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SPHERICAL SPACE FORMS REVISITED

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Abstract. We give a simplified proof of J. A. Wolf's classification of finite groups that can act freely and isometrically on a round sphere of some dimension. We slightly improve the classification by removing some nonobvious redundancy. The groups are the same as the Frobenius complements of finite group theory.

In chapters 4–7 of his famous *Spaces of Constant Curvature* [7], J. A. Wolf classified spherical space forms as connected Riemannian manifolds locally isometric to the n-sphere S^n. By passing to the action of the fundamental group on the universal covering space, this is equivalent to classifying the possible free isometric actions of finite groups on S^n. Then, by embedding S^n in Euclidean space, this is equivalent to classifying the real representations of finite groups that are "free" in the sense that no element except the identity fixes any vector except 0. This allowed Wolf to use the theory of finite groups and their representations.

Our first goal is to give a simplified proof of Wolf's classification of the finite groups G that can act freely and isometrically on spheres. Wolf's main result here was the list of presentations in theorems 6.1.11 and 6.3.1 of [7]. Our approach to Wolf's theorem leads to the most interesting example, the binary icosahedral group, with very little case analysis and no character theory. (The trick consists of the equalities (3.2) and (3.3) in the proof of Lemma 3.11. These rely on an elementary property of the binary tetrahedral group, stated in Lemma 3.10.)

Our second goal is to remove the redundancy from Wolf's list; this modest improvement appears to be new. Some groups appear repeatedly on Wolf's list because different presentations can define isomorphic groups. See Example 5.1 for some nonobvious isomorphisms. It would not be hard to just work out the isomorphisms among the groups defined by Wolf's presentations. But it is more natural to reformulate the classification in terms of intrinsically defined subgroups. Namely, a finite group G acts freely and isometrically on a sphere if and only if it has one of 6 possible "structures", in which case it has a unique such "structure" up to conjugation. See Theorems 1.1 and 1.2. In fact, we parameterize the possible G without redundancy, in terms of a fairly simple set of invariants. Namely, a type I–VI, two numbers $|G|$ and a, and a subgroup of the unit group of the ring $\mathbb{Z}/a$. In a special case one must also specify a second such subgroup. See section 5 for details.

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The full classification of spherical space forms requires not just the list of possible groups, but also their irreducible free actions on real vector spaces, and how their outer automorphism groups permute these representations. See [7, Thm 5.1.2] for why this is the right data to tabulate and [7, Ch. 7] for the actual data for each group. We expect that this data could be described cleanly in terms of our descriptions of the groups, but have not worked out the details. It would remain lengthy because of many cases and subcases to consider.

For many authors the phrase "spherical space form" means a quotient of a sphere by a free action of a finite group of homeomorphisms or diffeomorphisms, rather than isometries. In this paper we consider only isometric actions on round spheres. See [6] for the rich topology and group theory involved in the more general theory.

Expecting topologists and geometers rather than group theorists as readers, we have made the paper self-contained, with three exceptions. First, we omit proofs of Burnside's transfer theorem and the Schur–Zassenhaus theorem. Second, we use the fact that $\mathrm{SL}_2(\mathbb{F}_5)$ is the unique perfect central extension of the alternating group A_5 by $\mathbb{Z}/2$, giving a citation when needed. Third, we use $\operatorname{Aut}\mathrm{SL}_2(\mathbb{F}_5) = \mathrm{PGL}_2(\mathbb{F}_5) \cong S_5$, which is just an exercise.

Finite group theorists study the same groups Wolf did, from a different perspective. We will sketch the connection briefly because our descriptions of the groups may have some value in this context. A finite group G is called a Frobenius complement if it acts "freely" on some finite group H, meaning that no element of G except 1 fixes any element of H except 1. To our knowledge, the structure of Frobenius complements (in terms of presentations) is due to Zassenhaus [8]. Unfortunately his paper contains an error, and the first correct proof is due to Passman [5, Theorems 18.2 and 18.6]. See also Zassenhaus's later paper [9]. If G is a Frobenius complement, then after a preliminary reduction one can show that H may be taken to be a vector space over a finite field $\mathbb{F}_p$, where p is a prime not dividing $|G|$. Our arguments apply with few or no changes; see Remark 3.12 and also [4], especially Prop. 2.1.

We will continue to abuse language by speaking of free actions on vector spaces when really we mean that the action is free away from 0. We will use the standard notation G' for the commutator subgroup of a group G, and $O(G)$ for the unique maximal-under-inclusion odd normal subgroup when G is finite.

We will also use ATLAS notation for group structures [1]. That is, if a group G has a normal subgroup A, the quotient by which is B, then we may say "G has structure $A.B$". We sometimes write this as $G \sim A.B$. This usually does not completely describe G, because several nonisomorphic groups may have "structure $A.B$". Nevertheless it is a helpful shorthand, especially if A is characteristic. If the group extension splits, then we may write $A : B$ instead, and if it doesn't, then we may write $A \cdot B$. See Theorem 1.1 for a some examples. When we write $A : B$, we will regard B as a subgroup of G rather than just a quotient. (In all our uses of this notation, the complements to A turn out to be conjugate, so there is no real ambiguity in choosing one of them.)

1. The groups

It is well known that the groups of orientation-preserving isometries of the tetrahedron, octahedron (or cube), and icosahedron (or dodecahedron) are subgroups of $\mathrm{SO}(3)$ isomorphic to A_4, S_4, and A_5. The preimages of these groups in the double cover of SO(3) are called the binary tetrahedral, binary octahedral, and binary icosahedral groups. They have structures $2.A_4$, $2.S_4$, and $2.A_5$, where we are using another ATLAS convention: indicating a cyclic group of order n by simply writing n; here n is 2. These are the only groups with these structures that we will encounter in this paper. So we abbreviate them (again following ATLAS) to $2A_4$, $2S_4$, and $2A_5$, and specify that this notation refers to the binary polyhedral groups, rather than some other groups with structure $2.A_4$, $2.S_4$, or $2.A_5$. Alternate descriptions of the binary tetrahedral and binary icosahedral groups are $2A_4 \cong \mathrm{SL}_2(3)$ and $2A_5 \cong \mathrm{SL}_2(5)$.

It is also well known that the double cover of SO(3) may be identified with the unit sphere $\mathbb{H}^*$ in Hamilton's quaternions $\mathbb{H}$. A finite subgroup of $\mathbb{H}^*$ obviously acts by left multiplication on $\mathbb{H}^*$. So $2A_4$, $2S_4$, and $2A_5$ act freely on the unit sphere S^3.

Similarly, SO(3) contains dihedral subgroups, and their preimages in $\mathbb{H}^*$ are called binary dihedral. If we start with the dihedral group of order $2n$, then the corresponding binary dihedral group of order $4n$ can be presented by

$$\big\langle x, y \bigm| x^{2n} = 1,\ yxy^{-1} = x^{-1},\ y^2 = x^n \big\rangle.$$

This group may be identified with a subgroup of $\mathbb{H}^*$ by taking

$$x \mapsto (\text{any primitive } 2n\text{th root of unity in } \mathbb{R} \oplus \mathbb{R}i) \qquad y \mapsto j. \tag{1.1}$$

Left multiplication by these elements of $\mathbb{H}^*$ gives a free action on S^3. (Replacing the root of unity by its inverse gives an equivalent representation.) Restricting to the case $n = 2^{m-2}$, $m \geq 3$, one obtains the quaternion group Q_{2^m} of order 2^m. Some authors call this a generalized quaternion group, with "quaternion group" reserved for Q_8.

Now we can state our version of Wolf's theorems 6.1.11 and 6.3.1, supplemented by a uniqueness theorem.

Theorem 1.1 (Groups that act freely and isometrically on spheres). *Suppose G is a finite group that acts freely and isometrically on a sphere of some dimension. Then it has one of the following six structures, where A and B are cyclic groups whose orders are odd and coprime, every nontrivial Sylow subgroup of B acts nontrivially on A, and every prime-order element of B acts trivially on A.*

(I) $A : \big(B \times (\text{a cyclic 2-group } T)\big)$, *where if $T \neq 1$, then its involution fixes A pointwise.*

(II) $A : \big(B \times (\text{a quaternionic group } T)\big)$.

(III) $(Q_8 \times A) : (\Theta \times B)$, *where Θ is a cyclic 3-group which acts nontrivially on Q_8 and whose elements of order 3 centralize A, and $|A|$ and $|B|$ are prime to 3.*

(IV) $\big((Q_8 \times A) : (\Theta \times B)\big) \cdot 2$, *where Θ, $|A|$, and $|B|$ are as in* (III), *and the quotient $\mathbb{Z}/2$ is the image of a subgroup Φ of G, isomorphic to $\mathbb{Z}/4$, whose elements of order 4 act by an outer automorphism on Q_8, by inversion on Θ and trivially on B.*

(V) $2A_5 \times (A : B)$, *where $|A|$ and $|B|$ are prime to 15.*

(VI) $\big(2A_5 \times (A : B)\big) \cdot 2$, *where $|A|$ and $|B|$ are prime to 15, and the quotient $\mathbb{Z}/2$ is the image of a subgroup Φ of G, isomorphic to $\mathbb{Z}/4$, whose elements of order 4 act by an outer automorphism on $2A_5$ and trivially on B.*

Conversely, any group with one of these structures acts freely and isometrically on a sphere of some dimension.

These groups are parameterized in terms of simple invariants in Theorem 5.3. A binary dihedral group has type (I) or (II), $2A_4$ has type (III), $2S_4$ has type (IV), and $2A_5$ has type (V).

Theorem 1.2 (Uniqueness of structure)**.** *The structure in Theorem* 1.1 *is unique in the following sense:*

1. *Groups of different types* (I)–(VI) *cannot be isomorphic.*
2. *Suppose G has one of the types* (I)–(VI)*, with respect to some subgroups A, B (and whichever of T, Q_8, Θ, Φ, and $2A_5$ are relevant), and also with respect to some subgroups A^*, B^* (and T^*, Q_8^*, Θ^*, Φ^*, and $2A_5^*$, when relevant). Then some element of G conjugates every unstarred group to the corresponding starred group. In particular, $A^* = A$ (and $Q_8^* = Q_8$ and $2A_5^* = 2A_5$, when relevant).*

Remark 1.3 (Correspondence with Wolf's types)*.* Our types correspond in the obvious way to Wolf's in theorems 6.1.11 and 6.3.1 of [7]. However, his generator A might not generate our subgroup A and his generator B might not even lie in our subgroup B.

Remark 1.4 (Implied relations)*.* Some useful information is implicit. For example, B acts trivially on Q_8 for types (III) and (IV), because Q_8 has no automorphisms of odd order > 3. Also, Θ must act trivially on A for type (IV), because A has an abelian automorphism group and $\Theta \leq G'$. In light of these remarks, one could rewrite G's structure for type (IV) as $\big((Q_8 : \Theta) \times (A : B)\big) \cdot 2$. This would be more informative, but hide the relationship to type (III).

In all cases, G has at most one involution. This is obvious except for type (II). Then, T's central involution lies in T', which centralizes A because $\operatorname{Aut} A$ is abelian.

At this point we will prove the easy parts of the theorems, namely that the listed groups do act freely on spheres and that groups of different types cannot be isomorphic. The proof that every group acting freely on a sphere has structure as in Theorem 1.1 appears in sections 3–4, and Theorem 1.2's uniqueness statement is a byproduct of the proof.

Proof of "Conversely..." in Theorem 1.1. Define H as the normal subgroup of G that is generated by the elements of prime order. We claim that if H has a free action on a real vector space V, then G acts freely on its representation W induced from V. To see this, recall that as a vector space, W is a direct sum of copies of V, indexed by G/H. And H's actions on these copies of V are isomorphic to the representations obtained by precomposing $H \to \mathrm{GL}(V)$ by automorphisms $H \to H$ arising from conjugation in G. In particular, H acts freely on W. We claim that G also acts freely on W. Otherwise, some prime-order element of G would have a fixed point. But it would also lie in H, which acts freely.

Now it suffices to determine H and show that it always has a free action. For types (I)–(II), H is a cyclic group, and for types (V)–(VI) it is $2A_5 \times$(cyclic group of order prime to 30). For types (III)–(IV), H is either cyclic or $2A_4 \times$(cyclic group of order prime to 6), according to whether $|\Theta| > 3$ or $|\Theta| = 3$. These claims are all easy, using the fact that G's involution is central (if one exists) and the prime-order elements of B and (if relevant) Θ act trivially on A.

Cyclic groups obviously admit free actions. For a product of $2A_4$ or $2A_5$ by a cyclic group of coprime order, we identify each factor with a subgroup of $\mathbb{H}^*$ and make $2A_4$ or $2A_5$ act on $\mathbb{H}$ by left multiplication and the cyclic group act by right multiplication. If there were a nonidentity element of this group with a nonzero fixed vector, then there would be one having prime order, hence lying in one of the factors. But this is impossible since each factor acts freely. $\square$

Proof of Theorem 1.2(1.2). Suppose G has one of the types (I)–(VI). Then the subgroup H generated by A, B and the index 3 subgroup of Θ (for types (III)–(IV)) is normal in G and has odd order. The quotient G/H is a cyclic 2-group, a quaternionic group, $2A_4$, $2A_4 \cdot 2$, $2A_5$, or $2A_5 \cdot 2$ respectively. None of these has an odd normal subgroup larger than $\{1\}$. Therefore H is all of $O(G)$, and the isomorphism class of $G/O(G)$ distinguishes the types. $\square$

2. Preparation

In this section we suppose G is a finite group. We will establish general properties of G under hypotheses related to G having a free action on a sphere. The results before Lemma 2.8 are standard and are included for completeness. Lemma 2.8 is a refinement of a standard result.

Lemma 2.1 (Unique involution)**.** *Suppose G has a free action on a sphere. Then it has at most one involution.*

Proof. Choose a free (hence, faithful) action of G on a real vector space V. An involution has eigenvalues ± 1, but $+1$ cannot appear by freeness. So there can be only one involution, acting by negation. $\square$

Lemma 2.2. *Suppose G has a free action on a sphere. Then every abelian subgroup is cyclic, and so is every subgroup of order pq, where p and q are primes.*

Proof. Fix a free action of G on a real vector space V, and let $V_{\mathbb{C}}$ be its complexification. G also acts freely on $V_{\mathbb{C}}$. Otherwise, some nontrivial element has a nonzero fixed vector, hence has the real number 1 as an eigenvalue, and hence fixes a nonzero real vector.

Now suppose $A \leq G$ is abelian, and decompose $V_{\mathbb{C}}$ under A as a sum of 1-dimensional representations. By freeness, each of these is faithful. So A is a subgroup of the multiplicative group $\mathbb{C} - \{0\}$, hence cyclic, proving the first claim. Since any group of prime-squared order is abelian, this also proves the $p = q$ case of the second claim.

So suppose $p < q$ are primes and consider a subgroup of G with order pq; by discarding the rest of G we may suppose without loss that this subgroup is all of G. Write P, resp., Q, for a Sylow p-subgroup, resp., q-subgroup. By Sylow's theorem, P normalizes Q. If it acts trivially on Q, then G is abelian, so suppose P acts nontrivially on Q.

For purposes of this proof, a character of Q means a homomorphism $Q \to \mathbb{C}^*$. It is standard that any complex representation of Q is the direct sum of Q's character spaces, meaning the subspaces on which Q acts by its various characters. Fix a character χ of Q whose character space contains a nonzero vector $v \in V_{\mathbb{C}}$; χ is faithful since Q acts freely on $V_{\mathbb{C}}$. If $g \in P$, then $g(v)$ lies in the character space for the character $\chi \circ i_g^{-1} : Q \to \mathbb{C}^*$, where $i_g : x \mapsto gxg^{-1}$ means conjugation by g. Since χ is faithful and P acts faithfully on Q, the various characters $\chi \circ i_g^{-1}$ are all distinct. Therefore the terms in the sum $\sum_{g \in P} g(v)$ are linearly independent, so the sum is nonzero. But this contradicts freeness since the sum is obviously P-invariant. $\square$

Lemma 2.3 (Sylow subgroups)**.** *Suppose all of G's abelian subgroups are cyclic. Then its odd Sylow subgroups are cyclic and its Sylow 2-subgroups are cyclic or quaternionic.*

Proof. It suffices to treat the case of G a p-group, say of order p^n. We proceed by induction on n, with the cases $n \leq 2$ being trivial. So suppose $n > 2$.

First we treat the special case that G contains a cyclic group X of index p. If G acts trivially on X, then G is abelian, hence cyclic. So we may assume that G/X is identified with a subgroup of order p in $\operatorname{Aut} X$. Recall that $\operatorname{Aut} X$ is cyclic of order $(p-1)p^{n-2}$ if p is odd and is $\mathbb{Z}/2$ times a cyclic group of order 2^{n-3} if $p = 2$. This shows that some $y \in G - X$ acts on X by the λth power map, where

$$\lambda = \begin{cases} p^{n-2} + 1 & \text{if } p \text{ is odd,} \\ -1 \quad \text{or} \quad 2^{n-2} \pm 1 & \text{if } p = 2, \end{cases}$$

with the possibilities $2^{n-2} \pm 1$ considered only if $n > 3$. Write X_0 for the subgroup of X centralized by y. Now, $\langle X_0, y\rangle$ is abelian, hence cyclic. The index of its subgroup X_0 is p, because y^p lies in X and centralizes y. Since $y \notin X_0$, y generates $\langle X_0, y\rangle$. We write p^t for the index of X_0 in X, which can be worked out from y's action on X. Namely,

$$p^t = \begin{cases} 2^{n-2} & \text{if } p = 2, \text{ and } \lambda = -1 \text{ or } 2^{n-2} - 1, \\ p & \text{otherwise.} \end{cases}$$

We choose a generator x for X such that $y^p = x^{-p^t}$.

Since xy and y have the same centralizer in X, the same argument shows that $(xy)^p$ also generates X_0. Now,

$$\begin{aligned} (xy)^p &= x(yxy^{-1})(y^2xy^{-2}) \cdots (y^{p-1}xy^{1-p})y^p \\ &= x \cdot x^{\lambda} \cdot x^{\lambda^2} \cdots x^{\lambda^{p-1}} \cdot x^{-p^t}. \end{aligned}$$

Our two descriptions $\langle y^p\rangle$ and $\langle (xy)^p\rangle$ of $X_0 \leq X \cong \mathbb{Z}/p^{n-1}$ must coincide, so $\mu := 1 + \lambda + \cdots + \lambda^{p-1} - p^t$ generates the same subgroup of $\mathbb{Z}/p^{n-1}$ as p^t does. One computes

p^t	μ	
p	$\binom{p}{2}p^{n-2}$	if $\lambda = p^{n-2} + 1$ (including the case $2^{n-2} + 1$),
2^{n-2}	0	if $\lambda = 2^{n-2} - 1$,
2^{n-2}	2^{n-2}	if $\lambda = -1$.

Only in the last case do p^t and μ generate the same subgroup of $\mathbb{Z}/p^{n-1}$. So $p = 2$, y inverts X, and y^2 is the involution in X. That is, G is quaternionic. This finishes the proof in the special case.

Now we treat the general case. Take H to be a subgroup of index p. If p is odd, then H is cyclic by induction, so the special case shows that G is cyclic, too. So suppose $p = 2$. By induction, H is cyclic or quaternionic. If it is cyclic, then the special case shows that G is cyclic or quaternionic, as desired. So suppose H is quaternionic and take X to be a G-invariant index 2 cyclic subgroup of H. This is possible because H contains an odd number of cyclic subgroups of index 2 (three if $H \cong Q_8$ and one otherwise). Now we consider the action of G/X on X. If some element of $G-X$ acts trivially, then together with X it generates an abelian, hence cyclic, group, and the special case applies. So G/X is 2×2 or 4 and embeds in $\operatorname{Aut} X$. (These cases require $|X|\geq 8$ or 16, respectively, or equivalently $n\geq 5$ or 6.) Furthermore, $\operatorname{Aut} X$ contains just three involutions, and only one of them can be a square in $\operatorname{Aut} X$, namely the $(1+2^{n-2})$nd power map. Therefore, either possibility for G/X yields an element y of G which acts on X by this map and has square in X. But then $\langle X, y\rangle$ is neither cyclic nor quaternionic, contradicting the special case. □

Recall that a group is called perfect if its abelianization is trivial.

Lemma 2.4 ($2A_5$ recognition)**.** *Suppose G is perfect with center $\mathbb{Z}/2$, and every noncentral cyclic subgroup has binary dihedral normalizer. Then $G \cong 2A_5$.*

Proof. Write $2g$ for $|G|$. The hypothesis on normalizers shows that distinct maximal cyclic subgroups of G have intersection equal to $Z(G)$. So G is the disjoint union of $Z(G)$ and the subsets $C-Z(G)$ where C varies over the maximal cyclic subgroups of G. We choose representatives $C_1,\dots,C_n$ for the conjugacy classes of such subgroups and write $2c_1,\dots,2c_n$ for their orders. The numbers $c_1,\dots,c_n$ are pairwise coprime because each of $C_1,\dots,C_n$ is the centralizer of each of its subgroups of order >2. We number the C_i so that c_1 is divisible by 2, c_2 is divisible by the smallest prime involved in g but not c_1, c_3 is divisible by the smallest prime involved in g but neither c_1 nor c_2, and so on. In particular, c_i is at least as large as the ith prime number.

Each conjugate of $C_i-Z(G)$ has $2c_i-2$ elements, and the normalizer hypothesis tells us there are $g/2c_i$ many conjugates. Therefore $(2c_i-2)g/2c_i = g\big(1-\frac{1}{c_i}\big)$ elements of $G-Z(G)$ are conjugate into $C_i - Z(G)$. Our partition of G gives

$$2g = 2 + g\sum_{i=1}^{n}(1-\tfrac{1}{c_i}). \tag{2.1}$$

We can rewrite this as $g(2-n) = 2-\sum_{i=1}^{n} g/c_i$. Since G has no index 2 subgroups, each term g/c_i in the sum is larger than 2. Since the right side is negative, $g(2-n)$ is also. So $n>2$.

In fact $n=3$. Otherwise, we would use $c_1\geq 2$, $c_2\geq 3$, $c_3\geq 5$, and $c_4\geq 7$ to see that the sum on the right side of (2.1) is

$$(\text{at least } \frac{1}{2}) + (\text{at least } \frac{2}{3}) + (\text{at least } \frac{4}{5}) + (\text{at least } \frac{6}{7}) + \cdots > 2,$$

which is a contradiction. Now we rewrite (2.1) as $\frac{1}{c_1}+\frac{1}{c_2}+\frac{1}{c_3} = 1+2/g$. In particular, the left side must be larger than 1, which requires $c_1=2$, $c_2=3$, and $c_3=5$. Then $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}=1+2/g$ gives a formula for g, namely $g=60$, so $|G|=120$. So $G/Z(G)$ is nonsolvable of order 60. A Sylow's theorem exercise rules out the possibility that there are 15 Sylow 2-subgroups, so there must be 5, and it follows easily that $G/Z(G)\cong A_5$. So G has structure $2.A_5$. Finally, A_5 has a unique perfect central extension by $\mathbb{Z}/2$, namely the binary icosahedral group [5, Prop. 13.7]. □

Theorem 2.5 (Burnside's transfer theorem)**.** *Suppose G is a finite group, and P is a Sylow subgroup that is central in its normalizer. Then P maps faithfully to the abelianization G/G'.*

Proof. See [3, Thm. 5.13], [2, Thm. 4.3], or [7, Thm 5.2.9]. □

Corollary 2.6 (Cyclic transfer)**.** *Suppose G is a finite group, p is a prime, and G's Sylow p-subgroups are cyclic. If some nontrivial p-group is central in its normalizer or maps nontrivially to G/G', then every Sylow p-subgroup maps faithfully to G/G'. In particular, this holds if p is the smallest prime dividing $|G|$.*

Proof. Suppose that $P_0 \leq G$ is a p-group satisfying either of the two conditions, and choose a Sylow p-subgroup P containing it. It is cyclic by hypothesis, so P_0 is characteristic in P, so $N(P)$ lies in $N(P_0)$. The automorphisms of P with order prime to p act nontrivially on every nontrivial subquotient of P. Under either hypothesis, P_0 (hence P) has a nontrivial subquotient on which $N(P)$ acts trivially. Therefore the image of $N(P)$ in $\operatorname{Aut} P$ contains no elements of order prime to p. It contains no elements of order p either, since P is abelian. So P is central in its normalizer and we can apply Burnside's transfer theorem. Since P maps faithfully to G/G', so does every Sylow p-subgroup.

For the final statement, observe that $\operatorname{Aut} P$ has no elements of prime order $> p$. So its normalizer must act trivially on it, and we can apply the previous paragraph. □

Theorem 2.7 (Schur-Zassenhaus)**.** *Suppose G is a group, N is a normal subgroup, and $|N|$ and $|G/N|$ are coprime. Then there exists a complement to N in G, and all complements are conjugate.* □

As stated, this relies on the odd order theorem. But we only need the much more elementary case that N is abelian (Theorem 3.5 of [3]).

In determining the structure of his groups, Wolf used a theorem of Burnside: if all Sylow subgroups of a given group H are cyclic, then H' and H/H' are cyclic of coprime order, H' has a complement, and all complements are conjugate. We prefer the following decomposition $H = A : B$ because of its "persistence" property (IV). We only need this property for the imperfect case (section 4).

Lemma 2.8 (Metacyclic decomposition)**.** *Suppose H is a finite group, all of whose Sylow subgroups are cyclic. Define A as the subgroup generated by H' and all of H's Sylow subgroups that are central. Then A has the following two properties and is characterized by them:*

- (I) *A is normal, and A and H/A are cyclic of coprime orders.*
- (II) *Every nontrivial Sylow subgroup of H/A acts nontrivially on A.*

Furthermore,

- (III) *A has a complement B, and all complements are conjugate.*
- (IV) *Suppose a finite group G contains H as a normal subgroup, with $|G/H|$ coprime to $|H|$. Then there is a complement C to H such that the decomposition $H \sim A : B$ in* (III) *extends to $G \sim A : (B \times C)$. Furthermore, all complements of H that normalize B are conjugate under $N_H(B)$.*

Proof. First we show that H is solvable. If p is the smallest prime dividing $|H|$, and P is a Sylow p-subgroup, then Corollary 2.6 shows that H has a quotient group isomorphic to P. The kernel is solvable by induction, so H is too.

Now let F be the Fitting subgroup of H, i.e., the unique maximal normal nilpotent subgroup. Being nilpotent, it is the product of its Sylow subgroups. Since these are cyclic, so is F. Also, H/F acts faithfully on F, for otherwise F would lie in a strictly larger normal nilpotent subgroup. As a subgroup of the abelian group $\operatorname{Aut} F$, H/F is abelian. Therefore the cyclic group F contains H', so H' is cyclic.

If p is a prime dividing the order of H/H', then Corollary 2.6 shows that every Sylow p-subgroup meets H' trivially. It follows that the orders of H' and H/H' are coprime. A is obviously normal. Since A is the product of H' with the central Sylow subgroups of H, we see that A and H/A also have coprime orders. Since A contains H', H/A is abelian. Having cyclic Sylow subgroups, H/A is cyclic. We have proven (I).

Because $|A|$ and $|H/A|$ are coprime, the Schur–Zassenhaus theorem assures us that A has a complement B and that all complements are conjugate, proving (III). For (II), suppose a Sylow subgroup of B acts trivially on A. Then it is central in H, so A contains it by definition, which is a contradiction.

Next we prove the "persistence" property (IV), so we assume its hypotheses. Since A is characteristic in H, it is normal in G. Since all complements to A in H are conjugate in A, the Frattini argument shows that $N_G(B)$ maps onto G/H. Applying the Schur–Zassenhaus theorem to $N_H(B)$ inside $N_G(B)$ yields a complement C to H that normalizes B. That theorem also shows that all such complements are conjugate under $N_H(B)$

To prove (IV) it remains only to show that B and C commute. If C acted nontrivially on B, then G' would contain a nontrivial p-group for some prime p dividing $|B|$. Using Corollary 2.6 as before, it follows that G' contains a Sylow p-subgroup P of B. Since P lies in the commutator subgroup of G, it must act trivially on A. This contradicts (II). So B and C commute, completing the proof of (IV). (Remark: since $N_H(B) = C_A(B) \times B$, we could replace $N_H(B)$ in the statement of (IV) by $C_A(B)$.)

All that remains is to show that (I) and (II) characterize A; suppose $A^* \leq H$ has these properties. By (II), all the Sylow subgroups of H that act trivially on A^* lie in A^*. Since A^* is cyclic, $\operatorname{Aut} A^*$ is abelian, so H' acts trivially on it. We already saw that H' is the product of some of H's Sylow subgroups, so H' lies in A^*. The central Sylow subgroups of H also act trivially on A^*, so also lie in A^*. We have shown that A^* contains A. If A^* were strictly larger than A, then the coprimality of $|A^*|$ and $|H/A^*|$ would show that A^* contains a Sylow subgroup of H that is not in A. But then A^* is nonabelian by property (II) of A, and therefore property (I) fails for A^*. □

3. The perfect case

In this section and the next we prove Theorems 1.1 and 1.2 inductively. We suppose throughout that G is a finite group that acts freely on a sphere of some dimension. Under the assumption that every proper subgroup has one of the structures listed in Theorem 1.1, we will prove that G also has such a structure. In this section we also assume G is perfect. This includes base case $G = 1$ of the induction, which occurs in case (I). The only other perfect group in Theorem 1.1 is $2A_5$. Theorem 1.2 is trivial for $G \cong 1$ or $2A_5$. Therefore it will suffice to prove $G \cong 2A_5$ under the assumption $G \neq 1$.

Lemma 3.1. *G's Sylow 2-subgroups are quaternionic, in particular nontrivial.*

Proof. By Lemma 2.3, all the odd Sylow subgroups are cyclic. If the Sylow 2-subgroups were too, then Corollary 2.6, applied to the smallest prime dividing $|G|$, would contradict perfectness. Now Lemma 2.3 shows that the Sylow 2-subgroups must be quaternionic. □

Lemma 3.2. *G's elements of order 4 form a single conjugacy class.*

Proof. Let T be a Sylow 2-subgroup (quaternionic by the previous lemma) and let U be a cyclic subgroup of index 2. We consider the action of G on the coset space G/U, whose order is twice an odd number. By Lemma 2.1, G contains a unique involution, necessarily central. Since it lies in every conjugate of U, it acts trivially on G/U. Now let ϕ be any element of order 4. Since its square acts trivially, ϕ acts by exchanging some points in pairs. Since G is perfect, ϕ must act by an even permutation, so the number of these pairs is even. Since the size of G/U is not divisible by 4, ϕ must fix some points. The stabilizers of these points are conjugates of U, so ϕ is conjugate into U. Finally, the two elements of order 4 in U are conjugate since T contains an element inverting U. □

Lemma 3.3. *$O(G) = 1$.*

Proof. Suppose otherwise. Since G's odd Sylow subgroups are cyclic, Lemma 2.8(I) shows that $O(G)$ has a characteristic cyclic subgroup of prime order. Because this subgroup has an abelian automorphism group and G is perfect, G acts trivially on it. Now Corollary 2.6 shows that G has nontrivial abelianization, contradicting perfectness. □

Lemma 3.4. *Every maximal subgroup M of G has center of order 2.*

Proof. First, M contains G's central involution. Otherwise, adjoining it to M would yield G by maximality. Then G would have an index 2 subgroup, contrary to perfectness. Next we show that M has no central subgroup Y of order 4; suppose it did. By the conjugacy of $\mathbb{Z}/4$'s in G and the fact that G's Sylow 2-subgroups are quaternionic, $N(Y)$ contains an element inverting Y. So $N(Y)$ is strictly larger than M and hence coincides with G, and the map $G \to \mathrm{Aut}(Y) \cong \mathbb{Z}/2$ is nontrivial, contrary to perfectness.

Finally we show that M has no central subgroup Y of odd prime order > 1. The previous lemma shows that $N(Y)$ is strictly smaller than G. Since M normalizes Y and is maximal, it is Y's full normalizer. Since Y is central in M, we see that $N(Y)$ acts trivially on Y. Now Corollary 2.6 shows that G/G' is nontrivial, contradicting perfectness. □

The next lemma is where our development diverges from Wolf's.

Lemma 3.5 (Maximal subgroups)**.** *Every maximal subgroup M of G has one of the following structures, with $O(M)$ a cyclic group:*

- (I) $O(M)$: (cyclic 2-group of order > 2).
- (II) $O(M)$: (quaternionic group).
- (III) $2A_4$.
- (IV) $\big(O(M).2A_4\big) \cdot 2$, *where the elements of M outside $O(M).2A_4$ act on $O(M)$ by inversion and on the quotient $2A_4$ by outer automorphisms.*
- (V) $2A_5$.
- (VI) $\big(O(M) \times 2A_5\big) \cdot 2$, *where the elements of M outside $O(M) \times 2A_5$ act on $O(M)$ by inversion and on $2A_5$ by outer automorphisms.*

Furthermore, the G-normalizer of any nontrivial subgroup of $O(M)$ is M. If M_1 and M_2 are nonconjugate maximal subgroups, then $O(M_1)$ and $O(M_2)$ have coprime orders.

Remark. An alternate description of M in case (IV) is that it has structure $(O(M)\times Q).S_3$, where Q is quaternionic of order 8 and is the (unique) Sylow 2-subgroup of the commutator subgroup M'.

Proof. By induction, M has one of the structures in Theorem 1.1. In light of the previous lemma, we keep only those with center of order 2. Here are the details. In every case, the prime order elements of B are central in M, so they cannot exist, so $B = 1$. For types (I)–(II) this leaves $O(M) = A$ and establishes our claimed structure for M. For type (I) we must also show that the cyclic 2-group, call it T, has order > 2. Otherwise, A is central in M and hence trivial, so T is all of M and has order 2. That is, the center of G is a maximal subgroup of G, which is a contradiction because no group can have this property.

For type (V) we have shown $M \sim 2A_5 \times A$, so A is central, and hence trivial. For type (III) we know $M \sim (Q_8 \times A) : \Theta$, with Θ's elements of order 3 centralizing A. It follows that $|\Theta| = 3$, because otherwise Θ's elements of order 3 would also centralize Q_8, hence be central in M. From $|\Theta| = 3$ it follows that A is central in M, and hence trivial. So $M \sim Q_8 : 3$ with the $\mathbb{Z}/3$ acting nontrivially on Q_8. Since $\mathrm{Aut}(Q_8) \cong S_4$ has a unique class of elements of order 3, M is determined up to isomorphism, namely $M \cong 2A_4$.

For type (VI) we know $M \sim (2A_5 \times A)\cdot 2$ and $O(M) = A$. Also, A decomposes as the direct sum of its subgroup inverted by the nontrivial element t of $M/(2A_5 \times A) \cong \mathbb{Z}/2$ and its subgroup fixed pointwise by t. The latter subgroup is central in M and hence trivial, so t inverts A as claimed. Also, t's image in $\mathrm{Out}\, 2A_5$ is nontrivial by the definition of type (VI) groups.

Finally, for type (IV) we have $M \sim \big((Q_8 \times A) : \Theta\big) \cdot 2$. By Remark 1.4, Θ and A commute. So $O(M)$ is cyclic and generated by A and the index 3 subgroup of Θ, leaving $M \sim (O(M).2A_4) \cdot 2$. By the argument for type (VI), the elements of M mapping nontrivially to $\mathbb{Z}/2$ must invert $O(M)$ and act on Q_8 by outer automorphisms.

We have shown in each case that $O(M)$ is cyclic. So its subgroups are characteristic in $O(M)$ and hence normal in M. By Lemma 3.3 and maximality, M is the full normalizer of any nontrivial subgroup of $O(M)$. For the last claim of the theorem, suppose a prime p divides the orders of $O(M_1)$ and $O(M_2)$. We have just shown that M_1 is the normalizer of a cyclic group of order p and that M_2 is the normalizer of another. These cyclic groups are G-conjugate, so M_1 and M_2 are also. □

Lemma 3.6 (Done if $Q_{16} \not\leq G$)**.** *Suppose the Sylow 2-subgroups of G have order 8. Then $G \cong 2A_5$.*

Proof. Suppose M is a maximal subgroup of G. It cannot have type (IV) or (VI), because these contain copies of Q_{16}. If M has type (III) or (V), then $M \cong 2A_4$ or $2A_5$ by Lemma 3.5. We claim that in the remaining cases, M is binary dihedral.

If M has type (I), then it has structure $O(M) : T$, where T is a cyclic group of order ≥ 4. Since this is the largest a cyclic 2-group in G can be, T has order exactly 4. A generator for it must invert $O(M)$, or else M's center would have order > 2. We have shown that M is binary dihedral.

Now suppose M has type (II) and hence structure $O(M) : Q$, where $Q \cong Q_8$. If $O(M) = 1$, then $M \cong Q_8$, which is binary dihedral as claimed. So suppose $O(M) \neq 1$ and let P be any Sylow subgroup of $O(M)$. Since $\operatorname{Aut} P$ is cyclic and $Q/Q' \cong 2 \times 2$, some element of order 4 in Q acts trivially on P. Since all elements of order 4 are conjugate, the centralizer of any one of them has order divisible by $|P|$. Now fix a particular element ϕ of order 4. Letting P vary over all Sylow subgroups of $O(M)$ shows that $C(\phi)$ has order divisible by $|O(M)|$. The only maximal subgroup of G that could contain a cyclic group of order $4 \cdot |O(M)|$ is M, up to conjugacy. So after conjugation we may suppose that $O(M) \leq C(\phi) \leq M$. We have shown that some element ϕ of order 4 in Q centralizes $O(M)$. So Q acts on $O(M)$ via a quotient group of order ≤ 2. This quotient must be $\mathbb{Z}/2$, acting by negation, because otherwise M would have center larger than $\mathbb{Z}/2$. It follows that M is binary dihedral.

We have shown that every maximal subgroup of M is binary dihedral, binary tetrahedral, or binary icosahedral. By examining normalizers in these groups, one checks that every noncentral cyclic subgroup of G has binary dihedral normalizer. So $G \cong 2A_5$ by Lemma 2.4. □

To prove the perfect case of Theorems 1.1 and 1.2, it now suffices to rule out the case $Q_{16} \leq G$. We devote the rest of this section to this.

Lemma 3.7 ($\mathbb{Z}/4$ normalizers)**.** *For any subgroup $\Phi \cong \mathbb{Z}/4$ of G, $N(\Phi)$ has structure* (odd group).(Sylow 2-subgroup of G)*.*

Proof. We claim first that $N(\Phi)$ has structure (odd group).(2-group). Choosing a maximal subgroup M containing $N(\Phi)$, it suffices to show that $N_M(\Phi)$ has this structure. To prove this one considers each possible structure for M listed in Lemma 3.5, and each subgroup $\mathbb{Z}/4$ of it. To finish the proof we use the fact that some $\mathbb{Z}/4$ is normal in some Sylow 2-subgroup and that all $\mathbb{Z}/4$'s are conjugate. □

Lemma 3.8 (Q_8 normalizers)**.** *Suppose G contains a copy of Q_{16}. Choose $Q \leq G$ isomorphic to Q_8 and write N for its normalizer. Then*

(1) $N \sim \big(O(N).2A_4\big) \cdot 2$.
(2) *Q lies in a group $2A_4$ if and only if $3 \nmid |O(N)|$.*
(3) *G has more than one conjugacy class of Q_8 subgroups.*
(4) *G contains a subgroup $2A_4$.*

Proof. (3.8) We begin by exhibiting some elements of N. Choose any subgroup $\Phi \cong \mathbb{Z}/4$ of Q. There exists a Sylow 2-subgroup of $N(\Phi)$ that contains Q. So Q lies in some Q_{16} that normalizes Φ. This Q_{16} contains an element that inverts Φ and exchanges the other two $\mathbb{Z}/4$ subgroups of Q. In particular, N contains an element that normalizes our chosen $\mathbb{Z}/4$ and exchanges the other two $\mathbb{Z}/4$'s in Q. It follows that N acts by S_3 on the three $\mathbb{Z}/4$ subgroups of Q.

Now we fix a maximal subgroup M of G that contains N. The property of N just established forces M to have type (IV) or (VI). That is,

$$M \sim \big(O(M).(2A_4 \text{ or } 2A_5)\big) \cdot 2$$

with Q lying in the index 2 subgroup M'. In either case, N coincides with $N_M(Q)$, which has the stated structure. Also, $O(N) = O(M)$, which we will use later in the proof.

(3.8) "If" follows from the existence of a Sylow 3-subgroup isomorphic to $\mathbb{Z}/3$ in N. For "only if", suppose 3 divides $|O(N)|$. Then N cannot contain any $2A_4$ because all its elements of order 3 are central in N.

(3.8) Fix $\Phi \leq Q$ isomorphic to $\mathbb{Z}/4$. By Lemma 3.7 there is a subgroup $Q^* \cong Q_8$ of $N(\Phi)$ that is not $N(\Phi)$-conjugate to Q. We claim that Q^* is not G-conjugate to Q either; suppose to the contrary that some $g \in G$ conjugates Q^* to Q. By $\Phi \leq Q^*$ we see that g sends Φ into Q. By replacing g by its composition with an element of N that sends Φ^g back to Φ, we may suppose without loss that g normalizes Φ. That is, Q and Q^* are conjugate in $N(\Phi)$, which is a contradiction.

(3.8) Supposing that G contains no $2A_4$, we will show that every subgroup $Q^* \cong Q_8$ of G is conjugate to Q, which contradicts (3.8). Applying the argument for (3.8) to Q^*, we write N^* for its normalizer and M^* for a maximal subgroup containing N^*. By (3.8) and the nonexistence of $2A_4$'s, 3 divides the orders of $O(N)$ and $O(N^*)$. These groups are the same as $O(M)$ and $O(M^*)$. Since $|O(M)|$ and $|O(M^*)|$ have a common factor, the last part of Lemma 3.5 shows that M and M^* are conjugate. So their index 2 subgroups M' and ${M^*}'$ are conjugate, both of which have structure (odd group).($2A_4$ or $2A_5$). Since Q, resp., Q^*, is a Sylow 2-subgroup of M', resp., ${M^*}'$, it follows that Q and Q^* are conjugate. □

Lemma 3.9 (Free actions of binary dihedral groups)**.** *Suppose H is a binary dihedral group, U is an irreducible fixed-point-free real representation of H, and α is the natural map $\mathbb{R}[H] \to \mathrm{End}(U)$. Then $\alpha(\mathbb{R}[H]) \cong \mathbb{H}$. Furthermore, if J is any binary dihedral subgroup of H, then $\alpha(\mathbb{R}[J]) = \alpha(\mathbb{R}[H])$.*

Proof. It is easy to see that the lemma holds for the representations (1.1), using their description in terms of $\mathbb{H}$. So it will suffice to show that U is one of them. Let C be an index 2 cyclic subgroup of H. Under C, U decomposes as a direct sum of 2-dimensional spaces, on each of which C acts faithfully by rotations. Fix one, say T, and consider the induced representation $\mathrm{Ind}_C^H(T)$, of dimension 4. Its canonical image in U is H-invariant, hence all of U. This image is larger than 2-dimensional because a binary dihedral group has no 2-dimensional free real representations. (Every finite subgroup of $\mathrm{O}(2)$ is cyclic or dihedral.) Therefore $U \cong \mathrm{Ind}_C^H(T)$. And the representations in (1.1) are exactly the H-representations of this form. □

Lemma 3.10 (Free actions of the binary tetrahedral group)**.** *Assume $A \cong 2A_4$ acts freely on a real vector space V, and write Q for the copy of Q_8 in A. Then the image of $\mathbb{R}[A]$ in $\mathrm{End}(V)$ is isomorphic to $\mathbb{H}$ and equal to the image of $\mathbb{R}[Q]$. In particular, every Q-invariant subspace is also A-invariant.*

Proof. Fix an identification of A with

$$2A_4 = \Big\{\pm 1, \pm i, \pm j, \pm k, (\pm 1 \pm i \pm j \pm k)/2\Big\} \subseteq \mathbb{H}^*. \tag{3.1}$$

We claim that A has a unique irreducible free real representation. It follows that V is a direct sum of copies of A's left-multiplication action on $\mathbb{H}$, which proves the lemma.

For the claim, fix an irreducible free A-module U. Choose a Q-submodule T which is Q-irreducible. In the proof of Lemma 3.9 we showed that (1.1) accounts for all irreducible free actions of binary dihedral groups. For Q_8, there was only one. So we may identify T with the real vector space underlying $\mathbb{H}$, with $Q = \{\pm 1, \pm i, \pm j, \pm k\} \subseteq A$ acting by left multiplication. The image of the natural map

$\mathrm{Ind}_Q^A(T) \to U$ is A-invariant, and hence all of U. By definition, $\mathrm{Ind}_Q^A(T)$ is the real vector space underlying $\mathbb{H}^3$, with $i \in Q$ acting by $(x, y, z) \mapsto (ix, jy, kz)$ and similarly with i, j, k cyclically permuted, and $\theta := (-1 + i + j + k)/2 \in A$ acting by $(x, y, z) \mapsto (z, x, y)$. Obviously the kernel of $\mathrm{Ind}_Q^A(T) \to U$ contains the fixed points of the nonidentity elements of A. For θ and $i \circ \theta$ these are $\{(x, x, x)\}$ and $\{(x, jx, -ix)\}$, which together span an 8-dimensional subspace, hence the whole kernel. This proves the uniqueness of U: it is the quotient of $\mathrm{Ind}_Q^A(T)$ by the subspace generated by the fixed points of the nonidentity elements of A. □

Lemma 3.11 (The final contradiction)**.** *G contains no subgroup Q_{16}.*

Proof. Suppose otherwise, and fix an irreducible fixed-point free real representation V of G. We write α for the natural map $\mathbb{R}[G] \to \mathrm{End}\, V$.

By Lemma 3.8(3.8), G has a subgroup $A^* \cong 2A_4$. We write Q^* for its Q_8 subgroup. We fix a Q_{16}-subgroup of G that contains Q^*. This is the only group isomorphic to Q_{16} we will consider, so we write Q_{16} for it. Write Q for the other Q_8 subgroup of Q_{16}. We will distinguish two cases, according to whether there exists some $A \cong 2A_4$ containing Q. In each case we will prove $\alpha(\mathbb{R}[G]) \cong \mathbb{H}$, which implies that G is isomorphic to a subgroup of $\mathbb{H}^*$. We assume this for the moment.

A well-known property of $\mathbb{H}^*$ is that every noncentral cyclic subgroup has $\mathbb{H}^*$-normalizer isomorphic to the "continuous binary dihedral group", meaning the nonsplit extension $(\mathbb{R}/\mathbb{Z})\cdot 2$. It follows that the G-normalizer of any noncentral cyclic subgroup is cyclic or binary dihedral. The cyclic case is ruled out by Corollary 2.6 and the perfectness of G. So Lemma 2.4 applies, proving $G \cong 2A_5$.

It remains to prove $\alpha(\mathbb{R}[G]) \cong \mathbb{H}$. First suppose that A exists. G is generated by A and A^*, because no maximal subgroup contains two copies of $2A_4$, whose Q_8 subgroups generate a copy of Q_{16}. Now fix an irreducible Q_{16}-submodule U of V. It is obviously Q- and Q^*-invariant, and hence A- and A^*-invariant by Lemma 3.10. So it is G-invariant by $\langle A, A^* \rangle = G$, and the G-irreducibility of V implies $U = V$. Lemmas 3.9 and 3.10 also give us the equalities

$$\alpha(\mathbb{R}[A]) = \alpha(\mathbb{R}[Q]) = \alpha(\mathbb{R}[Q_{16}]) = \alpha(\mathbb{R}[Q^*]) = \alpha(\mathbb{R}[A^*]) \tag{3.2}$$

and say that this subalgebra of $\mathrm{End}(U)$ is isomorphic to $\mathbb{H}$. Since A and A^* generate G, $\alpha(\mathbb{R}[G])$ also equals this copy of $\mathbb{H}$, finishing the proof in the case that A exists.

Now suppose Q lies in no copy of $2A_4$, and set $N = N_G(Q)$ and $\Phi = Q \cap Q^* \cong \mathbb{Z}/4$. By (3.8) and (3.8) of Lemma 3.8 we have $N \sim (O(N).2A_4) \cdot 2$ with $|O(N)|$ divisible by 3. By construction Q^* normalizes Q, and by $Q^* \neq Q$ we see that Q^* contains an element q^* of order 4 which lies in N but outside its index 2 subgroup $O(N).2A_4$. So q^* inverts $O(N)$. Of course, q^* also inverts Φ. Also, $O(N)$ commutes with Q (hence Φ) by the known structure of N. So $H = \langle q^*, O(N), \Phi \rangle = \langle Q^*, O(N) \rangle$ is binary dihedral. The rest of the argument is similar to the previous case.

Namely, we have $G = \langle H, A^* \rangle$ because no maximal subgroup M of G contains a copy of Q_8 which normalizes one copy of $\mathbb{Z}/3$ in M and is normalized by a different copy of $\mathbb{Z}/3$ in M. Now we choose an irreducible H-submodule U of V. It is trivially Q^*-invariant. Then Lemma 3.10 shows that U is A^*-invariant, hence G-invariant, and hence all of V. Lemmas 3.9–3.10 give the equalities

$$\alpha(\mathbb{R}[H]) = \alpha(\mathbb{R}[Q^*]) = \alpha(\mathbb{R}[A^*]) \tag{3.3}$$

and say that this subalgebra of $\mathrm{End}(U)$ is isomorphic to $\mathbb{H}$. By $G = \langle H, A^* \rangle$, this subalgebra is also $\alpha(\mathbb{R}[G])$, finishing the proof. $\square$

Remark 3.12 (Frobenius complements)*.* As mentioned in the introduction, our arguments adapt easily to classify the Frobenius complements, which are the same groups. Supposing that G acts freely on a vector space over a finite field $\mathbb{F}_q$, where q is necessarily odd, we extend scalars to suppose without loss that all elements of G are diagonalizable. Then Lemmas 3.9–3.11 and their proofs still apply with $\mathbb{R}$ replaced by $\mathbb{F}_q$, $\mathbb{H}$ (the algebra) by $M_2\mathbb{F}_q$, $\mathbb{H}$ (the module) by $\mathbb{F}_q^2$, and $\mathbb{H}^*$ by $\mathrm{SL}_2\,\mathbb{F}_q$. (Although the proof of Lemma 3.10 looks $\mathbb{H}$-specific, one defines the Hurwitz integers as the $\mathbb{Z}$-span of the quaternions (3.1), and tensoring with $\mathbb{F}_q$ gives $M_2\mathbb{F}_q$.)

4. The imperfect case

In this section we will complete the proofs of Theorems 1.1 and 1.2. We suppose throughout that G is an imperfect finite group that acts freely and isometrically on a sphere of some dimension, and that every proper subgroup has one of the structures listed in the statement of Theorem 1.1. We will prove that G also has one of these structures, in fact a unique one in the sense of Theorem 1.2. We will prove Theorems 1.1 and 1.2 in three special cases (which don't actually use imperfectness), and then argue that these cases are enough.

Lemma 4.1. *Suppose $G/O(G)$ is a 2-group. Then G has type* (I) *or* (II) *from Theorem* 1.1*, for a unique-up-to-conjugation triple of subgroups* (A, B, T)*.*

Proof. We will construct A, B, and T such that $G \sim A : (B \times T)$ as in Theorem 1.1, along the way observing that the construction is essentially unique. Obviously we must choose A and B such that $O(G) = A : B$. Applying Lemma 2.8 to $O(G)$ proves the following. The required coprimality of $|A|$ and $|B|$, together with the requirement that each nontrivial Sylow subgroup of B acts nontrivially on A, can be satisfied in a unique way. That is, A is uniquely determined, and B is determined up to conjugacy in $O(G)$. Using the coprimality of $|O(G)|$ and $|G/O(G)| \cong T$, part (IV) of Lemma 2.8 shows that the decomposition $O(G) = A : B$ extends to a decomposition $G = A : (B \times T)$, where T is determined uniquely up to conjugacy in $N_G(B)$.

This finishes the proof of uniqueness; it remains to check a few assertions of Theorem 1.1. Lemma 2.3 says that T is cyclic or quaternionic. Lemma 2.2 assures us that every prime-order element of B or T acts trivially on every prime-order subgroup of A. An automorphism of the cyclic group A, of order prime to $|A|$, and acting trivially on every subgroup of prime order must act trivially on all of A. So the prime-order elements of B and T centralize A. $\square$

Lemma 4.2. *Suppose G contains a normal subgroup* $2A_5$*. Then G has type* (V) *or* (VI) *from Theorem* 1.1*, for a unique-up-to-conjugation tuple of subgroups* $(A, B, 2A_5)$ *or* $(A, B, 2A_5, \Phi)$*.*

Proof. We will show that there is a such a tuple, unique up to conjugation, whose $2A_5$ term is the given normal subgroup. The existence part of this assertion shows that $G \sim 2A_5 \times (A : B)$ or $G \sim \big(2A_5 \times (A : B)\big) \cdot 2$ as in Theorem 1.1. It follows that G has a unique normal subgroup isomorphic to $2A_5$. So every tuple has this particular subgroup as its $2A_5$ term. The uniqueness of the tuple up to conjugacy follows.

It remains to prove the lemma for the given $2A_5$ subgroup. We define I as the subgroup of G acting on $2A_5$ by inner automorphisms. Since $\mathrm{Out}(2A_5) = 2$, I has index 1 or 2 in G. By its definition, I is generated by $2A_5$ and $C(2A_5)$, whose intersection is the group Z generated by G's central involution. We claim that Z is the full Sylow 2-subgroup of $C(2A_5)$. Otherwise, $2A_5/Z \times \big(C(2A_5)/Z\big) \leq G/Z$ would contain an elementary abelian 2-group of rank 3. This is impossible because the Sylow 2-subgroups of G/Z are dihedral. From Corollary 2.6 and the fact that $C(2A_5)$ has cyclic Sylow 2-subgroup, we get $C(2A_5) = Z \times O(C(2A_5))$. It follows that $I = 2A_5 \times O(I)$.

Obviously we must choose A and B such that $A : B = O(I)$. As in the previous proof, Lemma 2.8 shows that there is an essentially unique way to satisfy the conditions that $|A|$ and $|B|$ are coprime and that every Sylow subgroup of B acts nontrivially on A. That is, A is uniquely determined and B is determined up to conjugacy in $O(I)$. Also, $|A|$ and $|B|$ are coprime to 15 because G has no subgroup 3×3 or 5×5 (Lemma 2.2). The prime-order elements of B act trivially on A by the same argument as in the previous proof.

We have shown that I has type (V), so if $I = G$, then the proof is complete. Otherwise, we know $G \sim \big(2A_5 \times (A : B)\big) \cdot 2$ and we must construct a suitable subgroup $\Phi \cong \mathbb{Z}/4$ of G. Because all complements to A in $A : B$ are conjugate, the Frattini argument shows that $N(B)$ surjects to G/A. So $N(B)$ contains a Sylow 2-subgroup $T \cong Q_{16}$ of G. By Sylow's theorem it is unique up to conjugacy in $N(B)$. Obviously $T \cap 2A_5$ is isomorphic to Q_8. Now, $T \cong Q_{16}$ contains exactly two subgroups isomorphic to $\mathbb{Z}/4$ that lie outside $T \cap 2A_5$. So we must take Φ to be one of them. They are conjugate under $T \cap 2A_5$, so Φ is uniquely defined up to a conjugation that preserves each of A, B, and $2A_5$. It remains only to check that Φ has the properties required for G to be a type (VI) group. It acts on $2A_5$ by an outer automorphism because it does not lie in I, by construction. To see that Φ commutes with B, suppose to the contrary. Then some subgroup of B of prime order p would lie in G'. Then Corollary 2.6 would show that the Sylow p-subgroup of B lies in G'. So it acts trivially on A, which is a contradiction. $\square$

Lemma 4.3. *Suppose G contains a normal subgroup $Q \cong Q_8$ that lies in G'. Then G has type* (III) *or* (IV) *from Theorem* 1.1*, for a unique-up-to-conjugation tuple of subgroups* (A, B, Q_8, Θ) *or* $(A, B, Q_8, \Theta, \Phi)$.

Proof. Mimicking the previous proof, we will show that there is a unique-up-to-conjugation tuple of such subgroups whose Q_8 term is Q. In particular, $G \sim (Q_8 \times A) : (B \times \Theta)$ or $G \sim \big((Q_8 \times A) : (B \times \Theta)\big) \cdot 2$ as in Theorem 1.1. This shows that G has a unique normal subgroup isomorphic to Q_8, namely Q, which will complete the proof for the same reason as before.

To show that there is a unique-up-conjugation tuple of subgroups whose Q_8 term is Q, consider the natural map $G \to \mathrm{Aut}\, Q \cong S_4$. We claim the image $\overline{G}$ contains an element of order 3. Otherwise, $\overline{G} \subseteq \mathrm{Aut}\, Q \cong S_4$ would lie in a Sylow 2-subgroup of $\mathrm{Aut}\, Q$, which is dihedral of order 8 with a commutator subgroup of order 2. Therefore $|\overline{G}'|$ would have order ≤ 2. But this is a contradiction because Q lies in G' and $\overline{Q} \cong 2 \times 2$. We have shown that the nontrivial 3-subgroup of $\mathrm{Out}\, Q \cong S_3$ lies in the image of G. (Now we can discard $\overline{G}$.)

Let J be the preimage in G of this copy of $\mathbb{Z}/3$. So $G \sim J.(1 \text{ or } 2)$. Obviously we must choose A, B, and Θ so that $J = (Q \times A) : (B \times \Theta)$. By construction, J has a nontrivial map to $\mathbb{Z}/3$. So Corollary 2.6 assures us that J's Sylow 3-subgroups map faithfully to J's abelianization. In particular, $J = I.(\text{cyclic 3-group})$, where $|I|$ is prime to 3. Mimicking the previous proof shows that $I = Q \times O(I)$. So we must choose A and B so that $A : B = O(I)$.

Continuing to follow the previous proof shows that there is an essentially unique way to satisfy the conditions that $|A|$ and $|B|$ are coprime and that every Sylow subgroup of B acts nontrivially on A. That is, A is uniquely determined and B is determined up to conjugacy in $O(I)$. The prime-order elements of B act trivially on A for the same reason as before. We have shown

$$G \sim \underbrace{\overbrace{\big(Q \times (A : B)\big)}^{I}.(\text{nontrivial cyclic 3-group})}_{J}.(1 \text{ or } 2).$$

Having worked our way to the "middle" of G, we now work outwards and construct Θ and (if required) Φ. Using the conjugacy of complements to A in $A : B$, the Frattini argument shows that $N_J(B)$ maps onto $J/(A : B)$. So $N_J(B)$ contains a Sylow 3-subgroup of J, indeed a unique one up to conjugacy in $N_J(B)$. So there is an essentially unique possibility for Θ. As observed above, Θ acts nontrivially on Q. Its elements of order 3 act trivially on A for the same reason that B's prime-order elements do. Finally, Θ commutes with B. Otherwise, some Sylow subgroup of B would lie in G', and hence act trivially on A, contrary to B's construction.

We have shown that J has type (III), so if $J = G$, then we are done. Otherwise, mimicking the previous proof shows that there exists a group $\Phi \cong \mathbb{Z}/4$ in $N(B \times \Theta)$ that does not lie in J, and that such a group is unique up to conjugacy in $N(B \times \Theta)$. Also as before, Φ commutes with B. By $\Phi \not\leq J$, Φ's elements of order 4 act on Q by outer automorphisms. Since the images of Φ and Θ in $\operatorname{Out} Q \cong S_3$ do not commute, Φ cannot commute with Θ. Therefore Φ's elements of order 4 must invert Θ. So G has type (IV) and the proof is complete. $\square$

To finish the proofs of Theorems 1.1 and 1.2 we will show that G satisfies the hypotheses of one of Lemmas 4.1–4.3. By imperfectness, G has a normal subgroup M of prime index p. By induction, M has one of the structures (I)–(VI). We will examine the various cases and see that one of the lemmas applies.

If M has type (V) or (VI), then it contains a unique subgroup $2A_5$, which is therefore normal in G, so Lemma 4.2 applies. If M has type (III) or (IV), then it contains a unique normal subgroup Q_8, which is therefore normal in G. Also, this Q_8 lies in M', and hence G', so Lemma 4.3 applies. Finally, suppose M has type (I) or (II), so $M \sim A : \big(B \times (\text{2-group } T)\big)$. If $p = 2$ then $O(G) = O(M) = A : B$ and $G/O(G)$ is a 2-group, so Lemma 4.1 applies.

So suppose $p > 2$, and observe that $G/O(M)$ has structure $T.p$. Choose a subgroup P of order p in $G/O(M)$. If it acts trivially on T, then we have $G/O(M) \cong T \times p$, so $G/O(G) \cong T$ and Lemma 4.1 applies. So suppose P acts nontrivially on T. The automorphism group of any cyclic or quaternionic 2-group is a 2-group, except for $\operatorname{Aut} Q_8 \cong S_4$. Since P acts nontrivially, we must have $T \cong Q_8$ and $p = 3$. The nontriviality of P's action also implies $T < G'$. From this and the fact that $\operatorname{Aut} A$ is abelian, it follows that T acts trivially on A. So T is normal in M. As M's unique Sylow 2-subgroup, it is normal in G. So Lemma 4.3 applies. This completes the proofs of Theorems 1.1 and 1.2.

5. Irredundant enumeration

The uniqueness expressed in Theorem 1.2 makes the isomorphism classification of groups in Theorem 1.1 fairly simple. Namely, one specifies such a group G up to isomorphism by choosing its type (I)–(VI), a suitable number a for the order of A, a suitable subgroup $\overline{G}$ of the unit group $(\mathbb{Z}/a)^*$ of the ring $\mathbb{Z}/a$, and a suitable number g for the order of G. In a special case one must also specify a suitable subgroup $\overline{G}_0$ of $\overline{G}$. Before developing this, we illustrate some redundancy in Wolf's list.

Example 5.1 (Duplication)**.** Wolf's presentations of type II in [7, theorem 6.1.11] have generators A, B, and R and relations

$$A^m = B^n = 1 \qquad BAB^{-1} = A^r$$
$$R^2 = B^{n/2} \qquad RAR^{-1} = A^l \qquad RBR^{-1} = B^k,$$

where (m, n, r, k, l) are numerical parameters satisfying nine conditions. It turns out that the six choices $(3, 20, -1, -1, \pm 1)$, $(5, 12, -1, -1, \pm 1)$ and $(15, 4, -1, -1, 4$ or $11)$ give isomorphic groups, namely $(3 \times 5) : Q_8$. Here one class of elements of order 4 in Q_8 inverts just the $\mathbb{Z}/3$ factor, another class inverts just the $\mathbb{Z}/5$ factor, and the third class inverts both. Wolf decomposes this group by choosing an index 2 subgroup (call it H) with a cyclic Sylow 2-subgroup, taking A to generate H' and taking B to generate a complement to H' in H (which always exists). Then he takes R to be an element of order 4 outside of H that normalizes $\langle B\rangle$. There are three ways to choose H, corresponding to the three $\mathbb{Z}/4$'s in Q_8. After H is chosen, there are essentially unique choices for $\langle A\rangle$ and $\langle B\rangle$, but two choices for $\langle R\rangle$. These choices lead to the six different presentations. Our form of this group is $A : (B \times Q_8) = 15 : (1 \times Q_8) = 15 : Q_8$. (Our A and B are subgroups, while Wolf's A and B are elements.)

A minor additional source of redundancy is that replacing Wolf's B by a different generator of $\langle B\rangle$ can change the parameter r in the presentation above.

Given a group G from Theorem 1.1, we record the following invariants. First we record its type (I)–(VI), which is well-defined by the easy part of Theorem 1.2, proven in section 1. Second we record $g := |G|$, $a := |A|$, and $\overline{G}$, where bars will indicate images in $\operatorname{Aut} A$. Finally, and only if G has type (II) and its order is divisible by 16, we record $\overline{G}_0$, where G_0 is the unique index 2 subgroup of G with cyclic Sylow 2-subgroups.

$\operatorname{Aut} A$ is canonically isomorphic to the group of units $(\mathbb{Z}/a)^*$ of the ring $\mathbb{Z}/a$, with $u \in (\mathbb{Z}/a)^*$ corresponding to the uth power map. Therefore we will regard $\overline{G}$ and $\overline{G}_0$ as subgroups of $(\mathbb{Z}/a)^*$. This is useful when comparing two groups G, G^* as follows.

Theorem 5.2 (Isomorphism recognition)**.** *Two finite groups G and G^*, that act freely and isometrically on spheres of some dimensions, are isomorphic if and only if they have the same type and satisfy $g = g^*$, $a = a^*$, $\overline{G} = \overline{G}^*$, and (if they are defined) $\overline{G}_0 = \overline{G}_0^*$.*

Proof. First suppose $G \cong G^*$. We have already mentioned that they have the same type, and they obviously have the same order. By the uniqueness of A and A^* (Theorem 1.2), any isomorphism $G \to G^*$ identifies A with A^*. In particular, $a = a^*$, and the action of G on A corresponds to that of G^* on A^*. So we must have $\overline{G} = \overline{G}^*$ and (when defined) $\overline{G}_0 = \overline{G}_0^*$.

Now suppose G and G^* have the same invariants; we must construct an isomorphism between them. Case analysis seems unavoidable, but the ideas are uniform and only the details vary.

Type (I). Since A and A^* are cyclic of the same order, we may choose an isomorphism $A \cong A^*$. By order considerations we have $|B| = |B^*|$ and $|T| = |T^*|$. Again by order considerations, the identification of $\overline{G}$ with $\overline{G}^*$ under the canonical isomorphism $\operatorname{Aut} A = \operatorname{Aut} A^*$ identifies $\overline{B}$ with $\overline{B}^*$ and $\overline{T}$ with $\overline{T}^*$. Because B and B^* are cyclic of the same order, we may lift the identification $\overline{B} \cong \overline{B}^*$ to an isomorphism $B \cong B^*$, and similarly for T and T^*. Together with $A \cong A^*$, these give an isomorphism $G \cong G^*$.

Type (II) *when* 16 *divides* $|G| = |G^*|$. Because $\overline{G}_0$ is identified with $\overline{G}_0^*$, the previous case gives an isomorphism $G_0 \cong G_0^*$ that identifies A with A^*, B with B^*, and the cyclic 2-group $T_0 := T \cap G_0$ with $T_0^* := T^* \cap G_0^*$. We will extend this to an isomorphism $G \cong G^*$. Choose an element ϕ of $T - T_0$. By the identification of $\overline{T}$ with $\overline{T}^*$ and $\overline{T}_0$ with $\overline{T}_0^*$, there exists an element ϕ^* of $T^* - T_0^*$ whose action on A^* corresponds to ϕ's action on A. As elements of $T - T_0$ and $T^* - T_0^*$, ϕ and ϕ^* have order 4. They commute with B and B^*, respectively. Their squares are the unique involutions in T_0 and T_0^*, which we have already identified with each other. So identifying ϕ with ϕ^* extends our isomorphism $\overline{G}_0 \cong \overline{G}_0^*$ to $\overline{G} \cong \overline{G}^*$.

Type (II) *when* 16 *does not divide* $|G| = |G^*|$. The argument for type (I) identifies A with A^* and B with B^*. Both G and G^* have Sylow 2-subgroups isomorphic to Q_8. $\overline{T}$ is the Sylow 2-subgroup of $\overline{G}$ and is elementary abelian of rank ≤ 2 because $Q_8/Q_8' \cong 2 \times 2$. The identification of $\overline{G}$ with $\overline{G}^*$ identifies $\overline{T}$ with $\overline{T}^*$. Because $\operatorname{Aut} Q_8$ acts as S_3 on Q_8/Q_8', it is possible to lift the identification $\overline{T} \cong \overline{T}^*$ to an isomorphism $T \cong T^*$. Now our identifications $A \cong A^*$, $B \cong B^*$, and $T \cong T^*$ fit together to give an isomorphism $G \cong G^*$.

Type (III). Choose an isomorphism $A \cong A^*$. By $|G| = |G^*|$ we get $|B| = |B^*|$ and $|\Theta| = |\Theta^*|$. The identification of $\overline{G} \leq \operatorname{Aut} A$ with $\overline{G}^* \leq \operatorname{Aut} A^*$ identifies $\overline{\Theta}$ with $\overline{\Theta}^*$ and $\overline{B}$ with $\overline{B}^*$. These identifications can be lifted to isomorphisms $\Theta \cong \Theta^*$ and $B \cong B^*$ by the same reasoning as before. Because all elements of order 3 in $\operatorname{Aut} Q_8 \cong S_4$ are conjugate, it is possible to choose an isomorphism $Q_8 \cong Q_8^*$ compatible with our isomorphism $\Theta \cong \Theta^*$ and the homomorphisms $\Theta \to \operatorname{Aut} Q_8$ and $\Theta^* \to \operatorname{Aut} Q_8^*$. Now our identifications $A \cong A^*$, $B \cong B^*$, $\Theta \cong \Theta^*$, and $Q_8 \cong Q_8^*$ fit together to give an isomorphism $G \cong G^*$.

Type (IV). Identify A, B, and Θ with A^*, B^*, and Θ^* as in the previous case. Choose generators ϕ and ϕ^* for Φ and Φ^*. Their actions on A and A^* correspond because they act by the unique involutions in $\overline{G}$ and $\overline{G}^*$ if these exist, and trivially otherwise. The image of ϕ in $\operatorname{Aut} Q_8 \cong S_4$ normalizes the image of Θ, so together they generate a copy of S_3, and similarly for their starred versions. Any isomorphism from one S_3 in $\operatorname{Aut} Q_8$ to another is induced by some conjugation in $\operatorname{Aut} Q_8$. (One checks this using $\operatorname{Aut} Q_8 \cong S_4$.) Therefore it is possible to identify Q_8 with Q_8^* such that the actions of Θ and Θ^* on them correspond and the actions of ϕ

and ϕ^* also correspond. Using this, and identifying ϕ with ϕ^*, gives an isomorphism $G \cong G^*$.

Type (V). Identify A and B with A^* and B^* as in the previous cases, and $2A_5$ with $2A_5^*$ however one likes.

Type (VI). Identify A and B with A^* and B^* as before, and choose generators ϕ and ϕ^* for Φ and Φ^*. As in the type (IV) case, their actions on A and A^* correspond. Next, ϕ and ϕ^* act on $2A_5$ and $2A_5^*$ by involutions which are not inner automorphisms. All such automorphisms of $2A_5$ are conjugate in $\mathrm{Aut}(2A_5)$. (They correspond to the involutions in $S_5 - A_5$.) So we may identify $2A_5$ with $2A_5^*$ in such a way that the actions of ϕ and ϕ^* on them correspond. Using this, and identifying ϕ with ϕ^*, gives an isomorphism $G \cong G^*$. □

We can now parameterize the isomorphism classes of finite groups G that admit free actions on spheres. First one specifies a type (I)–(VI). Then one specifies a positive integer a, a subgroup $\overline{G}$ of $(\mathbb{Z}/a)^*$, and possibly a subgroup $\overline{G}_0$ of $\overline{G}$, all satisfying some constraints. Then one chooses one or more auxiliary parameters, constrained in terms of properties of $\overline{G}$. Together with a, these specify g, and hence the isomorphism type of G. The following theorem is proven by combining Theorem 5.2 with an analysis of what possibilities can actually arise. We will write the parameters a, $\overline{G}$, and g in the order one chooses them, rather than in the order used in Theorem 5.2.

The constraints on the choices of parameters are difficult to express uniformly. But the constraint on one auxiliary parameter, called b, is uniform. For each type we obtain a positive integer $\overline{b}$ from the structure of $\overline{G}$, and b must be the product of $\overline{b}$ and nontrivial powers of all the primes dividing $\overline{b}$. We will express this by saying "b is as above".

Theorem 5.3 (Irredundant enumeration)**.** *Suppose G is a finite group admitting a free and isometric action on a sphere. Then there is exactly one tuple $((\mathrm{I})—(\mathrm{VI}), a, \overline{G}, g)$ or $((\mathrm{II}), a, \overline{G}, \overline{G}_0, g)$ listed below, whose corresponding group (defined at the end) is isomorphic to G.* □

$(\text{Type (I)}, a, \overline{G}, g = abt)$, where

(1) a is odd.
(2) $\overline{G}$ is a cyclic subgroup of $(\mathbb{Z}/a)^*$ of order prime to a. Define $\overline{b}$ and $\overline{t}$ as the odd and 2-power parts of $|\overline{G}|$.
(3) b is as above and t is a power of 2, larger than $\overline{t}$ if $\overline{t} \neq 1$.

$(\text{Type (II)}, a, \overline{G}, g = abt)$ or $(\text{Type (II)}, a, \overline{G}, \overline{G}_0, g = abt)$, where

(1) a is odd.
(2) $\overline{G}$ is a subgroup of $(\mathbb{Z}/a)^*$ which is the direct product of a cyclic group of order prime to $2a$ and an elementary abelian 2-group of rank ≤ 2. Define $\overline{b}$ and $\overline{t}$ as the orders of these factors.
(3) $\overline{G}_0$, if specified, is a subgroup of $\overline{G}$ of index 2 (if $\overline{t} = 4$) or index ≤ 2 (otherwise).
(4) b is as above and $t = 8$, unless $\overline{G}_0$ was specified, in which case t is a power of 2, larger than 8.

$(\text{Type (III)}, a, \overline{G}, g = 8ab\theta)$, where

(1) a is prime to 6.

(2) $\overline{G}$ is a subgroup of $(\mathbb{Z}/a)^*$ which is the direct product of a cyclic 3-group and a cyclic group of order prime to $6a$. Define $\bar{\theta}$ and $\overline{b}$ as the orders of these factors.
(3) b is as above and θ is a power of 3, larger than $\bar{\theta}$.

$(\text{Type (IV)}, a, \overline{G}, g = 16ab\theta)$, where
(1) a is prime to 6.
(2) $\overline{G}$ is a subgroup of $(\mathbb{Z}/a)^*$, which is the direct product of a cyclic group of order prime to $6a$ and a group of order 1 or 2. Define $\overline{b}$ as the order of the first factor.
(3) b is as above and θ is a nontrivial power of 3.

$(\text{Type (V)}, a, \overline{G}, g = 120ab)$, where
(1) a is prime to 30.
(2) $\overline{G}$ is a cyclic subgroup of $(\mathbb{Z}/a)^*$ of order prime to $30a$. Define $\overline{b}$ as its order.
(3) b is as above.

$(\text{Type (VI)}, a, \overline{G}, g = 240ab)$, where
(1) a is prime to 30.
(2) $\overline{G}$ is a subgroup of $(\mathbb{Z}/a)^*$, which is the direct product of a cyclic group of order prime to $30a$ and a group of order 1 or 2. Define $\overline{b}$ as the order of the first factor.
(3) b is as above.

Here are instructions for building G. In a sense this is a constructive version of the proof of Theorem 5.2. For all types, start by taking cyclic groups A, B with orders a, b. Up to isomorphism of the domain, B has a unique surjection to the subgroup of $\overline{G} \subseteq \operatorname{Aut} A$ of order $\overline{b}$. Form the corresponding semidirect product $A : B$. Now we consider the six cases.

The easiest is type (V)—just set $G = 2A_5 \times (A : B)$.

For type (I), we take a cyclic group T of order t. Just as for B, there is an essentially unique surjection from T to the subgroup of $\overline{G}$ of order $\overline{t}$. Then G is the semidirect product $A : (B \times T)$.

For type (III) one takes a cyclic group Θ of order θ. Just as for B, there is an essentially unique surjection from Θ to the subgroup of $\overline{G}$ of order $\bar{\theta}$. We also take Θ to act nontrivially on Q_8. (Up to conjugacy in $\operatorname{Aut} Q_8$ there is a unique nontrivial action.) Then G is the semidirect product $(Q_8 \times A) : (B \times \Theta)$. B acts trivially on Q_8; this is forced since $|B|$ is prime to 6.

A type (IV) or (VI) group is obtained from a type (III) or (V) group by adjoining a suitable element ϕ of order 4. In both cases, ϕ squares to the central involution, centralizes B, and acts on A by the nontrivial involution in $\overline{G}$ (if one exists) or trivially (otherwise). For type (VI), ϕ acts on $2A_5$ by an outer automorphism of order 2, which is unique up to $\operatorname{Aut} 2A_5$. For type (IV), ϕ inverts Θ and acts on Q_8 by an involution that inverts the action of Θ. Such an automorphism is unique up to an automorphism of Q_8 that respects the Θ-action.

One can describe type (II) groups in terms of type (I) in a similar way, but it is easier to build them directly. Take T to be a quaternion group of order t. First suppose $t = 8$. Then we did not specify $\overline{G}_0$. Up to automorphism of the domain there is a unique surjection from T to the subgroup $\overline{T}$ of $\overline{G}$ of order $\overline{t}$. We take $G = A : (B \times T)$. On the other hand, suppose $t > 8$, in which case we did specify

$\overline{G}_0$. We write T_0 for the index 2 cyclic subgroup of T. Up to automorphism of the domain, there is a unique surjection from T to $\overline{T}$ which carries T_0 onto the 2-part of $\overline{G}_0$ (which has order 1 or 2). And again $G = A : (B \times T)$.

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