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ON COMPACT 3-MANIFOLDS WITH NONNEGATIVE SCALAR CURVATURE WITH A CMC BOUNDARY COMPONENT

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Abstract. We apply the Riemannian Penrose inequality and the Riemannian positive mass theorem to derive inequalities on the boundary of a class of compact Riemannian 3-manifolds with nonnegative scalar curvature. The boundary of such a manifold has a CMC component, i.e., a 2-sphere with positive constant mean curvature; and the rest of the boundary, if nonempty, consists of closed minimal surfaces. A key step in our proof is the construction of a collar extension that is inspired by the method of Mantoulidis-Schoen.

1. Introduction and statement of results

In this paper, we are interested in a compact Riemannian 3-manifold Ω with nonnegative scalar curvature, with boundary $\partial\Omega$, such that $\partial\Omega$ has a component Σ_o that is a topological 2-sphere with positive mean curvature. When $\partial\Omega \setminus \Sigma_o \neq \emptyset$, we assume that $\partial\Omega\setminus\Sigma_o$ is the unique, closed minimal surface (possibly disconnected) in Ω; i.e., there are no other closed minimal surfaces in Ω. In this case, we denote $\partial\Omega \setminus \Sigma_o$ by Σ_h. In a relativistic context, such an Ω represents a finite body in a time-symmetric initial data set, surrounding the apparent horizon modeled by Σ_h.

Motivated by the quasi-local mass problem (cf. [18]), we want to understand the effect of nonnegative scalar curvature and the existence of Σ_h on the boundary geometry of Σ_o. To be more precise, let g denote the induced metric on Σ_o and let H be the mean curvature of Σ_o in Ω. We want to understand the restriction imposed by the scalar curvature and the horizon boundary Σ_h on the pair (g, H).

A special case of this question was studied in [15]. It was proved in [15] that

$$(\Sigma_o, g) \text{ is a round sphere} \Rightarrow \sqrt{\frac{|\Sigma_o|}{16\pi}}\left[1 - \frac{1}{16\pi|\Sigma_o|}\left(\int_{\Sigma_o} H d\sigma\right)^2\right] \geq \sqrt{\frac{|\Sigma_h|}{16\pi}},$$

where $|\Sigma_o|$, $|\Sigma_h|$ are the area of Σ_o, Σ_h, respectively, and $d\sigma$ denotes the area element on Σ_o. The left side of the above inequality closely resembles the Hawking mass [7] of Σ_o in Ω, given by

$$\mathfrak{m}_{_H}(\Sigma_o) = \sqrt{\frac{|\Sigma_o|}{16\pi}}\left[1 - \frac{1}{16\pi}\int_{\Sigma_o} H^2 \, d\sigma\right].$$

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The Hawking mass functional $\mathfrak{m}_{_H}(\cdot)$ played a key role in Huisken and Ilmanen's proof of the Riemannian Penrose inequality (cf. [2, 9]) when the horizon is connected. In particular, by the results in [9], if a weak solution $\{\Sigma_t\}$ consisting of connected surfaces to the inverse mean curvature flow with initial condition Σ_h exists in Ω and if Σ_o happens to be a leaf in $\{\Sigma_t\}$, then one would have $\mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}$.

In general, without imposing suitable conditions on Σ_o, one should not expect to have $\mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}$ since $\mathfrak{m}_{_H}(\Sigma_o)$ may even fail to be positive. On the other hand, if a 2-surface is a stable constant mean curvature (CMC) surface in a 3-manifold with nonnegative scalar curvature, Christodoulou and Yau [4] showed that its Hawking mass is always nonnegative.

In this paper, we consider an Ω in which Σ_o is a CMC surface. We have

Theorem 1.1. *Let Ω be a compact, orientable, Riemannian 3-manifold with boundary $\partial\Omega$. Suppose $\partial\Omega$ is the disjoint union of Σ_o and Σ_h such that*

(a) *Σ_o is a topological 2-sphere with constant mean curvature $H_o > 0$;*
(b) *Σ_h, which may have multiple components, is a minimal surface; and*
(c) *there are no other closed minimal surfaces in Ω.*

Suppose Ω has nonnegative scalar curvature and the induced metric g on Σ_o has positive Gauss curvature. There exists a quantity $0 < \eta(g) \leq \infty$, uniquely determined by (Σ_o, g) and invariant under scaling of g, such that if

$$\mathcal{W} := \frac{1}{16\pi}\int_{\Sigma_o} H_o^2 d\sigma < \eta(g),$$

then

$$\sqrt{\frac{|\Sigma_h|}{16\pi}} \leq \left[\frac{\mathcal{W}}{\eta(g) - \mathcal{W}}\right]^{\frac{1}{2}} \sqrt{\frac{|\Sigma_o|}{16\pi}} + \mathfrak{m}_{_H}(\Sigma_o). \tag{1.1}$$

Here $\eta(g) = \infty$ if g is a round metric. In this case, (1.1) reduces to $\sqrt{\frac{|\Sigma_h|}{16\pi}} \leq \mathfrak{m}_{_H}(\Sigma_o)$.

Theorem 1.1 has the following analogue when $\partial\Omega = \Sigma_o$.

Theorem 1.2. *Let Ω be a compact, Riemannian 3-manifold with nonnegative scalar curvature, with boundary Σ_o. Suppose Σ_o is a topological 2-sphere with constant mean curvature $H_o > 0$. Suppose the induced metric g on Σ_o has positive Gauss curvature. Let $\eta(g)$ be the scaling invariant of (Σ_o, g) stated in Theorem 1.1. If*

$$\mathcal{W} := \frac{1}{16\pi}\int_{\Sigma_o} H_o^2 d\sigma < \eta(g),$$

then

$$\left[\frac{\mathcal{W}}{\eta(g) - \mathcal{W}}\right]^{\frac{1}{2}} \sqrt{\frac{|\Sigma_o|}{16\pi}} + \mathfrak{m}_{_H}(\Sigma_o) \geq 0. \tag{1.2}$$

The quantity $\eta(g)$ measures how far g is different from a round metric on Σ_o. We will give its precise definition in Section 4. For now we give a few remarks on Theorems 1.1 and 1.2.

Remark 1.1. For a fixed $\delta \in (0, 1)$, it is proved in Proposition 4.1 that

$$\eta(g) \geq \frac{C}{||g - g_o||^2_{C^{0,\delta}(\Sigma_o)}} \tag{1.3}$$

for some positive constant C independent on g if g is $C^{2,\delta}$-close to a round metric g_o on Σ_o. In particular, $\eta(g)$ tends to ∞ as g approaches g_o in the $C^{2,\delta}$-norm. On the other hand, given an Ω in Theorem 1.1, by Shi and Tam's result [20, Theorem 1] (or, more precisely, by their proof), one has

$$\int_{\Sigma_o} H_o d\sigma < \int_{\Sigma_o} H_{_E} d\sigma,$$

where $H_{_E}$ is the mean curvature of the isometric embedding of Σ_o in $\mathbb{R}^3$. Consequently,

$$\mathcal{W} < \omega(g) := \frac{1}{16\pi|\Sigma_o|}\left(\int_{\Sigma_o} H_{_E} d\sigma\right)^2.$$

Therefore, the condition $\mathcal{W} < \eta(g)$ is automatically satisfied if $\omega(g) \le \eta(g)$. By (1.3), this is true if g is $C^{2,\delta}$-close to a round metric.

Remark 1.2. Given an Ω in Theorem 1.2, one knows that $\mathcal{W} < \eta(g)$ always holds if g is $C^{2,\delta}$-close to a round metric for the reason explained in Remark 1.1. Therefore, inequality (1.2) is true for any CMC surface Σ bounding a compact 3-manifold with nonnegative scalar curvature, provided the induced metric on Σ is sufficiently round. This may be compared with the result of Christodoulou and Yau [4] which gives $\mathfrak{m}_{_H}(\Sigma) \ge 0$ under the extrinsic curvature condition.

Remark 1.3. On an asymptotically flat 3-manifold M, there exist foliations by CMC spheres near infinity (cf. [5, 8, 10, 13, 16, 22]). For instance, Nerz [16] obtained the existence and uniqueness of such a foliation without assuming asymptotic symmetry conditions. Let $\{\Sigma_\sigma\}_{\sigma>\sigma_0}$ be a foliation of CMC spheres near infinity of M and suppose ∂M consists of outermost minimal surfaces. Let Ω_σ be the region bounded by Σ_σ and ∂M. Let g_σ be the induced metric on Σ_σ. If M is $C^{2,\delta}_\tau$-asymptotically flat with decay rate $\tau > \frac{1}{2}$, it follows from Nerz's work (cf. [16, Proposition 4.4]) that, upon pulling back to S^2, the rescaled metric $\tilde{g}_\sigma := \sigma^{-2} g_\sigma$ satisfies[1]

$$||\tilde{g}_\sigma - g_*||_{C^{2,\delta}(S^2)} \le C\sigma^{-\tau}$$

for some fixed round metric g_* of area 4π and a constant C independent on σ. Thus, along $\{\Sigma_\sigma\}$, $\mathcal{W} = 1 + O(\sigma^{-\tau})$ while $\eta(g_\sigma) \to \infty$ by (1.3). Hence, Theorem 1.1 is applicable to Ω_σ for large σ. However, our estimate of $\eta(g)$ in (1.3) is not strong enough to imply $\left[\frac{\mathcal{W}}{\eta(\tilde{g}_\sigma)-\mathcal{W}}\right]^{\frac{1}{2}} \sqrt{\frac{|\Sigma_\sigma|}{16\pi}} \to 0$ along $\{\Sigma_\sigma\}$. If this could be shown, then one would recover the Riemannian Penrose inequality by taking the limit of (1.1), since the Hawking mass $\mathfrak{m}_{_H}(\Sigma_\sigma)$ approaches the ADM mass [1] along $\{\Sigma_\sigma\}$.

When $\partial\Omega = \Sigma_o \cup \Sigma_h$, we have another result separate from Theorem 1.1.

Theorem 1.3. *Let Ω be a compact, orientable, Riemannian 3-manifold with boundary $\partial\Omega$. Suppose $\partial\Omega$ is the disjoint union of Σ_o and Σ_h such that*

(a) *Σ_o is a topological 2-sphere with constant mean curvature $H_o > 0$;*
(b) *Σ_h, which may have multiple components, is a minimal surface; and*
(c) *there are no other closed minimal surfaces in Ω.*

[1] We thank Christopher Nerz for explaining this estimate along the CMC foliation.

Suppose Ω has nonnegative scalar curvature and the induced metric g on Σ_o has positive Gauss curvature. There exist constants $0 < \beta_g \leq 1$ and $\alpha_g \geq 0$, determined by (Σ_o, g), such that if

$$\mathcal{W} := \frac{1}{16\pi} \int_{\Sigma_o} H_o^2 d\sigma < \frac{\beta_g}{1+\alpha_g},$$

then

$$\sqrt{\frac{|\Sigma_h|}{16\pi}} \leq \left[\left(\frac{\alpha_g \mathcal{W}}{\beta_g - (1+\alpha_g)\mathcal{W}} \right)^{\frac{1}{2}} + 1 \right] \mathfrak{m}_{_H}(\Sigma_o). \tag{1.4}$$

If g is a round metric, one can take $\beta_g = 1$ and $\alpha_g = 0$. In this case, (1.4) *reduces to* $\sqrt{\frac{|\Sigma_h|}{16\pi}} \leq \mathfrak{m}_{_H}(\Sigma_o)$.

Remark 1.4. Similar to $\eta(g)$, the constants α_g and β_g also measure how far g is different from a round metric. By the proof of Proposition 4.1 in Section 4, one can take $\alpha_g \to 0$ and $\beta_g \to 1$ as g approaches a round metric. As a result, suppose Ω is normalized so that $|\Sigma_o| = 4\pi$ and the mean curvature constant H_o satisfies $H_o < 2$. Then the condition $\mathcal{W} < \frac{\beta_g}{1+\alpha_g}$ is always met if g is sufficiently round.

Now we outline the idea of the proof of Theorems 1.1 – 1.3. When the intrinsic metric g on Σ_o is round, Theorems 1.1 and 1.3 follow from [15], and Theorem 1.2 follows from [14, 20]. Thus, the major case to prove is when g is not a round metric. In this case, our proof is inspired by the work of Mantoulidis-Schoen [12]. Suppose (Σ_o, g) is not isometric to a round sphere. We want to construct a collar extension (N, γ) of Ω, where $N = [0,1] \times \Sigma_o$ and γ is a suitably chosen metric, such that

a) γ has nonnegative scalar curvature;
b) the induced metric from γ on $\Sigma_0 := \{0\} \times \Sigma_o$ agrees with g, and the mean curvature of Σ_0 in (N, γ) equals the mean curvature H_o of Σ_o in Ω; and
c) the induced metric from γ on $\Sigma_1 := \{1\} \times \Sigma_o$ is a round metric, and the Hawking mass of Σ_1 in (N, γ) is suitably controlled by the pair (g, H_o).

We then attach (N, γ) to Ω (see Figure 1) to obtain a manifold $\hat{\Omega}$ whose (outer) boundary Σ_1 is a round sphere with constant mean curvature. Though $\hat{\Omega}$ may not be smooth across Σ_o, conditions a) and b) above ensure that the result in [15], which itself was proved using the Riemannian Penrose inequality [2, 9], can be applied to $\hat{\Omega}$ to obtain

$$\mathfrak{m}_{_H}(\Sigma_1) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}. \tag{1.5}$$

(If $\Sigma_h = \emptyset$, we apply the positive mass theorem [19, 21] instead to have $\mathfrak{m}_{_H}(\Sigma_1) \geq 0$.) This, combined with c), then implies the inequalities in Theorems 1.1 – 1.3.

In the construction of (N, γ), conditions on $\mathcal{W}$ are imposed so that γ has nonnegative scalar curvature and the introduction of $\eta(g)$, α_g, and β_g makes use of results from [12].

Remark 1.5. It is worth mentioning that the method described above indeed reveals information of the boundary component Σ_o in the non-CMC case as well. Without assuming that Σ_o is a CMC surface, Theorems 1.1 – 1.3 remain true if one lets $H_o = \min_{\Sigma_o} H$ in the expressions of $\mathcal{W}$ and $\mathfrak{m}_{_H}(\Sigma_o)$. With such a choice of H_o, the mean curvature of Σ_o in Ω, which is H, dominates the mean curvature of Σ_0

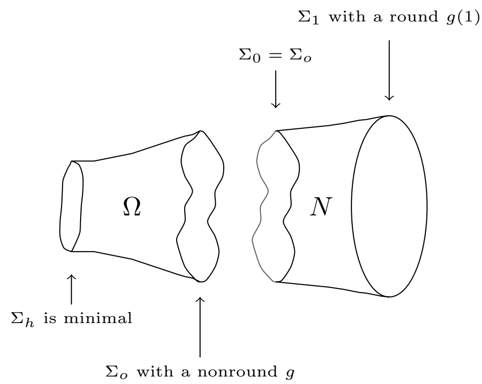

FIGURE 1. A neck N is attached to Ω.

in (N, γ), which is the constant H_o (cf. Figure 1). Therefore, by employing the techniques in [14], one knows that (1.5) (or $\mathfrak{m}_{_H}(\Sigma_1) \geq 0$) still holds on $\hat{\Omega}$.

This paper is organized as follows. In Section 2, we construct a suitable collar extension of Σ_o. In Section 3, we combine the collar extension and the Riemannian Penrose inequality (or the Riemannian positive mass theorem) to draw conclusions on $\partial\Omega$. In Section 4, we give the definition and estimate of $\eta(g)$ and prove Theorems 1.1 – 1.3. A comparison between inequalities (1.1) and (1.4) is included in an appendix.

2. Collar extensions

In this section, we let $\{g(t)\}_{t\in[0,1]}$ be a fixed, smooth path of metrics on $\Sigma = S^2$, satisfying

$$K(g(t)) > 0, \tag{2.1}$$

where $K(\cdot)$ denotes the Gauss curvature of a metric, and

$$\mathrm{tr}_{g(t)} g'(t) = 0 \tag{2.2}$$

for all $t \in [0,1]$, where $\mathrm{tr}_{g(t)}(\cdot)$ is taking trace on $(\Sigma, g(t))$. Let $|\Sigma|_{g(t)}$ be the area of $(\Sigma, g(t))$ which is a constant by (2.2). Let $r_o > 0$ be the corresponding constant given by

$$|\Sigma|_{g(t)} = 4\pi r_o^2. \tag{2.3}$$

We will be interested in a metric γ on $N = [0,1] \times \Sigma$ of the form

$$\gamma = A^2 dt^2 + E(t) g(t),$$

where $A > 0$ is a constant and $E(t) > 0$ is a function. To make a suitable choice of $E(t)$, we consider part of a spatial Schwarzschild metric

$$\gamma_m = \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 g_* \tag{2.4}$$

of mass $m \leq \frac{1}{2} r_o$ defined on $[r_o, \infty) \times S^2$. Here g_* denotes the standard metric on S^2 of area 4π. We emphasize that we do allow m to be negative in (2.4).

Making a change of variable

$$s = \int_{r_o}^{r} \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} dr,$$

we rewrite γ_m as

$$\gamma_m = ds^2 + u_m^2(s) g_*,$$

where $s \in [0, \infty)$ and $u_m(s) = r(s)$, which satisfies

$$u_m(0) = r_o, \qquad u_m'(s) = \left(1 - \frac{2m}{u_m(s)}\right)^{\frac{1}{2}}, \qquad u_m''(s) = \frac{m}{u_m(s)^2}. \tag{2.5}$$

Given any constants $A > 0$ and $k \geq 0$, we define

$$E(t) = r_o^{-2} u_m^2(Akt). \tag{2.6}$$

With such a choice of $E(t)$, the mean curvature $H(t)$ of $\Sigma_t := \{t\} \times \Sigma$ with respect to γ is

$$\begin{aligned} H(t) &= A^{-1} E^{-1} E' \\ &= 2k u_m^{-1} \left(1 - \frac{2m}{u_m}\right)^{\frac{1}{2}} \end{aligned} \tag{2.7}$$

by (2.2) and (2.5). The Hawking mass, $\mathfrak{m}_{H}(\Sigma_t)$, of Σ_t in (N, γ) is

$$\begin{aligned} \mathfrak{m}_{H}(\Sigma_t) &= \sqrt{\frac{|\Sigma_t|_{h(t)}}{16\pi}} \left[1 - \frac{1}{16\pi} \int_{\Sigma_t} H(t)^2 d\sigma_{h(t)}\right] \\ &= \frac{1}{2} u_m(Akt)(1 - k^2) + mk^2, \end{aligned} \tag{2.8}$$

where $h(t) := E(t) g(t)$ and $d\sigma_{h(t)}$ is the area element on $(\Sigma_t, h(t))$.

Next we consider the scalar curvature of γ, denoted by $R(\gamma)$. Direct calculation gives

$$R(\gamma) = 2K(h) + A^{-2} \left[-\mathrm{tr}_h h'' - \frac{1}{4}(\mathrm{tr}_h h')^2 + \frac{3}{4}|h'|_h^2\right],$$

where, by (2.2),

$$\mathrm{tr}_h h' = 2E^{-1}E',$$

$$|h'|_h^2 = E^{-2}\left[2(E')^2 + E^2 |g'|_g^2\right],$$

$$\mathrm{tr}_h h'' = 2E^{-1}E'' + \mathrm{tr}_g g'',$$

and

$$0 = [(\mathrm{tr}_g g')]' = \mathrm{tr}_g g'' - |g'|_g^2.$$

Hence,

$$R(\gamma) = E^{-1} 2K(g) + A^{-2}\left[-\frac{1}{4}|g'|_g^2 - 2E^{-1}E'' + \frac{1}{2}E^{-2}(E')^2\right]. \tag{2.9}$$

Plugging in $E(t) = r_o^{-2} u_m^2(Akt)$ and using (2.5), we have

$$\begin{aligned} & A^{-2}\left[-2E^{-1}E'' + \frac{1}{2}E^{-2}(E')^2\right] \\ =\ & k^2\left[-2u_m^{-2}(u_m')^2 - 4u_m^{-1}u_m''\right] \\ =\ & k^2\left[-2u_m^{-2}\left(1-\frac{2m}{u_m}\right) - 4u_m^{-3}m\right] \\ =\ & -k^2 2u_m^{-2}. \end{aligned} \tag{2.10}$$

Therefore, it follows from (2.9) and (2.10) that

$$\begin{aligned} R(\gamma) &= r_o^2 u_m^{-2} 2K(g) - k^2 2u_m^{-2} - \frac{1}{4}A^{-2}|g'|_g^2 \\ &= 2u_m^{-2}\left[r_o^2 K(g) - k^2 - u_m^2 A^{-2}\frac{1}{8}|g'|_g^2\right]. \end{aligned} \tag{2.11}$$

Now we define two quantities associated to the path $\{g(t)\}_{t\in[0,1]}$:

$$\beta := \min_{t\in[0,1],x\in\Sigma} r_o^2 K(g(t))(x) \tag{2.12}$$

and

$$\alpha := \max_{t\in[0,1],x\in\Sigma} \frac{1}{4}|g'|_g^2(t,x). \tag{2.13}$$

Clearly, $\alpha = 0$ if and only if $\{g(t)\}_{t\in[0,1]}$ is a constant path. Moreover, by the Gauss-Bonnet theorem and (2.3),

$$\int_\Sigma r_o^2 K(g(t))d\sigma_{g(t)} = 4\pi r_o^2 = \int_\Sigma 1 d\sigma_{g(t)},\ \forall\ t. \tag{2.14}$$

Therefore,

$$\beta \le 1, \text{ and } \beta = 1 \iff r_o^2 K(g(t))(x) = 1,\ \forall\ t, x. \tag{2.15}$$

In terms of β and α, it follows from (2.11) that

$$R(\gamma) \ge 2u_m^{-2}\left[\beta - k^2 - \frac{1}{2}u_m^2 A^{-2}\alpha\right]. \tag{2.16}$$

To further estimate $R(\gamma)$, we consider the cases of $m < 0$ and $m \ge 0$ separately.

Case 1 ($m < 0$). In this case, (2.5) and the fact that $u_m(s) \ge r_o$ imply that

$$u_m'(s) \le \left(1 - \frac{2m}{r_o}\right)^{\frac{1}{2}}, \tag{2.17}$$

and therefore

$$u_m(s) \le r_o + \left(1 - \frac{2m}{r_o}\right)^{\frac{1}{2}} s. \tag{2.18}$$

Hence, by (2.16) and (2.18),

$$\begin{aligned} R(\gamma) &\ge 2u_m^{-2}\left\{\beta - k^2 - \frac{1}{2}\left[\left(1-\frac{2m}{r_o}\right)^{\frac{1}{2}} kt + r_o A^{-1}\right]^2 \alpha\right\} \\ &\ge 2u_m^{-2}\left\{\beta - k^2 - \left[\left(1-\frac{2m}{r_o}\right)k^2 + (r_o A^{-1})^2\right]\alpha\right\}. \end{aligned} \tag{2.19}$$

Case 2 ($m \geq 0$). In this case, (2.5) implies that $u_m'(s) \leq 1$ and

$$u_m(s) \leq r_o + s. \tag{2.20}$$

Therefore, by (2.16) and (2.20),

$$\begin{aligned} R(\gamma) &\geq 2u_m^{-2}\left[\beta - k^2 - \frac{1}{2}\left(kt + r_o A^{-1}\right)^2 \alpha\right] \\ &\geq 2u_m^{-2}\left[\beta - k^2 - \left(k^2 + r_o^2 A^{-2}\right)\alpha\right]. \end{aligned} \tag{2.21}$$

We are led to the following proposition.

Proposition 2.1. *Given a smooth path of metrics* $\{g(t)\}_{t\in[0,1]}$ *on* Σ *satisfying* (2.1) *and* (2.2), *let* r_o, β, *and* α *be the constants defined by* (2.3), (2.12), *and* (2.13), *respectively. Suppose* $\alpha > 0$; *i.e.,* $\{g(t)\}_{t\in[0,1]}$ *is not a constant path. Let* $m \leq \frac{1}{2} r_o$ *and* $k \geq 0$ *be two constants satisfying*

$$\beta - \left[1 + \left(1 - \frac{2m}{r_o}\right)\alpha\right] k^2 > 0 \text{ if } m < 0, \tag{2.22}$$

or

$$\beta - (1+\alpha)k^2 > 0, \text{ if } m \geq 0. \tag{2.23}$$

Let $A_o > 0$ *be the constant given by*

$$A_o = r_o \left[\frac{\alpha}{\beta - \left[1 + \left(1 - \frac{2m}{r_o}\right)\alpha\right] k^2}\right]^{\frac{1}{2}} \text{ if } m < 0, \tag{2.24}$$

or

$$A_o = r_o \left[\frac{\alpha}{\beta - (1+\alpha)k^2}\right]^{\frac{1}{2}} \text{ if } m \geq 0. \tag{2.25}$$

Let $u_m(s)$ *be the function defined by* (2.5). *Then, for any constant* $A \geq A_o$, *the metric*

$$\gamma = A^2 dt^2 + r_o^{-2} u_m^2(Akt) g(t) \tag{2.26}$$

on $N = [0,1] \times \Sigma$ *satisfies*

(i) $R(\gamma) \geq 0$, *where* $R(\gamma)$ *is the scalar curvature of* γ;
(ii) *the induced metric on* $\Sigma_0 := \{0\} \times \Sigma$ *is* $g(0)$, *and the mean curvature of* Σ_0 *is* $H(0) = 2kr_o^{-1}(1 - \frac{2m}{r_o})^{\frac{1}{2}}$; *and*
(iii) $\Sigma_t := \{t\} \times \Sigma$ *has positive constant mean curvature for each* t, *and its Hawking mass is*

$$\mathfrak{m}_H(\Sigma_t) = \frac{1}{2}\left[u_m(Akt) - r_o\right](1 - k^2) + \mathfrak{m}_H(\Sigma_0).$$

Proof. (i) is a direct corollary of (2.19) and (2.21). (ii) follows from (2.7) and the fact that $u_m(0) = r_o$. (iii) is implied by (2.7) and (2.8). $\square$

Remark 2.1. In Proposition 2.1, one indeed has $R(\gamma) > 0$ on $[0,1) \times \Sigma$. This is because in both (2.19) and (2.21), the second inequality is a strict inequality unless $t = 1$. Now suppose $g(1)$ is a round metric and $g(0)$ is not round. Then $r_o^2 K(g(1)) = 1$ and $\beta < 1$ by (2.15). Thus, by (2.11), the inequality in (2.16) is strict at $t = 1$. Therefore, in this case, $R(\gamma) > 0$ everywhere on N.

Remark 2.2. When $\alpha = 0$, by (2.16), it suffices to require $\beta \geq k^2$ for γ to have $R(\gamma) \geq 0$. In particular, if $\{g(t)\}_{t\in[0,1]}$ consists of a fixed round metric and $k^2 = \beta = 1$, then γ reduces to the Schwarzschild metric γ_m.

3. Application

In this section, we let Ω be a compact Riemannian 3-manifold with the following properties:

- Ω has nonnegative scalar curvature;
- $\partial\Omega$ is the disjoint union of Σ_o and Σ_h, where Σ_o is a topological 2-sphere and Σ_h, if nonempty, is the unique, closed minimal surface (possibly disconnected) in Ω;
- the mean curvature of Σ_o in Ω is a positive constant H_o; and
- there exists a smooth path of metrics $\{g(t)\}_{t\in[0,1]}$ on $\Sigma := \Sigma_o$ satisfying (2.1) and (2.2) such that $g(0) = g$, which is the induced metric on Σ from Ω, and $g(1)$ is a round metric.

We will apply a suitable collar extension constructed in Proposition 2.1 and the Riemannian Penrose inequality (or the positive mass theorem) to draw information on the geometry of Σ_o.

First, we consider a result obtained by applying Proposition 2.1 with parameters $m < 0$. In this case, we impose a condition

$$\left(\frac{1}{4}H_o^2 r_o^2\right)\alpha < \beta \tag{3.1}$$

on Σ_o, where r_o is the area radius of (Σ_o, g) and β, α are the constants, associated to the path $\{g(t)\}_{t\in[0,1]}$, defined in (2.12), (2.13), respectively.

Theorem 3.1. *If* (3.1) *holds, then*

$$\frac{1}{2}r_o\left[\frac{\frac{1}{4}H_o^2 r_o^2\alpha}{\beta - \frac{1}{4}H_o^2 r_o^2\alpha}\right]^{\frac{1}{2}} + \mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}. \tag{3.2}$$

Proof. If $\alpha = 0$, then g is a round metric. In this case, the claim reduces to $\mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}$, which follows from [15, Theorem 1]. Therefore, it suffices to consider the case in which g is not round, i.e., $\alpha > 0$.

We will construct a suitable metric γ on $N = \Sigma \times [0,1]$ and attach (N, γ) to Ω along Σ_o. To do so, note that (3.1) implies there are constants $m < 0$ satisfying

$$\beta - \frac{1}{4}H_o^2 r_o^2\alpha - \frac{1}{4}H_o^2 r_o^2\left(1 - \frac{2m}{r_o}\right)^{-1} > 0. \tag{3.3}$$

For any such m, define

$$k = \frac{1}{2}H_o r_o\left(1 - \frac{2m}{r_o}\right)^{-\frac{1}{2}}. \tag{3.4}$$

Then (3.3) gives

$$\beta - \left[1 + \left(1 - \frac{2m}{r_o}\right)\alpha\right]k^2 > 0. \tag{3.5}$$

Now let

$$A_o = r_o \left[\frac{\alpha}{\beta - \left[1 + \left(1 - \frac{2m}{r_o}\right)\alpha\right] k^2} \right]^{\frac{1}{2}} \tag{3.6}$$

and consider the metric

$$\gamma = A_o^2 dt^2 + r_o^{-2} u_m^2 (A_o k t) g(t) \tag{3.7}$$

on N. Let $\Sigma_t := \{t\} \times \Sigma$. It follows from (3.5), (3.6), and Proposition 2.1 that (N, γ) has nonnegative scalar curvature, each Σ_t has positive constant mean curvature, the induced metric from γ on Σ_0 agrees with g, the mean curvature $H(0)$ of Σ_0 equals H_o, and the Hawking mass of Σ_1 in (N, γ) and the Hawking mass of Σ_o in Ω are related by

$$\mathfrak{m}_{_H}(\Sigma_1) = \frac{1}{2}\left[u_m(A_o k) - r_o\right](1 - k^2) + \mathfrak{m}_{_H}(\Sigma_0). \tag{3.8}$$

Now we glue (N, γ) and Ω along their common boundary component $\Sigma_0 = \Sigma_o$ to obtain a Riemannian manifold $\hat{\Omega}$. The metric $\hat{g}$ on $\hat{\Omega}$ is Lipschitz across Σ_o and smooth everywhere else, it has nonnegative scalar curvature away from Σ_o, and the mean curvature of Σ_o from both sides in $\hat{\Omega}$ agree. Moreover, $\partial\hat{\Omega} = \Sigma_h \cup \Sigma_1$ where Σ_1 is isometric to a round sphere and has constant mean curvature. Therefore, applying the mollification method used in [14, 15] which smooths out the corner of $\hat{g}$ at Σ_o, we know that [15, Theorem 1] applies to $\hat{\Omega}$ to give

$$\mathfrak{m}_{_H}(\Sigma_1) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}. \tag{3.9}$$

(A more precise and direct way to derive (3.9) is as follows. Since Σ_1 is both round and has constant mean curvature, we can again attach to $\hat{\Omega}$, along Σ_1, a manifold $N_\infty = \left([r_1, \infty) \times S^2, \gamma_m\right)$ with $4\pi r_1^2 = |\Sigma_1|$, γ_m given by (2.4) and $m = \mathfrak{m}_{_H}(\Sigma_1)$. Indeed, N_∞ is the region that is exterior to a rotationally symmetric sphere with area $|\Sigma_1|$ in the spatial Schwarzschild manifold whose mass is $\mathfrak{m}_{_H}(\Sigma_1)$. We denote the resulting manifold by $\hat{M}$, which consists of three pieces: Ω, N, and N_∞. The metric on $\hat{M}$ satisfies the mean curvature matching condition across both Σ_o and Σ_1. Therefore, one can repeat the same proof in [15], starting from Lemma 3 on page 278 and ending at equation (47) on page 280, to conclude that the Riemannian Penrose inequality still holds on such an $\hat{M}$, which proves (3.9).)

To proceed, we note that (3.8) and (3.9) imply that

$$\frac{1}{2}\left[u_m(A_o k) - r_o\right](1 - k^2) + \mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}. \tag{3.10}$$

By (3.5) and (2.15),

$$k^2 < \beta \leq 1, \tag{3.11}$$

and, by (2.18),

$$\begin{aligned} u_m(A_o k) - r_o &\leq \left(1 - \frac{2m}{r_o}\right)^{\frac{1}{2}} A_o k \\ &= \frac{1}{2} H_o r_o A_o. \end{aligned} \tag{3.12}$$

Therefore, (3.10) – (3.12) imply that

$$\frac{1}{4}H_o r_o A_o(1-k^2) + \mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}, \tag{3.13}$$

where

$$\frac{1}{4}H_o r_o A_o = \frac{1}{2} r_o \left[\frac{\frac{1}{4}H_o^2 r_o^2 \alpha}{\left(\beta - \frac{1}{4}H_o^2 r_o^2 \alpha\right) - k^2} \right]^{\frac{1}{2}}. \tag{3.14}$$

In summary, we have proved that

$$\frac{1}{2} r_o \left[\frac{\frac{1}{4}H_o^2 r_o^2 \alpha}{\left(\beta - \frac{1}{4}H_o^2 r_o^2 \alpha\right) - k^2} \right]^{\frac{1}{2}} (1-k^2) + \mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}} \tag{3.15}$$

for any $m < 0$ satisfying (3.3).

To obtain a result that does not involve m or k, we can let $m \to -\infty$ and (3.4) shows that

$$\lim_{m \to -\infty} k = 0. \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$\frac{1}{2} r_o \left[\frac{\frac{1}{4}H_o^2 r_o^2 \alpha}{\beta - \frac{1}{4}H_o^2 r_o^2 \alpha} \right]^{\frac{1}{2}} + \mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}, \tag{3.17}$$

which proves the theorem. □

Remark 3.1. If $\Sigma_h = \emptyset$, i.e., if Ω is merely a compact 3-manifold with nonnegative scalar curvature, with boundary $\partial\Omega = \Sigma_o$, then replacing the Riemannian Penrose inequality by the Riemannian positive mass theorem in the proof, one has $\mathfrak{m}_{_H}(\Sigma_1) \geq 0$ (cf. [14, 20]). In this case, the result becomes

$$\frac{1}{2} r_o \left[\frac{\frac{1}{4}H_o^2 r_o^2 \alpha}{\beta - \frac{1}{4}H_o^2 r_o^2 \alpha} \right]^{\frac{1}{2}} + \mathfrak{m}_{_H}(\Sigma_o) \geq 0. \tag{3.18}$$

Next, we consider a corresponding result obtained by applying Proposition 2.1 with parameters $m \geq 0$. In this case, we assume a condition

$$\frac{1}{4}H_o^2 r_o^2 < \frac{\beta}{1+\alpha}. \tag{3.19}$$

Theorem 3.2. *Suppose* (3.19) *holds. Given any constant* $m \in \left[0, \frac{1}{2}r_o\right)$ *satisfying*

$$\frac{1}{4}H_o^2 r_o^2 < \frac{\beta}{1+\alpha}\left(1 - \frac{2m}{r_o}\right), \tag{3.20}$$

define

$$k = \frac{1}{2}H_o r_o \left(1 - \frac{2m}{r_o}\right)^{-\frac{1}{2}}, \ A_o = r_o \left[\frac{\alpha}{\beta - (1+\alpha)k^2}\right]^{\frac{1}{2}}. \tag{3.21}$$

Then

$$\frac{1}{2}A_o k(1-k^2) + \mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}.$$

In particular, if one chooses $m = 0$, then

$$\left[\frac{\alpha \left(\frac{1}{4} H_o^2 r_o^2 \right)}{\beta - (1+\alpha) \left(\frac{1}{4} H_o^2 r_o^2 \right)} \right]^{\frac{1}{2}} \mathfrak{m}_{_H}(\Sigma_o) + \mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}, \tag{3.22}$$

and consequently

$$\left[\frac{\frac{1}{4} H_o^2 r_o^2}{\frac{\beta}{(1+\alpha)} - \frac{1}{4} H_o^2 r_o^2} \right]^{\frac{1}{2}} \mathfrak{m}_{_H}(\Sigma_o) + \mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}. \tag{3.23}$$

Proof. Again, it suffices to assume $\alpha > 0$. By (3.20) and (3.21),

$$\begin{aligned} & \beta - (1+\alpha) k^2 \\ = {} & \beta - (1+\alpha) \frac{1}{4} H_o^2 r_o^2 \left(1 - \frac{2m}{r_o} \right)^{-1} \\ > {} & 0. \end{aligned} \tag{3.24}$$

Consider the metric

$$\gamma = A_o^2 dt^2 + r_o^{-2} u_m^2 (A_o k t) g(t)$$

on $N = [0,1] \times \Sigma$. Let $\Sigma_t := \{t\} \times \Sigma$. It follows from (3.21), (3.24), and Proposition 2.1 that (N, γ) has nonnegative scalar curvature, the induced metric from γ on Σ_0 agrees with g, the mean curvature $H(0)$ of Σ_0 equals H_o, and the Hawking mass of Σ_1 in (N, γ) and the Hawking mass of Σ_o in Ω are related by

$$\mathfrak{m}_{_H}(\Sigma_1) = \frac{1}{2} \left[u_m(A_o k) - r_o \right] (1 - k^2) + \mathfrak{m}_{_H}(\Sigma_o). \tag{3.25}$$

Attaching (N, γ) to Ω, we have

$$\mathfrak{m}_{_H}(\Sigma_1) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}} \tag{3.26}$$

by the reason explained in the proof of Theorem 3.1. It follows from (3.25) and (3.26) that

$$\frac{1}{2} \left[u_m(A_o k) - r_o \right] (1 - k^2) + \mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}. \tag{3.27}$$

Again, since $\beta \leq 1$, (3.24) implies that $k^2 < 1$. Also, (2.20) shows that

$$u_m(A_o k) - r_o \leq A_o k.$$

Therefore, (3.27) implies that

$$\frac{1}{2} A_o k (1 - k^2) + \mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}, \tag{3.28}$$

where

$$A_o k = r_o \left[\frac{\alpha k^2}{\beta - (1+\alpha) k^2} \right]^{\frac{1}{2}}. \tag{3.29}$$

Thus, we have proved that

$$\frac{1}{2} r_o \left[\frac{\alpha k^2}{\beta - (1+\alpha) k^2} \right]^{\frac{1}{2}} (1 - k^2) + \mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}} \tag{3.30}$$

for any $m \in [0, \frac{1}{2} r_o)$ satisfying (3.20).

To obtain a result that does not involve m or k, we can take $m = 0$. In this case, $k = \frac{1}{2} H_o r_o$ and (3.30) becomes

$$\left[\frac{\alpha \frac{1}{4} H_o^2 r_o^2}{\beta - (1+\alpha) \frac{1}{4} H_o^2 r_o^2} \right]^{\frac{1}{2}} \mathfrak{m}_{_H}(\Sigma_o) + \mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}, \tag{3.31}$$

which proves (3.22). Inequality (3.23) follows from (3.22) simply by the fact that $\frac{\alpha}{1+\alpha} \leq 1$. This completes the proof. □

Remark 3.2. In the derivation of Theorems 3.1 and 3.2, besides taking $m = -\infty$ and $m = 0$, one can minimize the first term in (3.15) and (3.30), subject to the constraint m satisfies (3.3) and (3.20), respectively. We leave this calculation in Appendix A.

Remark 3.3. If g is not a round metric, i.e., $\alpha > 0$, the collar (N, γ) that we attached to Ω indeed has strictly positive scalar curvature by Remark 2.1. Therefore, by the rigidity statement of the Riemannian Penrose inequality, one naturally would expect that inequalities in (3.10) and (3.27) are indeed strict. Therefore, equalities in Theorems 3.1 and 3.2 should hold only if $\alpha = 0$, i.e., when g is a round metric on Σ_o. However, we do not have a rigorous proof of this claim.

4. Definition of $\eta(g)$

In this section, we define the quantity $\eta(g)$ and prove Theorems 1.1 – 1.3. Given a metric g with positive Gauss curvature on $\Sigma = S^2$, let $\{h(t)\}_{t\in[0,1]}$ denote a smooth path of metrics on Σ such that

- (i) $h(0)$ is isometric to g and $h(1)$ is a round metric;
- (ii) $h(t)$ has positive Gauss curvature, i.e., $K(h(t)) > 0$, $\forall\, t$; and
- (iii') $|\Sigma|_{h(t)} = |\Sigma|_g$, i.e., the area of $(\Sigma, h(t))$ is a constant, $\forall\, t$.

There are various ways to construct such a path. For instance, one may apply the uniformization theorem to write $g = e^{2w} g_o$ for some function w and a round metric g_o and to define $h(t) = e^{2(1-t)w} g_o$ (cf. [17]), followed by an area normalization.

Given such a path $\{h(t)\}_{t\in[0,1]}$, applying the proof of Lemma 1.2 in [12] to $\{h(t)\}_{t\in[0,1]}$, one can construct a new path of metrics $\{g(t)\}_{t\in[0,1]}$ satisfying (i) and (ii), with $h(t)$ replaced by $g(t)$, together with the following property that is stronger than (iii'):

- (iii) $\frac{d}{dt} d\sigma_{g(t)} = 0$ or equivalently $\mathrm{tr}_{g(t)} g'(t) = 0$, $\forall\, t$. Here $d\sigma_{g(t)}$ is the area form of $g(t)$.

We include this construction of $\{g(t)\}_{t\in[0,1]}$ by Mantoulidis and Schoen in the lemma below for the purpose of later obtaining estimates on $\eta(g)$.

Lemma 4.1 ([12])**.** *Given* $\{h(t)\}_{t\in[0,1]}$ *satisfying* (i), (ii), *and* (iii') *above, there exists* $\{g(t)\}_{t\in[0,1]}$ *satisfying* (i), (ii), *and* (iii).

Proof. Let $\nabla_{h(t)}$, $\Delta_{h(t)}$ denote the gradient, the Laplacian on $(\Sigma, h(t))$, respectively. Given a 1-parameter family of diffeomorphisms $\{\phi_t\}$ on Σ, define $g(t) := \phi_t^*(h(t))$. Then

$$g'(t) = \phi_t^*(h'(t)) + \phi_t^*\left(L_X h(t)\right), \tag{4.1}$$

$$\mathrm{tr}_{g(t)} g'(t) = \phi_t^*\left(\mathrm{tr}_{h(t)}\left(h'(t) + L_X h(t)\right)\right), \tag{4.2}$$

where $X = X(x,t)$ is the vector field satisfying $\frac{d}{dt}\phi_t = X(\phi_t, t)$ and L denotes the Lie derivative on Σ. Thus, to satisfy (iii), it suffices to demand $\mathrm{tr}_{h(t)} L_X h(t) = -\mathrm{tr}_{h(t)} h'(t)$, i.e.,

$$\mathrm{div}_{h(t)} X = -\frac{1}{2}\mathrm{tr}_{h(t)} h'(t). \tag{4.3}$$

A way to pick such an X is to assume that $X = \nabla_{h(t)} u$ for some function $u = u(x,t)$ satisfying

$$\Delta_{h(t)} u = -\frac{1}{2}\mathrm{tr}_{h(t)} h'(t) \quad \text{and} \quad \int_\Sigma u \, d\sigma_{h(t)} = 0. \tag{4.4}$$

Since

$$\int_\Sigma \mathrm{tr}_{h(t)} h'(t) d\sigma_{h(t)} = 0$$

by (iii'), (4.4) has a unique solution u that depends smoothly on t whenever $h(t)$ is smooth on t. This finishes the proof. □

Given any smooth path $\{g(t)\}_{t\in[0,1]}$ with properties (i), (ii), and (iii), let

$$\beta_{\{g(t)\}} := \min_{t\in[0,1], x\in\Sigma} \frac{1}{4\pi}|\Sigma|_{g(t)} K(g(t))(x)$$

and

$$\alpha_{\{g(t)\}} := \max_{t\in[0,1], x\in\Sigma} \frac{1}{4}|g'|_g^2(t,x),$$

where $|g'|_g^2$ denotes the square norm of $g'(t)$ with respect to $g(t)$.

Definition 4.1. Given a metric g with positive Gauss curvature on $\Sigma = S^2$, define

$$\eta(g) := \sup_{\{g(t)\}} \frac{\beta_{\{g(t)\}}}{\alpha_{\{g(t)\}}},$$

where the supremum is taken over all paths $\{g(t)\}_{t\in[0,1]}$ satisfying (i), (ii), and (iii). Similarly, one may also define

$$\kappa(g) := \sup_{\{g(t)\}} \frac{\beta_{\{g(t)\}}}{1+\alpha_{\{g(t)\}}}.$$

Clearly, $\eta(g)$ and $\kappa(g)$ satisfy

$$0 < \eta(g) \le \infty \quad \text{and} \quad 0 < \kappa(g) \le 1,$$

where the second inequality follows from (2.15). Moreover, for constant $c > 0$, it is straightforward to check that

$$\eta(c^2 g) = \eta(g) \quad \text{and} \quad \kappa(c^2 g) = \kappa(g). \tag{4.5}$$

If $g = g_o$ is a round metric, by taking $\{g(t)\}$ to be a constant path, one has $\alpha_{\{g(t)\}} = 0$ and $\beta_{\{g(t)\}} = 1$; hence

$$\eta(g_o) = \infty \quad \text{and} \quad \kappa(g_o) = 1. \tag{4.6}$$

Below, we give a lower bound of $\eta(g)$ and $\kappa(g)$ for g that is close to a round metric.

Proposition 4.1. *Let g_* be the standard metric of area 4π on $\Sigma = S^2$. There exists a constant $\epsilon_0 > 0$ such that if $||g - g_*||_{C^{2,\delta}(\Sigma)} < \epsilon_0$, then*

$$\eta(g) \geq \frac{C}{||g - g_*||^2_{C^{0,\delta}(\Sigma)}} \quad \text{and} \quad \kappa(g) \geq 1 - C||g - g_*||_{C^{2,\delta}(\Sigma)}. \tag{4.7}$$

Here C is some positive constant that is independent on g, and $||\cdot||_{C^{k,\delta}(\Sigma)}$ is the $C^{k,\delta}$ norm on (Σ, g_*) for an integer $k \geq 0$ and a constant $\delta \in (0,1)$.

Proof. Given any $\epsilon > 0$, let U_ϵ be the set of metrics g satisfying $||g - g_*||_{C^{2,\delta}(\Sigma)} < \epsilon$. First, choose a small ϵ_0 so that elements in U_{ϵ_0} all have positive Gauss curvature.

Given any $g \in U_{\epsilon_0}$, let $\tau = g - g_*$. Then $||\tau||_{C^{2,\delta}(\Sigma)} < \epsilon_0$. For each $t \in [0,1]$, define $\tilde{h}(t)$, $a(t)$, and $h(t)$, respectively, by

$$\tilde{h}(t) = g_* + (1-t)\tau, \qquad |\Sigma|_{\tilde{h}(t)} = a(t)|\Sigma|_g, \qquad h(t) = a^{-1}(t)\tilde{h}(t). \tag{4.8}$$

Then $|\Sigma|_{h(t)} = a^{-1}(t)|\Sigma|_{\tilde{h}(t)} = |\Sigma|_g$. Hence, $\{h(t)\}_{t\in[0,1]}$ is a path satisfying properties (i), (ii), and (iii'). Moreover,

$$||\tilde{h}(t) - g_*||_{C^{2,\delta}(\Sigma)} \leq ||\tau||_{C^{2,\delta}(\Sigma)}, \qquad |a(t) - 1| \leq C_1||\tau||_{C^{2,\delta}(\Sigma)}, \tag{4.9}$$

and

$$\begin{aligned} &||h(t) - g_*||_{C^{2,\delta}(\Sigma)} \\ =\ &||a^{-1}(t)(1-t)\tau + (a^{-1}(t) - 1)g_*||_{C^{2,\delta}(\Sigma)} \\ \leq\ &C_2||\tau||_{C^{2,\delta}(\Sigma)}. \end{aligned} \tag{4.10}$$

Here and below, C_1, $C_2, \ldots$ always denote constants that do not depend on τ and t.

Now let $\{g(t)\}_{t\in[0,1]}$ be the path of metrics constructed from $\{h(t)\}_{t\in[0,1]}$ in the proof of Lemma 4.1. It follows from (4.10) and the fact that $g(t) = \phi_t^*(h(t))$ that

$$\beta_{\{g(t)\}} = \frac{|\Sigma|_g}{4\pi} \min_{t\in[0,1], x\in\Sigma} K(h(t))(x) \geq 1 - C_3||\tau||_{C^{2,\delta}(\Sigma)}. \tag{4.11}$$

We next estimate $\alpha_{\{g(t)\}}$. By (4.1), $g'(t) = \phi_t^*(H(t))$, where

$$H(t) = h'(t) + L_X h(t).$$

Hence, $|g'|^2_g = \phi_t^*(|H|^2_h)$. Therefore,

$$\begin{aligned} \alpha_{\{g(t)\}} =\ & \max_{t\in[0,1], x\in\Sigma} \frac{1}{4}|H|^2_h(t,x) \\ \leq\ & \max_{t\in[0,1], x\in\Sigma} \frac{1}{2}\left[|h'|^2_h + |L_X h(t)|^2_h\right](t,x). \end{aligned} \tag{4.12}$$

Plugging in $X = \nabla_{h(t)} u$, we have

$$L_X h(t) = 2\nabla^2_{h(t)} u, \tag{4.13}$$

where $\nabla^2_{h(t)}$ denotes the Hessian on $(\Sigma, h(t))$. By (4.4), (4.10), and the standard linear elliptic estimates, we have

$$||u||_{C^{2,\delta}(\Sigma)} \leq C_4||\mathrm{tr}_{h(t)} h'(t)||_{C^{0,\delta}(\Sigma)}. \tag{4.14}$$

Therefore, by (4.13) and (4.14),

$$|L_X h(t)|_h \leq C_5||\mathrm{tr}_{h(t)} h'(t)||_{C^{0,\delta}(\Sigma)}. \tag{4.15}$$

It follows from (4.12) and (4.15) that

$$\alpha_{\{g(t)\}} \leq \max_{t\in[0,1],x\in\Sigma} \frac{1}{2}|h'|^2_h(t,x) + \max_{t\in[0,1]} C_6||\mathrm{tr}_{h(t)}h'(t)||^2_{C^{0,\delta}(\Sigma)}. \tag{4.16}$$

By (4.8), we have

$$\mathrm{tr}_{h(t)}h'(t) = -2a^{-1}a' - \mathrm{tr}_{\tilde{h}(t)}\tau, \tag{4.17}$$

$$|h'|^2_h = 2a^{-2}(a')^2 + |\tau|^2_{\tilde{h}} + 2a^{-1}a'\mathrm{tr}_{\tilde{h}(t)}\tau, \tag{4.18}$$

$$a'(t) = -\frac{1}{2|\Sigma|_g}\int_\Sigma \mathrm{tr}_{\tilde{h}(t)}\tau d\sigma_{\tilde{h}(t)}. \tag{4.19}$$

Thus, by (4.9) and (4.17) – (4.19), we have

$$|h'|^2_h \leq C_7||\tau||^2_{C^0(\Sigma)} \quad \text{and} \quad ||\mathrm{tr}_{h(t)}h'(t)||^2_{C^{0,\delta}(\Sigma)} \leq C_8||\tau||^2_{C^{0,\delta}(\Sigma)}. \tag{4.20}$$

Finally, by (4.16) and (4.20), we conclude that

$$\alpha_{\{g(t)\}} \leq C_9||\tau||^2_{C^{0,\delta}(\Sigma)}. \tag{4.21}$$

Estimate (4.7) then follows readily from (4.11) and (4.21). □

We now give the proof of Theorems 1.1 – 1.3.

Proof of Theorems 1.1 *and* 1.2. It suffices to assume that g is not a round metric. Let $\{g^{(j)}(t)\}_{t\in[0,1]}$, $j = 1, 2, \ldots$, be a sequence of paths of metrics satisfying (i), (ii), and (iii), such that

$$\frac{\beta_{\{g^{(j)}(t)\}}}{\alpha_{\{g^{(j)}(t)\}}} \to \eta(g), \text{ as } j \to \infty.$$

Suppose $\mathcal{W} < \eta(g)$. Then

$$\mathcal{W} < \frac{\beta_{\{g^{(j)}(t)\}}}{\alpha_{\{g^{(j)}(t)\}}}, \text{ for large } j.$$

For these j, by Theorem 3.1 and Remark 3.1,

$$\frac{1}{2}r_o\left[\frac{\mathcal{W}}{{\alpha_{\{g^{(j)}(t)\}}}^{-1}\beta_{\{g^{(j)}(t)\}} - \mathcal{W}}\right]^{\frac{1}{2}} + \mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}} \text{ when } \Sigma_h \neq \emptyset,$$

and

$$\frac{1}{2}r_o\left[\frac{\mathcal{W}}{{\alpha_{\{g^{(j)}(t)\}}}^{-1}\beta_{\{g^{(j)}(t)\}} - \mathcal{W}}\right]^{\frac{1}{2}} + \mathfrak{m}_{_H}(\Sigma_o) \geq 0 \text{ when } \partial\Omega = \Sigma_o.$$

Taking $j \to \infty$, Theorems 1.1 and 1.2 follow. □

Proof of Theorem 1.3. Assume that g is not a round metric. Pick any path $\{g(t)\}_{t\in[0,1]}$ used in Section 3 and choose α_g, β_g to be α, β associated to that path, respectively. Theorem 1.3 then follows directly from (3.22) in Theorem 3.2. □

It would be desirable to improve Theorem 1.3 in a way that Theorem 1.1 is proved from Theorem 3.1. However, due to the fact that (3.22) involves both $\frac{\beta}{1+\alpha}$ and $\frac{\alpha}{1+\alpha}$, we can only replace $\frac{\beta}{1+\alpha}$ by $\kappa(g)$ at the expense of giving up $\frac{\alpha}{1+\alpha}$. We record the following theorem.

Theorem 4.1. *Let Ω be a compact, orientable, Riemannian 3-manifold with boundary $\partial\Omega$. Suppose $\partial\Omega$ is the disjoint union of Σ_o and Σ_h such that*

(a) *Σ_o is a topological 2-sphere with constant mean curvature $H_o > 0$;*
(b) *Σ_h, which may have multiple components, is a minimal surface; and*
(c) *there are no other closed minimal surfaces in Ω.*

Suppose Ω has nonnegative scalar curvature and the induced metric g on Σ_o has positive Gauss curvature. Let $0 < \kappa(g) \leq 1$ be the scaling invariant of (Σ_o, g) defined in Definition 4.1. If

$$\mathcal{W} := \frac{1}{16\pi} \int_{\Sigma_o} H_o^2 d\sigma < \kappa(g),$$

then

$$\sqrt{\frac{|\Sigma_h|}{16\pi}} \leq \left[\left(\frac{\mathcal{W}}{\kappa(g) - \mathcal{W}} \right)^{\frac{1}{2}} + 1 \right] \mathfrak{m}_{_H}(\Sigma_o). \tag{4.22}$$

Proof. If g is round, we have $\sqrt{\frac{|\Sigma_h|}{16\pi}} \leq \mathfrak{m}_{_H}(\Sigma_o)$; in particular (4.22) holds. So we assume that g is not a round metric. Similar to the proof of Theorem 1.1 above, let $\{g^{(j)}(t)\}_{t\in[0,1]}$, $j = 1, 2, \ldots$, be a sequence of paths of metrics satisfying (i), (ii), and (iii), with

$$\frac{\beta_{\{g^{(j)}(t)\}}}{1 + \alpha_{\{g^{(j)}(t)\}}} \to \kappa(g) \text{ as } j \to \infty.$$

Suppose $\mathcal{W} < \kappa(g)$. Then

$$\mathcal{W} < \frac{\beta_{\{g^{(j)}(t)\}}}{1 + \alpha_{\{g^{(j)}(t)\}}} \quad \text{for large } j.$$

For these j, by (3.23) in Theorem 3.2,

$$\left[\frac{\mathcal{W}}{\frac{\beta_{\{g^{(j)}(t)\}}}{1+\alpha_{\{g^{(j)}(t)\}}} - \mathcal{W}} \right]^{\frac{1}{2}} \mathfrak{m}_{_H}(\Sigma_o) + \mathfrak{m}_{_H}(\Sigma_o) \geq \sqrt{\frac{|\Sigma_h|}{16\pi}}. \tag{4.23}$$

Taking $j \to \infty$, Theorem 4.1 follows. □

To end this paper, we remark that, besides employing the construction of Mantoulidis and Schoen in Lemma 4.1, there are other methods to obtain $\{g(t)\}_{t\in[0,1]}$ satisfying (i), (ii), and (iii) used in Definition 4.1. For instance, one may apply Hamilton's modified Ricci flow [6] on closed surfaces. Using results from [3,6], Lin and Sormani [11] introduced a concept of asphericity mass for a CMC surface normalized to have area 4π and used it to obtain upper bounds of the surface's Bartnik mass. It would be interesting to understand the relation between $\eta(g)$ or $\kappa(g)$ and the asphericity mass, since they are all determined solely by the intrinsic metric on the surface. It is also conceivably possible that the modified Ricci flow [6] may be used to obtain refined estimates of $\eta(g)$ and $\kappa(g)$. We leave these for the interested reader.

Appendix A

In this appendix, we give the calculation, stated in Remark 3.2, which minimizes the left side of (3.15) and (3.30), subject to the condition m satisfies (3.3) and (3.20), respectively.

We first consider the context of Theorem 3.2. Suppose $\alpha > 0$. Let $\mathcal{W} = \frac{1}{4}H_o^2 r_o^2$ and define

$$\kappa := \frac{\beta}{1+\alpha} \in (0,1). \tag{A.1}$$

Condition (3.19) becomes $\mathcal{W} < \kappa$, and the constraint (3.20) is

$$\mathcal{W} < \kappa\left(1 - \frac{2m}{r_o}\right),\ m \in [0, \frac{1}{2}r_o). \tag{A.2}$$

The quantity that we want to minimize is

$$\begin{aligned}\Phi &:= \frac{1}{2}r_o\left[\frac{\alpha k^2}{\beta - (1+\alpha)\,k^2}\right]^{\frac{1}{2}}(1-k^2)\\ &= \frac{1}{2}r_o\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}}\left[\frac{x}{\kappa - x}\right]^{\frac{1}{2}}(1-x),\end{aligned} \tag{A.3}$$

where $x := k^2 = \mathcal{W}\left(1 - \frac{2m}{r_o}\right)^{-1}$. In terms of x, the constraint (A.2) translates into $\mathcal{W} \le x < \kappa$. The solution to this calculus problem can be derived by considering

$$f(x) := \left(\frac{x}{\kappa - x}\right)(1-x)^2, \tag{A.4}$$

whose derivative is $f'(x) = \frac{(1-x)}{(\kappa-x)^2}\left(2x^2 - 3\kappa x + \kappa\right)$. We therefore have

Theorem 3.2'. *In the setting of Theorem* 3.2, *suppose $\alpha > 0$ and let κ be given by* (A.1). *Then* $\min_{\mathcal{W}\le x<\kappa}\Phi(x) + \mathfrak{m}_{_H}(\Sigma_o) \ge \sqrt{\frac{|\Sigma_h|}{16\pi}}$, *where*

(a) *if $\kappa \le \frac{8}{9}$ or if $\kappa > \frac{8}{9}$ and $x_2 := \frac{3\kappa + \sqrt{9\kappa^2 - 8\kappa}}{4} \le \mathcal{W}$, then $\min_{\mathcal{W}\le x<\kappa}\Phi(x) = \Phi|_{x=\mathcal{W}}$;*

(b) *if $\kappa > \frac{8}{9}$ and $x_1 := \frac{3\kappa - \sqrt{9\kappa^2 - 8\kappa}}{4} \le \mathcal{W} < x_2$, then $\min_{\mathcal{W}\le x<\kappa}\Phi(x) = \Phi|_{x=x_2}$;*

(c) *if $\kappa > \frac{8}{9}$ and $\mathcal{W} < x_1$, then $\min_{\mathcal{W}\le x<\kappa}\Phi(x) = \min\left\{\Phi|_{x=\mathcal{W}},\ \Phi|_{x=x_2}\right\}$. In particular, since $\Phi|_{x=x_2}$ is determined only by α and β, $\min_{\mathcal{W}\le x<\kappa}\Phi(x) = \Phi|_{x=\mathcal{W}}$ for small $\mathcal{W}$.*

Here $x_1, x_2 \in (0,\kappa)$ are the roots to $2x^2 - 3\kappa x + \kappa = 0$, and

$$\Phi|_{x=\mathcal{W}} = \Phi|_{m=0} = \left[\frac{\alpha\frac{1}{4}H_o^2 r_o^2}{\beta - (1+\alpha)\,\frac{1}{4}H_o^2 r_o^2}\right]^{\frac{1}{2}}\mathfrak{m}_{_H}(\Sigma_o).$$

Next we consider the context of Theorem 3.1. Suppose $\alpha > 0$. Define

$$b := \beta - \alpha\mathcal{W} \in (0,1), \tag{A.5}$$

where $\mathcal{W} = \frac{1}{4}H_o^2 r_o^2$. The condition (3.1) becomes $b > 0$, and the constraint (3.3) is

$$b > \mathcal{W}\left(1 - \frac{2m}{r_o}\right)^{-1},\ m < 0. \tag{A.6}$$

The quantity that we want to minimize is

$$\begin{aligned}\Psi &:= \frac{1}{2}r_o\left[\frac{\alpha\mathcal{W}}{(\beta-\alpha\mathcal{W})-k^2}\right]^{\frac{1}{2}}(1-k^2)\\ &= \frac{1}{2}r_o\,(\alpha\mathcal{W})^{\frac{1}{2}}\left[\frac{1}{b-x}\right]^{\frac{1}{2}}(1-x),\end{aligned}\tag{A.7}$$

where $x := k^2 = \mathcal{W}\left(1-\frac{2m}{r_o}\right)^{-1}$. There are two cases to consider when interpreting the constraint. If $b < \mathcal{W}$, (A.6) translates into $0 < x < b$. If $\mathcal{W} \le b$, (A.6) translates into $0 < x < \mathcal{W}$. In either case, the solution to this calculus problem can be derived by considering

$$\tilde{f}(x) := \left(\frac{1}{b-x}\right)(1-x)^2,\tag{A.8}$$

whose derivative is $\tilde{f}'(x) = \frac{(1-x)}{(b-x)^2}\left[x-(2b-1)\right]$. We therefore have

Theorem 3.1'. *In the setting of Theorem* 3.1, *suppose* $\alpha > 0$ *and let* b *be given by* (A.5).

(1) *If* $b < \mathcal{W}$, *then* $\min_{0<x<b}\Psi + \mathfrak{m}_H(\Sigma_o) \ge \sqrt{\frac{|\Sigma_h|}{16\pi}}$, *where*
   (a) *if* $b \le \frac{1}{2}$, $\min_{0<x<b}\Psi = \Psi|_{x=0+}$;
   (b) *if* $b > \frac{1}{2}$, $\min_{0<x<b}\Psi(x) = \Psi|_{x=2b-1}$.

(2) *If* $\mathcal{W} \le b$, *then* $\min_{0<x<\mathcal{W}}\Psi + \mathfrak{m}_H(\Sigma_o) \ge \sqrt{\frac{|\Sigma_h|}{16\pi}}$, *where*
   (a) *if* $b \le \frac{1}{2}$, $\min_{0<x<\mathcal{W}}\Psi = \Psi|_{x=0+}$;
   (b) *if* $\frac{1}{2} < b < \frac{1+\mathcal{W}}{2}$, $\min_{0<x<\mathcal{W}}\Psi(x) = \Psi|_{x=2b-1}$;
   (c) *if* $b \ge \frac{1+\mathcal{W}}{2}$, $\min_{0<x<\mathcal{W}}\Psi(x) = \Psi|_{x=\mathcal{W}-}$.

Here

$$\Psi|_{x=0+} := \lim_{x\to 0+}\Psi = \lim_{m\to-\infty}\Psi = \frac{1}{2}r_o\left[\frac{\frac{1}{4}H_o^2r_o^2\alpha}{\beta-\frac{1}{4}H_o^2r_o^2\alpha}\right]^{\frac{1}{2}},$$

and

$$\Psi|_{x=\mathcal{W}-} := \lim_{x\to\mathcal{W}-}\Psi = \lim_{m\to 0-}\Psi = \left[\frac{\alpha\frac{1}{4}H_o^2r_o^2}{\beta-(1+\alpha)\frac{1}{4}H_o^2r_o^2}\right]^{\frac{1}{2}}\mathfrak{m}_H(\Sigma_o).$$

It follows from Theorem 3.1' and Theorem 3.2'(2) that if $\mathcal{W} < \frac{\beta}{1+\alpha}$, there are cases, depending on $\mathcal{W}$, α, and β, in which the optimal values of Φ and Ψ both occur at $m = 0$ and they agree.

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