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SPECIAL VALUES OF HYPERGEOMETRIC FUNCTIONS AND PERIODS OF CM ELLIPTIC CURVES

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ABSTRACT. Let $X_0^6(1)/W_6$ be the Atkin–Lehner quotient of the Shimura curve $X_0^6(1)$ associated to a maximal order in an indefinite quaternion algebra of discriminant 6 over $\mathbb{Q}$. By realizing modular forms on $X_0^6(1)/W_6$ in two ways, one in terms of hypergeometric functions and the other in terms of Borcherds forms, and using Schofer's formula for values of Borcherds forms at CM-points, we obtain special values of certain hypergeometric functions in terms of periods of elliptic curves over $\overline{\mathbb{Q}}$ with complex multiplication.

1. Introduction

Let $X_0^D(N)$ be the Shimura curve associated to an Eichler order of level N in an indefinite quaternion algebra of discriminant D over $\mathbb{Q}$. When $D = 1$, the Shimura curve $X_0^1(N)$ is just the classical modular curve $X_0(N)$ and there are many different constructions of modular forms on $X_0(N)$ in literature, such as Eisenstein series, Dedekind eta functions, Poincare series, theta series, etc. These explicit constructions provide practical tools for solving problems related to classical modular curves. On the other hand, when $D \neq 1$, because of the lack of cusps, most of the methods for classical modular curves cannot possibly be extended to the case of general Shimura curves. As a result, even some of the most fundamental problems about Shimura curves, such as finding equations of Shimura curves, computing Hecke operators on explicitly given modular forms, etc., are not easy to answer. However, in recent years, there have been two realizations of modular forms on Shimura curves emerging in literature, and some progress toward the study of Shimura curves has already been made using these two methods.

The first method was due to the author of the present paper. In [35], we first observed that when a Shimura curve X has genus 0, all modular forms on X can be expressed in terms of solutions of the Schwarzian differential equation associated to a Hauptmodul of X. Then by utilizing the Jacquet–Langlands correspondence and explicit covers between Shimura curves, we devised a method to compute Hecke operators with respect to the explicitly given basis of modular forms. As applications of this computation of Hecke operators, we computed modular equations for Shimura curves, which can be regarded as equations for Shimura curves associated to Eichler orders of higher levels, in [34] and obtained Ramanujan-type identities for Shimura curves in [36]. In addition, since some Schwarzian differential equations are essentially hypergeometric differential equations, this realization of modular forms

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yields many beautiful identities among hypergeometric functions. This is discussed in [29, 31].

The second method is to realize meromorphic modular forms with divisors supported on CM-points as Borcherds forms associated to the lattice formed by the elements of trace zero in an Eichler order. Borcherds forms themselves are not easy to work with. What makes Borcherds forms useful in practice is Schofer's formula [24] for norms of (generalized) singular moduli of Borcherds forms, that is, norms of values of Borcherds forms at CM-points. Schofer's formula is based on an earlier work of Kudla [19], and the evaluation of derivatives of Fourier coefficients of Eisenstein series uses works of Kudla, Rapoport, and Yang [20, 22, 32]. An immediate consequence of Schofer's formula is a necessary condition for primes that can appear in the prime factorization of the norm of the difference of two singular moduli of different discriminants, which is analogous to Gross and Zagier's work [16] for the case of the classical modular curve $X_0(1)$. Also, Errthum [13] used Schofer's formula to determine singular moduli of $X_0^6(1)/W_6$ and $X_0^{10}(1)/W_{10}$, where W_D denotes the group of all Atkin–Lehner involutions on $X_0^D(1)$, verifying Elkies' numerical computation [12]. (However, we remark that Schofer's formula needs a slight correction when the Borcherds forms have nonzero weights. See Section 4 below.)

The realization of modular forms on Shimura curves in [35] is completely analytic, while Schofer's formula for singular moduli of Borcherds forms is more arithmetic in nature. (For example, the primary motivation of [19–22] was to obtain arithmetic Siegel–Weil formulas realizing generating series from arithmetic geometry as modular forms.) It is an interesting problem to see what results we can obtain by combining the two approaches. This is the main motivation of the present work.

In this paper, we will consider the Shimura curve $X = X_0^6(1)/W_6$. From [35], we know that every holomorphic modular form on X can be expressed in terms of hypergeometric functions. Now according to [26, Theorem 7.1] and [37, Theorem 1.2 and (1.4) of Chapter 3], if $t(\tau)$ is a modular function on X that takes algebraic values at all CM-points, then the value of $t'(\tau)$ at a CM-point of discriminant d is an algebraic multiple of the square of

$$\omega_d = e^{L'(0,\chi_{d_0})/2L(0,\chi_{d_0})} = \frac{1}{\sqrt{|d_0|}} \prod_{a=1}^{|d_0|-1} \Gamma\left(\frac{a}{|d_0|}\right)^{\chi_{d_0}(a)\mu_{d_0}/4h_{d_0}},$$

where d_0 is the discriminant of the field $\mathbb{Q}(\sqrt{d})$, χ_{d_0} is the Kronecker character associated to $\mathbb{Q}(\sqrt{d})$, μ_{d_0} is the number of roots of unity in $\mathbb{Q}(\sqrt{d})$, and h_{d_0} is the class number of $\mathbb{Q}(\sqrt{d})$. (See [2, Theorem 7] for some examples.) The significance of these numbers ω_d is that periods of any elliptic curve over $\overline{\mathbb{Q}}$ with CM by $\mathbb{Q}(\sqrt{d})$ lie in $\sqrt{\pi}\omega_d \cdot \overline{\mathbb{Q}}$. (See [14, 25].) In other words, the values of certain hypergeometric functions at singular moduli can be expressed in terms of periods of CM elliptic curves over $\overline{\mathbb{Q}}$.

Theorem 1. *Let $s(\tau)$ be the Hauptmodul of $X_0^6(1)/W_6$ that takes values 0, 1, and ∞ at the CM-points of discriminants -4, -24, and -3, respectively. Let τ_d be a CM-point of discriminant d such that $|s(\tau_d)| < 1$. Then*

$$ {}_2F_1\left(\frac{1}{24}, \frac{5}{24}; \frac{3}{4}; s(\tau_d)\right) \in \frac{\omega_d}{\omega_{-4}} \cdot \overline{\mathbb{Q}}, \quad {}_3F_2\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}; \frac{3}{4}, \frac{5}{4}; s(\tau_d)\right) \in \omega_d^2 \cdot \overline{\mathbb{Q}}.$$

Likewise, let $t(\tau) = 1/s(\tau)$. If τ_d is a CM-point of discriminant d such that $|t(\tau_d)| < 1$, then

$$ {}_2F_1\left(\frac{1}{24}, \frac{7}{24}; \frac{5}{6}; t(\tau_d)\right) \in \frac{\omega_d}{\omega_{-3}} \cdot \overline{\mathbb{Q}}, \quad {}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; \frac{5}{6}, \frac{7}{6}; t(\tau_d)\right) \in \omega_d^2 \cdot \overline{\mathbb{Q}}. $$

The proof of the theorem will be given at the end of Section 2. Note that the theorem says that if $d = -4r^2$ for some integer r, then ${}_2F_1(1/24, 5/24; 3/4; s(\tau_d)) \in \overline{\mathbb{Q}}$, and if $d = -3r^2$, then ${}_2F_1(1/24, 7/24; 5/6; t(\tau_d)) \in \overline{\mathbb{Q}}$. For instance, in [35], using the Jacquet–Langland correspondence, we find that for $d = -75$,

$$ {}_2F_1\left(\frac{1}{24}, \frac{7}{24}; \frac{5}{6}; -\frac{2^{10} \cdot 3^3 \cdot 5}{11^4}\right) = \sqrt{6}\sqrt[6]{\frac{11}{5^5}}. $$

The parallel results in the cases of classical modular curves can be described as follows. Let λ_1 and λ_2 be a basis for a lattice Λ in $\mathbb{C}$ with $\operatorname{Im}(\lambda_2/\lambda_1) > 0$, and for positive even integers $k \geq 4$, let

$$ G_k(\Lambda) = \sum_{\lambda \in \Lambda, \lambda \neq 0} \frac{1}{\lambda^k}. $$

Then Weierstrass's equation for the elliptic curve $\mathbb{C}/\Lambda$ over $\mathbb{C}$ is

$$ y^2 = 4x^3 - 40G_4(\Lambda)x - 140G_6(\Lambda). $$

From the relations

$$ G_4(\Lambda) = \frac{1}{45}\left(\frac{\pi}{\lambda_1}\right)^4 E_4(\tau), \qquad G_6(\Lambda) = \frac{2}{945}\left(\frac{\pi}{\lambda_1}\right)^6 E_6(\tau), $$

where $\tau = \lambda_2/\lambda_1$ and E_k are the normalized Eisentein series of weight k, we immediately see that for $\tau \in \mathbb{Q}(\sqrt{d}) \cap \mathbb{H}^+$, $\mathbb{H}^+ = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$,

$$ E_k(\tau) \in \left(\frac{\Omega_d}{\pi}\right)^k \cdot \overline{\mathbb{Q}}, $$

where Ω_d is any nonzero period of any elliptic curve over $\overline{\mathbb{Q}}$ with CM by $\mathbb{Q}(\sqrt{d})$. According to the Chowla–Selberg formula [14, 25], we may choose

$$ \Omega_d = \sqrt{\pi} \prod_{a=1}^{|d|-1} \Gamma\left(\frac{a}{|d|}\right)^{\chi_d(a)\mu_d/4h_d} = \sqrt{\pi|d|}\,\omega_d. $$

Now from the classical identity

$$ E_4(\tau) = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728}{j(\tau)}\right)^4, $$

we conclude that if $\tau \in \mathbb{Q}(\sqrt{d}) \cap \mathbb{H}^+$, then

$$ {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728}{j(\tau)}\right) \in \frac{\Omega_d}{\pi} \cdot \overline{\mathbb{Q}}. $$

For instance, for $\tau = i$, we have $j(i) = 1728$, and Gauss's formula for values of hypergeometric functions at 1 and the multiplication formula for the Gamma function yield

$$
{}_2F_1\left(\frac{1}{12},\frac{5}{12};1;1\right)=\frac{\Gamma(1/2)}{\Gamma(11/12)\Gamma(7/12)}=\frac{\sqrt{\pi}\Gamma(3/12)}{\Gamma(11/12)\Gamma(7/12)\Gamma(3/12)}
$$
$$
=\frac{\sqrt{\pi}\Gamma(1/4)}{(2\pi)3^{1/2-3/4}\Gamma(3/4)}=\frac{3^{1/4}}{2}\frac{\Omega_{-4}}{\pi}.
$$

For a fundamental discriminant $d<0$, one may use the Chowla–Selberg formula [25, Page 110]

$$
\prod_{j=1}^{h_d}a_j^{-6}\Delta(\tau_j)=\frac{\omega_d^{12h_d}}{(2\pi)^{6h_d}},
$$

where the product runs through the complete set of reduced primitive quadratic forms $a_jx^2+b_jxy+c_jy^2$ of discriminant d with $\tau_j=(-b_j+\sqrt{d})/2a_j$, along with its generalizations to determine special values of hypergeometric functions. See [1,4,11] for some examples.

Now to determine the precise values of the hypergeometric functions in Theorem 1 at singular moduli, we shall realize the modular forms involved as Borcherds forms. Then evaluating these modular forms at CM-points using Schofer's formula, we obtain formulas for special values of hypergeometric functions. The results in the cases where there exists exactly one CM-point of fundamental discriminant d are given in the next theorem. In Section 6, we will work out an example to illustrate a general technique to determine special values of the hypergeometric functions when there is more than one CM-point of discriminant d.

Theorem 2. *The evaluations*

$$
{}_2F_1\left(\frac{1}{24},\frac{5}{24};\frac{3}{4};\frac{M}{N}\right)=A_1\frac{\omega_d}{\omega_{-4}},
$$
$$
{}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};\frac{3}{4};\frac{5}{4};\frac{M}{N}\right)=A_2\omega_d^2
$$

hold for

d	M	N	A_1	A_2
-120	-7^4	$3^3\cdot 5^3$	$\frac{1}{2}\sqrt[8]{45}\sqrt{12+2\sqrt{30}}$	$\frac{45}{7}$
-52	$2^2\cdot 3^7$	5^6	$\frac{1}{2}\sqrt[4]{5}\sqrt{8+2\sqrt{13}}$	$\frac{25}{6}$
-132	$2^4\cdot 11^2$	5^6	$\frac{1}{2}\sqrt[8]{75}\sqrt{12+2\sqrt{33}}$	$\frac{75}{2\sqrt{22}}$
-43	$-3^7\cdot 7^4$	$2^{10}\cdot 5^6$	$\frac{1}{2}\sqrt[4]{10}\sqrt{7+\sqrt{43}}$	$\frac{100}{21}$
-88	$3^7\cdot 7^4$	$5^6\cdot 11^3$	$\frac{1}{2}\sqrt[8]{275}\sqrt{10+2\sqrt{22}}$	$\frac{275}{21\sqrt{2}}$
-312	$7^4\cdot 23^4$	$5^6\cdot 11^6$	$\frac{1}{2}\sqrt[8]{3}\sqrt[4]{55}\sqrt{18+2\sqrt{78}}$	$\frac{9075}{161\sqrt{2}}$
-148	$2^2\cdot 3^7\cdot 7^4\cdot 11^4$	$5^6\cdot 17^6$	$\frac{1}{2}\sqrt[4]{85}\sqrt{14+2\sqrt{37}}$	$\frac{7225}{231}$
-232	$-3^7\cdot 7^4\cdot 11^4\cdot 19^4$	$5^6\cdot 23^6\cdot 29^3$	$\frac{1}{2}\sqrt[8]{29}\sqrt[4]{115}\sqrt{16+2\sqrt{58}}$	$\frac{383525}{4389}$
-708	$2^8\cdot 7^4\cdot 11^4\cdot 47^4\cdot 59^2$	$5^6\cdot 17^6\cdot 29^6$	$\frac{1}{2}\sqrt[8]{3}\sqrt[4]{2465}\sqrt{30+2\sqrt{177}}$	$\frac{18228675}{3619\sqrt{118}}$
-163	$-3^{11}\cdot 7^4\cdot 19^4\cdot 23^4$	$2^{10}\cdot 5^6\cdot 11^6\cdot 17^6$	$\frac{1}{2}\sqrt[4]{1870}\sqrt{13+\sqrt{163}}$	$\frac{3496900}{27531}$

Also,

$$
{}_2F_1\left(\frac{1}{24},\frac{7}{24};\frac{5}{6};\frac{M}{N}\right)=B_1\frac{\omega_d}{\omega_{-3}},
$$
$$
{}_3F_2\left(\frac{1}{4},\frac{1}{2},\frac{3}{4};\frac{5}{6},\frac{7}{6};\frac{M}{N}\right)=B_2\omega_d^2
$$

hold for

d	M	N	B_1	B_2
-84	3^3	$2^2\cdot 7^2$	$\sqrt[12]{56}\sqrt{3+\sqrt{7}}$	$2\sqrt{\frac{14}{3}}$
-40	-5^3	3^7	$\sqrt[12]{\frac{4}{3}}\sqrt{2\sqrt{3}+\sqrt{10}}$	$\frac{6}{\sqrt{5}}$
-51	2^{10}	7^4	$\frac{1}{2}\sqrt[6]{7}\sqrt{10+2\sqrt{17}}$	$\frac{7}{2}$
-19	-2^{10}	3^7	$\frac{1}{2}\sqrt[12]{\frac{1}{3}}\sqrt{6\sqrt{3}+2\sqrt{19}}$	$\frac{3}{2}$
-168	5^6	$7^2\cdot 11^4$	$\sqrt[12]{7}\sqrt[6]{22}\sqrt{4+\sqrt{14}}$	$\frac{22}{5}\sqrt{7}$
-228	$-3^6\cdot 5^6$	$2^6\cdot 7^4\cdot 19^2$	$\sqrt[12]{38}\sqrt[6]{28}\sqrt{5+\sqrt{19}}$	$\frac{28}{15}\sqrt{114}$
-123	$2^{10}\cdot 5^6$	$7^4\cdot 19^4$	$\frac{1}{2}\sqrt[6]{133}\sqrt{14+2\sqrt{41}}$	$\frac{133}{10}$
-67	$-2^{16}\cdot 5^6$	$3^7\cdot 7^4\cdot 11^4$	$\sqrt[12]{\frac{1}{3}}\sqrt[6]{\frac{77}{8}}\sqrt{5\sqrt{3}+\sqrt{67}}$	$\frac{231}{20}$
-372	$3^3\cdot 5^6\cdot 11^6$	$2^2\cdot 7^4\cdot 19^4\cdot 31^2$	$\sqrt[12]{62}\sqrt[6]{266}\sqrt{7+\sqrt{31}}$	$\frac{266}{55}\sqrt{186}$
-408	$3^6\cdot 5^6\cdot 17^3$	$7^4\cdot 11^4\cdot 31^4$	$\sqrt[6]{4774}\sqrt{6+\sqrt{34}}$	$\frac{4774}{15\sqrt{17}}$
-267	$2^{16}\cdot 5^6\cdot 11^6$	$7^4\cdot 31^4\cdot 43^4$	$\frac{1}{2}\sqrt[6]{9331}\sqrt{22+2\sqrt{89}}$	$\frac{9331}{110}$

Remark 1. Let $F_1(s)={}_2F_1(1/24,5/24;3/4;s)$, $G_1(t)={}_2F_1(1/24,7/24;5/6;t)$, and

$$
F_2(s)={}_2F_1(7/24,11/24;5/4;s)={}_3F_2(1/3,1/2,2/3;3/4;5/4;s)/F_1(s),
$$
$$
G_2(t)={}_2F_1(5/24,11/24;7/6;t)={}_3F_2(1/4,1/2,3/4;5/6,7/6;t)/G_1(t)
$$

be the hypergeometric functions in Theorem 2. The Ramanujan-type identities obtained in [36] can be written as

$$
\left(R_1s\frac{d}{ds}F_1(s)^2+R_2F_1(s)^2\right)\Big|_{s=M/N}=\sqrt{R_3}|M|^{3/4}N^{1/4}C_1,
$$
$$
\left(R_1s\frac{d}{ds}F_2(s)^2+(R_1/2+R_2)F_2(s)\right)\Big|_{s=M/N}=\sqrt{R_3}|M|^{1/4}N^{3/4}C_1^{-1},
$$

and

$$
\left(R_1t\frac{d}{dt}G_1(t)^2+R_2G_1(t)^2\right)\Big|_{t=M/N}=\sqrt{R_3}|M|^{2/3}N^{1/3}C_2,
$$
$$
\left(R_1t\frac{d}{ds}G_2(t)^2+(R_1/3+R_2)G_2(t)\right)\Big|_{t=M/N}=\sqrt{R_3}|M|^{1/3}N^{2/3}C_2^{-1},
$$

for some rational numbers R_1, R_2, R_3 depending on d, where

$$C_1 = \frac{4}{\sqrt[4]{12}}\frac{\pi}{\Omega_{-4}^2} = \frac{4}{\sqrt[4]{12}}\frac{\Gamma(3/4)^2}{\Gamma(1/4)^2}, \qquad C_2 = \frac{3}{\sqrt[6]{2}}\frac{\pi}{\Omega_{-3}^2} = \frac{3}{\sqrt[6]{2}}\frac{\Gamma(2/3)^3}{\Gamma(1/3)^3}.$$

Combining these identities with the formulas in Theorem 2, we obtain special values for the functions

$$\begin{aligned}\frac{d}{ds}F_1(s)^2 &= \frac{d}{ds}{}_3F_2\left(\frac{1}{12},\frac{1}{4},\frac{5}{12};\frac{1}{2},\frac{3}{4};s\right) = \frac{5}{216}{}_3F_2\left(\frac{13}{12},\frac{5}{4},\frac{17}{12};\frac{3}{2},\frac{7}{4};s\right),\\ \frac{d}{ds}F_2(s)^2 &= \frac{d}{ds}{}_3F_2\left(\frac{7}{12},\frac{3}{4},\frac{11}{12};\frac{3}{2},\frac{5}{4};s\right) = \frac{77}{360}{}_3F_2\left(\frac{19}{12},\frac{7}{4},\frac{23}{12};\frac{5}{2},\frac{9}{4};s\right).\end{aligned}$$

For instance, for $d=-120$, we have

$$\begin{aligned}{}_3F_2\left(\frac{13}{12},\frac{5}{4},\frac{17}{12};\frac{3}{2},\frac{7}{4};-\frac{7^4}{15^3}\right) &= \frac{3^6\cdot 5^{9/4}}{2\cdot 7^3\cdot 19\cdot\omega_{-4}^2}\left((4\sqrt{3}+2\sqrt{10})\omega_{-120}^2-\sqrt{\frac{3}{2}}\right),\\ {}_3F_2\left(\frac{19}{12},\frac{7}{4},\frac{23}{12};\frac{5}{2},\frac{9}{4};-\frac{7^4}{15^3}\right) &= \frac{3^7\cdot 5^{23/4}\cdot\omega_{-4}^2}{7^7\cdot 11\cdot 19}\left(242(2\sqrt{3}-\sqrt{10})\omega_{-120}^2-7\sqrt{\frac{3}{2}}\right).\end{aligned}$$

There are similar formulas for the functions

$${}_3F_2\left(\frac{13}{12},\frac{4}{3},\frac{19}{12};\frac{11}{6},\frac{5}{3};t\right), \quad {}_3F_2\left(\frac{17}{12},\frac{5}{3},\frac{23}{12};\frac{13}{6},\frac{7}{3};t\right),$$

such as

$$\begin{aligned}{}_3F_2\left(\frac{13}{12},\frac{4}{3},\frac{19}{12};\frac{11}{6},\frac{5}{3};\frac{27}{196}\right) &= \frac{2^4\cdot 5\cdot 7^{7/6}}{3^2\cdot 13\cdot\omega_{-3}^2}\left(\frac{4}{\sqrt{3}}-\sqrt{2}(3+\sqrt{7})\omega_{-84}^2\right),\\ {}_3F_2\left(\frac{17}{12},\frac{5}{3},\frac{23}{12};\frac{13}{6};\frac{7}{3};\frac{27}{196}\right) &= \frac{2^5\cdot 7^{23/6}\cdot\omega_{-3}^2}{3^4\cdot 5\cdot 11\cdot 13}\left(4\sqrt{3}-55\sqrt{2}(3-\sqrt{7})\omega_{-84}^2\right).\end{aligned}$$

Remark 2. Notice that the numbers A_1 in the first table are all of the form $A^{1/8}(a+\sqrt{|d|})^{1/2}$ for some positive integer a and some rational number A whose denominator is 2 or 4. In other words, the special values ${}_2F_1(1/24,5/24;3/4;M/N)$ possess a certain integrality property. This integrality property is a consequence of Schofer's work [24] and our explicit realization of modular forms as Borcherds forms. On the other hand, if we can somehow manage to prove this integrality property without using Borcherds forms, then to obtain the identities in Theorem 2, we can just evaluate the hypergeometric functions to a high precision and identify the integers. Note that the prime factors of the numerator of A are either 2 or prime factors of N. This suggests that it may be possible to prove the integrality property using the moduli interpretation of the Shimura curve $X_0^6(1)$.

Remark 3. Note that the proof of Theorem 1 is certainly valid for other Shimura curves $X_0^D(N)/W$, W being a subgroup of the Atkin–Lehner groups, or even Shimura curves over totally real fields. However, other than the cases of arithmetic triangle groups, as classified in [28], there are only a very limited number of Shimura curves whose Schwarzian differential equations are known (see [12, 30]).

To obtain analogues of Theorem 2 for $X_0^D(N)/W$, one will need a method to construct Borcherds forms systematically. This is recently addressed in [17], so there is no problem in evaluating modular forms on $X_0^D(N)/W$ at CM-points. However, we remark that this only translates to analogues of the ${}_2F_1$-evaluations. To obtain analogues of the ${}_3F_2$-evaluations, we will need to determine the constant C such that the linear combination $f_1 + Cf_2$ of two solutions f_1 and f_2 of the Schwarzian differential equation is a modular form. In general, this is a difficult problem. (For the case of $X_0^6(1)/W_6$, the constant C is determined by using Gauss's formula ${}_2F_1(a, b; c; 1) = \Gamma(c)\Gamma(c-a-b)/\Gamma(c-a)\Gamma(c-b)$.)

If one wishes to further generalize Theorem 2 to Shimura curves over totally real fields, one will need the theory of Borcherds forms over totally real fields, developed recently by Bruinier and Yang [8, 9]. As far as we can see, it should in principle be possible to obtain explicit evaluations at least for the case of arithmetic triangle groups. We leave this problem for future investigation.

Remark 4. Notice that if a prime p divides M, then the hypergeometric series appearing in Theorem 2 converges p-adically and one may wonder what the limit is. Our computation suggests the following p-adic evaluation.

For a prime p, let $\Gamma_p(x)$ be the p-adic Gamma function defined by

$$\Gamma_p(n) = (-1)^n \prod_{0<j<n, p\nmid j} j$$

for positive integers n and extended continuously to $\mathbb{Z}_p$, and for a fundamental discriminant $d < 0$, set

$$\omega_{d,p} = \prod_{a=1}^{|d|-1} \Gamma_p\left(\frac{a}{|d|}\right)^{\chi_d(a)\mu_d/8h_d}.$$

Consider the two hypergeometric functions in the first set of identities in Theorem 2. Other than the cases $d = -52$ and $d = -132$, the series converge 7-adically. Then the numerical data suggest that

$$\begin{aligned} {}_2F_1\left(\frac{1}{24}, \frac{5}{24}; \frac{3}{4}; \frac{M}{N}\right) &= A_1 \frac{\omega_{d,7}}{\omega_{-4,7}}, \\ {}_3F_2\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}; \frac{3}{4}, \frac{5}{4}; \frac{M}{N}\right) &= A_2 \omega_{d,7}^2 \end{aligned}$$

hold with the same M and N, and

d	A_1	A_2
-120	$\frac{1}{2}\sqrt[8]{-\frac{1}{125}}\sqrt{\sqrt{-3}+\sqrt{-10}}$	$\frac{3}{8\sqrt{2}}$
-43	$\sqrt[4]{-10}\sqrt{\frac{1+\sqrt{43}}{43}}$	$-\frac{100}{129}$
-88	$\frac{1}{2}\sqrt[8]{-\frac{1}{11}}\sqrt[4]{\frac{5}{11}}\sqrt{1+\sqrt{22}}$	$\frac{25}{24\sqrt{2}}$
-312	$\sqrt[8]{-\frac{1}{3}}\sqrt[4]{-\frac{55}{8}}\sqrt{\frac{\sqrt{-6}+\sqrt{-13}}{13}}$	$-\frac{3025}{2392}$
-148	$\sqrt[4]{85}\sqrt{\frac{4+\sqrt{37}}{74}}$	$-\frac{7225}{9768}$
-232	$\sqrt[8]{\frac{1}{29}}\sqrt[4]{-\frac{115}{232}}\sqrt{2\sqrt{2}+\sqrt{29}}$	$\frac{13225}{5016}$
-708	$\sqrt[8]{-\frac{1}{3}}\sqrt[4]{2465}\sqrt{\frac{\sqrt{-3}+\sqrt{-59}}{118}}$	$\frac{6076225}{244024\sqrt{-59}}$
-163	$\sqrt[4]{1870}\sqrt{\frac{11+\sqrt{163}}{163}}$	$-\frac{3496900}{641079}$

(There are many places where we need to take square roots of p-adic numbers. The table above means that after taking suitable choices of square roots, the identities hold conjecturally.)

Note that for a prime p and a fundamental discriminant $d < 0$, the p-adic number $\omega_{d,p}^2$ appears in the matrix representation of the Frobenius automorphism on the de Rham cohomology $H^1_{\mathrm{dR}}(E/\overline{\mathbb{Q}}) \otimes K_{\mathfrak{p}}$ for an elliptic curve E over $\overline{\mathbb{Q}}$ with CM by $\mathbb{Q}(\sqrt{d})$, where $\mathfrak{p}$ is the prime of $\overline{\mathbb{Q}}$ lying over p and $K_{\mathfrak{p}}$ is the algebraic closure of $\mathbb{Q}_p$ in the completion of $\overline{\mathbb{Q}}$ at $\mathfrak{p}$. (See [23, Theorem 3.15].) Note also that if the prime p splits in $\mathbb{Q}(\sqrt{d})$, then $w_{d,p}$ is algebraic over $\mathbb{Q}$ since a suitable power of $\omega_{d,p}$ appears as the value of a certain p-adic Gaussian sum. (See [15, Theorem 1.12].) On the other hand, it is expected that when p is inert in $\mathbb{Q}(\sqrt{d})$, $\omega_{d,p}$ is transcendental over $\mathbb{Q}$. In our conjectural 7-adic formulas mentioned above, since the prime 7 is always inert in $\mathbb{Q}(\sqrt{d})$ (which is a consequence of Theorem 3.6 of [24]), we expect that $\omega_{d,7}$ is transcendental over $\mathbb{Q}$ for d given in the list.

2. Realization of modular forms in terms of Schwarzian differential equations

Here we briefly explain the realization of modular forms on Shimura curves using solutions of Schwarzian differential equations. For details, see [35].

Assume that a Shimura curve X has genus 0 with elliptic points and cusps $\tau_1, \ldots, \tau_r$ of order $e_1, \ldots, e_r$, respectively. (Here we set $e_j = \infty$ if τ_j is a cusp.) Let $t(\tau)$ be a Hauptmodul for X and set $a_j = t(\tau_j)$. Then Theorem 4 of [35] shows that a basis for the space of modular forms of even weight $k \geq 4$ is

$$t'(\tau)^{k/2}t(\tau)^j \prod_{i=1,a_i\neq\infty}^{r} (t(\tau)-a_i)^{-\lfloor k(1-1/e_i)/2\rfloor}, \quad j=0,\dots,d_k-1, \tag{1}$$

where

$$d_k = 1-k+\sum_{j=1}^{r}\left\lfloor \frac{k}{2}\left(1-\frac{1}{e_j}\right)\right\rfloor$$

is the dimension of the space of modular forms of weight k on X.

Now it is easy to check that $t'(\tau)$ is a meromorphic modular form of weight 2 on X. Thus, $t'(\tau)^{1/2}$ and $\tau t'(\tau)^{1/2}$, as functions of t, are solutions of a certain second-order linear differential equation with rational functions in t as coefficients. (See [27, Theorem 5.1] or [33, Theorem 1]. Here the coefficients of the differential equation are rational functions because t is a Hauptmodul.) In fact, this differential equation is

$$\frac{d^2}{dt^2}F + Q(t)F = 0, \tag{2}$$

where

$$Q(t) = -\frac{1}{2}\frac{\{t,\tau\}}{t'(\tau)^2}, \qquad \{t,\tau\} = \frac{t'''(\tau)}{t'(\tau)} - \frac{3}{2}\left(\frac{t''(\tau)}{t'(\tau)}\right)^2.$$

Because $\{t,\tau\}$ is classically known as the Schwarzian derivative, we call the differential equation satisfied by $t'(\tau)^{1/2}$ and $t(\tau)$ the *Schwarzian differential equation* associated to the Shimura curve. If we let $\{f_1, f_2\}$ be a basis for the solution of (2), then we have $t'(\tau) = (c_1f_1 + c_2f_2)^2$ for some complex numbers c_1 and c_2. Substituting this into (1), we obtain the realization of modular forms in terms of solutions of Schwarzian differential equations.

When a Shimura curve is of genus zero and has precisely three elliptic points or cusps, the Schwarzian differential equation is essentially a hypergeometric differential equation. In particular, for the curve $X = X_0^6(1)/W_6$, we can realize modular forms on X in terms of hypergeometric functions as follows.

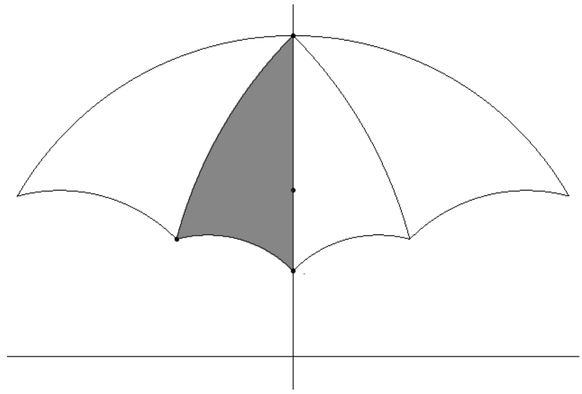
We let $B = \mathbb{Q} + \mathbb{Q}I + \mathbb{Q}J + \mathbb{Q}IJ$ with $I^2 = -1$, $J^2 = 3$, and $IJ = -JI$ be the quaternion algebra of discriminant 6 over $\mathbb{Q}$ and choose the embedding $\iota : B \hookrightarrow M(2,\mathbb{R})$ to be the one defined by

$$\iota(I) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \iota(J) = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}.$$

Fix a maximal order $\mathcal{O} = \mathbb{Z} + \mathbb{Z}I + \mathbb{Z}J + \mathbb{Z}(1+I+J+IJ)/2$ in B and choose representatives of CM-points of discriminants -3, -4, and -24 to be

$$P_{-3} = \frac{-1+i}{1+\sqrt{3}}, \quad P_{-4} = i, \quad P_{-24} = \frac{(\sqrt{6}-\sqrt{2})i}{2}.$$

They are the elliptic points of orders 6, 4, and 2, respectively. A fundamental domain is given by



Here the grey area represents a fundamental domain for $X_0^6(1)/W_6$. The four marked points on the boundary are P_{-4}, P_{-3}, P_{-24}, and $(2-\sqrt{3})i$.

We have the following bases for the spaces of modular forms on $X_0^6(1)/W_6$.

Proposition 5. *Let s be the Hauptmodul on $X = X_0^6(1)/W_6$ determined by $s(P_{-4}) = 0$, $s(P_{-24}) = 1$, and $s(P_{-3}) = \infty$. Then for an even integer $k \ge 4$, a basis for the space $S_k(X)$ of modular forms of weight k on X is*

$$s^{\{3k/8\}}(1-s)^{\{k/4\}}s^j\left({}_2F_1\left(\frac{1}{24},\frac{5}{24};\frac{3}{4};s\right)+\frac{1}{\sqrt[4]{12}\omega_{-4}^2}s^{1/4}{}_2F_1\left(\frac{7}{24},\frac{11}{24};\frac{5}{4};s\right)\right)^k,$$

$j = 0,\dots,d_k-1$, where $d_k = \dim S_k(X) = 1-k+\lfloor k/4\rfloor+\lfloor 3k/8\rfloor+\lfloor 5k/12\rfloor$.

Also, let $t = 1/s$. Then a basis for $S_k(X)$ is

$$t^{\{5k/12\}}(1-t)^{\{k/4\}}t^j\left({}_2F_1\left(\frac{1}{24},\frac{7}{24};\frac{5}{6};t\right)-\frac{e^{-2\pi i/8}}{\sqrt[6]{2}\omega_{-3}^2}t^{1/6}{}_2F_1\left(\frac{5}{24},\frac{11}{24};\frac{7}{6};t\right)\right)^k,$$

$j = 0,\dots,d_k-1$.

Proof. The first part is the content of Lemmas 3 and 4 of [36]. For the second part, the proof of Lemma 14 of [35] shows that

$$(3)\qquad t'(\tau)=\frac{6t^{5/6}(1-t)^{1/2}}{C(P_{-3}-\overline{P}_{-3})}\left({}_2F_1\left(\frac{1}{24},\frac{7}{24};\frac{5}{6};t\right)-Ct^{1/6}{}_2F_1\left(\frac{5}{24},\frac{11}{24};\frac{7}{6};t\right)\right)^2,$$

where

$$C=\frac{P_{-24}-P_{-3}}{P_{-24}-\overline{P}_{-3}}\frac{\Gamma(5/6)\Gamma(17/24)\Gamma(23/24)}{\Gamma(7/6)\Gamma(13/24)\Gamma(19/24)}.$$

Now

$$\frac{P_{-24}-P_{-3}}{P_{-24}-\overline{P}_{-3}}=(1-i)\left(1-\frac{1}{\sqrt{2}}\right)=e^{-2\pi i/8}(\sqrt{2}-1).$$

Also, by Euler's reflection formula and Gauss's multiplication formula, we have

$$\begin{aligned}\left(\frac{\Gamma(17/24)\Gamma(23/24)}{\Gamma(13/24)\Gamma(19/24)}\right)^2 &= \frac{\Gamma(17/24)\Gamma(23/24)\Gamma(5/24)\Gamma(11/24)}{\Gamma(13/24)\Gamma(19/24)\Gamma(1/24)\Gamma(7/24)} \\ &\qquad \times \frac{\sin(5\pi/24)\sin(11\pi/24)}{\sin(\pi/24)\sin(7\pi/24)} \\ &= 4^{-2/3}\frac{\Gamma(5/6)}{\Gamma(1/6)}(3+2\sqrt{2})\end{aligned}$$

and

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right) = (2\pi)^{1/2}2^{-1/6}\Gamma\left(\frac{2}{3}\right).$$

From these, we deduce that

$$\begin{aligned}\frac{\Gamma(5/6)}{\Gamma(7/6)}\frac{\Gamma(17/24)\Gamma(23/24)}{\Gamma(13/24)\Gamma(19/24)} &= 6\cdot 2^{-2/3}(\sqrt{2}+1)\frac{\Gamma(5/6)^{3/2}}{\Gamma(1/6)^{3/2}} \\ &= 6\cdot 2^{-2/3}(\sqrt{2}+1)\frac{1}{(2\pi)^{3/2}}\Gamma\left(\frac{5}{6}\right)^3 \\ &= 6\cdot 2^{-7/6}(\sqrt{2}+1)\frac{\Gamma(2/3)^3}{\Gamma(1/3)^3} = \frac{\sqrt{2}+1}{\sqrt[6]{2}\omega_{-3}^2},\end{aligned}$$

and hence

$$C = \frac{e^{-2\pi i/8}}{\sqrt[6]{2}\omega_{-3}^2}.$$

Then from (1), we conclude that the second set of functions in the lemma forms a basis for $S_k(X)$. □

For general Shimura curves, we can determine Schwarzian differential equations using Propositions 5 and 6 of [35] and explicit covers of Shimura curves. In [30], Tu determines Schwarzian differential equations for the cases when $X_0^D(1)/W_D$ and $X_0^D(N)/W_D$ both have genus zero.

We now give a proof of Theorem 1.

Proof of Theorem 1. Here we only prove the second half of the theorem; the proof of the first half is similar and is omitted.

Since $t(\tau)$ is a Hauptmodul that takes rational values at three distinct CM-points, it takes algebraic values at all CM-points. Thus, by [26, Theorem 7.1] and [37, Theorem 1.2 and (1.4) of Chapter 3], the value of $t'(\tau)$ at a CM-point of discriminant d is an algebraic multiple of $\omega_{d_0}^2$. Then, from (3), we see that

$${}_2F_1\left(\frac{1}{24},\frac{7}{24};\frac{5}{6};t(\tau_d)\right) - \frac{e^{-2\pi i/8}}{\sqrt[6]{2}\omega_{-3}^2}t(\tau_d)^{1/6}{}_2F_1\left(\frac{5}{24},\frac{11}{24};\frac{7}{6};t(\tau_d)\right) \in \frac{\omega_{d_0}}{\omega_{-3}}\cdot\overline{\mathbb{Q}}.$$

Without loss of generality, we may assume that τ_d lies in the fundamental domain depicted earlier. Then equation (22) of [35] implies that

$$\frac{{}_2F_1(5/24,11/24;7/6;t(\tau_d))}{{}_2F_1(1/24,7/24;5/6;t(\tau_d))} \in \omega_{-3}^2\cdot\overline{\mathbb{Q}}.$$

It follows that

$${}_2F_1\left(\frac{1}{24},\frac{7}{24};\frac{5}{6};t(\tau_d)\right) \in \frac{\omega_{d_0}}{\omega_{-3}}\cdot\overline{\mathbb{Q}}$$

and

$$ {}_3F_2\left(\frac14,\frac12,\frac34;\frac56,\frac76;t(\tau_d)\right) = {}_2F_1\left(\frac{1}{24},\frac{7}{24};\frac56;t(\tau_d)\right){}_2F_1\left(\frac{5}{24},\frac{11}{24};\frac76;t(\tau_d)\right) \in \omega_{d_0}^2\cdot\overline{\mathbb{Q}}. $$

This proves the theorem. □

3. Realization of modular forms as Borcherds forms

We first give a quick introduction to Borcherds forms. For details, see [5, 6].

Let L be an even lattice with a symmetric bilinear form $\langle\cdot,\cdot\rangle$ of signature (b^+,b^-), let $L^\vee=\{\gamma\in L\otimes\mathbb{Q}:\langle\gamma,\eta\rangle\in\mathbb{Z}\text{ for all }\eta\in L\}$ be its dual lattice, and let $\{e_\eta:\eta\in L^\vee/L\}$ be the standard basis for the vector space $\mathbb{C}[L^\vee/L]$. Let

$$ \widetilde{\mathrm{SL}}(2,\mathbb{Z})=\left\{\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},\pm\sqrt{c\tau+d}\right):\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z})\right\} $$

be the metaplectic double cover of $\mathrm{SL}(2,\mathbb{Z})$, which is generated by

$$ S=\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix},\sqrt{\tau}\right),\qquad T=\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},1\right). $$

Associated to the lattice L, we have the *Weil representation* $\rho_L:\widetilde{\mathrm{SL}}(2,\mathbb{Z})\to\mathrm{GL}(\mathbb{C}[L^\vee/L])$ defined by

$$ \rho_L(T)e_\eta=e^{-2\pi i\langle\eta,\eta\rangle/2}e_\eta, $$
$$ \rho_L(S)e_\eta=\frac{e^{2\pi i(b^+-b^-)/8}}{\sqrt{|L^\vee/L|}}\sum_{\gamma\in L^\vee/L}e^{2\pi i\langle\eta,\gamma\rangle}e_\gamma. $$

A holomorphic function $F:\mathbb{H}^+\to\mathbb{C}[L^\vee/L]$ is said to be a *weakly holomorphic vector-valued modular form* of weight $k\in\frac12\mathbb{Z}$ and type ρ_L if it satisfies

$$ F\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k\rho_L\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},\sqrt{c\tau+d}\right)F(\tau) $$

for all $\tau\in\mathbb{H}^+$ and all $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}(2,\mathbb{Z})$ and the principal part of its Fourier expansion $F(\tau)=\sum_\eta(\sum_{m\in\mathbb{Q}}c_\eta(m)q^m)e_\eta$, $q=e^{2\pi i\tau}$, has finitely many terms, i.e., the number of pairs (η,m) with $m<0$ and $c_\eta(m)\neq0$ is finite.

For $k=\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, let $V(k)=L\otimes k$ and extend the definition of $\langle\cdot,\cdot\rangle$ to $V(k)$ by linearity. Define the orthogonal groups

$$ O_V(\mathbb{R})=\{\sigma\in\mathrm{GL}(V(\mathbb{R})):\langle\sigma x,\sigma y\rangle=\langle x,y\rangle\text{ for all }x,y\in V(\mathbb{R})\} $$

and

$$ O_V^+(\mathbb{R})=\{\sigma\in O_V(\mathbb{R}):\mathrm{spin}\,\sigma=\mathrm{sgn}\,\det\sigma\}, $$

where if σ is equal to the product of n reflections with respect to the vectors $v_1,\dots,v_n$, then its spinor norm is defined by $\mathrm{spin}\,\sigma=(-1)^n\prod_{i=1}^n\mathrm{sgn}\,\langle v_i,v_i\rangle$. Also let

$$ O_L=\{\sigma\in O_V(\mathbb{R}):\ \sigma(L)=L\},\qquad O_L^+=O_L\cap O_V^+(\mathbb{R}). $$

(Note that the definition of spinor norms is different from that of [5] since the bilinear form in our setting differs from that of [5] by a factor of -1.)

From now on, we assume that the signature of L is $(b,2)$. Let $\mathrm{Gr}(V(\mathbb{R}))$ be the Grassmanian of oriented negative 2-planes in $V(\mathbb{R})$. For an element A in $\mathrm{Gr}(V(\mathbb{R}))$, we can find an oriented basis $\{x,y\}$ for A with $\langle x,x\rangle=\langle y,y\rangle=-1$ and $\langle x,y\rangle=0$.

Let $z = x + iy \in V(\mathbb{C})$. Then we have $\langle z, z\rangle = 0$ and $\langle z, \overline{z}\rangle < 0$. In fact, it is easy to show that $\mathrm{Gr}(V(\mathbb{R}))$ can be identified with the set

$$K = \{z \in V(\mathbb{C}) : \langle z, z\rangle = 0, \langle z, \overline{z}\rangle < 0\}/\mathbb{C}^\times.$$

The set K has two connected components which amount to the two choices of continuously varying orientation of negative 2-planes in $V(\mathbb{R})$. Pick one of them to be K^+. Then the orthogonal group $O_V^+(\mathbb{R})$ acts transitively on K^+. Let

$$\widetilde{K}^+ = \{z \in V(\mathbb{C}) : \langle z, z\rangle = 0, \langle z, \overline{z}\rangle < 0, [z] \in K^+\}.$$

Then for a subgroup Γ of O_L^+, a meromorphic function $\Psi : \widetilde{K}^+ \to \mathbb{P}^1(\mathbb{C})$ is called a modular form of weight k with character χ on Γ if Ψ satisfies

(1) $\Psi(cz) = c^{-k}\Psi(z)$ for all $c \in \mathbb{C}^\times$ and $z \in \widetilde{K}^+$, and
(2) $\Psi(gz) = \chi(g)\Psi(z)$ for all $g \in \Gamma$ and $z \in \widetilde{K}^+$.

Theorem A ([5, Theorem 13.3]). *Let L be an even lattice of signature $(b, 2)$ and let $F(\tau)$ be a weakly holomorphic vector-valued modular form of weight $1-b/2$ and type ρ_L with Fourier expansion $F(\tau) = \sum_{\eta\in L^\vee/L} F_\eta(\tau)e_\eta = \sum_\eta(\sum_{m\in\mathbb{Q}} c_\eta(m)q^m)e_\eta$. Suppose that $c_\eta(m) \in \mathbb{Z}$ whenever $m \le 0$. Then there corresponds a meromorphic function $\Psi_F(z)$, $z \in \widetilde{K}^+$, with the following properties:*

(1) *$\Psi_F(z)$ is a meromorphic modular form of weight $c_0(0)/2$ on the group*

$$O_{L,F}^+ = \{\sigma \in O_L^+ : \ F_{\sigma\eta} = F_\eta \text{ for all } \eta \in L^\vee/L\}$$

with respect to some unitary character χ of $O_{L,F}^+$.

(2) *The only zeros or poles of $\Psi_F(z)$ lie on the rational quadratic divisor $\lambda^\perp = \{z \in \widetilde{K}^+ : \langle z, \lambda\rangle = 0\}$ for $\lambda \in L$, $\langle\lambda, \lambda\rangle > 0$ and are of order*

$$\sum_{0<r\in\mathbb{Q}, r\lambda\in L^\vee} c_{r\lambda}(-r^2\langle\lambda, \lambda\rangle/2).$$

We call the function $\Psi_F(z)$ the *Borcherds form* associated to $F(\tau)$.

We now explain the idea of realizing modular forms on Shimura curves in terms of Borcherds forms. Even though this idea has been used in [13], it seems to us that some key properties were not explained very concretely there. For instance, it was not explained in [13] why the characters associated to the Borcherds forms constructed therein are trivial. Therefore, it is worthwhile to explain this approach in some detail.

Let $\mathcal{O}$ be an Eichler order of level N in an indefinite quaternion algebra B of discriminant D over $\mathbb{Q}$, $(N, D) = 1$, let $\mathcal{O}_1$ be the group of norm-one elements in $\mathcal{O}$, and let

$$L = \{\alpha \in \mathcal{O} : tr(\alpha) = 0\}$$

be the set of elements of trace zero in $\mathcal{O}$, where $tr(\alpha)$ and $n(\alpha)$ denote the trace and the norm of α, respectively. By setting $\langle\alpha, \beta\rangle = tr(\alpha\beta')$, L becomes a lattice of signature $(1, 2)$, where β' denote the quaternionic conjugate of β in B. We now determine O_L and O_L^+.

By the Cartan–Dieudonné theorem, every isometry σ in $O_V(\mathbb{R})$ is equal to the product of at most three reflections. Now it is clear that for an element of nonzero norm α in $V(\mathbb{R})$, the function $\tau_\alpha : \gamma \to -\alpha\gamma\alpha^{-1}$ sends α to $-\alpha$ and leaves any element of $V(\mathbb{R})$ orthogonal to α fixed. (Here we regard $V(\mathbb{R})$ as the set of trace-zero elements in the quaternion algebra $B\otimes\mathbb{R}$ and define multiplication and inverse accordingly.) In other words, τ_α is the reflection with respect to α. Thus, σ has determinant 1, i.e., σ is the product of an even number of reflections if and only if σ is the isometry $\sigma_\beta : \gamma \to \beta\gamma\beta^{-1}$ induced by the conjugation by an element β of nonzero norm in $B \otimes \mathbb{R}$ and σ has determine -1 if and only if $\sigma = -\sigma_\beta$ for some β. From this, we deduce that

$$O_V(\mathbb{R}) = \{\sigma_\beta : \beta \in (B \otimes \mathbb{R})/\mathbb{R}^\times,\ n(\beta) \neq 0\} \times \{\pm 1\}$$

and

$$O_V^+(\mathbb{R}) = \{\sigma_\beta : \beta \in (B \otimes \mathbb{R})/\mathbb{R}^\times,\ n(\beta) > 0\} \times \{\pm 1\}.$$

In addition, by the Noether–Skolem theorem, if σ_β, $\beta \in B \otimes \mathbb{R}$, satisfies $\sigma_\beta(V(\mathbb{Q})) = V(\mathbb{Q})$, then β can be chosen from B. It follows that

$$O_L = \{\sigma_\beta : \beta \in N_B(\mathcal{O})/\mathbb{Q}^\times\} \times \{\pm 1\}$$

and

$$O_L^+ = \{\sigma_\beta : \beta \in N_B^+(\mathcal{O})/\mathbb{Q}^\times\} \times \{\pm 1\},$$

where $N_B(\mathcal{O})$ denotes the normalizer of $\mathcal{O}$ in B and $N_B^+(\mathcal{O})$ is the subgroup of elements of positive norm in $N_B(\mathcal{O})$.

Now assume that the quaternion algebra B is represented by $B = \left(\frac{a,b}{\mathbb{Q}}\right)$ with $a, b > 0$. That is, $B = \mathbb{Q} + \mathbb{Q}I + \mathbb{Q}J + \mathbb{Q}IJ$ with $I^2 = a$, $J^2 = b$, and $IJ = -JI$. Fix an embedding $\iota : B \to M(2, \mathbb{R})$ by

$$\iota : I \to \begin{pmatrix} 0 & \sqrt{a} \\ \sqrt{a} & 0 \end{pmatrix}, \quad J \to \begin{pmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{pmatrix}.$$

We can show that each class in $K = \{z \in V(\mathbb{C}) : \langle z, z\rangle = 0, \langle z, \overline{z}\rangle < 0\}/\mathbb{C}^\times$ contains a unique representative of the form

$$z(\tau) = \frac{1-\tau^2}{2\sqrt{a}} I + \frac{\tau}{\sqrt{b}} J + \frac{1+\tau^2}{2\sqrt{ab}} IJ \tag{4}$$

for some $\tau \in \mathbb{H}^\pm$, the union of the upper and lower half-planes, and the mapping $\tau \to z(\tau) \bmod \mathbb{C}^\times$ is a bijection between $\mathbb{H}^\pm$ and K. Let K^+ be the image of $\mathbb{H}^+$ under this mapping. Now the group $N_B^+(\mathcal{O})/\mathbb{Q}^\times$ acts on $\mathbb{H}^+$ by linear fractional transformation through the embedding ι and also on K^+ by conjugation. By a straightforward computation, we can verify that the actions are compatible. To be more concrete, for $\alpha \in N_B^+(\mathcal{O})$, if we write $\iota(\alpha) = \left(\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right)$, then for all $\tau \in \mathbb{H}^+$, we have

$$\alpha z(\tau)\alpha^{-1} = \frac{(c_3\tau + c_4)^2}{n(\alpha)} z\left(\frac{c_1\tau + c_2}{c_3\tau + c_4}\right). \tag{5}$$

Thus, if $\Psi(z)$ is a meromorphic modular form of weight k on O_L^+ with character χ, then the function $\psi(\tau)$ defined by $\psi(\tau) = \Psi(z(\tau))$ is a meromorphic modular form of weight $2k$ with character on the Shimura curve $N_B^+(\mathcal{O})\backslash\mathbb{H}^+$. Since the group $N_B^+(\mathcal{O})/(\mathbb{Q}^\times\mathcal{O}_1)$ contains the Atkin–Lehner group, we find that $\psi(\tau)$ is a modular form on $X_0^D(N)/W_{D,N}$, the quotient of the Shimura curve $X_0^D(N)$ by the group $W_{D,N}$ of all Atkin–Lehner involutions. In particular, we have the following lemma.

Lemma 6. *Let $F(\tau) = \sum_\eta(\sum_m c_\eta(m)q^m)e_\eta$ be a weakly holomorphic vector-valued modular form of weight $1/2$ and type ρ_L such that $O_{L,F}^+ = O_L^+$ and $c_\eta(m) \in \mathbb{Z}$ whenever $m \le 0$. Then the function $\psi_F(\tau)$ defined by $\psi_F(\tau) = \Psi_F(z(\tau))$ is a* *meromorphic modular form of weight* $c_0(0)$ *with certain unitary character* χ *on the Shimura curve* $X_0^D(N)/W_{D,N}$.

We now determine the divisor of $\psi_F(\tau)$. According to Borcherds's theorem, the divisor of $\Psi_F(z)$ is supported on $\lambda^\perp$ for $\lambda \in L$ with positive norm such that $c_{r\lambda}(-r^2 n(\lambda)) \neq 0$ for some positive rational number r. Now suppose that λ is such an element of L. The condition $\langle \lambda, z \rangle = 0$ implies that $\lambda z \lambda^{-1} = -z = z \bmod \mathbb{C}^\times$. That is, $\lambda^\perp/\mathbb{C}^\times$ consists of the point z_λ in K^+ fixed by the action of λ, and the corresponding point τ_λ in $\mathbb{H}^+$ is a CM point. Let $E = \mathbb{Q}(\sqrt{-n(\lambda)})$ and let $\phi : E \to B$ be the embedding determined by $\phi(\sqrt{-n(\lambda)}) = \lambda$. Then the discriminant of this CM-point is the discriminant of the quadratic order R in E such that $\phi(E) \cap \mathcal{O} = \phi(R)$. Note, however, that if the CM-point τ_λ happens to be an elliptic point of order e, then the projection $K^+ \simeq \mathbb{H}^+ \to X_0^D(N)/W_{D,N}$ is locally e-to-1 at τ_λ. Thus, the order of the modular form $\psi_F(\tau)$ at τ_λ is $1/e$ of that of $\Psi_F(z)$ at z_λ.

In practice, to have a simpler description of the divisor of $\psi_F(\tau)$, we often assume that the weakly holomorphic vector-valued modular form F has the property that the only $\eta \in L^\vee/L$ such that $c_\eta(m) \neq 0$ for some $m < 0$ is 0. In such a case, if we assume that λ is primitive, that is, $\lambda/n \notin \mathcal{O}$ for any positive integer $n \geq 2$, then the discriminant of the CM-point τ_λ is either $-n(\lambda)$ or $-4n(\lambda)$, depending on whether or not $(1+\lambda)/2$ is in $\mathcal{O}$. In summary, the divisor of $\psi_F(\tau)$ can be described as follows.

Lemma 7. *Let* $F(\tau)$, $\Psi_F(z)$, *and* $\psi_F(\tau)$ *be as in the previous lemma. Assume in addition that the only* $\eta \in L^\vee/L$ *such that* $c_\eta(m) \neq 0$ *for some* $m < 0$ *is* 0. *Then we have*

$$\operatorname{div} \psi_F = \sum_{m<0} c_0(m) \sum_{r \in \mathbb{Z}^+, 4m/r^2 \text{ is a discriminant}} \frac{1}{e_{4m/r^2}} \sum_{\tau \in \mathrm{CM}(4m/r^2)} \tau,$$

where for a negative discriminant d, $\mathrm{CM}(d)$ *denotes the set of CM-points of discriminant* d *(which might be empty) and* e_d *is the cardinality of the stabilizer of* $\tau \in \mathrm{CM}(d)$ *in* $N_B^+(\mathcal{O})/\mathbb{Q}^\times$.

We next determine when the character of a Borcherds form $\psi_F(\tau)$ is trivial, under the assumption that the genus of $N_B^+(\mathcal{O})\backslash\mathbb{H}^+$ is zero.

Lemma 8. *Assume that the genus of* $X = N_B^+(\mathcal{O})\backslash\mathbb{H}^+$ *is zero. Let* $\tau_1, \ldots, \tau_r$ *be the elliptic points of* X *and assume that their orders are* $b_1, \ldots, b_r$, *respectively. Assume further that, as CM-points, the discriminants of* $\tau_1, \ldots, \tau_r$ *are* $d_1, \ldots, d_r$, *respectively. Let* $F(\tau) = \sum_\eta (\sum_m c_\eta(m) q^m) e_\eta$ *be a weakly holomorphic vector-valued modular form of weight* $1/2$ *and type* ρ_L *such that* $O^+_{L,F} = O^+_L$ *and* $c_\eta(m) \in \mathbb{Z}$ *whenever* $m \leq 0$. *Assume that* $c_0(0)$ *is even. Then the character associated to the modular form* $\psi_F(\tau)$ *is trivial if and only if for all* j *such that* $b_j \neq 3$, *the order of* $\Psi_F(z)$ *at* $z(\tau_j)$ *has the same parity as* $c_0(0)/2$.

Proof. Let $\gamma_1, \ldots, \gamma_r$ be generators of the stabilizer subgroups of $\tau_1, \ldots, \tau_r$ in the group $\Gamma = N_B^+(\mathcal{O})/\mathbb{Q}^\times$. Since X is assumed to be of genus zero, the group Γ is generated by $\gamma_1, \ldots, \gamma_r$ with a single relation

$$\gamma_1 \ldots \gamma_r = 1, \tag{6}$$

after a suitable reindexing. (See [18, Chapter 4].)

Recall that the order of an elliptic point can only be 2, 3, 4, or 6. Also, an elliptic point of order 3 or 6 is necessarily a CM-point of discriminant -3 and a CM-point of discriminant -3 is an elliptic point of order 3 or 6 depending on whether $3 \nmid DN$ or $3|DN$. In particular, an elliptic point of order 3 and an elliptic point of order 6 cannot exist at the same time. Moreover, on $X_0^D(N)/W_{D,N}$, there can be at most one CM-point of discriminant -3. Likewise, an elliptic point of order 4 is necessarily a CM-point of discriminant -4, and on $X_0^D(N)/W_{D,N}$ there can be at most one such point. Therefore, there are at most two elliptic points whose orders are different from 2.

Consider the case where there is one or zero elliptic point whose order is different from 2 first. By (6), to show that the character χ associated to the modular form $\psi_F(\tau)$ is trivial, it suffices to prove that $\chi(\gamma_j) = 1$ for j with $b_j = 2$.

Observe that for j with $b_j = 2$, γ_j is an element of order 2 in Γ and hence of trace zero and positive norm. Now by the compatibility relation (5), if we write $\iota(\gamma_j) = \left(\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right)$ and set $k = c_0(0)$, then

$$\psi_F\left(\frac{c_1\tau + c_2}{c_3\tau + c_4}\right) = \Psi_F\left(\frac{n(\gamma_j)}{(c_3\tau + c_4)^2}\gamma_j z(\tau)\gamma_j^{-1}\right) = \frac{(c_3\tau + c_4)^k}{n(\gamma_j)^{k/2}}\Psi_F\left(\gamma_j z(\tau)\gamma_j^{-1}\right).$$

Let σ_j be the element of O_L^+ that corresponds to the reflection with respect to γ_j. We have $\sigma_j : z \to -\gamma_j z\gamma_j^{-1}$. Being a reflection, σ_j acts on $\Psi_F(z)$ as $+1$ or -1, depending on whether $\Psi_F(z)$ has an even order or an odd order at the fixed point $z(\tau_j)$ of σ_j. Thus, assuming the order of $\Psi_F(z)$ at $z(\tau_j)$ has the same parity as $k/2 = c_0(0)/2$, we have

$$\begin{aligned}\psi_F\left(\frac{c_1\tau + c_2}{c_3\tau + c_4}\right) &= \frac{(c_3\tau + c_4)^k}{n(\gamma_j)^{k/2}}\Psi_F(-\sigma_j z(\tau)) \\ &= (-1)^{k/2}\frac{(c_3\tau + c_4)^k}{n(\gamma_j)^{k/2}}\Psi_F(\sigma_j z(\tau)) \\ &= \frac{(c_3\tau + c_4)^k}{n(\gamma_j)^{k/2}}\Psi_F(z(\tau)) = \frac{(c_3\tau + c_4)^k}{n(\gamma_j)^{k/2}}\psi_F(\tau).\end{aligned}$$

Therefore, if the order of $\Psi_F(z)$ at $z(\tau_j)$ has the same parity as $k/2 = c_0(0)/2$ for all j with $b_j = 2$, then $\psi_F(\tau)$ is a modular form with trivial character on X.

Now consider the remaining case where there are two elliptic points of order different from 2. By the remark made earlier, the orders of these two elliptic points can only be 3 and 4 or 4 and 6. By the same argument in the previous paragraph, we find that, under the assumption of the lemma, for all j with b_j even, we have $\chi(\gamma_j^{b_j/2}) = 1$. It follows that if $b_j = 4$, then $\chi(\gamma_j)^2 = 1$, and if $b_j = 3$ or $b_j = 6$, then $\chi(\gamma_j)^3 = 1$. Since $\chi(\gamma_1)\cdots\chi(\gamma_r) = 1$, we conclude that $\chi(\gamma_j) = 1$ for all j. This proves the lemma. $\square$

For the case of $X_0^6(1)/W_6$ under consideration, there are three elliptic points of order 2, 4, and 6, respectively. They are CM-points of discriminants -24, -4, and -3, respectively. The proof of the above lemma gives us the following criterion for a Borcherds form $\psi_F(\tau)$ to be a modular form with trivial character on $X_0^6(1)/W_6$.

Corollary 9. *Let $\mathcal{O}$ be a maximal order in the quaternion algebra of discriminant 6 over $\mathbb{Q}$ and let L be the lattice formed by the elements of trace zero in $\mathcal{O}$. Suppose that $F(\tau) = \sum_\eta(\sum_m c_\eta(m)q^m)e_\eta$ is a weakly holomorphic vector-valued modular* *form of weight* $1/2$ *and type* ρ_L *such that* $c_\eta(m) \in \mathbb{Z}$ *whenever* $m \le 0$ *and* $O^+_{L,F} = O^+_L$. *Assume in addition that*

(1) *the only* $\eta \in L^\vee/L$ *such that* $c_\eta(m) \neq 0$ *for some* $m < 0$ *is* 0, *and*
(2) $c_0(0)$ *is even and*

$$\sum_{m=-r^2} c_0(m) \equiv \sum_{m=-3r^2} c_0(m) \equiv c_0(0)/2 \bmod 2.$$

Then the Borcherds form $\psi_F(\tau) = \Psi_F(z(\tau))$ *is a modular form of weight* $c_0(0)$ *and trivial character on the Shimura curve* $X_0^6(1)/W_6$.

Finally, we introduce Errthum's method for constructing $F(\tau)$ satisfying the conditions in the lemma above [13]. Here we consider general Eichler orders in an indefinite quaternion algebra over $\mathbb{Q}$.

The first lemma shows that we can construct $F(\tau)$ out of a scalar-valued modular form with suitable properties. To state the required properties, we let χ_θ denote the character associated to the Jacobi theta function $\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$. That is, χ_θ is the character satisfying

$$\theta(\gamma\tau) = \chi_\theta(\gamma)(c\tau + d)^{1/2}\theta(\tau)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$ and all $\tau \in \mathbb{H}^+$. For a scalar-valued modular form $f(\tau)$ of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_0(M)$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M)$, we let

$$f\big|_\gamma(\tau) = (c\tau + d)^{-k} f(\gamma\tau).$$

We observe that the level M of the lattice L is always a multiple of 4 for any D and N.

Lemma 10 ([3, Theorem 4.2.9]). *Let* M *be the level of the lattice* L. *Suppose that* $f(\tau)$ *is a weakly holomorphic scalar-valued modular form of weight* $1/2$ *such that*

$$f(\gamma\tau) = \chi_\theta(\gamma)(c\tau + d)^{1/2} f(\tau)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M)$. *Then the function* $F_f(\tau)$ *defined by*

$$F_f(\tau) = \sum_{\gamma \in \widetilde{\Gamma}_0(M)\backslash \widetilde{\mathrm{SL}}(2,\mathbb{Z})} f\big|_\gamma(\tau)\rho_L(\gamma^{-1})e_0 \tag{7}$$

is a weakly holomorphic vector-valued modular form of weight $1/2$ *and type* ρ_L.

Lemma 11 ([13, Proposition 5.4]). *Suppose that the weakly holomorphic modular form* $f(\tau)$ *in the above lemma has a pole only at the infinity cusp. Then the Fourier expansion* $F_f(\tau) = \sum_\eta(\sum_m c_\eta(m)q^m)e_\eta$ *satisfies* $c_\eta(m) = 0$ *whenever* $\eta \neq 0$ *and* $m < 0$.

Lemma 12 ([13, Theorem 5.8]). *Let* $f(\tau)$ *and* $F_f(\tau)$ *be given as in the previous lemmas. Then for any* $\eta, \eta' \in L^\vee$ *with* $\langle \eta, \eta\rangle = \langle \eta', \eta'\rangle$, *the* e_η*-component and the* $e_{\eta'}$*-component of* $F_f(\tau)$ *are equal. Consequently, we have* $O^+_{L,F_f} = O^+_L$.

It remains to construct scalar-valued modular forms $f(\tau)$ satisfying the condition in Lemma 10.

Lemma 13 ([6, Theorem 6.2]). *Let M be the level of the lattice L. Suppose that r_d, $d|M$, are integers satisfying the conditions*

(1) $\sum_{d|M} r_d = 1$,
(2) $|L^\vee/L|\prod_{d|M} d^{r_d}$ *is a square in* $\mathbb{Q}^\times$,
(3) $\sum_{d|M} dr_d \equiv 0 \bmod 24$, *and*
(4) $\sum_{d|M}(M/d)r_d \equiv 0 \bmod 24$.

Then $\prod_{d|M}\eta(d\tau)^{r_d}$ *is a weakly holomorphic modular form satisfying the condition for* $f(\tau)$ *in Lemma* 10.

We now consider the case of $X_0^6(1)/W_6$.

Proposition 14. *Consider the case* $X_0^6(1)/W_6$. *Let*

$$f(\tau) = 2\frac{\eta(2\tau)\eta(3\tau)^2\eta(4\tau)^4\eta(6\tau)^4}{\eta(12\tau)^{10}} + 2\frac{\eta(\tau)\eta(2\tau)^3\eta(6\tau)^2}{\eta(3\tau)\eta(4\tau)\eta(12\tau)^3}$$

and

$$g(\tau) = 2\frac{\eta(\tau)\eta(2\tau)^3\eta(6\tau)^2}{\eta(3\tau)\eta(4\tau)\eta(12\tau)^3}.$$

Let $F_f(\tau)$ *and* $F_g(\tau)$ *be defined as in* (7). *Then* $\psi_{F_f}(\tau)$ *and* $\psi_{F_g}(\tau)$ *span the one-dimensional spaces of holomorphic modular form on* $X_0^6(1)/W_6$ *of weight* 8 *and* 12, *respectively.*

Proof. The two eta-products were found in [13, page 848]. Here we give a quick explanation.

In the case of $X_0^6(1)/W_6$, the lattice L has level 12 and $|L^\vee/L| = 72$. The two eta-products clearly satisfy the four conditions in Lemma 13. Now the congruence subgroup $\Gamma_0(12)$ has 6 cusps, represented by $1/c$, $c|12$. The orders of the eta functions $\eta(d\tau)$ at these cusps, multiplied by 24, are given by the following table:

	1/1	1/2	1/4	1/3	1/6	1/12
$\eta(\tau)$	12	3	3	4	1	1
$\eta(2\tau)$	6	6	6	2	2	2
$\eta(4\tau)$	3	3	12	1	1	4
$\eta(3\tau)$	4	1	1	12	3	3
$\eta(6\tau)$	2	2	2	6	6	6
$\eta(12\tau)$	1	1	4	3	3	12

From the table, we see that the two eta-products have only a pole at the cusp $1/12 \sim \infty$. Thus, by Lemma 11, the divisors of $\psi_{F_f}(\tau)$ and $\psi_{F_g}(\tau)$ are determined by the e_0-components of the Fourier expansions of $F_f(\tau)$ and $F_g(\tau)$. Since $f(\tau) = 2q^{-3} - 6 - 18q + \cdots$ and $g(\tau) = 2q^{-1} - 2 - 8q + 8q^2 + \cdots$, the e_0-components of $F_f(\tau)$ and $F_g(\tau)$ are

$$2q^{-3} + c_0 + \cdots, \qquad 2q^{-1} + d_0 + \cdots$$

for some c_0 and d_0, respectively. The numbers c_0 and d_0 are complicated to compute directly from the definition of F_f and F_g. Here we observe that, by Lemma 7,

$$\operatorname{div}\psi_{F_f}(\tau) = \frac{1}{3}P_{-3}, \qquad \operatorname{div}\psi_{F_g}(\tau) = \frac{1}{2}P_{-4},$$

where P_{-3} and P_{-4} denote the unique CM-points of discriminants -3 and -4, respectively. (Note that there does not exist a CM-point of discriminant -12 on $X_0^6(1)/W_6$.) Therefore, the weight of $\psi_{F_f}(\tau)$ must be 8 and the weight of $\psi_{F_g}(\tau)$ must be 12. In other words, we have $c_0 = 8$ and $d_0 = 12$. Then, by Corollary 9, $\psi_{F_f}(\tau)$ and $\psi_{G_f}(\tau)$ must be modular forms on $X_0^6(1)/W_6$ with trivial characters. This proves the proposition. □

Combining Proposition 5 and Proposition 14, we find that

$$\psi_{F_f}(\tau) = C_1 \left({}_2F_1\left(\frac{1}{24}, \frac{5}{24}; \frac{3}{4}; s\right) + \frac{1}{\sqrt[4]{12}\omega_{-4}^2} s^{1/4} {}_2F_1\left(\frac{7}{24}, \frac{11}{24}; \frac{5}{4}; s\right)\right)^8 \tag{8}$$

and

$$\psi_{F_g}(\tau) = C_2 \left({}_2F_1\left(\frac{1}{24}, \frac{7}{24}; \frac{5}{6}; t\right) - \frac{e^{-2\pi i/8}}{\sqrt[6]{2}\omega_{-3}^2} t^{1/6} {}_2F_1\left(\frac{5}{24}, \frac{11}{24}; \frac{7}{6}; t\right)\right)^{12} \tag{9}$$

for some complex numbers C_1 and C_2. To determine the absolute values of these two numbers, we shall use Schofer's formula for values of Borcherds forms at CM-points.

4. Schofer's formula for values of Borcherds forms at CM-points

Let $\mathcal{O}$ be an Eichler order of level N in an indefinite quaternion algebra of discriminant D over $\mathbb{Q}$. Throughout this section, we assume that the level N is squarefree and the symbol d always denotes a negative fundamental discriminant. Let $L = \{\alpha \in \mathcal{O} : tr(\alpha) = 0\}$ be the lattice of signature $(1,2)$ formed by the elements of trace 0 in $\mathcal{O}$. We retain all the notation $\langle \cdot, \cdot \rangle$, $V(\mathbb{Q})$, $V(\mathbb{R})$, $V(\mathbb{C})$, K, $\widetilde{K}^+$, O_L, $O_{L,F}$, etc. used in the previous section. Here let us summarize Schofer's formula [24] for average values of Borcherds forms at CM-points first. The explanation of the terms involved will be given later.

Theorem B ([24, Corollaries 1.2 and 3.5]). *Let $F(\tau) = \sum_\eta (\sum_m c_\eta(m) q^m) e_\eta$ be a weakly holomorphic vector-valued modular form of weight $1/2$ and type ρ_L such that $O_{L,F}^+ = O_L^+$ and $c_\eta(m) \in \mathbb{Z}$ whenever $m \le 0$. Let $\Psi_F(z)$ be the Borcherds form associated $F(\tau)$ and let $\psi_F(\tau) = \Psi_F(z(\tau))$ be the modular form of weight $c_0(0)$ on $X_0^D(N)/W_{D,N}$ as described in Lemma 6, where $z(\tau)$ is given by (4). Let $d < 0$ be a fundamental discriminant such that the set $\mathrm{CM}(d)$ of CM-points of discriminant d is not empty and that the support of $\operatorname{div}\psi(\tau)$ does not intersect $\mathrm{CM}(d)$. Then we have*

$$\sum_{\tau \in \mathrm{CM}(d)} \log \left| \psi_F(\tau) (\operatorname{Im}\tau)^{c_0(0)/2} \right| = -\frac{|\mathrm{CM}(d)|}{4} \left(\sum_{\eta \in L^\vee/L} \sum_{m \ge 0} c_\eta(-m)\kappa_\eta(m) + c_0(0)(\Gamma'(1) + \log(2\pi)) \right).$$

Remark 15.

(1) Note that the formula given in [24] is valid for Borcherds forms associated to lattices of general signature $(n,2)$. Here we have specialized the formulas to the cases under our consideration. Note also that in [24], the left-hand side of the formula has $\Psi_F(z)|y|^{c_0(0)/2}$ in place of $\psi_F(\tau)(\operatorname{Im}\tau)^{c_0(0)/2}$, where

$z = x+iy \in \widetilde{K}^+$ and $|y| = \sqrt{|\langle y, y\rangle|}$. By a direct computation, we find that for $z = z(\tau)$ given in (4), we have $|y| = \operatorname{Im}\tau$. Notice that in general, for any modular form $\psi(\tau)$ of weight k on a Fuchsian subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$, we have $|\psi(\gamma\tau)(\operatorname{Im}\gamma\tau)^{k/2}| = |\psi(\tau)(\operatorname{Im}\tau)^{k/2}|$ for any $\tau \in \mathbb{H}^+$ and $\gamma \in \Gamma$. Thus, the left-hand side of the formula does not depend on the choice of representatives of the CM-points.

(2) Let χ_d be the Kronecker character associated to the field $\mathbb{Q}(\sqrt{d})$ and let

$$\Lambda(s,\chi_d) = \left(\frac{\pi}{|d|}\right)^{-(1+s)/2} \Gamma\left(\frac{1+s}{2}\right) L(s,\chi_d)$$

be the complete L-function associated to χ_d. In [24], the term $\kappa_0(0)$ was given as

$$\kappa_0(0) = 2\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)}$$

under a certain assumption. (Note that our definition of $\Lambda(s,\chi_d)$ is different from that in [24].) Later on, we will prove that for the cases under our consideration, we have

$$\kappa_0(0) = 2\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)} + \sum_{p|D/(D,d)} \frac{p-1}{p+1}\log p + \sum_{p|N/(N,d)} \log p,$$

where the last two summations run over all prime divisors p of $D/(D,d)$ and $N/(N,d)$, respectively.

(3) From the functional equation $\Lambda(s,\chi_d) = \Lambda(1-s,\chi_d)$ for $\Lambda(s,\chi_d)$, we deduce that

$$2\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)} = \log\frac{4\pi}{|d|} - \Gamma'(1) - 2\frac{L'(0,\chi_d)}{L(0,\chi_d)}. \tag{10}$$

By the Chowla–Selberg formula, we have

$$e^{L'(0,\chi_d)/2L(0,\chi_d)} = \frac{1}{\sqrt{|d|}}\prod_{a=1}^{|d|-1}\Gamma\left(\frac{a}{|d|}\right)^{\chi_d(a)\mu_d/4h_d} = \omega_d.$$

This shows that the value of a suitable modular form of weight k on $X_0^D(N)/W_{D,N}$ at a CM-point of discriminant d will be an algebraic multiple of ω_d^k, agreeing with the results of [26] and [37].

We now explain the terms $\kappa_\eta(m)$. Recall that each CM-point τ of discriminant d corresponds to an embedding $\phi : \mathbb{Q}(\sqrt{d}) \to B$ such that $\phi(\mathbb{Q}(\sqrt{d})) \cap \mathcal{O} = \phi(R_d)$, where R_d is the imaginary quadratic order of discriminant d. To be more precise, τ is the common fixed point of $\phi(R_d)$ in the upper half-plane. Let $\lambda = \phi(\sqrt{d})$. Then λ is an element of positive norm in L and the set $U = \lambda^\perp = \{\alpha \in V(\mathbb{Q}) : \langle\lambda,\alpha\rangle = 0\}$ is a negative 2-plane isomorphic to $\mathbb{Q}(\sqrt{d})$ in the sense that there is an isomorphism $h : U \to \mathbb{Q}(\sqrt{d})$ as vector spaces over $\mathbb{Q}$ and a negative rational number c such that $c\langle\alpha,\beta\rangle = tr_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{d})}(h(\alpha)\overline{h(\beta)})$ for all $\alpha,\beta \in U$.

Let $L_+ = L \cap \mathbb{Q}\lambda$ and $L_- = L \cap U$. We have

$$L_+ + L_- \subset L \subset L^\vee \subset L_+^\vee + L_-^\vee.$$

For $\mu \in L_-^\vee/L_-$, let $\varphi_\mu : U \to \mathbb{C}$ be the characteristic function of $\mu + L_-$. Then for each $\mu \in L_-^\vee/L_-$, we have the incoherent Eisenstein series $E(\tau,s;\varphi_\mu,1)$ of weight

1 [20–22, 24]. Write $\tau = u + iv$ and let

$$E(\tau, s; \varphi_\mu, 1) = \sum_m A_\mu(s, m, v) q^m, \quad q = e^{2\pi i \tau},$$

be the Fourier expansion of $E(\tau, s; \varphi_\mu, 1)$. The Eisenstein series $E(\tau, s; \varphi_\mu, 1)$ vanishes at $s = 0$. Thus,

$$A_\mu(s, m, v) = b_\mu(m, v)s + O(s^2)$$

near $s = 0$ for some function $b_\mu(m, v)$. We define

$$\kappa_\mu^-(m) := \begin{cases} \lim_{v\to\infty} b_\mu(m, v) & \text{if } m > 0, \\ \lim_{v\to\infty} (b_0(0, v) - \log v) & \text{if } m = 0 \text{ and } \mu = 0, \\ 0 & \text{else.} \end{cases} \tag{11}$$

Then the term $\kappa_\eta(m)$ in Schofer's formula is defined by

$$\kappa_\eta(m) = \sum_{\mu \in L/(L_+ + L_-)} \sum_{x \in \eta_+ + \mu_+ + L_+} \kappa^-_{\eta_- + \mu_-}(m - \langle x, x\rangle/2), \tag{12}$$

where for $\mu \in L/(L_+ + L_-)$ and $\eta \in L^\vee/L$, we write $\mu = \mu_+ + \mu_-$ and $\eta = \eta_+ + \eta_-$ with $\mu_+, \eta_+ \in \mathbb{Q}\lambda$ and $\mu_-, \eta_- \in U$. The terms $\kappa_\eta(m)$ look very complicated, but nonetheless are computable using the fact that $A_\mu(s, m, v)q^m$ can be written as a product of local Whittaker functions [20–22], which can be computed using formulas in [22, 32]. Here we briefly describe a general strategy to compute $A_\mu(s, m, v)$ and $\kappa_\mu^-(m)$ efficiently, following [13, 22].

In general, for $\mu \in L_-^\vee/L_-$, we have $A_\mu(s, m, v) = 0$ unless $\langle \mu, \mu\rangle/2 - m \in \mathbb{Z}$, and when $\langle \mu, \mu\rangle/2 - m \in \mathbb{Z}$ holds, we have

$$A_\mu(s, m, v)q^m = \delta_{\mu,m} v^{s/2} + W_{m,\infty}(\tau, s) \prod_{p<\infty} W_{m,p}(s, \varphi_{\mu,p}),$$

where

$$\delta_{\mu,m} = \begin{cases} 1 & \text{if } \mu = 0 \text{ and } m = 0, \\ 0 & \text{else} \end{cases}$$

and $W_{m,\infty}(\tau, s)$ and $W_{m,p}(s, \varphi_{\mu,p})$ are the local Whittaker factors at ∞ and p, respectively. (See [20, Section 2].) Let Δ be the discriminant of the lattice L_-. When a prime p does not divide Δ and the p-adic valuation $v_p(m)$ is zero, equation (4.4) and Theorems 4.3 and 4.4 of [22] yield

$$W_{m,p}(s, \varphi_{\mu,p}) = \gamma_p(1 - \chi_d(p)p^{-1-s}),$$

where γ_∞ and γ_p are certain explicit constants that do not have any effect on the calculation since $\gamma_\infty \prod_p \gamma_p = 1$. Therefore, assuming $m > 0$, letting

$$S_{m,\mu} = \{p : \ p|\Delta \text{ or } v_p(m) > 0\}, \tag{13}$$

and using the formula for $W_{m,\infty}$ in Proposition 2.3 of [22], we find

$$A_\mu(m, s, v) = -\frac{2\pi}{L(s+1, \chi_d)} \prod_{p \in S_{m,\mu}} \frac{W_{m,p}(s, \varphi_{\mu,p})}{1 - \chi_d(p)p^{-1-s}}.$$

As $A_\mu(m, 0, v) = 0$, there exists at least a prime p' in $S_{m,\mu}$ such that $W_{m,p'}(0, \varphi_{\mu,p'}) = 0$. Taking the derivative of the above expression and evaluating at $s = 0$, we obtain the following lemma.

Lemma 16. *Assume that $m > 0$ and let all the notation be given as in the discussion. We have*

$$\kappa_\mu^-(m) = -\frac{\mu_d\sqrt{|d|}}{h_d}\frac{W'_{m,p'}(0,\varphi_{\mu,p'})}{1-\chi_d(p')/p'}\prod_{p\in S_{m,\mu},p\neq p'}\frac{W_{m,p}(0,\varphi_{\mu,p})}{1-\chi_d(p)/p},$$

where μ_d and h_d denote the number of roots of unity and the class number of $\mathbb{Q}(\sqrt{d})$, respectively.

This is essentially Theorem 6.3 of [13]. We now consider the term $\kappa_0^-(0)$. If the discriminant Δ of L_- is precisely $|d|$, then, again, Theorems 4.3 and 4.4 of [22] show that

$$W_{0,p}(s,\varphi_{0,p}) = \gamma_p\frac{1-\chi_d(p)p^{-1-s}}{1-\chi_d(p)p^{-s}}$$

so that

$$A_0(s,0,v) = v^{s/2} - v^{-s/2}\frac{\Lambda(s,\chi_d)}{\Lambda(s+1,\chi_d)}$$

and

$$b_0(0,v) = \log v + \frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)} - \frac{\Lambda'(0,\chi_d)}{\Lambda(0,\chi_d)} = \log v + 2\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)}.$$

(See [24, Lemma 2.20].)

In general, the discriminant Δ of L_- may not be exactly $|d|$. Let

$$S = \{p:\ p|(\Delta/d)\}.$$

Then we have

$$A_0(s,0,v) = v^{s/2} - v^{-s/2}\frac{\Lambda(s,\chi_d)}{\Lambda(s+1,\chi_d)}\prod_{p\in S}\frac{(1-\chi_d(p)p^{-s})W_{0,p}(s,\varphi_{0,p})}{1-\chi_d(p)p^{-1-s}}. \tag{14}$$

Let $G(s)$ denote the product on the right. Since $A_0(0,0,v)$ is identically 0, we must have $G(0) = 1$. From this, we deduce the following lemma.

Lemma 17. *Let all the notation be given as above. We have*

$$\kappa_0^-(0) = 2\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)} - \frac{d}{ds}\log G(s)\Big|_{s=0}.$$

We now determine $G(s)$ for the cases under our consideration. In the following lemma, for a prime p, we let $L_p = L_-\otimes_{\mathbb{Z}}\mathbb{Z}_p$.

Lemma 18. *Let all the notation be given as in the preceding discussion. Assume that the level N of the Eichler order $\mathcal{O}$ is squarefree and that $d < 0$ is a fundamental discriminant.*

(1) *Let p be an odd prime. Then there exists a basis $\{\ell_1,\ell_2\}$ for L_p and $\epsilon_1,\epsilon_2\in\mathbb{Z}_p$ with $\epsilon_1\epsilon_2 = -d$ such that the Gram matrix $(\langle\ell_i,\ell_j\rangle)$ is equal to*

$$\begin{cases}\begin{pmatrix}\epsilon_1 & 0\\ 0 & \epsilon_2\end{pmatrix} & \text{if } p|(DN,d) \text{ or if } p\nmid DN,\\ p\begin{pmatrix}\epsilon_1 & 0\\ 0 & \epsilon_2\end{pmatrix} & \text{if } p|DN \text{ but } p\nmid d.\end{cases}$$

(2) *Assume that* $d \equiv 0 \bmod 4$. *Then there exists a basis* $\{\ell_1, \ell_2\}$ *for* L_2 *and* $\epsilon_1, \epsilon_2 \in \mathbb{Z}_2$ *with* $\epsilon_1\epsilon_2 = -d/4$ *such that the Gram matrix is*

$$2\begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}.$$

(3) *Assume that* $d \equiv 1 \bmod 8$ *(and* $2 \nmid D$*). Then there exists a basis* $\{\ell_1, \ell_2\}$ *for* L_2 *and* $\epsilon \in \mathbb{Z}_2^\times$ *such that the Gram matrix is*

$$\begin{cases} 2\epsilon\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } 2|N, \\ \epsilon\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } 2 \nmid N. \end{cases}$$

(4) *Assume that* $d \equiv 5 \bmod 8$ *(and* $2 \nmid N$*). Then there exists a basis* $\{\ell_1, \ell_2\}$ *for* L_2 *and* $\epsilon \in \mathbb{Z}_2^\times$ *such that the Gram matrix is*

$$\begin{cases} 2\epsilon\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{if } 2|D, \\ \epsilon\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{if } 2 \nmid D. \end{cases}$$

Proof. Assume that p is an odd prime. Consider the case when p divides DN first. There exists a basis $\{e_1, e_2, e_3\}$ for $L \otimes \mathbb{Z}_p$ such that

$$(\langle e_i, e_j \rangle) = \begin{pmatrix} 2\mu_1 & 0 & 0 \\ 0 & 2\mu_2 p & 0 \\ 0 & 0 & 2\mu_1\mu_2 p \end{pmatrix},$$

where μ_1 and μ_2 are elements in $\mathbb{Z}_p^\times$ with the property that the Hilbert symbol $(-\mu_1, -\mu_2 p)_p$ is 1 or -1 depending on whether $p|N$ or $p|D$.

Assume that $\lambda = c_1e_1 + c_2e_2 + c_3e_3$. If $p|d$, then we have $p|c_1$ and at least one of c_2 and c_3 must be in $\mathbb{Z}_p^\times$. Without loss of generality, we assume that $c_2 \in \mathbb{Z}_p^\times$. Then L_p is spanned by $-c_2\mu_2e_1 + (c_1/p)\mu_1e_2$ and $c_3\mu_1e_2 - c_2e_3$. The Gram matrix of L_p with respect to this basis has determinant $-(2c_2\mu_1\mu_2)^2d$. It follows that there is a basis $\{\ell_1, \ell_2\}$ for L_p such that $(\langle \ell_i, \ell_j \rangle) = \left(\begin{smallmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{smallmatrix}\right)$ with the properties $\epsilon_1, \epsilon_2 \in \mathbb{Z}_p$ and $\epsilon_1\epsilon_2 = -d$.

If $p \nmid d$, then $p \nmid c_1$. We find that L_p is spanned by $-c_2\mu_2pe_2 + c_1\mu_1e_2$ and $-c_3\mu_2p + c_1e_3$. The Gram matrix of L_p with respect to this basis is inside $pM(2, \mathbb{Z}_p)$ and its determinant is $-(2c_1\mu_1\mu_2p)^2d$. It follows that there exists a basis $\{\ell_1, \ell_2\}$ for L_p such that $(\langle \ell_i, \ell_j \rangle) = \left(\begin{smallmatrix} \epsilon_1 p & 0 \\ 0 & \epsilon_2 p \end{smallmatrix}\right)$ with $\epsilon_1, \epsilon_2 \in \mathbb{Z}_p$ and $\epsilon_1\epsilon_2 = -d$. The proof of the case $p \nmid DN$ is similar and is omitted.

Now consider the case $p = 2$. If $2 \nmid DN$, then $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is isomorphic to $M(2, \mathbb{Z}_2)$. Thus, we may assume that $L \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is $\{\alpha \in M(2, \mathbb{Z}_2) : tr(\alpha) = 0\}$ so that $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and $e_3 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ form a basis for $L \otimes_{\mathbb{Z}} \mathbb{Z}_2$. Let c_1, c_2, c_3 be the elements in $\mathbb{Z}_2$ such that

$$c_1e_1 + c_2e_2 + c_3e_3 = \begin{cases} \lambda & \text{if } d \equiv 1 \bmod 4, \\ \lambda/2 & \text{if } d \equiv 0 \bmod 4. \end{cases}$$

When $d \equiv 1 \bmod 4$, the element λ satisfies $(1 + \lambda)/2 \in \mathcal{O} \otimes \mathbb{Z}_2$, which implies that $2 \nmid c_1$ and $2|c_2, c_3$. Therefore, the lattice $L_2 = L_- \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is spanned by

$-(c_2/2)e_1 + c_1e_3$ and $-(c_3/2)e_1 + c_1e_2$. The Gram matrix relative to this basis is

$$\begin{pmatrix} -c_2^2/2 & -c_1^2 - c_2c_3/2 \\ -c_1^2 - c_2c_3/2 & -c_3^2/2 \end{pmatrix}$$

with determinant $-c_1^2(c_1^2 + c_2c_3) \equiv -d \bmod 8$. By Lemma 8.4.1 of [10], there is a basis $\{\ell_1, \ell_2\}$ for L_2 and $\epsilon \in \mathbb{Z}_2^\times$ such that the Gram matrix is

$$\begin{cases} \epsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } d \equiv 1 \bmod 8, \\ \epsilon \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{if } d \equiv 5 \bmod 8. \end{cases}$$

If $d \equiv 0 \bmod 4$, then c_2 and c_3 cannot both be even since $-c_1^2 - c_2c_3 = -d/4 \equiv 1, 2 \bmod 4$. Assume that $2 \nmid c_2$. Then L_2 is spanned by $c_2e_1 - 2c_1e_3$ and $c_2e_2 - c_3e_3$. The Gram matrix with respect to this basis is

$$\begin{pmatrix} -2c_2^2 & 2c_1c_2 \\ 2c_1c_2 & 2c_2c_3 \end{pmatrix}.$$

It follows from Lemma 8.4.1 of [10] that there exists a basis $\{\ell_1, \ell_2\}$ for L_2 such that the Gram matrix is

$$2\begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}$$

with $\epsilon_1, \epsilon_2 \in \mathbb{Z}_2$ and $\epsilon_1\epsilon_2 = -d/4$. This proves the case $2 \nmid DN$.

The proof of the case $2|DN$ is similar. We remark that when $2|N$, we have $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \simeq \left(\begin{smallmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 2\mathbb{Z}_2 & \mathbb{Z}_2 \end{smallmatrix}\right)$ and when $2|D$, we have $B \otimes \mathbb{Q}_2 \simeq \left(\frac{-1,-1}{\mathbb{Q}_2}\right)$, and the maximal order in $\left(\frac{-1,-1}{\mathbb{Q}_2}\right)$ is $\mathbb{Z}_2 + \mathbb{Z}_2 I + \mathbb{Z}_2 J + \mathbb{Z}_2(1 + I + J + IJ)/2$. The rest of proof is similar to that in the other cases and is omitted. □

Corollary 19. *Let all the notation and assumptions be given as before. Let*

$$r = \prod_{p|DN/(DN,d)} p.$$

Then the Gram matrix of L_- is equivalent to $-rM$ for some positive definite integral matrix M of determinant $|d|$. In particular, the discriminant of L_- is $r^2|d|$.

Lemma 20. *Assume that N is squarefree and $d < 0$ is a fundamental discriminant. Let χ_d, $\Lambda(s, \chi_d)$, λ, L^+, and L_- be defined as above. Let $\kappa_\mu^-(m)$ and $\kappa_\eta(m)$ be defined as in* (11) *and* (12)*, respectively. Then we have*

$$\kappa_0^-(0) = \kappa_0(0) = 2\frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)} + \sum_{p|D/(D,d)} \frac{p-1}{p+1} \log p + \sum_{p|N/(N,d)} \log p,$$

where the two sums run over prime divisors of $D/(D, d)$ and $N/(N, d)$, respectively.

Proof. Consider the case when an odd prime p divides $DN/(DN, d)$, i.e., $p|DN$, but $p \nmid d$. By Lemma 18, the Gram matrix of $L_p = L_- \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is equivalent to $p\left(\begin{smallmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{smallmatrix}\right)$ for some $\epsilon_1, \epsilon_2 \in \mathbb{Z}_p$ with $\epsilon_1\epsilon_2 = -d$. We shall apply Theorem 4.3 of [22] with $\mu = 0$ and $m = 0$. Using the notation in Section 4.2 of [22], we have $H_\mu = \{1, 2\}$, $K_0(\mu) = \infty$,

$$L_\mu(k) = \begin{cases} \{1, 2\} & \text{if } k \text{ is even}, \\ \emptyset & \text{if } k \text{ is odd}, \end{cases}$$

$d_\mu(k) = 1$ for all k, $\epsilon_\mu(k) = \chi_d(p)^{k-1}$, $t_\mu(m) = 0$, and $a_\mu(m) = \infty$. Thus, the combination of (4.4) and Theorem 4.3 of [22] yields

$$\frac{W_{0,p}(s, \varphi_{0,p})}{\gamma_p p^{-1}} = 1 + \left(1 - \frac{1}{p}\right) \sum_{k=1}^{\infty} p\chi_d(p)^{k-1} p^{-ks} = 1 + (p-1)\frac{p^{-s}}{1 - \chi_d(p)p^{-s}}.$$

That is,

$$W_{0,p}(s, \varphi_{0,p}) = \gamma_p \cdot \frac{1 - \chi_d(p)p^{-1-s}}{1 - \chi_d(p)p^{-s}} \cdot \frac{1 + (p - 1 - \chi_d(p))p^{-s}}{p - \chi_d(p)p^{-s}}. \tag{15}$$

For the case $2|DN/(DN, d)$, i.e., $2|DN$ and $d \equiv 1 \bmod 4$, we use the results in Section 4.3 of [22]. Consider the case $d \equiv 1 \bmod 8$ and $2|N$ first. By Lemma 18, the Gram matrix of L_2 is equivalent to $2\epsilon\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Following the notation in Section 4.3 of [22], we have $H_\mu = N_\mu = \emptyset$, $M_\mu = \{1\}$, $L_\mu(k) = \emptyset$, $d_\mu(k) = p_\mu(k) = \epsilon_\mu(k) = \delta_\mu(k) = 1$ for all $k \geq 1$, $K_0(\mu) = \infty$, and $t_\mu = \nu = 0$. Thus, Theorem 4.4 and (4.4) of [22] yield

$$W_{0,2}(s, \varphi_{0,p}) = \frac{\gamma_2}{2}\left(1 + 2^{-s} + 2^{-2s} + \cdots\right) = \gamma_2 \cdot \frac{1 - 2^{-1-s}}{1 - 2^{-s}} \cdot \frac{1}{2 - 2^{-s}}. \tag{16}$$

For the case $d \equiv 5 \bmod 8$ and $2|D$, Lemma 18 shows that the Gram matrix is equivalent to $2\epsilon\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ for some $\epsilon \in \mathbb{Z}_2^\times$. In this case, we have $H_\mu = M_\mu = \emptyset$, $N_\mu = \{1\}$, $L_\mu(k) = \emptyset$, $d_\mu(k) = \epsilon_\mu(k) = \delta_\mu(k) = 1$ for $k \geq 1$, $p_\mu(k) = (-1)^{k-1}$, $K_0(\mu) = \infty$, and $t_\mu = \nu = 0$. Then Theorem 4.4 of [22] shows that

$$W_{0,2}(s, \varphi_{0,p}) = \frac{\gamma_2}{2}\left(1 + 2^{-s} - 2^{-2s} + 2^{-3s} - \cdots\right) = \gamma_2 \cdot \frac{1 + 2^{-1-s}}{1 + 2^{-s}} \cdot \frac{1 + 2^{1-s}}{2 + 2^{-s}}. \tag{17}$$

From (14), (15), (16), and (17), we see that

$$A_0(s, 0, v) = v^{s/2} - v^{-s/2}\frac{\Lambda(s, \chi_d)}{\Lambda(s+1, \chi_d)} \prod_{p|D/(D,d)} \frac{1 + p^{1-s}}{p + p^{-s}} \prod_{p|N/(N,d)} \frac{1 + (p-2)p^{-s}}{p - p^{-s}}.$$

By Lemma 17

$$\begin{aligned}
\kappa_0^-(0) &= 2\frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)} - \sum_{p|D/(D,d)} \left(\frac{-p^{1-s}\log p}{1 + p^{1-s}} - \frac{-p^{-s}\log p}{p + p^{-s}}\right)\Big|_{s=0} \\
&\quad - \sum_{p|N/(N,d)} \left(\frac{-(p-2)p^{-s}\log p}{1 + (p-2)p^{-s}} - \frac{p^{-s}\log p}{p - p^{-s}}\right)\Big|_{s=0} \\
&= 2\frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)} + \sum_{p|D/(D,d)} \frac{p-1}{p+1}\log p + \sum_{p|N/(N,d)} \log p,
\end{aligned}$$

and the proof of the lemma is complete. □

Example 21. Let $\psi_{F_f}(\tau)$ and $\psi_{F_g}(\tau)$ be the Borcherds forms given in Proposition 14. In this example, we shall utilize Lemmas 16 and 20 to determine the absolute values of $\psi_{F_f}(\tau)$ at the CM-point of discriminant -4 and that of $\psi_{F_g}(\tau)$ at the CM-point of discriminant -3.

Let $B = \left(\frac{-1,3}{\mathbb{Q}}\right)$, let $\mathcal{O} = \mathbb{Z} + \mathbb{Z}I + \mathbb{Z}J + \mathbb{Z}IJ$, and let the embedding $\iota : B \hookrightarrow M(2, \mathbb{R})$ be chosen as in Section 2. Choose $\lambda = I$. Then $\phi : i \to I$ defines an optimal embedding relative to $(\mathcal{O}, \mathbb{Z}[i])$ and the fixed point τ_d of $\iota(\phi(I))$ in the upper half-plane is a CM-point of discriminant $d = -4$. By Theorem B, Lemma 20, and (10), we have

$$\begin{aligned}\log\left|\psi_{F_f}(\tau_d)(\operatorname{Im}\tau_d)^4\right| &= -\frac{1}{4}\left(2\kappa_0(3)+8\kappa_0(0)+8\Gamma'(1)+8\log(2\pi)\right)\\ &= -\frac{1}{2}\kappa_0(3)-4\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)}-\log 3-2\Gamma'(1)-2\log(2\pi)\\ &= -\frac{1}{2}\kappa_0(3)+4\frac{L'(0,\chi_d)}{L(0,\chi_d)}-\log 3+2\log|d|-2\log(8\pi^2).\end{aligned}$$

The term that needs some work is $\kappa_0(3)$.

We have $L_+ = \mathbb{Z}I$ and $L_- = \mathbb{Z}J + \mathbb{Z}IJ$. Thus, $L = L_+ + L_-$ and by (12), we have

$$\kappa_0(3) = \sum_{x\in L_+}\kappa_0^-(3-\langle x,x\rangle/2) = \kappa_0^-(3)+2\kappa_0^-(2).$$

With respect to the basis $\{J, IJ\}$, the Gram matrix of L_- is $\left(\begin{smallmatrix}-6 & 0\\ 0 & -6\end{smallmatrix}\right)$. Thus, the sets $S_{m,\mu}$ in (13) is $\{2,3\}$ for both $\kappa_0^-(3)$ and $\kappa_0^-(2)$. Using results in Section 4 of [22], we find that

$$W_{3,2}(s,\varphi_{0,2}) = \frac{1}{2}(1-2^{-2s}), \qquad W_{3,3}(s,\varphi_{0,3}) = \frac{1}{3}(1+2\cdot 3^{-s}+3^{-2s})$$

and

$$W_{2,2}(s,\varphi_{0,2}) = \frac{1}{2}(1+2^{-3s}), \qquad W_{2,3}(s,\varphi_{0,3}) = \frac{1}{3}(1-3^{-s}).$$

Therefore, by Lemma 16,

$$\kappa_0^-(3) = -8\log 2, \qquad \kappa_0^-(2) = -2\log 3$$

and $\kappa_0(3) = -8\log 2 - 4\log 3$. It follows that

$$\left|\psi_{F_f}(\tau_d)(\operatorname{Im}\tau_d)^4\right| = 48\cdot\frac{|d|^2}{64\pi^4}e^{4L'(0,\chi_d)/L(0,\chi_d)}. \tag{18}$$

We next determine the value of $\psi_{F_g}(\tau)$ at the CM-point of discriminant $d = -3$. Choose $\lambda = 3I - J + IJ$ so that $\phi : \sqrt{-3} \to \lambda$ defines an optimal embedding of discriminant -3. By Theorem B, Lemma 20, and (10) again, we have

$$\begin{aligned}\log\left|\psi_{F_g}(\tau_d)(\operatorname{Im}\tau_d)^6\right| &= -\frac{1}{2}\kappa_0(1)-6\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)}-\log 2-2\Gamma'(1)-2\log(2\pi)\\ &= -\frac{1}{2}\kappa_0(1)+6\frac{L'(0,\chi_d)}{L(0,\chi_d)}-\log 2+3\log|d|-3\log(8\pi^2).\end{aligned}$$

By Corollary 19, the lattice L_- has discriminant 12 and its Gram matrix must be equivalent to $\left(\begin{smallmatrix}-4 & -2\\ -2 & -4\end{smallmatrix}\right)$. Since the discriminant of the lattice $L_+ + L_-$ is equal to that of L, $L/(L_+ + L_-)$ is trivial. Consequently,

$$\kappa_0(1) = \sum_{x\in L_+}\kappa_0^-(1-\langle x,x\rangle/2) = \kappa_0^-(1).$$

The set $S_{m,\mu}$ in (13) is $\{2,3\}$ for $\kappa_0^-(1)$. Using Theorems 4.3 and 4.4 of [22], we find

$$W_{1,2}(s,\varphi_{0,2}) = \frac{1}{2}(1-2^{-s}), \qquad W_{1,3}(s,\varphi_{0,3}) = \frac{1}{\sqrt{3}}(1+3^{-s}).$$

Then, Lemma 16 yields

$$\kappa_0^-(1) = -6\sqrt{3} \cdot \frac{\log 2}{3} \cdot \frac{2}{\sqrt{3}} = -4 \log 2.$$

Finally, we arrive at

$$\left|\psi_{F_g}(\tau_d)(\operatorname{Im}\tau_d)^6\right| = 2 \cdot \frac{|d|^3}{512\pi^6} e^{6L'(0,\chi_d)/L(0,\chi_d)}. \tag{19}$$

Corollary 22. *The absolute values of the constants* C_1 *and* C_2 *in* (8) *and* (9) *are*

$$|C_1| = \frac{12}{\pi^4} e^{4L'(0,\chi_{-4})/L(0,\chi_{-4})}, \qquad |C_2| = \frac{27(1+\sqrt{3})^6}{256\pi^6} e^{6L'(0,\chi_{-3})/L(0,\chi_{-3})},$$

respectively.

Proof. The CM-point of discriminant -4 in the example above is $\tau_{-4} = i$. According to our choice of $s(\tau)$ in Proposition 5, we have $s(i) = 0$. Therefore, the right-hand side of (8) is simply C_1. Then (18) gives us the absolute value of C_1. The determination of $|C_2|$ is similar. □

Remark 23. The values of $|C_1|$ and $|C_2|$ can also be determined by considering the values of the Borcherds forms at the CM-point τ_{-24} of discriminant -24. At the point τ_{-24}, the functions $s(\tau)$ and $t(\tau)$ take value 1. Thus, the right-hand sides of (8) and (9) can be expressed in terms of Gamma values using Gauss's formula $2F_1(a, b; c; 1) = \Gamma(c)\Gamma(c-a-b)/(\Gamma(c-a)\Gamma(c-b))$. By repeatedly applying Euler's reflection formula and Gauss's multiplication formula, we arrive at the same expressions for $|C_1|$ and $|C_2|$.

Example 24. Consider the case $d = -163$. By Theorem B and Lemma 20, we have

$$\log\left|\psi_{F_f}(\tau_d)(\operatorname{Im}\tau_d)^4\right| = -\frac{1}{2}\kappa_0(3) + 4\frac{L'(0,\chi_d)}{L(0,\chi_d)} - \log 3 - \frac{2}{3}\log 2 + 2\log|d| - 2\log(8\pi^2).$$

On page 851 of [13], it is computed that

$$\kappa_0(3) = -\frac{40}{3}\log 2 - 4\log 3 - 4\log 5 - 4\log(11) - 4\log(17).$$

Thus,

$$\left|\psi_{F_f}(\tau_d)(\operatorname{Im}\tau_d)^4\right| = 2^6 \cdot 3 \cdot 5^2 \cdot 11^2 \cdot 17^2 \cdot \frac{|d|^2}{64\pi^4} e^{4L'(0,\chi_d)/L(0,\chi_d)}.$$

We now give the values of the Borcherds forms $\psi_{F_f}(\tau)$ and $\psi_{F_g}(\tau)$ at various CM-points. The computation is done using Magma [7]. (The use of Magma is not essential. We use Magma only because it has built-in functions for computation about quaternion algebras.) The Magma code is available as a supplement to the arXiv version of this paper (arXiv:1503.07971).[1]

Lemma 25. *For a fundamental discriminant* $d < 0$ *appearing in Theorem* 2, *let* $\tau_d \in \mathbb{H}^+$ *be a CM-point of discriminant* d, *and let*

$$\omega_d = e^{L'(0,\chi_d)/2L(0,\chi_d)} = \frac{1}{\sqrt{|d|}} \prod_{a=1}^{|d|-1} \Gamma\left(\frac{a}{|d|}\right)^{\chi_d(a)\mu_d/4h_d}.$$

[1]The interested reader should download the source file instead of the pdf file.

Let A_d be the number such that

$$\left|\psi_{F_f}(\tau_d)(\operatorname{Im}\tau_d)^4\right| = A_d\frac{|d|^2}{64}\left(\frac{\omega_d}{\sqrt{\pi}}\right)^8. \tag{20}$$

Then we have

d	A_d	d	A_d	d	A_d
-4	$2^4\cdot 3$	-132	$2^4\cdot 3^2\cdot 5^2$	-148	$2^4\cdot 3\cdot 5^2\cdot 17^2$
-24	$2^4\cdot 3^2$	-43	$2^6\cdot 3\cdot 5^2$	-232	$2^4\cdot 3\cdot 5^2\cdot 23^2\cdot 29$
-120	$2^4\cdot 3^3\cdot 5$	-88	$2^4\cdot 3\cdot 5^2\cdot 11$	-708	$2^4\cdot 3^2\cdot 5^2\cdot 17^2\cdot 29^2$
-52	$2^4\cdot 3\cdot 5^2$	-312	$2^4\cdot 3^2\cdot 5^2\cdot 11^2$	-163	$2^6\cdot 3\cdot 5^2\cdot 11^2\cdot 17^2$

Also, let B_d be the number such that

$$\left|\psi_{F_g}(\tau_d)(\operatorname{Im}\tau_d)^6\right| = B_d\frac{|d|^3}{512}\left(\frac{\omega_d}{\sqrt{\pi}}\right)^{12}. \tag{21}$$

We have

d	B_d	d	B_d	d	B_d
-3	2	-19	$2\cdot 3^2$	-67	$2\cdot 3^2\cdot 7^2\cdot 11^2$
-84	$2^4\cdot 7$	-168	$2^3\cdot 7\cdot 11^2$	-372	$2^4\cdot 7^2\cdot 19^2\cdot 31$
-40	$2^3\cdot 3^2$	-228	$2^6\cdot 7^2\cdot 19$	-408	$2^3\cdot 7^2\cdot 11^2\cdot 31^2$
-51	$2\cdot 7^2$	-123	$2\cdot 7^2\cdot 19^2$	-267	$2\cdot 7^2\cdot 31^2\cdot 43^2$

5. Proof of Theorem 2

In this section, we shall convert informations from Lemma 25 into special-value formulas for hypergeometric functions.

We retain our choices of B, $\mathcal{O}$, ι, the fundamental domain, etc. from Section 2. In the following discussion, we let s be the Hauptmodul of $X_0^6(1)/W_6$ that takes values 0, 1, and ∞ at the CM-points of discriminants -4, -24, and -3, respectively. According to the choice of the fundamental domain in Section 2, these CM-points are represented by i, $(\sqrt{6}-\sqrt{2})i/2$, and $(-1+i)/(1+\sqrt{3})$, respectively. Let also $t = 1/s$. For a CM-point τ_d of a fundamental discriminant $d < 0$ inside the fundamental domain, we let $\phi : \mathbb{Q}(\sqrt{d}) \hookrightarrow B$ be the corresponding optimal embedding and assume that $\phi(\sqrt{d}) = a_1 I + a_2 J + a_3 IJ$. Then we have

$$\tau_d = \frac{a_2\sqrt{3}+\sqrt{d}}{a_1 + a_3\sqrt{3}}. \tag{22}$$

We first recall a technical lemma from [36].

Lemma 26 ([36, Lemma 5])**.** *If $s(\tau_d)$ takes a value in the line segment $[0,1]$, then $a_2 = 0$. If $s(\tau_d)$ takes a value in $[1,\infty)$, then $a_1 = 3a_3$. If $s(\tau_d)$ takes a negative value, then $a_2 = -a_3$.*

Recall that $\psi_{F_f}(\tau)$ and $\psi_{F_g}(\tau)$ are the Borcherds forms defined in (8) and (9), respectively.

Proposition 27. *Assume that* $-1 < s(\tau_d) < 1$. *Let* A_d *be the real number such that* (20) *holds. Then we have*

$$ {}_2F_1\left(\frac{1}{24},\frac{5}{24};\frac{3}{4};s(\tau_d)\right)^8 = \frac{A_d}{2^{12}\cdot 3}(a_1+\sqrt{|d|})^4\left(\frac{\omega_d}{\omega_{-4}}\right)^8 \tag{23}$$

and

$$ {}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};\frac{3}{4};\frac{5}{4};s(\tau_d)\right)^4 = \frac{3^2A_d}{2^{10}|s(\tau_d)|}(a_2^2+a_3^2)^2\omega_d^8. \tag{24}$$

Assume that $-1 < t(\tau_d) < 1$. *Let* B_d *be the real number such that* (21) *holds. Then we have*

$$\begin{aligned} {}_2F_1\left(\frac{1}{24},\frac{7}{24};\frac{5}{6};t(\tau_d)\right)^{12} &= \frac{B_d}{2^7\cdot 3^3}\left(\frac{\omega_d}{\omega_{-3}}\right)^{12} \\ &\quad\times\begin{cases}((a_2+2a_3)\sqrt{3}+\sqrt{|d|})^6 & \text{if } t(\tau_d)>0,\\ ((a_1-2a_3)\sqrt{3}+\sqrt{|d|})^6 & \text{if } t(\tau_d)<0\end{cases}\end{aligned} \tag{25}$$

and

$$ {}_3F_2\left(\frac{1}{4},\frac{1}{2},\frac{3}{4};\frac{5}{6},\frac{7}{6};t(\tau_d)\right)^6 = \frac{B_d}{216|t|}\omega_d^{12}\times\begin{cases}27(a_2+a_3)^6 & \text{if } t(\tau_d)>0,\\ (a_1-3a_3)^6 & \text{if } t(\tau_d)<0.\end{cases} \tag{26}$$

Proof. For convenience, set

$$F_1(s) = {}_2F_1(1/24,5/24;3/4;s), \qquad F_2(s) = {}_2F_1(7/24,11/24;5/4;s),$$

and $s_d = s(\tau_d)$. Note that we have $F_1(s)F_2(s) = {}_3F_2(1/3,1/2,2/3;3/4,5/4,s)$. Let $C = -1/\sqrt[4]{12}\omega_{-4}^2$. By Lemma 5 of [36], we have

$$\frac{Cs_d^{1/4}F_2(s_d)}{F_1(s_d)} = \frac{\tau_d-i}{\tau_d+i}. \tag{27}$$

Combining (8), (20), (22), and Corollary 22, we find

$$\begin{aligned} A_d\frac{|d|^2}{64}\left(\frac{\omega_d}{\sqrt{\pi}}\right)^8 &= \frac{12\omega_{-4}^8|d|^2}{\pi^4(a_1+a_3\sqrt{3})^4}F_1(s_d)^8\left|1-\frac{\tau_d-i}{\tau_d+i}\right|^8 \\ &= \frac{12\omega_{-4}^8|d|^2}{\pi^4(a_1+a_3\sqrt{3})^4}F_1(s_d)^8\left(\frac{2(a_1+a_3\sqrt{3})(a_1-\sqrt{|d|})}{3(a_2^2+a_3^2)}\right)^4.\end{aligned}$$

Simplifying the identity, we get (23). To prove (24), we observe that from (27) we obtain

$$F_2(s_d) = \sqrt[4]{12}\omega_{-4}^2\frac{F_1(s_d)}{|s_d|^{1/4}}\left|\frac{\tau_d-i}{\tau_d+i}\right| = \sqrt[4]{12}\omega_{-4}^2\frac{F_1(s_d)}{|s_d|^{1/4}}\frac{a_1-\sqrt{|d|}}{\sqrt{3(a_2^2+a_3^2)}}.$$

Combining this with (23), we obtain

$$\begin{aligned} F_1(s_d)^8F_2(s_d)^8 &= 2^4\cdot 3^2\cdot\omega_{-4}^{16}\frac{F_1(s_d)^{16}}{s_d^2}\frac{(a_1-\sqrt{|d|})^8}{3^4(a_2^2+a_3^2)^4} \\ &= \frac{A_d^2}{2^{20}\cdot 3^4\cdot s_d^2}\frac{(a_1+\sqrt{|d|})^8(a_1-\sqrt{|d|})^8}{(a_2^2+a_3^2)^4}\omega_d^{16} = \frac{3^4A_d^2}{2^{20}s_d^2}(a_2^2+a_3^2)^4\omega_d^{16}.\end{aligned}$$

Simplifying the equality, we obtain (24).

Similarly, we write $t_d = t(\tau_d)$, and

$$G_1(t) = {}_2F_1\left(\frac{1}{24}, \frac{7}{24}; \frac{5}{6}; t\right), \qquad G_2(t) = {}_2F_1\left(\frac{5}{24}, \frac{11}{24}; \frac{7}{6}; t\right).$$

Then $G_1(t)G_2(t) = {}_3F_2(1/4, 1/2, 3/4; 5/6, 7/6; t)$. Let $C' = e^{-2\pi i/8}/\sqrt[6]{2}\omega_{-3}^2$. We have

$$\frac{C' t_d^{1/6} G_2(t_d)}{G_1(t_d)} = \frac{\tau_d - \tau_{-3}}{\tau_d - \overline{\tau}_{-3}}, \qquad \tau_{-3} = \frac{-1+i}{1+\sqrt{3}}.$$

Using

$$\left|\frac{\tau_d - \tau_{-3}}{\tau_d - \overline{\tau}_{-3}}\right|^2 = \frac{\sqrt{3}(a_1+a_2-a_3) - \sqrt{|d|}}{\sqrt{3}(a_1+a_2-a_3) + \sqrt{|d|}},$$

$$\left|1 - \frac{\tau_d - \tau_{-3}}{\tau_d - \overline{\tau}_{-3}}\right|^2 = \frac{2}{1+\sqrt{3}} \frac{a_1 + a_3\sqrt{3}}{\sqrt{3}(a_1+a_2-a_3) + \sqrt{|d|}},$$

(9), (19), and Corollary 22, we deduce that

$$\begin{aligned} B_d \frac{|d|^3}{512}\left(\frac{\omega_d}{\sqrt{\pi}}\right)^{12} &= \frac{27(1+\sqrt{3})^6 \omega_{-3}^{12} |d|^3}{256\pi^6 (a_1 + a_3\sqrt{3})^6} G_1(t_d)^{12} \left|1 - \frac{\tau_d - \tau_{-3}}{\tau_d - \overline{\tau}_{-3}}\right|^{12} \\ &= \frac{27\omega_{-3}^{12}|d|^3}{4\pi^6(\sqrt{3}(a_1+a_2-a_3) + \sqrt{|d|})^6} G_1(t_d)^{12} \end{aligned}$$

so that

$$G_1(\tau_d)^{12} = \frac{B_d(\sqrt{3}(a_1+a_2-a_3) + \sqrt{|d|})^6}{2^7 \cdot 3^3} \left(\frac{\omega_d}{\omega_{-3}}\right)^{12}$$

and

$$\begin{aligned} G_1(\tau_d)^{12} G_2(\tau_d)^{12} &= \frac{4\omega_{-3}^{24}}{t_d^2} G_1(t_d)^{24} \left(\frac{\sqrt{3}(a_1+a_2-a_3) - \sqrt{|d|}}{\sqrt{3}(a_1+a_2-a_3) + \sqrt{|d|}}\right)^6 \\ &= \frac{B_d^2}{2^{12} \cdot 3^6} \left(3(a_1+a_2-a_3)^2 - |d|\right)^6 \omega_d^{24} \\ &= \frac{B_d^2}{2^6 \cdot 3^6} \left(a_1^2 + 3a_2^2 + 3a_3^2 + 3a_1a_2 - 3a_2a_3 - 3a_1a_3\right)^6 \omega_d^{24}. \end{aligned}$$

With Lemma 26, these two identities reduce to (25) and (26), respectively. This completes the proof. □

Proof of Theorem 2. The values of $s(\tau)$ and $t(\tau)$ at CM-points were computed in [13]. They are the rational numbers M/N from the two tables in Theorem 2. The optimal embeddings corresponding to the CM-points inside the fundamental domain are given in the two tables below.

d	$\phi(\sqrt{d})$	d	$\phi(\sqrt{d})$
-52	$8I + 2IJ$	-120	$12I - 2J + 2IJ$
-88	$10I + 2IJ$	-43	$7I - J + IJ$
-132	$12I + 2IJ$	-232	$16I - 2J + 2IJ$
-312	$18I + 2IJ$	-163	$13I - J + IJ$
-148	$14I + 4IJ$		
-708	$30I + 8IJ$		

d	$\phi(\sqrt{d})$	d	$\phi(\sqrt{d})$
-84	$12I-2J+4IJ$	-40	$8I-2J+2IJ$
-51	$9I-J+3IJ$	-19	$5I-J+IJ$
-168	$18I-4J+6IJ$	-228	$18I-4J+4IJ$
-123	$15I-3J+5IJ$	-67	$11I-3J+3IJ$
-372	$24I-2J+8IJ$		
-408	$30I-8J+10IJ$		
-267	$21I-3J+7IJ$		

Here the left columns of the two tables are for discriminants d with $s(\tau_d)>0$ and $t(\tau_d)>0$, respectively. Combining information from Lemma 25, Proposition 27, and the above two tables, we obtain the identities in Theorem 2. □

6. Further examples

Observe that for each discriminant d appearing in Theorem 2, there is only one CM-point of discriminant d on the Shimura curve $X_0^6(1)/W_6$. In such cases, Schofer's formula readily tells us the absolute value of a Borcherds form at the unique CM-point of discriminant d. However, in general, we can only read from Schofer's formula the products of values of Borcherds forms at CM-points. In this section, we introduce a technique to separate the value at a CM-point from those at the other CM-points of the same discriminant using Hecke operators. This technique relies on the method developed in [35] for computing Hecke operators. Here we will work out the case $d=-276$. In principle, the method works at least for any imaginary quadratic number field whose ideal class group, after quotient by the prime ideals lying above 2 and 3, is an elementary 2-group.

Let $E=\mathbb{Q}(\sqrt{-276})$ and let R be the ring of integers in E. There are two CM-points of discriminant $d=-276$ on $X_0^6(1)/W_6$, represented by the two points

$$\tau_1=\frac{\sqrt{-69}}{9+2\sqrt{3}},\qquad \tau_2=\frac{-3\sqrt{3}+\sqrt{-69}}{12+4\sqrt{3}}$$

in the fundamental domain. The corresponding optimal embeddings ϕ_1 and ϕ_2 are

$$\lambda_1=\phi_1(\sqrt{-276})=18I+4IJ,\qquad \lambda_2=\phi_2(\sqrt{-276})=24I-6J+8IJ,$$

respectively. According to the table at the end of Section 5 of [34], the values of the Hauptmodul $s(\tau)$ at these two points are $(166139596\pm 95538528\sqrt{3})/1771561$. (The values can also be determined using Borcherds forms and Schofer's formula.) From Lemma 26, we deduce that

$$s(\tau_1)=\frac{166139596-95538528\sqrt{3}}{1771561},\qquad s(\tau_2)=\frac{166139596+95538528\sqrt{3}}{1771561}.$$

Call these two numbers s_1 and s_2, respectively. Let $\mathfrak{p}_2$ and $\mathfrak{p}_3$ be the prime ideals of R lying above 2 and 3, respectively, and let $\mathfrak{p}_5$ be any prime above 5. Then the ideal class group of R is isomorphic to $(\mathbb{Z}/2Z)\times(\mathbb{Z}/4\mathbb{Z})$ generated by the element $\mathfrak{p}_2$ of order 2 and the element $\mathfrak{p}_5$ of order 4. Moreover, the product $\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_5^2$ is a principal ideal. It follows that the ideal class group, after quotient by the subgroup generated by $\mathfrak{p}_2$ and $\mathfrak{p}_3$, is cyclic of order 2 and generated by $\mathfrak{p}_5$. In terms of CM-points on $X_0^6(1)/W_6$, this means that there should exist an element α of norm 5, 10, 15, or 30 in $\mathcal{O}$ such that $\iota(\alpha)\tau_1 = \tau_2$. (Here we retain the notation $\mathcal{O}$, ι, etc. used in Section 2.) Indeed, such an element is

$$\alpha = 3 - 2I - IJ.$$

(Another element is $\alpha' = (3 - 9I + J - 3IJ)/2$.) In other words, we have $\lambda_2 = \alpha\lambda_1\alpha^{-1}$.

Now let $F(\tau) = \psi_{F_f}(\tau)$ be the modular form of weight 8 defined in Proposition 14 and set

$$\widetilde{F}(\tau) := F\big|_8\iota(\alpha) = \frac{10^4}{((2+\sqrt{3})\tau - 3)^8} F\left(\frac{3\tau + 2 - \sqrt{3}}{(-2-\sqrt{3})\tau + 3}\right).$$

In general, we have

$$\frac{10^4}{|(2+\sqrt{3})\tau - 3|^8} = \left(\frac{\operatorname{Im}\iota(\alpha)\tau}{\operatorname{Im}\tau}\right)^4.$$

Thus,

$$|F(\tau_2)| = \left(\frac{\operatorname{Im}\tau_1}{\operatorname{Im}\tau_2}\right)^4 \left|\widetilde{F}(\tau_1)\right|. \tag{28}$$

On the other hand, Schofer's formula yields

$$|F(\tau_1)F(\tau_2)|\,(\operatorname{Im}\tau_1)^4(\operatorname{Im}\tau_2)^4 = 2^8 \cdot 3^4 \cdot 11^2 \left(\frac{|d|^2}{64\pi^4}\omega_d^8\right)^2.$$

Substituting (28) into this, we obtain

$$\left|F(\tau_1)(\operatorname{Im}\tau_1)^4\right|^2 \left|\frac{\widetilde{F}(\tau_1)}{F(\tau_1)}\right| = 2^8 \cdot 3^4 \cdot 11^2 \left(\frac{|d|^2}{64\pi^4}\omega_d^8\right)^2. \tag{29}$$

The main task remaining is to determine the value of $\widetilde{F}(\tau_1)/F(\tau_1)$.

Let Γ be the discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ such that $X_0^6(1)/W_6 = \Gamma\backslash\mathbb{H}^+$, i.e., $\Gamma := \{\iota(\gamma)/(\det\gamma)^{1/2} : \gamma \in N_B^+(\mathcal{O})\}$. Let γ_j, $j = 1, \ldots, 5$, be elements in $\Gamma\iota(\alpha)\Gamma$ such that $\gamma_0 = \iota(\alpha)$ and γ_j, $j = 1, \ldots, 5$, form a complete set of coset representatives of $\Gamma\backslash\Gamma\iota(\alpha)\Gamma$. In Section 4 of [36], by using results from [35], we find that

$$\begin{aligned}\prod_{j=0}^{5}\left(y - \frac{F\big|_8\gamma_j}{F}\right) = y^6 &+ \frac{114}{125}y^5 - \frac{6333}{78125}y^4 + \frac{4}{5^{11}}(8640000s - 5177953)y^3 \\ &+ \frac{3}{5^{15}}(8467200000s + 1804020097)y^2 \\ &+ \frac{726}{5^{20}}(93744000000s - 3501556201)y \\ &+ \frac{1}{5^{16}}(138240s + 14641)^2.\end{aligned}$$

Substituting s by $s_1 = (166139596 - 95538528\sqrt{3})/1771561$, we deduce that $\widetilde{F}(\tau_1)/F(\tau_1)$ is a zero of

$$(9150625y^2 + (40464094y - 20903960\sqrt{3})y + 82650625 - 47425000\sqrt{3})g(y),$$

where $g(y) \in \mathbb{Q}(\sqrt{3})[y]$ is an irreducible polynomial of degree 4 over $\mathbb{Q}(\sqrt{3})$. In fact, we can show that it is a zero of the factor of degree 2 shown above. Hence, we have

$$\left|\frac{\widetilde{F}(\tau_1)}{F(\tau_1)}\right| = \left(\frac{82650625 - 47425000\sqrt{3}}{9150625}\right)^{1/2} = \left(\frac{14-5\sqrt{3}}{11}\right)^2. \tag{30}$$

(It is possible to determine the precise value, not just the absolute value. The two zeros of the factor of degree 2 are $\widetilde{F}(\tau_1)/F(\tau_1)$ and the value of $(F\big|_8\iota(\alpha'))/F$ at τ_1, where $\alpha' = (3-9I+J-3IJ)/2$. It is easy to find the ratio of the two values and hence determine $\widetilde{F}(\tau_1)/F(\tau_1)$.) Substituting (30) into (29), we obtain

$$\left|F(\tau_1)(\operatorname{Im}\tau_1)^4\right| = 144(14+5\sqrt{3})\frac{|d|^2}{64\pi^4}\omega_d^8.$$

By Proposition 27, this implies that

$$_2F_1\left(\frac{1}{24},\frac{5}{24};\frac{3}{4};s_1\right)^8 = \frac{3(14+5\sqrt{3})}{16}(9+\sqrt{69})^4\left(\frac{\omega_{-276}}{\omega_{-4}}\right)^8$$

and

$$_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};\frac{3}{4},\frac{5}{4};s_1\right)^4 = \left(\frac{3(16+23\sqrt{3})}{11}\right)^4\left(\frac{2+3\sqrt{3}}{23}\right)^2(2+\sqrt{3})\omega_{-276}^8.$$

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