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AN EXPLICIT WALDSPURGER FORMULA FOR HILBERT MODULAR FORMS

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ABSTRACT. We describe a construction of preimages for the Shimura map on Hilbert modular forms and give an explicit Waldspurger type formula relating their Fourier coefficients to central values of twisted L-functions. Our construction is inspired by that of Gross and applies to any nontrivial level and arbitrary base field, subject to certain conditions on the Atkin-Lehner eigenvalues and on the weight.

INTRODUCTION

Computing central values of L-functions attached to modular forms is interesting because of the arithmetic information they encode. These values are related to Fourier coefficients of half-integral weight modular forms and the Shimura correspondence, as shown in great generality in [Wal81]. For classical modular forms, explicit formulas of Waldspurger type can be found in [Gro87], [BSP90], [MRVT07], among other works. In the Hilbert setting there are Waldspurger type formulas available in [Shi93], [BM07]. More explicit formulas for computing central values in terms of Fourier coefficients can be found in [HI13] in the case of trivial level and in [Xue11], where the result is restricted to modular forms of prime power level over fields with odd class number. In [CST14] the authors give a formula in terms of heights in the case of parallel weight $\mathbf{2}$.

In this article we prove a formula relating central values of twisted L-functions attached to a Hilbert cuspidal newform g to Fourier coefficients of certain modular forms of half-integral weight, which are constructed explicitly as theta series and map to g under the Shimura correspondence. Our result applies to any nontrivial level and arbitrary base field and to a broader family of twists than the one considered in [Xue11]. In the classical case it is more general than [Gro87] and [BSP90], where the authors consider prime and square-free levels respectively.

Let g be a normalized Hilbert cuspidal newform over a totally real number field F, of level $\mathfrak{N} \subsetneq \mathcal{O}_F$, weight $\mathbf{2} + 2\mathbf{k}$ and trivial central character. For each $\mathfrak{p} \mid \mathfrak{N}$ denote by $\varepsilon_g(\mathfrak{p})$ the eigenvalue of the $\mathfrak{p}$-th Atkin-Lehner involution acting on g, and let $\mathcal{W}^- = \{\mathfrak{p} \mid \mathfrak{N} : \varepsilon_g(\mathfrak{p}) = -1\}$. We make the following hypotheses on $\mathcal{W}^-$ and $\mathbf{k}$:

H1. $|\mathcal{W}^-|$ and $[F : \mathbb{Q}]$ have the same parity.
H2. $v_{\mathfrak{p}}(\mathfrak{N})$ is odd for every $\mathfrak{p} \in \mathcal{W}^-$.
H3. $(-1)^{\mathbf{k}} = 1$.

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For $D \in F^+$ denote by $L_D(s,g) = L(s,g)L(s,g\otimes\chi_D)$ the Rankin-Selberg convolution L-function of g by the quadratic character χ_D associated to the extension $F(\sqrt{-D})/F$, normalized with center of symmetry at $s = 1/2$. The main result of this article is stated in Theorem 5.3; in the simpler form given by Corollary 5.7 it claims that there exists a Hilbert cuspidal form f of weight $\mathbf{3/2}+\mathbf{k}$ whose Fourier coefficients $\lambda(D,\mathfrak{a};f)$ satisfy

$$L_D(1/2,g) = \frac{c_D}{D^{\mathbf{k}+\mathbf{1/2}}}\left|\lambda(D,\mathfrak{a};f)\right|^2,$$

for every D such that $\chi_D(\mathfrak{p}) = \varepsilon_g(\mathfrak{p})$ whenever $v_{\mathfrak{p}}(\mathfrak{N})$ is odd and such that the conductor of χ_D is prime to $2\mathfrak{N}$. Here c_D and $\mathfrak{a}$ are, respectively, an explicit positive rational number and a fractional ideal of F, both depending only on D. If $L(1/2,g) \neq 0$, then $f \neq 0$ and it maps to g under the Shimura correspondence. Actually, in this case we construct a linearly independent family of preimages for the Shimura correspondence, as shown in Corollary 5.6.

A different generalization of Gross's formula in [MRVT07, Mao12] gets rid of the restriction on D for classical modular forms of prime level. In future work we will combine this idea with our methods to obtain a formula without restrictions on D.

This article is organized as follows. In Section 1 we state some basic facts about the space of quaternionic modular forms. In Section 2 we show how to obtain half-integral weight Hilbert modular forms out of quaternionic modular forms and give a formula for their Fourier coefficients in terms of special points and the height pairing on the space of quaternionic modular forms. In Section 3 we relate central values of twisted L-functions to the height pairing, using results of [Zha01] and [Xue06] about central values of Rankin L-functions. In Section 4 we state an auxiliary result needed for the proof of the main result of this article, which we give in Section 5.

Notation summary. We fix a totally real number field F of discriminant d_F, with ring of integers $\mathcal{O}_F$. We denote by $\mathcal{J}_F$ the group of fractional ideals of F, and we write $\mathrm{Cl}(F)$ for the class group and h_F for the class number. We denote by $\mathbf{a}$ the set of embeddings $\tau : F \hookrightarrow \mathbb{R}$, and we let $F^+ = \{\xi \in F : \tau(\xi) > 0\ \forall \tau \in \mathbf{a}\}$. Given $\mathbf{k} = (k_\tau) \in \mathbb{Z}^{\mathbf{a}}$ and $\xi \in F$, we let $\xi^{\mathbf{k}} = \prod_{\tau\in\mathbf{a}} \tau(\xi)^{k_\tau}$. By $\mathfrak{p}$ we always denote a prime ideal of $\mathcal{O}_F$, and we use $\mathfrak{p}$ as a subindex to denote completions of global objects at $\mathfrak{p}$. Given an integral ideal $\mathfrak{N} \subseteq \mathcal{O}_F$ we let $\omega(\mathfrak{N}) = |\{\mathfrak{p} : \mathfrak{p} \mid \mathfrak{N}\}|$. Given $\mathfrak{p}$ we denote by $\pi_{\mathfrak{p}}$ a local uniformizer at $\mathfrak{p}$, and we let $v_{\mathfrak{p}}$ denote the $\mathfrak{p}$-adic valuation.

Given a totally imaginary quadratic extension K/F we let $\mathcal{O}_K$ be the maximal order and let $\mathfrak{D}_K \subseteq \mathcal{O}_F$ denote the relative discriminant. We let $t_K = [\mathcal{O}_K^\times : \mathcal{O}_F^\times]$, and let $m_K \in \{1,2\}$ be the order of the kernel of the natural map $\mathrm{Cl}(F) \to \mathrm{Cl}(K)$.

Given a quaternion algebra B/F we denote by $\mathcal{N} : B^\times \to F^\times$ and $\mathcal{T} : B \to F$ the reduced norm and trace maps, and we use $\mathcal{N}$ and $\mathcal{T}$ to denote other norms and traces as well. We denote by $\widehat{B} = \prod'_{\mathfrak{p}} B_{\mathfrak{p}}$ and $\widehat{B}^\times = \prod'_{\mathfrak{p}} B_{\mathfrak{p}}^\times$ the corresponding restricted products, and we let $B_\infty = \prod_{\tau\in\mathbf{a}} B_\tau$. Finally, given a level $\mathfrak{N} \subseteq \mathcal{O}_F$, an integral or half-integral weight $\mathbf{k}$, and a Hecke character χ, we denote by $\mathcal{M}_{\mathbf{k}}(\mathfrak{N},\chi)$ and $\mathcal{S}_{\mathbf{k}}(\mathfrak{N},\chi)$ the corresponding spaces of Hilbert modular and cuspidal forms.

1. Quaternionic modular forms

Let B be a totally definite quaternion algebra over F. Let (V,ρ) be an irreducible unitary right representation of $B^\times/F^\times$, which we denote by $(v,\gamma) \mapsto v\cdot\gamma$. Let R be an order of (reduced) discriminant $\mathfrak{N}$ in B. A *quaternionic modular form* of weight ρ and level R is a function $\varphi : \widehat{B}^\times \to V$ such that for every $x \in \widehat{B}^\times$ the following transformation formula is satisfied:

$$\varphi(ux\gamma) = \varphi(x) \cdot \gamma \qquad \forall\, u \in \widehat{R}^\times,\, \gamma \in B^\times .$$

The space of all such functions is denoted by $\mathcal{M}_\rho(R)$. We let $\mathcal{E}_\rho(R)$ be the subspace of functions that factor through the map $\mathcal{N} : \widehat{B}^\times \to \widehat{F}^\times$. These spaces come equipped with the action of Hecke operators $T_\mathfrak{m}$, indexed by integral ideals $\mathfrak{m} \subseteq \mathcal{O}_F$, and given by

$$T_\mathfrak{m}\, \varphi(x) = \sum_{h \in \widehat{R}^\times \backslash H_\mathfrak{m}} \varphi(hx)\,,$$

where $H_\mathfrak{m} = \Big\{ h \in \widehat{R} \,:\, \widehat{\mathcal{O}}_F\, \mathcal{N}(h) \cap \mathcal{O}_F = \mathfrak{m} \Big\}$.

Given $x \in \widehat{B}^\times$, we let

$$\widehat{R}_x = x^{-1} \widehat{R}\, x, \qquad R_x = B \cap \widehat{R}_x, \qquad \Gamma_x = R_x^\times / \mathcal{O}_F^\times, \qquad w_x = |\Gamma_x|\,.$$

The sets Γ_x are finite since B is totally definite. Let $\mathrm{Cl}(R) = \widehat{R}^\times \backslash \widehat{B}^\times / B^\times$. We define an inner product on $\mathcal{M}_\rho(R)$, called the *height pairing*, by

$$\langle \varphi, \psi \rangle = \sum_{x \in \mathrm{Cl}(R)} \tfrac{1}{w_x} \langle \varphi(x), \psi(x) \rangle\,.$$

The space of *cuspidal forms* $\mathcal{S}_\rho(R)$ is defined as the orthogonal complement of $\mathcal{E}_\rho(R)$ with respect to this pairing.

Let $N(\widehat{R}) = \{ z \in \widehat{B}^\times \,:\, \widehat{R}_z = \widehat{R} \}$ be the normalizer of $\widehat{R}$ in $\widehat{B}^\times$. We let $\widetilde{\mathrm{Bil}}(R) = \widehat{R}^\times \backslash N(\widehat{R}) / F^\times$. We have an embedding $\mathrm{Cl}(F) \hookrightarrow \widetilde{\mathrm{Bil}}(R)$. The group $\widetilde{\mathrm{Bil}}(R)$, and in particular $\mathrm{Cl}(F)$, acts on $\mathcal{M}_\rho(R)$ by letting $(\varphi \cdot z)(x) = \varphi(zx)$. This action restricts to $\mathcal{S}_\rho(R)$ and is related to the height pairing by the equality

$$\langle \varphi \cdot z, \psi \cdot z \rangle = \langle \varphi, \psi \rangle\,.$$

The action of $\widetilde{\mathrm{Bil}}(R)$ commutes with the action of the Hecke operators. The adjoint of $T_\mathfrak{m}$ with respect to the height pairing is given by $\varphi \mapsto T_\mathfrak{m}\, \varphi \cdot \mathfrak{m}^{-1}$.

The subspaces of $\mathcal{M}_\rho(R)$ and $\mathcal{S}_\rho(R)$ fixed by the action of $\mathrm{Cl}(F)$ are denoted by $\mathcal{M}_\rho(R, \mathbb{1})$ and $\mathcal{S}_\rho(R, \mathbb{1})$. Let $\mathrm{Bil}(R) = \widehat{R}^\times \backslash N(\widehat{R}) / \widehat{F}^\times$. Then $\mathrm{Bil}(R)$ acts on $\mathcal{M}_\rho(R, \mathbb{1})$ and $\mathcal{S}_\rho(R, \mathbb{1})$, and we have a short exact sequence

$$1 \longrightarrow \mathrm{Cl}(F) \longrightarrow \widetilde{\mathrm{Bil}}(R) \longrightarrow \mathrm{Bil}(R) \longrightarrow 1\,. \tag{1.1}$$

Forms with minimal support. Given $x \in \widehat{B}^\times$ and $v \in V$, let $\varphi_{x,v} \in \mathcal{M}_\rho(R)$ be the quaternionic modular form given by

$$\varphi_{x,v}(y) = \sum_{\gamma \in \Gamma_{x,y}} v \cdot \gamma\,,$$

where $\Gamma_{x,y} = (B^\times \cap x^{-1} \widehat{R}^\times y) / \mathcal{O}_F^\times$. Note that $\varphi_{x,v}$ is supported in $\widehat{R}^\times x B^\times$. Furthermore, we have that

$$\varphi_{ux\gamma, v} = \varphi_{x, v \cdot \gamma^{-1}} \qquad \forall\, u \in \widehat{R}^\times,\, \gamma \in B^\times, \tag{1.2}$$

$$\varphi_{x,v} \cdot z = \varphi_{z^{-1}x, v} \qquad \forall\, z \in \widetilde{\mathrm{Bil}}(R)\,. \tag{1.3}$$

Given $\varphi \in \mathcal{M}_\rho(R)$, using that $\varphi(x) \in V^{\Gamma_x}$ for every $x \in \widehat{B}^\times$ we get that

$$\varphi = \sum_{x \in \mathrm{Cl}(R)} \tfrac{1}{w_x} \varphi_{x,\varphi(x)} \,. \tag{1.4}$$

Proposition 1.5. *Let $x \in \widehat{B}^\times$ and $v \in V$. Then $T_\mathfrak{m}\, \varphi_{x,v} = \sum_{h \in H_\mathfrak{m}/\widehat{R}^\times} \varphi_{h^{-1}x,v}$.*

Proof. Given $y \in \widehat{B}^\times$, let $\Gamma = (B^\times \cap x^{-1} H_\mathfrak{m}\, y)/\mathcal{O}_F^\times$. Then

$$\Gamma = \coprod_{h \in \widehat{R}^\times \backslash H_\mathfrak{m}} \Gamma_{x,hy} = \coprod_{h \in H_\mathfrak{m}/\widehat{R}^\times} \Gamma_{h^{-1}x,y} \,.$$

The first decomposition implies that

$$T_\mathfrak{m}\, \varphi_{x,v}(y) = \sum_{h \in \widehat{R}^\times \backslash H_\mathfrak{m}} \varphi_{x,v}(hy) = \sum_{h \in \widehat{R}^\times \backslash H_\mathfrak{m}} \sum_{\gamma \in \Gamma_{x,hy}} v \cdot \gamma = \sum_{\beta \in \Gamma} v \cdot \beta \,,$$

whereas the second decomposition implies that

$$\sum_{h \in H_\mathfrak{m}/\widehat{R}^\times} \varphi_{h^{-1}x,v}(y) = \sum_{h \in H_\mathfrak{m}/\widehat{R}^\times} \sum_{\gamma \in \Gamma_{h^{-1}x,y}} v \cdot \gamma = \sum_{\beta \in \Gamma} v \cdot \beta \,,$$

which completes the proof. □

Proposition 1.6. *Given $x, y \in \widehat{B}^\times$ and $v, w \in V$, we have*

$$\langle \varphi_{x,v} \,, \varphi_{y,w} \rangle = \sum_{\gamma \in \Gamma_{x,y}} \langle v \cdot \gamma, w \rangle.$$

Proof. Since $\Gamma_y = \Gamma_{y,y}$ acts on $\Gamma_{x,y}$ on the right, using that (ρ, V) is unitary we get that

$$\begin{aligned} \langle \varphi_{x,v}, \varphi_{y,w} \rangle &= \tfrac{1}{w_y} \langle \varphi_{x,v}(y), \varphi_{y,w}(y) \rangle = \tfrac{1}{w_y} \sum_{\alpha \in \Gamma_{x,y}} \sum_{\beta \in \Gamma_y} \langle v \cdot \alpha, w \cdot \beta \rangle \\ &= \tfrac{1}{w_y} \sum_{\alpha \in \Gamma_{x,y}} \sum_{\beta \in \Gamma_y} \left\langle v \cdot \alpha\beta^{-1}, w \right\rangle = \sum_{\gamma \in \Gamma_{x,y}} \langle v \cdot \gamma, w \rangle \,. \end{aligned}$$ □

2. Half-integral weight modular forms and special points

From now on we specify the representation (V, ρ). Let $\mathbf{k} = (k_\tau) \in \mathbb{Z}_{\geq 0}^{\mathbf{a}}$. For each $\tau \in \mathbf{a}$ we consider the real vector space $W_\tau = B_\tau / F_\tau$, with inner product induced by the totally positive definite quadratic form $-\Delta(x) = 4\mathcal{N}(x) - \mathcal{T}(x)^2$. By letting $B_\tau^\times / F_\tau^\times$ act on W_τ by conjugation we get an orthogonal representation. This gives naturally an orthogonal representation of $B_\tau^\times / F_\tau^\times$ on $\mathbb{R}_{k_\tau}[W_\tau] = \mathrm{Sym}^{k_\tau}(\mathrm{Hom}_\mathbb{R}(W_\tau, \mathbb{R}))$, the space of homogeneous polynomials on W_τ of degree k_τ with coefficients in $\mathbb{R}$, and hence a unitary representation of $B_\tau^\times / F_\tau^\times$ on $\mathbb{C}_{k_\tau}[W_\tau] = \mathbb{R}_{k_\tau}[W_\tau] \otimes_\mathbb{R} \mathbb{C}$. We let V_{k_τ} denote the $B_\tau^\times / F_\tau^\times$-submodule of $\mathbb{C}_{k_\tau}[W_\tau]$ of harmonic polynomials with respect to $-\Delta$. This is, up to isomorphism, the unique irreducible unitary representation of $B_\tau^\times / F_\tau^\times$ of dimension $2k_\tau + 1$. We let $V_\mathbf{k} = \bigotimes_{\tau \in \mathbf{a}} V_{k_\tau}$, and through the embedding $B^\times \hookrightarrow B_\infty^\times$ we get an irreducible unitary representation $(V_\mathbf{k}, \rho_\mathbf{k})$ of $B^\times / F^\times$. We denote the corresponding spaces of quaternionic modular forms by $\mathcal{M}_\mathbf{k}(R)$, etc.

Denote by $\mathcal{H}$ the complex upper half-plane. Let $e_F : F \times \mathcal{H}^\mathbf{a} \to \mathbb{C}$ be the exponential function given by $e_F(\xi, z) = \exp\left(2\pi i \sum_{\tau \in \mathbf{a}} \tau(\xi)\, z_\tau\right)$. Given $x \in \widehat{B}^\times$, let $L_x \subseteq B/F$ be the lattice given by $L_x = R_x/\mathcal{O}_F$. Given $P \in V_{\mathbf{k}}$, we let $\vartheta_{x,P} : \mathcal{H}^{\mathbf{a}} \to \mathbb{C}$ be the function given by

$$\vartheta_{x,P}(z) = \sum_{y \in L_x} P(y)\, e_F(-\Delta(y), z/2)\,.$$

These theta series satisfy

$$\vartheta_{x,P} = (-1)^{\mathbf{k}}\, \vartheta_{x,P}\,, \tag{2.1}$$

$$\vartheta_{zx\gamma,P\cdot\gamma} = \vartheta_{x,P}\,, \qquad \forall z \in N(\widehat{R}),\ \gamma \in B^\times. \tag{2.2}$$

The following two propositions extend the results about theta series from [Sir14, Section 4] to arbitrary weights.

Proposition 2.3. *Let $x \in \widehat{B}^\times$ and let $P \in V_{\mathbf{k}}$. Then $\vartheta_{x,P} \in \mathcal{M}_{\mathbf{3/2+k}}(4\mathfrak{N}, \chi_1)$, where χ_1 is the Hecke character associated to the extension $F(\sqrt{-1})/F$. Furthermore, for $D \in F^+ \cup \{0\}$ and $\mathfrak{a} \in \mathcal{J}_F$, the D-th Fourier coefficient of $\vartheta_{x,P}$ at the cusp $\mathfrak{a}$ is given by*

$$\lambda(D, \mathfrak{a}; \vartheta_{x,P}) = \frac{1}{\mathcal{N}(\mathfrak{a})} \sum_{y \in \mathcal{A}_{D,\mathfrak{a}}(x)} P(y)\,, \tag{2.4}$$

where $\mathcal{A}_{D,\mathfrak{a}}(x) = \{y \in \mathfrak{a}^{-1} L_x \,:\, \Delta(y) = -D\}$.

Proof. Let $\mathcal{B} = \{v_1, v_2, v_3\}$ be a basis of B/F such that there exist $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3 \in \mathcal{J}_F$ satisfying $L_x = \bigoplus_{i=1}^3 \mathfrak{a}_i\, v_i$. Let $S \in \mathrm{GL}_3(F)$ be the matrix of the quadratic form $-\Delta$ with respect to this basis. For each $\tau \in \mathbf{a}$ write $\tau(S) = A_\tau^t\, A_\tau$, with $A_\tau \in \mathrm{GL}_3(\mathbb{R})$. Then there exist homogeneous harmonic polynomials $Q_\tau(X)$ of degree k_τ such that

$$P(y) = \prod_{\tau \in \mathbf{a}} Q_\tau(A_\tau\, \tau([y]_{\mathcal{B}}))\,,$$

where we denote by $[y]_{\mathcal{B}}$ the coordinates of y with respect to the basis $\mathcal{B}$. For each τ we may assume that $Q_\tau(X) = (z_\tau^t\, X)^{k_\tau}$, with $z_\tau \in \mathbb{C}^3$ such that $z_\tau^t\, z_\tau = 0$ (see [Iwa97, Theorem 9.1]). Let $\rho_\tau = (A_\tau^{-1}\, z_\tau)^t$. Then we have that $\rho_\tau^t\, \tau(S)\, \rho_\tau = 0$. Let $\sigma : F^3 \to \mathbb{C}$ be the function given by $\sigma(\xi) = \prod_{\tau \in \mathbf{a}} (\rho_\tau^t\, \tau(S)\, \tau(\xi))^{k_\tau}$, so that $P(y) = \sigma([y]_{\mathcal{B}})$, and let $\eta : F^3 \to \mathbb{C}$ be the characteristic function of $\mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3$. Then we have that

$$\vartheta_{x,P}(x) = \sum_{\xi \in F^3} \eta(\xi)\, \sigma(\xi)\, e_F(\xi^t\, S\, \xi, z/2)\,.$$

The modularity of $\vartheta_{x,P}$ follows applying [Shi87, Proposition 11.8]. Finally, (2.4) follows as in the case $\mathbf{k} = \mathbf{0}$ considered in [Sir14, Proposition 4.4]. □

The theta series $\vartheta_{x,P}$ defines a linear map $\vartheta : \mathcal{M}_{\mathbf{k}}(R) \to \mathcal{M}_{\mathbf{3/2+k}}(4\mathfrak{N}, \chi_1)$, given by $\vartheta(\varphi_{x,P}) = \vartheta_{x,P}$. This map is well-defined by (1.4) and (2.2) and satisfies

$$\vartheta(\varphi) = \sum_{x \in \mathrm{Cl}(R)} \tfrac{1}{w_x} \vartheta_{x,\varphi(x)}\,.$$

Note that if hypothesis H3 does not hold, then $\vartheta = 0$ by (2.1).

Proposition 2.5. *The map ϑ is Hecke-linear and satisfies that*

$$\vartheta(\varphi \cdot z) = \vartheta(\varphi) \quad \forall\, z \in \widetilde{\mathrm{Bil}}(R)\,.$$

Furthermore, $\vartheta(\varphi)$ is cuspidal if and only if φ is cuspidal.

Proof. For $z \in \widetilde{\mathrm{Bil}}(R)$ we have, by (1.3) and (2.2), that

$$\vartheta(\varphi_{x,P} \cdot z) = \vartheta(\varphi_{z^{-1}x,P}) = \vartheta(\varphi_{x,P}) .$$

The Hecke-linearity was proved in [Sir14, Theorem 4.11] when $\mathbf{k} = \mathbf{0}$ and can be proved in the general case following the same lines. The assertion on the cuspidality was proved in [Sir14, Theorem 4.11] when $\mathbf{k} = \mathbf{0}$, and in the remaining cases follows from the facts that $\mathcal{M}_{\mathbf{k}}(R) = \mathcal{S}_{\mathbf{k}}(R)$ and that every theta series is cuspidal. □

From now on D denotes an element in F^+, and we denote $K = F(\sqrt{-D})$. Assume that there exists an embedding $K \hookrightarrow B$, which we fix. Let $P_D \in V_{\mathbf{k}}$ be the polynomial characterized by the property

$$P(\omega) = \langle P, P_D \rangle \qquad \forall\, P \in V_{\mathbf{k}}\,, \tag{2.6}$$

where $\omega \in K/F$ is such that $\Delta(\omega) = -D$. Note that ω is uniquely determined up to sign. By hypothesis H3 we have $P(-\omega) = P(\omega)$ for every $P \in V_{\mathbf{k}}$, which implies that P_D does not depend on ω. Since $(P \cdot a)(\omega) = P(\omega)$ for every $a \in K^\times$, we have that

$$P_D \cdot a = P_D \quad \forall\, a \in K^\times/F^\times. \tag{2.7}$$

Proposition 2.8. *The polynomial P_D satisfies*

$$\langle P_D, P_D \rangle = D^{\mathbf{k}} \prod_{\tau \in \mathbf{a}} s_{k_\tau}\,, \tag{2.9}$$

where for $k \in \mathbb{Z}_{\geq 0}$ we denote by s_k the positive rational number given by

$$s_k = \frac{1}{\Gamma(k+1/2)} \sum_{q=0}^{\lfloor \frac{k}{2} \rfloor} \frac{\Gamma(k+1/2-q)}{q!\,(k-2q)!\,2^{2q}}\,. \tag{2.10}$$

Proof. We have that $P_D = \bigotimes_{\tau\in\mathbf{a}} P_{D,\tau}$, where $P_{D,\tau} \in V_{k_\tau}$ is the polynomial characterized by the property

$$P_\tau(\omega) = \langle P_\tau, P_{D,\tau} \rangle \qquad \forall\, P_\tau \in V_{k_\tau}\,.$$

Identifying B_τ with Hamilton quaternions $\langle 1, i, j, ij\rangle_{\mathbb{R}}$ and letting $X_1 = i/2$, $X_2 = j/2$, $X_3 = ij/2$, we have that $\{X_1, X_2, X_3\}$ is an orthonormal basis for W_τ with respect to $-\Delta$. Then the monomials $X_1^a\, X_2^b\, X_3^{k_\tau - a - b} \in \mathbb{C}_{k_\tau}[W_\tau]$ are orthogonal and have norm equal to $a!\, b!\, (k_\tau - a - b)!$, which implies that the inner product $\langle\,,\rangle$ we consider on V_{k_τ} is related to the inner product $\langle\!\langle\,,\rangle\!\rangle$ considered in [Geb09, Section 4.1] by $\langle\,,\rangle = k_\tau!\ \langle\!\langle\,,\rangle\!\rangle$. Hence (2.9) follows from the explicit formulas for the Gegenbauer polynomials given in [Geb09, Proposition 4.1.9], which imply that

$$\langle\!\langle P_{D,\tau}, P_{D,\tau} \rangle\!\rangle = \frac{\tau(D)^{k_\tau} s_{k_\tau}}{k_\tau!}\,.$$ □

Given $\mathfrak{a} \in \mathcal{J}_F$, we say that the pair $(-D, \mathfrak{a})$ is a *discriminant* if there exists $\omega \in K$ with $\Delta(\omega) = -D$ such that $\mathcal{O}_F \oplus \mathfrak{a}\,\omega$ is an order in K. In this case it is the unique order in K of discriminant $D\mathfrak{a}^2$, and in particular it does not depend on ω. We denote it by $\mathcal{O}_{D,\mathfrak{a}}$. We say that the discriminant $(-D, \mathfrak{a})$ is *fundamental* if $\mathcal{O}_{D,\mathfrak{a}} = \mathcal{O}_K$.

Proposition 2.11. *Let $\mathcal{O}$ be an $\mathcal{O}_F$-order in K. Then there exists a unique $\mathfrak{a} \in \mathcal{J}_F$ such that $(-D, \mathfrak{a})$ is a discriminant with $\mathcal{O}_{D,\mathfrak{a}} = \mathcal{O}$.*

Proof. Let r be an $\mathcal{O}_F$-linear retraction for the embedding $\mathcal{O}_F \hookrightarrow \mathcal{O}$, which we extend to an F-linear map $r : K \to F$. Let $\omega' \in K$ be any element satisfying $\Delta(\omega') = -D$, and let $\omega = \omega' - r(\omega')$. Then, since $\omega \notin F$, we have that $\ker r = F\omega$. Hence letting $\mathfrak{a} = (\mathcal{O}\omega^{-1}) \cap F$ we have that $\mathcal{O} = \mathcal{O}_F \oplus \mathfrak{a}\,\omega$. Note that the ideal $\mathfrak{a}$ is uniquely determined since $D\mathfrak{a}^2$ is the discriminant of $\mathcal{O}$. □

Proposition 2.12. *Let $\mathfrak{p}$ be a prime ideal. If $(-D, \mathfrak{a})$ is a discriminant, then $(-D, \mathfrak{p}\mathfrak{a})$ is a discriminant, and the converse is true if $\mathfrak{p} \nmid 2$ and $D \in \mathfrak{a}^{-2}$.*

Proof. The first statement is trivial. To prove the converse, assume that $\mathcal{O}_F \oplus \mathfrak{p}\mathfrak{a}\,\omega$ is an order in K, with $\Delta(\omega) = -D$. In particular, we have that $\mathcal{T}(\omega) \in (\mathfrak{p}\mathfrak{a})^{-1}$ and $\mathcal{N}(\omega) \in (\mathfrak{p}\mathfrak{a})^{-2}$. Since $\mathfrak{p} \nmid 2$, there exists $\xi \in \mathcal{O}_F$ such that $1 - 2\xi \in \mathfrak{p}$. Then, changing ω by $\omega - \xi\,\mathcal{T}(\omega)$, we may assume that $\mathcal{T}(\omega) \in \mathfrak{a}^{-1}$.

By hypothesis we have that $\Delta(\omega) = \mathcal{T}(\omega)^2 - 4\,\mathcal{N}(\omega) \in \mathfrak{a}^{-2}$. In particular, since $\mathcal{T}(\omega) \in \mathfrak{a}^{-1}$ we have that $4\,\mathcal{N}(\omega) \in \mathfrak{a}^{-2}$. Since $\mathfrak{p} \nmid 2$ and $\mathcal{N}(\omega) \in (\mathfrak{p}\mathfrak{a})^{-2}$ we have that $\mathcal{N}(\omega) \in \mathfrak{a}^{-2}$, which allows us to conclude that $\mathcal{O}_F \oplus \mathfrak{a}\,\omega$ is an order in K. □

Let $(-D, \mathfrak{a})$ be a discriminant, and let $\widetilde{X}_{D,\mathfrak{a}} = \{x \in \widehat{B}^\times : \mathcal{O}_{D,\mathfrak{a}} \subseteq R_x\}$. We define a set $X_{D,\mathfrak{a}}$ of *special points* associated to the discriminant $(-D, \mathfrak{a})$ by

$$X_{D,\mathfrak{a}} = \widehat{R}^\times \backslash \widetilde{X}_{D,\mathfrak{a}} / K^\times .$$

If $(-D, \mathfrak{a})$ is not a discriminant, we let $X_{D,\mathfrak{a}} = \varnothing$. Let

$$\eta_{D,\mathfrak{a}} = \sum_{x \in X_{D,\mathfrak{a}}} \frac{1}{[\mathcal{O}_x^\times : \mathcal{O}_F^\times]}\, \varphi_{x,P_D} \qquad \in \mathcal{M}_{\mathbf{k}}(R)\,,$$

where $\mathcal{O}_x = R_x \cap K$. This is well-defined by (1.2) and (2.7). When $(-D, \mathfrak{a})$ is fundamental then

$$\eta_{D,\mathfrak{a}} = \tfrac{1}{t_K} \sum_{x \in X_{D,\mathfrak{a}}} \varphi_{x,P_D}\,, \tag{2.13}$$

because in this case $\mathcal{O}_x = \mathcal{O}_K$ for every $x \in X_{D,\mathfrak{a}}$. It can be proved that $\eta_{D,\mathfrak{a}}$ does not depend on the choice of the embedding $K \hookrightarrow B$. When there does not exist such an embedding, we let $\eta_{D,\mathfrak{a}} = 0$.

Proposition 2.14. *Let $\varphi \in \mathcal{M}_{\mathbf{k}}(R)$. Let $D \in F^+$ and let $\mathfrak{a} \in \mathcal{J}_F$. Then the D-th Fourier coefficient of $\vartheta(\varphi)$ at the cusp $\mathfrak{a}$ is given by*

$$\lambda(D, \mathfrak{a}; \vartheta(\varphi)) = \frac{1}{\mathcal{N}(\mathfrak{a})}\, \langle \varphi, \eta_{D,\mathfrak{a}} \rangle\,. \tag{2.15}$$

Proof. By (1.4) we can assume that $\varphi = \varphi_{x,P}$ with $P \in V_{\mathbf{k}}^{\Gamma_x}$, so that $\vartheta(\varphi) = \vartheta_{x,P}$. If K does not embed into B, then $\mathcal{A}_{D,\mathfrak{a}}(x) = \varnothing$ and both sides of (2.15) vanish.

Fix $\omega \in K$ with $\Delta(\omega) = -D$, and let Γ_x act on $\mathcal{A}_{D,\mathfrak{a}}(x)$ by conjugation. Given $y \in \mathcal{A}_{D,\mathfrak{a}}(x)$, since $\Delta(y) = \Delta(\omega)$, we can assume there exists $\gamma \in B^\times$ such that $y = \gamma\omega\gamma^{-1}$. In particular $\mathcal{O}_{D,\mathfrak{a}} = \mathcal{O}_F \oplus \mathfrak{a}\,\omega$. The map $y \mapsto x\gamma$ induces an injection

$$\Gamma_x \backslash \mathcal{A}_{D,\mathfrak{a}}(x) \longrightarrow X_{D,\mathfrak{a}}\,.$$

Note that $\operatorname{Stab}_{\Gamma_x} y = (R_x \cap F(y))/\mathcal{O}_F^\times \simeq (R_{x\gamma} \cap K)/\mathcal{O}_F^\times = \mathcal{O}_{x\gamma}^\times/\mathcal{O}_F^\times$. Note also that $P(y) = \langle P \cdot \gamma, P_D \rangle = \frac{1}{w_x} \langle \varphi_{x,P}, \varphi_{x\gamma,P_D} \rangle$, using that P is fixed by Γ_x. Hence

$$\begin{aligned}\mathcal{N}(\mathfrak{a})\, \lambda(D,\mathfrak{a};\vartheta_{x,P}) &= \sum_{y \in \mathcal{A}_{D,\mathfrak{a}}(x)} P(y) = \sum_{y \in \Gamma_x \backslash \mathcal{A}_{D,\mathfrak{a}}(x)} [\Gamma_x : \operatorname{Stab}_{\Gamma_x} y]\, P(y) \\ &= \sum_{x\gamma \in X_{D,\mathfrak{a}}} \frac{w_x}{[\mathcal{O}_{x\gamma}^\times : \mathcal{O}_F^\times]} \langle P \cdot \gamma, P_D \rangle \\ &= \sum_{z \in X_{D,\mathfrak{a}}} \frac{1}{[\mathcal{O}_z^\times : \mathcal{O}_F^\times]} \langle \varphi_{x,P}, \varphi_{z,P_D} \rangle = \langle \varphi_{x,P}, \eta_{D,\mathfrak{a}} \rangle .\end{aligned}$$

Note that in the last sum $\langle \varphi_{x,P}, \varphi_{z,P_D} \rangle = 0$ unless $z = x\gamma$. □

By analogy with the case $F = \mathbb{Q}$ (see [Koh82]), we consider the *plus subspace* of $\mathcal{M}_{\mathbf{3/2+k}}(4\mathfrak{N},\chi_1)$, which, under hypothesis H3, is given by

$$\mathcal{M}^+_{\mathbf{3/2+k}}(4\mathfrak{N},\chi_1) = \big\{ f \in \mathcal{M}_{\mathbf{3/2+k}}(4\mathfrak{N},\chi_1) \ : \\ \lambda(D,\mathfrak{a};f) = 0 \text{ unless } (-D,\mathfrak{a}) \text{ is a discriminant} \big\},$$

and we let $\mathcal{S}^+_{\mathbf{3/2+k}}(4\mathfrak{N},\chi_1) = \mathcal{M}^+_{\mathbf{3/2+k}}(4\mathfrak{N},\chi_1) \cap \mathcal{S}_{\mathbf{3/2+k}}(4\mathfrak{N},\chi_1)$. Using the formula for the action of the Hecke operators in terms of Fourier coefficients (see [Shi87, Proposition 5.4]) together with Proposition 2.12 it is easy to prove that $\mathcal{M}^+_{\mathbf{3/2+k}}(4\mathfrak{N},\chi_1)$ is stable by the Hecke operators $T_\mathfrak{p}$ with $\mathfrak{p} \nmid 2$.

Corollary 2.16. *The Hecke-linear map ϑ sends $\mathcal{M}_\mathbf{k}(R)$ into $\mathcal{M}^+_{\mathbf{3/2+k}}(4\mathfrak{N},\chi_1)$.*

3. Height and geometric pairings

We start this section by comparing the geometric pairing on CM-cycles of [Zha01] (see [Xue06] for the case of higher weight) with the height pairing introduced in Section 1.

Let K/F be a totally imaginary quadratic extension. As in Section 2 we assume that there exists an embedding $K \hookrightarrow B$, which we fix. Furthermore, we assume $\mathcal{O}_K \subseteq R$. Let $\mathcal{C} = (\widehat{B}^\times/\widehat{F}^\times)/(K^\times/F^\times)$, and let $\pi : \widehat{B}^\times/\widehat{F}^\times \to \mathcal{C}$ be the projection map. We fix a Haar measure μ on $\widehat{B}^\times/\widehat{F}^\times$. On $K^\times/F^\times$ we consider the discrete measure, and we let $\bar{\mu}$ be the quotient measure on $\mathcal{C}$. We write $\mu_R = \mu\big(\widehat{R}^\times/\widehat{\mathcal{O}}_F^\times\big)$.

We consider the space $\mathcal{D}(\mathcal{C})$ of *CM-cycles* on $\mathcal{C}$. These are locally constant functions on $\mathcal{C}$ with compact support. This space comes equipped with the action of Hecke operators $T_\mathfrak{m}$ given by

$$T_\mathfrak{m}\, \alpha(x) = \tfrac{1}{\mu_R} \int_{H_\mathfrak{m}/\widehat{\mathcal{O}}_F^\times} \alpha(hx)\, dh. \tag{3.1}$$

Given $v \in V$ which is fixed by $K^\times/F^\times$, we let $M_v : B^\times/F^\times \to \mathbb{C}$ be the matrix coefficient given by $\gamma \mapsto \langle v \cdot \gamma, v \rangle$. Then M_v is bi-$K^\times/F^\times$-invariant and satisfies that $\overline{M_v(\gamma)} = M_v(\gamma^{-1})$. We call M_v a *multiplicity function.* We let $k_v : \mathcal{C} \times \mathcal{C} \to \mathbb{C}$ be the map given by

$$k_v(x,y) = \textstyle\sum_{\gamma \in \Gamma'_{x,y}} M_v(\gamma),$$

where for $x, y \in \widehat{B}^\times$ we denote $\Gamma'_{x,y} = (B^\times \cap x^{-1}\widehat{F}^\times \widehat{R}^\times y)/F^\times$. We consider the *geometric pairing* on $\mathcal{D}(\mathcal{C})$ induced by M_v, which for $\alpha, \beta \in \mathcal{D}(\mathcal{C})$ that are left invariant by $\widehat{R}^\times/\widehat{\mathcal{O}}_F^\times$ is given by

$$\langle \alpha, \beta \rangle_v = \tfrac{1}{\mu_R} \iint_{\mathcal{C}\times\mathcal{C}} \alpha(x)\,\overline{\beta(y)}\,k_v(x,y)\,dx\,dy\,. \tag{3.2}$$

Lemma 3.3. *Let $x, y \in \widehat{B}^\times$. The natural map $\Gamma_{x,y} \to \Gamma'_{x,y}$ is injective, and*

$$\Gamma'_{x,y} = \coprod_{\xi \in \mathrm{Cl}(F)} \Gamma_{\xi x, y}.$$

Proof. Let $u, v \in \widehat{R}^\times$ be such that there exists $\eta \in F^\times$ with $x^{-1}uy\eta = x^{-1}vy$. Then $\eta = u^{-1}v \in F^\times \cap \widehat{R}^\times = \mathcal{O}_F^\times$. This proves the first statement.

It is clear that the union gives all of $\Gamma'_{x,y}$. To see that it is disjoint, suppose that $\xi, \zeta \in \widehat{F}^\times$ are such that there exist $u, v \in \widehat{R}^\times$ and $\eta \in F^\times$ with $x^{-1}\xi u y \eta = x^{-1}\zeta v y$. Then $\xi\zeta^{-1}\eta = vu^{-1} \in \widehat{F}^\times \cap \widehat{R}^\times = \widehat{\mathcal{O}}_F^\times$, and hence $\xi = \zeta$ in $\mathrm{Cl}(F)$. □

The following result is immediate from this lemma and Proposition 1.6.

Proposition 3.4. *Let $x, y \in \widehat{B}^\times$. Then $k_v(x,y) = \sum_{\xi\in\mathrm{Cl}(F)} \langle \varphi_{\xi x, v}\,, \varphi_{y,v}\rangle$.*

Given $a \in \widehat{K}^\times$, we let $\alpha_a \in \mathcal{D}(\mathcal{C})$ be the characteristic function of $\pi\big(\widehat{R}^\times a\big) \subseteq \mathcal{C}$. Since $\mathcal{O}_K \subseteq R$, the CM-cycle α_a depends only on the element in $\mathrm{Cl}(K)$ determined by a. The same holds for the quaternionic modular form $\varphi_{a,v}$ by (1.2).

Proposition 3.5. *Let $\mathfrak{m} \subseteq \mathcal{O}_F$ be an ideal. For $a, b \in \mathrm{Cl}(K)$ we have that*

$$\frac{\langle T_\mathfrak{m}\,\alpha_a, \alpha_b\rangle_v}{\mu_R} = \tfrac{1}{t_K^2} \sum_{\xi\in\mathrm{Cl}(F)} \langle T_\mathfrak{m}\,\varphi_{\xi a, v}, \varphi_{b,v}\rangle\,.$$

Proof. Using (3.1) and (3.2), we obtain that

$$\begin{aligned}\frac{\langle T_\mathfrak{m}\,\alpha_a, \alpha_b\rangle_v}{\mu_R} &= \tfrac{1}{\mu_R^3} \iint_{\mathcal{C}\times\mathcal{C}} \int_{H_\mathfrak{m}/\widehat{\mathcal{O}}_F^\times} \alpha_a(hx)\,\alpha_b(y)\,k_v(x,y)\;dh\,dx\,dy\\ &= \tfrac{1}{\mu_R^3} \iint_{\pi(\widehat{R}^\times a)\times\pi(\widehat{R}^\times b)} \int_{H_\mathfrak{m}/\widehat{\mathcal{O}}_F^\times} k_v(h^{-1}x,y)\;dh\,dx\,dy.\end{aligned}$$

Note that

$$\tfrac{1}{\mu_R} \int_{H_\mathfrak{m}/\widehat{\mathcal{O}}_F^\times} k_v(h^{-1}x,y)\;dh = \sum_{h\in H_\mathfrak{m}/\widehat{R}^\times} k_v(h^{-1}x,y)$$

is constant on $\pi(\widehat{R}^\times a)\times\pi(\widehat{R}^\times b)$, and $\mu_R/\bar{\mu}\big(\pi(\widehat{R}^\times)\big) = |K^\times/F^\times \cap \widehat{R}^\times/\widehat{\mathcal{O}}_F^\times| = t_K$.

Using this and Proposition 3.4 we get that

$$\frac{\langle T_\mathfrak{m}\,\alpha_a, \alpha_b\rangle_v}{\mu_R} = \tfrac{1}{t_K^2} \sum_{h\in H_\mathfrak{m}/\widehat{R}^\times} k_v(h^{-1}a, b) = \tfrac{1}{t_K^2} \sum_{\xi\in\mathrm{Cl}(F)} \sum_{h\in H_\mathfrak{m}/\widehat{R}^\times} \big\langle \varphi_{h^{-1}\xi a, v}, \varphi_{b,v}\big\rangle.$$

Then the result follows from Proposition 1.5. □

Let $\alpha_K \in \mathcal{D}(\mathcal{C})$ be the characteristic function of $\pi(\widehat{R}^\times \widehat{K}^\times)$. We have that

$$\alpha_K = \tfrac{m_K}{h_F} \sum_{a\in\mathrm{Cl}(K)} \alpha_a\,.$$

Similarly we define

$$\psi_v = \tfrac{1}{t_K} \sum_{a\in\mathrm{Cl}(K)} \varphi_{a,v} \qquad \in \mathcal{M}_\rho(R, \mathbb{1})\,. \tag{3.6}$$

After these definitions and Proposition 3.5 we get the following result, analogous to [Xue06, Corollary 3.5], where the author only considers the case when $F = \mathbb{Q}$ and $\mathfrak{N}$ is square-free.

Corollary 3.7. *Let* $\mathfrak{m} \subseteq \mathcal{O}_F$ *be an ideal. Then*

$$\frac{\langle T_{\mathfrak{m}}\, \alpha_K, \alpha_K \rangle_v}{\mu_R} = \frac{m_K^2}{h_F}\, \langle T_{\mathfrak{m}}\, \psi_v, \psi_v \rangle \,.$$

Central values. Let g be a normalized Hilbert cuspidal newform over F of level $\mathfrak{N}$ and trivial central character as in the introduction. Write $K = F(\sqrt{-D})$ with $D \in F^+$, and denote by χ_D the Hecke character corresponding to the extension K/F. We assume that

$$\Sigma_D = \mathbf{a} \cup \left\{ \mathfrak{p} \mid \mathfrak{N} \,:\, \chi_D(\mathfrak{p})^{v_{\mathfrak{p}}(\mathfrak{N})} = -1 \right\} \tag{3.8}$$

is of even cardinality. For the rest of this section we let B be the quaternion algebra ramified exactly at Σ_D. Note that this satisfies the assumption that K embeds into B.

Let T_g be a polynomial in the Hecke operators prime to $\mathfrak{N}$ giving the g-isotypical projection. The following result is [Xue06, Theorem 1.2], which was originally proved for parallel weight **2** in [Zha01].

Theorem 3.9. *Assume* $\mathfrak{N} \subsetneq \mathcal{O}_F$ *and* $\mathfrak{D}_K$ *is prime to* $2\mathfrak{N}$. *Then*

$$L_D(1/2, g) = \langle g, g \rangle \, \frac{d_F^{1/2}\, C(\mathfrak{N})}{\mathcal{N}(\mathfrak{D}_K)^{1/2}} \, \frac{\langle\!\langle T_g\, \alpha_K, \alpha_K \rangle\!\rangle}{\mu_R} \,, \tag{3.10}$$

where $C(\mathfrak{N})$ *is the positive rational constant given by*

$$C(\mathfrak{N}) = \prod_{\mathfrak{p} \mid \mathfrak{N}} (\mathcal{N}(\mathfrak{p}) + 1)\, \mathcal{N}(\mathfrak{p})^{v_{\mathfrak{p}}(\mathfrak{N}) - 1} \,,$$

and where $\langle\!\langle \,,\, \rangle\!\rangle$ *denotes the geometric pairing in* $\mathcal{D}(\mathcal{C})$ *given in* [Xue06, (3.4)].

Remark 3.11. The constant C_1 mentioned in [Xue06, Theorem 1.2] contains a wrong factor, so we refer to [Xue06, (3.65)], The constants $\mu_{\mathfrak{N}\,\mathfrak{D}_K}$, μ_{Δ^*}, and μ_{Δ} appearing in the latter satisfy

$$\mu_{\mathfrak{N}\,\mathfrak{D}_K}^{-1} = C(\mathfrak{N}\,\mathfrak{D}_K) = C(\mathfrak{N})\, C(\mathfrak{D}_K) \,, \quad \mu_{\Delta^*} = C(\mathfrak{D}_K)\, \mu_{\Delta} = 2^{|S|}\, \mu_R \,.$$

Using this we obtain (3.10).

Remark 3.12. The proof given in [Xue06] is valid for a particular order in B containing $\mathcal{O}_K$. Since by [Gro88, Proposition 3.4] any two orders in B containing $\mathcal{O}_K$ are locally conjugate by an element of $\widehat{K}^\times$ and the right hand side of (3.10) is invariant by such a conjugation, it follows that Theorem 3.9 holds for any order in B containing $\mathcal{O}_K$.

Corollary 3.13. *Under the hypotheses above, assume that* $V = V_{\mathbf{k}}$ *as in Section 2, and let* $P_D \in V_{\mathbf{k}}$ *as in* (2.6). *Then*

$$L_D(1/2, g) = \langle g, g \rangle \, \frac{d_F^{1/2}}{h_F} \, \frac{c(\mathbf{k})\, C(\mathfrak{N})}{\mathcal{N}(\mathfrak{D}_K)^{1/2}} \, \frac{m_K^2}{D^{\mathbf{k}}} \, \langle T_g\, \psi_{P_D}, \psi_{P_D} \rangle \,.$$

Here $c(\mathbf{k})$ *is the positive rational constant given by* $c(\mathbf{k}) = \prod_{\tau\in\mathbf{a}} \frac{r_{k_\tau}}{s_{k_\tau}}$*, where for* $k \in \mathbb{Z}_{\geq 0}$ *we denote*

$$r_k = \frac{2^{2k+1}(k!)^2}{(2k)!}\,,$$

and s_k *is given by* (2.10)*.*

Proof. Follows from Corollary 3.7, Theorem 3.9, and the next lemma. □

Lemma 3.14. *Assume that* $V = V_{\mathbf{k}}$ *as in Section* 2*, and let* $P_D \in V_{\mathbf{k}}$ *as in* (2.6)*. Then*

$$\langle\!\langle \, , \rangle\!\rangle = \frac{c(\mathbf{k})}{D^{\mathbf{k}}} \langle \, , \rangle_{P_D}\,.$$

Proof. Let $M_\infty : B_\infty^\times/F_\infty^\times \to \mathbb{R}$ denote the multiplicity function considered in [Xue06, (3.9)]. Note that M_{P_D} factors through $B_\infty^\times/F_\infty^\times$, since the representation $(\rho_{\mathbf{k}}, V_{\mathbf{k}})$ does. Furthermore, M_{P_D} and M_∞ are, locally, the matrix coefficient of the (up to multiplication by scalars) unique vector in V_{k_τ} fixed by the action of $K_\tau^\times/F_\tau^\times$: the first claim follows by definition; for the second, see [Xue06, Lemma 3.13]. This implies $M_\infty = \frac{M_\infty(1)}{M_{P_D}(1)}\, M_{P_D}$.

Since $\langle\!\langle \, , \rangle\!\rangle$ is defined in the same fashion as $\langle \, , \rangle_{P_D}$ but using M_∞ instead of M_{P_D}, we have that

$$\langle\!\langle \, , \rangle\!\rangle = \frac{M_\infty(1)}{M_{P_D}(1)} \langle \, , \rangle_{P_D}\,.$$

Since $M_\infty(1) = \prod_{\tau\in\mathbf{a}} r_{k_\tau}$ and $M_{P_D}(1) = \langle P_D, P_D\rangle$, this together with (2.9) completes the proof. □

4. A result for certain orders

Assume in this section that $R \subseteq B$ is an order of discriminant $\mathfrak{N}$ satisfying that for every $\mathfrak{p} \mid \mathfrak{N}$ the Eichler invariant $e(R_\mathfrak{p})$ is not zero. If $e(R_\mathfrak{p}) = 1$, then

$$R_\mathfrak{p} \simeq \left\{ \begin{pmatrix} a & b \\ \pi_\mathfrak{p}^r c & d \end{pmatrix} \,:\, a,b,c,d \in \mathcal{O}_{F_\mathfrak{p}} \right\}, \tag{4.1}$$

where $r = v_\mathfrak{p}(\mathfrak{N})$. If $e(R_\mathfrak{p}) = -1$ and we let L be the unique unramified quadratic extension of $F_\mathfrak{p}$, then

$$R_\mathfrak{p} \simeq \left\{ \begin{pmatrix} a & \pi_\mathfrak{p}^r b \\ \pi_\mathfrak{p}^{r+t}\bar{b} & \bar{a} \end{pmatrix} \,:\, a,b \in \mathcal{O}_L \right\}, \tag{4.2}$$

where $t \in \{0,1\}$ and $2r + t = v_\mathfrak{p}(\mathfrak{N})$.

Proposition 4.3. *Let $\mathfrak{p}$ be a prime ideal of F, and let* $\mathrm{Bil}(R_\mathfrak{p}) = R_\mathfrak{p}^\times \backslash N(R_\mathfrak{p})/F_\mathfrak{p}^\times$.

(1) *If $\mathfrak{p} \nmid \mathfrak{N}$, then $\mathrm{Bil}(R_\mathfrak{p})$ is the trivial group.*
(2) *If $\mathfrak{p} \mid \mathfrak{N}$, then $\mathrm{Bil}(R_\mathfrak{p})$ is a group of order two generated by the equivalence class of an element $w_\mathfrak{p} \in R_\mathfrak{p} \cap N(R_\mathfrak{p})$, which, in terms of the identifications given by* (4.1) *and* (4.2), *is given by*

$$w_\mathfrak{p} = \begin{cases} \begin{pmatrix} 0 & 1 \\ \pi_\mathfrak{p}^r & 0 \end{pmatrix}, & \text{if } e(R_\mathfrak{p}) = 1, \\ \begin{pmatrix} 0 & \pi_\mathfrak{p}^r \\ \pi_\mathfrak{p}^{r+t} & 0 \end{pmatrix}, & \text{if } e(R_\mathfrak{p}) = -1. \end{cases}$$

Proof.

(1) See [Vig80, II.§4, Théorème 2.3].
(2) See [Hij74, (2.2)] and [Piz76, Proposition 3] for the cases $e(R_\mathfrak{p}) = 1$ and $e(R_\mathfrak{p}) = -1$ respectively. In the latter the author considers the case when $t = 1$, but the proof is valid in the general case. □

From these local facts and (1.1) we get the following statement.

Proposition 4.4. *The group $\mathrm{Bil}(R)$ is isomorphic to $\prod_{\mathfrak{p}\mid\mathfrak{N}} \mathbb{Z}/2\mathbb{Z}$, and $\widetilde{\mathrm{Bil}}(R)$ is a finite group of order $h_F\, 2^{\omega(\mathfrak{N})}$.*

Let $D \in F^+$. Let $K = F(\sqrt{-D})$. By Proposition 2.11 there exists a unique $\mathfrak{a} \in \mathcal{J}_F$ such that $(-D, \mathfrak{a})$ is a fundamental discriminant. Since $\mathfrak{a}$ is determined by D, we omit it in the subindexes for the rest of this section.

As in Section 3, we assume that there exists an embedding $K \hookrightarrow B$ such that $\mathcal{O}_K \subseteq R$, i.e., such that $1 \in \widetilde{X}_D$. There is a left action of $\widetilde{\mathrm{Bil}}(R)$ on X_D, induced by the action of $N(\widehat{R})$ on $\widetilde{X}_D$ by left multiplication. There is also a right action of $\mathrm{Cl}(K) = \widehat{\mathcal{O}}_K^\times \backslash \widehat{K}^\times / K^\times$ on X_D, induced by the action of $\widehat{K}^\times$ on $\widetilde{X}_D$ by right multiplication.

Lemma 4.5. *Let* $X_{D,\mathfrak{p}} = \{x_\mathfrak{p} \in B_\mathfrak{p}^\times \,:\, K_\mathfrak{p} \cap x_\mathfrak{p}^{-1} R_\mathfrak{p} x_\mathfrak{p} = \mathcal{O}_{K_\mathfrak{p}}\}$.

(1) *The action of $\mathcal{O}_{K_\mathfrak{p}}^\times \backslash K_\mathfrak{p}^\times$ on $R_\mathfrak{p}^\times \backslash X_{D,\mathfrak{p}}$ is free.*
(2) $X_{D,\mathfrak{p}} = N(R_\mathfrak{p}) K_\mathfrak{p}^\times$.

Proof.

(1) Let $a_\mathfrak{p} \in K_\mathfrak{p}^\times$ and $x_\mathfrak{p} \in X_{D,\mathfrak{p}}$ be such that there exists $u_\mathfrak{p} \in R_\mathfrak{p}^\times$ with $x_\mathfrak{p} a_\mathfrak{p} = u_\mathfrak{p} x_\mathfrak{p}$. Then $a_\mathfrak{p} = x_\mathfrak{p}^{-1} u_\mathfrak{p} x_\mathfrak{p} \in K_\mathfrak{p} \cap x_\mathfrak{p}^{-1} R_\mathfrak{p}^\times x_\mathfrak{p} = \mathcal{O}_{K_\mathfrak{p}}^\times$.
(2) Given $x_\mathfrak{p} \in X_{D,\mathfrak{p}}$, let $Q_\mathfrak{p} = x_\mathfrak{p}^{-1} R_\mathfrak{p} x_\mathfrak{p}$. Since $R_\mathfrak{p}$ and $Q_\mathfrak{p}$ contain $\mathcal{O}_{K_\mathfrak{p}}$ and have the same discriminant, by [Gro88, Proposition 3.4] there exists $a_\mathfrak{p} \in K_\mathfrak{p}^\times$ such that $a_\mathfrak{p}^{-1} R_\mathfrak{p} a_\mathfrak{p} = Q_\mathfrak{p}$. Then $x_\mathfrak{p} \in N(R_\mathfrak{p}) a_\mathfrak{p}$. □

Lemma 4.6. *Let $\mathfrak{p} \mid \mathfrak{N}$. Let $w_\mathfrak{p} \in N(R_\mathfrak{p})$ be as in Proposition* 4.3. *If $w_\mathfrak{p} \in R_\mathfrak{p}^\times K_\mathfrak{p}^\times$, then the extension $K_\mathfrak{p}/F_\mathfrak{p}$ is ramified.*

Proof. Write $w_\mathfrak{p} = u_\mathfrak{p}\, a_\mathfrak{p}$ with $u_\mathfrak{p} \in R_\mathfrak{p}^\times$ and $a_\mathfrak{p}$ in $K_\mathfrak{p}^\times$. Then $a_\mathfrak{p} \in \mathcal{O}_{K_\mathfrak{p}}$. Using the explicit description of $w_\mathfrak{p}$ given in Proposition 4.3 we see that $\pi_\mathfrak{p} \nmid a_\mathfrak{p}$ in $\mathcal{O}_{K_\mathfrak{p}}$. Furthermore, we see that $\pi_\mathfrak{p} \mid \mathcal{T}(a_\mathfrak{p}), \mathcal{N}(a_\mathfrak{p})$ in $\mathcal{O}_{F_\mathfrak{p}}$, hence $\pi_\mathfrak{p} \mid a_\mathfrak{p}^2$ in $\mathcal{O}_{K_\mathfrak{p}}$. Thus $\pi_\mathfrak{p}$ is ramified in $K_\mathfrak{p}$. □

As a consequence of these lemmas and Proposition 4.3 we obtain the following result.

Proposition 4.7. *The group* $\mathrm{Cl}(K)$ *acts freely on* X_D*, and the action of* $\mathrm{Bil}(R)$ *on* $X_D/\mathrm{Cl}(K)$ *is transitive. Furthermore, the latter action is free if* $(\mathfrak{D}_K : \mathfrak{N}) = 1$.

Let $\eta_D \in \mathcal{M}_{\mathbf{k}}(R)$ be as in (2.13) and let $\psi_{P_D} \in \mathcal{M}_{\mathbf{k}}(R, \mathbb{1})$ be as in (3.6). We conclude this section by relating these quaternionic modular forms.

Proposition 4.8. *Assume that* $(\mathfrak{D}_K : \mathfrak{N}) = 1$. *Then*

$$\eta_D = \sum_{z \in \mathrm{Bil}(R)} \psi_{P_D} \cdot z \,.$$

In particular, $\eta_D \in \mathcal{M}_{\mathbf{k}}(R, \mathbb{1})^{\mathrm{Bil}(R)}$.

Proof. Since $1 \in \widetilde{X}_D$, using (1.3) and Proposition 4.7 we get that

$$\sum_{z \in \mathrm{Bil}(R)} \psi_{P_D} \cdot z = \tfrac{1}{t_K} \sum_{z \in \mathrm{Bil}(R)} \sum_{a \in \mathrm{Cl}(K)} \varphi_{z^{-1}a, P_D} = \tfrac{1}{t_K} \sum_{x \in X_D} \varphi_{x, P_D} = \eta_D \,.$$

□

The following statement follows from this result and Proposition 4.4.

Corollary 4.9. *Assume that* $(\mathfrak{D}_K : \mathfrak{N}) = 1$. *If* $\varphi \in \mathcal{M}_{\mathbf{k}}(R, \mathbb{1})^{\mathrm{Bil}(R)}$, *then*

$$\langle \varphi, \eta_D \rangle = 2^{\omega(\mathfrak{N})} \langle \varphi, \psi_{P_D} \rangle \,.$$

5. Main theorem

Let $\mathbf{k} \in \mathbb{Z}_{\geq 0}^{\mathbf{a}}$, let $\mathfrak{N} \subsetneq \mathcal{O}_F$ be an integral ideal, and let $g \in \mathcal{S}_{\mathbf{2}+2\mathbf{k}}(\mathfrak{N}, \mathbb{1})$ be a normalized cuspidal newform with Atkin-Lehner eigenvalues $\varepsilon_g(\mathfrak{p})$ for $\mathfrak{p} \mid \mathfrak{N}$, as in the introduction. Let $\mathscr{E}$ denote the set of functions $\varepsilon : \{\mathfrak{p} : \mathfrak{p} \mid \mathfrak{N}\} \to \{\pm 1\}$ satisfying

$$\varepsilon(\mathfrak{p})^{v_\mathfrak{p}(\mathfrak{N})} = \varepsilon_g(\mathfrak{p}) \qquad \forall\, \mathfrak{p} \mid \mathfrak{N} \,. \tag{5.1}$$

Note that this set is not empty. This is equivalent to hypothesis H2.

Given $D \in F^+$ we let $K = F(\sqrt{-D})$, and we denote by χ_D the Hecke character corresponding to the extension K/F. Given $\varepsilon \in \mathscr{E}$ we say that D is of *type* ε when $\chi_D(\mathfrak{p}) = \varepsilon(\mathfrak{p})$ for all $\mathfrak{p} \mid \mathfrak{N}$. In particular the conductor of χ_D is prime to $\mathfrak{N}$. Hypothesis H1 implies that for such D the sign of the functional equation for $L_D(s, g)$ equals 1.

Let B be the quaternion algebra over F ramified exactly at $\mathbf{a} \cup \mathcal{W}^-$, which is possible by hypothesis H1. Fix $\varepsilon \in \mathscr{E}$, and let $R = R_\varepsilon \subseteq B$ be an order with discriminant $\mathfrak{N}$ and Eichler invariant $e(R_\mathfrak{p}) = \varepsilon(\mathfrak{p})$ for every $\mathfrak{p} \mid \mathfrak{N}$. Such order exists by (5.1) and belongs to the class of orders considered in Section 4.

Note that for D of type ε the set Σ_D given in (3.8) is precisely the ramification of B and moreover $\mathcal{O}_K \subseteq R$, as required by Theorem 3.9 and Corollary 3.13.

Let π be the irreducible automorphic representation of GL_2 corresponding to g. For every prime $\mathfrak{p}$ where B is ramified $v_\mathfrak{p}(\mathfrak{N})$ is odd by hypothesis H2; hence the local component of π at $\mathfrak{p}$ is square integrable. It follows that there is an irreducible automorphic representation π_B of $\widehat{B}^\times$ which corresponds to π under the Jacquet-Langlands map.

In [Gro88, Proposition 8.6] it is shown that $\widehat{R}^\times$ fixes a unique line in the representation space of π_B. This line gives an explicit quaternionic modular form $\varphi_\varepsilon \in \mathcal{S}_{\mathbf{k}}(R, \mathbb{1})$, which is well-defined up to a constant.

Lemma 5.2. *The quaternionic modular form φ_ε is fixed by the action of* $\mathrm{Bil}(R)$.

Proof. Let $\mathfrak{p}$ be a prime dividing $\mathfrak{N}$, and let $w_\mathfrak{p} \in N(R_\mathfrak{p})$ be the generator for $\mathrm{Bil}(R_\mathfrak{p})$ given in Proposition 4.3. Since $w_\mathfrak{p}$ has order two and normalizes $\widehat{R}^\times$, it acts on φ_ε by multiplication by $\delta_\mathfrak{p} \in \{\pm 1\}$.

When B is split at $\mathfrak{p}$ we have $\delta_\mathfrak{p} = \varepsilon_g(\mathfrak{p})$, and $\delta_\mathfrak{p} = -\varepsilon_g(\mathfrak{p})$ when B is ramified at $\mathfrak{p}$ (for instance, see [Rob89, Theorem 2.2.1]). Thus $\delta_\mathfrak{p} = 1$ for every $\mathfrak{p} \mid \mathfrak{N}$ by our choice of B, and the result follows since $\{w_\mathfrak{p} : \mathfrak{p} \mid \mathfrak{N}\}$ generates $\mathrm{Bil}(R)$. □

Let c_g be the positive real number given by

$$c_g = \langle g, g\rangle \, \frac{d_F^{1/2}}{h_F} \, \frac{c(\mathbf{k})\, C(\mathfrak{N})}{2^{2\omega(\mathfrak{N})}} ,$$

where $\langle g, g\rangle$ is the Petersson norm of g, $c(\mathbf{k})$ is as in Corollary 3.13, and $C(\mathfrak{N})$ is as in Theorem 3.9.

Theorem 5.3. *Let $f_\varepsilon = \vartheta(\varphi_\varepsilon) \in \mathcal{S}^+_{\mathbf{3/2+k}}(4\mathfrak{N}, \chi_1)$. For every $D \in F^+$ of type ε such that the conductor of χ_D is prime to $2\mathfrak{N}$ we have*

$$L_D(1/2, g) = c_g \, \frac{c_D}{D^{\mathbf{k+1/2}}} \, \frac{|\lambda(D, \mathfrak{a}; f_\varepsilon)|^2}{\langle \varphi_\varepsilon, \varphi_\varepsilon\rangle} , \tag{5.4}$$

where $\mathfrak{a} \in \mathcal{J}_F$ is the unique ideal such that $(-D, \mathfrak{a})$ is a fundamental discriminant, c_D is the positive rational number given by $c_D = m_K^2 \mathcal{N}(\mathfrak{a})$, and $\lambda(D, \mathfrak{a}; f_\varepsilon)$ is the D-th Fourier coefficient of f_ε at the cusp $\mathfrak{a}$.

Remark 5.5. Hypothesis H1 implies that the sign of the functional equation for $L(s, g)$ equals $(-1)^{\mathbf{k}}$. If hypothesis H3 does not hold, then both sides of (5.4) vanish trivially, since $\vartheta = 0$ and $L(1/2, g) = 0$. In particular (5.4) still holds, but it cannot be used to compute $L(1/2, g \otimes \chi_D)$. This issue will be addressed in a future work by the authors.

Proof. Let T_g be the polynomial in the Hecke operators prime to $\mathfrak{N}$ giving the g-isotypical projection. Let ψ_{P_D} and η_D be as in Corollary 4.9. Since $T_g\, \psi_{P_D}$ is the φ_ε-isotypical projection of ψ_{P_D} we have that $T_g\, \psi_{P_D} = \frac{\langle \psi_{P_D}, \varphi_\varepsilon\rangle}{\langle \varphi_\varepsilon, \varphi_\varepsilon\rangle} \varphi_\varepsilon$. Combining this with Proposition 2.14, Corollary 4.9, and Lemma 5.2 we get that

$$\langle T_g\, \psi_{P_D}, \psi_{P_D}\rangle = \frac{|\langle \psi_{P_D}, \varphi_\varepsilon\rangle|^2}{\langle \varphi_\varepsilon, \varphi_\varepsilon\rangle} = \frac{|\langle \eta_D, \varphi_\varepsilon\rangle|^2}{2^{2\omega(\mathfrak{N})}\, \langle \varphi_\varepsilon, \varphi_\varepsilon\rangle} = \frac{\mathcal{N}(\mathfrak{a})^2}{2^{2\omega(\mathfrak{N})}} \, \frac{|\lambda(D, \mathfrak{a}; f_\varepsilon)|^2}{\langle \varphi_\varepsilon, \varphi_\varepsilon\rangle} .$$

Then (5.4) follows from Corollary 3.13. □

Corollary 5.6. *Assume that $L(1/2, g) \neq 0$. Then $f_\varepsilon \neq 0$ and it maps to g under the Shimura correspondence. Moreover, the set $\{f_\varepsilon : \varepsilon \in \mathscr{E}\}$ is linearly independent.*

In particular, this proves [Sir14, Conjecture 5.6].

Proof. By hypotheses H1 and H3 the sign of the functional equation for $L(s, g)$ equals 1. Hence by [Wal91, Théorème 4] for every $\varepsilon \in \mathscr{E}$ there exists $D_\varepsilon \in F^+$ of type ε with $\mathfrak{D}_\varepsilon = \mathfrak{D}_{F(\sqrt{-D_\varepsilon})}$ prime to $2\mathfrak{N}$ such that $L(1/2, g \otimes \chi_{D_\varepsilon}) \neq 0$. Then by (5.4) we have that $\lambda(D_\varepsilon, \mathfrak{a}_\varepsilon; f_\varepsilon) \neq 0$, where $(-D_\varepsilon, \mathfrak{a}_\varepsilon)$ is the discriminant satisfying $D_\varepsilon \mathfrak{a}_\varepsilon^2 = \mathfrak{D}_\varepsilon$. This, together with the Hecke-linearity of the map ϑ, proves the the first assertion. The second assertion follows from the fact that if $\varepsilon' \neq \varepsilon$, then $\lambda(D_\varepsilon, \mathfrak{a}_\varepsilon; f_{\varepsilon'}) = 0$. □

We say that $D \in F^+$ is *permitted* if the conductor of χ_D is prime to $2\mathfrak{N}$ and $\chi_D(\mathfrak{p}) = \varepsilon_g(\mathfrak{p})$ for all $\mathfrak{p} \mid \mathfrak{N}$ such that $v_{\mathfrak{p}}(\mathfrak{N})$ is odd. By hypothesis H2, every permitted D is of type ε for some $\varepsilon \in \mathscr{E}$.

Corollary 5.7. *There exists $f \in \mathcal{S}^+_{\mathbf{3/2+k}}(4\mathfrak{N}, \chi_1)$ whose Fourier coefficients satisfy*

$$L_D(1/2, g) = \frac{c_D}{D^{\mathbf{k+1/2}}} \left|\lambda(D, \mathfrak{a}; f)\right|^2$$

for every permitted D, where $\mathfrak{a} \in \mathcal{J}_F$ is the unique ideal such that $(-D, \mathfrak{a})$ is a fundamental discriminant. Moreover, if $L(1/2, g) \neq 0$, then $f \neq 0$ and it maps to g under the Shimura correspondence.

In particular, this proves [RTV14, Conjecture 2.8].

Proof. This follows from Theorem 5.3 and Corollary 5.6, letting

$$f = c_g^{1/2} \sum_{\varepsilon \in \mathscr{E}} \frac{f_\varepsilon}{\langle \varphi_\varepsilon, \varphi_\varepsilon \rangle^{1/2}} . \qquad \square$$

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