1, \dots, c_k$. Then any $c_{k+1} \in \mathscr{B} \setminus (q_1\mathbb{Z} \cup \dots \cup q_m\mathbb{Z})$ is coprime with each of $c_1, \dots, c_k$.

(g) $\Rightarrow$ (a). If (g) holds, then, by Remark 3.6, we have $X_\eta \subset \widetilde{X}_{\eta'}$. By Proposition 3.3, $\widetilde{X}_{\eta'}$ is proximal. Hence, by Remark 2.11, we obtain (a).

(d) $\Rightarrow$ (h) Condition (d) implies that $\mathcal{M}_{\mathscr{B}}$ contains intervals of integers of arbitrary length, whence (h) follows.

(h) $\Rightarrow$ (f). Suppose that (f) does not hold and let $q_1, \dots, q_k$, $k \geq 1$, be such that $\mathscr{B} \subset \bigcup_{i=1}^k \mathbb{Z}q_i$. Let $M := q_1 \cdot \ldots \cdot q_k$. We claim that $b \nmid \ell M + 1$ for every $b \in \mathscr{B}$, i.e., $\ell M + 1 \in \mathcal{F}_{\mathscr{B}}$ for every $\ell \in \mathbb{Z}$. Indeed, given $b \in \mathscr{B}$, there exists q_i $(1 \leq i \leq k)$ such that $q_i \mid b$. If $b \mid \ell M + 1$, then $q_i \mid \ell M + 1$. This is however impossible since $q_i \mid M$. ☐

Proof of Theorem B. The assertion is an immediate consequence of Theorem 3.7 and Remark 2.9. ☐

Let us see some consequences of Theorem 3.7. By Remark 3.6 and Theorem 3.7, we have the following:

Corollary 3.8. *If (X_η, S) is proximal, then $X_\eta \subset \widetilde{X}_{\eta'}$ with $\mathscr{B}'$ coprime.*

Remark 3.9. Recall (see [1]) that if $\mathscr{B}$ is Erdös, then $X_\eta = X_{\mathscr{B}}$. In particular, X_η is hereditary.

If $\mathscr{B}$ is Behrend, then $\ldots 0.00\ldots \in X_\eta$, so by the implication (d) $\Rightarrow$ (a), we obtain that (X_η, S) is proximal; hence $\mathcal{F}_{\mathscr{B}}$ contains an infinite subset which is pairwise coprime. The latter can be seen directly from the implication (f) $\Rightarrow$ (g).

By the implication (d) $\Rightarrow$ (c) in Theorem 3.7, we obtain the following (see also Example 4.16):

Corollary 3.10. *If X_η is hereditary, then (X_η, S) is proximal.*

Question 3.11. Is it possible that $X_\eta \subsetneq \widetilde{X}_\eta = X_{\mathscr{B}}$ with (X_η, S) proximal?

We will now give one more characterization of the proximality of (X_η, S), in terms of the maximal equicontinuous factor (cf. Corollary 3.4):

Theorem 3.12. *The system (X_η, S) is proximal if and only if its maximal equicontinuous factor is trivial.*

For the proof, we will need the following lemma:

Lemma 3.13. *Let $d \geq 1$ and let $A \subset \{0, 1, \ldots, d-1\}$. Suppose that for any $k \geq 1$ there exist $n_k \in \mathbb{Z}$ and $0 \leq r_k \leq d-1$ such that*

$$A + md + r_k \subset \mathcal{F}_{\mathscr{B}} \quad \text{for} \quad n_k \leq m \leq n_k + k. \tag{26}$$

Then, for any $0 \leq r \leq d-1$ such that there are infinitely many $k \geq 1$ with $r_k = r$, we have

$$A + \mathbb{Z}d + r \subset \mathcal{F}_{\mathscr{B}}. \tag{27}$$

Proof. Let $0 \leq r \leq d-1$ be such that there are infinitely many $k \geq 1$ satisfying (26) with $r_k = r$, i.e.,

$$A + md + r \subset \mathcal{F}_{\mathscr{B}} \quad \text{for} \quad n_k \leq m \leq n_k + k. \tag{28}$$

Suppose that (27) fails. Then, for some $a \in A$ and $k \in \mathbb{Z}$, we have $a + kd + r \in \mathcal{M}_{\mathscr{B}}$. In other words, for some $b \in \mathscr{B}$, we have $b \mid a + kd + r$. It follows that for any $\ell \in \mathbb{Z}$, we have $b \mid a + (k + \ell b)d + r$. This, however, contradicts (28). □

Proof of Theorem 3.12. Since proximality implies that the maximal equicontinuous factor is trivial, we only need to show the converse implication. Suppose that (X_η, S) is not proximal. Let $d \geq 1$ be the smallest number such that $\mathcal{F}_{\mathscr{B}}$ contains an infinite arithmetic progression with difference d (such d exists by Theorem 3.7(h)). Let $F \subset \{0, \ldots, d-1\}$ be the maximal set such that

$$F + \mathbb{Z}d \subset \mathcal{F}_{\mathscr{B}} \tag{29}$$

($F \neq \emptyset$ by the definition of d). We claim that for any $y \in X_\eta$, there exists a unique $0 \leq r < d$ such that

$$y(a + md + r) = 1 \quad \text{for all} \quad a \in F \quad \text{and} \quad m \in \mathbb{Z}. \tag{30}$$

Since $y \in X_\eta$, it follows from (29) that such r exists and we only need to show uniqueness. Suppose that (30) holds for $r = r_1, r_2$, where $d \nmid (r_1 - r_2)$; i.e., we have

$$y(a + md) = 1 \quad \text{for all} \quad a \in (F + r_1) \cup (F + r_2) \quad \text{and} \quad m \in \mathbb{Z}.$$

Since $y \in X_\eta$, each block from y appears in η, and it follows that the assumptions of Lemma 3.13 hold for $A := (F + r_1) \cup (F + r_2) \bmod d$. Therefore, using additionally (29),

$$[F \cup (F + r_1 + s) \cup (F + r_2 + s)] + \mathbb{Z}d \subset \mathcal{F}_{\mathscr{B}} \quad \text{for some } s.$$

Note that by the minimality of d, we have $F+i \neq F \bmod d$ for $0 < i < d$. Therefore,

$$F \subsetneq F \cup (F + r_1 + s) \cup (F + r_2 + s).$$

This contradicts the maximality of F and thus indeed implies the uniqueness of r. It follows that

$$X_\eta = \bigcup_{i=0}^{d-1} X_\eta^{(i)},\ X_\eta^{(i)} = \{y \in X_\eta : (30) \text{ holds for } r = i\}$$

is a decomposition of X_η into d pairwise disjoint sets. Clearly, each $X_\eta^{(i)}$ is closed and $SX_\eta^{(i)} = X_\eta^{(i-1)}$, where $X_\eta^{(-1)} = X_\eta^{(d-1)}$. It follows that (X_η, S) has the (minimal) rotation on d points as a topological factor, which completes the proof. □

The following natural question arises:

Question 3.14. Given $\mathscr{B} \subset \mathbb{N}$, what is the maximal equicontinuous factor of (X_η, S)?

A partial answer to Question 3.14 is given by the following (see also Section 4.3.1):

Proposition 3.15. *Suppose that $X_\eta = X_\eta \cap Y$. Then (G, T) is the maximal equicontinuous factor of (X_η, S). In particular, if we additionally assume that $\mathscr{B}$ is infinite, then the maximal equicontinuous factor of (X_η, S) is infinite.*

Proof. Notice first that, by Remark 2.45, $\theta\colon X_\eta \to G$ is well-defined and continuous. Thus, (G, T) is an equicontinuous factor of (X_η, S) and we only need to show its maximality. Notice that the (discrete) spectrum of the maximal equicontinuous factor of (X_η, S) is always included in the discrete part of the spectrum of (X_η, ν, S) for any $\nu \in \mathcal{P}(X_\eta, S)$. Therefore, to prove the maximality of (G, T), it suffices to find ν such that the discrete part of the spectrum of (X_η, ν, S) agrees with the (discrete) spectrum of $(G, \mathbb{P}, T)$. We have

$$(G, \mathbb{P}, T) \xrightarrow{\varphi} (X_\eta, \nu_\eta, S) \xrightarrow{\theta} (G, \mathbb{P}, T).$$

It follows from the coalescence of $(G, \mathbb{P}, T)$ that $\theta \circ \varphi$ yields an isomorphism of $(G, \mathbb{P}, T)$ with itself, whence φ yields an isomorphism of $(G, \mathbb{P}, T)$ and (X_η, ν_η, S). In particular, the (discrete) spectrum of $(G, \mathbb{P}, T)$ is the same as the (discrete) spectrum of (X_η, ν_η, S), and the claim follows. □

Example 3.16. Let $\mathscr{B}$ be as in Example 3.1. Then $\sum_{i\geq 1} \frac{1}{2^i b_i} \leq \sum_{i\geq 1} \frac{1}{2^i}$ is thin and it follows from (5), (7), and Corollary 4.32 that $\eta \in Y$. Moreover, by the minimality of (X_η, S), for each $0 \leq s_k \leq b_k$, $k \geq 1$, we have that either $X_\eta \cap Y^k_{\geq s_k} = X_\eta$ or $X_\eta \cap Y^k_{\geq s_k} = \emptyset$. Since $\eta \in Y$, it follows that $X_\eta \cap Y^k_{\geq s_k} = \emptyset$ whenever $s_k \geq 2$. Since $X_\eta = X_\eta \cap (\bigcup_{1 \leq s_k \leq b_k} Y^k_{s_k})$ for each $k \geq 1$, it follows that $X_\eta = X_\eta \cap Y$. By Proposition 3.15, the associated canonical odometer (G, T) is the maximal equicontinuous factor of (X_η, S).

3.3. **Transitivity.**

3.3.1. *Transitivity of* $(\widetilde{X}_\eta, S)$ *and* $(X_{\mathscr{B}}, S)$.

Proposition 3.17. *For any* $\mathscr{B} \subset \mathbb{N}$ *such that the support of* η *is infinite, the following conditions are equivalent:*

(a) $(\widetilde{X}_\eta, S)$ *is transitive.*
(b) $(\widetilde{X}_\eta, S)$ *does not have open wandering sets of positive diameter.*
(c) *For any block* B *that appears in* η *there exists a block* $B' \geq B$ (*coordinatewise*) *that appears in* η*, infinitely often.*

Proof. Since the implication (a) $\Rightarrow$ (b) is obvious, it remains to show (b) $\Rightarrow$ (c) $\Rightarrow$ (a). We will prove first (b) $\Rightarrow$ (c). Suppose that (c) does not hold. Let B be a block in η such that all blocks $B' \geq B$ appear in η (at most) finitely many times. Let

$$K := \min\{k \in \mathbb{Z} : \eta[k, k + |B| - 1] \geq B\},$$
$$L := \max\{k + |B| - 1 : \eta[k, k + |B| - 1] \geq B\}$$

(in particular, blocks $B' \geq B$ do not appear in η outside $\eta[K, L]$). We claim that, for any $x \in \widetilde{X}_\eta$, the block $C := \eta[K, L]$ appears in x at most once. Suppose that, for some $x \in \widetilde{X}_\eta$, C appears in x twice. It follows that a block of the form $C'DC''$, where $C', C'' \geq C$, appears in η, and this is impossible by the choice of C. Thus, the cylinder set $\mathcal{C} := \{x \in \widetilde{X}_\eta : x[K, L] = C\}$ corresponding to C is an open wandering set. Clearly, we have $\eta \in \mathcal{C}$. Moreover, since the support of η is infinite, we also have $x \in \mathcal{C}$ for x given by $x(n) = \eta(n)$ for $n \in [K, L]$; $x(n) = 0$ otherwise. It follows that $|\mathcal{C}| \geq 2$; i.e., the diameter of $\mathcal{C}$ is positive and we conclude that (b) fails.

We will now prove (c) $\Rightarrow$ (a). By Remark 2.1, given blocks B', C' that appear in η and $B \leq B', C \leq C'$, it suffices to show that there exists $x \in \widetilde{X}_\eta$ such that both B and C appear in x. It follows from (c) that there exists $B'' \geq B'$ that appears in η infinitely often. Therefore for some block D, a block of the form $C'DB''$ or a block of the form $B''DC'$ appears in η. Hence, $x := \dots 00B0^{|D|}C00 \dots \in \widetilde{X}_\eta$, and the result follows. □

As an immediate consequence of Proposition 3.17, we obtain the following:

Corollary 3.18. *Let* $\mathscr{B} \subset \mathbb{N}$ *be such that* η *is recurrent. Then* $(\widetilde{X}_\eta, S)$ *is transitive.*

In particular, by Corollary 3.18, Theorem 4.1, and Theorem G, we have the following (cf. Example 2.42):

Corollary 3.19. *The subshift* $(\widetilde{X}_\eta, S)$ *is transitive whenever* $\mathscr{B}$ *has light tails.*

Clearly, if $X_\eta = X_{\mathscr{B}}$, then $(X_{\mathscr{B}}, S)$ is transitive (recall that $X_\eta = X_{\mathscr{B}}$ holds for $\mathscr{B}$ Erdös). We will now give an example where $(X_{\mathscr{B}}, S)$ fails to be transitive.

Example 3.20. Let $\mathscr{B}$ be as in Example 2.43, i.e., $4, 6 \in \mathscr{B}$ and $b > 12$ for $b \in \mathscr{B} \setminus \{4, 6\}$. Let

$$A_1 := 110011100110,$$
$$A_2 := 011101010111 = \eta[0, 11].$$

Suppose that both A_1, A_2 appear in $x \in \{0,1\}^{\mathbb{Z}}$. We will show that $x \notin X_{\mathscr{B}}$. Indeed, we have

$$\mathbb{Z}/4\mathbb{Z} \setminus (\text{supp } A_1 \bmod 4) = \{3\},$$
$$\mathbb{Z}/4\mathbb{Z} \setminus (\text{supp } A_2 \bmod 4) = \{0\}.$$

Let $k, \ell \in \mathbb{Z}$ be such that $x[k, k+11] = A_1$ and $x[\ell, \ell+11] = A_2$. It follows that if x is $\{4\}$-admissible, then $4 \mid k+3-\ell$. In a similar way, if x is $\{6\}$-admissible, then $6 \mid k+2-\ell$. Since one of the numbers $k+3-\ell$ and $k+2-\ell$ is odd, we conclude that x is not $\{4,6\}$-admissible, so all the more, it is not $\mathscr{B}$-admissible.

3.3.2. *Topological non-disjointness of* (X_η, S) *and* (G, T).

Proposition 3.21. *There exists a proper, closed, and $T \times S$-invariant subset $N \subset G \times X_\eta$ with full projections on both coordinates. In other words, there is a non-trivial topological joining between (G, T) and (X_η, S).*

Proof. Let $N := \overline{\mathcal{O}_{T\times S}(\underline{0}, \eta)}$, where $\underline{0} = (0, 0, \dots)$; i.e., N is the closure of the graph of φ along the orbit of $\underline{0}$ (indeed, we have $S^n\eta = S^n\varphi(\underline{0}) = \varphi(T^n\underline{0})$). Since the orbit of $\underline{0}$ under T is dense in G and the orbit of η under S is dense in X_η, it follows that N has full projection on both coordinates. Moreover, N is closed and $T \times S$-invariant. It remains to show that $N \neq G \times X_\eta$. Take $\dots 0.00 \dots \neq x \in X_\eta$. We claim that $\{g \in G : (g, x) \in N\} \neq G$. Indeed, let $k_0 \in \mathbb{Z}$ be such that $x(k_0) = 1$ and suppose that $(T^{n_i} \times S^{n_i})(\underline{0}, \eta) \to (g, x)$. Then $S^{n_i}\eta \to x$, whence, for i sufficiently large, $\eta(k_0 + n_i) = S^{n_i}\eta(k_0) = x(k_0) = 1$. It follows that $n_i + k_0 \in \mathcal{F}_{\mathscr{B}}$, i.e., $n_i + k_0 \neq 0 \bmod b_k$ for each $k \geq 1$. On the other hand, we have $T^{n_i}\underline{0} \to g$, i.e., $(n_i, n_i, \dots) \to (g_1, g_2, \dots)$. Thus, $g_k \neq -k_0 \bmod b_k$ for each $k \geq 1$. Hence, $\{g \in G : (g, x) \in N\} \neq G$ for $x \neq \dots 0.00 \dots$, which completes the proof. □

Remark 3.22. Suppose that $\mathscr{B}$ is taut and $1 \notin \mathscr{B}$. By Corollary 4.32, $\eta \in Y$, i.e., for each $k \geq 1$, we have $\mathcal{F}_{\mathscr{B}} \bmod b_k = (\mathbb{Z}/b_k\mathbb{Z}) \setminus \{0\}$. It follows from the above proof that $\{g \in G : (g, \eta) \in N\} = \{\underline{0}\}$. In a similar way, if $x \in Y$, then $\{g \in G : (g, x) \in N\}$ is a singleton; in particular, for each $n \in \mathbb{Z}$, the set $\{g \in G : (g, S^n\eta) \in N\}$ is a singleton.

We can now use the theorem about disjointness of topologically weakly mixing systems with (minimal) equicontinuous systems (see Theorem II.3 in [24]) to deduce the following:

Corollary 3.23. *The product system $(X_\eta \times X_\eta, S \times S)$ is not transitive.*

Remark 3.24. It follows that whenever (X_η, S) is proximal, we have:

- (X_η, S) is transitive with trivial maximal equicontinuous factor,
- $(X_\eta \times X_\eta, S \times S)$ has trivial equicontinuous factor, but it is not transitive.

It is well-known that when the product system $(X \times X, T \times T)$ is transitive then all non-zero powers T^m are transitive. Hence, Corollary 3.23 can also be deduced from the following result.

Proposition 3.25. *A $\mathscr{B}$-free system (X_η, S) is not totally transitive. More precisely, if $c := \min \mathscr{B}$, then (X_η, S^c) is not transitive.*

Proof. Since the set $\{1, \dots, c-1\}$ consists of $\mathscr{B}$-free numbers, we have $C := 1^{c-1}0 = \eta[1, c]$. Moreover, $\{j \in \mathbb{Z} : C = \eta[j, j+c-1]\} \subset c\mathbb{Z} + 1$ (otherwise, we obtain

c consecutive integers from which none is divisible by c). It follows that for each $y \in X_\eta$, there exists $0 \leq s < c$ such that $\{j \in \mathbb{Z} : C = y[j, j+c-1] \subset c\mathbb{Z} + s$, whence S^c cannot be transitive. $\square$

4. Tautness

4.1. η is quasi-generic for ν_η (proof of Proposition E).

Theorem 4.1. *Given $\mathscr{B} \subset \mathbb{N}$, let (N_k) be such that*

$$\underline{d}(\mathcal{M}_\mathscr{B}) = \lim_{k\to\infty} \frac{1}{N_k} |[1, N_k] \cap \mathcal{M}_\mathscr{B}|.$$

Then η is quasi-generic for ν_η along (N_k). In particular, if $\mathscr{B}$ is Besicovitch, then η is generic for ν_η.

Proof. If $\mathscr{B}$ is finite, then the result follows from Proposition 4.25. Hence, we may assume that $\mathscr{B}$ is infinite. According to [1], by a pure measure theory argument, we only need to prove that

$$\frac{1}{N_k} \sum_{n \leq N_k} \mathbb{1}_{\varphi^{-1}(A)}(T^n \underline{0}) \to \mathbb{P}(\varphi^{-1}(A)) \quad \text{as } k \to \infty,$$

for each $A = \{x \in \{0,1\}^\mathbb{Z} : x(j_s) = 0, s = 1, \ldots, r\}$, where $j_1 < \ldots < j_r$ and $r \geq 1$. Recall that

$$C = \{(g_1, g_2, \ldots) \in G : g_k \not\equiv 0 \bmod b_k \quad \text{for } k \geq 1\}$$

and, for $K \geq 1$, define

$$C_K := \{(g_1, g_2, \ldots) \in G : g_k \not\equiv 0 \bmod b_k \quad \text{for } 1 \leq k \leq K\}.$$

Then each C_K is clopen and $C_K \searrow C$ when $K \to \infty$. We have $\varphi^{-1}(A) = \bigcap_{s=1}^r T^{-j_s} C^c$, whence

$$\bigcap_{s=1}^r T^{-j_s} C_K^c \subset \varphi^{-1}(A) \subset \bigcap_{s=1}^r T^{-j_s} C_K^c \cup \bigcup_{s=1}^r T^{-j_s}(C^c \setminus C_K^c). \tag{31}$$

Moreover, since $\mathbb{1}_{\bigcap_{s=1}^r T^{-j_s} C_K^c}$ is continuous, by the unique ergodicity of T in Example 2.3, we have

$$\frac{1}{N_k} \sum_{n \leq N_k} \mathbb{1}_{\bigcap_{s=1}^r T^{-j_s} C_K^c}(T^n \underline{0}) \to \mathbb{P}(\bigcap_{s=1}^r T^{-j_s} C_K^c) \tag{32}$$

and, given $\varepsilon > 0$, for K sufficiently large, we have

$$\mathbb{P}(\bigcap_{s=1}^r T^{-j_s} C_K^c) \geq \mathbb{P}(\bigcap_{s=1}^r T^{-j_s} C^c) - \varepsilon. \tag{33}$$

Notice that

$$T^n \underline{0} \in C^c \setminus C_K^c \iff n \in \mathcal{M}_\mathscr{B} \setminus \mathcal{M}_{\{b_1, \ldots, b_K\}}.$$

By Theorem 2.23, if K is large enough, then

$$d(\mathcal{M}_{\{b_1, \ldots, b_K\}}) \geq \underline{d}(\mathcal{M}_\mathscr{B}) - \varepsilon.$$

Therefore, and by the choice of (N_k),

$$\limsup_{k\to\infty} \frac{1}{N_k} \sum_{n \leq N_k} \mathbb{1}_{\bigcup_{s=1}^r (C^c \setminus C_K^c)}(T^n \underline{0}) \leq \varepsilon. \tag{34}$$

Putting together (31), (32), (33), and (34) completes the proof. $\square$

Remark 4.2. Notice that by Theorem 4.1, we have

$$\nu_\eta(\{x \in \{0,1\}^{\mathbb{Z}} : x(0) = 1\}) = \lim_{k\to\infty} \frac{1}{N_k} |\{1 \le n \le N_k : \eta(n) = 1\}| = \overline{d}(\mathcal{F}_{\mathscr{B}}).$$

It follows immediately that $\mathscr{B}$ is Behrend if and only if $\nu_\eta = \delta_{\dots 0.00\dots}$.

By Theorem 4.1, if a block does not appear in η, then the Mirsky measure of the corresponding cylinder set is zero. As a consequence, we obtain the following:

Corollary 4.3. *We have $\nu_\eta(X_\eta) = 1$.*

Proof of Proposition E. The assertion follows from Theorem 4.1 and Corollary 4.3. □

Remark 4.4. As an immediate consequence of Theorem 4.1, we have

$$\overline{d}(\mathcal{F}_{\mathscr{B}}) > 0 \iff \nu_\eta \neq \delta_{\dots 0.00\dots}.$$

In particular, if $\mathscr{B} \neq \{1\}$ is taut, then $\nu_\eta \neq \delta_{\dots 0.00\dots}$, as such a $\mathscr{B}$ is not Behrend in view of Proposition 2.30.

4.2. **Tautness and Mirsky measures (Theorem C – first steps).** In this section our main goal is to prove the following:

Theorem 4.5. *For each $\mathscr{B} \subset \mathbb{N}$, there exists a taut set $\mathscr{B}' \subset \mathbb{N}$ such that $\mathcal{F}_{\mathscr{B}'} \subset \mathcal{F}_{\mathscr{B}}$ and $\nu_\eta = \nu_{\eta'}$.*

We will see later that, in fact, the equality $\nu_\eta = \nu_{\eta'}$ determines $\mathscr{B}'$; cf. Corollary 4.36. In the course of the construction of $\mathscr{B}'$ and to prove that $\mathscr{B}'$ satisfies the required properties, we will use the following general lemmas:

Lemma 4.6. *Suppose that $\mathscr{B} \subset \mathbb{N}$ is primitive. Then $\mathscr{B}$ is taut if and only if there exists a cofinite subset of $\mathscr{B}$ that is taut.*

Proof. Let $\mathscr{B} \subset \mathbb{N}$ be primitive. It suffices to show that if $\mathscr{B} \setminus \{b\}$ is taut for some $b \in \mathscr{B}$, then $\mathscr{B}$ is taut. Suppose that $\mathscr{B}$ fails to be taut. By Proposition 2.30, there exist $c \in \mathbb{N}$ and a Behrend set $\mathscr{A}$ such that $c\mathscr{A} \subset \mathscr{B}$. Then $c\mathscr{A}' \subset \mathscr{B} \setminus \{b\}$, where $\mathscr{A}' = \mathscr{A} \setminus \{b/c\}$ and $\mathscr{A}'$ is Behrend by Proposition 2.28. Applying again Proposition 2.30, we conclude that $\mathscr{B} \setminus \{b\}$ also fails to be taut. □

Lemma 4.7. *Suppose that $\mathscr{B} \subset \mathbb{N}$ is primitive. If $\mathscr{B}$ is not taut, then, for some $c \in \mathbb{N}$, the set*

$$\mathscr{A}_c := \left\{ \frac{b}{c} : b \in \mathscr{B} \text{ and } c \mid b \right\} \tag{35}$$

is Behrend.

Proof. Clearly, for any $c \in \mathbb{N}$, we have $c\mathscr{A}_c \subset \mathscr{B}$, where $\mathscr{A}_c$ (possibly empty) is as in (35). By Proposition 2.30, we have

$$C := \{c \in \mathbb{N} : c\mathscr{A}'_c \subset \mathscr{B} \quad \text{for some Behrend set } \mathscr{A}'_c\} \neq \emptyset$$

and, for any $c \in C$, we have $\mathscr{A}'_c \subset \mathscr{A}_c$, whence $\mathscr{A}_c$ is Behrend. □

Lemma 4.8. *Let $\mathscr{B}_1, \mathscr{B}_2 \subset \mathbb{N}$ be disjoint and such that $\mathscr{B} := \mathscr{B}_1 \cup \mathscr{B}_2$ is primitive. Then $\mathscr{B}$ is taut if and only if both $\mathscr{B}_1$ and $\mathscr{B}_2$ are taut.*

Proof. If $\mathscr{B}_i$ is not taut for some $i \in \{1,2\}$, then, by Proposition 2.30, there exist $c \in \mathbb{N}$ and a Behrend set $\mathscr{A}$ such that $c\mathscr{A} \subset \mathscr{B}_i \subset \mathscr{B}$. Applying again Proposition 2.30, we deduce that $\mathscr{B}$ also fails to be taut. On the other hand, if $\mathscr{B}$ is not taut, then, by Proposition 2.30, there exist $c \in \mathbb{N}$ and a Behrend set $\mathscr{A}$ such that $c\mathscr{A} \subset \mathscr{B}$. Let

$$\mathscr{A}_i := \left\{ \frac{b}{c} : b \in \mathscr{B}_i \right\}, \qquad i = 1,2.$$

Clearly, $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$. Moreover, by Proposition 2.28, $\mathscr{A}_i$ is Behrend for some $i \in \{1,2\}$. We obtain $c\mathscr{A}_i \subset \mathscr{B}_i$ for this i and, by Proposition 2.30, we conclude that $\mathscr{B}_i$ fails to be taut. □

Construction. We may assume without loss of generality that $\mathscr{B}$ is primitive (cf. Remark 2.20).

Step 1. If $1 \in \mathscr{B}$, we set $\mathscr{B}' := \{1\}$.

Step 2. Suppose now that $1 \notin \mathscr{B}$ and suppose that $\mathscr{B}$ is not taut. Let $c_1 \in \mathbb{N}$ be the smallest natural number such that

$$\mathscr{A}^1 := \left\{ \frac{b}{c_1} : b \in \mathscr{B} \text{ and } c_1 \mid b \right\}$$

is Behrend (such c_1 exists by Lemma 4.7). By the definition of $\mathscr{A}^1$, we have $\mathscr{B} \setminus c_1\mathscr{A}^1 = \mathscr{B} \setminus c_1\mathbb{Z}$. Let

$$\mathscr{B}^1 := (\mathscr{B} \setminus c_1\mathbb{Z}) \cup \{c_1\} = (\mathscr{B} \setminus c_1\mathscr{A}^1) \cup \{c_1\}. \tag{36}$$

We claim that $\mathscr{B}^1$ is primitive. Indeed, if this is not the case, then, by the primitivity of $\mathscr{B}$, for some $b \in \mathscr{B} \setminus c_1\mathbb{Z}$, we have $b \mid c_1$ or $c_1 \mid b$. The latter is impossible for $b \notin c_1\mathbb{Z}$, whence $b \mid c_1$. This implies $b \mid c_1a_1 \in \mathscr{B}$ for any $a_1 \in \mathscr{A}^1$. By the primitivity of $\mathscr{B}$, it follows that $b = c_1a_1$ for infinitely many a_1, which is impossible, and we obtain that $\mathscr{B}^1$ is indeed primitive. If $\mathscr{B}^1$ is taut, we stop the procedure here and set $\mathscr{B}' := \mathscr{B}^1$. Otherwise, we continue inductively.

Step n. Suppose that from the previous step we have

$$\begin{aligned}\mathscr{B}^{n-1} &= (\mathscr{B} \setminus (c_1\mathbb{Z} \cup \ldots \cup c_{n-1}\mathbb{Z})) \cup \{c_1, \ldots, c_{n-1}\}\\ &= (\mathscr{B} \setminus (c_1\mathscr{A}^1 \cup \ldots \cup c_{n-1}\mathscr{A}^{n-1})) \cup \{c_1, \ldots, c_{n-1}\}\end{aligned}$$

that is primitive but not taut. Then, by Lemma 4.6, $\mathscr{B} \setminus (c_1\mathbb{Z} \cup \ldots \cup c_{n-1}\mathbb{Z})$ is not taut. Let $c_n \in \mathbb{N}$ be the smallest number such that

$$\mathscr{A}^n := \left\{ \frac{b}{c_n} : b \in \mathscr{B} \setminus (c_1\mathbb{Z} \cup \ldots \cup c_{n-1}\mathbb{Z}) \quad \text{and} \quad c_n \mid b \right\}$$

is Behrend (such c_n exists by Lemma 4.7). Note that (by the definition of $c_1, \ldots, c_n$)

$$c_n > c_{n-1} \quad \text{and} \quad c_n \notin c_1\mathbb{Z} \cup \cdots \cup c_{n-1}\mathbb{Z}. \tag{37}$$

Moreover,

$$\mathscr{B} \setminus (c_1\mathbb{Z} \cup \ldots \cup c_n\mathbb{Z}) = \mathscr{B} \setminus (c_1\mathscr{A}^1 \cup \ldots \cup c_n\mathscr{A}^n). \tag{38}$$

Let

$$\begin{aligned}\mathscr{B}^n &:= \mathscr{B} \setminus (c_1\mathbb{Z} \cup \ldots \cup c_n\mathbb{Z}) \cup \{c_1, \ldots, c_n\}\\ &= \mathscr{B} \setminus (c_1\mathscr{A}^1 \cup \ldots \cup c_n\mathscr{A}^n) \cup \{c_1, \ldots, c_n\}.\end{aligned} \tag{39}$$

We claim that $\mathscr{B}^n$ is primitive. Indeed, if this is not the case, then, by the primitivity of $\mathscr{B}^{n-1}$ and (37), for some $b \in \mathscr{B} \setminus (c_1\mathbb{Z} \cup \ldots \cup c_n\mathbb{Z})$, we have $b \mid c_n$ or $c_n \mid b$. The latter is impossible for $b \notin c_n\mathbb{Z}$, whence $b \mid c_n$. This implies $b \mid c_n a_n$ for any $a_n \in \mathscr{A}^n$. By the primitivity of $\mathscr{B}$, it follows that $b = c_n a_n$ for infinitely many a_n, which is impossible, and we obtain that $\mathscr{B}^n$ is indeed primitive.

If $\mathscr{B}^n$ is taut, we stop the procedure and set $\mathscr{B}' := \mathscr{B}^n$.

Step ∞. If $\mathscr{B}^n$ is not taut for all $n \geq 1$, we set

$$\mathscr{B}' := (\mathscr{B} \setminus \bigcup_{n\geq 1} c_n\mathbb{Z}) \cup \{c_n : n \geq 1\} = (\mathscr{B} \setminus \bigcup_{n\geq 1} c_n\mathscr{A}^n) \cup \{c_n : n \geq 1\}, \tag{40}$$

where the above equality follows from (38). Note that for any $b, b' \in \mathscr{B}'$ there exists $n \geq 1$ with $b, b' \in \mathscr{B}^n$. Therefore, by the primitivity of $\mathscr{B}^n$, $n \geq 1$, also $\mathscr{B}'$ is primitive.

From now on, for the sake of readability, we will restrict ourselves to the case when $\mathscr{B}'$ is defined by (40). When $\mathscr{B} = \mathscr{B}^n$ for some $n \geq 1$, the proof goes along the same lines, with some simplifications.

Remark 4.9. It follows from (38) that

$$\mathscr{B} = (\mathscr{B} \setminus \bigcup_{n\geq 1} c_n\mathbb{Z}) \cup \bigcup_{n\geq 1} c_n\mathscr{A}^n.$$

Therefore, $\mathcal{M}_{\mathscr{B}} \subset \mathcal{M}_{\mathscr{B}'}$. Moreover, $\eta' \leq \eta$ and $\widetilde{X}_{\eta'} \subset \widetilde{X}_\eta$.

Lemma 4.10. *The set* $\mathscr{B}'$ *is taut.*

Proof. Recall that $\mathscr{B}'$ is primitive. In view of Lemma 4.8, it suffices to show that $\mathscr{B} \setminus \bigcup_{n\geq 1} c_n\mathbb{Z}$ and $\{c_n : n \geq 1\}$ are taut. Suppose that $\mathscr{B} \setminus \bigcup_{n\geq 1} c_n\mathbb{Z}$ fails to be taut. Then, by Proposition 2.30, for some $c \in \mathbb{N}$ and a Behrend set $\mathscr{A}$, we have $c\mathscr{A} \subset \mathscr{B} \setminus \bigcup_{n\geq 1} c_n\mathbb{Z}$. Therefore, for any $n \geq 1$, we have $c\mathscr{A} \subset \mathscr{B} \setminus (c_1\mathbb{Z} \cup \ldots \cup c_n\mathbb{Z})$. By the definition of c_{n+1}, we obtain $c \geq c_{n+1}$. Since $n \geq 1$ is arbitrary and the sequence $(c_n)_{n\geq 1}$ is strictly increasing, this yields a contradiction.

Suppose now that $\mathscr{C} := \{c_n : n \geq 1\}$ fails to be taut. Then, for some $n_0 \geq 1$, we have $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{C}}) = \boldsymbol{\delta}(\mathcal{M}_{\mathscr{C}\setminus\{c_{n_0}\}})$. Note that by (37), we have $c_{n_0} \notin \bigcup_{n\neq n_0} c_n\mathbb{Z}$.

Therefore, by Proposition 2.29, the set $\{c_n/\gcd(c_n, c_{n_0}) : n \neq n_0\}$ is Behrend. We have

$$\left\{\frac{c_n}{\gcd(c_n, c_{n_0})} : n \neq n_0\right\} = \bigcup_{d_{n_0}\mid c_{n_0}} \left\{\frac{c_n}{d_{n_0}} : n \neq n_0, \gcd(c_n, c_{n_0}) = d_{n_0}\right\}.$$

It follows from Proposition 2.28 that at least one of the sets in the union above is Behrend. Denote this set by $\mathscr{A}(d_{n_0})$ and, for $m > n_0$, define

$$\mathscr{A}_m := \left\{\frac{c_n}{d_{n_0}} : n \geq m, \gcd(c_n, c_{n_0}) = d_{n_0}\right\}.$$

Since each $\mathscr{A}_m$ differs from $\mathscr{A}_{d_{n_0}}$ by at most finitely many elements, it follows from Proposition 2.28 that $\mathscr{A}_m$ is Behrend for $m > n_0$. Let

$$\mathscr{A}'_m := \bigcup_{\substack{n\geq m \\ \gcd(c_n, c_{n_0}) = d_{n_0}}} \frac{c_n}{d_{n_0}}\mathscr{A}^n.$$

Using Theorem 2.23, Corollary 2.25, and the fact that $\mathscr{A}^n$ is Behrend (the sets $\mathscr{A}^n$ are the same as in the construction of $\mathscr{B}'$), we obtain

$$\begin{aligned}\boldsymbol{\delta}(\mathcal{M}_{\mathscr{A}'_m}) &= \lim_{K\to\infty} \boldsymbol{\delta}\Big(\mathcal{M}_{\bigcup_{\substack{m\leq n\leq K\\ \gcd(c_n,c_{n_0})=d_{n_0}}} \frac{c_n}{d_{n_0}}\mathscr{A}^n}\Big)\\ &= \lim_{K\to\infty} \boldsymbol{\delta}\Big(\mathcal{M}_{\big\{\frac{c_n}{\gcd(c_n,c_{n_0})}:m\leq n\leq K, \gcd(c_n,c_{n_0})=d_{n_0}\big\}}\Big) = \boldsymbol{\delta}(\mathcal{M}_{\mathscr{A}_m}) = 1,\end{aligned}$$

since $\mathscr{A}_m$ is Behrend. By the definition of $\mathscr{A}'_m$ and $\mathscr{A}^n$, $n \geq m > n_0$, it follows that

$$d_{n_0}\mathscr{A}'_m \subset \bigcup_{n\geq m} c_n\mathscr{A}^n \subset \mathscr{B} \setminus \bigcup_{n<m} c_n\mathbb{Z}.$$

Moreover, by the definition of c_m, it follows that $d_{n_0} \geq c_m$, which is impossible as $m \geq n_0$ is arbitrary. This completes the proof. □

Lemma 4.11. *We have $\nu_\eta = \nu_{\eta'}$.*

Proof. We will show first that $\underline{d}(\mathcal{M}_{\mathscr{B}}) = \underline{d}(\mathcal{M}_{\mathscr{B}'})$. Let

$$\mathscr{A}_1 := \mathscr{B} \setminus \bigcup_{n\geq 1} c_n\mathbb{Z},\ \ \mathscr{A}_k := c_{k-1}\mathscr{A}^{k-1} \quad \text{for } k \geq 2,$$
$$\mathscr{A}'_1 := \mathscr{B} \setminus \bigcup_{n\geq 1} c_n\mathbb{Z},\ \ \mathscr{A}'_k := \{c_{k-1}\} \quad \text{for } k \geq 2.$$

Then

$$\mathscr{B} = \bigcup_{n\geq 1} \mathscr{A}_n \quad \text{and} \quad \mathscr{B}' = \bigcup_{n\geq 1} \mathscr{A}'_n. \tag{41}$$

Since each of the sets $\mathscr{A}^k$, $k \geq 1$, is Behrend, we have

$$\boldsymbol{\delta}(\mathcal{M}_{\mathscr{A}_1\cup\ldots\cup\mathscr{A}_K}) = \boldsymbol{\delta}(\mathcal{M}_{\mathscr{A}'_1\cup\ldots\cup\mathscr{A}'_K}) \quad \text{for each } K \geq 1. \tag{42}$$

It follows from (41), (42), and by Corollary 2.25 that

$$\begin{aligned}\underline{d}(\mathcal{M}_{\mathscr{B}}) = \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}}) &= \lim_{K\to\infty} \boldsymbol{\delta}(\mathcal{M}_{\mathscr{A}_1\cup\ldots\cup\mathscr{A}_K})\\ &= \lim_{K\to\infty} \boldsymbol{\delta}(\mathcal{M}_{\mathscr{A}'_1\cup\ldots\cup\mathscr{A}'_K}) = \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}'}) = \underline{d}(\mathcal{M}_{\mathscr{B}'}).\end{aligned}$$

Moreover, since $\mathcal{M}_{\mathscr{B}} \subset \mathcal{M}_{\mathscr{B}'}$, it follows that whenever $(N_k)_{k\geq 1}$ satisfies

$$\lim_{k\to\infty} \frac{1}{N_k}|\mathcal{M}_{\mathscr{B}'} \cap [1, N_k]| = \underline{d}(\mathcal{M}_{\mathscr{B}'}),$$

then

$$\lim_{k\to\infty} \frac{1}{N_k}|\mathcal{M}_{\mathscr{B}} \cap [1, N_k]| = \underline{d}(\mathcal{M}_{\mathscr{B}}).$$

Since η and η' differ, along $(N_k)_{k\geq 1}$, on a subset of zero density, it follows from Theorem 4.1 that η and η' are generic along $(N_k)_{k\geq 1}$ for the same measure, i.e., $\nu_\eta = \nu_{\eta'}$. □

Theorem 4.5 follows from Lemmas 4.10 and 4.11.

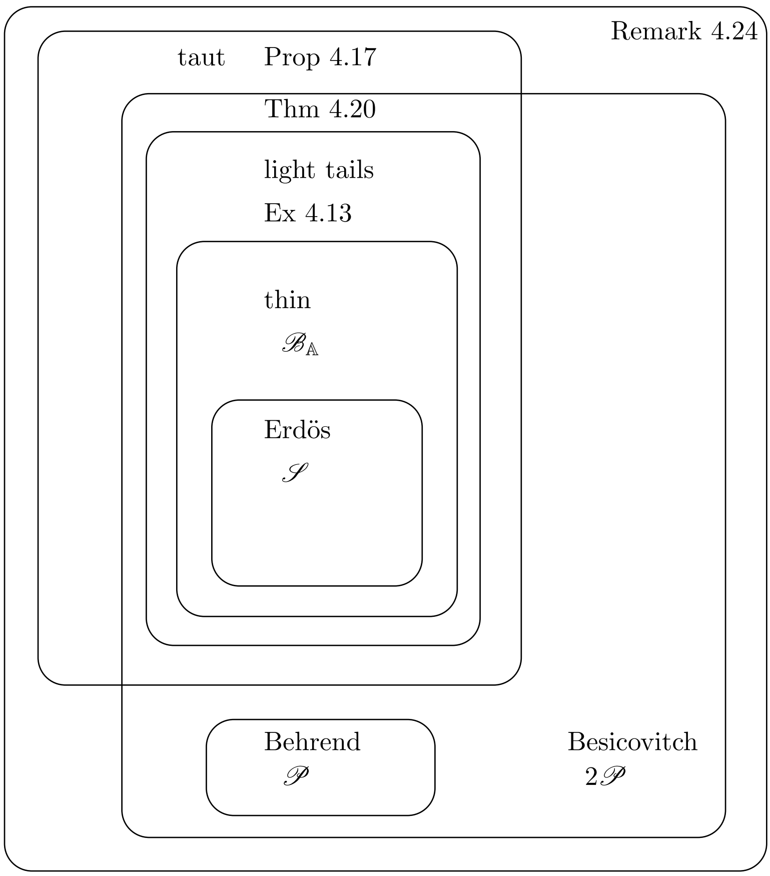
4.3. **Different classes of $\mathscr{B}$-free numbers.** In Section 2.6, we defined several classes of $\mathscr{B}$-free numbers and described some basic relations between them. In particular, we showed that

$$\mathscr{B} \text{ is thin } \Rightarrow \mathscr{B} \text{ has light tails}$$

and

$$\mathscr{B} \text{ has light tails (and is primitive) } \Rightarrow \mathscr{B} \text{ is taut.}$$

We will now continue this discussion. In particular, we will show that the implications converse to the above do not hold. The relations between various classes of $\mathscr{B}$-free numbers for primitive $\mathscr{B} \subset \mathbb{N}$ are summarized in this diagram (all depicted regions are non-empty):



Remark 4.12. Let $\mathscr{B}, \mathscr{B}' \subset \mathbb{N}$ be such that:

- for each $b' \in \mathscr{B}'$ there exists $b \in \mathscr{B}$ such that $b \mid b'$,
- for each $b \in \mathscr{B}$ there exists $b' \in \mathscr{B}'$ such that $b \mid b'$.

Then, clearly, $\mathcal{F}_{\mathscr{B}} \subset \mathcal{F}_{\mathscr{B}'}$. Suppose additionally that $\mathscr{B}$ has light tails and for each $b \in \mathscr{B}$ the set $\{b' \in \mathscr{B}' : b \mid b'\}$ is finite. Then, given $K \geq 1$, there exists $N_K \geq 1$ such that

$$\text{if } b \in \mathscr{B}, b' \in \mathscr{B}',\ b \mid b', \text{ and } b' > N_K, \text{ then } b > K.$$

It follows that

$$\bigcup_{b'>N_K} b'\mathbb{Z} \subset \bigcup_{b>K} b\mathbb{Z}.$$

Therefore, if $\mathscr{B}$ has light tails, then also $\mathscr{B}'$ has light tails. In particular, this applies when $\mathscr{B}$ is thin (see Example 4.13 below).

Example 4.13 ($\mathscr{B}$ has light tails $\not\Rightarrow$ $\mathscr{B}$ is thin)**.** Let $(q_n)_{n\geq 1}$ be a thin sequence of primes, i.e., $\sum_{n\geq 1}\frac{1}{q_n} < +\infty$. We arrange the remaining primes into countably many finite pairwise disjoint sets of the form $\{p_{n,1}, p_{n,2}, \dots, p_{n,k_n}\}$ such that

$$\frac{1}{p_{n,1}} + \frac{1}{p_{n,2}} + \ldots + \frac{1}{p_{n,k_n}} \geq q_n \text{ for any } n.$$

Let $\mathscr{B} := \{q_n p_{n,j} : n \in \mathbb{N}, j = 1, \dots, k_n\}$. By Remark 4.12, $\mathscr{B}$ has light tails. We will show now that $\mathscr{B}$ is not thin. Indeed,

$$\sum_{b\in\mathscr{B}} \frac{1}{b} = \sum_{n\geq 1}\left(\frac{1}{q_n p_{n,1}} + \frac{1}{q_n p_{n,2}} + \ldots + \frac{1}{q_n p_{n,k_n}}\right) \geq \sum_{n\geq 1} 1 = +\infty.$$

Remark 4.14. Notice that $\mathscr{B}$ from Example 4.13 is not coprime ($q_n p_{n,1}$ and $q_n p_{n,2}$ are clearly not coprime). This is not surprising – if $\mathscr{B}$ is coprime, then it has light tails if and only if it is thin. (Indeed, in the coprime case the density of $\mathcal{F}_{\mathscr{B}}$ exists and it is equal to $\prod_{k\geq 1}(1-\frac{1}{b_k})$; see, e.g., [28]). Note however that $\mathscr{B}$ above is primitive.

Remark 4.15. Let $\mathscr{B}$ be as in Example 4.13. It follows from Remark 4.4 that $\nu_\eta \neq \delta_{\ldots 0.00\ldots}$.

Example 4.16. Let $\mathscr{B} := 2\mathscr{P} \cup (\mathscr{S} \setminus \{2\})$. Then, by Corollary 2.22, $\mathscr{B}$ is Besicovitch but, in view of Propositions 2.28 and 2.30, it is neither taut nor Behrend. Moreover, by Theorem 3.7, (X_η, S) is proximal. We claim that it is not hereditary. Indeed, we have $\eta[-3,3] = 1110111$ and $\eta(2n) = 0$ for each $|n| \neq 1$. Hence the block 0110111 does not appear in η and the claim follows.

Proposition 4.17. *There exists a taut $\mathscr{B}$ which is not Besicovitch.*

In the proof, we will use the following lemma:

Lemma 4.18. *Let $\mathscr{B} \subset \mathbb{N}$ and let $\mathscr{B}'$ be as in the proof of Theorem* 4.5*. Then $\mathscr{B}$ is Besicovitch whenever $\mathscr{B}'$ is Besicovitch.*

Proof. Recall that in the notation from the proof of Theorem 4.5, we have

$$\mathscr{B} = (\mathscr{B} \setminus \bigcup_{n\geq 1} c_n\mathscr{A}^n) \cup \bigcup_{n\geq 1} c_n\mathscr{A}^n$$

and

$$\mathscr{B}' = (\mathscr{B} \setminus \bigcup_{n\geq 1} c_n\mathscr{A}^n) \cup \{c_n : n \geq 1\}.$$

It follows from Theorem 2.23, by the fact that the sets $\mathscr{A}^n$ for $n \geq 1$ are Behrend, and by Corollary 2.25 that we have

$$\underline{d}(\mathcal{M}_{\mathscr{B}}) = \lim_{K\to\infty} \boldsymbol{\delta}(\mathcal{M}_{(\mathscr{B}\setminus\bigcup_{n\geq 1} c_n\mathscr{A}^n)\cup\bigcup_{n\leq K} c_n\mathscr{A}_n})$$
$$= \lim_{K\to\infty} \boldsymbol{\delta}(\mathcal{M}_{(\mathscr{B}\setminus\bigcup_{n\geq 1} c_n\mathscr{A}^n)\cup\{c_n : n\leq K\}}) = \underline{d}(\mathcal{M}_{\mathscr{B}'}).$$

Therefore, if $\mathscr{B}'$ is Besicovitch, we obtain $\underline{d}(\mathcal{M}_{\mathscr{B}}) = d(\mathcal{M}_{\mathscr{B}'})$. On the other hand, by Theorem 4.5, we have $\mathcal{M}_{\mathscr{B}} \subset \mathcal{M}_{\mathscr{B}'}$, and it follows that $\overline{d}(\mathcal{M}_{\mathscr{B}}) \leq d(\mathcal{M}_{\mathscr{B}'})$. We obtain $\overline{d}(\mathcal{M}_{\mathscr{B}}) \leq \underline{d}(\mathcal{M}_{\mathscr{B}})$ and conclude that also $\mathscr{B}$ must be Besicovitch. □

Proof of Proposition 4.17. Consider $\mathscr{B}$ that fails to be Besicovitch. By Lemma 4.18, the associated set $\mathscr{B}'$ defined as in (40) also fails to be Besicovitch. Moreover, in view of Lemma 4.10, $\mathscr{B}'$ is taut. □

Since, as noted in Section 2.6, each $\mathscr{B}$ with light tails is automatically Besicovitch, we have the following immediate consequence of Proposition 4.17:

Corollary 4.19. *There is a taut $\mathscr{B}$ which does not have light tails.*

The rest of this section is devoted to the proof of the following more subtle result:

Theorem 4.20. *There is a taut and Besicovitch $\mathscr{B}$ which does not have light tails.*

To prove Theorem 4.20, we will need three lemmas.

Lemma 4.21. *Let $\mathscr{R}$ be a union of finitely many arithmetic progressions with differences $d_1, ..., d_r$. Then $\mathscr{R}$ is a union of finitely many pairwise disjoint arithmetic progressions with differences $d' = \operatorname{lcm}(d_1, \dots, d_r)$.*

Proof. Let $\mathscr{R} = \bigcup_{i=1}^{r}(d_i\mathbb{Z} + a_i)$ and $\{a'_1, \dots, a'_s\} = \mathscr{R} \cap [0, d')$. Clearly, $\mathscr{R}' = \bigcup_{i=1}^{s}(d'\mathbb{Z} + a'_i)$ is a union of finitely many arithmetic sequences with differences d', and it is easy to see that $\mathscr{R} = \mathscr{R}'$. □

Lemma 4.22. *Assume that $B, C \subset \mathbb{N}$ are thin, with $\gcd(b, c) = 1$ for any $b \in B$, $c \in C$. Let $BC := \{bc : b \in B, c \in C\}$. Then*

$$d(\mathcal{M}_{BC}) = d(\mathcal{M}_B \cap \mathcal{M}_C) = d(\mathcal{M}_B)d(\mathcal{M}_C). \tag{43}$$

Proof. Since $\operatorname{lcm}(b, c) = bc$ for any $b \in B$ and $c \in C$, it follows that

$$\mathcal{M}_{BC} = \mathcal{M}_B \cap \mathcal{M}_C.$$

It remains to show the right hand side equality in (43), and it is enough to show its validity for finite sets B, C (since BC is thin, it is Besicovitch, and we can use Theorem 2.23 to pass to a limit).

Let $B = \{b_1, \dots, b_n\}$, $C = \{c_1, \dots, c_m\}$, and set $b' := \operatorname{lcm}(b_1, \dots, b_n)$, $c' := \operatorname{lcm}(c_1, \dots, c_m)$. Then, by Lemma 4.21,

$$\mathcal{M}_B = \bigcup_{r \in R}(b'\mathbb{Z} + r) \quad \text{and} \quad \mathcal{M}_C = \bigcup_{s \in S}(c'\mathbb{Z} + s)$$

for some finite sets $R, S \subset \mathbb{N}$. Note that

$$d(\mathcal{M}_B) = \frac{|R|}{b'} \quad \text{and} \quad d(\mathcal{M}_C) = \frac{|S|}{c'}. \tag{44}$$

Since $\gcd(b', c') = 1$, we get

$$d((b'\mathbb{Z} + r) \cap (c'\mathbb{Z} + s)) = \frac{1}{b'c'} \tag{45}$$

for any $r \in R$, $s \in S$. Hence, by (45) and (44), we obtain

$$d(\mathcal{M}_B \cap \mathcal{M}_C) = d(\bigcup_{(r,s) \in R \times S}(b'\mathbb{Z} + r) \cap (c'\mathbb{Z} + s)) = \frac{|R \times S|}{b'c'} = d(\mathcal{M}_B)d(\mathcal{M}_C),$$

and the result follows. □

Lemma 4.23. *Let $P \subset \mathbb{N}$ be pairwise coprime with $\sum_{p\in P} 1/p = +\infty$. For any $0 < \beta < 1$ there exists a finite (resp. infinite and thin) set $P' \subset P$ such that*

$$\beta < d(\mathcal{M}_{P'}) < 1.$$

Proof. For $n \geq 1$, let $P_n := \{p \in P : p \leq n\}$. By Theorem 2.23, we have

$$\lim_{n\to\infty} d(\mathcal{M}_{P_n}) = d(\mathcal{M}_P) = 1.$$

Therefore, for $n \geq 1$ large enough, we have $\beta < d(\mathcal{M}_{P_n}) < 1$ and we can take $P' := P_n$ to obtain a finite set satisfying the assertion. To obtain an infinite set P', let the sequence $(p_m)_{m\geq 1} \subset P$ be such that $d(\mathcal{M}_{P_n}) + \sum_{m\geq 1} 1/p_m < 1$ and take $P' := P_n \cup \{p_m : m \geq 1\}$. □

Construction. Fix $0 < \gamma < 1$ and choose a sequence $(\gamma_k)_{k\geq 1} \subset (0, 1)$ such that $\prod_{k\geq 1} \gamma_k = \gamma$ (for instance, $\gamma_k = \gamma^{1/2^k}$). Applying Lemma 4.23, we construct a collection $\{B_k, C_k : k \in \mathbb{N}\}$ of pairwise disjoint thin sets of primes such that

$$\gamma_k < d(\mathcal{M}_{B_k}) < 1 \quad \text{for } k \geq 1 \tag{46}$$

and

$$1 - \frac{1}{k} < d(\mathcal{M}_{C_k}) \quad \text{for } k \geq 1. \tag{47}$$

Let

$$\mathscr{B} := B_1C_1 \cup B_1B_2C_2 \cup \ldots \cup B_1 \ldots B_nC_n \cup \ldots. \tag{48}$$

Notice that $B_1C_1 \cup B_1B_2C_2 \cup \ldots \cup B_1 \ldots B_nC_n$ is thin for any $n \in \mathbb{N}$.

Proof of Theorem 4.20. Let $\mathscr{B}$ be defined as in (48). We claim the following:

(a) $\mathscr{B}$ is Besicovitch,
(b) $\mathscr{B}$ does not have light tails,
(c) $\mathscr{B}$ is taut.

We will first prove (a). For $k \geq m$, we have

$$\mathcal{M}_{B_1\ldots B_kC_k} \subset \mathcal{M}_{B_1\ldots B_k} \subset \mathcal{M}_{B_1\ldots B_m}.$$

Thus,

$$\overline{d}(\mathcal{M}_{\bigcup_{k\geq m+1} B_1\ldots B_kC_k} \setminus \mathcal{M}_{B_1\ldots B_mC_m}) \leq d(\mathcal{M}_{B_1\ldots B_m} \setminus \mathcal{M}_{B_1\ldots B_mC_m}). \tag{49}$$

By Lemma 4.22 and (47), we get

$$d(\mathcal{M}_{B_1\ldots B_mC_m}) = d(\mathcal{M}_{B_1\ldots B_m})d(\mathcal{M}_{C_m}) \geq d(\mathcal{M}_{B_1\ldots B_m})(1 - 1/m),$$

whence

$$d(\mathcal{M}_{B_1\ldots B_m} \setminus \mathcal{M}_{B_1\ldots B_mC_m}) \leq (1/m)d(\mathcal{M}_{B_1\ldots B_m}) \leq 1/m. \tag{50}$$

Using (49) and (50), we obtain

$$\overline{d}(\mathcal{M}_{\bigcup_{i=m+1}^{\infty} B_1\ldots B_iC_i} \setminus \mathcal{M}_{B_1\ldots B_mC_m}) \leq 1/m.$$

In view of Remark 2.26, this implies that $\mathscr{B}$ is Besicovitch.

We will now show (b). By Lemma 4.22, (46), and (47), we have

$$d(\mathcal{M}_{B_1\ldots B_mC_m}) \geq \gamma_1 \ldots \gamma_m(1 - 1/m) \to \gamma > 0$$

as $m \to +\infty$, which yields (b).

It remains to prove (c). Suppose that $\mathscr{B}$ is not taut. Since $\mathscr{B}$ is primitive, it follows from Proposition 2.30 that for some $c \in \mathbb{N}$ and a Behrend set $\mathscr{A} \subset \mathbb{N} \setminus \{1\}$, we have $c\mathscr{A} \subset \mathscr{B}$. Let $m \in \mathbb{N}$ be such that c is coprime to all elements of B_{m+1} (such m exists since B_n, $n \in \mathbb{N}$, are pairwise disjoint sets of primes). Let

$$\mathscr{B}_1 := B_1C_1 \cup \ldots \cup B_1B_2 \ldots B_mC_m \quad \text{and} \quad \mathscr{B}_2 := \bigcup_{n>m} B_1B_2 \ldots B_nC_n.$$

Then clearly, $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$. Moreover, let

$$\mathscr{A}_1 := \left\{\frac{b}{c} : b \in c\mathscr{A} \cap \mathscr{B}_1\right\} \quad \text{and} \quad \mathscr{A}_2 := \left\{\frac{b}{c} : b \in c\mathscr{A} \cap \mathscr{B}_2\right\}.$$

Then clearly, $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$. Since $\mathscr{B}_1$ is thin, it follows from (5) and (7) that $\mathscr{B}_1$ is taut. Therefore, since $c\mathscr{A}_1 \subset \mathscr{B}_1$, it follows from Proposition 2.30 that $\mathscr{A}_1$ is not Behrend. Since $\mathscr{A}$ is Behrend, we obtain by Proposition 2.28 that $\mathscr{A}_2$ must be Behrend. Moreover, we have $c\mathscr{A}_2 \subset \mathscr{B}_2$. Take $a \in \mathscr{A}_2$. Since c is coprime to each element of B_{m+1}, it follows that $a \in \mathcal{M}_{B_{m+1}}$. Hence, $\mathcal{M}_{\mathscr{A}_2} \subset \mathcal{M}_{B_{m+1}}$, which is impossible since $d(\mathcal{M}_{\mathscr{A}_2}) = 1$, whereas $d(\mathcal{M}_{B_{m+1}}) < 1$ since B_{m+1} is thin. We obtain that $\mathscr{B}$ is taut, which completes the proof. □

Remark 4.24. There exists a set which is not Besicovitch nor taut. In order to see this let $\mathscr{B}$ be any set which is not Besicovitch [6]; that is, $\underline{d}(\mathcal{M}_{\mathscr{B}}) < \overline{d}(\mathcal{M}_{\mathscr{B}})$ and let $m \in \mathbb{N}$ be a number such that $\frac{1}{m} < \overline{d}(\mathcal{M}_{\mathscr{B}}) - \underline{d}(\mathcal{M}_{\mathscr{B}})$. Then it is not hard to see that $\mathscr{B} \cup m\mathscr{P}$ is not Besicovitch and clearly it is not taut.

4.3.1. *When $\mathscr{B}$ is finite, $(\widetilde{X}_\eta, S)$ and $(X_{\mathscr{B}}, S)$ are sofic.*

Proposition 4.25. *Let $\mathscr{B} \subset \mathbb{N}$ be primitive. Then $\mathscr{B}$ is finite if and only if η is periodic, with the minimal period $m = \mathrm{lcm}(\mathscr{B})$.*

Proof. If $\mathscr{B}$ is finite, then η is periodic with period $\mathrm{lcm}\,\mathscr{B}$. Suppose now that η is periodic and denote its period by m. Let $1 \le r_1 < r_2 < \ldots < r_s \le m$ be such that $(\mathrm{supp}\ \eta) \cap [1, m] = \{1, \ldots, m\} \setminus \{r_1, \ldots, r_s\}$. Then

$$\bigcup_{b \in \mathscr{B}} b\mathbb{Z} = \bigcup_{\ell=1}^{s} (m\mathbb{Z} + r_\ell).$$

For $1 \leqslant \ell \leqslant s$, let $d_\ell := \gcd(m, r_\ell)$. By the definition of d_ℓ, we have

$$d_\ell\mathbb{Z} \supset m\mathbb{Z} + r_\ell. \tag{51}$$

Then, $\gcd(m, r_\ell) \mid d_\ell$; hence the equation $r_\ell x \equiv d_\ell \bmod m$ has a solution for every ℓ, which implies that there exists $k_\ell \in \mathbb{Z}$ such that $r_\ell k_\ell \equiv d_\ell \mod m$. Since $\eta(r_\ell) = 0$, we have $\eta(r_\ell k_\ell) = 0$, which, by periodicity, yields $\eta(d_\ell) = 0$. This and (51) imply that

$$\bigcup_{b \in \mathscr{B}} b\mathbb{Z} = \bigcup_{\ell=1}^{s} d_\ell\mathbb{Z}. \tag{52}$$

Fix $b \in \mathscr{B}$. It follows from (52) that $d_\ell \mid b$ for some $1 \le \ell \le s$. On the other hand, there exists $b' \in \mathscr{B}$ such that $b' \mid d_\ell$. By the primitivity of $\mathscr{B}$, we have $b' \mid b$, whence $b = b'$ and $d_\ell = b$. We conclude that $\mathscr{B} \subset \{d_\ell : 1 \le \ell \le s\}$; i.e., $\mathscr{B}$ is finite. Moreover, since $d_\ell \mid m$ for $1 \le \ell \le s$, we obtain $b \mid m$ for each $b \in \mathscr{B}$. This yields $\mathrm{lcm}(\mathscr{B}) \mid m$. □

Assume now that $\mathscr{B} = \{b_1, \dots, b_n\}$. It is folklore that both subshifts $(\widetilde{X}_\eta, S)$ and $(X_\mathscr{B}, S)$ are sofic, but we provide a proof of these facts for completeness (the subshifts $(X_\mathscr{B}, S)$, for any finite $\mathscr{B} \subset \mathscr{S}$, were shown to be sofic in [47]). Thus, for a finite $\mathscr{B}$, the results of the paper or answers to questions stated in the paper follow from the theory of sofic systems, see, e.g., [39,53]. Recall that, by Proposition 3.25, sofic systems obtained from finite $\mathscr{B}$ are not totally transitive; cf. also Example 2.43 to see that $\widetilde{X}_\eta$ can be smaller than $X_\mathscr{B}$.

For any n-tuple $r = (r_1, \dots, r_n) \in \mathbb{Z}^n$, we say that a block $x = x_1 \dots x_\ell \in \{0,1\}^\ell$ satisfies the formula ϕ_r if and only if

$$\mathrm{supp}(x) \cap \bigcup_{i=1}^{n} (b_i\mathbb{Z} + r_i) = \emptyset.$$

In this case, we write $\phi_r(x)$. We say that ϕ_r *is equivalent to* ϕ_s provided $\phi_r(x) \Leftrightarrow \phi_s(x)$ for any block x.

Remark 4.26. Clearly, the formulas ϕ_r, ϕ_s, where $r = (r_1, \dots, r_n)$, $s = (s_1, \dots, s_n)$ are equivalent if $r_i = s_i \mod b_i$ for any $i = 1, \dots, n$. Thus, there are only finitely many equivalence classes of the formulas ϕ_r.

Given $m \in \mathbb{Z}$ and $r = (r_1, \dots, r_n) \in \mathbb{Z}^n$, we set $r - m := (r_1 - m, \dots, r_n - m)$. Let $C := \{(r_1, \dots, r_n) \in \mathbb{Z}^n : \forall_{i,j=1,\dots,n} \gcd(b_i, b_j) | r_i - r_j\}$. Let $F_{X_\mathscr{B}}(x)$ (resp. $F_{\widetilde{X}_\eta}(x)$) denote the set of the *followers* of x in $X_\mathscr{B}$ (resp. in $\widetilde{X}_\eta$), that is,

$$F_{X_\mathscr{B}}(x) := \{y \in \bigcup_{q=0}^{\infty} \{0,1\}^q : xy \text{ appears in } X_\mathscr{B}\}.$$

To show that $(X_\mathscr{B}, S)$ is sofic, following [53], we need to show that the family $\{F_{X_\mathscr{B}}(x) : x \in X_\mathscr{B}\}$ is finite. Further, we denote

$$\Phi(x) = \{r \in \mathbb{Z}^n : \phi_r(x)\}, \quad \Psi(x) = \{r \in C : \phi_r(x)\}.$$

Lemma 4.27. *We have the following:*

1. *A block x appears in $X_\mathscr{B}$ if and only if $\phi_r(x)$ for some $r \in \mathbb{Z}^n$.*
2. *A block x appears in $\widetilde{X}_\eta$ if and only if $\phi_r(x)$ for some $r \in C$.*
3. *A block $y \in F_{X_\mathscr{B}}(x)$ if and only if $\phi_{r-|x|}(y)$ for some $r \in \Phi(x)$.*
4. *A block $y \in F_{\widetilde{X}_\eta}(x)$ if and only if $\phi_{r-|x|}(y)$ for some $r \in \Psi(x)$.*

Proof. (1) follows by the definition of $X_\mathscr{B}$, whereas (2) follows by the classical Lemma 5.15.

Now, we show (3). Let $y = y_1 \dots y_k$ be a block. In view of (1), the following conditions are equivalent:

- the concatenation $xy = x_1 \dots x_\ell y_1 \dots y_k$ appears in $X_\mathscr{B}$,
- $\phi_r(xy)$ for some $r \in \mathbb{Z}^n$,
- $\mathrm{supp}(xy) \cap \bigcup_{i=1}^{n} (b_i\mathbb{Z} + r_i) = \emptyset$ for some $r = (r_1, \dots, r_n) \in \mathbb{Z}^n$,
- $\mathrm{supp}(x) \cap \bigcup_{i=1}^{n} (b_i\mathbb{Z} + r_i) = \emptyset$ and $\mathrm{supp}(y) \cap \bigcup_{i=1}^{n} (b_i\mathbb{Z} + r_i - |x|) = \emptyset$ for some $r \in \mathbb{Z}^n$,
- $\phi_{r-|x|}(y)$ and $r \in \Phi(x)$ for some $r \in \mathbb{Z}^n$.

Thus, (3) follows, and the proof of (4) is analogous. □

Proposition 4.28. *The systems $X_{\mathscr{B}}$ and $\widetilde{X}_\eta$ are sofic.*

Proof. By Lemma 4.27, the follower sets $F_{X_{\mathscr{B}}}(x)$, $F_{\widetilde{X}_\eta}(x)$ are completely determined by the sets $\Phi(x)$, $\Psi(x)$, respectively. But there are only finitely many such sets, since the number of equivalence classes of the formulas ϕ_r is finite by Remark 4.26. □

4.4. **Tautness and combinatorics (Theorem L – first steps).** Since $\eta \in X_{\mathscr{B}}$, a natural question arises how many residue classes are missing on supp η mod b_k, $k \geq 1$. We will answer this question under the assumption that $\mathscr{B}$ is taut. Recall first the following result:

Theorem 4.29 (Dirichlet)**.** *Let $a, r \in \mathbb{N}$. If $\gcd(a, r) = 1$, then $a\mathbb{Z} + r$ contains infinitely many primes. Moreover, $\sum_{p\in(a\mathbb{Z}+r)\cap\mathscr{P}} 1/p = +\infty$.*

Since each set containing a pairwise coprime set with divergent sum of reciprocals is automatically Behrend and Proposition 2.28 holds, we obtain the following:

Corollary 4.30. *Let $a, r \in \mathbb{N}$. If $\gcd(a, r) = 1$, then, for each $N \geq 1$, the set $(a\mathbb{Z} + r) \cap [N, \infty) \cap \mathscr{P}$ is Behrend.*

Proposition 4.31. *Assume that $\mathscr{B} \subset \mathbb{N}$ is taut, $a \in \mathbb{N}$, $1 \leq r \leq a$, and $N \geq 1$. If*

$$(a\mathbb{Z} + r) \cap [N, \infty) \subset \bigcup_{b\in\mathscr{B}} b\mathbb{Z}, \tag{53}$$

then there exists $b \in \mathscr{B}$ such that $b \mid \gcd(a, r)$. In particular, if $a \in \mathscr{B}$, then $r = a$.

Proof. Suppose that $a \in \mathbb{N}$ and $1 \leq r \leq a$ are such that (53) holds. Let $d := \gcd(a, r)$, $a' := a/d$, $r' := r/d$; i.e., we have

$$d \cdot (a'\mathbb{Z} + r') \cap [N, \infty) \subset \bigcup_{b\in\mathscr{B}} b\mathbb{Z}.$$

Applying Corollary 4.30 to a' and r', we obtain $d(\mathcal{M}_{(a'\mathbb{Z}+r')\cap[N,\infty)}) = 1$, whence $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}}) = \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}\cup\{d\}})$. In view of Corollary 2.31, $d \in \mathcal{M}_{\mathscr{B}}$. Then there exists $b \in \mathscr{B}$ such that $b \mid d$, whence $b \mid \gcd(a, r)$.

Suppose now that $a \in \mathscr{B}$ and (53) holds. By the first part of the proof, we have $b \mid \gcd(a, r)$ for some $b \in \mathscr{B}$. It follows that $b \mid a$ and, since $a, b \in \mathscr{B}$, by the primitivity of $\mathscr{B}$, we obtain $a = b$. Therefore, using the relation $b \mid \gcd(a, r)$, we obtain that $b \mid r$, and, since $1 \leq r \leq b$, this yields $r = b$. □

Corollary 4.32. *Assume that $\mathscr{B} \subset \mathbb{N}$ is taut. Then, for each $b \in \mathscr{B}$ and $1 \leq r \leq b - 1$, there exist infinitely many $m \in \mathcal{F}_{\mathscr{B}}$ such that $m \equiv r \bmod b$. In particular, $\eta \in Y$.*

Proof. Fix $N \geq 1$, $b \in \mathscr{B}$, and consider $b\mathbb{Z}+r$ for $1 \leq r \leq b-1$. By Proposition 4.31, $(b\mathbb{Z} + r) \cap [N, \infty) \not\subset \mathcal{M}_{\mathscr{B}}$, i.e.,

$$(\mathcal{F}_{\mathscr{B}} \cap [N, \infty)) \bmod b = \{1, \dots, r - 1\},$$

and the result follows. □

Remark 4.33. Note that if $\eta \in Y$, then $\mathscr{B}$ is primitive. Indeed, if $\mathscr{B}$ is not primitive, then, for some $b, b' \in \mathscr{B}$, we have $b \mid b'$. If |supp η mod $b'| = b'-1$, then |supp η mod $b| = b$. The latter is impossible as $\eta \in X_{\mathscr{B}}$, and it follows that $\eta \notin Y$.

The following example shows that the converse of Corollary 4.32 does not hold:

Example 4.34. Consider $\{(p_i, r_i) : i \geq 1\} = \{(p, r) : p \in \mathscr{P}, 0 < r < p\}$. Every progression $p_i\mathbb{Z} + r_i$ contains infinitely many primes; given $i \geq 1$ let, for $n \geq 1$,

$$q_i^n \in (p_i\mathbb{Z} + r_i) \cap \mathscr{P} \text{ be such that } q_i^n > 2^n \cdot i^2.$$

We set $\mathscr{B} := \mathscr{P} \setminus \{q_i^n : i, n \geq 1\}$. Since $\sum_{i,n\geq 1} \frac{1}{q_i^n} < \infty$, it follows that $\mathscr{B}$ is Behrend, so, in particular, $\mathscr{B}$ is not taut.

Let $b \in \mathscr{B}$ and $0 < r < b$ and let $i \geq 1$ be such that $(b, r) = (p_i, r_i)$. Then, for each $n \geq 1$, $q_i^n \equiv r \bmod b$ by the choice of q_i^n. Moreover, $q_i^n \in \mathcal{F}_{\mathscr{B}}$ since it is a prime not belonging to $\mathscr{B}$.

The following corollary extends an analogous result proved in [36] for $\mathscr{B}, \mathscr{B}' \subset \mathbb{N}$ thin and coprime.

Corollary 4.35. *Let $\mathscr{B}, \mathscr{B}' \subset \mathbb{N}$ and suppose that $\mathscr{B}$ is taut. Then the following conditions are equivalent:*

(a) $X_{\mathscr{B}} \subset X_{\mathscr{B}'}$,
(b) *for each $b' \in \mathscr{B}'$ there exists $b \in \mathscr{B}$ with $b \mid b'$,*
(c) $\eta \leq \eta'$,
(d) $\widetilde{X}_\eta \subset \widetilde{X}_{\eta'}$,
(e) $\eta \in \widetilde{X}_{\eta'}$,
(f) $\eta \in X_{\mathscr{B}'}$.

Proof. Clearly, we have (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) and (a) $\Rightarrow$ (f). Therefore, to complete the proof it suffices to show (b) $\Rightarrow$ (a) and (f) $\Rightarrow$ (b).

Suppose that (b) holds and let $A \subset \mathbb{N}$ be $\mathscr{B}$-admissible. Take $b' \in \mathscr{B}$ and let $b \in \mathscr{B}$ be such that $b \mid b'$. It follows from the $\{b\}$-admissibility of A that for some $0 \leq r \leq b-1$, we have $(b\mathbb{Z}+r) \cap A = \emptyset$, so all the more, we have $(b'\mathbb{Z}+r) \cap A = \emptyset$; i.e., A is $\{b'\}$-admissible and (a) follows.

Suppose that (f) holds. Then, for each $b' \in \mathscr{B}'$ there exists $1 \leq r' \leq b'$ such that $r' \notin \mathcal{F}_{\mathscr{B}} \bmod b'$, i.e.,

$$b'\mathbb{Z} + r' \subset \bigcup_{b\in\mathscr{B}} b\mathbb{Z}.$$

It follows from Proposition 4.31 that there exists $b \in \mathscr{B}$ such that $b \mid \gcd(b', r')$, so, in particular, $b \mid b'$; i.e., (b) holds. □

Corollary 4.36. *Suppose that $\mathscr{B}, \mathscr{B}'$ are taut. Then the following conditions are equivalent:*

(a) $X_{\mathscr{B}} = X_{\mathscr{B}'}$,
(b) $\mathscr{B} = \mathscr{B}'$,
(c) $\eta = \eta'$,
(d) $\widetilde{X}_\eta = \widetilde{X}_{\eta'}$,
(e) $\eta \in \widetilde{X}_{\eta'}$ *and* $\eta' \in \widetilde{X}_\eta$,
(f) $\eta \in X_{\mathscr{B}'}$ *and* $\eta' \in X_{\mathscr{B}}$,
(g) $X_\eta = X_{\eta'}$.

Proof. We have immediately (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f), (b) $\Rightarrow$ (a) $\Rightarrow$ (f), and (c) $\Rightarrow$ (g) $\Rightarrow$ (d). It remains to show (f) $\Rightarrow$ (b). By the corresponding implication in Corollary 4.35, for any $b \in \mathscr{B}$ there exist $b' \in \mathscr{B}'$ and $b'' \in \mathscr{B}$ such that $b'' \mid b' \mid b$. Since $\mathscr{B}$ is taut, it is, in particular, primitive, which yields $b = b' = b''$, i.e., $\mathscr{B} \subset \mathscr{B}'$. Reversing the roles of $\mathscr{B}$ and $\mathscr{B}'$, we obtain $\mathscr{B} = \mathscr{B}'$. □

5. Heredity (proofs of Theorems D and G)

By Corollary 3.10, (X_η, S) is proximal whenever X_η is hereditary. The converse to that does not hold; cf. Example 2.42. In this section, we will show however that the proximality of (X_η, S) and the heredity of X_η are equivalent when $\mathscr{B}$ has light tails.

Definition 5.1. We say that $A \subset \mathbb{N}$ is *η-admissible* whenever

$$\{k+1, \dots, k+n\} \cap \mathcal{F}_{\mathscr{B}} = A + k \tag{54}$$

for some $k, n \in \mathbb{N}$ (in other words, $\operatorname{supp} \eta[k+1, k+n] = A + k$).

Definition 5.2. We say that $A \subset \mathbb{N}$ satisfies condition $(\mathrm{T_{her}})$ whenever

$(\mathrm{T_{her}})$ there exists $\{n_b \in \mathbb{Z} \colon b \in \mathscr{B}\}$ such that $A \cap (b\mathbb{Z} + n_b) = \emptyset$ and $\gcd(b, b') \mid n_b - n_{b'}$ for any $b, b' \in \mathscr{B}$.

Our main goal in this section is to prove the following:

Theorem 5.3. *Assume that $\mathscr{B} \subset \mathbb{N}$ has light tails and satisfies* (Au). *Let $n \in \mathbb{N}$ and $A \subset \{1, \dots, n\}$. The following conditions are equivalent:*

(a) *A satisfies* $(\mathrm{T_{her}})$,
(b) *A is η-admissible.*

In particular, X_η is hereditary, i.e., $X_\eta = \widetilde{X}_\eta$.

Remark 5.4 (*Proof of Theorem* D). Clearly, if $A' \subset A \subset \mathbb{N}$ and A satisfies $(\mathrm{T_{her}})$, then also A' satisfies $(\mathrm{T_{her}})$. Thus, Theorem D, i.e., the assertion that X_η is hereditary in Theorem 5.3, follows immediately by the equivalence of (a) and (b).

As an immediate consequence of Theorems 3.7 and 5.3, we have:

Corollary 5.5. *Assume that $\mathscr{B} \subset \mathbb{N}$ has light tails. Then X_η is hereditary if and only if (X_η, S) is proximal.*

Example 5.6. Let $\mathscr{B} \subset \mathbb{N}$ be as in Example 2.43. If additionally $\mathscr{B}$ has light tails and satisfies (Au), then, by Theorem 5.3, $X_\eta = \widetilde{X}_\eta$. For example, one can take $\mathscr{B} = \{4, 6\} \cup \{p^2 : p \in \mathscr{P},\ p > 12\}$.

On the other hand, if (Au) fails, then, by Theorem 3.7, (X_η, S) fails to be proximal. Hence, by Corollary 3.10, X_η also fails to be hereditary. For example, one can take $\mathscr{B} = \{4, 6\} \cup \{5p^2 : p \in \mathbb{P},\ p > 12\}$.

We leave the following question open:

Question 5.7. Are the heredity of X_η and proximality of (X_η, S) the same whenever $\mathscr{B}$ is taut?

Remark 5.8. Notice that $\mathscr{B}$ from the construction on page 5459 satisfies condition (Au) whenever B_1, C_1 are infinite; i.e., (X_η, S) is proximal. We do not know whether in this example $X_\eta = \widetilde{X}_\eta$.

For the proof of Theorem 5.3, we will need several auxiliary results.

Lemma 5.9. *Let $n \in \mathbb{N}$ and suppose that $A \subset \{1, \dots, n\}$ is η-admissible. Then A satisfies* $(\mathrm{T_{her}})$.

Proof. Suppose that $\{k+1,\dots,k+n\}\cap\mathcal{F}_{\mathscr{B}}=A+k$ for some k. For $b\in\mathscr{B}$, let $n_b:=-k$. Since for any $i\in A$, $i+k\in\mathcal{F}_{\mathscr{B}}$, we have $i+k\notin b\mathbb{Z}$ for any $b\in\mathscr{B}$. This means that $i\notin b\mathbb{Z}-k=b\mathbb{Z}+n_b$. Clearly, $\gcd(b,b')|(n_b-n_{b'})$ for $b,b'\in\mathscr{B}$; hence A satisfies ($\mathrm{T}_{\mathrm{her}}$). □

Lemma 5.9 gives the implication (b) ⇒ (a) in the assertion of Theorem 5.3. Now, we will prove the converse implication. For $n\geq 1$, let

$$\mathscr{B}^{(n)}:=\{b\in\mathscr{B}: p\leq n \text{ for any } p\in\mathrm{Spec}(b)\},$$

where Spec(b) stands for the set of all prime divisors of b. For $A\subset\mathbb{N}$ the set Spec(A) is defined as the union of Spec(a), $a\in A$. Our main tools are the following two results:

Proposition 5.10. *Assume that $\mathscr{B}\subset\mathbb{N}$ satisfies* (Au) *and $\mathscr{B}^{(n)}\subset\mathscr{A}\subset\mathscr{B}$. Suppose that*

$$\{k+1,\dots,k+n\}\cap\mathcal{M}_{\mathscr{A}}=\{k+i_1,k+i_2,\dots,k+i_r\} \tag{55}$$

for some $1\leq i_1,\dots,i_r\leq n$, $r<n$ (if $r=0$, we interpret the right hand side of (55) *as the empty set). Then, for arbitrary $i_0\in\{1,\dots,n\}$, there exist $\mathscr{B}^{(n)}\subset\mathscr{A}'\subset\mathscr{B}$ and $k'\in\mathbb{Z}$ such that*

$$\{k'+1,\dots,k'+n\}\cap\mathcal{M}_{\mathscr{A}'}=\{k'+i_0,k'+i_1,\dots,k'+i_r\}.$$

Proposition 5.11. *Assume that $\mathscr{B}\subset\mathbb{N}$ has light tails and $\mathscr{B}^{(n)}\subset\mathscr{A}\subset\mathscr{B}$. Suppose that*

$$\{k+1,\dots,k+n\}\cap\mathcal{M}_{\mathscr{A}}=\{k+i_0,k+i_1,\dots,k+i_r\} \tag{56}$$

for some $1\leq i_0,\dots,i_r\leq n$, $r<n$. Then the density of $k'\in\mathbb{N}$ such that

$$\{k'+1,\dots,k'+n\}\cap\mathcal{M}_{\mathscr{B}}=\{k'+i_0,k'+i_1,\dots,k'+i_r\}$$

is positive.

Remark 5.12. For the purposes of this section it would be sufficient to know that such k' exists. We will use this result in its full form later.

Before we give the proofs of Propositions 5.10 and 5.11, we will show how these two results yield the implication (a) ⇒ (b) in Theorem 5.3. Notice first that iterating Propositions 5.10 and 5.11, we obtain immediately the following:

Corollary 5.13. *Assume that $\mathscr{B}$ has light tails and satisfies* (Au)*. Assume that $\mathscr{B}^{(n)}\subset\mathscr{A}\subset\mathscr{B}$. Suppose that*

$$\{k+1,\dots,k+n\}\cap\mathcal{M}_{\mathscr{A}}=k+C \tag{57}$$

for some $C\subset\{1,\dots.,n\}$. Then, for arbitrary set C' such that $C\subset C'\subset\{1,\dots,n\}$, the density of the set of $k'\in\mathbb{Z}$ such that $\{k'+1,\dots,k'+n\}\cap\mathcal{M}_{\mathscr{B}}=k'+C'$ is positive.

We will now present some auxiliary results.

Lemma 5.14. *Let $A\subset\mathbb{N}$ be primitive, with Spec(A) finite. Then A is also finite.*

Proof. The proof will use induction on $|\mathrm{Spec}(A)|$. Clearly, if $|\mathrm{Spec}(A)|=1$, then also $|A|=1$. Suppose that the assertion holds for any set A with $|\mathrm{Spec}(A)|\leq n-1$.

Now let A be primitive with $|\mathrm{Spec}(A)| = n$, i.e., $\mathrm{Spec}(A) = \{p_1, \dots, p_n\} \subset \mathscr{P}$. For $k \geq 0$, let

$$A^{(k)} := \{a \in A : k = \max\{\ell \geq 0 : (p_1 \cdot \ldots \cdot p_n)^\ell \mid a\}\},$$
$$B^{(k)} := \{a/(p_1 \cdot \ldots \cdot p_n)^k : a \in A^{(k)}\}.$$

For $1 \leq i \leq n$, let $B_i^{(k)} := \{b \in B^{(k)} : p_i \nmid b\}$. By the induction hypothesis, each of the sets $B_i^{(k)}$ is finite. Therefore $B^{(k)}$ is finite because $B^{(k)} = \bigcup_{1\leq i\leq n} B_i^{(k)}$. It follows immediately that also $A^{(k)}$ is finite. Suppose that $|\{k \geq 0 : A^{(k)} \neq \emptyset\}| = \infty$. Choose $a = p_1^{\alpha_1} \cdot \ldots \cdot p_n^{\alpha_n} \in A$. Let $k_0 > \max\{\alpha_i : 1 \leq i \leq n\}$ be such that $A^{(k_0)} \neq \emptyset$ and take $a' \in A^{(k_0)}$. Then $a \mid a'$; however $a \neq a'$, which yields a contradiction; i.e., we have $|\{k \geq 0 : A^{(k)} \neq \emptyset\}| < \infty$. Since $A = \bigcup_{k\geq 0} A^{(k)}$ is a finite union of finite sets, we obtain $|A| < \infty$. □

Lemma 5.15 (See, e.g., [40]). *Let $b_1, \dots, b_k \in \mathbb{N}$, $n_1 \dots, n_k \in \mathbb{Z}$. The system of congruences*

$$m \equiv n_i \bmod b_i,\ 1 \leq i \leq k,$$

has a solution $m \in \mathbb{N}$ if and only if $\gcd(b_i, b_j) \mid (n_i - n_j)$ for any $i, j = 1, \dots, k$.

Proof of Theorem 5.3. In view of Lemma 5.9, we have (b) ⇒ (a). We will now show that (a) ⇒ (b). Assume that $A \subset \{1, \dots, n\}$ satisfies condition ($\mathrm{T_{her}}$) with $\{n_b : b \in \mathscr{B}\}$. Since $\mathscr{B}$ is primitive, it follows from Lemma 5.14 that $\mathscr{B}^{(n)}$ is finite. Therefore, by Lemma 5.15, there exists $m \in \mathbb{N}$ such that $m \equiv -n_b \bmod b$ for each $b \in \mathscr{B}^{(n)}$. It follows that

$$\{m+1, \dots, m+n\} \cap \mathcal{M}_{\mathscr{B}^{(n)}}$$
$$= (\{1, \dots, n\} \cap \bigcup_{b\in\mathscr{B}^{(n)}} (b\mathbb{Z} + n_b)) + m \subset (\{1, \dots, n\} \setminus A) + m.$$

Applying Corollary 5.13 to $\mathscr{A} = \mathscr{B}^{(n)}$, $k = m$, $C = \{1, \dots, n\} \cap \bigcup_{b\in\mathscr{B}^{(n)}} (b\mathbb{Z} + n_b)$, and $C' = \{1, \dots, n\} \setminus A$, we conclude that there exists m' such that

$$\{m'+1, \dots, m'+n\} \cap \mathcal{M}_{\mathscr{B}} = (\{1, \dots, n\} \setminus A) + m'.$$

Equivalently, $\{m'+1, \dots, m'+n\} \cap \mathcal{F}_{\mathscr{B}} = A + m'$, which yields (a) ⇒ (b). □

What remains to be proved is Propositions 5.10 and 5.11.

Proof of Proposition 5.10. For $u = 1, \dots, r$, let j_u be such that $b_{j_u} \in \mathscr{A}$ and $b_{j_u} \mid k + i_u$. Let $B := \mathscr{B}^{(n)} \cup \{b_{j_1}, \dots, b_{j_r}\}$. Then any $b \in \mathscr{B} \setminus B$ has a prime divisor $p > n$ and, by Lemma 5.14, the set B is finite. Let $\beta_1 := \mathrm{lcm}\, B$. Using $b_{j_u} \mid k + i_u$ and the assumption (55), we obtain

$$\{i_1, \dots, i_r\} \subset (\{k + \beta_1\ell + 1, \dots, k + \beta_1\ell + n\} \cap \mathcal{M}_B) - (k + \beta_1\ell)$$
$$= (\{k+1, \dots, k+n\} \cap \mathcal{M}_B) - k \subset (\{k+1, \dots, k+n\} \cap \mathcal{M}_{\mathscr{A}}) - k = \{i_1, \dots, i_r\};$$

i.e., for any $\ell \in \mathbb{Z}$ we have

$$(58)\qquad (\{k + \beta_1\ell + 1, \dots, k + \beta_1\ell + n\} \cap \mathcal{M}_B) - (k + \beta_1\ell) = \{i_1, \dots, i_r\}.$$

Using (Au), we can find j_0 such that $\gcd(b_{j_0}, \beta_1) = 1$. It follows that there are $\ell_0 \in \mathbb{Z}$ and $s \in \mathbb{Z}$ such that $\beta_1\ell_0 - sb_{j_0} = -i_0 - k$. Hence, for $k' := k + \beta_1\ell_0$, we have $b_{j_0} \mid k' + i_0$. Since $b_{j_0} \notin B$, we have $b_{j_0} > n$. It follows that

$$(59)\qquad b_{j_0} \nmid k' + i \text{ for any } 1 \leq i \neq i_0 \leq n$$

(indeed, if $b_{j_0} \mid k'+i$, then $n < b_{j_0} \mid (i-i_0)$). Let $\beta := \beta_1 b_{j_0}$. It follows from (59) and (58) (with $l := l_0 + mb_{j_0}$) that

$$(\{k'+\beta m+1,\ldots,k'+\beta m+n\} \cap \mathcal{M}_{B\cup\{b_{j_0}\}}) - (k'+\beta m)$$

$$= \{i_0, i_1, \ldots, i_r\}$$

for any $m \in \mathbb{N}$. Hence, it suffices to take $\mathscr{A}' = B \cup \{b_{j_0}\}$. ☐

The proof of Proposition 5.11 will be proceeded by several lemmas.

Lemma 5.16. *Let $\mathscr{R}$ be the intersection of finitely many arithmetic progressions with differences $d_1, \ldots, d_r$. Then either $\mathscr{R} = \emptyset$ or $\mathscr{R}$ is equal to an arithmetic progression of difference* $\operatorname{lcm}(d_1, \ldots, d_r)$*.*

Proof. It suffices to notice that if $a \in \mathscr{R}$, then $\mathscr{R} = \operatorname{lcm}(d_1, \ldots, d_r)\mathbb{Z} + a$. ☐

Lemma 5.17. *Let $\beta, r, n \in \mathbb{N}$, and assume that $p > n$ is a prime that does not divide β. Assume that $\mathscr{R}$ is a union of finitely many arithmetic progressions with steps not divisible by p. Then*

$$d\left((\beta\mathbb{Z}+r) \cap \left(\bigcup_{i=1}^{n} (p\mathbb{Z}-i)\right) \cap \mathscr{R}\right) = \frac{n}{p} d((\beta\mathbb{Z}+r)\cap\mathscr{R}) \tag{60}$$

and

$$d\left((\beta\mathbb{Z}+r) \setminus \left(\bigcup_{i=1}^{n} (p\mathbb{Z}-i) \cup \mathscr{R}\right)\right) = \left(1-\frac{n}{p}\right) d\left((\beta\mathbb{Z}+r)\setminus\mathscr{R}\right). \tag{61}$$

Proof. By Lemma 4.21, in order to prove (60), it suffices to prove it for $\mathscr{R} = b\mathbb{Z}+j$, where $p \nmid b$. Moreover, since the progressions $p\mathbb{Z}-i$ are pairwise disjoint for $1 \le i \le n$, what we need to show is

$$d((\beta\mathbb{Z}+r)\cap(p\mathbb{Z}-i)\cap(b\mathbb{Z}+j)) = \frac{1}{p} d((\beta\mathbb{Z}+r)\cap(b\mathbb{Z}+j)) \tag{62}$$

for each $1 \le i \le n$. Clearly, the above equality holds if $(\beta\mathbb{Z}+r)\cap(b\mathbb{Z}+j) = \emptyset$. Otherwise, let $\beta' := \operatorname{lcm}(\beta, b)$ and take $a \in (\beta\mathbb{Z}+r)\cap(b\mathbb{Z}+j)$. Then, by Lemma 5.16, $(\beta\mathbb{Z}+r)\cap(b\mathbb{Z}+j) = \beta'\mathbb{Z}+a$ and (62) is equivalent to

$$d((\beta'\mathbb{Z}+a)\cap(p\mathbb{Z}-i)) = \frac{1}{p} d(\beta'\mathbb{Z}+a). \tag{63}$$

Since $\gcd(\beta', p) = 1$, it follows that $(\beta'\mathbb{Z}+a)\cap(p\mathbb{Z}-i) \neq \emptyset$ and (63) is a straightforward consequence of Lemma 5.16.

In order to prove (61), note that

$$\begin{aligned}
&d\left((\beta\mathbb{Z}+r)\setminus\left(\bigcup_{i=1}^{n}(p\mathbb{Z}-i)\cup\mathscr{R}\right)\right)\\
&=d\left(\beta\mathbb{Z}+r\right)-d\left((\beta\mathbb{Z}+r)\cap\mathscr{R}\right)-d\left((\beta\mathbb{Z}+r)\cap\left(\bigcup_{i=1}^{n}(p\mathbb{Z}-i)\right)\right)\\
&\quad+d\left((\beta\mathbb{Z}+r)\cap\left(\bigcup_{i=1}^{n}(p\mathbb{Z}-i)\cap\mathscr{R}\right)\right)\\
&=d\left(\beta\mathbb{Z}+r\right)-d\left((\beta\mathbb{Z}+r)\cap\mathscr{R}\right)-\frac{n}{p}d\left(\beta\mathbb{Z}+r\right)+\frac{n}{p}d\left((\beta\mathbb{Z}+r)\cap\mathscr{R}\right)\\
&=\left(1-\frac{n}{p}\right)d((\beta\mathbb{Z}+r)\setminus\mathscr{R}),
\end{aligned}$$

where the second equality follows from (60). □

Lemma 5.18. *Let $\beta, r, n, c_1, \ldots, c_m \in \mathbb{N}$. Assume that $p > n$ is a prime, p divides $c_1, \ldots, c_k$, and p does not divide $c_{k+1}, \ldots, c_m$ nor β. Then*

$$d\left((\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{c_1,\ldots,c_m\}}-i\right)\right) \geq \left(1-\frac{n}{p}\right)d\left((\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{c_{k+1},\ldots,c_m\}}-i\right)\right). \tag{64}$$

Proof. Notice first that

$$(A-i)^c = A^c - i \quad \text{for any } A\subset\mathbb{Z}, i\in\mathbb{Z}. \tag{65}$$

Therefore,

$$(\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{c_1,\ldots,c_m\}}-i\right) = (\beta\mathbb{Z}+r)\setminus\left(\bigcup_{i=1}^{n}\left(\mathcal{M}_{\{c_1,\ldots,c_m\}}-i\right)\right). \tag{66}$$

Since

$$\mathcal{M}_{\{c_1,\ldots,c_m\}}\subset\mathcal{M}_{\{p,c_{k+1},\ldots,c_m\}}=p\mathbb{Z}\cup\mathcal{M}_{\{c_{k+1},\ldots,c_m\}},$$

using (66), we obtain

$$(\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{c_1,\ldots,c_m\}}-i\right)\supset(\beta\mathbb{Z}+r)\setminus\left(\bigcup_{i=1}^{n}(p\mathbb{Z}-i)\cup\bigcup_{i=1}^{n}\left(\mathcal{M}_{\{c_{k+1},\ldots,c_m\}}-i\right)\right).$$

To complete the proof, we apply Lemma 5.17 to $\mathscr{R}=\bigcup_{i=1}^{n}(\mathcal{M}_{\{c_{k+1},\ldots,c_m\}}-i)$ and use again (65). □

Remark 5.19. In Lemma 5.18 we allow $k=m$ (then $\{c_{k+1},\ldots,c_m\}=\emptyset$, $\mathcal{F}_\emptyset=\mathbb{Z}$, and $\mathcal{M}_\emptyset=\emptyset$).

Lemma 5.20. *Let $\beta, r, n \in \mathbb{N}$. Suppose that $\{c_m : m\geq 1\}\subset\mathbb{N}$ is Besicovitch. Assume that $p > n$ is a prime and p divides c_1 but does not divide β. Then the* *densities of* $(\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{c_m:m\geq 1\}}-i\right)$ *and* $(\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{c_m:m\geq 2\}}-i\right)$ *exist and*

$$d\left((\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{c_m:m\geq 1\}}-i\right)\right)\geq\left(1-\frac{n}{p}\right)d\left((\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{c_m:m\geq 2\}}-i\right)\right).$$

Proof. Fix $M\in\mathbb{N}$ and assume that $c_{l_1},\ldots,c_{l_t}$ are the elements of the set $\{c_1,\ldots,c_M\}$ which are not divisible by p (t can be equal to 0; cf. Remark 5.19). By Lemma 5.18, it follows that

$$d\left((\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{c_1,\ldots,c_M\}}-i\right)\right)\geq\left(1-\frac{n}{p}\right)d\left((\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{c_{l_1},\ldots,c_{l_t}\}}-i\right)\right).$$

On the other hand, $\mathcal{F}_{\{c_2,\ldots,c_M\}}\subset\mathcal{F}_{\{c_{l_1},\ldots,c_{l_t}\}}$. Thus, we obtain

$$d\left((\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{c_1,\ldots,c_M\}}-i\right)\right)\geq\left(1-\frac{n}{p}\right)d\left((\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{c_2,\ldots,c_M\}}-i\right)\right).$$

To finish the proof use Theorem 2.23 and let $M\to\infty$. □

Lemma 5.21. *Suppose that* $\mathscr{B}$ *has light tails. Assume that* $\beta,r,n\in\mathbb{N}$ *and* $b_{k_1},b_{k_2},\ldots\in\mathscr{B}$ *are such that each* b_{k_j} *has a prime divisor greater than* n *and not dividing* β. *Then the density of*

$$(\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{b_{k_j}:j\geq 1\}}-i\right)$$

exists and is positive.

Proof. Observe that by Lemma 5.20, for any $m\geq 1$, we have

$$d\left((\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{b_{k_m},b_{k_{m+1}},\ldots\}}-i\right)\right)\geq\left(1-\frac{n}{p}\right)d\left((\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{b_{k_{m+1}},\ldots\}}-i\right)\right),$$

where $p>n$ is a prime divisor of b_{k_m}. It follows that

$$d\left((\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{b_{k_1},b_{k_2},\ldots\}}-i\right)\right)\geq\rho(m)\,d\left((\beta\mathbb{Z}+r)\cap\bigcap_{i=1}^{n}\left(\mathcal{F}_{\{b_{k_m},b_{k_{m+1}},\ldots\}}-i\right)\right),$$

where $\rho(m) > 0$ depends only on m. Since $\mathscr{B}$ has light tails, for m large enough so that $d\left(\mathcal{M}_{\{b_{k_m}, b_{k_{m+1}}, \ldots\}}\right) < \frac{1}{n\beta}$, we have

$$d\left((\beta\mathbb{Z}+r) \cap \bigcap_{i=1}^{n} \left(\mathcal{F}_{\{b_{k_{m+1}}, b_{k_{m+2}}, \ldots\}} - i\right)\right) > 0,$$

and the assertion follows. □

Proof of Proposition 5.11. For $u = 1, \ldots, r$, let j_u be such that $b_{j_u} \in \mathscr{A}$ and $b_{j_u} \mid k+i_u$. Without loss of generality, we may assume that $\mathscr{A} = \{b_{j_u} : 0 \leq u \leq r\} \cup B^{(n)}$. Then, by Lemma 5.14, $\mathscr{A}$ is finite, and we set $\beta := \gcd(\mathscr{A})$. It follows from (56) that

$$(67) \qquad (\{k+\beta m+1, \ldots, k+\beta m+n\} \cap \mathcal{M}_{\mathscr{A}}) - (k+\beta m) = \{i_0, \ldots, i_r\}$$

for any $m \in \mathbb{N}$. Let

$$B := \{b \in \mathscr{B} \setminus \mathscr{A} : \text{ all prime divisors of } b \text{ greater than } n \text{ divide } \beta\}$$

(B may be empty) and notice that B is finite. Indeed, if p is a prime divisor of $b \in B$, then either $p \leq n$ or $p > n$ and divides β. Hence $|\mathrm{Spec}(B)| < \infty$ and we use Lemma 5.14. Since $\mathscr{B}^{(n)} \subset \mathscr{A}$, we have $B \subset \mathscr{B} \setminus \mathscr{B}^{(n)}$, and it follows that any $b \in B$ has a prime divisor $p > n$. Let $b \in B$ and take a prime $p \mid b$, $p > n$. By the definition of B, we have $p \mid \beta$, whence $p \mid b_{j_u}$ for some $0 \leq u \leq r$. It follows that if $b \mid k+\beta m+i$ for some $1 \leq i \leq n$, then $i \in \{i_0, \ldots, i_r\}$ (otherwise, $b_{j_u} \mid k+i_u$ would imply $p \mid i_u - i$, which is impossible). Thus, by (67), we obtain

$$(68) \qquad (\{k+\beta m+1, \ldots, k+\beta m+n\} \cap \mathcal{M}_{\mathscr{A} \cup B}) - (k+\beta m) = \{i_0, i_1, \ldots, i_r\}$$

for any $m \in \mathbb{N}$. Let

$$(\mathscr{B} \setminus \mathscr{A}) \setminus B =: B' = \{b_{k_1}, b_{k_2}, \ldots\};$$

i.e., each b_{k_j} has a prime divisor greater than n, not dividing β. By Lemma 5.21, the density of the set

$$(69) \qquad (\mathbb{Z}\beta + k) \cap (\bigcap_{i=1}^{n} \mathcal{F}_{\{b_{k_j} : j \geq 1\}} - i)$$

exists and is positive. Therefore, for $m \in \mathbb{N}$ from some positive density set, we have $\beta m + k + i \in \mathcal{F}_{\{b_{k_j} : j \geq 1\}}$ for any $i = 1, \ldots, n$. Using (68), we obtain that for each such $m \in \mathbb{N}$, we have

$$\begin{aligned}(\{k+\beta m+1, \ldots, k+\beta m+n\} \cap \mathcal{M}_{\mathscr{B}}) - (k+\beta m)\\ = (\{k+\beta m+1, \ldots, k+\beta m+n\} \cap \mathcal{M}_{\mathscr{A} \cup B}) - (k+\beta m) = \{i_0, \ldots, i_r\},\end{aligned}$$

as required. □

Proof of Theorem G. The assertion is an immediate consequence of Theorem 4.1 and of Proposition 5.11 (applied to $\mathscr{A} := \mathscr{B}$). □

6. Entropy

6.1. **Entropy of $\widetilde{X}_\eta$ and $X_{\mathscr{B}}$ (proof of Proposition K).** In this section our main goal is to prove Proposition K. To fix attention, we will restrict ourselves to the case when $\mathscr{B}$ is infinite. The proof will be very similar to the proof of Theorem 5.3 in [1]. However, since our $\mathscr{B}$ are no longer Erdös, we cannot use the Chinese Remainder Theorem directly and we will need an additional ingredient:

Lemma 6.1 (Rogers, see [27], p. 242)**.** *For any b_k, $k \geq 1$, any $r_k \in \mathbb{Z}/b_k\mathbb{Z}$, and $K \geq 1$, we have*

$$d\Big(\bigcup_{k\leq K}(b_k\mathbb{Z}+r_k)\Big) \geq d\left(\mathcal{M}_{\{b_1,\dots,b_K\}}\right). \tag{70}$$

Remark 6.2. Clearly, for any $n \in \mathbb{N}$,

$$d\Big(\bigcup_{k\leq K}(b_k\mathbb{Z}+r_k)\Big) = \frac{1}{n\cdot b_1\cdot\ldots\cdot b_K}\Big|[1, n\cdot b_1\cdot\ldots\cdot b_K]\cap\Big(\bigcup_{k\leq K}(b_k\mathbb{Z}+r_k)\Big)\Big|.$$

Proof of Proposition K. In view of Theorem 2.23, the result will follow once we show that

$$h_{top}(\widetilde{X}_\eta, S) = h_{top}(X_{\mathscr{B}}, S) = \overline{d}(\mathcal{F}_{\mathscr{B}}).$$

For $n \in \mathbb{N}$, let $\gamma(n) := |\{B \in \{0,1\}^n : B \text{ is } \mathscr{B}\text{-admissible}\}|$ and, for $K \geq 1$, let

$$\gamma_K(n) := |\{B \in \{0,1\}^n : B \text{ is } \{b_1,\dots,b_K\}\text{-admissible}\}|.$$

Clearly, $\gamma(n) \leq \gamma_K(n)$ for any $K \geq 1$. Moreover, any $\{b_1,\dots,b_K\}$-admissible $n\cdot b_1\cdot\ldots\cdot b_K$-block $B \in \{0,1\}^{[1,n\cdot b_1\cdot\ldots\cdot b_K]}$ can be obtained in the following way:

(a) choose $(r_1,\dots,r_K) \in \prod_{k\leq K}\mathbb{Z}/b_k\mathbb{Z}$ and set $B(j) := 0$ for $1 \leq j \leq n\cdot b_1\cdot\ldots\cdot b_K$ satisfying $j \equiv r_k \bmod b_k$ for some $1 \leq k \leq K$,

(b) complete the word by choosing arbitrarily $B(j) \in \{0,1\}$ for all other $1 \leq j \leq n\cdot b_1\cdot\ldots\cdot b_K$.

(Clearly, $(\text{supp } B)\cap(b_i\mathbb{Z}+r_i) = \emptyset$.) Notice that once $(r_1,\dots,r_K) \in \prod_{k\leq K} b_k\mathbb{Z}$ is fixed, the freedom in step (b) gives $2^{n\cdot b_1\cdot\ldots\cdot b_K\left(1-d\left(\bigcup_{k\leq K}b_k\mathbb{Z}+r_k\right)\right)}$ pairwise distinct $\{b_1,\dots,b_K\}$-admissible $n\cdot b_1\cdot\ldots\cdot b_K$-blocks (cf. Remark 6.2). Moreover, in view of Lemma 6.1, this number does not exceed

$$2^{n\cdot b_1\cdot\ldots\cdot b_K(1-d_K)}, \tag{71}$$

where $d_K = d(\mathcal{M}_{\{b_1,\dots,b_K\}})$.

We will show that $h_{top}(X_{\mathscr{B}}, S) \leq \overline{d}(\mathcal{F}_{\mathscr{B}})$. Fix $\varepsilon > 0$. In view of Theorem 2.23, if K is large enough, then $d_K \geq 1 - \overline{d}(\mathcal{F}_{\mathscr{B}}) - \varepsilon$. Fix such K. It follows from Lemma 6.1, Remark 6.2, and the discussion preceding (71) that

$$\gamma_K(n\cdot b_1\cdot\ldots\cdot b_K) \leq 2^{n\cdot b_1\cdot\ldots\cdot b_K\cdot(1-d_K)}\cdot\prod_{k\leq K} b_k$$

whenever $n = n(K,\varepsilon)$ is sufficiently large. Thus (since the number of possible choices in step (a) equals $b_1\cdot\ldots\cdot b_K$), for such n, we obtain

$$\gamma_K(n\cdot b_1\cdot\ldots\cdot b_K) \leq 2^{n\cdot b_1\cdot\ldots\cdot b_K\cdot(\overline{d}(\mathcal{F}_{\mathscr{B}})+\varepsilon)}\cdot\prod_{k\leq K} b_k.$$

Therefore, $h_{top}(X_{\mathscr{B}}, S) = \lim_{n\to\infty}\frac{1}{n}\log\gamma(n) \leq \lim_{n\to\infty}\frac{1}{n}\log\gamma_K(n) \leq \overline{d}(\mathcal{F}_{\mathscr{B}})$.

We will now show that $h_{top}(\widetilde{X}_\eta, S) \geq \overline{d}(\mathcal{F}_{\mathscr{B}})$. For $n \geq 1$, denote by $p(n)$ the number of n-blocks occurring on $\widetilde{X}_\eta$. Let (N_k) be such that

$$\lim_{k\to\infty} \frac{1}{N_k}|[0,N_k] \cap \mathcal{F}_{\mathscr{B}}| = \overline{d}(\mathcal{F}_{\mathscr{B}}).$$

Since $p(N_k) \geq 2^{|[0,N_k]\cap\mathcal{F}_{\mathscr{B}}|}$, it follows that $h_{top}(\widetilde{X}_\eta, S) = \lim_{k\to\infty} \frac{1}{N_k}\log p(N_k) \geq \overline{d}(\mathcal{F}_{\mathscr{B}})$, which completes the proof. □

Remark 6.3. Recall that a hereditary system has zero entropy if and only if $\delta_{\dots 0.\underline{0}0\dots}$ is the unique invariant measure (for the proof, see [38]). Therefore, since both $\widetilde{X}_\eta$ and $X_{\mathscr{B}}$, are hereditary, it follows from Proposition K that the following conditions are equivalent:

- $\mathcal{P}(X_{\mathscr{B}}, S) = \{\delta_{\dots 0.00\dots}\}$,
- $\mathcal{P}(\widetilde{X}_\eta, S) = \{\delta_{\dots 0.00\dots}\}$,
- $\boldsymbol{\delta}(\mathcal{F}_{\mathscr{B}}) = 0$; i.e. $\mathscr{B}$ is Behrend,
- the upper Banach density

$$\overline{bd}(\mathcal{F}_{\mathscr{B}}) := \lim_{N\to\infty}\sup_{k\geq 1}\frac{1}{N}|\mathcal{F}_{\mathscr{B}} \cap [k, k+N-1]|$$

  of $\mathcal{F}_{\mathscr{B}}$ is zero.

Note that the last condition follows directly from Theorem 2.23. Indeed, for each $A \subset \mathbb{Z}$, we have $\overline{bd}(A) = 1 - \underline{bd}(A)$. Moreover, for each $K \geq 1$, $bd(\mathcal{M}_{\{b\in\mathscr{B}: b\leq K\}}) = d(\mathcal{M}_{\{b\in\mathscr{B}: b\leq K\}})$. Since $\mathcal{M}_{\{b\in\mathscr{B}: b\leq K\}} \cap \mathcal{F}_{\mathscr{B}} = \emptyset$, we have

$$\overline{bd}(\mathcal{F}_{\mathscr{B}}) \leq \overline{bd}(\mathbb{Z} \setminus \mathcal{M}_{\{b\in\mathscr{B}: b\leq K\}}) = 1 - \underline{bd}(\mathcal{M}_{\{b\in\mathscr{B}: b\leq K\}}) \to 0,$$

when $K \to \infty$, and our claim follows.

In particular (cf. (1)), we obtain one more proof of the fact that $\overline{bd}(\mathscr{P}) = 0$.

6.2. **Entropy of some invariant subsets of $\widetilde{X}_\eta$.** In this section we will prove the following:

Proposition 6.4. *If $\mathscr{B}$ is taut, then*

$$h_{top}(Y_{\geq s_1,\geq s_2,\dots} \cap \widetilde{X}_\eta, S) < h_{top}(\widetilde{X}_\eta, S)$$

whenever $s_k > 1$ for some $k \geq 1$.

For this, we will need some tools.

Lemma 6.5 (Cf. Lemma 1.17 in [28] and Theorem 2.23)**.** *Let $\mathscr{B} \subset \mathbb{N}$. For any $q \in \mathbb{N}$ and $0 \leq r \leq q-1$ the logarithmic density of $\mathcal{M}_{\mathscr{B}} \cup (q\mathbb{Z}+r)$ exists and*

$$\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}} \cup (q\mathbb{Z}+r)) = \underline{d}(\mathcal{M}_{\mathscr{B}} \cup (q\mathbb{Z}+r)) = \lim_{k\to\infty} d(\mathcal{M}_{\{b_1,\dots,b_k\}} \cup (q\mathbb{Z}+r)).$$

Proof. Since

$$\mathcal{M}_{\mathscr{B}} \cup (q\mathbb{Z}+r) = (q\mathbb{Z}+r) \cup \bigcup_{0\leq s\neq r\leq q-1} \mathcal{M}_{\mathscr{B}} \cap (q\mathbb{Z}+s),$$

it suffices to prove that the logarithmic density of $\mathcal{M}_{\mathscr{B}} \cap (q\mathbb{Z}+s)$ exists and

$$\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}} \cap (q\mathbb{Z}+s)) = \underline{d}(\mathcal{M}_{\mathscr{B}} \cap (q\mathbb{Z}+s)) = \lim_{k\to\infty} d(\mathcal{M}_{\{b_1,\dots,b_k\}} \cap (q\mathbb{Z}+s)) \tag{72}$$

for each $0 \leq s \leq q-1$. Indeed, if (72) holds, we have

$$\begin{aligned}\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}} \cup (q\mathbb{Z}+r)) &\geq \underline{d}(\mathcal{M}_{\mathscr{B}} \cup (q\mathbb{Z}+r))\\ &\geq d(q\mathbb{Z}+r) + \bigcup_{0\leq s\neq r\leq q-1} \underline{d}(\mathcal{M}_{\mathscr{B}} \cap (q\mathbb{Z}+s))\\ &= d(q\mathbb{Z}+r) + \bigcup_{0\leq s\neq r\leq q-1} \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}} \cap (q\mathbb{Z}+s)) = \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}} \cup (q\mathbb{Z}+r)).\end{aligned}$$

To show (72), notice first that, for each $k \geq 1$, we have

$$\underline{d}(\mathcal{M}_{\mathscr{B}} \cap (q\mathbb{Z}+s)) \geq d(\mathcal{M}_{\{b_1,\dots,b_k\}} \cap (q\mathbb{Z}+s)),$$

whence

$$\underline{d}(\mathcal{M}_{\mathscr{B}} \cap (q\mathbb{Z}+s)) \geq \lim_{k\to\infty} d(\mathcal{M}_{\{b_1,\dots,b_k\}} \cap (q\mathbb{Z}+s)). \tag{73}$$

On the other hand, for each $k \geq 1$,

$$\overline{\boldsymbol{\delta}}(\mathcal{M}_{\mathscr{B}} \cap (q\mathbb{Z}+s)) \leq d(\mathcal{M}_{\{b_1,\dots,b_k\}} \cap (q\mathbb{Z}+s)) + \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}} \setminus \mathcal{M}_{\{b_1,\dots,b_k\}}),$$

whence, by Theorem 2.23,

$$\overline{\boldsymbol{\delta}}(\mathcal{M}_{\mathscr{B}} \cap (q\mathbb{Z}+s)) \leq \lim_{k\to\infty} d(\mathcal{M}_{\{b_1,\dots,b_k\}} \cap (q\mathbb{Z}+s)). \tag{74}$$

The claim follows from (73) and (74). □

Lemma 6.6. *Assume that $\mathscr{B}$ is taut. Fix $k_0 \geq 1$ and let $0 < r < b_{k_0}$. Then*

$$\underline{d}\left(\mathcal{M}_{\mathscr{B}} \cup (b_{k_0}\mathbb{Z}+r)\right) > \underline{d}\left(\mathcal{M}_{\mathscr{B}}\right).$$

Proof. By Lemma 6.5, we have

$$\underline{d}(\mathcal{M}_{\mathscr{B}} \cup (b_{k_0}\mathbb{Z}+r)) = \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}} \cup (b_{k_0}\mathbb{Z}+r)) = \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}}) + \boldsymbol{\delta}((b_{k_0}\mathbb{Z}+r) \setminus \mathcal{M}_{\mathscr{B}}), \tag{75}$$

where

$$\boldsymbol{\delta}((b_{k_0}\mathbb{Z}+r) \setminus \mathcal{M}_{\mathscr{B}}) = \boldsymbol{\delta}((b_{k_0}\mathbb{Z}+r) \setminus \mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}}), \tag{76}$$

since $(b_{k_0}\mathbb{Z}+r) \cap b_{k_0}\mathbb{Z} = \emptyset$. Moreover, since $(b_{k_0}\mathbb{Z}+r) \cup \mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}}$ is a disjoint union of $\mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}}$ and $(b_{k_0}\mathbb{Z}+r) \setminus \mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}}$ (and the logarithmic density of $(b_{k_0}\mathbb{Z}+r) \cup \mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}}$ and $\mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}}$ exists by Lemma 6.5 and Theorem 2.23, respectively), we obtain

$$\boldsymbol{\delta}((b_{k_0}\mathbb{Z}+r) \setminus \mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}}) = \boldsymbol{\delta}((b_{k_0}\mathbb{Z}+r) \cup \mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}}) - \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}}). \tag{77}$$

By the tautness of $\mathscr{B}$, $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}}) > \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}})$. Hence, by (75), (76), (77),

$$\begin{aligned}&\underline{d}(\mathcal{M}_{\mathscr{B}} \cup (b_{k_0}\mathbb{Z}+r))\\ &\quad> \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}}) + \boldsymbol{\delta}((b_{k_0}\mathbb{Z}+r) \cup \mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}}) - \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}})\\ &\quad= \boldsymbol{\delta}((b_{k_0}\mathbb{Z}+r) \cup \mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}}).\end{aligned} \tag{78}$$

Moreover, applying consecutively Lemmas 6.5 and 6.1 and Theorem 2.23 we obtain

$$\begin{aligned}\boldsymbol{\delta}((b_{k_0}\mathbb{Z}+r) \cup \mathcal{M}_{\mathscr{B}\setminus\{b_{k_0}\}}) &= \lim_{k\to\infty} d((b_{k_0}\mathbb{Z}+r) \cup \mathcal{M}_{\{b_i:1\leq i\leq k, i\neq k_0\}})\\ &\geq \lim_{k\to\infty} d(\mathcal{M}_{\{b_i:1\leq i\leq k\}}) = \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}}) = \underline{d}(\mathcal{M}_{\mathscr{B}}).\end{aligned}$$

This, together with (78), completes the proof. □

Proof of Proposition 6.4. Fix $k_0 \geq 1$ such that $s_{k_0} > 1$. For $0 < r < b_{k_0}$, let

$$D_r := \underline{d}\left(\mathcal{M}_{\mathscr{B}} \cup (b_{k_0}\mathbb{Z} + r)\right)$$

and $D := \min_{0<r<b_{k_0}} D_r$. In view of Lemma 6.6, there exist $\varepsilon > 0$, $c > 0$ such that

$$D - \underline{d}(\mathcal{M}_{\mathscr{B}}) - 2\varepsilon > c > 0. \tag{79}$$

By Lemma 6.5, there exists $K \geq k_0$ such that

$$d(\mathcal{M}_{\{b_1,\ldots,b_K\}} \cup (b_{k_0}\mathbb{Z} + r)) \geq \underline{d}(\mathcal{M}_{\mathscr{B}} \cup (b_{k_0}\mathbb{Z} + r)) - \varepsilon. \tag{80}$$

Finally, let $N_0 \in \mathbb{N}$ be sufficiently large, so that for $N > N_0$, we have

$$\begin{aligned} \frac{1}{N \cdot b_1 \cdot \ldots \cdot b_K} &\left|[0, N \cdot b_1 \cdot \ldots \cdot b_K - 1] \cap (\mathcal{M}_{\{b_1,\ldots,b_K\}} \cup (b_{k_0}\mathbb{Z} + r))\right| \\ &\geq d(\mathcal{M}_{\{b_1,\ldots,b_K\}} \cup (b_{k_0}\mathbb{Z} + r)) - \varepsilon. \end{aligned} \tag{81}$$

Fix $N > N_0$ and take B which appears in $Y_{\geq s_1, \geq s_2, \ldots} \cap \widetilde{X}_\eta$, with $|B| = N \cdot b_1 \cdot \ldots \cdot b_K$. Then there exists $k \in \mathbb{Z}$ such that

$$B + k \leq \eta[k, k + N \cdot b_1 \cdot \ldots \cdot b_K - 1]. \tag{82}$$

It follows from (82) and the choice of k_0 that there exists $0 < r_0 < b_{k_0}$ such that

$$\text{supp } \eta \cap [k, k + N \cdot b_1 \cdot \ldots \cdot b_K - 1] \cap (b_{k_0}\mathbb{Z} + r_0) = \emptyset. \tag{83}$$

Therefore, using (83), (81), (80), the definition of D_{r_0} and D, and (79), we obtain

$$\begin{aligned} &\frac{|B| - |\text{supp } B|}{|B|} \\ &\geq \frac{1}{N \cdot b_1 \cdot \ldots \cdot b_K} \left|[k, k + N \cdot b_1 \cdot \ldots \cdot b_K - 1] \cap (\mathcal{M}_{\mathscr{B}} \cup (b_{k_0}\mathbb{Z} + r_0))\right| \\ &\geq \frac{1}{N \cdot b_1 \cdot \ldots \cdot b_K} \left|[k, k + N \cdot b_1 \cdot \ldots \cdot b_K - 1] \cap (\mathcal{M}_{\{b_1,\ldots,b_K\}} \cup (b_{k_0}\mathbb{Z} + r_0))\right| \\ &= \frac{1}{N \cdot b_1 \cdot \ldots \cdot b_K} \left|[0, N \cdot b_1 \cdot \ldots \cdot b_K - 1] \cap (\mathcal{M}_{\{b_1,\ldots,b_K\}} \cup (b_{k_0}\mathbb{Z} + r_0))\right| \\ &\geq d(\mathcal{M}_{\{b_1,\ldots,b_K\}} \cup (b_{k_0}\mathbb{Z} + r)) - \varepsilon \geq \underline{d}(\mathcal{M}_{\mathscr{B}} \cup (b_{k_0}\mathbb{Z} + r_0)) - 2\varepsilon \\ &= D_{r_0} - 2\varepsilon \geq D - 2\varepsilon > \underline{d}(\mathcal{M}_{\mathscr{B}}) + c. \end{aligned}$$

Thus

$$\frac{|\text{supp } B|}{|B|} < \overline{d}(\mathcal{F}_{\mathscr{B}}) - c. \tag{84}$$

We will now proceed as in the proof of Proposition K. For $n \in \mathbb{N}$, let

$$\gamma^{s_1,s_2,\cdots}(n) := |\{B \in \{0,1\}^n : B \text{ appears in } Y_{\geq s_1, \geq s_2, \ldots} \cap \widetilde{X}_\eta\}|$$

and, for $K \geq 1$, let

$$\gamma_K^{s_1,s_2,\ldots,s_K}(n) := |\{B \in \{0,1\}^n : |\text{supp } B| \leq b_k - s_k \text{ for } 1 \leq k \leq K\}|.$$

Clearly,

$$\gamma^{s_1,s_2,\cdots}(n) \leq \gamma_K^{s_1,s_2,\ldots,s_K}(n) \quad \text{for any } K \geq 1.$$

Consider the following procedure of defining a block $B \in \{0,1\}^n$:

(a) Choose $(r_1, \dots, r_K) \in \prod_{k \le K} \mathbb{Z}/b_k\mathbb{Z}$, set $B(j) := 0$ for $1 \le j \le n$ such that $j \equiv r_k \bmod b_k$ for some $1 \le k \le K$, choose $r'_{k_0} \not\equiv r_{k_0} \bmod b_{k_0}$, and set $B(j) := 0$ for $1 \le j \le n$ such that $j \equiv r_{k_0} \bmod b_{k_0}$.

(b) Complete the block by choosing arbitrarily $B(j) \in \{0,1\}$ for all other $1 \le j \le n$.

Notice that all $B \in \{0,1\}^n$ satisfying

$$|(\text{supp } B) \bmod b_k| \le \begin{cases} b_k - 1 & \text{for } k \ne k_0, \\ b_k - 2 & \text{for } k = k_0 \end{cases} \tag{85}$$

can be obtained this way. In particular, we obtain all blocks $B \in \{0,1\}^n$ such that

$$|(\text{supp } B) \bmod b_k| \le b_k - s_k \text{ for } k \ge 1.$$

Notice also that once the parameters $(r_1, \dots, r_K)$ and r'_{k_0} in step (a) are fixed, the freedom in step (b) gives, for $n = N \cdot b_1 \cdot \ldots \cdot b_K$, in view of (84), at most $2^{N \cdot b_1 \cdot \ldots \cdot b_K (\overline{d}(\mathcal{F}_{\mathscr{B}}) - c)}$ $N \cdot b_1 \cdot \ldots \cdot b_K$-blocks. It follows that

$$\begin{aligned} h_{top}(Y_{\ge s_1, \ge s_2, \dots} \cap \widetilde{X}_\eta, S) &= \lim_{n \to \infty} \frac{1}{n} \log \gamma^{s_1, s_2, \cdots}(n) \\ &\le \lim_{n \to \infty} \frac{1}{n} \log \gamma_K^{s_1, s_2, \dots, s_K}(n) \le \overline{d}(\mathcal{F}_{\mathscr{B}}) - c = h_{top}(\widetilde{X}_\eta, S) - c, \end{aligned}$$

which completes the proof. □

Corollary 6.7. *Suppose that* $\mathscr{B} \subset \mathbb{N}$ *is taut. Let* $\nu \in \mathcal{P}(\widetilde{X}_\eta, S)$ *be such that* $h(\widetilde{X}_\eta, \nu, S) = h_{top}(\widetilde{X}_\eta, S)$. *Then* $\nu(Y \cap \widetilde{X}_\eta) = 1$.

Proof. By considering the ergodic decomposition, we may restrict ourselves to $\nu \in \mathcal{P}^e(\widetilde{X}_\eta, S)$. Fix such ν and suppose that $h(\widetilde{X}_\eta, \nu, S) = h_{top}(\widetilde{X}_\eta, S)$ but $\nu(Y \cap \widetilde{X}_\eta) = 0$ (by the ergodicity of ν, we have $\nu(Y \cap \widetilde{X}_\eta) \in \{0,1\}$). Note that, for each $k \ge 1$, there exists $1 \le s_k < b_k$ such that $\nu(Y^k_{s_k} \cap \widetilde{X}_\eta) = 1$; i.e., we obtain $(s_k)_{k \ge 1}$ such that $\nu(Y_{s_1, s_2, \dots} \cap \widetilde{X}_\eta) = 1$, so, all the more, $\nu(Y_{\ge s_1, \ge s_2, \dots} \cap \widetilde{X}_\eta) = 1$. Since $\nu(Y \cap \widetilde{X}_\eta) = 0$, there exists $k \ge 1$ such that $s_k \ge 2$. But then, by Proposition 6.4 and the variational principle, $h(\widetilde{X}_\eta, \nu, S) = h(Y_{\ge s_1, \ge s_2, \dots} \cap \widetilde{X}_\eta, \nu, S) \le h_{top}(Y_{\ge s_1, \ge s_2, \dots} \cap \widetilde{X}_\eta, S) < h_{top}(\widetilde{X}_\eta, S)$. This contradicts our assumption. □

7. Tautness and support of ν_η (proof of Theorem H)

We will now use Theorem C and Proposition E to prove Theorem H.

Proof of Theorem H. Note that (a) ⇒ (b) by Corollary 6.7. To prove (b) ⇒ (c), we claim that (b) implies $\nu_\eta(\varphi(\theta(Y \cap \widetilde{X}_\eta))) = 1$. Then, since by Remark 2.46 we have $\varphi(\theta(Y \cap \widetilde{X}_\eta)) \subset Y$, it will follow that $\nu_\eta(Y) = 1$. Moreover, since, by Proposition E, we have $\nu_\eta(X_\eta) = 1$, we obtain (c). Thus, we are left to prove the claim. Recall that by Remark 2.46, we have $\theta_*(\nu) = \mathbb{P}$ for any $\nu \in \mathcal{P}(Y \cap \widetilde{X}_\eta, S)$ and, by definition, $\nu_\eta = \varphi_*(\mathbb{P})$. Therefore,

$$\begin{aligned} \nu_\eta(\varphi(\theta(Y \cap \widetilde{X}_\eta))) &= \mathbb{P}(\varphi^{-1}(\varphi(\theta(Y \cap \widetilde{X}_\eta)))) \ge \mathbb{P}(\theta(Y \cap \widetilde{X}_\eta)) \\ &= \theta_*\nu(\theta(Y \cap \widetilde{X}_\eta)) = \nu(\theta^{-1}(\theta(Y \cap \widetilde{X}_\eta))) \ge \nu(Y \cap \widetilde{X}_\eta) = 1, \end{aligned}$$

and the claim holds.

It remains to show that (c) implies (a). Suppose that $\mathscr{B}$ is not taut. Let $\mathscr{B}'$ be as in the proof of Theorem 4.5. For simplicity, we assume that $\mathscr{B}'$ is given by (40), i.e.,

$$\mathscr{B}' = (\mathscr{B} \setminus \bigcup_{n\geq 1} c_n\mathbb{Z}) \cup \{c_n : n \geq 1\} = (\mathscr{B} \setminus \bigcup_{n\geq 1} c_n\mathscr{A}^n) \cup \{c_n : n \geq 1\},$$

where $\mathscr{A}^n$, $n \geq 1$, are Behrend sets. By Theorem 4.5, $\mathscr{B}'$ is taut and we have $\nu_\eta = \nu_{\eta'}$. Let

$$Y' := \{x \in \{0,1\}^{\mathbb{Z}} : |\text{supp } x \bmod b'_k| = b'_k - 1 \quad \text{for each } k \geq 1\}.$$

Since (a) holds for $\mathscr{B}'$ and we have proved that (a) implies (c), we have $\nu_{\eta'}(Y' \cap X_{\eta'}) = 1$. We will show that $\nu_\eta(Y \cap X_\eta) = 0$. Since $\nu_\eta = \nu_{\eta'}$, it suffices to show that $Y \cap Y' = \emptyset$. Take $a \geq 2$ such that $c_1 a \in \mathscr{B}$ and $c_1 \in \mathscr{B}'$ and consider the natural projections

$$\mathbb{Z} \xrightarrow{\pi_1} \mathbb{Z}/c_1a\mathbb{Z} \xrightarrow{\pi_2} \mathbb{Z}/c_1\mathbb{Z},$$

where $\pi_1(n) = n \bmod c_1a$ for $n \in \mathbb{Z}$ and $\pi_2(n) = n \bmod c_1$ for $n \in \mathbb{Z}/c_1a\mathbb{Z}$. Then, for any $A \subset \mathbb{Z}$, we have $\pi_1(A) \subset \pi_2^{-1}(\pi_2(\pi_1(A)))$. Moreover, for any $B \subset \mathbb{Z}/c_1\mathbb{Z}$, we have $|\pi_2^{-1}(B)| = a|B|$. Therefore, for $x \in \{0,1\}^{\mathbb{Z}}$, we have

$$\begin{aligned}|\text{supp } x \bmod c_1a| = |\pi_1(\text{supp } x)| &\leq |\pi_2^{-1}(\pi_2(\pi_1(\text{supp } x)))| \\ &= a|\pi_2(\pi_1(\text{supp } x))| = a|\text{supp } x \bmod c_1|.\end{aligned}$$

Therefore,

$$\begin{aligned}Y' &\subset \{x \in \{0,1\}^{\mathbb{Z}} : |\text{supp } x \bmod c_1| = c_1 - 1\} \\ &\subset \{x \in \{0,1\}^{\mathbb{Z}} : |\text{supp } x \bmod c_1a| \leq c_1a - a\}.\end{aligned}$$

On the other hand, we have

$$Y \subset \{x \in \{0,1\}^{\mathbb{Z}} : |\text{supp } x \bmod c_1a| = c_1a - 1\}.$$

Since $c_1a - a < c_1a - 1$, we have $Y \cap Y' = \emptyset$. This completes the proof. □

Remark 7.1. If $\mathscr{B} \subset \mathbb{N}$ has light tails, then $\nu_\eta(Y \cap X_\eta) = 1$ can be shown directly. Namely, fix $K \geq 1$ and let

$$Y_K := \{x \in \{0,1\}^{\mathbb{Z}} : |\text{supp } x \bmod b_k| = b_k - 1 \quad \text{for } 1 \leq k \leq K\}.$$

Then $Y_K \cap X_\eta$ is S-invariant and $\eta \in Y_K \cap X_\eta$ (by Corollary 4.32); thus $Y_K \cap X_\eta \neq \emptyset$. Furthermore, $Y_K \cap X_\eta$ is open in X_η (indeed, if $x \in Y_K \cap X_\eta$ and $M \in \mathbb{N}$ is such that $\text{supp } x \bmod b_k = (\text{supp } x \cap [0, M]) \bmod b_k$ for each $k \geq 1$, then for each $y \in X_\eta$ with $y[0, M] = x[0, M]$, we have $y \in Y_K \cap X_\eta$).

In view of Theorem G, since $Y_K \cap X_\eta$ is open and non-empty, we have $\nu_\eta(Y_K \cap X_\eta) > 0$. By ergodicity and S-invariance, we obtain $\nu_\eta(Y_K \cap X_\eta) = 1$. It follows that $\nu_\eta(Y \cap X_\eta) = \nu_\eta(\bigcap_{K\geq 1} Y_K \cap X_\eta) = 1$.

8. Invariant measures (proof of Theorem I)

In [36], a description of $\mathcal{P}(X_\eta, S)$ was given in case of $\mathscr{B}$ Erdös (recall that in this case we have $X_\eta = \widetilde{X}_\eta$). Theorem I extends this result, yielding a description of $\mathcal{P}(\widetilde{X}_\eta, S)$ for all $\mathscr{B}$ (in particular, when $X_\eta = \widetilde{X}_\eta$, we obtain a description of $\mathcal{P}(X_\eta, S)$).

Remark 8.1. Notice that Theorem I is stated in a more compact form than in [36] (cf. Theorem 1.2 therein). What corresponds directly to [36] are Theorems 8.2 and 8.4. Notice that Theorem 8.4 is an immediate consequence of Theorem I (it suffices to take $b'_k = b_k$ for all $k \geq 1$). The role of $b'_k \mid b_k$, $k \geq 1$, will become more clear later when we discuss the discrete rational part of the spectrum of $(\widetilde{X}_\eta, \nu, S)$; see Section 8.3.

We will present only sketches of the proofs, referring the reader to [36] for the remaining details (which can be repeated word by word). We will restrict ourselves to the case when $\mathscr{B}$ is infinite (cf. Section 4.3.1).

8.1. Invariant measures on $Y \cap \widetilde{X}_\eta$ (Theorem I – first steps).

Theorem 8.2. *For any $\nu \in \mathcal{P}^e(Y \cap \widetilde{X}_\eta, S)$, there exists $\widetilde{\rho} \in \mathcal{P}^e(X_\eta \times \{0,1\}^{\mathbb{Z}}, S \times S)$ such that $\widetilde{\rho}|_{X_\eta} = \nu_\eta$ and $M_*(\widetilde{\rho}) = \nu$, where $M\colon X_\eta \times \{0,1\}^{\mathbb{Z}} \to \widetilde{X}_\eta$ stands for the coordinatewise multiplication.*

Proof. Fix $\nu \in \mathcal{P}^e(Y \cap \widetilde{X}_\eta, S)$. Note that $\nu \neq \delta_{\dots 0.00\dots}$. Note also that it suffices to find $\widetilde{\rho} \in \mathcal{P}(X_\eta \times \{0,1\}^{\mathbb{Z}}, S \times S)$ such that $\widetilde{\rho}|_{X_\eta} = \nu_\eta$ and $M_*(\widetilde{\rho}) = \nu$ and use the ergodic decomposition. Let

$$Y_\infty := \{y \in Y : |\text{supp } y \cap (-\infty, 0)| = |\text{supp } y \cap (0, \infty)| = \infty\}$$

(notice that the definition of Y_∞ is different from the one in [36]; we have changed the notation to simplify the proof). Since $\nu \neq \delta_{\dots 0.00\dots}$, we have $\nu(Y_\infty) = 1$. For $x \in \{0,1\}^{\mathbb{Z}}$, $z \in Y_\infty$, let $\widehat{x}_z$ be the sequence obtained by reading consecutive coordinates of x which are in supp z and such that

$$\widehat{x}_z(0) = x(\min\{k \geq 0 : k \in \operatorname{supp} z\}).$$

Step 1. We define $\widetilde{T}\colon G \times \{0,1\}^{\mathbb{Z}} \to G \times \{0,1\}^{\mathbb{Z}}$ by

$$\widetilde{T}(g, x) = \begin{cases} (Tg, x) & \text{if } \varphi(g)(0) = 0, \\ (Tg, Sx) & \text{if } \varphi(g)(0) = 1. \end{cases}$$

By Remark 2.46, for $y \in Y_\infty \cap \widetilde{X}_\eta$, we have $\varphi(\theta(y)) \in Y_\infty$. Let $\Theta\colon Y_\infty \cap \widetilde{X}_\eta \to G \times \{0,1\}^{\mathbb{Z}}$ be given by $\Theta(y) = (\theta(y), \widehat{y}_{\varphi(\theta(y))})$. One can show that

$$\widehat{Sx}_{Sz} = \begin{cases} \widehat{x}_z & \text{if } z(0) = 0, \\ S\widehat{x}_z & \text{if } z(0) = 1. \end{cases} \tag{86}$$

Hence, it follows from $\varphi \circ T = S \circ \varphi$ and Remark 2.46 that $\Theta \circ S = \widetilde{T} \circ \Theta$ on Y_∞.

Let $\Phi\colon \varphi^{-1}(Y_\infty) \times \{0,1\}^{\mathbb{Z}} \to X_{\mathscr{B}}$ be the unique element in $X_{\mathscr{B}}$ such that

$$\Phi(g, x) \leq \varphi(g) \quad \text{and} \quad \widehat{(\Phi(g, x))}_{\varphi(g)} = x.$$

Since $\nu(Y\cap\widetilde{X}_\eta)=1$, by Theorem H, we have that $\nu_\eta(Y\cap\widetilde{X}_\eta)=1$, so, in particular, $\nu_\eta\neq\delta_{\dots0.00\dots}$. It follows that Φ is well-defined a.e. with respect to any $\widetilde{T}$-invariant measure. Moreover, using (86), one can show that $S\circ\Phi=\Phi\circ\widetilde{T}$ on $\varphi^{-1}(Y_\infty)\times\{0,1\}^{\mathbb{Z}}$. Therefore, the following diagram commutes:

$$\begin{array}{ccc}
Y\cap\widetilde{X}_\eta & \xrightarrow{\;S\;} & Y\cap\widetilde{X}_\eta \\
\Big\downarrow \Theta & & \Big\downarrow \Theta \\
G\times\{0,1\}^{\mathbb{Z}} & \xrightarrow{\;\widetilde{T}\;} & G\times\{0,1\}^{\mathbb{Z}} \\
\Big\downarrow \Phi & & \Big\downarrow \Phi \\
X_{\mathscr{B}} & \xrightarrow{\;S\;} & X_{\mathscr{B}}
\end{array}$$

In particular, $\Phi\circ\Theta$ is well-defined a.e. with respect to any $\nu\in\mathcal{P}(Y\cap\widetilde{X}_\eta,S)$. Moreover, by the choice of Θ and Φ, we obtain

$$\Phi\circ\Theta=id \text{ a.e. with respect to any } \nu\in\mathcal{P}^e(Y\cap\widetilde{X}_\eta,S). \tag{87}$$

This gives, for any $\nu\in\mathcal{P}^e(Y\cap\widetilde{X}_\eta,S)$, the equality $\nu=\Phi_*\Theta_*\nu$, with $\Theta_*\nu\in\mathcal{P}^e(G\times\{0,1\}^{\mathbb{Z}},\widetilde{T})$.

Step 2. Let $\Psi\colon\varphi^{-1}(Y_\infty)\times\{0,1\}^{\mathbb{Z}}\to\varphi^{-1}(Y_\infty)\times\{0,1\}^{\mathbb{Z}}$ be given by $\Psi(g,x)=(g,\widehat{x}_{\varphi(g)})$. Note that Ψ is defined a.e. with respect to any $T\times S$-invariant measure, so that Ψ is onto a.e. with respect to any $\widetilde{T}$-invariant measure. Using again (86), one can show that diagram (88) commutes:

$$\begin{array}{ccc}
G\times\{0,1\}^{\mathbb{Z}} & \xrightarrow{\;T\times S\;} & G\times\{0,1\}^{\mathbb{Z}} \\
\Big\downarrow \Psi & & \Big\downarrow \Psi \\
G\times\{0,1\}^{\mathbb{Z}} & \xrightarrow{\;\widetilde{T}\;} & G\times\{0,1\}^{\mathbb{Z}}
\end{array} \tag{88}$$

Notice that $\emptyset\neq\Psi^{-1}(g,y)\subset\{g\}\times\{0,1\}^{\mathbb{Z}}$. Moreover, given $(g,x)\in\Psi^{-1}(g,y)$, all other points in $\Psi^{-1}(g,y)$ are obtained by changing in an arbitrary way these coordinates in x which are not in the support of $\varphi(g)$. In particular, each fiber $\Psi^{-1}(g,y)$ is infinite. For $k_1<\dots<k_s$ and $(i_1,\dots,i_s)\in\{0,1\}^s$, we define the following cylinder set:

$$C=C^{i_1,\dots,i_s}_{k_1,\dots,k_s}:=\{x\in\{0,1\}^{\mathbb{Z}}:x(k_j)=i_j,1\leq j\leq s\}. \tag{89}$$

For each such C and for $A\in\mathcal{B}(G)$, we put

$$\lambda_{(g,y)}(A\times C):=\mathbb{1}_A(g)\cdot 2^{-m},\ \text{ where } m=|\{1\leq j\leq s:\varphi(g)(k_j)=0\}|,$$

if $\Phi(g,y)(k_j)=i_j$ whenever $\varphi(g)(k_j)=1$ (otherwise we set $\lambda_{(g,y)}(A\times C):=0$). Now, as in [36], we can prove the following:

(a) the map $F\colon(g,y)\mapsto\lambda_{(g,y)}$ is measurable,
(b) $(T\times S)_*\lambda_{(g,y)}=\lambda_{\widetilde{T}(g,y)}$.

Then for any $\rho\in\mathcal{P}^e(G\times\{0,1\}^{\mathbb{Z}},\widetilde{T})$, we obtain

$$\widetilde{\rho}:=\int\lambda_{(g,y)}\,d\rho(g,y)\in\mathcal{P}(T\times S,G\times\{0,1\}^{\mathbb{Z}})\text{ with }\Psi_*\widetilde{\rho}=\rho.$$

Step 3. By the choice of Φ and Ψ, it follows that $M\circ(\varphi\times id_{\{0,1\}^{\mathbb{Z}}})=\Phi\circ\Psi$. Then, for any $\nu\in\mathcal{P}^e(Y\cap\widetilde{X}_\eta,S)$,

$$\nu=\Phi_*\Theta_*\nu=\Phi_*\Psi_*\widetilde{\Theta_*\nu}=M_*(\varphi\times id_{\{0,1\}^{\mathbb{Z}}})_*\widetilde{\Theta_*\nu},$$

with $\widetilde{\Theta_*\nu}\in\mathcal{P}(T\times S,G\times\{0,1\}^{\mathbb{Z}})$.

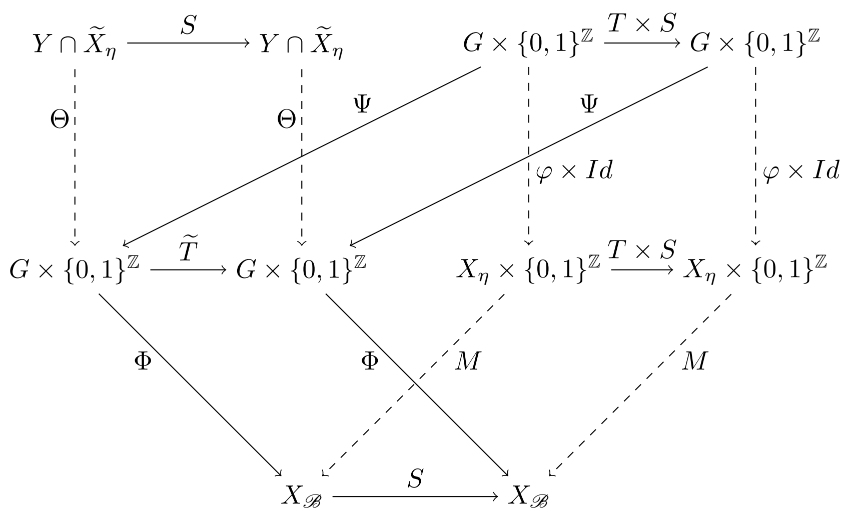
Step 4. To conclude it suffices to notice that

$$\varphi\times id_{\{0,1\}^{\mathbb{Z}}}\colon G\times\{0,1\}^{\mathbb{Z}}\to X_\eta\times\{0,1\}^{\mathbb{Z}}$$

induces a map from $\mathcal{P}(T\times S,G\times\{0,1\}^{\mathbb{Z}})$ to the simplex of probability $S\times S$-invariant measures on $X_\eta\times\{0,1\}^{\mathbb{Z}}$ whose projection onto the first coordinate is ν_η.

□

Remark 8.3. The above proof can be summarized on the following commuting diagram:



8.2. **Invariant measures on $\widetilde{X}_\eta$ (proof of Theorem I).** The main ingredient in the proof of Theorem I is the following result:

Theorem 8.4. *For any $\nu\in\mathcal{P}^e(\widetilde{X}_\eta,S)$ there exist $b_k'\mid b_k$, $k\geq1$, and $\widetilde{\rho}\in\mathcal{P}^e(X_{\eta'}\times\{0,1\}^{\mathbb{Z}},S\times S)$ such that $\widetilde{\rho}|_{X_{\eta'}}=\nu_{\eta'}$ and $M_*(\widetilde{\rho})=\nu$, where η' corresponds to $\mathscr{B}'=\{b_k':k\geq1\}$ and $M\colon X_{\eta'}\times\{0,1\}^{\mathbb{Z}}\to\widetilde{X}_{\eta'}$ stands for the coordinatewise multiplication.*

For the proof of Theorem 8.4 we will need several tools. Notice first that if $\nu=\delta_{\dots0.00\dots}$, then the above assertion holds true since $M_*(\delta_{\dots0.00\dots}\otimes\kappa)=\delta_{\dots0.00\dots}$ for any $\kappa\in\mathcal{P}(\{0,1\}^{\mathbb{Z}},S)$ and $\delta_{\dots0.00\dots}=\nu_{\eta'}$ for η' associated to $\mathscr{B}'=\{1\}$. Thus, we only need to cover the case $\nu\neq\delta_{\dots0.00\dots}$.

Recall that

$$\widetilde{X}_\eta=\bigcup_{k\geq1}\bigcup_{0\leq s_k\leq b_k}Y_{s_1,s_2,\dots}\cap\widetilde{X}_\eta$$

is a partition of $\widetilde{X}_\eta$ into Borel, S-invariant sets. Proceeding in a similar way as in Section 3.2 in [36], we will now further refine this partition.

Fix $\underline{s} = (s_k)_{k\geq 1}$ with $1 \leq s_k \leq b_k - 1$, $\underline{a} = (a_1^k, \dots, a_{s_k}^k)_{k\geq 1}$ with $a_i^k \in \mathbb{Z}/b_k\mathbb{Z}$ for $1 \leq i \leq s_k$ and $|\{a_1^k, \dots, a_{s_k}^k\}| = s_k$. Let

$$Y_{k,s_k;a_1,\dots,a_{s_k}} := \{x \in \{0,1\}^{\mathbb{Z}} : \text{supp } x \bmod b_k = \mathbb{Z}/b_k\mathbb{Z} \setminus \{a_1, \dots, a_{s_k}\}\}.$$

For each $k \geq 1$, any two sets of such form are either disjoint or they coincide. Since $\text{supp } Sx = \text{supp } x - 1$, we have

$$SY_{k,s_k;a_1^k,\dots,a_{s_k}^k} = Y_{k,s_k;a_1^k-1,\dots,a_{s_k}^k-1} \tag{90}$$

(note that subtraction is taken mod b_k). Let

$$b_k' := \min\{j \geq 1 : \{a_1^k, \dots, a_{s_k}^k\} = \{a_1^k - j, \dots, a_{s_k}^k - j\}\} \tag{91}$$

and note that $b_k' \geq 2$. Clearly, $S^{b_k'} Y_{k,s_k;a_1^k,\dots,a_{s_k}^k} = Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}$ and the sets

$$Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}, SY_{k,s_k;a_1^k,\dots,a_{s_k}^k}, \dots, S^{b_k'-1}Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}$$

are pairwise disjoint. Finally, we define

$$Y_{\underline{s},\underline{a}} := \bigcap_{k\geq 1} \bigcup_{j=0}^{b_k'-1} S^j Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}$$

(notice that if $s_k = 1$ for all $k \geq 1$, we have $Y_{\underline{s},\underline{a}} = Y$ for any choice of $\underline{a}$).

Fix $\underline{s}, \underline{a}$ and suppose that $\mathcal{P}(Y_{\underline{s},\underline{a}} \cap \widetilde{X}_\eta, S) \neq \emptyset$. Let

$$G_{\underline{s},\underline{a}} := \overline{\{\underline{n}_{\mathscr{B}'} : n \in \mathbb{Z}\}} \subset G_{\mathscr{B}'} = \prod_{k\geq 1} \mathbb{Z}/b_k'\mathbb{Z},$$

where b_k', $k \geq 1$, are as in (91); cf. (10). Define $\varphi_{\underline{s},\underline{a}} \colon G_{\underline{s},\underline{a}} \to \{0,1\}^{\mathbb{Z}}$ (cf. (16)) by

$$\varphi_{\underline{s},\underline{a}}(g)(n) = \begin{cases} 1 & \text{if } g_k - a_i^k + n \neq 0 \bmod b_k \quad \text{for all } k \geq 1, 1 \leq i \leq s_k, \\ 0 & \text{otherwise.} \end{cases}$$

We also define $\theta_{\underline{s},\underline{a}} \colon Y_{\underline{s},\underline{a}} \cap \widetilde{X}_\eta \to G_{\mathscr{B}'}$ in the following way; cf. (18):

$$\theta_{\underline{s},\underline{a}}(y) = g \iff -g_k + a_i^k \notin \text{supp}(y) \bmod b_k \quad \text{for all } 1 \leq i \leq s_k.$$

Notice that given $y \in Y_{\underline{s},\underline{a}}$ and $k_0 \geq 1$, there exists $N \geq 1$ such that

$$(\text{supp } y) \cap [-N, N] \bmod b_k = \mathbb{Z}/b_k\mathbb{Z} \setminus \{-g_k + a_i^k : 1 \leq i \leq s_k\} \quad \text{for } 1 \leq k \leq k_0. \tag{92}$$

Furthermore, we claim that $\theta_{\underline{s},\underline{a}}(Y_{\underline{s},\underline{a}} \cap \widetilde{X}_\eta) \subset G_{\underline{s},\underline{a}}$. Indeed, take $y \in Y_{\underline{s},\underline{a}} \cap \widetilde{X}_\eta$. Given $k_0 \geq 1$, let $N \geq 1$ be such that (92) holds and let $M \in \mathbb{Z}$ be such that $y[-N, N] \leq \eta[-N + M, N + M]$. It follows that $\theta(y) = (g_1, g_2, \dots)$, where $g_k \equiv -M \bmod b_k$ for $1 \leq k \leq k_0$. This yields the claim.

Remark 8.5. Note that $\theta_{\underline{s},\underline{a}}$ is continuous. Indeed, given $y \in Y_{\underline{s},\underline{a}}$ and $k_0 \geq 1$, let N be such that (92) holds. Then, if $y' \in Y_{\underline{s},\underline{a}}$ is sufficiently close to y, (92) holds for y' as well. Therefore, if $y_n \to y$ in $Y_{\underline{s},\underline{a}}$, then $\theta_{\underline{s},\underline{a}}(y_n) \to \theta_{\underline{s},\underline{a}}(y)$.

Denote by $T_{\underline{s},\underline{a}} \colon G_{\underline{s},\underline{a}} \to G_{\underline{s},\underline{a}}$ the map given by $T_{\underline{s},\underline{a}} g = g + \underline{1}_{\mathscr{B}'} = (g_1 + 1, g_2 + 1, \dots)$, where $g = (g_1, g_2, \dots)$.

Remark 8.6 (cf. Remark 2.46). We have:

- $T_{\underline{s},\underline{a}} \circ \theta_{\underline{s},\underline{a}} = \theta_{\underline{s},\underline{a}} \circ S$,
- for each $y \in Y_{\underline{s},\underline{a}} \cap \widetilde{X}_\eta$, $y \leq \varphi_{\underline{s},\underline{a}}(\theta_{\underline{s},\underline{a}}(y))$,
- for any $\nu \in \mathcal{P}(Y_{\underline{s},\underline{a}} \cap \widetilde{X}_\eta, S)$, $(\theta_{\underline{s},\underline{a}})_*(\nu) = \mathbb{P}_{\underline{s},\underline{a}}$.

Lemma 8.7. *Suppose that* $\mathcal{P}(Y_{\underline{s},\underline{a}} \cap \widetilde{X}_\eta, S) \neq \emptyset$. *Then* $(\varphi_{\underline{s},\underline{a}})_*(\mathbb{P}_{\underline{s},\underline{a}})(Y_{\underline{s},\underline{a}}) = 1$. *In particular,* $(\varphi_{\underline{s},\underline{a}})_*(\mathbb{P}_{\underline{s},\underline{a}}) \neq \delta_{\dots 0.00\dots}$.

Proof. Take $\nu \in \mathcal{P}(Y_{\underline{s},\underline{a}} \cap \widetilde{X}_\eta, S)$. It follows from Remark 8.6 that

$$(\varphi_{\underline{s},\underline{a}})_*(\mathbb{P}_{\underline{s},\underline{a}})(Y_{\underline{s},\underline{a}}) = (\varphi_{\underline{s},\underline{a}})_*(\theta_{\underline{s},\underline{a}})_*(\nu)(Y_{\underline{s},\underline{a}}) \geq \nu(Y_{\underline{s},\underline{a}}) = 1.$$

Since $\dots 0.00 \dots \notin Y_{\underline{s},\underline{a}}$, we conclude. □

For $n \in \mathbb{N}$, let $M^{(n)} \colon (\{0,1\}^{\mathbb{Z}})^{\times n} \to \{0,1\}^{\mathbb{Z}}$ be given by

$$M^{(n)}((x_i^{(1)})_{i\in\mathbb{Z}}, \dots, (x_i^{(n)})_{i\in\mathbb{Z}}) = (x_i^{(1)} \cdot \ldots \cdot x_i^{(n)})_{i\in\mathbb{Z}}.$$

Moreover, we define $M^{(\infty)} \colon (\{0,1\}^{\mathbb{Z}})^{\mathbb{N}} \to \{0,1\}^{\mathbb{Z}}$ as

$$M^{(\infty)}((x_i^{(1)})_{i\in\mathbb{Z}}, (x_i^{(2)})_{i\in\mathbb{Z}}, \dots) = (x_i^{(1)} \cdot x_i^{(2)} \cdot \ldots)_{i\in\mathbb{Z}}.$$

Lemma 8.8 (Cf. Lemma 2.2.22 in [36])**.** *We have* $(\varphi_{\underline{s},\underline{a}})_*(\mathbb{P}_{\underline{s},\underline{a}}) = M_*^{(\infty)}(\rho)$, *where* ρ *is a joining of a countable number of copies of* $(\{0,1\}^{\mathbb{Z}}, \nu_{\eta'}, S)$.

Proof. The proof is the same as in [36]. □

Lemma 8.9 (Lemma 2.2.23 in [36])**.** *Let* $\nu_1, \dots, \nu_n, \nu_{n+1} \in \mathcal{P}(\{0,1\}^{\mathbb{Z}}, S)$. *Then for any joinings*

- $\rho_{1,n} \in J((\{0,1\}^{\mathbb{Z}}, \nu_1, S), \dots, (\{0,1\}^{\mathbb{Z}}, \nu_n, S))$,
- $\rho_{(1,n),n+1} \in J((\{0,1\}^{\mathbb{Z}}, M_*^{(n)}(\rho_{1,n}), S), (\{0,1\}^{\mathbb{Z}}, \nu_{n+1}, S))$

there exist

- $\rho_{2,n+1} \in J((\{0,1\}^{\mathbb{Z}}, \nu_2, S), \dots, (\{0,1\}^{\mathbb{Z}}, \nu_n, S), (\{0,1\}^{\mathbb{Z}}, \nu_{n+1}, S))$,
- $\rho_{1,(2,n+1)} \in J((\{0,1\}^{\mathbb{Z}}, \nu_1, S), (\{0,1\}^{\mathbb{Z}}, M_*^{(n)}(\rho_{2,n+1}), S))$

such that $M_*^{(2)}(\rho_{(1,n),n+1}) = M_*^{(2)}(\rho_{1,(2,n+1)})$.

Remark 8.10. We could write the equality $M_*^{(2)}(\rho_{(1,n),n+1}) = M_*^{(2)}(\rho_{1,(2,n+1)})$ as $M_*^{(2)}(M_*^{(n)}(\nu_1 \vee \dots \vee \nu_n) \vee \nu_{n+1}) = M_*^{(2)}(\nu_1 \vee M_*^{(n)}(\nu_2 \vee \ldots \vee \nu_n \vee \nu_{n+1}))$. However, until we say which joining we mean by each symbol $\vee$, this expression has no concrete meaning.

Remark 8.11. The above lemma remains true when we consider infinite joinings; i.e., instead of $\nu_1, \dots, \nu_n$ we have $\nu_1, \nu_2, \dots$, and instead of $M^{(n)}$ we consider $M^{(\infty)}$.

Proof of Theorem 8.4. Fix $\delta_{\dots 0.00\dots} \neq \nu \in \mathcal{P}^e(\widetilde{X}_\eta, S)$ and let $\underline{s}, \underline{a}$ be such that $\nu(Y_{\underline{s},\underline{a}} \cap \widetilde{X}_\eta) = 1$. In view of Lemma 8.8, Lemma 8.9, and Remark 8.11, it suffices to show that there exists $\widetilde{\rho} \in \mathcal{P}(\{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}, S \times S)$ such that the projection of $\widetilde{\rho}$ onto the first coordinate equals $(\varphi_{\underline{s},\underline{a}})_*(\mathbb{P}_{\underline{s},\underline{a}})$ and $M_*(\widetilde{\rho}) = \nu$.

By Lemma 8.7, we have $(\varphi_{\underline{s},\underline{a}})_*(\mathbb{P}_{\underline{s},\underline{a}}) \neq \delta_{\dots 0.00\dots}$. The remaining part of the proof goes exactly along the same lines as the proof of Theorem 8.2, with the following modification: we need to replace some objects related to Y by their counterparts related to $Y_{\underline{s},\underline{a}}$. Namely, instead of G, Θ, Y_∞, $\widetilde{T}$, Φ, and Ψ, we use

$$G_{\underline{s},\underline{a}},\ \Theta_{\underline{s},\underline{a}},\ (Y_{\underline{s},\underline{a}})_\infty,\ \widetilde{T}_{\underline{s},\underline{a}},\ \Phi_{\underline{s},\underline{a}},\ \text{and } \Psi_{\underline{s},\underline{a}},$$

where

- $\Theta_{\underline{s},\underline{a}}\colon Y_{\underline{s},\underline{a}}\cap\widetilde{X}_\eta \to G_{\underline{s},\underline{a}}\times\{0,1\}^{\mathbb{Z}}$ is given by $\Theta_{\underline{s},\underline{a}}(y):=(\theta_{\underline{s},\underline{a}}(y),\widehat{y}_{\varphi_{\underline{s},\underline{a}}(\theta_{\underline{s},\underline{a}}y)})$,
- $(Y_{\underline{s},\underline{a}})_\infty := \{y\in Y_{\underline{s},\underline{a}} : |\text{supp } y\cap(-\infty,0)| = |\text{supp } y\cap(0,\infty)|=\infty\}$,
- $\widetilde{T}_{\underline{s},\underline{a}}\colon G_{\underline{s},\underline{a}}\times\{0,1\}^{\mathbb{Z}}\to G_{\underline{s},\underline{a}}\times\{0,1\}^{\mathbb{Z}}$ given by
$$\widetilde{T}_{\underline{s},\underline{a}}(g,x)=\begin{cases}(T_{\underline{s},\underline{a}}g,x) & \text{if } \varphi_{\underline{s},\underline{a}}(g)(0)=0,\\ (T_{\underline{s},\underline{a}}g,Sx) & \text{if } \varphi_{\underline{s},\underline{a}}(g)(0)=1,\end{cases}$$
- $\Phi_{\underline{s},\underline{a}}(g,x)$ is the unique element in $X_{\mathscr{B}}$ such that
  (i) $\Phi_{\underline{s},\underline{a}}(g,x)\leq\varphi_{\underline{s},\underline{a}}(g)$,
  (ii) $(\Phi_{\underline{s},\underline{a}}(g,x))\widehat{\ }_{\varphi_{\underline{s},\underline{a}}(g)}=x$; i.e., the consecutive coordinates of x can be found in $\Phi_{\underline{s},\underline{a}}(g,x)$ along $\varphi_{\underline{s},\underline{a}}(g)$,
- $\Psi_{\underline{s},\underline{a}}(g,x)=(g,\widehat{x}_{\varphi_{\underline{s},\underline{a}}(g)})$. ☐

Lemma 8.12 (Cf. the proof of Lemma 2.2.22 in [36])**.** *Fix $b_k'\mid b_k$ for $k\geq 1$. Then there exists $\rho\in J((X_\eta,\nu_\eta,S),(X_\eta,\nu_\eta,S),\dots)$ such that $\nu_{\eta'}=M_*^{(\infty)}(\rho)$.*

Proof. For $i\geq 1$, let $R^{(i)}\colon G\to G$ be given by $R^{(i)}(g)=(g_k+ib_k')_{k\geq1}$. We claim that
$$(M^{(\infty)})_*(\varphi\circ R^{(1)}\times\varphi\circ R^{(2)}\times\dots)_*(\mathbb{P})=\nu_{\eta'}.$$
In order to prove this, we will first show that
$$M^{(\infty)}\circ(\varphi\circ R^{(1)}\times\varphi\circ R^{(2)}\times\dots)=\varphi'\circ p,$$
where $p\colon G\to G'$ is the natural projection and φ' is defined analogously as φ, using $\mathscr{B}'$ instead of $\mathscr{B}$. Indeed, we have
$$\begin{aligned}&M^{(\infty)}\circ(\varphi\circ R^{(1)}\times\varphi\circ R^{(2)}\times\dots)(g)(n)=1\\ &\iff \varphi\circ R^{(i)}(g)(n)=1\quad\text{for all } i\geq1\\ &\iff g_k+ib_k'+n\not\equiv 0 \bmod b_k\quad \text{for all } i\geq1,k\geq1\\ &\iff g_k+n\not\equiv 0 \bmod b_k'\quad\text{for all } k\geq1\end{aligned}$$
and, on the other hand,
$$\begin{aligned}\varphi'\circ p(g)(n)=1 &\iff (g_k \bmod b_k')+n\not\equiv 0\bmod b_k'\quad\text{for all } k\geq1\\ &\iff g_k+n\not\equiv0\bmod b_k'\quad\text{for all }k\geq1.\end{aligned}$$
This completes the proof as $(\varphi'\circ p)_*(\mathbb{P})=\varphi'_*(p_*(\mathbb{P}))=\varphi'_*(\mathbb{P}')=\nu_{\eta'}$ and $R_*^{(i)}(\mathbb{P})=\mathbb{P}$ for each $i\geq1$. ☐

Proof of Theorem I. The assertion is a consequence of Theorem 8.2, Theorem 8.4, Lemma 8.12, Lemma 8.9, and Remark 8.11. ☐

8.3. **Rational discrete spectrum (proof of Theorem F).**

Remark 8.13. Let $\underline{s},\underline{a}$ be such that $\mathcal{P}(Y_{\underline{s},\underline{a}},S)\neq\emptyset$ and fix $\nu\in\mathcal{P}(Y_{\underline{s},\underline{a}},S)$. Let $b_k'\mid b_k$, $k\geq1$, be as in the proof of Theorem 8.4. Recall (from the proof of Theorem 8.4) that there is an equivariant map $\Theta_{\underline{s},\underline{a}}\colon Y_{\underline{s},\underline{a}}\to G_{\underline{s},\underline{a}}\times\{0,1\}^{\mathbb{Z}}$. It follows that $(G_{\underline{s},\underline{a}},\mathbb{P}_{\underline{s},\underline{a}},T_{\underline{s},\underline{a}})$ is a factor of $(Y_{\underline{s},\underline{a}},\nu,S)$. In particular, the rational discrete spectrum of $(Y_{\underline{s},\underline{a}},\nu,S)$ includes all b_k'-roots of unity.

Theorem 8.14. *Suppose that $\mathcal{P}(Y_{\underline{s},\underline{a}}\cap\widetilde{X}_\eta,S)\neq\emptyset$. Then $\varphi_{\underline{s},\underline{a}}$ yields an isomorphism of $(G_{\underline{s},\underline{a}},\mathbb{P}_{\underline{s},\underline{a}},T_{\underline{s},\underline{a}})$ and $(Y_{\underline{s},\underline{a}}\cap\widetilde{X}_{\underline{s},\underline{a}},(\varphi_{\underline{s},\underline{a}})_*(\mathbb{P}_{\underline{s},\underline{a}}),S)$.*

Proof. Since, by Lemma 8.7, we have $(\varphi_{\underline{s},\underline{a}})_*(\mathbb{P}_{\underline{s},\underline{a}})(Y_{\underline{s},\underline{a}}) = 1$, we obtain the following equivariant maps:

$$(G_{\underline{s},\underline{a}}, \mathbb{P}_{\underline{s},\underline{a}}, T_{\underline{s},\underline{a}}) \xrightarrow{\varphi_{\underline{s},\underline{a}}} (Y_{\underline{s},\underline{a}} \cap \widetilde{X}_\eta, (\varphi_{\underline{s},\underline{a}})_*(\mathbb{P}_{\underline{s},\underline{a}}), S) \xrightarrow{\theta_{\underline{s},\underline{a}}} (G_{\underline{s},\underline{a}}, \mathbb{P}_{\underline{s},\underline{a}}, T_{\underline{s},\underline{a}}).$$

It follows from the coalescence of $(G_{\underline{s},\underline{a}}, \mathbb{P}_{\underline{s},\underline{a}}, T_{\underline{s},\underline{a}})$ that $\varphi_{\underline{s},\underline{a}}$ yields an isomorphism of $(G_{\underline{s},\underline{a}}, \mathbb{P}_{\underline{s},\underline{a}}, T_{\underline{s},\underline{a}})$ and $(Y_{\underline{s},\underline{a}}, (\varphi_{\underline{s},\underline{a}})_*(\mathbb{P}_{\underline{s},\underline{a}}), S)$. □

Proof of Theorem F. The assertion follows from the above and Corollary 6.7. □

9. Tautness revisited

9.1. **Tautness and combinatorics revisited (proof of Theorem L).** We will prove an extension of Corollaries 4.35 and 4.36.

Corollary 9.1. *Let $\mathscr{B}, \mathscr{B}' \subset \mathbb{N}$ and suppose that $\mathscr{B}$ is taut. Conditions* (a)–(f) *from Corollary* 4.35 *are equivalent to each of the following:*

- (g) $\nu_\eta \in \mathcal{P}(\widetilde{X}_{\eta'}, S)$,
- (h) $\mathcal{P}(\widetilde{X}_\eta, S) \subset \mathcal{P}(\widetilde{X}_{\eta'}, S)$.

Proof. Notice first that (e) from Corollary 4.35 implies (g). Suppose now that (g) holds. In view of Theorem I and Lemma 8.9, this yields (h). Suppose that (h) holds. By the variational principle, we have $h_{top}(\widetilde{X}_\eta, S) = h_{top}(\widetilde{X}_\eta \cap \widetilde{X}_{\eta'}, S)$. Moreover, since $\widetilde{X}_\eta \cap \widetilde{X}_{\eta'} \subset X_{\mathscr{B}} \cap X_{\mathscr{B}'} = X_{\mathscr{B}\cup\mathscr{B}'} \subset X_{\mathscr{B}}$, we have

$$h_{top}(\widetilde{X}_\eta \cap \widetilde{X}_{\eta'}, S) \le h_{top}(X_{\mathscr{B}\cup\mathscr{B}'}, S) \le h_{top}(X_{\mathscr{B}}, S).$$

By Proposition K, it follows that $h_{top}(\widetilde{X}_\eta, S) = h_{top}(X_{\mathscr{B}}, S)$. By the above, we obtain

$$h_{top}(X_{\mathscr{B}}, S) = h_{top}(X_{\mathscr{B}\cup\mathscr{B}'}, S). \tag{93}$$

Moreover, since $X_{\mathscr{B}\cup\mathscr{B}'} \subset X_{\mathscr{B}\cup\{b'\}} \subset X_{\mathscr{B}}$ for any $b' \in \mathscr{B}'$, (93) yields

$$h_{top}(X_{\mathscr{B}}, S) = h_{top}(X_{\mathscr{B}\cup\{b'\}}, S) \quad \text{for any } b' \in \mathscr{B}'.$$

It follows from Proposition K that $\boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}}) = \boldsymbol{\delta}(\mathcal{M}_{\mathscr{B}\cup\{b'\}})$. In view of Corollary 2.31, either $b' \in \mathcal{M}_{\mathscr{B}}$ or $\mathscr{B}$ is not taut. The latter is impossible; hence $b \mid b'$ for some $b \in \mathscr{B}$, and we conclude that (b) from Corollary 4.35 holds. □

Corollary 9.2. *Suppose that $\mathscr{B}, \mathscr{B}' \subset \mathbb{N}$ are taut. Conditions* (a)–(g) *from Corollary* 4.36 *are equivalent to each of the following:*

- (h) $\nu_\eta = \nu_{\eta'}$,
- (i) $\nu_\eta \in \mathcal{P}(\widetilde{X}_{\eta'}, S)$ *and* $\nu_{\eta'} \in \mathcal{P}(\widetilde{X}_\eta, S)$,
- (j) $\mathcal{P}(\widetilde{X}_\eta, S) = \mathcal{P}(\widetilde{X}_{\eta'}, S)$.

Proof. Clearly, (c) from Corollary 4.36 together with Proposition E implies (h). Moreover, (h) implies (i) and, by Corollary 9.1, (i) implies (j). Suppose now that (j) holds. So, (h) from Corollary 9.1 also holds. Applying again Corollary 9.1, we obtain (a) from Corollary 4.35. Moreover, (a) from Corollary 4.35 still holds when we exchange the roles of $\mathscr{B}$ and $\mathscr{B}'$. Therefore, we conclude that $X_{\mathscr{B}'} = X_{\mathscr{B}}$; i.e., (a) from Corollary 4.36 holds. □

Proof of Theorem L. The result follows directly from Corollary 9.2. □

9.2. **Tautness and invariant measures (proof of Theorem C).**

Proof of Theorem C. The assertion is an immediate consequence of Theorem 4.5, Theorem I, and Theorem L. □

We will now prove Corollary 1.5. For this, we will need the following folklore result (cf. Theorem 2.3 in [37]):

Lemma 9.3. *Let (X,T) be a topological dynamical system and let $X' \subset X$ be compact and T-invariant. Then the following are equivalent:*

(a) $\mathcal{P}(X,T) = \mathcal{P}(X',T)$,
(b) *for each $x \in X$, we have* $\lim_{n\to\infty, n\notin E_x} d(T^n x, X') = 0$, *where* $d(E_x) = 0$,
(c) *the set $X \setminus X'$ is universally null; that is, $\mu(X\setminus X') = 0$ for each $\mu \in \mathcal{P}(X,T)$.*

Definition 9.4. When (b) of Lemma 9.3 holds, we say that X' is a *quasi-attractor* in (X,T). Sometimes, the smallest possible quasi-attractor is called the *measure center.*

Proof of Corollary 1.5. The assertion follows immediately from Theorem C and Lemma 9.3. □

It can be rephrased as follows:

Corollary 9.5. *For any $\mathscr{B} \subset \mathbb{N}$, the subshift $(\widetilde{X}_\eta, S)$ has a quasi-attractor of the form $\widetilde{X}_{\eta'}$ for some taut set $\mathscr{B}'$ such that $\mathcal{F}_{\mathscr{B}'} \subset \mathcal{F}_{\mathscr{B}}$. Moreover, such $\mathscr{B}'$ is unique.*

Remark 9.6. Using Theorems D and G, we can deduce that if $\mathscr{B}'$ above has light tails and contains an infinite coprime set, then $X_{\eta'}$ is the measure center of $(\widetilde{X}_\eta, S)$.

10. Intrinsic ergodicity (proof of Theorem J)

Theorem 10.1. *Let $\mathscr{B} \subset \mathbb{N}$ and suppose that $\mathscr{B}$ is taut. Then $(\widetilde{X}_\eta, S)$ is intrinsically ergodic. In particular, if $X_\eta = \widetilde{X}_\eta$, then (X_η, S) is intrinsically ergodic.*

The above theorem extends Theorem 1.1 from [36] for $\mathscr{B}$ Erdös to the case when $\mathscr{B}$ is taut. The main ideas for the proof of Theorem 10.1 come from [36]. We will present the sketch of the proof only, referring the reader to [36] for the remaining details.

Proof of Theorem 10.1. We will use the notation introduced in the proof of Theorem 8.2. There exists $C_0 \subset C$ (recall that C was defined in (14)) such that every point from C_0 returns to C infinitely often under T and $\mathbb{P}(C_0) = \mathbb{P}(C)$. It follows that every point from $C_0 \times \{0,1\}^{\mathbb{Z}}$ returns to $C \times \{0,1\}^{\mathbb{Z}}$ infinitely often under $\widetilde{T}$ and $\nu(C_0 \times \{0,1\}^{\mathbb{Z}}) = \nu(C \times \{0,1\}^{\mathbb{Z}})$ for every $\nu \in \mathcal{P}(G \times \{0,1\}^{\mathbb{Z}}, \widetilde{T})$. Thus, the induced transformation $\widetilde{T}_{C\times\{0,1\}^{\mathbb{Z}}}$ is well-defined. Recall that

$$\widetilde{T}(g,x) = \begin{cases} (Tg,x) & \text{if } g \notin C,\\ (Tg,Sx) & \text{if } g \in C.\end{cases}$$

It follows that $\widetilde{T}_{C\times\{0,1\}^{\mathbb{Z}}} = T_C \times S$ a.e. for any $\widetilde{T}$-invariant measure (cf. the definitions of $\widetilde{T}$ and C).

We will show now that $\widetilde{T}$ has a unique measure of maximal (measure-theoretic) entropy. In view of Abramov's formula, for this it suffices to show that $\widetilde{T}_{C\times\{0,1\}} = T_C \times S$ has a unique measure of maximal entropy. For any $T_C\times S$-invariant measure κ, by the Pinsker formula, we have

$$\begin{aligned}h(\{0,1\}^{\mathbb{Z}}, \kappa|_{\{0,1\}^{\mathbb{Z}}}, S) &\leq h(C\times\{0,1\}^{\mathbb{Z}}, \kappa, T_C\times S)\\ &\leq h(C,\kappa|_C, T_C) + h(\{0,1\}^{\mathbb{Z}}, \kappa|_{\{0,1\}^{\mathbb{Z}}}, S) = h(\{0,1\}^{\mathbb{Z}}, \kappa|_{\{0,1\}^{\mathbb{Z}}}, S).\end{aligned} \tag{94}$$

Since κ can be arbitrary, it follows that a measure κ has the maximal entropy among all $T_C\times S$-invariant measures if and only if $h(C\times\{0,1\}^{\mathbb{Z}}, \kappa, T_C\times S) = h_{top}(S)$. Moreover, κ is a measure of maximal entropy for $T_C\times S$ if and only if $\kappa|_{\{0,1\}^{\mathbb{Z}}}$ is the measure of maximal entropy for S; that is, $\kappa|_{\{0,1\}^{\mathbb{Z}}}$ is the Bernoulli measure $B(1/2,1/2)$, i.e., when κ is a joining of the unique invariant measure for T_C and $B(1/2,1/2)$. Since the unique invariant measure for T_C is of zero entropy, it follows from the disjointness of K-automorphisms with zero entropy automorphisms [24] that κ is the product measure. In particular, κ is unique.

It follows from (87) that Θ is 1-1. Hence, $\Theta_*\colon \mathcal{P}(Y\cap\widetilde{X}_\eta, S)\to\mathcal{P}(G\times\{0,1\}^{\mathbb{Z}},\widetilde{T})$ is also 1-1, and for any $\nu\in\mathcal{P}(Y\cap\widetilde{X}_\eta,S)$, we have

$$h(Y\cap\widetilde{X}_\eta,\nu,S) = h(G\times\{0,1\}^{\mathbb{Z}},\Theta_*\nu,\widetilde{T}).$$

The result follows now from Corollary 6.7. □

Remark 10.2. Suppose that $\mathscr{B}\subset\mathbb{N}$ is taut. Notice that we have $\Psi_*(\mathbb{P}\otimes B(1/2,1/2)) = \mathbb{P}\otimes B(1/2,1/2)$. Moreover,

$$(\mathbb{P}\otimes B(1/2,1/2))_{C\times\{0,1\}^{\mathbb{Z}}} = \mathbb{P}_C\otimes B(1/2,1/2).$$

Since $h(C\times\{0,1\}^{\mathbb{Z}},\mathbb{P}_C\otimes B(1/2,1/2),T_C\times S) = \log 2$, it follows from the above proof of Theorem 10.1 that

$$\begin{aligned}\Phi_*\Psi_*(\mathbb{P}\otimes B(1/2,1/2)) &= M_*(\varphi\times id)_*(\mathbb{P}\otimes B(1/2,1/2))\\ &= M_*(\nu_\eta\otimes B(1/2,1/2))\end{aligned}$$

is the unique measure of maximal entropy for $(\widetilde{X}_\eta, S)$.

Proof of Theorem J. The assertion is an immediate consequence of Theorem C and Theorem 10.1. □

11. Remarks on abundant numbers

For $n\in\mathbb{N}$, consider the aliquot sum $\sigma(n) := \sum_{d|n} d$. Then, $n\in\mathbb{N}$ is called *abundant* if $\sigma(n)\geq 2n$, *perfect* if $\sigma(n)=2n$, and *deficient* if $\sigma(n)<2n$. We denote the set of abundant, perfect, and deficient numbers by $\mathbf{A}$, $\mathbf{P}$, and $\mathbf{D}$, respectively. Notice that $\mathbf{A}$ is closed under taking multiples. It follows that $\mathbf{A}=\mathbb{N}\cap\mathcal{M}_{\mathscr{B}_{\mathbf{A}}}$ and $\mathbf{D}=\mathbb{N}\cap\mathcal{F}_{\mathscr{B}_{\mathbf{A}}}$ for some primitive $\mathscr{B}_{\mathbf{A}}\subset\mathbb{N}$.

Lemma 11.1. *The set $\mathscr{B}_{\mathbf{A}}$ is thin. In particular, $\mathscr{B}_{\mathbf{A}}$ has light tails and is Besicovitch.*

Proof. Erdös [19] showed that $|\mathscr{B}_{\mathbf{A}} \cap [0,n]| = o(n/\log^2 n)$. Let j_n be the n-th $\mathscr{B}_{\mathbf{A}}$-free natural number. Therefore, for n sufficiently large, $n \leq j_n(\log^2 j_n)^{-1}$. It follows that, for large n, we have $n\log^2 n \leq n \log^2 j_n \leq j_n$, whence

$$\sum_{b\in\mathscr{B}_{\mathbf{A}}} {}^1\!/b = \sum_{n\geq 1} {}^1\!/j_n \leq \sum_{n\geq 1} {}^1\!/n\log^2 n < \infty; \tag{95}$$

i.e., $\mathscr{B}_{\mathbf{A}}$ is thin. But thin sets are Besicovitch. □

Lemma 11.2 (E.g. [23], Chapter 8.14, p. 243)**.** *We have $d(\mathbf{P}) = 0$.*

Corollary 11.3. *Suppose that $A, D \subset \mathbb{N}$ are finite sets consisting of abundant and deficient numbers, respectively. Then the density of the set of $n \in \mathbb{N}$ such that $A+n$ and $D+n$ consist of abundant and deficient numbers, respectively, is positive.*

Proof. By Lemma 11.1 and Corollary 1.9, we have

$$d(\{n \in \mathbb{N} : A + n \subset \mathbf{A} \text{ and } F + n \subset \mathbf{D} > 0.$$

□

Since $\{1, 2, 3, 4, 5\} \subset \mathbf{D}$, the following result immediately follows.

Corollary 11.4. *The set of $n \in \mathbb{N}$ such that the numbers $n+1, n+2, \ldots, n+5$ are deficient has positive density.*

Remark 11.5. Notice that Corollary 11.4 yields an independent proof and strengthens the result from [51] that there are infinitely many sequences of 5 consecutive deficient numbers.

Lemma 11.6. *The set $\mathscr{B}_{\mathbf{A}}$ contains an infinite coprime subset.*

Proof. It follows from [20] that $\bigcap_{1\leq k\leq K}(\mathcal{M}_{\mathscr{B}_{\mathbf{A}}} - k) \neq \emptyset$ for any $K \geq 1$, i.e., $\ldots 0.00\ldots \in X_\eta$. To conclude, it suffices to use Theorem B. □

Remark 11.7. Another way to prove the above lemma is to use the algorithm presented in [31], outputting the smallest element of $\mathbf{A} \setminus \mathbf{P}$ not divisible by the first k primes.

Corollary 11.8. *Denote by (n_j) the sequence of consecutive deficient numbers. Then, for any $K \geq 1$,*

$$\limsup_{j\to\infty} \min_{0\leq k\leq K} (n_{j+k+1} - n_{j+k}) = \infty.$$

Proof. The assertion is an immediate consequence of Proposition M, Lemma 11.1, and Lemma 11.6. □

Remark 11.9. Remembering that $X_\eta = \widetilde{X}_\eta$, we will show that $X_\eta \subsetneq X_{\mathscr{B}_{\mathbf{A}}}$.

Let us note that $6, 12, 18, 20, 24, 28, 30 \in \mathscr{B}_{\mathbf{A}}$ are first seven abundant numbers. Consider the block $B = 10011111011111011111001111011111011$ of length 35 (we enumerate the entries from 0 to 34). The eight zeros are at positions: 1, 2, 8, 14, 20, 21, 26 and 32. Now, the block B is admissible since 2 is a missing residue class mod 6 (and consequently mod 12, mod 18, mod 24 and mod 30), 1 is a missing residue class mod 20 and 8 is a missing residue class mod 28. We claim that B does not appear in $\eta_{\mathscr{B}_{\mathbf{A}}}$. Indeed, suppose that for some m, we have $B = \eta[m, m+34]$. Then 6 divides exactly one number from the set of six consecutive numbers $\{m, m+1, \ldots, m+5\}$. Since $\mathcal{M}_{\mathscr{B}_{\mathbf{A}}} \cap \{m, m+1, \ldots, m+5\} = \{m+1, m+2\}$, so $6 \mid (m+1)$ or $6 \mid (m+2)$. If

$6|(m+1)$, then $6|(m+7)$, a contradiction with $B[7]=1$. Thus $6|(m+2)$. Similarly, 20 divides exactly one of the numbers $m+1, m+2, m+8, m+14$. If $20 \mid (m+2)$, then $20|(m+22)$, which contradicts to $B[22]=1$. Similarly, if $20 \mid (m+8)$ or $20 \mid (m+14)$, then we obtain a contradiction with $B[28]=B[34]=1$. It follows that $20|(m+1)$. But it is impossible to have simultaneously $6 \mid (m+2)$ and $20 \mid (m+1)$.

Acknowledgment

Our special thanks go to the referee for a detailed and profound review. The referee's remarks allowed us to improve and clarify the text and helped us to make the presentation more complete.

References

[1] El Houcein El Abdalaoui, Mariusz Lemańczyk, and Thierry de la Rue, *A dynamical point of view on the set of $\mathcal{B}$-free integers*, Int. Math. Res. Not. IMRN **16** (2015), 7258–7286, DOI 10.1093/imrn/rnu164. MR3428961

[2] Ethan Akin, Joseph Auslander, and Eli Glasner, *The topological dynamics of Ellis actions*, Mem. Amer. Math. Soc. **195** (2008), no. 913, vi+152, DOI 10.1090/memo/0913. MR2437846

[3] Emre Alkan and Alexandru Zaharescu, *Consecutive large gaps in sequences defined by multiplicative constraints*, Canad. Math. Bull. **51** (2008), no. 2, 172–181, DOI 10.4153/CMB-2008-019-x. MR2414205

[4] Michael Baake and Christian Huck, *Ergodic properties of visible lattice points*, Proc. Steklov Inst. Math. **288** (2015), no. 1, 165–188, DOI 10.1134/S0081543815010137. MR3485709

[5] Michael Baake, Robert V. Moody, and Peter A. B. Pleasants, *Diffraction from visible lattice points and kth power free integers*, Discrete Math. **221** (2000), no. 1-3, 3–42, DOI 10.1016/S0012-365X(99)00384-2. Selected papers in honor of Ludwig Danzer. MR1778906

[6] A. S. Besicovitch, *On the density of certain sequences of integers*, Math. Ann. **110** (1935), no. 1, 336–341, DOI 10.1007/BF01448032. MR1512943

[7] E. Bessel-Hagen, *Zahlentheorie*, Teubner, 1929.

[8] F. Cellarosi and Ya. G. Sinai, *Ergodic properties of square-free numbers*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 4, 1343–1374, DOI 10.4171/JEMS/394. MR3055764

[9] Francesco Cellarosi and Ilya Vinogradov, *Ergodic properties of k-free integers in number fields*, J. Mod. Dyn. **7** (2013), no. 3, 461–488, DOI 10.3934/jmd.2013.7.461. MR3296562

[10] S. Chowla, *On abundant numbers*, J. Indian Math. Soc., New Ser., 1 (1934), 41–44.

[11] Jesse Paul Clay, *Proximity relations in transformation groups*, Trans. Amer. Math. Soc. **108** (1963), 88–96, DOI 10.2307/1993827. MR0154269

[12] H. Davenport, *Über numeri abundantes*, Sitzungsber. Preuss. Akad. Wiss. (1933), 830–837.

[13] H. Davenport and P. Erdös, *On sequences of positive integers*, Acta Arithmetica 2 (1936), 147–151.

[14] H. Davenport and P. Erdös, *On sequences of positive integers*, J. Indian Math. Soc. (N.S.) **15** (1951), 19–24. MR0043835

[15] L. E. Dickson, *A new extension of dirichlet's theorem on prime numbers*, Messenger of Math. 33 (1904), 155–161.

[16] Tomasz Downarowicz, *Survey of odometers and Toeplitz flows*, Algebraic and topological dynamics, Contemp. Math., vol. 385, Amer. Math. Soc., Providence, RI, 2005, pp. 7–37, DOI 10.1090/conm/385/07188. MR2180227

[17] Tomasz Downarowicz, *Entropy in dynamical systems*, New Mathematical Monographs, vol. 18, Cambridge University Press, Cambridge, 2011. MR2809170

[18] T. Downarowicz and A. Iwanik, *Quasi-uniform convergence in compact dynamical systems*, Studia Math. **89** (1988), no. 1, 11–25. MR951081

[19] Paul Erdös, *On the Density of the Abundant Numbers*, J. London Math. Soc. **S1-9** (1934), no. 4, 278, DOI 10.1112/jlms/s1-9.4.278. MR1574879

[20] ———, *Note on consecutive abundant numbers*, J. London Math. Soc.,**10** (1935), pp. 128–131. MR1574240

[21] P. Erdös, *On the density of some sequences of integers*, Bull. Amer. Math. Soc. **54** (1948), 685–692, DOI 10.1090/S0002-9904-1948-09057-7. MR0026088

[22] P. Erdős, *On the difference of consecutive terms of sequences defined by divisibility properties*, Acta Arith **12** (1966/1967), 175–182. MR0207673

[23] Paul Erdős and János Surányi, *Topics in the theory of numbers*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2003. Translated from the second Hungarian edition by Barry Guiduli. MR1950084

[24] Harry Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory **1** (1967), 1–49, DOI 10.1007/BF01692494. MR0213508

[25] Eli Glasner, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, vol. 101, American Mathematical Society, Providence, RI, 2003. MR1958753

[26] Frank Hahn and William Parry, *Some characteristic properties of dynamical systems with quasi-discrete spectra*, Math. Systems Theory **2** (1968), 179–190, DOI 10.1007/BF01692514. MR0230877

[27] Heini Halberstam and Klaus Friedrich Roth, *Sequences*, 2nd ed., Springer-Verlag, New York-Berlin, 1983. MR687978

[28] Richard R. Hall, *Sets of multiples*, Cambridge Tracts in Mathematics, vol. 118, Cambridge University Press, Cambridge, 1996. MR1414678

[29] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 5th ed., The Clarendon Press, Oxford University Press, New York, 1979. MR568909

[30] C. Huck and M. Baake, *Dynamical properties of k-free lattice points*, Acta Phys. Pol. A 126 (2014), 482–485.

[31] Douglas E. Iannucci, *On the smallest abundant number not divisible by the first k primes*, Bull. Belg. Math. Soc. Simon Stevin **12** (2005), no. 1, 39–44. MR2134854

[32] Emily Jennings, Paul Pollack, and Lola Thompson, *Variations on a theorem of Davenport concerning abundant numbers*, Bull. Aust. Math. Soc. **89** (2014), no. 3, 437–450, DOI 10.1017/S0004972713000695. MR3254753

[33] G. Keller and C. Richard, *Dynamics on the graph of the torus parametrisation.* Ergodic Theory and Dynamical Systems (2016), 1–38. doi:10.1017/etds.2016.53

[34] David Kerr and Hanfeng Li, *Independence in topological and C^*-dynamics*, Math. Ann. **338** (2007), no. 4, 869–926, DOI 10.1007/s00208-007-0097-z. MR2317754

[35] Mitsuo Kobayashi, *A new series for the density of abundant numbers*, Int. J. Number Theory **10** (2014), no. 1, 73–84, DOI 10.1142/S1793042113500814. MR3189967

[36] Joanna Kułaga-Przymus, Mariusz Lemańczyk, and Benjamin Weiss, *On invariant measures for $\mathscr{B}$-free systems*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 6, 1435–1474, DOI 10.1112/plms/pdv017. MR3356811

[37] Marcin Kulczycki, Dominik Kwietniak, and Piotr Oprocha, *On almost specification and average shadowing properties*, Fund. Math. **224** (2014), no. 3, 241–278, DOI 10.4064/fm224-3-4. MR3194417

[38] Dominik Kwietniak, *Topological entropy and distributional chaos in hereditary shifts with applications to spacing shifts and beta shifts*, Discrete Contin. Dyn. Syst. **33** (2013), no. 6, 2451–2467, DOI 10.3934/dcds.2013.33.2451. MR3007694

[39] Douglas Lind and Brian Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995. MR1369092

[40] Kurt Mahler, *On the Chinese remainder theorem*, Math. Nachr. **18** (1958), 120–122, DOI 10.1002/mana.19580180112. MR0096565

[41] Yves Meyer, *Algebraic numbers and harmonic analysis*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1972. North-Holland Mathematical Library, Vol. 2. MR0485769

[42] Y. Meyer, *Adèles et séries trigonométriques spéciales* (French), Ann. of Math. (2) **97** (1973), 171–186, DOI 10.2307/1970880. MR0316977

[43] L. Mirsky, *Note on an asymptotic formula connected with r-free integers*, Quart. J. Math., Oxford Ser. **18** (1947), 178–182. MR0021566

[44] L. Mirsky, *Arithmetical pattern problems relating to divisibility by rth powers*, Proc. London Math. Soc. (2) **50** (1949), 497–508, DOI 10.1112/plms/s2-50.7.497. MR0028334

[45] Piotr Oprocha and Guohua Zhang, *Topological aspects of dynamics of pairs, tuples and sets*, Recent progress in general topology. III, Atlantis Press, Paris, 2014, pp. 665–709, DOI 10.2991/978-94-6239-024-9_16. MR3205496

[46] Ryan Peckner, *Uniqueness of the measure of maximal entropy for the squarefree flow*, Israel J. Math. **210** (2015), no. 1, 335–357, DOI 10.1007/s11856-015-1255-8. MR3430278

[47] ———, *Uniqueness of the measure of maximal entropy for the squarefree flow*. Unpublished, `http://arxiv.org/abs/1205.2905v1`.

[48] Peter A. B. Pleasants and Christian Huck, *Entropy and diffraction of the k-free points in n-dimensional lattices*, Discrete Comput. Geom. **50** (2013), no. 1, 39–68, DOI 10.1007/s00454-013-9516-y. MR3070541

[49] P. Sarnak, *Three lectures on the Möbius function, randomness and dynamics*, `http://publications.ias.edu/sarnak/`.

[50] A. Schinzel, *Remarks on the paper "Sur certaines hypothèses concernant les nombres premiers"*, Acta Arith. **7** (1961/1962), 1–8. MR0130203

[51] C. R. Wall and Lorraine L. Foster, *Problems and Solutions: Solutions of Advanced Problems: 6356*, Amer. Math. Monthly **90** (1983), no. 3, 215–216, DOI 10.2307/2975563. MR1540167

[52] B. Weiss, *Intrinsically ergodic systems*, Bull. Amer. Math. Soc. **76** (1970), 1266–1269, DOI 10.1090/S0002-9904-1970-12632-5. MR0267076

[53] Benjamin Weiss, *Subshifts of finite type and sofic systems*, Monatsh. Math. **77** (1973), 462–474, DOI 10.1007/BF01295322. MR0340556

[54] T. S. Wu, *Proximal relations in topological dynamics*, Proc. Amer. Math. Soc. **16** (1965), 513–514, DOI 10.2307/2034685. MR0179775

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

Email address: `aurbart@mat.umk.pl`

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

Email address: `skasjan@mat.umk.pl`

Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00-956 Warszawa, Poland–and–Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

Email address: `joanna.kulaga@gmail.com`

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

Email address: `mlem@mat.umk.pl`