TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 9, September 2018, Pages 6245–6295
http://dx.doi.org/10.1090/tran/7119
Article electronically published on February 1, 2018

ON THE IRREDUCIBILITY OF GLOBAL DESCENTS FOR EVEN UNITARY GROUPS AND ITS APPLICATIONS

KAZUKI MORIMOTO

Abstract. In this paper, we prove the irreducibility of global descents for even unitary groups. More generally, through Fourier-Jacobi coefficients of automorphic forms, we give a bijection between a certain set of irreducible cuspidal automorphic representations of $\mathrm{U}(n,n)(\mathbb{A})$ and a certain set of irreducible square-integrable automorphic representations of $\mathrm{U}(2n,2n)(\mathbb{A})$. We also give three applications of the irreducibility of global descents. As a global application, we prove a rigidity theorem for irreducible generic cuspidal automorphic representations of $\mathrm{U}(n,n)$. Moreover, as a local application, we prove the irreducibility of explicit local descents for a couple of supercuspidal representations and a local converse theorem for generic representations in the case of $\mathrm{U}(n,n)$.

1. Introduction

Functorial lifts of automorphic representations for classical groups to appropriate general linear groups with respect to standard representations have been studied in various situations. For example, in Kim-Krishnamurthy [KK04], [KK05], they constructed such functorial lifts for irreducible generic cuspidal automorphic representations of quasi-split unitary groups using the converse theorem as in Cogdell–Kim–Piatetski-Shapiro–Shahidi [CKPSS01], [CKPSS04] and Cogdell–Piatetski-Shapiro–Shahidi [CPSS11]. Recently, as an analogue of Arthur [Ar13], Mok [Mo15] established the endoscopic classification for irreducible tempered cuspidal automorphic representations of quasi-split unitary groups using trace formulas, and he obtained functorial lifts for such automorphic representations under certain assumptions.

Because of these functorial lifts, we can transfer some questions, such as analytic properties of L-functions, of automorphic representations of classical groups to that of general linear groups. On the other hand, in order to pull back a result of automorphic representations of $\mathrm{GL}(n)$ to that of classical groups, we need an inverse map of the functorial lifts. As a sort of such inverse maps, Ginzburg–Rallis–Soudry [GRS11] established the theory of global descent, which gives a generic cuspidal automorphic representation of quasi-split classical groups for given automorphic representations of general linear groups satisfying certain conditions. At the present time, the global descent method becomes one of the most important theories to study automorphic representations for classical groups and their L-functions. However, it seems that the theory of global descents is not complete for unitary groups. Indeed, we do not have the irreducibility of global descents in this

Received by the editors October 3, 2016, and, in revised form, November 3, 2016.
2010 *Mathematics Subject Classification.* Primary 11F70; Secondary 11F30.
The research of the author was supported in part by Grant-in-Aid for JSPS Fellow (26-1158) and Grant-in-Aid for Young Scientists (B) 26800021.

case. One of our main purposes of the present paper is to prove this irreducibility in the case of even unitary groups.

In order to state our main global results, let us introduce some notation. Let F be a number field and let E be a quadratic extension of F. We denote the adeles of F and E by $\mathbb{A}_F$ and $\mathbb{A}_E$, respectively. We often simply write $\mathbb{A} = \mathbb{A}_F$. Let $\omega_{E/F}$ be the quadratic character of $\mathbb{A}_F^\times/F^\times$ corresponding to E/F. We denote by $G_n := \mathrm{U}(n,n) \subset \mathrm{Res}_{E/F}\mathrm{GL}_{2n}$ the quasi-split unitary group of degree $2n$ with respect to E/F. Let ψ_F be a non-trivial additive character of $\mathbb{A}_F/F$ and define a character ψ of $\mathbb{A}_E/E$ by $\psi(x) = \psi_F\left(\frac{x+\bar{x}}{2}\right)$ where $x \mapsto \bar{x}$ is the action of the non-trivial element of $\mathrm{Gal}(E/F)$.

Let $[n_i] = (n_1, \dots, n_r)$ be a partition of $2n$; i.e., n_i are positive integers such that $2n = n_1 + \cdots + n_r$, and $\bar{\tau} = (\tau_1, \dots, \tau_r)$ with an irreducible unitary cuspidal automorphic representation τ_i of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ such that $\tau_i \not\simeq \tau_j$ if $i \neq j$, and $L(s, \tau_i, \mathrm{Asai})$ has a pole at $s = 1$. Then we consider Eisenstein series $E(f_{\bar{\tau},\bar{s}}, g)$ on $G_{2n}(\mathbb{A})$ with respect to a parabolic subgroup whose Levi part is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{E/F}\mathrm{GL}_{n_r}$. We know that this Eisenstein series has a pole at $\bar{s} = (\frac{1}{2}, \dots, \frac{1}{2})$. Then we define an irreducible residual representation $\mathcal{E}_{\bar{\tau}}$ of $G_{2n}(\mathbb{A})$ by the residues of those Eisenstein series.

For an automorphic form φ on $G_{2n}(\mathbb{A})$, a Fourier-Jacobi coefficient $\mathrm{FJ}_{\phi,n}^{\psi,\eta}(\varphi)(\cdot)$ (see Section 3.2 for the definition) gives an automorphic form on $G_n(\mathbb{A})$ where η is a character of $\mathbb{A}_E^\times/E^\times$ such that $\eta|_{\mathbb{A}_F^\times} = \omega_{E/F}$ and $\phi \in \mathcal{S}(\mathbb{A}_E^n)$. Then for an automorphic representation Π of $G_{2n}(\mathbb{A})$, we may define a global descent $\mathcal{D}_{2n,\psi}^{4n,\eta}(\Pi)$ which is the automorphic representation of $G_n(\mathbb{A})$ generated by $\mathrm{FJ}_{\phi,n}^{\psi,\eta}(\varphi_\Pi)(\cdot)$ for any $\varphi_\Pi \in \Pi$ and $\phi \in \mathcal{S}(\mathbb{A}_E^n)$. In the case $\Pi = \mathcal{E}_{\bar{\tau}}$, Ginzburg–Rallis–Soudry [GRS11, Theorem 3.1] showed fundamental properties of $\mathcal{D}_{2n,\psi}^{4n,\eta}(\mathcal{E}_{\bar{\tau}})$. For example, it is cuspidal and ψ^{-1}-generic. See Section 3.1 for the definition of ψ^{-1}-generic representation. One of the main results of this paper is the irreducibility of global descents.

Theorem 1.1. *The global descent $\mathcal{D}_{2n,\psi}^{4n,\eta}(\mathcal{E}_{\bar{\tau}})$ is an irreducible cuspidal ψ^{-1}-generic automorphic representation of $G_n(\mathbb{A})$.*

We note that in the case of metaplectic groups and odd special orthogonal groups, Jiang and Soudry [JS03] proved the irreducibility of global descents as a consequence of a local converse theorem.

In this paper, we shall prove this irreducibility by a similar argument as [GJS12]. Indeed, as in [GJS12], we shall prove such irreducibility in a more general setting; namely, we shall study the global descents not only for $\mathcal{E}_{\bar{\tau}}$ but also for a certain family of automorphic representations of $G_{2n}(\mathbb{A})$ which are nearly equivalent to $\mathcal{E}_{\bar{\tau}}$.

Let $\mathcal{N}_{2n}(\bar{\tau}, \eta, \psi)$ be the set of irreducible automorphic representations of $G_{2n}(\mathbb{A})$ such that they have non-trivial image under the descent map $\mathcal{D}_{2n,\psi}^{4n,\eta}(\cdot)$, appear in the discrete spectrum, and are nearly equivalent to $\mathcal{E}_{\bar{\tau}}$. We note that from [GRS11, Theorem 3.1], $\mathcal{E}_{\bar{\tau}}$ is contained in $\mathcal{N}_{2n}(\bar{\tau}, \eta, \psi)$. We denote by $\mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi)$ the subset consisting of cuspidal representations in $\mathcal{N}_{2n}(\bar{\tau}, \eta, \psi)$ and $\mathcal{E}_{\bar{\tau}}$. Then Theorem 1.1 is a special case of the following result (see Theorem 8.1).

Theorem 1.2. *Let $\Pi \in \mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi)$. Then the global descent $\mathcal{D}_{2n,\psi}^{4n,\eta}(\Pi)$ is an irreducible cuspidal automorphic representation of $G_n(\mathbb{A})$.*

In [GJS12], they proved a similar result for metaplectic groups when $r = 1$. The case of $r = 1$ is not enough to study any automorphic representation of classical groups and metaplectic groups. Indeed, the image of functorial lifts for those groups is an isobaric sum of certain cuspidal automorphic representations, and it is not cuspidal in general. We shall prove the above theorem extending their argument to the case $r \geq 1$. Main ingredients of our proof are computations of Fourier-Jacobi coefficients and Fourier coefficients of residual representations.

Our proof is roughly as follows. Let $\mathcal{N}_n(\bar{\tau}, \eta)$ be the set of irreducible cuspidal automorphic representations of $G_n(\mathbb{A})$, which weakly lift to $\boxplus(\tau_i \otimes \eta^{-1})$. Then we can consider a residual automorphic representation $\mathcal{E}_{\bar{\tau},\pi}$ of $G_{3n}(\mathbb{A})$ associated to $\bar{\tau}$ and $\pi \in \mathcal{N}_n(\bar{\tau}, \eta)$ with respect to a parabolic subgroup whose Levi part is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{E/F}\mathrm{GL}_{n_r} \times G_n$. By an explicit computation of Fourier-Jacobi coefficients, we shall show that (see Proposition 7.2)

$$\mathcal{D}^{4n,\eta}_{2n,\psi}\left(\mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi})\right) = \pi \otimes \eta^{-1},$$

where $\mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi})$ (resp. $\mathcal{D}^{4n,\eta}_{2n,\psi}\left(\mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi})\right)$) is the automorphic representation of $G_{2n}(\mathbb{A})$ (resp. $G_n(\mathbb{A})$) given by certain Fourier-Jacobi coefficients. Further, by an explicit computation of Fourier-Jacobi coefficients and Fourier coefficients, we shall show that for a given $\Pi \in \mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi)$, there exists $\pi_0 \in \mathcal{N}_n(\bar{\tau}, \eta)$ such that (see Proposition 7.7 and Theorem 7.8)

$$\Pi \otimes \eta^{-1} \subset \mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi_0}). \tag{1.1}$$

From these equations, the above irreducibility readily follows.

Because of the irreducibility in Theorem 1.2, we can define a map from $\mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi)$ to $\mathcal{N}_n(\bar{\tau}, \eta)$. We shall refine the above result under a certain assumption; indeed we shall prove that this map becomes a bijection. For simplicity, we write $\Phi(\pi) := \mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi})$ with $\pi \in \mathcal{N}_n(\bar{\tau}, \eta)$ and $\Psi(\Pi) = \mathcal{D}^{4n,\eta}_{2n,\psi}(\Pi)$ with $\Pi \in \mathcal{N}_{2n}(\bar{\tau}, \eta, \psi)$. We also write $\Phi'(\pi) := \Phi(\pi) \otimes \eta$.

Theorem 1.3. *For each* $\Pi \in \mathcal{N}_{2n}(\bar{\tau}, \eta, \psi)$, Π *is a subrepresentation of* $\Phi'(\Psi(\Pi))$, *which is an inclusion as spaces of square-integrable automorphic forms. Moreover, we assume that* $\mathcal{E}_{\bar{\tau},\pi}$ *is irreducible for any* $\pi \in \mathcal{N}_n(\bar{\tau}, \eta)$. *Then for* $\Pi \in \mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi)$,

$$\Phi(\Psi(\Pi)) = \Pi \otimes \eta^{-1}, \quad \textit{i.e.,} \quad \Phi'(\Psi(\Pi)) = \Pi.$$

In particular, $\Phi(\pi)$ *is irreducible, and thus the mappings*

$$\Psi : \mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi) \to \mathcal{N}_n(\bar{\tau}, \eta)$$

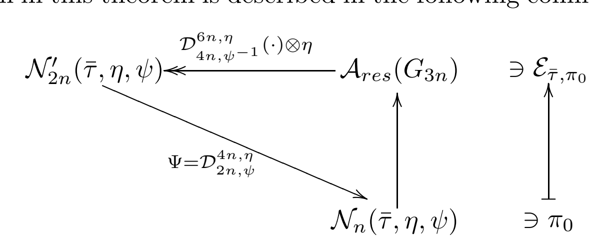
and

$$\Phi' : \mathcal{N}_n(\bar{\tau}, \eta) \to \mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi)$$

are bijective and satisfy

$$\Psi \circ \Phi' = \mathrm{Id}_{\mathcal{N}_n(\bar{\tau},\eta)}, \quad \Phi' \circ \Psi = \mathrm{Id}_{\mathcal{N}'_{2n}(\bar{\tau},\eta,\psi)}.$$

The bijection in this theorem is described in the following commutative diagram:



where $\mathcal{A}_{res}(G_{3n})$ is the space of residual automorphic forms of $G_{3n}(\mathbb{A}_F)$, and the composition of the vertical map and the horizontal map is Φ'. We also note that the existence of automorphic representation in $\mathcal{N}_n(\bar{\tau}, \eta, \psi)$ satisfying (1.1) is the surjectivity of the horizontal map.

As we noted above, Ginzburg–Jiang–Soudry [GJS12] proved similar results for metaplectic groups when $r = 1$; namely, $\bar{\tau}$ is an irreducible cuspidal automorphic representation. In this paper, we shall follow their argument to prove the above results. We note that in their proof, some results on vanishing and non-vanishing of Fourier coefficients were crucial (e.g. see [GJS12, Theorem 2.1]). In the present paper, we also prove similar results on Fourier coefficients using a similar argument as their proof except for one step. Indeed, in our case, we should study a certain integral over $\mathbb{A}_E/\mathbb{A}_F$, and it may not be exchanged by other root subgroups using a root exchange. Then we shall show that its constant term only survives by using root exchange in a different manner and an observation on an unramified component (see proof of Theorem 5.2).

Another main purpose of the present paper is to study a fundamental problem in the representation theory of p-adic reductive groups that is a local converse theorem. As we remarked above, the irreducibility of global descents for metaplectic groups and odd special orthogonal groups is already known, and it is a consequence of the local converse theorem for generic representations by Jiang–Soudry [JS03]. On the other hand, in this paper, we shall prove a local converse theorem for generic representations of even unitary groups as an application of the irreducibility of global descents.

In [JS03], they proved the local converse theorem using the local descents for a couple of supercuspidal representations. However, in our case, the irreducibility of the local descents has not been proved yet. Moreover, it does not seem easy to prove the irreducibility in a similar argument used in the case of metaplectic groups for a couple of supercuspidal representations (see Soudry–Tanay [ST15, p. 561]). In order to prove the local converse theorem, we shall prove the irreducibility of local descents first. In our proof, the following global application of the irreducibility of global descents plays an important role, which is called the rigidity theorem.

Theorem 1.4. *Let σ and σ' be irreducible ψ^{-1}-generic cuspidal automorphic representations of $G_n(\mathbb{A})$. Suppose that σ and σ' are nearly equivalent; i.e., for almost all places v of a number field F, $\sigma_v \simeq \sigma'_v$. Then*

$$\sigma = \sigma'.$$

In particular, the multiplicity one theorem for the generic spectrum holds for G_n.

Using the rigidity theorem, the irreducibility of global descents, and globalization of supercuspidal representations, we can prove the following irreducibility of local descents.

Theorem 1.5. *Let η_0 be a character of $(k')^\times$ such that $\eta_0|_k = \omega_{k'/k}$. Let $\bar{\tau} = (\tau_1, \ldots, \tau_r)$ with τ_i being an irreducible supercuspidal representation of $\mathrm{GL}_{n_i}(k')$ such that $\tau_i \not\simeq \tau_j$ if $i \neq j$ and $L(s, \tau_i, \mathrm{Asai})$ has a pole at $s = 0$. Then the explicit local descent $D^{\eta_0}_{\psi_{k'}}(\bar{\tau})$ is an irreducible $\psi_{k'}^{-1}$-generic supercuspidal representation of $G_n(k)$ (see Section 9.2 for the definition of explicit local descents).*

In our future work, we will extend this result to the case where τ_i are discrete series representations using the globalization by Ichino–Lapid–Mao [ILM17, Corollary A.6]. We note that the irreducibility of local descents was studied by Soudry–Tanay [ST15] when $r = 1$. They computed several Jacquet modules explicitly and showed the irreducibility under a certain assumption. In the above theorem, we do not need any assumption. As a consequence of the above applications and globalizations of supercuspidal representations, we can prove the following local converse theorem.

Theorem 1.6. *Let k be a non-archimedean local field of characteristic zero and let k' be a quadratic extension of k. Let ψ_k be a non-trivial additive character of k and define an additive character $\psi_{k'}$ of k' by $\psi_{k'}(x) = \psi_k(\frac{x+\bar{x}}{2})$ for $x \in k'$. Here, $x \mapsto \bar{x}$ is the non-trivial element of* $\mathrm{Gal}(k'/k)$. *Let π_1 and π_2 be irreducible $\psi_{k'}^{-1}$-generic representations of $G_n(k)$ such that*

$$\gamma^{Sh}(s, \pi_1 \times \sigma, \psi_{k'}) = \gamma^{Sh}(s, \pi_2 \times \sigma, \psi_{k'})$$

holds for any irreducible supercuspidal representation σ of $\mathrm{GL}_i(k')$ *with $1 \leq i \leq n$. Then*

$$\pi_1 \simeq \pi_2.$$

Here, local γ-factors are the ones studied in Shahidi [Sh90a] *and* [Sh90b].

We would like to remark that recently, Zhang [Zh17a], [Zh17b] studied a local converse theorem for G_1 and G_2 with respect to local gamma factors defined by Shimura type integrals. His proof is purely local and he uses Howe vectors as in the proof of a local converse theorem for GSp_4 by Baruch [Ba95]. Further, we should mention that in our proof of the above theorem, we shall use a local converse theorem for $\mathrm{GL}_{2n}(k')$. A local converse theorem for GL_m was first proved by Henniart [He93], and his result needed a twist by supercuspidal representations of GL_i for $1 \leq i \leq m-1$. Recently Jacquet and Liu [JL16] and independently Chai [Ch16] weakened this condition; indeed they proved a local converse theorem with a twist by supercuspidal representations of GL_i for $1 \leq i \leq [\frac{m}{2}]$.

From our local converse theorem, we obtain a characterization of local base change lift for supercuspidal representations (see Corollary 9.10), and thus we get the uniqueness of local Langlands correspondence for even unitary groups in [Mo15]. We also obtain a characterization of $(\mathrm{GL}_{2n}(k), \omega_{k'/k})$-distinguished supercuspidal representations of $\mathrm{GL}_{2n}(k')$ in terms of local base change lifts (see Corollary 9.12).

Finally, we would like to mention that in Lapid–Mao [LM16], they used the theory of global descents in order to reduce their conjecture on Whittaker Fourier coefficients of automorphic forms on unitary groups to certain local identities assuming the irreducibility of global descents and certain properties on local zeta integrals. Required properties of the local zeta integrals in the assumption were recently proved by Ben-Artzi–Soudry [BAS]. Further, because of our irreducibility result, this reduction holds without any assumption. In our future work, we shall give a rigorous proof of conjectural local identities in [LM16].

This paper is organized as follows. In Section 2, we prepare some notation. In Section 3, we define some unipotent subgroups. We also define Fourier coefficients and Fourier-Jacobi coefficients with respect to these unipotent subgroups. In Section 4, we firstly construct residual representations of G_{2n} and recall the definition of global descents. Secondly, we construct several residual representations of G_{3n}

and G_{4n}. In Section 5, by computing Fourier coefficients explicitly, we shall give some vanishing and non-vanishing of Fourier coefficients of residual representations. Using this result, we shall study a global descent of a residual representation of G_{3n}. In Section 6, we give definitions of certain nearly equivalent classes and their properties. In Section 7, we give basic identities of double descents, which is crucial to proving the irreducibility of global descents. Also, we prove a key result, namely Theorem 7.8. In Section 8, we prove the irreducibility of global descents and its generalization to certain nearly equivalent sets. In Section 9, we give applications of the irreducibility of global descents. In Appendix A, we shall show some facts on unramified representations, which are used to compute Fourier coefficients of residual representations.

2. Preliminaries

Let F be a number field and let E be a quadratic extension of F. We denote by $x \mapsto \bar{x}$ the action of the non-trivial element of $\mathrm{Gal}(E/F)$. Let us take $\iota \in E$ such that $\bar{\iota} = -\iota$ and $E = F(\iota)$. We denote the ring of adeles of F and E by $\mathbb{A}_F$ and $\mathbb{A}_E$, respectively. We denote by $\omega_{E/F}$ the quadratic character of $\mathbb{A}_F^\times / F^\times$ corresponding to the quadratic extension E.

Let ψ_F be a non-trivial character of $\mathbb{A}_F/F$, and define a character ψ of $\mathbb{A}_E/E$ by

$$\psi(x) = \psi_F\left(\frac{x+\bar{x}}{2}\right).$$

Further, for $a \in E^\times$, we define

$$\psi^a(x) := \psi(ax), \quad x \in \mathbb{A}_E.$$

In particular, when $a \in F^\times$, we have

$$\psi^a(x) = \psi_F\left(\frac{a(x+\bar{x})}{2}\right).$$

Let A_r be the subgroup of diagonal elements of $\mathrm{Res}_{E/F}\mathrm{GL}_r$ and let Z_r be the subgroup of upper triangular unipotent matrices of $\mathrm{Res}_{E/F}\mathrm{GL}_r$. Then $\mathcal{P}_r = A_r Z_r$ is the Borel subgroup of $\mathrm{Res}_{E/F}\mathrm{GL}_r$. Further, for a partition $[n_i]$ of r, namely for positive integers n_i such that $n_1 + \cdots + n_s = r$, let $P^{\mathrm{GL}}_{[n_i]}$ denote the standard parabolic subgroup of $\mathrm{Res}_{E/F}\mathrm{GL}_r$ whose Levi component is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{E/F}\mathrm{GL}_{n_s}$.

Define an even unitary group $G_n := \mathrm{U}(n,n)$, which is an algebraic group defined over F such that

$$G_n(F) = \left\{ g \in \mathrm{GL}(2n, E) : {}^t\bar{g} J_{2n}^{-} g = J_{2n}^{-} \right\}.$$

Here, J_{2n}^{-} is a $2n \times 2n$ skew-symmetric matrix defined by

$$J_{2n}^{-} := \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix},$$

where J_n is the $n \times n$ symmetric matrix defined inductively by

$$J_n = \begin{pmatrix} 0 & 1 \\ J_{n-1} & 0 \end{pmatrix} \quad \text{and} \quad J_1 = (1).$$

For $g \in \mathrm{Res}_{E/F}\mathrm{GL}_i$, we write

$$g^* = J_i {}^t\bar{g}^{-1} J_i,$$

and we define an element of G_{2m} by

$$\hat{g}_{2m} := \begin{pmatrix} g & & \\ & 1_{2m-2k} & \\ & & g^* \end{pmatrix}.$$

For simplicity, we often denote $\hat{g}_{2m}$ by $\hat{g}$.

Let T_n be the group of diagonal matrices in G_n and let T_n° be the subgroup of T_n such that $T_n^\circ(F) = T_n(F) \cap \mathrm{GL}_{2n}(F)$. Then T_n is a maximal torus of G_n and T_n° is a maximal F-split torus. Let B_n be the subgroup of upper triangular matrices of G_n. Then B_n is the Borel subgroup and it has the Levi decomposition $B_n = T_n U_n$, where U_n is the subgroup of upper triangular unipotent matrices.

Let us denote by Φ the set of roots associated to (B_n, T_n). For $\alpha \in \Phi$, we denote by X_α the one-parameter subgroup corresponding to α. Further, we denote the Weyl group $N_{G_n}(T)/Z_{G_n}(T)$ as W_{2n} and we will fix its representatives for the elements of the Weyl group.

Let us define several parabolic subgroups of unitary groups. Let Q_r^{2k} be the standard (non-maximal) parabolic subgroup of G_k whose Levi part M_r^{2k} is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_1^r \times G_{k-r}$. We denote its unipotent radical by U_r^{2k}.

Let P_r^{2k} be the standard maximal parabolic subgroup of G_k whose Levi part D_r^{2k} is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_r \times G_{k-r}$. We denote its unipotent radical by N_r^{2k}. More generally, let $[n_i] = (n_1, \ldots, n_r)$ be an r-tuple of positive integers and let n be a positive integer. Then we denote by $P_{[n_i],n}$ the standard parabolic of $G_{n_1+\cdots+n_r+n}$ whose Levi component $M_{[n_i],n}$ is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{E/F}\mathrm{GL}_{n_r} \times G_n$. We denote its unipotent radical by $V_{[n_i],n}$.

Let us define L-groups of G_n and $\mathrm{Res}_{E/F}\mathrm{GL}_n$ as follows. Let ${}^L G_n$ be the L-group of G_n which is isomorphic to $\mathrm{GL}_{2n}(\mathbb{C}) \rtimes \mathbb{Z}_2$ with the action of non-trivial element $\mathbb{Z}_2$ on $\mathrm{GL}_{2n}(\mathbb{C})$ given by

$$g \mapsto J_{2n} \cdot {}^t g^{-1} J_{2n}.$$

Let ${}^L\mathrm{Res}_{E/F}\mathrm{GL}_n$ be the L-group of $\mathrm{Res}_{E/F}\mathrm{GL}_n$, which is isomorphic to $(\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) \rtimes \mathbb{Z}_2$ with the action of non-trivial element σ of $\mathbb{Z}_2$ on $(\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}))$ given by

$$(g_1, g_2)^\sigma = (g_2, g_1).$$

Then we define an Asai representation Asai^+ of ${}^L\mathrm{Res}_{E/F}\mathrm{GL}_n$ by

$$\mathrm{Asai}^+(g_1, g_2)(x) = g_1 x {}^t g_2 \quad \text{and} \quad \mathrm{Asai}^+(\sigma)(x) = {}^t x$$

for $x \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. Similarly, we may define another Asai representation Asai^- of ${}^L\mathrm{Res}_{E/F}\mathrm{GL}_n$ by

$$\mathrm{Asai}^-(g_1, g_2)(x) = g_1 x {}^t g_2 \quad \text{and} \quad \mathrm{Asai}^-(\sigma)(x) = -{}^t x$$

for $x \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. We note that for an irreducible cuspidal automorphic representation τ of $\mathrm{GL}_n(\mathbb{A}_E)$, we have

$$L(s, \tau \otimes \eta, \mathrm{Asai}^+) = L(s, \tau, \mathrm{Asai}^-)$$

for any character η of $\mathbb{A}_E^\times/E^\times$ such that $\eta|_{\mathbb{A}_F^\times} = \omega_{E/F}$. Hereafter, we shall simply write the representation Asai^+ by Asai.

For an automorphic representation (τ, V_τ) of $\mathrm{GL}_n(\mathbb{A}_E)$, we write by τ^θ the automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ given by $g \mapsto \tau(\bar{g})$ and $V_{\tau^\theta} = \{g \mapsto \varphi(\bar{g}) : \varphi \in V_\tau\}$. Then

$$L(s, \tau \otimes \tau^\theta) = L(s, \tau, \mathrm{Asai})L(s, \tau, \mathrm{Asai}^-).$$

For an automorphic form φ on $G_n(\mathbb{A})$ and a unipotent subgroup U of G_n, we write

$$\mathcal{C}_U(\varphi) = \int_{U(F)\backslash U(\mathbb{A})} \varphi(u)\, du.$$

Finally, we let $\mathcal{A}_d(G_n)$ be the set of all irreducible automorphic representations of $G_n(\mathbb{A})$ occurring as subrepresentations in the space of square-integrable automorphic forms of $G_n(\mathbb{A})$.

3. Fourier coefficients and Fourier-Jacobi coefficients

3.1. **Fourier coefficients.** Following [CM93], we define unipotent subgroups attached to symplectic partitions. We only need a few kinds of unipotent subgroups. Hence, for the exposition, we do not give a definition of such unipotent subgroups in general, but we shall define such unipotent subgroups explicitly in particular cases.

Let $\mathcal{O}_{2r,2(k-r)} := [(2r)1^{2(k-r)}]$ be a partition of $2k$, i.e.,

$$2k = 2r + \overbrace{1 + \cdots + 1}^{2(k-r)}.$$

For any $a \in F^\times$, let $t_{\mathcal{O}_{2r,2(k-r)}}(a)$ be an element of T_n° defined by

$$t_{\mathcal{O}_{2r,2(k-r)}}(a) = \mathrm{diag}(a^{2r-1}, a^{2r-3}, \dots, a, 1, \dots, 1, a^{-1}, \dots a^{-2r+1}).$$

Let us define a unipotent subgroup of G_k by

$$\begin{aligned} &V_r^{2k} \\ =&V_{[(2r)1^{2(k-r)}]} \\ :=&\langle x_\alpha(r) \in X_\alpha : t_{\mathcal{O}_{2r,2(k-r)}}(a) x_\alpha(r) t_{\mathcal{O}_{2r,2(k-r)}}(a)^{-1} = x_\alpha(a^i r) \subset G_k \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{with } i \geq 2,\ \alpha \in \Phi\rangle. \end{aligned}$$

In matrices, V_r^{2k} is the unipotent group consisting of the following elements:

$$v = v(u, x, z) = \begin{pmatrix} u & x & z \\ & 1_{2(k-r)} & x' \\ & & u^* \end{pmatrix} \in G_k, \tag{3.1}$$

where $u \in Z_r(F)$ and $x \in \mathrm{Mat}_{r\times 2(k-r)}(E)$ is such that its last row is zero. We note that this is a subgroup of the unipotent radical U_r^{2k} of the parabolic subgroup Q_r^{2k}.

Let us define characters of V_r^{2k}. When we write an element of $V_r^{2k}(\mathbb{A})$ in the form (3.1), for $a \in F^\times$, we define

$$\psi_{V_r^{2k},a}(v(u, x, z)) := \psi(u_{1,2} + \cdots + u_{r-1,r} + a z_{r,1}),$$

where we note that $z_{r,1} \in \mathbb{A}_F$. Under the conjugation by $M_r^{2k}(F)$, the orbit of $\psi_{V_r^{2k},a}$ is open in the group of characters of $V_r^{2k}/[V_r^{2k}, V_r^{2k}]$. Conversely, such a character of $V_r^{2k}(\mathbb{A})$ is given by $\psi_{V_r^{2k},a}$ for $a \in F^\times/N_{E/F}(E^\times)$ up to a conjugation by $M_r^{2k}(F)$. Then for an automorphic representation π of G_k, we define a Fourier coefficient of $\varphi_\pi \in \pi$ associated to the partition $[(2r)1^{2(k-r)}]$ by

$$\mathcal{F}^{\psi_{V_r^{2k},a}}(\varphi_\pi) := \int_{V_r^{2k}(F)\backslash V_r^{2k}(\mathbb{A})} \varphi_\pi(v)\psi^{-1}_{V_r^{2k},a}(v)\, dv.$$

We note that when $k = r$, V_r^{2k} is the upper triangular matrices U_k of G_k and the character $\psi_{V_r^{2k},a}$ gives a non-degenerate character of $U_k(\mathbb{A})$:

$$\psi(v_{1,2} + \cdots + v_{k-1,k} + av_{k,k+1}), \quad v \in V_k^{2k}(\mathbb{A}) = U_k(\mathbb{A}). \tag{3.2}$$

When no confusion occurs we shall write this character simply by ψ^a. Then we call $\mathcal{F}^{\psi_{V_k^{2k},a}}(\varphi_\pi)$ the ψ^a-Whittaker coefficient of φ_π, and we say that π is ψ^a-generic if ψ^a-Whittaker coefficients are not identically zero on V_π.

Similarly, for a partition $[(2n)^2 1^{2r}]$ of $4n + 2r$, we define a unipotent subgroup $V_{[(2n)^2 1^{2r}]}$ of G_{2n+r}. Then it is the subgroup of G_{2n+r} consisting of the following matrices:

$$\begin{pmatrix} 1_2 & * & * & * & * & * & * & * \\ & 1_2 & * & * & * & * & * & * \\ & & \ddots & * & * & * & * & * \\ & & & 1_2 & 0 & * & * & * \\ & & & & 1_{2r} & 0 & * & * \\ & & & & & 1_2 & * & * \\ & & & & & & \ddots & * \\ & & & & & & & 1_2 \end{pmatrix}.$$

To define characters of $V_{[(2n)^2 1^{2r}]}(\mathbb{A})$, we identify

$$V_{[(2n)^2 1^{2r}]}(F)/[V_{[(2n)^2 1^{2r}]}(F), V_{[(2n)^2 1^{2r}]}(F)] \simeq \mathrm{Mat}_{2\times 2}(E)^{n-1} \times \mathrm{Herm}^0_{2\times 2}(F), \tag{3.3}$$

where

$$\begin{aligned} \mathrm{Herm}^0_{2\times 2}(F) &= \{A \in \mathrm{Mat}_{2\times 2}(E) : -J_2 A + {}^t\bar{A} J_2 = 0\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & \bar{a} \end{pmatrix} : a \in E,\, b, c \in F \right\}. \end{aligned}$$

Then, through the above isomorphism, the Levi subgroup $\mathrm{GL}_2(E)^{n-1} \times G_r(F)$ acts on $V_{[(2n)^2 1^{2r}]}(F)/[V_{[(2n)^2 1^{2r}]}(F), V_{[(2n)^2 1^{2r}]}(F)]$ by

$$(h_1, \dots, h_n : g) \cdot (X_1, \dots, X_{n-1}; Y) = (h_1 X_1 h_2^{-1}, \dots, h_{n-1} X_{n-1} h_n^{-1}; h_n Y (h_n^*)^{-1}).$$

Here, recall that $h_n^* = J_2 {}^t\bar{h}^{-1} J_2$. Representatives of generic $\mathrm{GL}_2(E)^{n-1} \times G_r(F)$-orbits on the quotient (3.3) are given by

$$\left(1_2, \dots 1_2; \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\right), \text{ where } a, b \in F^\times/N_{E/F}(E^\times).$$

Hereafter, we identify the representative of a class of $F^\times/N_{E/F}(E^\times)$ with its class. Define a character $\psi_{[(2n)^2 1^{2r}];b,a}$ of $V_{[(2n)^2 1^{2r}]}(\mathbb{A})$ by

$$\psi_{[(2n)^2 1^{2r}];b,a}(v) = \psi\left(\mathrm{tr}\left(X_1 + \cdots + X_n + \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} Y\right)\right),$$

where $\left(X_1 + \cdots + X_n + \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} Y\right)$ is the image of v of the following map:

$$V_{[(2n)^2 1^{2r}]}(\mathbb{A}) \to V_{[(2n)^2 1^{2r}]}(\mathbb{A}) / [V_{[(2n)^2 1^{2r}]}(\mathbb{A}), V_{[(2n)^2 1^{2r}]}(\mathbb{A})] \\ \simeq \mathrm{Mat}_{2\times 2}(\mathbb{A}_E)^{n-1} \times \mathrm{Herm}^0_{2\times 2}(\mathbb{A}).$$

For an automorphic form φ on $G_{2n+r}(\mathbb{A})$, we define a Fourier coefficient of φ associated to the partition $[(2n)^2 1^{2r}]$ by

$$\mathcal{F}^{\psi_{[(2n)^2 1^{2r}];b,a}}(\varphi) = \int_{V_{[(2n)^2 1^{2r}]}(F) \backslash V_{[(2n)^2 1^{2r}]}(\mathbb{A})} \varphi(v) \psi^{-1}_{[(2n)^2 1^{2r}];b,a}(v)\, dv.$$

3.2. **Fourier-Jacobi coefficients.** Let us recall the definition of Fourier-Jacobi coefficients.

Let us quickly review the Weil representation (see [GRS11, pp. 8, 9]). Let Y be a non-degenerate $2n$-dimensional anti-Hermitian space with the anti-Hermitian form $(\ ,\ )$. Assume that its Witt index is n; i.e., the dimension of a maximal isotropic subspace of Y is n over E. Let $\mathrm{U}(Y)$ be the corresponding unitary group.

Define an F-bilinear form

$$\langle , \rangle = \frac{1}{2} \mathrm{Tr}_{E/F}(\ ,\ )$$

on Y. Then it is a non-degenerate symplectic form on Y when we regard Y as a $4n$-dimensional vector space over F. Denote this $4n$-dimensional space by Y'. We may define the corresponding symplectic group $\mathrm{Sp}(Y')$, and we have an F-embedding,

$$\mathrm{U}(Y) \hookrightarrow \mathrm{Sp}(Y').$$

Further, we note that the metaplectic cover of $\mathrm{Sp}(Y')$ splits over $\mathrm{U}(Y)$. Remark that this splitting is not canonical. Fix a character η of $\mathbb{A}_E^\times / E^\times$ such that its restriction to $\mathbb{A}_F^\times$ is the character $\omega_{E/F}$ corresponding to the quadratic extension E/F. Then we choose the splitting as in Moeglin–Vigneras–Waldspurger [MVW87] and Kudla [Ku94] corresponding to η. We denote the corresponding Weil representation of $\mathrm{U}(Y)(\mathbb{A}) \ltimes H(Y')(\mathbb{A})$ by $\omega_{\psi,\eta}$. Here, $H(Y') := Y' \oplus F$ denotes the Heisenberg group associated to Y' with the multiplication

$$(w, z) \cdot (w', z') = (w + w', z + z' + \frac{1}{2}\langle w, w' \rangle), \quad w, w' \in Y', z, z' \in F. \tag{3.4}$$

Then its explicit action is given as follows.

We fix a polarization

$$Y = Y^+ + Y^-,$$

where $Y^\pm$ are maximal isotropic subspaces of Y. Hereafter, we shall write any element of $H(Y')$ by $(y_1, y_2; z)$ with $y_1 \in Y^+, y_2 \in Y^-$, and $z \in F$. We have for $\phi \in \mathcal{S}(Y^+(\mathbb{A}_E))$ (see [GRS11, (1.5)])

$$\begin{aligned} \omega_{\psi,\eta}((y^+, 0; 0))\phi(\xi) &= \phi(\xi + y), \\ \omega_{\psi,\eta}((0, y^-; 0))\phi(\xi) &= \psi(2\langle \xi, y_- \rangle)\phi(\xi), \\ \omega_{\psi,\eta}((0, 0; t))\phi(\xi) &= \psi(t)\phi(\xi), \\ \omega_{\psi,\eta}\left(\begin{pmatrix} g & \\ & g^* \end{pmatrix}\right)\phi(\xi) &= \gamma_\psi(N_{E/F}(\det g))\eta(\det g)|\det g|^{1/2}\phi(\xi \cdot g), \\ \omega_{\psi,\eta}\left(\begin{pmatrix} 1_n & X \\ & 1_n \end{pmatrix}\right)\phi(\xi) &= \psi(\langle \xi, \xi \cdot X \rangle)\phi(\xi), \end{aligned} \tag{3.5}$$

where $y^+ \in Y^+(\mathbb{A}_E)$, $y^- \in Y^-(\mathbb{A}_E)$, $t \in \mathbb{A}_F$, and we write the elements of $\mathrm{U}(Y)$ as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a \in \mathrm{Hom}_E(Y^+, Y^+)$, $b \in \mathrm{Hom}_E(Y^+, Y^-)$, $c \in \mathrm{Hom}_E(Y^-, Y^+)$, and $d \in \mathrm{Hom}_E(Y^-, Y^-)$.

Recall that for $a \in F^\times$, we define $\psi^a(x) = \psi_F(\frac{a}{2} \cdot \mathrm{Tr}_{E/F}(x))$. Then for the Weil representation $\omega_{\psi^a,\eta}$ and $\phi \in \mathcal{S}(\mathbb{A}_E^{n-r})$, we define the theta function by

$$\theta_{\phi,n-r}^{\psi^a,\eta}(vh) := \sum_{\xi \in E^{n-r}} \omega_{\psi^a,\eta}(vh)\phi(\xi), \quad h \in U(Y)(\mathbb{A}), v \in H(Y')(\mathbb{A}).$$

We shall use the matrix form to study Fourier-Jacobi coefficients. We fix a basis to realize $Y \simeq E^{2n}$ and $\mathrm{U}(Y) \simeq G_n$. Recall that $Q_r^{2n} = M_r^{2n} U_r^{2n}$ is the standard parabolic subgroup of G_n whose Levi subgroup M_r^{2n} is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_1^r \times G_{n-r}$. Then the map

$$\ell_{n-r} : v \mapsto \left(v_{r,r+1}, \dots, v_{r,2n-r}, \frac{1}{2}\mathrm{Tr}_{E/F}(v_{n,4n-r+1}) \right)$$

gives an isomorphism from U_r^{2n}/U_{r-1}^{2n} to the Heisenberg group $H(E^{2(n-r)})$. Define a character $\psi_{U_r^{2n}}$ of $U_r^{2n}(\mathbb{A})$ by

$$\psi_{U_r^{2n}}(u) = \psi(u_{1,2} + \dots + u_{r-1,r}).$$

For an automorphic form φ on $G_n(\mathbb{A})$, we define the Fourier-Jacobi coefficient of φ with respect to $n-r$ and ψ^a by the following integral:

(3.6)

$$\mathrm{FJ}_{\phi,n-r}^{\psi^a,\eta}(\varphi)(h) = \int_{U_r^{2n}(F)\backslash U_r^{2n}(\mathbb{A})} \varphi(uh)\theta_{\phi,n-r}^{\psi^a,\eta^{-1}}(\ell_{n-r}(u)h)\psi_{U_r^{2n}}(u)\, du, \quad h \in G_{n-r}(\mathbb{A}).$$

We know that it defines an automorphic form on $G_{n-r}(\mathbb{A})$.

Remark 3.1. We note that U_r^{2n} contains the unipotent subgroup $V_{[(2r)1^{2(k-r)}]}$. In particular, the Fourier-Jacobi coefficient $\mathrm{FJ}_{\phi,n-r}^{\psi^a,\eta}(\varphi)(1)$ contains the Fourier-coefficient $\mathcal{F}^{\psi_{V_r^{2k},a}}(\varphi)$ as an inner integral.

4. Certain residual representations

Let us construct some residual representations and define global descents. Let τ_i be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ ($1 \leq i \leq r$) such that $2n = n_1 + \dots + n_r$, $\tau_i \not\simeq \tau_j$ if $i \neq j$ and $L(s, \tau_i, \mathrm{Asai})$ has a pole at $s = 1$. Then we write $\bar{\tau} = (\tau_1, \dots, \tau_r)$, and we regard $\bar{\tau}$ as an automorphic representation of $\prod_{i=1}^r \mathrm{GL}_{n_i}(\mathbb{A}_E)$. Consider the parabolic induction

$$I(\bar{\tau}, \bar{s}) := \mathrm{Ind}_{P_{[n_i]}(\mathbb{A})}^{G_{2n}(\mathbb{A})}(\tau_1 |\det|_E^{s_1} \otimes \dots \otimes \tau_r |\det|_E^{s_r}),$$

where $\bar{s} := (s_1, \dots, s_r) \in \mathbb{C}^r$ and $P_{[n_i]} := P_{[n_i],0}$ is the standard parabolic subgroup of G_{2n} whose Levi component is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_{n_1} \times \dots \times \mathrm{Res}_{E/F}\mathrm{GL}_{n_r}$. Then for a holomorphic section $f_{\bar{\tau},\bar{s}} \in I(\bar{\tau}, \bar{s})$, we form an Eisenstein series on $G_{2n}(\mathbb{A})$ by

$$E(h, f_{\bar{\tau},\bar{s}}) := \sum_{\gamma \in P_{[n_i]}(F)\backslash G_{2n}(F)} f_{\bar{\tau},\bar{s}}(\gamma h),$$

which converges absolutely for $\mathrm{Re}(s_i) \gg 0$ and has a meromorphic continuation to $\mathbb{C}^r$ (Langlands [La76]). Then by [GRS11, Theorem 2.1], this Eisenstein series has a pole at $\bar{s} = (\frac{1}{2}, \dots, \frac{1}{2})$ for some $f_{\bar{\tau},\bar{s}}$, and we write $\mathcal{E}_{\bar{\tau}}$ for the automorphic representation of $G_{2n}(\mathbb{A})$ generated by these residues. We note that $\mathcal{E}_{\bar{\tau}}$ is square-integrable and irreducible.

Now, let us recall the definition of the global descents. Let π be an automorphic representation of $G_n(\mathbb{A})$. Then we denote by $\mathcal{D}^{2n,\eta}_{2(n-r),\psi^a}(\pi)$ the automorphic representation of $G_{n-r}(\mathbb{A})$ generated by all Fourier-Jacobi coefficients $\mathrm{FJ}^{\psi^a,\eta}_{\phi,n-r}(\varphi_\pi)$ for $\varphi_\pi \in \pi$ and $\phi \in \mathcal{S}(\mathbb{A}_E^{n-r})$.

Theorem 4.1 ([GRS11, Theorem 3.1])**.** *The global descent $\mathcal{D}^{4n,\eta}_{2n,\psi}(\mathcal{E}_{\bar{\tau}})$ from $G_{2n}(\mathbb{A})$ to $G_n(\mathbb{A})$ of $\mathcal{E}_{\bar{\tau}}$ is a direct sum of irreducible cuspidal ψ^{-1}-generic automorphic representations which weakly lift to $(\tau_1 \otimes \eta^{-1}) \boxplus \cdots \boxplus (\tau_r \otimes \eta^{-1})$.*

Let us define a residual automorphic representation of $G_{3n}(\mathbb{A})$. First, let us recall the following result on the functoriality of generic cuspidal automorphic representations of $G_n(\mathbb{A})$.

Theorem 4.2 (Theorems 1.1 and 8.10 in [KK05])**.** *Let π be an irreducible cuspidal automorphic representation of $G_n(\mathbb{A})$ which is ψ^{-1}-generic. Then there is a base change lift of π to $\mathrm{GL}_{2n}(\mathbb{A}_E)$, and it is written as $(\tau_1 \otimes \eta^{-1}) \boxplus \cdots \boxplus (\tau_r \otimes \eta^{-1})$, where τ_i is an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ such that $\tau_i \not\simeq \tau_j$ if $i \neq j$ and $L(s, \tau_i, \mathrm{Asai})$ has a simple pole at $s = 1$. Thus the residual representation $\mathcal{E}_{\bar{\tau}}$ exists for such $\bar{\tau} = (\tau_1, \ldots, \tau_r)$.*

The above theorem should hold for cuspidal automorphic representations which have a tempered A-parameter (see [Mo15]). We note that it can be non-generic. When we study the non-generic case, we assume the following.

Assumption 1. *Let π be an irreducible cuspidal automorphic representation of $G_n(\mathbb{A})$. Then it has a base change lift to an irreducible isobaric automorphic representation $\boxplus_{i=1}^r(\tau_i \otimes \eta^{-1})$ of $\mathrm{GL}_{2n}(\mathbb{A}_E)$ such that $\tau_i \not\simeq \tau_j$ if $i \neq j$ and $L(s, \tau_i, \mathrm{Asai})$ has a simple pole at $s = 1$. In this case, we can also construct the residual representation $\mathcal{E}_{\bar{\tau}}$.*

Let $\bar{\tau}$ and π be as in Assumption 1 or Theorem 4.2. Then we consider the parabolic induction

$$\mathrm{Ind}^{G_{3n}(\mathbb{A})}_{P_{[n_i],n}(\mathbb{A})}(\tau_1|\det|_E^{s_1} \otimes \cdots \otimes \tau_r|\det|_E^{s_r} \otimes \pi), \quad s_i \in \mathbb{C},$$

where $P_{[n_i],n}$ is the standard parabolic subgroup of G_{3n} whose Levi component $M_{[n_i],n}$ is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{E/F}\mathrm{GL}_{n_r} \times G_n$. Then for a holomorphic section $f_{\bar{\tau},\pi,\bar{s}}(\cdot)$ in the above parabolic induction, we define an Eisenstein series on $G_{3n}(\mathbb{A})$ by

$$E(h, f_{\bar{\tau},\pi,\bar{s}}) := \sum_{\gamma \in P_{[n_i],n}(F) \backslash G_{3n}(F)} f_{\bar{\tau},\pi,\bar{s}}(\gamma h),$$

which converges absolutely for $\mathrm{Re}(s_i) \gg 0$ and has a meromorphic continuation to $\mathbb{C}^r$ (Langlands [La76]).

Lemma 4.3. *Let $\bar{\tau}$ and π be as in Assumption 1 (or as in Theorem 4.2 when π is ψ^{-1}-generic). Then the function*

$$\bar{s} \mapsto (s_1 - 1) \cdots (s_r - 1) E(h, f_{\bar{\tau},\pi,\bar{s}})$$

is holomorphic at $\bar{s} = \mathbf{1}$. Moreover, for some $f_{\bar{\tau},\pi,\bar{s}}$, the residue is non-trivial. We write the representation of $G_{3n}(\mathbb{A})$ generated by these residues as $\mathcal{E}_{\bar{\tau},\pi}$.

Proof. We shall prove this lemma by a similar argument as in the proof of [GRS11, Theorem 2.1]. First, suppose that $r = 1$; namely $\bar{\tau} = \tau$ is cuspidal. Then since $P_{2n,n}$ is a maximal parabolic subgroup, we know that constant terms of $E(h, f_{\tau,\pi,s})$ along N_k^{6n} are zero except for $k = 2n$. In the case $k = 2n$, the constant term is written as

$$\mathcal{C}_{N_{2n}^{6n}}(E(\cdot, f_{\tau,\pi,s}))(g) = f_{\tau,\pi,s}(g) + M(w_0)(f_{\tau,\pi,s})(g),$$

where $M(w_0)$ is the intertwining operator corresponding to the Weyl element

$$w_0 = \begin{pmatrix} & & 1_{2n} \\ & 1_{2n} & \\ -1_{2n} & & \end{pmatrix}.$$

Then the normalizing factor of $M(w_0)(f_{\bar{\tau},\pi,\bar{s}})(g)$ (outside a finite set S of places) is

$$\frac{L^S(s, \tau \times \pi^\vee)}{L^S(s+1, \tau \times \pi^\vee)} \cdot \frac{L^S(2s, \tau, \mathrm{Asai})}{L^S(2s+1, \tau, \mathrm{Asai})}.$$

Note that we have

$$L^S(s, \tau \times \pi^\vee) = L^S(s, \tau, \mathrm{Asai}^-)L^S(s, \tau, \mathrm{Asai}).$$

Since $L^S(s, \tau, \mathrm{Asai})$ has a simple pole at $s = 1$ and $L^S(s, \tau, \mathrm{Asai}^-)$ is holomorphic and does not vanish at $s = 1$, $L^S(s, \tau \times \pi^\vee)$ has a simple pole at $s = 1$. Further, remaining L-functions are holomorphic and non-zero at $s = 1$. Hence,

$$s \mapsto (s-1)E(h, f_{\tau,\pi,s})$$

is holomorphic at $s = 1$, and it is non-trivial for some $f_{\tau,\pi,s}$.

Suppose that $r > 1$. Let $\{n_{a(1)}, \ldots, n_{a(l)}\}$ be a subset of $\{n_1, \ldots, n_r\}$. We write $\bar{\tau}' = (\tau_{a(1)}, \ldots, \tau_{a(l)})$ and $\bar{s}' = (s_{a(1)}, \ldots s_{a(l)})$. Then we consider an Eisenstein series $E(h, f_{\bar{\tau}',\pi,\bar{s}'})$ on $G_{m(a)+n}(\mathbb{A})$ corresponding to the parabolic induction

$$\mathrm{Ind}_{P_{[n_i]',n}(\mathbb{A})}^{G_{(n+m(a))}(\mathbb{A})}(\tau_{a(1)}|\det|^{s_{a(1)}} \otimes \cdots \otimes \tau_{a(l)}|\det|^{s_{a(l)}} \otimes \pi),$$

where $m(a) = n_{a(1)} + \cdots + n_{a(l)}$ and $[n_i]'$ is the corresponding partition of $m(a)$. Let us study constant terms of $E(h, f_{\bar{\tau}',\pi,\bar{s}'})$.

By [MW95, II.1.7], for $g \in D_k^{2(n+m(a))}(\mathbb{A})$,

$$\mathcal{C}_{N_k^{2(n+m(a))}}(E(\cdot, f_{\bar{\tau}',\pi,\bar{s}}))(g) = \sum_{w \in W_{[n_i],k}} E_{D_k^{2(n+m(a))}}(g, M(w)f_{\bar{\tau}',\pi,\bar{s}}), \tag{4.1}$$

where

(4.2)

$$W_{[n_i]',k} = \left\{ w \in W_{3n} : \begin{array}{l} \text{(i) } w(\alpha) > 0 \text{ for all positive roots } \alpha \text{ inside } M_{[n_j]',2n}, \\ \text{(ii) } w^{-1}(\alpha) > 0 \text{ for all positive roots } \alpha \text{ inside } D_k^{2(n+m(a))}, \\ \text{(iii) } wM_{[n_j]',2n}w^{-1} \text{ is a standard Levi subgroup of } D_k^{2(n+m(a))} \end{array} \right\}$$

and $E_{D_k^{2(n+m(a))}}(g, M(w)f_{\bar{\tau}',\pi,\bar{s}'})$ denotes the Eisenstein series on $D_k^{2(n+m(a))}(\mathbb{A})$ defined by the section $M(w)f_{\bar{\tau}',\pi,\bar{s}}|_{D_k^{2(n+m(a))}}$ which lies in the parabolic induction

$$\mathrm{Ind}_{Q_{[n_i]',D}}^{D_k^{2(n+m(a))(\mathbb{A})}(\mathbb{A})}(\tau_1|\det|_E^{s_1} \otimes \cdots \otimes \tau_{a(l)}|\det|_E^{s_{a(l)}} \otimes \pi)^w \tag{4.3}$$

with the standard parabolic subgroup $Q_{[n_i]',D}$ of $D_k^{2(n+m(a))}$ whose Levi part is $wM_{[n_j]',2n}w^{-1}$. Then as in [GRS11, p. 33], we may decompose $w \in W_{[n_i]',k}$ into the form

$$w = w_1 w_2 w_3$$

where

$$w_1 = \begin{pmatrix} 1_{n_{a(1)}+\cdots+n_{a(i-1)}} &&&& \\ & \omega &&& \\ && 1 && \\ &&& \omega^* & \\ &&&& 1_{n_{a(1)}+\cdots+n_{a(i-1)}} \end{pmatrix}$$

with

$$\omega = \begin{pmatrix} & 1_{n_{a(i)}} \\ 1_{n_{a(i+1)}} & \end{pmatrix},$$

$$w_2 = \begin{pmatrix} 1_{n_{a(1)}+\cdots+n_{a(i-1)}} && \\ & \mu & \\ && 1_{n_{a(1)}+\cdots+n_{a(i-1)}} \end{pmatrix}$$

with

$$\mu = \begin{pmatrix} && 1_{n_{a(i)}+\cdots+n_{a(k)}} \\ & 1_{2n} & \\ 1_{n_{a(i)}+\cdots+n_{a(k)}} && \end{pmatrix},$$

and w_3 is a certain Weyl element contained in $D_{2n}^{2(n+m(a))}(F)$ which preserves $M_{[n_i]',2n}^{2(n+m(a))}$. Before proceeding with the proof, we note that in the case of $l = 1$, we can show that

$$(s_{a(1)} - 1)E_{D_k^{2(n+m(a))}}(g, M(w)f_{\bar{\tau}',\pi,\bar{s}'})$$

is holomorphic at $\bar{s}' = \mathbf{1}$ as in the case of $r = 1$.

In order to study the analytic properties of each Eisenstein series in (4.3), we shall first study the analytic behavior of the intertwining operator $M(w)$. First, we note that

$$M(w) = M(w_1) \circ M(w_2) \circ M(w_3).$$

Then from the definition of w_3, [GRS11, Lemma 2.1] shows that $M(w_3)(f_{\bar{\tau}',\pi,\bar{s}})$ is holomorphic at $\bar{s}' = \mathbf{1}$.

Let us consider $M(w_2) \circ M(w_3)(f_{\bar{\tau}',\pi,\bar{s}'})$. From the decomposition [GRS11, (2.56)] of $M(w_2)$, in a similar argument as [GRS11, p. 35] we can reduce an analytic behavior of $M(w_2) \circ M(w_3)(f_{\bar{\tau}',\pi,\bar{s}'})$ to the case $M(w[a(i)]) \circ M(w_3)(f_{\bar{\tau}',\pi,\bar{s}'})$ where

$$w[a(i)] = \begin{pmatrix} 1_{n_{a(1)}+\cdots+n_{a(i-1)}} &&&& \\ &&& 1_{n_{a(i)}} & \\ && 1 && \\ & 1_{n_{a(i)}} &&& \\ &&&& 1_{n_{a(1)}+\cdots+n_{a(i-1)}} \end{pmatrix}.$$

Then $(s_{a(i)} - 1)M(w[a(i)]) \circ M(w_3)(f_{\bar{\tau}',\pi,\bar{s}'})$ is holomorphic at $\bar{s}' = \mathbf{1}$ by the same argument as in the case of $r = 1$. Then the argument in [GRS11, p. 35] shows that

$$\prod_{j=p}^{q}(s_{a(i_j)} - 1)M(w_2) \circ M(w_3)(f_{\bar{\tau}',\pi,\bar{s}'}) \text{ is holomorphic at } \bar{s}' = \mathbf{1}$$

with $1 \leq p \leq q \leq l$. Here, p and q are integers such that $M(w)$ maps the original inducing data to define the parabolic induction (4.3) to the inducing data

$$\left(\bigotimes_{j=1}^{p-1} \tau_{a(i_j)} |\det|^{s_{a(i_j)}}\right) \otimes \tau_{a(i_p)} |\det|^{-s_{a(i_p)}} \otimes \cdots \otimes \tau_{a(i_q)} |\det|^{-s_{a(i_q)}} \otimes \left(\bigotimes_{j=q+1}^{l} \tau_{a(i_j)} |\det|^{s_{a(i_j)}}\right) \otimes \pi.$$

We note that the case $p = 1$ and $q = l$ corresponds to the Weyl element

$$w_0^l = \begin{pmatrix} & & 1_{n_{a(1)}+\cdots+n_{a(l)}} \\ & 1_{2n} & \\ -1_{n_{a(1)}+\cdots+n_{a(l)}} & & \end{pmatrix}.$$

Finally, by the argument in the case of $r = 1$, we see that

$$F_{\bar{\tau}', \pi, \bar{s}'} := \prod_{j=p}^{q} (s_{a(i_j)} - 1) M(w_1) \circ M(w_2) \circ M(w_3)(f_{\bar{\tau}, \pi, \bar{s}}) \text{ is holomorphic at } \bar{s}' = \mathbf{1}.$$

Let us consider the Eisenstein series

$$E_{D_k^{2(n+m(a))}}(g, F_{\bar{\tau}', \pi, \bar{s}'}).$$

As in [GRS11, pp. 36, 37], we may write this Eisenstein series as a finite sum of automorphic forms of the form $E^{\mathrm{GL}} \times E^{\mathrm{G}}$ where E^{GL} (resp E^{G}) is an Eisenstein series on $\mathrm{GL}_{n_{a(i_1)}+\cdots+n_{a(i_{p-1})}}(\mathbb{A}_E)$ (resp. $G_{n+n_{a(i_{q+1})}+n_{a(i_l)}}(\mathbb{A})$) defined by the induced representation corresponding to $\bigotimes_{j=1}^{p-1} \tau_{a(i_j)} |\det|^{s_{a(i_j)}}$ (resp. $\left(\bigotimes_{j=q+1}^{l} \tau_{a(i_j)} |\det|^{s_{a(i_j)}}\right) \otimes \pi$). Since $n_{a(i_{q+1})} + n_{a(i_l)} \leq m(a)$, in an inductive argument, we can show that

$$(s_{a(q+1)} - 1) \cdots (s_{a(l)} - 1) E^{\mathrm{G}}$$

is holomorphic at $\bar{s}' = \mathbf{1}$, and E^{GL} is holomorphic at the same point by [GRS11, Lemma 2.1]. Therefore,

$$\prod_{j=p}^{l} (s_{a(i_j)} - 1) E_{D_k^{2(n+m(a))}}(g, M(w)(f_{\bar{\tau}, \pi, \bar{s}}))$$

is holomorphic.

Suppose that $l = r$. As a consequence of the above argument, we see that unless $w = w_0^r$ (and thus $k = 2n$ and $p = 1$),

$$\prod_{i=1}^{r} (s_i - 1) E_{D_k^{6n}}(g, M(w) f_{\bar{\tau}', \pi, \bar{s}})$$

is zero. In this case, the corresponding term in (4.1) is

$$E_{D_{2n}^{6n}}(g, M(w_0^r) f_{\bar{\tau}, \pi, \bar{s}}),$$

which is the Eisenstein series on $D_{2n}^{6n}(\mathbb{A}) \simeq \mathrm{GL}_{2n}(\mathbb{A}_E) \times G_n(\mathbb{A})$ associated to $M(w_0^r) f_{\bar{\tau}, \pi, \bar{s}}$. Indeed, it is an Eisenstein series corresponding to the parabolic induction $\mathrm{Ind}_{P_{[n_i]}^{\mathrm{GL}}}^{\mathrm{GL}_{2n}(\mathbb{A}_E)} (\tau_1 |\det|^{-s_1} \otimes \cdots \otimes \tau_r |\det|^{-s_r}) \otimes \pi$.

It is easy to see that the normalizing factor of $M(w_0^r)$ outside S is given by

$$\prod_{i=1}^r \frac{L^S(s_i,\tau_i\times\pi^\vee)}{L^S(s_i+1,\tau_i\times\pi^\vee)}\cdot\prod_{i=1}^r \frac{L^S(2s_i,\tau_i,\mathrm{Asai})}{L^S(2s_i+1,\tau_i,\mathrm{Asai})}.$$

Then, as in the case of $r=1$, we see that the product of this function with $\prod_{i=1}^r(s_i-1)$ is non-trivial at $\bar{s}=\mathbf{1}$. In summary, $\prod_{i=1}^r(s_i-1)E(h,f_{\bar{\tau},\pi,\bar{s}})$ is holomorphic, and it is non-trivial at $\bar{s}=\mathbf{1}$ for some $f_{\bar{\tau},\pi,\bar{s}}$. □

For $s_0\in\mathbb{C}$ and $\mathbf{s}_0=(s_0,\dots,s_0)\in\mathbb{C}^r$, we define automorphic representations $E(\bar{\tau},\bar{s}_0)$ of $\mathrm{GL}_{2n}(\mathbb{A}_E)$ as follows. For a holomorphic section $f^{\mathrm{GL}}_{\bar{\tau},\bar{s}}$ of the induced representation $\mathrm{Ind}_{P^{\mathrm{GL}}_{[n_i]}}^{\mathrm{GL}_{2n}(\mathbb{A}_E)}(\tau_1|\det|^{s_1}\otimes\cdots\otimes\tau_r|\det|^{s_r})$, we form an Eisenstein series

$$E(h,f^{\mathrm{GL}}_{\bar{\tau},\bar{s}})=\sum_{\gamma\in P^{\mathrm{GL}}_{[n_i]}(F)\backslash\mathrm{GL}_{2n}(E)} f^{\mathrm{GL}}_{\bar{\tau},\bar{s}}(\gamma h),$$

which converges absolutely for $\mathrm{Re}(s_i)\gg 0$ and has a meromorphic continuation to $\mathbb{C}^r$ (Langlands [La76]). Then we define an automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_E)$ by

$$E(\bar{\tau},\mathbf{s}_0)=\left\langle E(\cdot,f^{\mathrm{GL}}_{\bar{\tau},\bar{s}}):f^{\mathrm{GL}}_{\bar{\tau},\bar{s}} \text{ such that } E(h,f^{\mathrm{GL}}_{\bar{\tau},\bar{s}}) \text{ is holomorphic at } \bar{s}=\mathbf{s}_0\right\rangle.$$

We note that this space may be zero. By the computation in [Sh10, Chapter 7], we can determine whether $E(\bar{\tau},\mathbf{s}_0)$ is zero or not by an analytic behavior of certain L-functions. We note that in [GRS11, Lemma 2.1], they give a sufficient condition so that the Eisenstein series $E(\bar{\tau},\bar{s})$ becomes holomorphic at $s=\mathbf{s}_0$. We can easily show that this space is not zero if

$$\prod_{i<j} L(s_i-s_j,\tau_i\times\tau_j^\vee)$$

is holomorphic and non-zero at $\bar{s}=\mathbf{s}_0$. Hence, this is equivalent to $\tau_i\not\simeq\tau_j$ for $i\neq j$. From our assumption on $\bar{\tau}$, $E(\bar{\tau},\mathbf{s}_0)$ is non-zero. We also note that in this case, any non-zero automorphic form in this space is generic.

From the proof of the above lemma, we obtain the following relation.

Corollary 4.4. *Let $\bar{\tau}$ and π be as in the previous lemma. Then we have*

$$\mathcal{C}_{N_{2n}^{6n}}(\mathcal{E}_{\bar{\tau},\pi})\subset\left(\delta_{P_{2n}^{6n}}^{\frac{1}{2}}\cdot E\left(\bar{\tau},-\mathbf{1}\right)\right)\otimes\pi$$

as automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A}_E)\times G_n(\mathbb{A})$.

Proof. We note that for $\xi_{\bar{\tau},\pi}\in\mathcal{E}_{\bar{\tau},\pi}$, we have

$$\begin{aligned}C_{N_{2n}^{6n}}(\xi_{\bar{\tau},\pi})&=\lim_{s_i\to1}\prod_{i=1}^r(s_i-1)E_{D_{2n}^{6n}}(g,M(w_0^r)f_{\bar{\tau},\pi,\bar{s}})\\&=\lim_{s_i\to1}E_{D_{2n}^{6n}}\left(g,\prod_{i=1}^r(s_i-1)M(w_0^r)f_{\bar{\tau},\pi,\bar{s}}\right).\end{aligned}$$

Since $M(w_0^r)f_{\bar{\tau},\pi,\bar{s}}\in\mathrm{Ind}_{P^{\mathrm{GL}}_{[n_i]}}^{\mathrm{GL}_{2n}(\mathbb{A}_E)}(\tau_1|\det|^{-s_1}\otimes\cdots\otimes\tau_r|\det|^{-s_r})\otimes\pi$, our claim follows from the above equation. □

Finally, we construct a residual representation of $G_{4n}(\mathbb{A})$ given by non-cuspidal representations.

Definition 4.5. Let $\bar{\tau}$ be as above. Then we say that an irreducible cuspidal automorphic representation π of $G_{2n}(\mathbb{A})$ is a CAP representation with respect to the CAP-datum

$$(\mathrm{GL}_{2n}, \bar{\tau}, \frac{1}{2})$$

if at almost all finite places v where the local components Π_v and $\tau_{i,v}$ are unramified, Π_v is isomorphic to the unramified irreducible constituent of

$$\mathrm{Ind}_{P_{[n_i]}(F_v)}^{G_{2n}(F_v)}(\tau_{1,v}|\det|_E^{\frac{1}{2}} \otimes \cdots \otimes \tau_{r,v}|\det|_E^{\frac{1}{2}}).$$

More generally, if $\Pi \in \mathcal{A}_d(G_{2n})$ (possibly non-cuspidal) satisfies this condition, we also say that Π is of type $(\mathrm{GL}_{2n}, \bar{\tau}, \frac{1}{2})$.

Let $\bar{\tau}$ be as above and let $\Pi \in \mathcal{A}_d(G_{2n})$ be of type $(\mathrm{GL}_{2n}, \bar{\tau}, \frac{1}{2})$. We consider the Eisenstein series on $G_{4n}(\mathbb{A})$ given by

$$E(h, f_{\bar{\tau},\Pi,\bar{s}}) = \sum_{\gamma \in P_{[n_i],2n}(F)\backslash G_{4n}(F)} f_{\bar{\tau},\Pi,\bar{s}}(\gamma g), \quad g \in G_{4n}(\mathbb{A}),$$

where $f_{\bar{\tau},\Pi,\bar{s}}$ is a holomorphic section of

$$\mathrm{Ind}_{P_{[n_i],2n}(\mathbb{A})}^{G_{4n}(\mathbb{A})}(\tau_1|\det|^{s_1} \otimes \cdots \otimes \tau_r|\det|^{s_r} \otimes \Pi).$$

This series converges absolutely for $\mathrm{Re}(s_i) \gg 0$ and has a meromorphic continuation to $\mathbb{C}^r$ (Langlands [La76]).

Proposition 4.6. *Let $\bar{\tau}$ and Π be as above. Then the function*

$$\bar{s} \mapsto \left(s_1 - \frac{3}{2}\right) \cdots \left(s_r - \frac{3}{2}\right) E(h, f_{\bar{\tau},\Pi,\bar{s}})$$

is holomorphic at $\bar{s} = \frac{3}{2} \cdot \mathbf{1}$, and its residue is non-trivial for some $f_{\bar{\tau},\Pi,\bar{s}}$. Let us denote by $\mathcal{E}_{\bar{\tau},\Pi}$ the automorphic representation of $G_{4n}(\mathbb{A})$ generated by these residues. Moreover we have, as spaces of automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A}_E) \times G_{2n}(\mathbb{A})$,

$$\mathcal{C}_{N_{2n}^{8n}}(\mathcal{E}_{\bar{\tau},\Pi}) \subset \left(\delta_{P_{2n}^{8n}}^{\frac{1}{2}} \cdot E\left(\bar{\tau}, -\frac{3}{2} \cdot \mathbf{1}\right)\right) \otimes \Pi.$$

Proof. We use the same argument as in the proof of Lemma 4.3 (see also the proof of [GRS11, Theorem 2.1]). Suppose that $r = 1$. We know that $\mathcal{C}_{N_k^{8n}}(E(\cdot, f_{\tau,\Pi;s}))(g) = 0$ unless $k = 2n$ and

$$\mathcal{C}_{N_{2n}^{8n}}(E(\cdot, f_{\tau,\Pi;s}))(g) = \sum_{w \in W_{2n,8n}} \int_{(wN_{2n}^{8n}(F)w^{-1} \cap N_{2n}^{8n}(F))\backslash N_{2n}^{8n}(\mathbb{A})} f_{\tau,\Pi;s}(w^{-1}ng)\, dn,$$

where $W_{2n,8n}$ is a subset of the Weyl group W_{8n} given as in (4.2). Then the longest Weyl element in $W_{2n,8n}$ is given by

$$\tilde{w}_0 = \begin{pmatrix} & & 1_{2n} \\ & 1_{4n} & \\ 1_{2n} & & \end{pmatrix} \in G_{4n}(F).$$

We note that the constant term of Π along a parabolic subgroup $P_0 \subset G_{2n}$ is zero except for $P_0 = P_{2n}^{4n}$. Then by the cuspidality of τ and this fact, we may rewrite this constant term as

$$f_{\tau,\Pi;s}(g) + M(\tilde{w}_0) f_{\tau,\Pi;s}(g) + \sum_w \int_{(wP_{2n}^{8n}(F)w^{-1} \cap N_{2n}^{8n}(F))\backslash N_{2n}^{8n}(\mathbb{A})} f_{\tau,\Pi;s}(w^{-1}ng)\, dn,$$

where w ranges over an element in $W_{2n,8n}$ such that $P_{2n}^{8n}(F) \cap w^{-1}N_{2n}^{8n}(F)w = N_{2n}^{4n}(F)$. Each term in the last part is written as

$$\int_{wN_{2n}^{4n}(F)w^{-1}\backslash N_{2n}^{8n}(\mathbb{A})} f_{\tau,\Pi;s}(w^{-1}ng)\,dn = \int_{N_{2n}^{4n}(\mathbb{A})\backslash w^{-1}N_{2n}^{8n}(\mathbb{A})w} \int_{N_{2n}^{4n}(F)\backslash N_{2n}^{4n}(\mathbb{A})} f_{\tau,\Pi;s}(nw^{-1}ug)\,dn\,du.$$

The inner integral is the constant term of Π, and thus it is easy to see that this integral is holomorphic at $s=\frac{3}{2}$. On the other hand, the normalizing factor of $M(\tilde{w}_0)f_{\tau,\Pi;s}(g)$ is given by

$$\frac{L^S(s,\Pi^\vee\times\tau)}{L^S(s+1,\Pi^\vee\times\tau)}\times\frac{L^S(2s,\tau,\mathrm{Asai})}{L^S(2s+1,\tau,\mathrm{Asai})}.$$

Since Π is of type $(\mathrm{GL}_{2n},\tau,\frac{1}{2})$, this is equal to

$$\frac{L^S(s+\frac{1}{2},\tau\times\tau^\vee)L^S(s-\frac{1}{2},\tau\times\tau^\vee)}{L^S(s+\frac{1}{2},\tau\times\tau^\vee)L^S(s+\frac{3}{2},\tau\times\tau^\vee)}\times\frac{L^S(2s,\tau,\mathrm{Asai})}{L^S(2s+1,\tau,\mathrm{Asai})}.$$

We know that $L^S(s-\frac{1}{2},\tau\times\tau^\vee)$ has a pole at $s=\frac{3}{2}$ and remaining factors are holomorphic and non-zero at $s=\frac{3}{2}$. Therefore, $M(\tilde{w}_0)f_{\tau,\Pi,\bar{s}}$ has a pole at $s=\frac{3}{2}$, and thus $(s-\frac{3}{2})E(\cdot,f_{\tau,\Pi;s})$ is holomorphic and non-zero at $s=\frac{3}{2}$. Finally, we note that the last assertion follows as in the proof of Corollary 4.4.

Suppose that $r>1$. We know that

$$\mathcal{C}_{N_k^{8n}}(E(\cdot,f_{\bar{\tau},\Pi,\bar{s}}))(g)=\sum_{w\in W_{[n_i],k,8n}} E_{D_k^{4n}}(g,M(w)f_{\bar{\tau},\Pi,\bar{s}}),$$

where $W_{[n_i],k,8n}$ is a subset of the Weyl group W_{8n} given as in (4.2) and $E_{D_k^{4n}}(g,M(w)f_{\bar{\tau},\Pi,\bar{s}})$ is a certain Eisenstein series of D_k^{4n}. Then as in the proof of Lemma 4.3 using the case of $r=1$, we see that only the term corresponding to $\tilde{w}_0$ contributes to the residue. We note that in this case, we should have $k=2n$. Further, the normalizing factor of $M(\tilde{w}_0)f_{\bar{\tau},\Pi,\bar{s}}$ is

$$\prod_{i<j}\frac{L^S(s_i+s_j,\tau_i\times\tau_j^\vee)}{L^S(s_i+s_j+1,\tau_i\times\tau_j^\vee)}\prod_i\left(\frac{L^S(s_i,\Pi^\vee\times\tau_i)}{L^S(s_i+1,\Pi^\vee\times\tau_i)}\times\frac{L^S(2s_i,\tau_i,\mathrm{Asai})}{L^S(2s_i+1,\tau_i,\mathrm{Asai})}\right).$$

Since Π is of type $(\mathrm{GL}_{2n},\bar{\tau},\frac{1}{2})$, this is equal to

$$\prod_{i<j}\frac{L^S(s_i+s_j,\tau_i\times\tau_j^\vee)}{L^S(s_i+s_j+1,\tau_i\times\tau_j^\vee)}\times\prod_{i,j}\frac{L^S(s_i+\frac{1}{2},\tau_i\times\tau_j^\vee)L^S(s-\frac{1}{2},\tau_i\times\tau_j^\vee)}{L^S(s_i+\frac{1}{2},\tau_i\times\tau_j^\vee)L^S(s+\frac{3}{2},\tau_i\times\tau_j^\vee)}\times\prod_i\frac{L^S(2s_i,\tau_i,\mathrm{Asai})}{L^S(2s_i+1,\tau_i,\mathrm{Asai})}.$$

Then $\prod_{i=1}^r L^S(s_i-\frac{1}{2},\tau_i\times\tau_i^\vee)$ has a pole at $\bar{s}=\frac{3}{2}\cdot\mathbf{1}$, and remaining factors are holomorphic and non-zero at $\bar{s}=\frac{3}{2}\cdot\mathbf{1}$. Therefore,

$$\left(s_1-\frac{3}{2}\right)\cdots\left(s_r-\frac{3}{2}\right)E(\cdot,f_{\bar{\tau},\Pi,\bar{s}})$$

is holomorphic, and it is non-zero at $\bar{s}=\frac{3}{2}\cdot\mathbf{1}$ for some $f_{\bar{\tau},\Pi,\bar{s}}$. The remaining part is also proved as in the proof of Lemma 4.3 and Corollary 4.4. □

5. Vanishing and non-vanishing of Fourier coefficients

In this section, we shall study Fourier coefficients and Fourier-Jacobi coefficients of residual representations. Indeed, we give some vanishing results and non-vanishing results on these coefficients. Further, using these results, we shall study certain descent maps of G_{3n}.

The following useful lemma is proved in the same way as [GRS03, Lemma 1.1] (see also [JL15, Corollary 2.6]).

Lemma 5.1. *Let φ be an automorphic form on $G_k(\mathbb{A})$. Then the following conditions are equivalent:*

(1)

$$\int_{V_r^{2k}(F)\backslash V_r^{2k}(\mathbb{A})} \varphi(vg)\psi_{V_r^{2k},a}(v)\,dv = 0$$

for all $g \in G_k(\mathbb{A})$,

(2)

$$\int_{Y(F)\backslash Y(\mathbb{A})} \int_{V_r^{2k}(F)\backslash V_r^{2k}(\mathbb{A})} \varphi(vyg)\psi_{V_r^{2k},a}(v)\,dv\,dy = 0$$

for all $g \in G_k(\mathbb{A})$,

(3)

$$\int_{U_r^{2k}(F)\backslash U_r^{2k}(\mathbb{A})} \varphi(ug)\theta_{\phi,k-r}^{\psi^a,\eta}(\ell_{k-r}(u))\psi_{U_r^{2k}}(u)\,du = 0$$

for all $g \in G_k(\mathbb{A})$.

Here, Y is a maximal abelian subgroup of U_r^{2k}/V_r^{2k}. Similarly, for an automorphic form φ $G_{2n+r}(\mathbb{A})$ and any $a, b \in F^\times$, the following conditions are equivalent:

(1)

$$\int_{V_{[(2n)^2 1^{2r}]}(F)\backslash V_{[(2n)^2 1^{2r}]}(\mathbb{A})} \varphi(vg)\psi_{V_{[(2n)^2 1^{2r}]};b,a}(v)\,dv = 0$$

for all $g \in G_k(\mathbb{A})$,

(2)

$$\int_{Y_0(F)\backslash Y_0(\mathbb{A})} \int_{V_{[(2n)^2 1^{2r}]}(F)\backslash V_{[(2n)^2 1^{2r}]}(\mathbb{A})} \varphi(yvg)\psi_{V_{[(2n)^2 1^{2r}]};b,a}(v)\,dv\,dy = 0$$

for all $g \in G_k(\mathbb{A})$.

Here, $V'_{[(2n)^2 1^{2r}]}$ is the unipotent radical of the standard parabolic subgroup of G_{2n+r} whose Levi part is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_2^n \times G_r$, and Y_0 is a maximal abelian subgroup of $V'_{[(2n)^2 1^{2r}]}/V_{[(2n)^2 1^{2r}]}$.

In the second case, we may give an equivalent condition in terms of Fourier-Jacobi coefficient as in the first case. Since these two conditions are sufficient for our purpose, we only give here an equivalence of these two conditions.

Now, let us prove a crucial result on vanishing and non-vanishing of Fourier coefficients on a residual representation. We shall prove the following theorem in a similar way as the proof of [GJS12, Theorem 2.1] except for one step as explained in the Introduction.

Theorem 5.2. *Let $\bar{\tau}$ and π be as in Assumption* 1 *or Theorem* 4.2 *when π is ψ^{-1}-generic. Then for all l such that $n < l \leq 3n$ and any $a \in F^\times$, $\mathcal{F}^{\psi_{V_l^{6n}},a}$ is zero on the residual representation $\mathcal{E}_{\bar{\tau},\pi}$. Moreover, for all $a \in F^\times$, the Fourier coefficient $\mathcal{F}^{\psi_{V_n^{6n}},a}$ is non-trivial on $\mathcal{E}_{\bar{\tau},\pi}$.*

Proof. Let π_0 be an irreducible subquotient of $\mathcal{E}_{\bar{\tau},\pi}$. Let v be a place of F such that $\pi_{0,v}$ is unramified. Then $\pi_{0,v}$ is the unramified quotient of the induced representation

$$\mathrm{Ind}_{P_{2n}^{6n}}^{G_{3n}(F_v)}(\tau_v|\det|_E \otimes \pi_v),$$

where τ_v is the local functorial lift of π_v to $\mathrm{GL}_{2n}(E_v)$. Then from Lemma A.3, it does not have a linear functional ℓ on $V_{\pi_{0,v}}$ such that $\ell(\pi_{0,v}(u)w) = \psi_{V_l^{6n}}(u)\ell(w)$ for any $u \in V_l^{6n}(F_v)$. Hence, the Fourier coefficient $\mathcal{F}^{\psi_{V_l^{6n}},a}$ is zero on π_0, and thus this Fourier coefficient is zero on $\mathcal{E}_{\bar{\tau},\pi}$.

Let us prove a non-triviality of the Fourier coefficient $\mathcal{F}^{\psi_{V_n^{6n}},a}$ on $\mathcal{E}_{\bar{\tau},\pi}$. Because of Lemma 5.1, for $\xi_{\bar{\tau},\pi} \in \mathcal{E}_{\bar{\tau},\pi}$, the non-vanishing of

$$\mathcal{F}^{\psi_{V_n^{6n}},a}(\xi_{\bar{\tau},\pi}) = \int_{V_n^{6n}(F)\backslash V_n^{6n}(\mathbb{A})} \xi_{\bar{\tau},\pi}(v)\psi_{V_n^{6n},a}(v)\,dv$$

is equivalent to the non-vanishing of

$$\mathcal{F}^{\psi_{\tilde{V}_n^{6n}},a}(\xi_{\bar{\tau},\pi}) = \int_{\tilde{V}_n^{6n}(F)\backslash \tilde{V}_n^{6n}(\mathbb{A})} \xi_{\bar{\tau},\pi}(v)\psi_{\tilde{V}_n^{6n},a}(v)\,dv,$$

where $\tilde{V}_n^{6n}$ is the group consisting of the following elements in $G_{3n}(F)$:

$$v(u,x,z) = \begin{pmatrix} u & x & z \\ & 1_{4n} & x' \\ & & u^* \end{pmatrix}$$

where $u \in Z_n(F)$ (recall that this is the group of upper unipotent matrices of $\mathrm{GL}_n(E)$), $x \in \mathrm{Mat}_{n\times 4n}(E)$ is such that $x_{n,1} = \cdots = x_{n,3n} = 0$, and

$$\psi_{\tilde{V}_n^{6n},a}(v(u,x,z)) = \psi(u_{1,2} + \cdots + u_{n-1,n} + az_{n,1}).$$

Clearly, the non-triviality of this Fourier coefficient follows from the non-triviality of the following Fourier coefficient on the residual representation $\mathcal{E}_{\bar{\tau},\pi}$:

(5.1)

$$\int_{V_n^{4n}(F)\backslash V_n^{4n}(\mathbb{A})} \int_{\tilde{V}_n^{6n}(F)\backslash \tilde{V}_n^{6n}(\mathbb{A})} \xi_{\bar{\tau},\pi}(vv_1)\psi_{\tilde{V}_n^{6n},a}(v)\psi_{\tilde{V}_n^{4n},-a}(v_1)\,dv\,dv_1, \quad \xi_{\bar{\tau},\pi} \in \mathcal{E}_{\bar{\tau},\pi}.$$

Here we regard a subgroup of G_{2n} as a subgroup of G_{3n} via the embedding

$$g \mapsto \begin{pmatrix} 1_n & & \\ & g & \\ & & 1_n \end{pmatrix}.$$

Let $\tilde{\omega}$ be a Weyl element of $\mathrm{GL}_{2n}(E)$ defined by

$$\tilde{\omega}_{2i,i} = \tilde{\omega}_{2i-1,i+n} = 1, \quad i = 1,\ldots,n,$$
$$\tilde{\omega}_{i,j} = 0 \quad \text{otherwise}.$$

Put

$$\omega = \begin{pmatrix} \tilde{\omega} & & \\ & 1_{2n} & \\ & & \tilde{\omega}^* \end{pmatrix} \in G_{3n}(F). \tag{5.2}$$

Let $R = V_n^{4n}\bar{V}_n^{6n}$, and consider

$$B = \omega R \omega^{-1}.$$

Then the integral (5.1) is written as

$$\int_{B(F)\backslash B(\mathbb{A})} \xi_{\bar{\tau},\pi}(v\omega)\chi_{\psi,a}(v)\, dv. \tag{5.3}$$

Here, B denotes the group consisting of the following elements of G_{3n}:

$$v(T,C,Z) = \begin{pmatrix} T & C & Z \\ & 1_{2n} & C' \\ & & T^* \end{pmatrix}, \tag{5.4}$$

where the last two rows of C are zero and $T \in \mathrm{GL}_{2n}(E)$ such that when we write T as an $n \times n$ matrix of 2×2 block matrices $T = ([T]_{i,j}), 1 \leq i,j \leq n$,

(1) $[T]_{n,1} = \cdots = [T]_{n,n-1} = 0, [T]_{n,n} = 1_2$;
(2) $[T]_{i,i}$ is lower unipotent, for $i < n$;
(3) $[T]_{i,j}$ is lower triangular, for $i < j$;
(4) $[T]_{i,j}$ is lower nilpotent, for $j < i < n$.

With this notation, $\chi_{\psi,a}$ is the character of $B(\mathbb{A})$ given by

$$\psi\left(\mathrm{tr}\left([T]_{1,2} + [T]_{2,3} + \cdots + [T]_{n-1,n}\right) + a(Z_{2n,1} - Z_{2n-1,2})\right).$$

Then applying the exact same argument in [GJS12, pp. 965–968], we can show that the integral (5.3) is equal to

$$\int_{Y(\mathbb{A})}\int_{L(F)\backslash L(\mathbb{A})} \xi_{\bar{\tau},\pi}(vy\omega)\psi'_{L,a}(v)\, dv, \tag{5.5}$$

where Y is the subgroup of lower unipotent matrices in B, $L = V_{[(2n)^2 1^{2n}]}$, and $\psi'_{L,a} = \psi_{[(2n)^2 1^{2n}];a,-a}$. Further, from [GRS11, Corollary 7.2], for a given $\xi_{\bar{\tau},\pi} \in \mathcal{E}_{\bar{\tau},\pi}$, there is $\xi'_{\bar{\tau},\pi}$ such that

$$\int_{Y(\mathbb{A})}\int_{L(F)\backslash L(\mathbb{A})} \xi_{\bar{\tau},\pi}(vyh\omega)\psi'_{L,a}(v)\, dv\, dy = \int_{L(F)\backslash L(\mathbb{A})} \xi'_{\bar{\tau},\pi}(vyh)\psi'_{L,a}(v)\, dv.$$

Let

$$b = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and define

$$\hat{b} = \mathrm{diag}(b, \ldots, b, 1_{2n}, b^*, \ldots, b^*) \in G_{3n}(F). \tag{5.6}$$

Then we may write

$$\mathcal{F}^{\psi_{[(2n)^2 1^{2n}];a,-a}}(\xi_{\bar{\tau},\pi}) = \int_{L(F)\backslash L(\mathbb{A})} \xi_{\bar{\tau},\pi}(v\hat{b})\psi_{L,a}(v)\, dv,$$

where $\psi_{L,a}$ is the character given by $v \mapsto \psi'_{L,a}(\hat{b}^{-1}v\hat{b})$. We write an element of the unipotent subgroup L in the form

$$v(A,C,Z) = \begin{pmatrix} A & C & Z \\ & 1_{2n} & C' \\ & & A^* \end{pmatrix} \tag{5.7}$$

where the last two rows of C are zero, and when we write A as an $n \times n$ matrix of 2×2 blocks, A should be of the form

$$A = \begin{pmatrix} 1_2 & A_{1,2} & \cdots\cdots & A_{1,n} \\ & 1_2 & \cdots\cdots & A_{2,n} \\ & & & \vdots \\ & & \ddots & A_{n-1,n} \\ & & & 1_2 \end{pmatrix}.$$

Let L' be the subgroup consisting of $v(A, C, Z)$ with A as above and C such that only its last row is zero. Then by the second case of Lemma 5.1, $\mathcal{F}^{\psi_{[(2n)^2 1^{2n}];a,-a}}(\xi_{\bar{\tau},\pi})$ is not identically zero on $\mathcal{E}_{\bar{\tau},\pi}$ if and only if the integral

$$\mathcal{F}^{\psi_{L',a}}(\xi_{\bar{\tau},\pi}) = \int_{L'(F)\backslash L'(\mathbb{A})} \xi_{\bar{\tau},\pi}(v)\psi_{L',a}(v)\,dv \tag{5.8}$$

is not identically zero on $\mathcal{E}_{\bar{\tau},\pi}$. Here, $\psi_{L',a}(v)$ is the character defined as in the definition of $\psi_{L,a}$.

Let ν be the Weyl element in $G_{2n}(F)$ defined as in [GJS12, p. 970]:

$$\begin{aligned} \nu_{i,2i-1} &= 1, \quad i = 1, \ldots, 2n, \\ \nu_{2n+i,2i} &= -1, \quad i = 1, \ldots, n, \\ \nu_{2n+i,2i} &= 1, \quad i = n+1, \ldots, n, \\ \nu_{i,j} &= 0 \quad \text{otherwise}. \end{aligned}$$

Write

$$\nu = \begin{pmatrix} \nu_1 & \nu_2 \\ \nu_3 & \nu_4 \end{pmatrix}$$

where ν_i are $2n \times 2n$ matrices, and let

$$\nu' = \begin{pmatrix} \nu_1 & & \nu_2 \\ & 1_{2n} & \\ \nu_3 & & \nu_4 \end{pmatrix}. \tag{5.9}$$

Define

$$B' = \nu' L' (\nu')^{-1}.$$

Then elements in B' have the following form:

$$v = \begin{pmatrix} u_1 & u_2 & c & z_1 & z_2 \\ 0 & u_3 & 0 & 0 & z_1' \\ 0 & d' & 1_{2n} & 0 & c' \\ y_1 & y_2 & d & u_3^* & u_2' \\ 0 & y_1' & 0 & 0 & u_1^* \end{pmatrix} \in G_{3n}, \tag{5.10}$$

where u_1, u_3 (and also u_1^*, u_3^*) are $n \times n$ upper unipotent, z_1, y_1 (and also z_1', y_1') are upper nilpotent, the last row of d is zero, and the first column of d' is zero. We have

$$\mathcal{F}^{\psi_{L',a}}(\xi_{\tau,\pi}) = \int_{B'(F)\backslash B'(\mathbb{A})} \xi_{\tau,\pi}(v\nu')\psi_{B',a}(v)\,dv \tag{5.11}$$

where

$$\psi_{B',a}(v) = \psi((u_1)_{1,2} + (u_1)_{n-1,n} - a(u_2)_{n,1} - (u_3)_{1,2} - \cdots - (u_3)_{n-1,n}). \tag{5.12}$$

Lemma 5.3. *The right-hand side of* (5.11) *is equal to*

$$\int_{L_0(\mathbb{A})}\int_{V_{2n}^{6n}(F)\backslash V_{2n}^{6n}(\mathbb{A})}\xi_{\tau,\pi}(vy\nu')\psi'_a(v)\,dv\,dy, \tag{5.13}$$

where L_0 is the group consisting of lower unipotent matrices in B', and the character $\psi'_a(v)$ is the character of $V_{2n}^{6n}(\mathbb{A})$ defined by the same formula of (5.12)*.*

Proof. Before proceeding with a proof, we note that in the proof of this lemma, we will encounter an essentially different part (e.g., see (5.19)) from the proof of [GJS12, Theorem 2.1].

Let

$$\mathcal{Z}_{2n}=v(Z_{2n},0,0)\subset B'.$$

Here, recall that Z_{2n} denotes the group of upper triangular unipotent matrices in $\mathrm{Res}_{E/F}\mathrm{GL}_{2n}$. Define $1\le i\le j\le 2n$ such that $i+j\neq 2n+1$,

$$\begin{aligned}X_{i,j}=&\{v(I_{2n},0,te_{i,j}+\bar{t}e_{2n+1-j,2n+1-i}):t\in\mathrm{Res}_{E/F}\mathbb{G}_a\},\\ Y_{i,j}=&\{\bar{v}(I_{2n},0,te_{i,j}+\bar{t}e_{2n+1-j,2n+1-i}):t\in\mathrm{Res}_{E/F}\mathbb{G}_a\},\end{aligned}$$

where $v(A,C,Z)$ is as in (5.7), and

$$\bar{v}(A,C,Z)=\begin{pmatrix}A&&\\C&1_{2n}&\\Z&C'&A^*\end{pmatrix}\in G_{3n},\quad A\in\mathrm{Res}_{E/F}\mathrm{GL}_{2n}.$$

Further, for $1\le i\le 2n$, define

$$X_{i,2n+1-i}=\{v(I_{2n},0,te_{i,2n+1-i}:t\in\mathbb{G}_a\}$$

and

$$Y_{i,2n+1-i}=\{\bar{v}(I_{2n},0,te_{i,2n+1-i}:t\in\mathbb{G}_a\}.$$

We note that

$$X_{i,j}=X_{2n+1-j,2n+1-i}\quad\text{and}\quad Y_{i,j}=Y_{2n+1-j,2n+1-i}.$$

Similarly, for $1\le i,j\le 2n$, define

$$X'_{i,j}=\{v(1_{2n},te_{i,j},0):t\in\mathrm{Res}_{E/F}\mathbb{G}_a\}$$

and

$$Y'_{i,j}=\{\bar{v}(1_{2n},te_{i,j},0):t\in\mathrm{Res}_{E/F}\mathbb{G}_a\}.$$

For simplicity, we shall denote the group of F-rational points of $X_{i,j},Y_{i,j},X'_{i,j},Y'_{i,j}$ by the same symbol. Then we have

$$B'=\langle\mathcal{Z}_{2n},X_{i,j},Y_{i,j},X'_{p,q},Y'_{r,s};1\le i<j\le 2n,1\le p\le n,1\le q,r\le 2n,\\ n+2\le s\le 2n\rangle.$$

For $1\le i<j\le n+1$, we define a subgroup $C_{i,j}$ of B' as follows. First, define

$$T(i,j)=\{(a,b)\in\mathbb{N}^2:b\le a\le j-2,\text{ or }a=j-1\text{ and }i+1\le b\le j-1\}$$

and

$$S(i,j)=\left\{(a,b)\in\mathbb{N}^2:2\le b<j\text{ and }1\le a\le b-1,\text{ or }b=j\text{ and }i\le a\le j-1\right\}.$$

Then we define

$$C_{i,j}=\left\langle\mathcal{Z}_{2n},X'_{p,q},Y'_{r,s},X_{k,\ell},Y_{s,t}\;\middle|\;\begin{array}{l}1\le p\le n,1\le q,\quad r\le 2n,n+2\le s\le 2n,\\(k,\ell)\in T(i,j),\quad(s,t)\notin S(i,j)\end{array}\right\rangle\cap B'.$$

Further, we define

$$A_{i,j} := D_{i,j}Y_{i,j},\ B_{i,j} := C_{i,j}Y_{i,j},\ \text{and } D_{i,j} := C_{i,j}X_{j-1,i}$$

and we put

$$D^0_{n,n+1} = C_{n,n+1}X^0_{n,n},$$

where

$$X^0_{n,n} = X_{n,n} \cap \mathrm{GL}_{6n/F}.$$

Then we note that

$$B' = B_{1,2} \tag{5.14}$$

and

$$D_{i,j} = B_{i-1,j} \quad (2 \le i < j \le n+1) \quad \text{and} \quad D_{1,j} = B_{j,j+1} \quad (1 \le j \le n). \tag{5.15}$$

We can easily show that the character $\eta_{\psi,a} := \psi_{B',a}|_{\mathcal{Z}_{2n}(\mathbb{A})}$ of $\mathcal{Z}_{2n}(\mathbb{A})$ can be extended to $C_{i,j}(\mathbb{A}), B_{i,j}(\mathbb{A})$, and $D_{i,j}(\mathbb{A})$ so that it is trivial on the corresponding subgroups $X_{p,q}(\mathbb{A}), Y_{p,q}(\mathbb{A}), X'_{p,q}(\mathbb{A})$, and $Y'_{p,q}(\mathbb{A})$. We denote each such extension by $\eta^{(i,j)}_{\psi,a}$.

Define for $h \in G_{3n}(\mathbb{A})$,

$$R_{i,j}(\xi_{\bar{\tau},\pi})(h) = \int_{B_{i,j}(F)\backslash B_{i,j}(\mathbb{A})} \xi_{\bar{\tau},\pi}(vh\nu')\eta^{(i,j)}_{\psi,a}(v)\,dv$$

and

$$R'_{i,j}(\xi_{\bar{\tau},\pi})(h) = \int_{D_{i,j}(F)\backslash D_{i,j}(\mathbb{A})} \xi_{\bar{\tau},\pi}(vh\nu')\eta^{(i,j)}_{\psi,a}(v)\,dv. \tag{5.16}$$

Then the right-hand side of (5.11) is equal to $R_{1,2}(\xi_{\bar{\tau},\pi})(1)$ by (5.14).

For $1 \le i < j \le n$, applying [GRS11, Lemma 7.1] for $A_{i,j}, B_{i,j}, C_{i,j}, D_{i,j}, X_{j-1,i}$, and $Y_{i,j}$ with $\eta^{(i,j)}_{\psi,a}$, we obtain

$$R_{i,j}(\xi_{\bar{\tau},\pi})(1) = \int_{Y_{i,j}(\mathbb{A})} R'_{i,j}(\xi_{\bar{\tau},\pi})(y)\,dy. \tag{5.17}$$

Hence,

$$\begin{aligned} R_{1,2}(\xi_{\bar{\tau},\pi})(1) &= \int_{Y_{1,2}(\mathbb{A})} R'_{1,2}(\xi_{\bar{\tau},\pi})(y)\,dy \\ &= \int_{Y_{1,2}(\mathbb{A})} \int_{D_{1,2}(F)\backslash D_{1,2}(\mathbb{A})} \xi_{\bar{\tau},\pi}(v\nu')\eta^{(1,2)}_{\psi,a}(v)\,dv. \end{aligned}$$

By (5.15), this is equal to

$$\int_{Y_{1,2}(\mathbb{A})} \int_{B_{2,3}(F)\backslash B_{2,3}(\mathbb{A})} \xi_{\bar{\tau},\pi}(v\nu')\eta^{(2,3)}_{\psi,a}(v)\,dv = \int_{Y_{1,2}(\mathbb{A})} R_{2,3}(\xi_{\bar{\tau},\pi})(y)\,dy.$$

Using (5.17) again, we obtain

$$R_{1,2}(\xi_{\bar{\tau},\pi})(1) = \int_{Y_{1,2}(\mathbb{A})} \int_{Y_{2,3}(\mathbb{A})} R'_{2,3}(\xi_{\bar{\tau},\pi})(y_1y_2)\,dy_1\,dy_2.$$

Repeating this argument, we obtain

$$R_{1,2}(\xi_{\bar{\tau},\pi})(1) = \int_{Y_0(\mathbb{A})} R'_{1,n}(\xi_{\bar{\tau},\pi})(y)\,dy, \tag{5.18}$$

where Y_0 is the group generated by

$$Y_{1,n} \cdots Y_{n-1,n} \cdots Y_{1,3}, Y_{2,3}, Y_{1,2}.$$

Similarly, we may apply [GRS11, Lemma 7.1] for

$$A_{n,n+1}, B_{n,n+1}, C_{n,n+1}, D^0_{n,n+1}, X^0_{n,n}, \text{ and } Y_{n,n+1}$$

with $\eta^{(n,n+1)}_{\psi,a}$, and thus we obtain

$$R_{1,2}(\xi_{\bar{\tau},\pi})(1) = \int_{Y_{00}(\mathbb{A})} (R'_{1,n})^0(\xi_{\bar{\tau},\pi})(y)\, dy,$$

where

$$(R'_{1,n})^0(\xi_{\bar{\tau},\pi})(h) = \int_{D^0_{n,n+1}(F)\backslash D^0_{n,n+1}(\mathbb{A})} \xi_{\bar{\tau},\pi}(vh\nu')\eta^{(n,n+1)}_{\psi,a}(v)\, dv$$

and

$$Y_{00} = \langle Y_{n,n+1}, Y_0\rangle.$$

Now, we shall show that

$$\int_{F\backslash\mathbb{A}} (R'_{1,n})^0(\xi_{\bar{\tau},\pi})(v(1_{2n}, 0, \iota a e_{n,n} - \iota a e_{n+1,n+1})h)\psi_F(\lambda a)\, da = 0 \tag{5.19}$$

for any $\lambda \in F^\times$. Define a character $\eta^{(n,n+1),0}_{\psi,a}$ on $D_{n,n+1}(\mathbb{A}) = C_{n,n+1}(\mathbb{A})X_{n,n}(\mathbb{A})$ by

$$\eta^{(n,n+1),0}_{\psi,a}(cv(1_{2n}, 0, te_{n,n} - \bar{t}e_{n+1,n+1})) := \eta^{(n,n+1)}_{\psi,a}(c)\psi(\lambda\iota^{-1}t)$$

for $c \in C_{n,n+1}(\mathbb{A})$ and $t \in \mathbb{A}_E$. Then we see that the integral (5.19) is equal to

$$\int_{D_{n,n+1}(F)\backslash D_{n,n+1}(\mathbb{A})} \xi_{\bar{\tau},\pi}(vh\nu')\eta^{(n,n+1),0}_{\psi,a}(v)\, dv.$$

We note that as an inner integral, this integral contains the integral

$$\int_{D^*_{n,n+1}(F)\backslash D^*_{n,n+1}(\mathbb{A})} \xi_{\bar{\tau},\pi}(vh\nu')\eta^{(n,n+1),0}_{\psi,a}(vv)\, dv, \tag{5.20}$$

where $D^*_{n,n+1}$ denotes the group

$$\{d \in D_{n,n+1} : \text{when we consider } d \text{ as a block matrix of size } n \times n, \text{ its } (2,2)\text{-block is } 1_n\}.$$

Then for our purpose, it suffices to show that the integral (5.20) is identically zero.

For $1 \le i \le n-1$, let us define subgroups $A^*_{i,n+1}(\mathbb{A}), B^*_{i,n+1}(\mathbb{A}), C^*_{i,n+1}(\mathbb{A})$, and $D^*_{i,n+1}(\mathbb{A})$ of $A_{i,n+1}(\mathbb{A})$, $B_{i,n+1}(\mathbb{A})$, $C_{i,n+1}(\mathbb{A})$, and $D_{i,n+1}(\mathbb{A})$ satisfying the same condition in the definition of $D^*_{n,n+1}$, respectively. Then we can extend the character $\eta^{(n,n+1),0}_{\psi,a}$ to $B^*_{i,n+1}(\mathbb{A}), C^*_{i,n+1}(\mathbb{A})$, and $D^*_{i,n+1}(\mathbb{A})$ so that it is trivial on corresponding $X_{p,q}(\mathbb{A})$, $Y_{p,q}(\mathbb{A})$, $X'_{p,q}(\mathbb{A})$, $Y'_{p,q}(\mathbb{A})$. We denote each such character by $\eta^{(i,n+1),0}_{\psi,a}$.

Repeatedly using [GRS11, Lemma 7.1] for groups $A^*_{i,n+1}, B^*_{i,n+1}, C^*_{i,n+1}, D^*_{i,n+1}$, $Y_{i,n+1}$ and $X_{n,i}$ for $1 \le i \le n-1$ with the character $\eta^{(i,n+1),0}_{\psi,a}$, as in (5.18), we see that the integral (5.20) is equal to

$$\int_{Y^*(\mathbb{A})} \int_{D^*_{1,n+1}(F)\backslash D^*_{1,n+1}(\mathbb{A})} \xi_{\tau,\pi}(vy\nu')\eta^{(1,n+1),0}_{\psi,a}(vy)\, dv\, dy,$$

where Y^* is the group generated by

$$Y_{1,n+1}Y_{2,n+1}\cdots Y_{n-1,n+1}.$$

This integral contains the following integral as an inner integral:

$$\int_{U_n^{6n}(F)\backslash U_n^{6n}(\mathbb{A})}\xi_{\tau,\pi}(u\nu')\eta_{\psi,a}^{(1,n+1),0}(u)\,du.$$

As in the beginning of the proof of this theorem, this should be zero by Lemma A.2. In summary, the integral (5.19) is zero. Further, we see that taking the Fourier expansion of $(R'_{1,n})^0(\xi_{\bar\tau,\pi})(h)$ along $X_{n,n}$,

$$(R'_{1,n})^0(\xi_{\bar\tau,\pi})(h)=\int_{D_{n,n+1}(F)\backslash D_{n,n+1}(\mathbb{A})}\xi_{\bar\tau,\pi}(vh\nu')\eta_{\psi,a}^{(n,n+1)}(v)\,dv$$

and

$$R_{1,2}(\xi_{\bar\tau,\pi})(1)=\int_{Y_{00}(\mathbb{A})}R'_{n,n+1}(\xi_{\bar\tau,\pi})(y)\,dy.$$

Repeating the above argument, we see that

$$R_{1,2}(\xi_{\bar\tau,\pi})(1)=\int_{Y(\mathbb{A})}R'_{1,n+1}(\xi_{\bar\tau,\pi})(y)\,dy, \tag{5.21}$$

where Y is the group generated by

$$Y_{1,n+1}\cdots Y_{n,n+1}\cdots Y_{1,3},Y_{2,3},Y_{1,2}.$$

Let us consider the Fourier expansion of $R'_{1,n+1}(\xi_{\tau,\pi})(h)$ along $X_{n+1,n}$. Then each Fourier coefficient with respect to a non-trivial character contains a Fourier coefficient $\mathcal{F}^{\psi_{V_{n+1}^{6n}},a}(\xi_{\bar\tau,\pi})$ of $\xi_{\bar\tau,\pi}$ as an inner integral, and it is zero by the first part of this theorem. Thus, only trivial character contributes to the Fourier expansion, and we get

$$\begin{aligned}R'_{1,n+1}(\xi_{\bar\tau,\pi})(h)&=\int_{X_{n+1,n}(F)\backslash X_{n+1,n}(\mathbb{A})}R'_{1,n+1}(\xi_{\bar\tau,\pi})(xh)\,dx\\&=\int_{X_{n+1,n}(F)\backslash X_{n+1,n}(\mathbb{A})}\int_{D_{1,n+1}(F)\backslash D_{1,n+1}(\mathbb{A})}\xi_{\bar\tau,\pi}(vxh\nu')\eta_{\psi,a}^{(1,n+1)}(v)\,dv\,dx.\end{aligned}\tag{5.22}$$

For $1\leq i\leq n-1$, let us define $C_{i,n+2}$ and $D_{i,n+2}$ as follows. First, we let $C_{n-1,n+2}$ be the subgroup of $D_{1,n+1}X_{n+1,n}$ generated by $\mathcal{Z}_{2n}$ and all roots in $D_{1,n+1}X_{n+1,n}$ except $Y_{n-1,n+2}$. Then we put $D_{n-1,n+2}=C_{n-1,n+2}X_{n+1,n-1}$. Inductively, we may define the subgroup $C_{i,n+2}$ of $D_{i+1,n+2}$ generated by $\mathcal{Z}_{2n}$ and all roots except $Y_{i,n+2}$ and

$$D_{i,n+2}=C_{i,n+2}X_{n+1,i}.$$

As in the proof of (5.17), by [GRS11, Lemma 7.1] we may exchange $Y_{n-1,n+2}$ and $X_{n+1,n-1}$. Further, we may use root exchange [GRS11, Lemma 7.1] to exchange

$$Y_{n-2,n+2},\dots,Y_{1,n+2}$$

with

$$X_{n+1,n-2},\dots,X_{n+1,1},$$

respectively. In a similar manner for $X'_{n+1,2n-i}$ and $Y'_{2n-i,n+2}$, $i=0,1,\dots,2n-1$, we may define a unipotent group $D'_{1,n+2}$. Similarly, we may use [GRS11, Lemma 7.1] to exchange the roots

$$Y'_{2n,n+2},\dots,Y'_{1,n+2}$$

with

$$X'_{n+1,2n}, \ldots, X'_{n+1,1},$$

respectively. Then we see that

$$R'_{1,n+1}(\xi_{\bar{\tau},\pi})(1) = \int_{L_{n+2}(\mathbb{A})} R'_{1,n+2}(\xi_{\bar{\tau},\pi})(y)\,dy, \tag{5.23}$$

where L_{n+2} is the group generated by

$$Y'_{2n-i,n+2}\,(i=0,1,\ldots,2n-1) \quad \text{and} \quad Y_{i,n+2}\,(i=1,\ldots,n-1)$$

and

$$R'_{1,n+2}(\xi_{\bar{\tau},\pi})(h) = \int_{D'_{1,n+2}(F)\backslash D'_{1,n+2}(\mathbb{A})} \xi_{\tau,\pi}(vh\nu')\eta'_{\psi,a}(v)\,dv. \tag{5.24}$$

Here, we extend that character $\eta^{(1,n+1)}_{\psi,a}$ from $D_{1,n+1}(\mathbb{A})$ to $D_{1,n+1}(\mathbb{A})X_{n+1,n}(\mathbb{A})$, and similarly we extend the character to $D_{i,n+2}(\mathbb{A})$, $C_{i,n+2}(\mathbb{A})$, and $D'_{1,n+2}(\mathbb{A})$, and we denote such character by $\eta'_{\psi,a}$. In particular, we see that as a functional on $\mathcal{E}_{\bar{\tau},\pi}$,

$$R_{1,2}(\cdot)(1) \not\equiv 0 \Longleftrightarrow R'_{1,n+2}(\cdot)(1) \not\equiv 0.$$

We repeat the same argument in a proof of the identity (5.22). Indeed, Fourier coefficients of $R'_{1,n+2}(\xi_{\bar{\tau},\pi})(h)$ along $X_{n+2,n-1}$ with respect to a non-trivial character give Fourier coefficients corresponding to $\mathcal{F}^{\psi_{V^{6n}_{n+2}},a}(\xi_{\bar{\tau},\pi})$, and it is zero by the first part of this theorem. Thus, we get

$$R'_{1,n+2}(\xi_{\bar{\tau},\pi})(h) = \int_{X_{n+2,n-1}(F)\backslash X_{n+2,n-1}(\mathbb{A})} R'_{1,n+2}(\xi_{\bar{\tau},\pi})(xh)\,dx.$$

Repeat the above argument by exchanging the roots

$$Y_{n-i+1,n+i}, Y_{n-i,n+i}, \ldots, Y_{1,n+i}, Y'_{2n,n+i}, Y'_{2n-1,n+i}, \ldots, Y'_{1,n+i} \tag{5.25}$$

with

$$X_{n+1-i,n-i+1}, X_{n+1-i,n-i}, \ldots, X_{n+1-i,1}, X'_{n+i-1,2n}, X_{n+i-1,2n-1}, \ldots, X_{n+i-1,1}$$

for $3 \leq i \leq n$. We denote the resulting unipotent group by $D'_{1,n+i}$. Define $R'_{1,n+i}(\xi_{\bar{\tau},\pi})$ as in (5.24). Further, from the first part of this theorem, $\mathcal{E}_{\bar{\tau},\pi}$ has no non-zero Fourier coefficient along V^{6n}_l with respect to $\psi_{V^{6n}_l,a}$ for all $n < l < 3n$, and thus

$$R'_{1,n+i}(\xi_{\bar{\tau},\pi})(h) = \int_{X_{n+i,n-i+1}(F)\backslash X_{n+i,n-i+1}(\mathbb{A})} R'_{1,n+i}(\xi_{\bar{\tau},\pi})(xh)\,dx. \tag{5.26}$$

Let L_i be the group generated by (5.25). Then as in (5.23), we see that

$$R'_{1,n+1}(\xi_{\bar{\tau},\pi})(1) = \int_{L_{2n}(\mathbb{A})} \cdots \int_{L_{n+1}(\mathbb{A})} R'_{1,2n}(\xi_{\bar{\tau},\pi})(y_1\cdots y_n)\,dy_1\cdots dy_n. \tag{5.27}$$

Now, we note that

$$V^{6n}_{2n} = D'_{1,2n}X_{2n,1}.$$

Because of the property (5.26), the inner integral of (5.13) is $R'_{1,2n}(\xi_{\tau,\pi})(h)$. In summary, since L is generated by L_i $(n+1 \leq i \leq 2n)$ and Y, (5.21) and (5.27) show that (5.11) is equal to (5.13).

We remark that from [GRS11, Corollary 7.2] (see also [GJS12, p. 967]), we find that for any $h \in G_{3n}(\mathbb{A})$ and $\xi_{\bar{\tau},\pi} \in \mathcal{E}_{\bar{\tau},\pi}$, there exists $\xi'_{\bar{\tau},\pi} \in \mathcal{E}_{\bar{\tau},\pi}$:

$$\int_{L(\mathbb{A})} \int_{V_{2n}^{6n}(F)\backslash V_{2n}^{6n}(\mathbb{A})} \xi_{\bar{\tau},\pi}(vyh\nu')\psi'_a(v)\, dv\, dy = \int_{V_{2n}^{6n}(F)\backslash V_{2n}^{6n}(\mathbb{A})} \xi'_{\bar{\tau},\pi}(vh)\psi'_a(v)\, dv.$$

In particular, (5.11) is non-zero if and only if the integral

$$\int_{V_{2n}^{6n}(F)\backslash V_{2n}^{6n}(\mathbb{A})} \xi_{\bar{\tau},\pi}(v)\psi'_a(v)\, dv \tag{5.28}$$

is not identically zero. □

Let us show that (5.28) is equal to

$$\int_{U_{2n}^{6n}(F)\backslash U_{2n}^{6n}(\mathbb{A})} \xi_{\bar{\tau},\pi}(v)\psi''_a(v)\, dv. \tag{5.29}$$

Here, we recall that U_{2n}^{6n} is the unipotent radical of the standard parabolic subgroup whose Levi part M_{2n}^{6n} is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_1^{2n} \times G_n$, and ψ''_a is the character of $U_{2n}^{6n}(\mathbb{A})$ defined by the same formula as (5.12). First, consider the Fourier expansion of (5.28) along $X_{2n,1}$. Then from the first part of this theorem, the constant term only contributes to the Fourier expansion. Further, we shall consider the Fourier expansion along

$$x \mapsto \begin{pmatrix} 1_{2n-1} & & & & & \\ & 1 & x & n(x) & & \\ & & 1_{2n} & x' & & \\ & & & 1 & & \\ & & & & 1_{2n-1} \end{pmatrix}$$

where

$$x' = {}^t(\overline{x_4}, \overline{x_3}, -\overline{x_2}, -\overline{x_1}) \quad \text{and} \quad n(x) = \frac{1}{2}(\overline{x_1}x_4 - x_1\overline{x_4} + \overline{x_2}x_3 - x_2\overline{x_3}).$$

Then we can write (5.28) as

$$\begin{aligned} &\sum_{\gamma \in \mathcal{P}^1\backslash G_n(F)} \int_{U_{2n}^{6n}(F)\backslash U_{2n}^{6n}(\mathbb{A})} \xi_{\tau,\pi}(\hat{\gamma}v)\psi_a^1(v)\, dv \\ &\quad + \sum_{\gamma \in \mathcal{P}^0\backslash G_n(F)} \int_{U_{2n}^{6n}(F)\backslash U_{2n}^{6n}(\mathbb{A})} \xi_{\tau,\pi}(\hat{\gamma}v)\psi_a^0(v)\, dv \\ &\quad + \int_{U_{2n}^{6n}(F)\backslash U_{2n}^{6n}(\mathbb{A})} \xi_{\tau,\pi}(v)\psi''_a(v)\, dv, \end{aligned}$$

where we define characters ψ_a^1 and ψ_a^0 by

$$\psi_a^1(v) = \psi(v_{1,2} + \cdots + v_{n-1,n} - av_{n,n+1} - v_{n+1,n+2} \cdots - v_{2n,2n+1})$$

and

$$\psi_a^0(v) = \psi(v_{1,2} + \cdots + v_{n-1,n} - av_{n,n+1} - v_{n+1,n+2} \cdots - v_{2n,2n+1} - v_{2n,4n}),$$

and we denote the stabilizer of ψ_a^1 and ψ_a^0 in $G_n(F) \subset M_{2n}^{6n}(F)$ by $\mathcal{P}^1$ and $\mathcal{P}^0$, respectively. The first sum is zero from the following lemma, and the second sum should be zero from Lemma A.2. Then we see that (5.28) is equal to (5.29).

Lemma 5.4. *The following integral is identically zero on $\mathcal{E}_{\bar{\tau},\pi}$:*

$$\int_{U_{2n}^{6n}(F)\backslash U_{2n}^{6n}(\mathbb{A})} \xi_{\bar{\tau},\pi}(v)\psi_a^1(v)\,dv, \quad \xi_{\bar{\tau},\pi}\in\mathcal{E}_{\bar{\tau},\pi}.$$

Proof. For $0\le\ell\le n$ and any $\phi\in\mathcal{E}_{\bar{\tau},\pi}$, let us define

$$p_\ell(\phi)=\int_{U_{3n-\ell}^{6n}(F)\backslash U_{3n-\ell}^{6n}(\mathbb{A})}\phi(v)\psi_{a;\ell}^1(v)\,dv,$$

where $U_{3n-\ell}^{6n}$ is the unipotent radical of the standard parabolic subgroup whose Levi part is $\mathrm{Res}_{E/F}\mathrm{GL}_{3n-\ell}\times G_\ell$, and we define a character $\psi_{a;\ell}^1$ of $U_{3n-\ell}^{6n}(\mathbb{A})$ by

$$\psi_{a;\ell}^1(v)=\psi(v_{1,2}+\cdots+v_{n-1,n}-av_{n,n+1}-v_{n+1,n+2}\cdots-v_{3n-\ell,3n-\ell+1}).$$

Our assertion is that $p_n(\cdot)$ is identically zero on $\mathcal{E}_{\bar{\tau},\pi}$. Indeed, we shall show that $p_\ell(\cdot)$ is identically zero on $\mathcal{E}_{\bar{\tau},\pi}$ for any $0\le\ell\le n$.

We note that $p_0(\cdot)$ is identically zero on $\mathcal{E}_{\tau,\pi}$ by Lemma A.3. Then we shall prove our assertion by an induction. Suppose that $p_\ell(\cdot)$ is identically zero for any $0\le\ell<\ell_0$. Then we shall show that $p_{\ell_0}(\cdot)$ is identically zero.

Define

$$R_{\ell_0}=\left\{u(x,t):=\begin{pmatrix}1_{3n-\ell_0}&&&&\\&1&x&t&\\&&1_{2\ell_0-2}&x'&\\&&&1&\\&&&&1_{3n-\ell_0}\end{pmatrix}\right\}.$$

Then consider the function

$$(x,t)\mapsto\int_{U_{3n-\ell_0}^{6n}(F)\backslash U_{3n-\ell_0}^{6n}(\mathbb{A})}\phi(u(x,t)v)\psi_{a;\ell}^1(v)\,dv.$$

Each non-constant term of Fourier expansion along $\{u(0,t):t\in\mathbb{G}_a\}$ is zero by Lemma A.3. Further, consider the Fourier expansion along $\{u(x,n(x)):x\in\mathrm{Res}_{E/F}\mathbb{G}_a^{2\ell_0-2}\}$. First, its constant term is written by

$$\int_{\mathcal{Z}_{3n-\ell_0+1}(F)\backslash\mathcal{Z}_{3n-\ell_0+1}(\mathbb{A})}\left(\int_{U_{3n-\ell_0+1}^{6n}(F)\backslash U_{3n-\ell_0+1}^{6n}(\mathbb{A})}\phi(un)\,du\right)\psi_{a;\ell_0}^1(n)\,dn.$$

The inner integral is zero because it is a constant term of a maximal parabolic subgroup which is not conjugate to P_{2n}^{6n}. Secondly, each Fourier coefficient associated to $\eta\in E^{2\ell_0-2}$ such that ${}^t\overline{\eta}J_{2n}^-\eta=0$ is of the form

$$\int_{U_{3n-\ell_0+1}^{6n}(F)\backslash U_{3n-\ell_0+1}^{6n}(\mathbb{A})}\phi(v\gamma)\psi_{a;\ell_0-1}^1(v)\,dv$$

with some $\gamma\in G_{\ell_0-1}(F)$. This is zero by our assumption. Finally, each Fourier coefficient associated to $\eta\in E^{2\ell_0-2}$ such that ${}^t\overline{\eta}J_{2n}^-\eta\ne0$ is of the form

$$\int_{U_{3n-\ell_0+1}^{6n}(F)\backslash U_{3n-\ell_0+1}^{6n}(\mathbb{A})}\phi(v\gamma)\psi_{a,b;\ell_0-1}^0(v)\,dv,$$

where $\gamma\in G_{\ell_0-1}(F)$ and for $b\in E^\times$, we define

$$\begin{aligned}&\psi_{a,b;\ell_0-1}^0(v)\\&=\psi(v_{1,2}+\cdots+v_{n-1,n}-av_{n,n+1}-v_{n+1,n+2}\cdots-v_{3n-\ell+1,3n-\ell+2}+bv_{3n-\ell_0+1,3n+\ell_0}).\end{aligned}$$

This is zero by Lemma A.2, and this completes our proof. □

As a consequence, it suffices to show that (5.29) is not identically zero. First, we note that (5.29) is equal to

$$\int_{\mathcal{Z}_{2n}(F)\backslash\mathcal{Z}_{2n}(\mathbb{A})}\left(\int_{N_{2n}^{6n}(F)\backslash N_{2n}^{6n}(\mathbb{A})}\xi_{\bar{\tau},\pi}(nv)\,dn\right)\psi''_a(v)\,dv,$$

and it is the residue of

$$\int_{\mathcal{Z}_{2n}(F)\backslash\mathcal{Z}_{2n}(\mathbb{A})}\left(\int_{N_{2n}^{6n}(F)\backslash N_{2n}^{6n}(\mathbb{A})}E(nv,f_{\bar{\tau},\pi,\bar{s}})\,dn\right)\psi''_a(v)\,dv.$$

From the proof of Lemma 4.3, this is equal to

$$\int_{\mathcal{Z}_{2n}(F)\backslash\mathcal{Z}_{2n}(\mathbb{A})}(M(w_0)f_{E(\bar{\tau},\bar{s}),\pi,\bar{s}'})(v)\psi''_a(v)\,dv, \tag{5.30}$$

where $E(\bar{\tau},\bar{s})$ is the Eisenstein series on $\mathrm{GL}_{2n}(\mathbb{A}_E)$ corresponding to $\tau_1|\det|_E^{s_1}\otimes\cdots\otimes\tau_r|\det|_E^{s_r}$. Then the residue of the image of the intertwining operator is in

$$|\det|_E^{2n-1}E(\bar{\tau},-\mathbf{1})\otimes\pi$$

with $\mathbf{1}=(1,\ldots,1)$. Since $E(\bar{\tau},-\mathbf{1})$ is generic with respect to ψ''_a, (5.30) is not identically zero. □

Theorem 5.5. *Let π be an irreducible (possibly non-generic) cuspidal automorphic representation of $G_n(\mathbb{A})$, which has a weak lift to an irreducible isobaric automorphic representation $\boxplus_{i=1}^r(\tau_i\otimes\eta^{-1})$ of $\mathrm{GL}_{2n}(\mathbb{A}_E)$. Then the following hold.*

1. *As an automorphic representation of $G_{2n}(\mathbb{A})$, the descent $\mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi})$ is non-trivial and square-integrable. Moreover, it is a subrepresentation of the space of the automorphic discrete spectrum of $G_{2n}(\mathbb{A})$.*
2. *The descent $\mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi})$ is cuspidal if and only if π is not ψ^{-1}-generic.*
3. *If π is ψ^{-1}-generic, then the descent $\mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi})$ is a direct sum of residual representation $\mathcal{E}_{\bar{\tau}}$ and a cuspidal automorphic representation of $G_{2n}(\mathbb{A})$.*

Proof. This theorem is proved in a similar argument as in the proof of [GRS11, Theorem 2.5].

The non-vanishing of the descent $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi})$ in the first part follows from Theorem 5.2 and the first part of Lemma 5.1.

In order to show other statements, we shall compute constant terms along all standard parabolic subgroups. Recall that $P_r^{4n}(1\le r\le 2n)$ is the standard maximal parabolic subgroup of G_{2n} whose Levi part is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_r\times G_{2n-r}$ and its unipotent radical is denoted by N_r^{4n}.

Let $\phi=\phi_1\otimes\phi_2$ with $\phi_1\in\mathcal{S}(\mathbb{A}_E^r)$ and $\phi_2\in\mathcal{S}(\mathbb{A}_E^{2n-r})$ and regard it as the element of $\mathcal{S}(\mathbb{A}_E^{2n})$. Let us study the constant term of the Fourier-Jacobi coefficient $\mathrm{FJ}^{\psi^{-1},\eta}_{\phi,2n}(\xi_{\bar{\tau},\pi})$ along P_r^{4n}:

$$\mathcal{C}_{N_r^{4n}}(\mathrm{FL}^{\psi^{-1},\eta}_{\phi,2n}(\xi_{\bar{\tau},\pi})).$$

Then by [GRS11, Theorem 7.8], this constant is written as

$$\sum_{j=0}^{r}\sum_{\gamma\in P_{r-j,1^j}(F)\backslash\mathrm{GL}_r(E)}\int_{S_r(\mathbb{A})}\phi_1(i(\lambda))\mathrm{FJ}^{\psi^{-1},\eta}_{\phi_2,2n-r}(\mathcal{C}_{N^{6n}_{r-j}}(\xi_{\bar{\tau},\pi}))(\hat{\gamma}\lambda\beta_r)\,d\lambda, \tag{5.31}$$

where $\mathrm{FJ}^{\psi^{-1},\eta}_{\phi_2,2n-r}(\mathcal{C}_{N^{6n}_{r-j}}(\xi_{\bar\tau,\pi}))$ denotes the Fourier-Jacobi coefficient of an automorphic form $\mathcal{C}_{N^{6n}_{r-j}}(\xi_{\bar\tau,\pi})$ on G_{3n-r+j}. We put

$$\beta_r = \begin{pmatrix} & 1_r \\ 1_n & \end{pmatrix}^{\wedge}$$

and we define

$$P_{r-j,1^j} = \left\{ \begin{pmatrix} g & x \\ 0 & z \end{pmatrix} \in \mathrm{Res}_{E/F}\mathrm{GL}_r : z \in Z_r \right\},$$

$$S_r = \left\{ \lambda(x) = \begin{pmatrix} 1_r & \\ x & 1_n \end{pmatrix}^{\wedge} \in G_{3n} : x \in \mathrm{Res}_{E/F}\mathrm{Mat}_{r\times n} \right\},$$

and $i(\lambda) = i(\lambda(x))$ denotes the last row of x. Then from the computation in the proof of Lemma 4.3, the constant term $(\mathcal{C}_{N^{6n}_{r-j}}(\xi_{\bar\tau,\pi}))(\hat\gamma\lambda\beta_r)$ is zero unless $r = j$ or $r - j = 2n$. Let us consider the first case. In this case, the corresponding term in (5.31) is an integral of $\mathrm{FJ}^{\psi^{-1},\eta}_{2n-r}(\xi_{\bar\tau,\pi})$. Since this Fourier-Jacobi coefficient contains the Fourier coefficient $\mathcal{F}^{\psi_{V^{6n}_{2(n+r)}},a}$ as an inner integral, it is zero by the first part of Theorem 5.2. Hence, we should have $r - j = 2n$. Moreover, since $r \le 2n$, we should have $j = 0$ and $r = 2n$. Then we can write the Fourier coefficient of (5.31) along P^{4n}_{2n} as

$$\int_{S_n(\mathbb{A})} \phi_1(i(\lambda))\mathrm{FJ}^{\psi^{-1}}_{\phi_2,0}(\mathcal{C}_{N^{6n}_{2n}}(\xi_{\bar\tau,\pi}))(\lambda\beta_{2n})\, d\lambda. \tag{5.32}$$

We note that by [GRS11, Corollary 7.2], (5.32) is not identically zero if and only if $\mathrm{FJ}^{\psi^{-1}}_{\phi_2,0}(\mathcal{C}_{N^{6n}_{2n}}(\xi_{\bar\tau,\pi}))$ is not identically zero.

As we explained in the proof of Theorem 5.2, the constant term $\mathcal{C}_{N^{6n}_{2n}}(\xi_{\bar\tau,\pi})$ is equal to the residue at $s = 1$ of the intertwining operator corresponding to the longest Weyl element, and it takes values in the space of

$$(\delta^{\frac{1}{2}}_{P^{6n}_{2n}} \otimes E(\bar\tau, -\mathbf{1})) \otimes \pi. \tag{5.33}$$

In (5.32), $\mathrm{FJ}^{\psi^{-1}}_{\phi_2,0}(\mathcal{C}_{N^{6n}_{2n}}(\xi_{\bar\tau,\pi}))$ is the ψ^{-1}-Whittaker Fourier coefficient of $\mathcal{C}_{N^{6n}_{2n}}(\xi_{\bar\tau,\pi})$ when we regard this as an automorphic form on $G_n(\mathbb{A})$. Hence, (5.32) is not identically zero if and only if π is ψ^{-1}-generic.

Let U be a unipotent radical of a standard parabolic subgroup of G_{2n}, which may not be maximal. Then from the above discussion, we see that the constant term of $\mathrm{FL}^{\psi^{-1},\eta}_{\phi,2n}(\xi_{\bar\tau,\pi})$ along U is zero unless $N^{4n}_{2n} \subset U$. In this case, we may write $U = N^{4n}_{2n} \rtimes (U^1 \times U^2)$ where U^1 (resp. U^2) is a unipotent radical of a standard parabolic subgroup of $\mathrm{Res}_{E/F}\mathrm{GL}_{2n}$ (resp. G_n). Since π is cuspidal it suffices to consider the case $U^2 = 1$. Then the constant term of $\mathrm{FJ}^{\psi^{-1},\eta}_{\phi,2n}(\xi_{\bar\tau,\pi})$ along U is given by

$$\int_{U_1(F)\backslash U_1(\mathbb{A})}\int_{S_n(\mathbb{A})} \phi_1(i(\lambda))\mathrm{FJ}^{\psi^{-1}}_{\phi_2,0}(\mathcal{C}_{N^{6n}_{2n}}(\xi_{\bar\tau,\pi}))(u\lambda\beta_{2n})\, d\lambda\, du. \tag{5.34}$$

If π is not ψ^{-1}-generic, then (5.32) is zero, and thus this integral is zero. On the other hand, if this integral is zero for any U, then taking $U^1 = 1$, (5.32) is not identically zero and thus π is ψ^{-1}-generic. Therefore, $\mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\bar\tau,\pi})$ is cuspidal if and only if π is not ψ^{-1}-generic. This proves the second part.

Suppose that π is ψ^{-1}-generic. In this case, $\mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi})$ has non-zero constant term (5.34). In order to prove the square integrability, we should show that $\mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi})$ has cuspidal support with negative exponent. From (5.34) and (5.33), it suffices to study the exponent of $E(\bar{\tau}, -\mathbf{1})$. Since its exponent is $-\mathbf{1}$, in particular negative, by the square-integrability criterion of Langlands (see [MW95, I.4.11 Lemma]), $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi})$ is square integrable.

Let $U_{[n_i],\mathrm{GL}}$ be the unipotent radical of the standard parabolic subgroup of $\mathrm{Res}_{E/F}\mathrm{GL}_{2n}$ whose Levi part is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{E/F}\mathrm{GL}_{n_r}$. Then using (5.34) for $U_1 = U_{[n_i],\mathrm{GL}}$, we see that its constant term is

$$\delta^{\frac{1}{2}}_{P^{4n}_{2n}} |\det|_E^{-1}\tau_1 \otimes \cdots \otimes |\det|_E^{-1}\tau_r \otimes \pi.$$

This implies that $\mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\tau,\pi})$ appears in the discrete spectrum.

It is easy to see that $\mathcal{E}_{\bar{\tau}}$ (which is non-zero because of Theorem 4.2 in the generic case and Assumption 1 in the non-generic case) has the same exponent as $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi})$ by (5.33) and the proof of [GRS11, Theorem 2.1]. Recall that $\mathcal{E}_{\bar{\tau}}$ is irreducible by [GRS11, Theorem 2.1]. Therefore this irreducibility implies that $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi})$ is a direct sum of $\mathcal{E}_{\bar{\tau}}$ and cuspidal representations. □

6. Certain nearly equivalent sets

Recall that two irreducible automorphic representations $\pi_i = \bigotimes \pi_{i,v}$ for $i = 1, 2$ of $G(\mathbb{A})$ are said to be nearly equivalent if the local components $\pi_{1,v}$ and $\pi_{2,v}$ are equivalent as representations of $G(F_v)$, at almost all places v of F. We are going to define certain nearly equivalent subsets in $\mathcal{A}_d(G_n)$, which will be main objects in the following sections. Throughout the remainder of this paper, we fix $\bar{\tau} = (\tau_1, \dots, \tau_r)$ where τ_i is an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ such that $\tau_i \not\simeq \tau_j$ if $i \neq j$, and $L(s, \tau_i, \mathrm{Asai})$ has a pole at $s = 1$.

Definition 6.1. We denote by $\mathcal{N}_n(\bar{\tau}, \eta)$ the set of all irreducible cuspidal automorphic representations of $G_n(\mathbb{A})$ which weakly lift to $\boxplus_{i=1}^r \tau_i \otimes \eta^{-1}$.

We note that for a given $\pi \in \mathcal{N}_n(\bar{\tau}, \eta)$, if an irreducible cuspidal automorphic representation π' of $G_n(\mathbb{A})$ is nearly equivalent to π, then π' is also in $\mathcal{N}_n(\bar{\tau}, \eta)$.

Lemma 6.2. *The set $\mathcal{N}_n(\bar{\tau}, \eta)$ is not empty. Moreover, there is a ψ^{-1}-generic element in $\mathcal{N}_n(\bar{\tau}, \eta)$.*

Proof. By Theorem 4.2, the residual representation $\mathcal{E}_{\bar{\tau}}$ exists. Then by Theorem 4.1, the descent $\mathcal{D}^{4n,\eta}_{2n,\psi}(\mathcal{E}_{\bar{\tau}})$ is non-trivial and any irreducible constituent of this descent is a ψ^{-1}-generic element of $\mathcal{N}_n(\bar{\tau}, \eta)$. □

Let us define a nearly equivalent set of an automorphic representation of $G_{2n}(\mathbb{A})$.

Definition 6.3. We denote by $\mathcal{N}_{2n}(\bar{\tau}, \eta, \psi)$ the set of all irreducible automorphic representations π of $G_{2n}(\mathbb{A})$ such that

(1) it appears in the discrete spectrum $\mathcal{A}_d(G_{2n})$,
(2) it is of type $(\mathrm{GL}_{2n}, \bar{\tau}, \frac{1}{2})$,
(3) the Fourier coefficient $\mathcal{F}^{\psi_{V^{4n}_n,1}}$ is not identically zero on π, which means $\mathcal{D}^{4n,\eta}_{2n,\psi}(\pi) \neq 0$.

From the definition and Theorem 4.1, we have $\mathcal{E}_{\bar{\tau}} \in \mathcal{N}_{2n}(\bar{\tau}, \eta, \psi)$. In particular, this nearly equivalent set is not empty.

Let $\mathcal{N}^0_{2n}(\bar{\tau}, \eta, \psi)$ be the subset of cuspidal representations in $\mathcal{N}_{2n}(\bar{\tau}, \eta, \psi)$, and define

$$\mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi) := \mathcal{N}^0_{2n}(\bar{\tau}, \eta, \psi) \cup \{\mathcal{E}_{\bar{\tau}}\}.$$

Then we have

$$\mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi) \subset \mathcal{N}_{2n}(\bar{\tau}, \eta, \psi),$$

and it is expected that (see Mok [Mo15])

$$\mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi) = \mathcal{N}_{2n}(\bar{\tau}, \eta, \psi).$$

For $\pi \in \mathcal{N}_n(\bar{\tau}, \eta)$, by Lemma 4.3, we may define the residual representation $\mathcal{E}_{\bar{\tau},\pi}$, and we define

$$\Phi(\pi) := \mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi}),$$

which is an automorphic representation of $G_{2n}(\mathbb{A})$. We also write

$$\Phi'(\pi) = \Phi(\pi) \otimes \eta.$$

Lemma 6.4. $\Phi'(\pi)$ *is a non-trivial square integrable automorphic representation, and every irreducible constituent of* $\Phi'(\pi)$ *satisfies the conditions* (1) *and* (2) *in the definition of* $\mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi)$.

Proof. By Theorem 5.5, $\Phi(\pi)$ is non-trivial and a square integrable automorphic representation of $G_{2n}(\mathbb{A})$, and so is $\Phi'(\pi)$. Again by Theorem 5.5, $\Phi'(\pi)$ is a direct sum of $\mathcal{E}_{\bar{\tau}}$ and cuspidal automorphic representations. From Lemma A.4, any irreducible constituent of $\Phi'(\pi)$ is of type $(\mathrm{GL}_{2n}, \bar{\tau}, \frac{1}{2})$. ☐

We define a sort of inverse map of Φ'. For $\pi \in \mathcal{N}_{2n}(\bar{\tau}, \eta, \psi)$, we define

$$\Psi(\pi) := \mathcal{D}^{4n,\eta}_{2n,\psi}(\pi).$$

Lemma 6.5. $\Psi(\pi)$ *is a non-trivial cuspidal automorphic representation of* $G_n(\mathbb{A})$.

Proof. By the definition of $\mathcal{N}_{2n}(\bar{\tau}, \eta, \psi)$, the non-triviality follows. We may prove the cuspidality in a similar way as the proof of the cuspidality of global descents. Indeed, by [GRS11, Theorem 7.10], the cuspidality of $\Psi(\pi)$ follows from

$$\mathcal{D}^{4n,\eta}_{\ell,\psi}(\pi) = 0 \tag{6.1}$$

for any $\ell > 2n$. When $\pi = \mathcal{E}_{\bar{\tau}}$, this result was proved in [GRS11, Proposition 7.4]. Recall their proof of this fact.

Let σ be an irreducible constituent of $\mathcal{D}^{4n,\eta}_{\ell,\psi}(\mathcal{E}_{\bar{\tau}})$. Let v be a finite place such that σ_v and $\tau_{i,v}$ are unramified, and let E_v be an unramified quadratic extension of F_v. Then σ_v is the unramified constituent of

$$\mathrm{Ind}^{G_{2n}(F_v)}_{P_{[n_i]}(F_v)}(\tau_{1,v}|\det|^{\frac{1}{2}} \otimes \cdots \otimes \tau_{r,v}|\det|^{\frac{1}{2}}).$$

By [GRS11, Theorem 6.4], this induced representation does not have a linear functional corresponding to the character $\psi_{V^{4n}_{\ell,1}}$. Therefore, the triviality (6.1) follows. On the other hand, when $\pi \neq \mathcal{E}_{\bar{\tau}}$, there is a finite place satisfying the above condition since π is in $\mathcal{N}_{2n}(\bar{\tau}, \eta, \psi)$. Therefore, in this case, the same argument shows the triviality (6.1), and this gives the cuspidality of $\Psi(\pi)$. ☐

7. Some basic identities

7.1. Fourier-Jacobi coefficients of residual representations of G_{3n} and G_{4n}.

The following identity is proved in the same way as the proof of [GJS12, Theorem 5.1].

Theorem 7.1. *Let π be an element of $\mathcal{N}_n(\bar{\tau},\eta)$. Let $\phi_1 \in \mathcal{S}(\mathbb{A}_E^n)$ and $\phi_2 \in \mathcal{S}(\mathbb{A}_E^{2n})$, and let $\xi_{\bar{\tau},\pi} \in \mathcal{E}_{\bar{\tau},\pi}$. Assume that $\phi_2 = \phi_{21} \otimes \phi_{22}$ with $\phi_{21}, \phi_{22} \in \mathcal{S}(\mathbb{A}_E^n)$. Then the following identity holds as functions in $h \in G_n(\mathbb{A})$:*

$$\begin{aligned}(\mathrm{FJ}_{\phi_1,n}^{\psi,\eta} \circ \mathrm{FJ}_{\phi_2,2n}^{\psi^{-1},\eta})(\xi_{\bar{\tau},\pi})(h) &= \eta^{-1}(\det h) \cdot \int_{\mathbb{A}_E^n} \int_{Y(\mathbb{A})} \int_{\mathbb{A}_E^{2n}} \int_{L_0(\mathbb{A})} \int_{U_{2n}^{6n}(F)\backslash U_{2n}^{6n}(\mathbb{A})} \xi_{\bar{\tau},\pi}(vhy'\nu'\hat{l}_2\hat{b}y\omega\hat{l}_1) \\ &\qquad \cdot \psi''_{-1}(v)\, \phi'_3(l_2)\phi_{21}(l_1)\, dv\, dy'\, dl_2\, dy\, dl_1.\end{aligned}$$

Here, $\hat{b}$, ω, and ν' are as in (5.6), (5.2), and (5.9), respectively; ψ''_{-1} is the character of $U_{2n}^{6n}(\mathbb{A})$ defined as in (5.29); and we put

$$\phi'_3 = \omega_{\psi^{-1},\eta^{-1}}(\gamma^{-1})(\phi_{22} \otimes \phi_1) \in \mathcal{S}(\mathbb{A}_E^{3n})$$

and

$$\hat{l}_1 = v(1_{2n} + r_1 e_{n,1} + \cdots + r_n e_{n,n}, 0_{2n}, 0) \text{ and } \hat{l}_2 = v(1_{2n}, m_1 e_{2n,2n+1} + \cdots + m_{2n} e_{2n,4n}).$$

Finally, Y and L_0 are unipotent subgroups of G_{3n} given in (5.5) and Lemma 5.3, respectively.

Proof. By the same argument as in the proof of [GJS12, Theorem 5.1], practically word for word, we can show that $(\mathrm{FJ}_{\phi_1,n}^{\psi} \circ \mathrm{FJ}_{\phi_2,2n}^{\psi^{-1}})(\xi_{\bar{\tau},\pi})(h)$ is equal to

$$\begin{aligned}&\int_{\mathbb{A}_E^n} \int_{Y(\mathbb{A})} \int_{L'(F)\backslash L'(\mathbb{A}_E)} \int_{\mathbb{A}_E^{2n}} \omega_{\psi^{-1},\eta^{-1}}(\ell(v)h(l_2, 0_{2n}; 0))\phi'_3(0) \\ &\qquad\qquad \times \xi_{\bar{\tau},\pi}(vh\hat{b}y\omega\hat{l}_1)\psi_1(v)\phi_{21}(l_1)\, dl_2\, dv\, dy\, dl_1\end{aligned}$$

where L' is the one in (5.8). Then by the direct computation of the explicit action of the Weil representation (3.5), we find that this is equal to

$$\begin{aligned}&\eta^{-1}(\det h) \\ &\quad \cdot \int_{\mathbb{A}_E^n} \int_{Y(\mathbb{A})} \int_{\mathbb{A}_E^{2n}} \int_{L'(F)\backslash L'(\mathbb{A})} \xi_{\bar{\tau},\pi}(vh\hat{l}_2\hat{b}y\omega\hat{l}_1)\psi_{L',-1}(v)\phi'_3(l_2)\phi_{21}(l_1)\, dv\, dl_2\, dy\, dl_1,\end{aligned}$$

where for $l_2 = (m_1, \ldots, m_{2n}) \in \mathbb{A}_E^{2n}$, we define

$$\hat{l}_2 = v(1_{2n}, m_1 e_{2n,2n+1} + \cdots + m_{2n} e_{2n,4n}).$$

Then by (5.29) and Lemma 5.3, this integral is equal to

$$\begin{aligned}&\eta^{-1}(\det h) \cdot \int_{\mathbb{A}_E^n} \int_{Y(\mathbb{A})} \int_{\mathbb{A}_E^{2n}} \int_{L_0(\mathbb{A})} \int_{U_{2n}^{6n}(F)\backslash U_{2n}^{6n}(\mathbb{A})} \\ &\qquad\qquad \xi_{\bar{\tau},\pi}(vhy'\nu'\hat{l}_2\hat{b}y\omega\hat{l}_1)\psi''_{-1}(v)\, \phi'_3(l_2)\phi_{21}(l_1)\, dv\, dy'\, dl_2\, dy\, dl_1.\end{aligned}$$

□

Proposition 7.2. *The space of automorphic forms on $G_n(\mathbb{A})$ generated by all elements*

$$(\mathrm{FJ}^{\psi,\eta}_{\phi_1,n} \circ \mathrm{FJ}^{\psi^{-1},\eta}_{\phi_2,2n})(\xi_{\bar{\tau},\pi})(h),$$

for $\phi_1 \in \mathcal{S}(\mathbb{A}^n_E), \phi_2 \in \mathcal{S}(\mathbb{A}^{2n}_E), \xi_{\bar{\tau},\pi} \in \mathcal{E}_{\bar{\tau},\pi}$, is equal to the space of the automorphic representation $\pi \otimes \eta^{-1}$. In other words, the space of the double descent of the residual representation $\mathcal{E}_{\bar{\tau},\pi}$ is equal to $\pi \otimes \eta^{-1}$:

$$\mathcal{D}^{4n,\eta}_{2n,\psi} \circ \mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi}) = \pi \otimes \eta^{-1}. \tag{7.1}$$

Proof. By [GRS11, Corollary 7.2], we see that $(\mathrm{FJ}^{\psi,\eta}_{\phi_1,n} \circ \mathrm{FJ}^{\psi^{-1},\eta}_{\phi_2,2n})(\xi_{\bar{\tau},\pi})(h)$ is equal to

$$\eta^{-1}(\det h) \cdot \int_{U^{6n}_{2n}(F)\backslash U^{6n}_{2n}(\mathbb{A})} \varepsilon_{\bar{\tau},\pi}(vh)\psi''_{-1}(v)\, dv$$

for some $\varepsilon_{\bar{\tau},\pi} \in \mathcal{E}_{\bar{\tau},\pi}$. Then as in the end of the proof of Theorem 7.1, this is in $\pi \otimes \eta^{-1}$ as an automorphic form on $G_n(\mathbb{A})$. Since π is irreducible, our assertion follows. □

We may restate the above results in the following way. Recall that for $\pi \in \mathcal{N}_n(\bar{\tau},\eta)$, we may write $\Phi'(\pi) = \Pi_1 \oplus \cdots \oplus \Pi_r$ with irreducible automorphic representations Π_i of $G_n(\mathbb{A})$. We note that by Lemma 6.4, each Π_i satisfies conditions (1) and (2) in Definition 6.3.

Remark 7.3. We may also check the condition (3) as in the proof of [GJS12, Proposition 3.4]. However, this fact is not necessary for our purpose and we do not consider it.

For simplicity, we write

$$\Psi(\Phi'(\pi)) := \Psi(\Pi_i) + \cdots + \Psi(\Pi_r),$$

where $\Psi(\Pi_i)$ is zero if Π_i does not satisfy the condition (3). Then the above theorem can be stated as follows.

Corollary 7.4. *For any $\pi \in \mathcal{N}_n(\bar{\tau},\eta)$, we have the equality*

$$\Psi(\Phi(\pi)) = \pi \otimes \eta^{-1}, \quad i.e., \quad \Psi(\Phi'(\pi)) = \pi,$$

as subspaces in the space of square-integrable automorphic functions on $G_n(\mathbb{A})$. In particular, for each $\pi \in \mathcal{N}_n(\bar{\tau},\eta)$, there is $\Pi \in \mathcal{N}_{2n}(\bar{\tau},\eta,\psi)$ such that $\Psi(\Pi) = \pi \otimes \eta^{-1}$.

We study a similar double decent for residual representations of another type. Let $\Pi \in \mathcal{N}_{2n}(\bar{\tau},\eta,\psi)$. Then there is a residual representation $\mathcal{E}_{\bar{\tau},\Pi}$ of $G_{4n}(\mathbb{A})$ by Proposition 4.6.

Theorem 7.5. *For all integers l such that $n < l \leq 4n$ and for any $a \in F^\times$, Fourier coefficient $\mathcal{F}^{\psi_{V^{8n}_n},a}(\cdot)$ is trivial on the residual representation $\mathcal{E}_{\bar{\tau},\Pi}$. Also, the Fourier coefficient $\mathcal{F}^{\psi_{V^{8n}_n},a}(\cdot)$ is non-trivial on $\mathcal{E}_{\bar{\tau},\Pi}$ for all $a \in F^\times$.*

Proof. In a similar way as in the proof of Theorem 5.2, we can prove all assertions. See also the proof of [GJS12, Theorem 5.4]. □

In a similar way as in the proof of Theorem 7.1, we can prove the following identity. We omit the proof. See also [GJS12, Theorems 5.1 and 5.5].

Theorem 7.6. *Let* Π *be an element of* $\mathcal{N}_{2n}(\bar{\tau},\eta,\psi)$. *Let* $\phi_1 \in \mathcal{S}(\mathbb{A}_E^{2n})$ *and let* $\phi_2 \in \mathcal{S}(\mathbb{A}_E^{3n})$ *and* $\xi_{\bar{\tau},\pi} \in \mathcal{E}_{\bar{\tau},\pi}$. *Assume that* $\phi_2 = \phi_{21} \otimes \phi_{22}$ *with* $\phi_{21} \in \mathcal{S}(\mathbb{A}_E^{n})$ *and* $\phi_{22} \in \mathcal{S}(\mathbb{A}_E^{2n})$. *Then the following identity holds as functions in* $h \in G_{2n}(\mathbb{A})$:

$$\begin{aligned}
&(\mathrm{FJ}^{\psi^{-1},\eta}_{\phi_1,2n} \circ \mathrm{FJ}^{\psi,\eta}_{\phi_2,3n})(\xi_{\bar{\tau},\Pi})(h) \\
&\quad = \int_{U_n^{6n}(F)\backslash U_n^{6n}(\mathbb{A})} \int_{U_n^{8n}(F)\backslash U_n^{8n}(\mathbb{A})} \xi_{\bar{\tau},\Pi}(uvh)\theta^{\psi^{-1},\eta^{-1}}_{\phi_2,3n}(\ell_{3n}(u)h) \\
&\qquad\qquad \times \psi_{U_n^{8n}}(u)\theta^{\psi,\eta^{-1}}_{\phi_1,2n}(\ell_{2n}(v)h)\psi_{U_n^{6n}}(v)\,du\,dv \\
&\quad = \eta^{-1}(\det h)\cdot \int_{\mathbb{A}_E^n}\int_{Y'(\mathbb{A})}\int_{\mathbb{A}_E^{4n}}\int_{L_0'(\mathbb{A})}\int_{U_{2n}^{8n}(F)\backslash U_{2n}^{8n}(\mathbb{A})} \xi_{\bar{\tau},\Pi}(vhy'\nu''\hat{l}_2\hat{b}'y\omega'\hat{l}_1) \\
&\qquad\qquad \times \psi^{\star}_{-1}(v)\,\phi(l_1,l_2)\,dv\,dy'\,dl_2\,dy\,dl_1.
\end{aligned}$$

Here, Y' *is the subgroup of lower unipotent matrices of the form*

$$v'(T,C,Z) := \begin{pmatrix} T & C & Z \\ & 1_{4n} & C' \\ & & T^* \end{pmatrix},$$

where the last row of C *is zero and* $T \in \mathrm{Res}_{E/F}\mathrm{GL}_{2n}$ *satisfies the conditions right after* (5.4), *and* L_0' *is the lower unipotent matrices of the form* (5.10), *replacing* 1_{2n} *by* 1_{4n}. *We define the character* $\psi^{\star}_{-1}$ *of* $U_{2n}^{8n}(\mathbb{A})$ *by*

$$\psi^{\star}_{-1}(v) = \psi(v_{1,2} + \cdots + v_{n,n+1} - v_{n,n+1} - \cdots - v_{2n-1,2n}).$$

Let ν'' *(resp.* ω'*) be the Weyl element of* $G_{4n}(F)$ *obtained by replacing* 1_{2n} *by* 1_{4n} *in* ν' *(resp.* ω*) defined in* (5.9) *(resp.* (5.2)*). Similarly,* $\hat{b}'$ *is the matrix of* $G_{4n}(F)$ *obtained by replacing* 1_{2n} *by* 1_{4n} *in* $\hat{b}$ *defined in* (5.6). *For* $l_1 = (m_1,\ldots,m_n) \in \mathbb{A}_E^n$ *and* $l_2 = (m_1,\ldots,m_{4n}) \in \mathbb{A}_E^{4n}$, *we define*

$$\hat{l}_1 = v'(1_{2n} + r_1 e_{n,1} + \cdots + r_n e_{n,n}, 0_{4n\times 2n}, 0_{2n\times 2n})$$

and

$$\hat{l}_2 = v'(1_{2n}, m_1 e_{2n,2n+1} + \cdots + m_{4n}e_{2n,6n}, 0_{2n\times 2n}).$$

Finally, $\phi \in \mathcal{S}(\mathbb{A}_E^{5n})$ *is defined explicitly from* ϕ_1, ϕ_{21}, *and* ϕ_{22}.

In a similar way as in the proof of Proposition 7.2, this theorem gives the following result.

Proposition 7.7. *Let* $\bar{\tau}$ *and* Π *be as above. Then the space of automorphic forms on* $G_{2n}(\mathbb{A})$ *generated by the elements*

$$(\mathrm{FJ}^{\psi^{-1},\eta}_{\phi_1,2n} \circ \mathrm{FJ}^{\psi,\eta}_{\phi_2,3n})(\xi_{\bar{\tau},\Pi})(h), \quad \xi_{\bar{\tau},\Pi} \in \mathcal{E}_{\bar{\tau},\Pi},$$

is equal to the space of $\Pi \otimes \eta^{-1}$. *In other words, the space of the double descent is identically equal to the space of* $\Pi\otimes\eta^{-1}$, *i.e.,*

$$\mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}\left(\mathcal{D}^{8n,\eta}_{6n,\psi}(\mathcal{E}_{\bar{\tau},\Pi})\right) = \Pi \otimes \eta^{-1}.$$

Finally, we give a relation between several global descents. It is proved in a similar argument as in the proof of [GJS12, Theorem 5.8], and we omit a proof.

Theorem 7.8. *Let* $\Pi \in \mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi)$. *Then* $\mathcal{D}^{8n,\eta}_{6n,\psi}(\mathcal{E}_{\bar{\tau},\Pi})$ *is a square-integrable automorphic representation. Moreover, there is an irreducible subrepresentation* π *of* $\Psi(\Pi)$ *such that there is an irreducible automorphic representation* σ *of* $G_{3n}(\mathbb{A})$ *satisfying the conditions:*

(1) $\sigma \subset \mathcal{D}^{8n,\eta}_{6n,\psi}(\mathcal{E}_{\bar{\tau},\Pi})$,
(2) $\sigma \subset \mathcal{E}_{\bar{\tau},\pi}$,
(3) $\mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\sigma) \neq 0$.

8. Irreducibility of global descents and a bijection between $\mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi)$ and $\mathcal{N}_n(\bar{\tau}, \eta)$

Theorem 8.1. *Let* Π *be an element of* $\mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi)$. *Then the descent*

$$\Psi(\Pi) = \mathcal{D}^{4n,\eta}_{2n,\psi}(\Pi)$$

is an irreducible cuspidal automorphic representation of $G_n(\mathbb{A})$. *In particular, the global descent* $\mathcal{D}^{4n,\eta}_{2n,\psi}(\mathcal{E}_{\bar{\tau}})$ *is irreducible.*

Proof. This theorem is proved in the same way as the proof of [GJS12, Theorem 4.1] using Proposition 7.7, Theorem 7.8, and Corollary 7.4 instead of Theorem 4.1, Theorem 5.7, and Theorem 4.2 of [GJS12], respectively. For the convenience of the reader, we shall prove it when $\Pi = \mathcal{E}_{\bar{\tau}}$ taking Assumption 1 and the character η into account.

By Proposition 7.7, we have

$$\mathcal{E}_{\bar{\tau}} \otimes \eta^{-1} = \mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}\left(\mathcal{D}^{8n,\eta}_{6n,\psi}(\mathcal{E}_{\bar{\tau},\mathcal{E}_{\bar{\tau}}})\right).$$

From Theorem 4.1, any irreducible constituent of $\Psi(\mathcal{E}_{\bar{\tau}}) = \mathcal{D}^{4n,\eta}_{2n,\psi}(\mathcal{E}_{\bar{\tau}})$ is ψ^{-1}-generic and cuspidal. Then we note that for any such irreducible constituent π_0, we have the residual representation $\mathcal{E}_{\bar{\tau},\pi_0}$, where we do not need Assumption 1. Let π and σ be automorphic representations of $G_n(\mathbb{A})$ and $G_{3n}(\mathbb{A})$ given in Theorem 7.8 for $\Pi = \mathcal{E}_{\bar{\tau}}$. Then we have

$$\mathcal{E}_{\bar{\tau}} \otimes \eta^{-1} = \mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\sigma)$$

since $\mathcal{E}_{\bar{\tau}}$ is irreducible. From condition (2) in Theorem 7.8, we have

$$\mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\sigma) \subset \mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\mathcal{E}_{\bar{\tau},\pi}), \text{ i.e., } \mathcal{D}^{6n,\eta}_{4n,\psi^{-1}}(\sigma) \subset \Phi(\pi).$$

Therefore, we obtain

$$\mathcal{E}_{\bar{\tau}} \otimes \eta^{-1} \subset \Phi(\pi). \tag{8.1}$$

From the definition of the descent, we have

$$\Psi(\mathcal{E}_{\bar{\tau}} \otimes \eta^{-1}) = \Psi(\mathcal{E}_{\bar{\tau}}) \otimes \eta^{-1}. \tag{8.2}$$

Hence, (8.1), (8.2), and Corollary 7.4 show that

$$\Psi(\mathcal{E}_{\bar{\tau}}) \otimes \eta^{-1} = \Psi(\mathcal{E}_{\bar{\tau}} \otimes \eta^{-1}) \subset \Psi(\Phi(\pi)) = \pi \otimes \eta^{-1},$$

and thus $\Psi(\mathcal{E}_{\bar{\tau}}) = \pi$ is irreducible without Assumption 1. □

Theorem 8.2. *For each representation* $\Pi \in \mathcal{N}_{2n}(\bar{\tau}, \eta, \psi)$, Π *is a subrepresentation of* $\Phi(\Psi(\Pi))$, *which is an inclusion as spaces of square-integrable automorphic forms.*

Proof. This theorem is proved in the same way as the proof of [GJS12, Theorem 4.3] using Proposition 7.7 and Theorem 7.8. □

In order to consider the opposite direction, let us consider the following assumption.

Assumption 2. *For any* $\pi \in \mathcal{N}_n(\bar{\tau}, \eta)$*, the residual representation* $\mathcal{E}_{\bar{\tau},\pi}$ *of* $G_{3n}(\mathbb{A})$ *is irreducible.*

We remark that this assumption should follow from Mok [Mo15] if assumptions there are proved. When we assume Assumption 2, we can refine the above result.

Theorem 8.3. *Suppose that Assumption 2 holds. Then for* $\Pi \in \mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi)$*,*

$$\Phi(\Psi(\Pi)) = \Pi \otimes \eta^{-1}, \quad \textit{i.e.,} \quad \Phi'(\Psi(\Pi)) = \Pi.$$

In particular, $\Phi'(\pi)$ *and* $\Phi(\pi)$ *are irreducible. Moreover, the mappings*

$$\Psi : \mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi) \to \mathcal{N}_n(\bar{\tau}, \eta)$$

and

$$\Phi' : \mathcal{N}_n(\bar{\tau}, \eta) \to \mathcal{N}'_{2n}(\bar{\tau}, \eta, \psi)$$

are bijective and satisfy

$$\Psi \circ \Phi' = \mathrm{Id}_{\mathcal{N}_n(\bar{\tau},\eta)}, \quad \Phi' \circ \Psi = \mathrm{Id}_{\mathcal{N}'_{2n}(\bar{\tau},\eta,\psi)}.$$

Proof. All assertions are proved in the same way as the proofs of [GRS11, Theorem 4.4, 4.5] using Assumption 2. □

9. Applications of the irreducibility of global descents

In this section, as an application of the irreducibility of global descents, we shall show the rigidity theorem and a local converse theorem for generic representations and the irreducibility of explicit local descents.

9.1. **Global application.**

In this section, we use the same notation as in the previous sections.

Theorem 9.1. *Let* σ *and* σ' *be irreducible* ψ^{-1}*-generic cuspidal automorphic representations of* $G_n(\mathbb{A})$*. Suppose that* σ *and* σ' *are nearly equivalent; i.e., for almost all places* v *of* F*,* $\sigma_v \simeq \sigma'_v$*. Then*

$$\sigma = \sigma'.$$

Hence, the multiplicity one theorem for the generic spectrum holds for G_n*.*

Proof. Let τ and τ' be base change lifts of σ and σ' to $\mathrm{GL}_{2n}(\mathbb{A}_E)$, respectively (see Theorem 4.2). Since $\sigma_v \simeq \sigma'_v$ for almost all v, we have $\tau_v \simeq \tau'_v$ at these places v. Then by the strong multiplicity one theorem for $\mathrm{GL}_{2n}(\mathbb{A}_E)$, we have

$$\tau = \tau'.$$

Let us write $\tau = \boxplus_{i=1}^{n} \tau_i \otimes \eta^{-1}$ and denote the global descent for $\bar{\tau} = (\tau_1, \ldots, \tau_r)$ by $\mathcal{D}_{2n,\psi}^{4n,\eta}(\mathcal{E}_{\bar{\tau}})$. From [GRS11, Theorem 3.1(6)], the complex conjugates of σ and σ' have L^2-pairings with $\mathcal{D}_{2n,\psi}^{4n,\eta}(\mathcal{E}_{\bar{\tau}})$. Since $\mathcal{D}_{2n,\psi}^{4n,\eta}(\mathcal{E}_{\bar{\tau}})$ is irreducible by Theorem 8.1, we obtain

$$\sigma = \mathcal{D}_{2n,\psi}^{4n,\eta}(\mathcal{E}_{\bar{\tau}}) = \sigma'.$$

□

Remark 9.2. This theorem should follow from Ma [Mo15] if we admit certain assumptions in his paper.

Form Theorem 9.1, the following corollary readily follows.

Corollary 9.3. *Let* $\mathcal{U}_{2n,\psi}^{\mathrm{igca}}$ *be the set of all equivalence classes of irreducible* ψ^{-1}*-generic cuspidal automorphic representations of* $G_n(\mathbb{A})$ *and let* $\mathcal{GL}_{2n}^{\mathrm{ia}}(\mathbb{A}_E)$ *be the set of all equivalence classes of irreducible automorphic representations of* $\mathrm{GL}_{2n}(\mathbb{A}_E)$*. Then the base change lift from* $\mathcal{U}_{2n,\psi}^{\mathrm{igca}}$ *to* $\mathcal{GL}_{2n}^{\mathrm{ia}}(\mathbb{A}_E)$ *is injective. More precisely, when we have*

$$\sigma \mapsto \bar{\pi} \otimes \eta^{-1}$$

under the base change lift from $G_n(\mathbb{A})$ *to* $\mathrm{GL}_{2n}(\mathbb{A}_E)$*,*

$$\sigma = \mathcal{D}_{2n,\psi}^{4n,\eta}(\mathcal{E}_{\bar{\pi}}).$$

9.2. **Local applications.** In this section, we shall consider local applications of the irreducibility of global descents. Let us set some notation. Let k be a non-archimedean local field of characteristic zero and let k' be a quadratic extension of k. Let ψ_k be a non-trivial additive character of k, and define an additive character $\psi_{k'}$ of k' by

$$\psi_{k'}(x) = \psi_k\left(\frac{x+\bar{x}}{2}\right), \quad x \in k',$$

where $x \mapsto \bar{x}$ is the action of the non-trivial element of $\mathrm{Gal}(k'/k)$. Then we define a generic character $\psi_{k'}^{\mathrm{GL}}$of upper unipotent matrices $Z_m(k')$ of $\mathrm{GL}_m(k')$ by

$$u \mapsto \psi_{k'}(u_{1,2} + \cdots + u_{m-1,m}).$$

First of all, as a local counterpart of the irreducibility of global descents, we shall prove the irreducibility of local descents. Recall that we have two kinds of local descents, called explicit local descents and abstract local descents (e.g. see [LM16, Section 5.1]). In this section, we shall show the irreducibility of explicit local descents, which we call simply local descents from now on.

Let us recall the definition of local descents. Let n_i $(1 \le i \le r)$ be positive integers such that $n_1 + \cdots + n_r$ is even, say $2n$. Let $\bar{\tau} = (\tau_1, \ldots, \tau_r)$ with an irreducible supercuspidal representation τ_i of $\mathrm{GL}_{n_i}(k)$ such that $L(s, \tau_i, \mathrm{Asai})$ has a pole at $s = 0$, and if $i \neq j$, then $\tau_i \not\simeq \tau_j$. Here, the $L(s, \tau_i, \mathrm{Asai})$ is the Asai L-function defined by the Langlands-Shahidi method [Sh90b] or the L-function introduced by Flicker [Fl88], [Fl93] since both L-functions match (see [AR05]). Then we consider a parabolic induction of $\mathrm{GL}_{2n}(k')$:

$$\pi := \boxplus \tau_i = \mathrm{Ind}_{P_{[n_i]}^{\mathrm{GL}}(k')}^{\mathrm{GL}_{2n}(k')}(\tau_1 \otimes \cdots \otimes \tau_r).$$

Let $\mathbb{W}^{\psi_{k'}^{\mathrm{GL}}}(\pi)$ be the Whittaker model of π with respect to $\psi_{k'}^{\mathrm{GL}}$ and let $\mathrm{Ind}(\mathbb{W}^{\psi_{k'}^{\mathrm{GL}}}(\pi))$ be the space of $G_{2n}(k)$-smooth left $N_n^{2n}(k)$-invariant functions $W : G_{2n}(k) \to \mathbb{C}$ such that for all $g \in G_{2n}(k)$, the function $m \mapsto \delta_{P_n^{2n}}(m)^{-\frac{1}{2}} W(mg)$ on $M_n^{2n}(k)$ belongs to $\mathbb{W}^{\psi_{k'}}(\pi)$.

Let $K_{\mathrm{GL}_{2n}}$ be the standard maximal compact subgroup of $\mathrm{GL}_{2n}(k')$ and let $K_{2n} = K_{\mathrm{GL}_{2n}} \cap G_{2n}(F)$. Then the function on $D_{2n}^{4n}(k)$ defined by

$$\begin{pmatrix} g & \star \\ & g^* \end{pmatrix} \mapsto |\det g|$$

extends to the function on $G_{2n}(k)$ so that it is trivial on K_{2n}. We denote this function by $\nu(g)$ for $g \in G_{2n}(k)$.

For $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{k'}^{\mathrm{GL}}}(\pi))$ and $s \in \mathbb{C}$, we define

$$W_s(g) = \nu(g)^s W(g).$$

Then we consider the integral

$$A^{\psi,\eta}(W,\Phi,g,s)=\int_{(U_n^{4n})_\gamma\backslash U_n^{4n}(k)} W_s(\gamma v g)\omega_{\psi_{k'}^{-1},\eta^{-1}}(vg)\Phi(\xi_n),$$

where $\gamma=\begin{pmatrix} & 1_n & & \\ -1_n & & & 1_n \\ & & 1_n & \end{pmatrix}$, $\xi_n=(0,\ldots,0,1)\in(k')^n$, and $(U_n^{4n})_\gamma=\gamma^{-1}U_{2n}(k)\gamma\cap U_n^{4n}(k)$. We know that for each $g\in G_{2n}(k)$, $A^{\psi,\eta}(W,\Phi,g,s)$ is entire as a function of s, and it is $(U_n(k),\psi_{k'}^{-1})$-invariant (see [LM16, Lemma 5.1]). Here, $\psi_{k'}^{-1}$ denotes the generic character of $U_n(k)$ defined as in (3.2).

Let us define an intertwining operator by

$$M(s)W(g)=\nu(g)^s\int_U W_s(w_U ug)\,du \quad \text{with} \quad w_U=\begin{pmatrix} & 1_{2n}\\ -1_{2n} & \end{pmatrix}.$$

We note that by [LM16, Proposition 2.1], $M(s)$ is holomorphic at $s=\frac{1}{2}$. Then we denote by $\mathcal{D}^{\eta}_{\psi_{k'}}(\bar\tau)$ the space of Whittaker functions on $G_n(k)$ generated by $A^{\psi,\eta}\left(M(\frac{1}{2})W,\Phi,\cdot,-\frac{1}{2}\right)$ for $W\in\mathrm{Ind}(\mathbb{W}^{\psi_{k'}^{\mathrm{GL}}}(\pi))$. We call $\mathcal{D}^{\eta}_{\psi_{k'}}(\bar\tau)$ the local descent of $\bar\tau$. We know that $\mathcal{D}^{\eta}_{\psi_{k'}}(\bar\tau)$ is non-zero (see [LM16, Section 5.1]). Further, it is easy to see that

$$\mathcal{D}^{\eta}_{\psi_{k'}}(\bar\tau)\simeq\mathcal{D}^{\eta}_{\psi_{k'}^a}(\bar\tau) \tag{9.1}$$

for any $a\in(k^\times)^2$. Here, $\psi_{k'}^a(x)=\psi_{k'}(ax)$ for $x\in k'$.

Theorem 9.4. *The local descent $D^{\eta}_{\psi_{k'}}(\bar\tau)$ is an irreducible $\psi_{k'}^{-1}$-generic supercuspidal representation. Moreover, $D^{\eta}_{\psi_{k'}}(\bar\tau)$ is the unique $\psi_{k'}^{-1}$-generic supercuspidal representation such that*

$$\gamma^{Sh}\left(s,D^{\eta}_{\psi_{k'}}(\bar\tau)\times(\tau_i\otimes\eta^{-1}),\psi_{k'}\right) \tag{9.2}$$

has a simple pole at $s=1$ for each $1\le i\le r$. Here, local γ-factors are the ones defined in Shahidi [Sh90a], [Sh90b].

Proof. Let us prove the irreducibility. Let K be a number field and let K' be a quadratic extension of K such that for some finite place v, $K_v\simeq k$ and $K'_v\simeq k'$. We know that there is a character Υ of $\mathbb{A}_{K'}/K'$ such that $\Upsilon_v\simeq\eta$ and $\Upsilon|_{\mathbb{A}_K^\times}$ is the quadratic character of $\mathbb{A}_K^\times/K^\times$ corresponding to the quadratic extension K'/K (e.g. see [GI16, Section 6.6]). Further, from (9.1), we may suppose that $\psi_{k'}$ is a v-component of some character $\psi_{\mathbb{A}_{K'}}$ of $\mathbb{A}_{K'}/K'$ such that $\psi_{\mathbb{A}_{K'}}(\bar x)=\psi_{\mathbb{A}_{K'}}(x)$ for any $x\in\mathbb{A}_{K'}$.

From [Ka04, Theorem 4], we know that each τ_i is of unitary type; i.e., it has a non-trivial $\mathrm{GL}_{n_i}(k)$-invariant linear form. Then by [HM02, Theorem 1], there exists an irreducible cuspidal automorphic representation Π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_{K'})$ such that $\Pi_{i,v}\simeq\tau_i$ and the following linear functional is not identically zero on Π_i:

$$\varphi\mapsto\int_{\mathbb{A}_K^\times\mathrm{GL}_{n_i}(K)\backslash\mathrm{GL}_{n_i}(\mathbb{A}_K)}\varphi(g)\,dg,\quad \varphi\in\Pi_i.$$

Therefore, by [FZ95, Theorem], a partial L-function $L^S(s,\Pi_i,\mathrm{Asai})$ has a pole at $s=1$. Then we have the residual representation $\mathcal{E}_{\bar\Pi}$ by [GRS11, Theorem 2.1] where $\bar\Pi=(\Pi_1,\ldots,\Pi_r)$.

Lapid–Mao [LM16, Theorem 5.5] showed that under two working assumptions, $D^{\eta}_{\psi_{k'}}(\bar{\tau}) \simeq D^{4n,\Upsilon}_{2n,\psi}(\mathcal{E}_{\bar{\Pi}})_v$ is irreducible. One of their working assumptions is the irreducibility of global descents, namely Theorem 8.1. The other working assumption on analytic properties of local zeta integrals was proved by Ben-Artzi and Soudry [BAS]. Hence, we obtain the above irreducibility without any assumption.

On the other hand, by Corollary 9.3, the base change lift of $D^{4n,\Upsilon}_{2n,\psi}(\mathcal{E}_{\bar{\Pi}})$ to $\mathrm{GL}_{2n}(\mathbb{A}_K)$ is $\boxplus\Pi_i \otimes \Upsilon^{-1}$. In particular, the local base change lift of $D^{\eta}_{\psi}(\bar{\tau})$ to $\mathrm{GL}_{2n}(k')$ is $\boxplus\Pi_{i,v} \otimes \Upsilon_v^{-1} \simeq \boxplus\tau_i \otimes \eta^{-1}$. In Kim–Krishnamurthy [KK05, Proposition 8.4 - 8.7], they constructed explicitly local base change lifts for $\psi_{k'}^{-1}$-generic representations. Indeed, their construction implies that if an irreducible admissible $\psi_{k'}^{-1}$-generic representation of $G_n(k)$ lifts to $\boxplus\tau_i \otimes \eta^{-1}$, then it is supercuspidal. Hence, the local descent $D^{\eta}_{\psi_{k'}}(\bar{\tau})$ is supercuspidal.

We note that the above local base change lift by Kim–Krishnamurthy [KK05] is strong; in particular, we have the following identity:

$$\gamma^{Sh}\left(s, D^{\eta}_{\psi_{k'}}(\bar{\tau}) \times (\tau_i \otimes \eta^{-1}), \psi_{k'}\right) = \lambda(E/F, \psi_{k'})^{2nn_i} \prod_{j=1}^{r} \gamma^{RS}(s, (\tau_j \otimes \eta^{-1}) \times (\tau_i \otimes \eta^{-1}), \psi_{k'}).$$

Here, the γ-factor on the right-hand side is the one defined by the Rankin-Selberg method by Jacquet, Piatetski-Shapiro, and Shalika [JPSS83]. Since $\tau_i \not\simeq \tau_j$ if $i \neq j$, $\gamma^{RS}(s, (\tau_j \otimes \eta^{-1}) \times (\tau_i \otimes \eta^{-1}), \psi_{k'})$ does not have a pole and zero at $s = 1$. Hence, (9.2) has a pole at $s = 1$ for each i. Conversely, let σ be a $\psi_{k'}^{-1}$-generic supercuspidal representation of $G_n(k)$ such that

$$\gamma^{Sh}\left(s, \sigma \times (\tau_i \otimes \eta^{-1}), \psi_{k'}\right)$$

has a simple pole at $s = 1$ for each i. Let ρ be the base change lift of σ to $\mathrm{GL}_{2n}(k')$ given by [KK05, Proposition 8.4]. Then we can write

$$\rho = \rho_1 \boxplus \cdots \boxplus \rho_{r'}$$

with irreducible supercuspidal representation ρ_i of $\mathrm{GL}_{n_i}(k')$ such that $\rho_i \not\simeq \rho_j$ if $i \neq j$. By our assumption, we have that

$$\prod_{j=1}^{r} \gamma^{RS}\left(s, \rho_j \times (\tau_i \otimes \eta^{-1}), \psi_{k'}\right)$$

has a pole at $s = 1$. Then there is $1 \leq j_0 \leq r'$ such that

$$\gamma^{RS}\left(s, \rho_{j_0} \times (\tau_i \otimes \eta^{-1}), \psi_{k'}\right)$$

has a pole at $s = 1$ and thus we should have $\rho_{j_0} \simeq \tau_i \otimes \eta^{-1}$. Therefore we should have

$$\rho = \boxplus(\tau_i \otimes \eta^{-1}).$$

From the proof of the irreducibility, we should have

$$\sigma = \mathcal{D}^{\eta}_{\psi}(\bar{\tau}),$$

and this completes our proof. □

Remark 9.5. In Soudry–Tanay [ST15], they studied the local descent when $r = 1$. In this case, they gave a characterization of local descent and proved the irreducibility under a certain assumption.

As another application, we prove a local converse theorem for generic representations of $G_n(k)$.

Theorem 9.6. *Let π and π' be irreducible $\psi_{k'}^{-1}$-generic representations of $G_n(k)$ such that*

$$\gamma^{Sh}(s, \pi \times \sigma, \psi_{k'}) = \gamma^{Sh}(s, \pi' \times \sigma, \psi_{k'})$$

holds for any irreducible supercuspidal representation σ of $\mathrm{GL}_i(k')$ with $1 \leq i \leq n$. Then

$$\pi \simeq \pi'.$$

Proof. First, suppose that π and π' are supercuspidal. Let K and K' be as in the proof of Theorem 9.4. First, we note that we may suppose that $\psi_{k'}$ is a v-component of some character $\psi_{\mathbb{A}_{K'}}$ of $\mathbb{A}_{K'}/K'$ such that $\psi_{\mathbb{A}_{K'}}(\bar{x}) = \psi_{\mathbb{A}_{K'}}(x)$ for any $x \in \mathbb{A}_{K'}$. This clearly follows from the identity

$$\gamma^{Sh}(s, \pi \times \sigma, \psi_{k'}^{a}) = \gamma^{Sh}(s, \pi' \times \sigma, \psi_{k'}^{a}) \tag{9.3}$$

for $a \in (k^{\times})^2$ where $\psi_{k'}^{a}(x) := \psi_{k'}(ax)$. From the definition [Sh90b, Theorem 3.5] of γ-factors, this identity follows from a similar relation for local coefficients. Further from [Sh90b, (1.2)], such identity follows from an invariance of Whittaker functionals, and it is easy to check this invariance from the definition.

By a standard argument as in [Sh90b, Proposition 5.1] (see also a proof of [ST15, Theorem 7.3]), we see that there are irreducible cuspidal $\psi_{K'}^{-1}$-generic automorphic representations Π and Π' of $G_n(\mathbb{A}_K)$ such that $\Pi_v = \pi$ and $\Pi'_v = \pi'$. Let $\Xi = \boxplus \Xi_i$ and $\Xi' = \boxplus \Xi'_i$ be the base change lift of Π and Π' to $\mathrm{Res}_{K'/K}\mathrm{GL}_{2n}(\mathbb{A}_K) \simeq \mathrm{GL}_{2n}(\mathbb{A}_{K'})$ established by [KK05], respectively. Since this lift is strong, we have

$$\lambda(k'/k, \psi_{k'})^{2ni} \gamma^{RS}(s, \Xi_v \times \sigma, \psi_{k'}) = \gamma^{Sh}(s, \pi \times \sigma, \psi_{k'})$$

and

$$\lambda(k'/k, \psi_{k'})^{2ni} \gamma^{RS}(s, \Xi'_v \times \sigma, \psi_{k'}) = \gamma^{Sh}(s, \pi' \times \sigma, \psi_{k'})$$

for any irreducible supercuspidal representation σ of $\mathrm{GL}_i(k')$ with $1 \leq i \leq n$. Hence, we have

$$\gamma^{RS}(s, \Xi_v \times \sigma, \psi_{k'}) = \gamma^{RS}(s, \Xi'_v \times \sigma, \psi_{k'}).$$

By [JNS15, Corollary 2.7], Ξ_v and Ξ'_v have the same central character. Then from the local converse theorem for $\mathrm{GL}_{2n}(k')$ by [JL16, Theorem 1.3] and [Ch16, Theorems 4.3, 4.4], we obtain

$$\Xi_v \simeq \Xi'_v. \tag{9.4}$$

Let $\bar{\Xi} = (\Xi_1, \ldots, \Xi_r)$ and $\bar{\Xi}' = (\Xi'_1, \ldots, \Xi_{r'})$. Then by Corollary 9.3, we have

$$\mathcal{D}_{2n,\psi}^{4n,\eta}(\mathcal{E}_{\bar{\Xi} \otimes \eta}) = \Pi \quad \text{and} \quad \mathcal{D}_{2n,\psi}^{4n,\eta}(\mathcal{E}_{\bar{\Xi}' \otimes \eta}) = \Pi'. \tag{9.5}$$

On the other hand, as in the proof of the previous theorem,

$$\mathcal{D}_{2n,\psi}^{4n,\eta}(\mathcal{E}_{\bar{\Xi} \otimes \eta})_v \simeq \mathcal{D}_{\psi_v}^{\eta_v}(\bar{\Xi}_v \otimes \eta_v) \quad \text{and} \quad \mathcal{D}_{2n,\psi}^{4n,\eta}(\mathcal{E}_{\bar{\Xi}' \otimes \eta})_v \simeq \mathcal{D}_{\psi_v}^{\eta_v}(\bar{\Xi}'_v \otimes \eta_v). \tag{9.6}$$

Therefore, since $\Pi_v \simeq \pi$ and $\Pi'_v \simeq \pi'$, (9.4), (9.5), (9.6), and Theorem 9.4 show that

$$\pi \simeq \pi'.$$

Let us consider the general case. First, we note that any irreducible $\psi_{k'}^{-1}$-generic representation π is written as a subquotient of

$$\mathrm{Ind}_{P_{[a_i],a}}^{G_n(k)}(\tau_1 |\det|^{r_1} \otimes \cdots \otimes \tau_r |\det|^{r} \otimes \rho_0),$$

where τ_i is an irreducible supercuspidal representation of $\mathrm{GL}_{a_i}(k')$, ρ_0 is an irreducible supercuspidal $\psi_{k'}^{-1}$-generic representation of $G_a(k)$, and z_i is a real number such that $z_1 \geq \cdots \geq z_r \geq 0$. Then we say that π has supercuspidal support $(P_{[a_i],a}, \tau_1, \ldots, \tau_r; \rho_0)$ and exponents $(z_1, \ldots, z_r)$.

In the same argument as in the proofs of [JS03, Proposition 3.2] and [Li11, Lemma 3.3], we can prove the following lemma using the multiplicativity of γ-factors by Shahidi [Sh90a] and the local base change lifts given by Kim and Krishnamurthy [KK05]. We omit a proof.

Lemma 9.7. *Let π be an irreducible admissible $\psi_{k'}^{-1}$-generic representation of $G_n(k)$ with supercuspidal support $(P_{[a_i],a}, \tau_1, \ldots, \tau_r; \rho_0)$ and exponents $(z_1, \ldots, z_r)$. Then $s = 1 + z_1$ is the rightmost real point at which the twisted local gamma factor $\gamma^{Sh}(s, \pi \times \sigma, \psi_{k'})$ can possibly have a pole where σ is an irreducible unitary supercuspidal representation of $\mathrm{GL}_{a_i}(k')$ with $l \in \mathbb{Z}_{\geq 0}$. If the pole at $s = 1 + z_1$ occurs for some (l, σ), then $\sigma \simeq \tau_{i_0}$ for some $1 \leq i_0 \leq r$ such that $z_{i_0} = z_1$.*

Moreover, in the same argument as in [JS03, Theorem 5.1] (cf. [Li11, Theorem 3.5]), we can prove the following result, which obviously finishes a proof of Theorem 9.6 by the above argument. We omit a proof.

Proposition 9.8. *Let π and π' be irreducible $\psi_{k'}^{-1}$-generic representations of $G_n(k)$, with supercuspidal support $(P_{[a_i],a}, \tau_1, \ldots, \tau_r; \rho_0)$ and $(P_{[a_i'],a'}, \tau_1', \ldots, \tau_{r'}'; \rho_0')$, exponents $(z_1, \ldots, z_r)$ and exponents $(z_1', \ldots, z_{r'}')$, respectively. Suppose that $\gamma^{Sh}(s, \pi \times \sigma, \psi_{k'}) = \gamma^{Sh}(s, \pi' \times \sigma, \psi_{k'})$ for any irreducible supercuspidal representation σ of $\mathrm{GL}_l(k')$, with $1 \leq l \leq n$. Then after a possible rearrangement of $(\tau_1', z_1'; \ldots; \tau_{r'}', z_{r'}')$, without affecting the decreasing order of $z_1', \ldots, z_{r'}'$,*

(1) *$r = r'$ and $a_i = a_i'$ for $1 \leq i \leq r$,*
(2) *$z_i = z_i'$ and $\tau_i \simeq \tau_i'$ for $1 \leq i \leq r$,*
(3) *$\gamma^{Sh}(s, \rho_0 \times \sigma, \psi_{k'}) = \gamma^{Sh}(s, \rho_0' \times \sigma, \psi_{k'})$ for any irreducible supercuspidal representation σ of $\mathrm{GL}_l(k')$ with $1 \leq l \leq n$.*

□

Remark 9.9. We expect that a precise study of local theta correspondences between even unitary groups and odd unitary groups should give a local converse theorem for odd unitary groups.

By the above two theorems, we obtain the following characterization of local base change lifts for supercuspidal representations.

Corollary 9.10. *Let $\mathrm{Irr}_{cusp}^{gen}(G_n(k))$ be the set of isomorphic classes of irreducible $\psi_{k'}^{-1}$-generic supercuspidal representations of $G_n(k)$ and let $\mathrm{Irr}_{cusp}^{\mathrm{isob}}(\mathrm{GL}_{2n}(k'))$ be the set of isomorphic classes of irreducible representations of $\mathrm{GL}_{2n}(k')$ of the form $\rho_1 \boxplus \cdots \boxplus \rho_r$ such that $L(s, \rho_i \otimes \eta, \mathrm{Asai})$ has a pole at $s = 0$ and $\rho_i \not\simeq \rho_j$ if $i \neq j$. Then there is a unique bijection between*

$$\ell : \mathrm{Irr}_{cusp}^{gen}(G_n(k)) \to \mathrm{Irr}_{cusp}^{\mathrm{isob}}(\mathrm{GL}_{2n}(k'))$$

satisfying

$$\gamma^{Sh}(s, \pi \times \tau, \psi_{k'}) = \lambda(E/F, \psi_{k'})^{2ni} \gamma^{RS}(s, \ell(\pi) \times \tau, \psi_{k'}) \tag{9.7}$$

for any irreducible supercuspidal representation τ of $\mathrm{GL}_i(k')$ with $1 \leq i \leq n$.

Proof. The base change lift given by Kim–Krishnamurthy [KK05] gives this bijection. Indeed, the condition (9.7) is proved in [KK05], and the injectivity follows from Theorem 9.6. Moreover, in the proof of Theorem 9.4, for a given $\boxplus\tau_i \otimes \eta^{-1} \in \mathrm{Irr}_{cusp}^{isob}(\mathrm{GL}_{2n}(k'))$, we showed that the base change lift of $\mathcal{D}_\psi^\eta(\bar{\tau})$ to $\mathrm{GL}_{2n}(k')$ is $\boxplus\tau_i \otimes \eta^{-1}$, and thus the surjectivity follows.

Finally, the uniqueness of the bijection satisfying the above condition follows from Theorem 9.6. □

Remark 9.11. From this corollary, we obtain the uniqueness of a local Langlands correspondence for even unitary groups given in [Mo15].

This corollary gives a characterization of $(\mathrm{GL}_{2n}(k), \omega_{k'/k})$-distinguished supercuspidal representations of $\mathrm{GL}_{2n}(k')$ in terms of local base change lifts, which is an affirmative answer to a conjecture by Flicker and Rallis (see [Fl91]) in the case of supercuspidal representations.

Corollary 9.12. *Let π be an irreducible supercuspidal representation of $\mathrm{GL}_{2n}(k')$. Then π is $(\mathrm{GL}_{2n}(k), \omega_{k'/k})$-distinguished if and only if π is a local base change lift of an irreducible $\psi_{k'}^{-1}$-generic supercuspidal representation of $G_n(k)$.*

Proof. By [AKT04, Corollary 1.5], π is $(\mathrm{GL}_{2n}(k), \omega_{k'/k})$-distinguished if and only if $L(s, \pi \otimes \eta, \mathrm{Asai})$ has a pole at $s = 0$. Then the above corollary gives the required equivalence. □

Appendix A. Linear functionals and Jacquet modules of unramified representations

In this appendix, we shall prove some results on unramified representations. This is crucial for our computations of Fourier coefficients and Fourier-Jacobi coefficients. Since we consider a local situation, we let the base F be a non-archimedean local field of characteristic zero and let E be an unramified quadratic extension of F. Let η be an unramified character of $E^\times$ such that $\eta|_{F^\times}$ is the quadratic character of $F^\times$ corresponding to E/F. Further, other notation is the same as in the main body of this paper.

Let σ be an irreducible unramified representation of $G_n(F)$. Assume that σ is the unramified constituent of the parabolic induction

$$\mathrm{Ind}_{B_n(F)}^{G_n(F)}(\chi), \quad \chi = \prod_{i=1}^n \chi_i.$$

Here χ_i are unramified characters of $E^\times$. Let τ be an unramified representation of $\mathrm{GL}_{2n}(E)$. We suppose that τ is the base change lift of σ; namely, τ is the unramified constituent of

$$\mathrm{Ind}_{\mathcal{P}_{2n}(F)}^{\mathrm{GL}_{2n}(E)}(\chi).$$

Here, we regard χ as a character of $\mathcal{P}_{2n}(F)$ by

$$\chi(\mathrm{diag}(t_1, \dots, t_{2n})u) = \prod_{i=1}^n \chi_i(t_i) \cdot \prod_{i=n+1}^{2n} \chi_{2n+1-i}(\bar{t}_i)^{-1}, \quad t_i \in E^\times, u \in Z_{2n}.$$

Let θ_τ denote the unramified constituent of $I(\tau) := \mathrm{Ind}_{Q_k}^{G_{2nk}}(\tau|\det|^{k-1/2} \otimes \cdots \otimes \tau|\det|^{1/2})$. Here, Q_k is the standard parabolic subgroup of $G_{2nk}(F)$ with the Levi decomposition $Q_k = (\mathrm{GL}_{2n}(E) \times \cdots \times \mathrm{GL}_{2n}(E))U(Q_k)$.

Similarly, let $\theta_{\tau,\sigma}$ denote the unramified constituent of the induced representation $I(\tau,\sigma) := \mathrm{Ind}_{L_k}^{G_{n(2k+1)}(F)}(\tau|\det|^k \otimes \cdots \otimes \tau|\det| \otimes \sigma)$. Here, L_k is the standard parabolic subgroup of $G_{n(2k+1)}(F)$ with the Levi decomposition $L_k = (\mathrm{GL}_{2n}(E) \times \cdots \times \mathrm{GL}_{2n}(E) \times G_n(F))U(L_k)$.

Let $\hat{Q}_k$ denote the standard parabolic subgroup of $G_{2nk}(F)$ with the Levi decomposition $\hat{Q}_k = (\mathrm{GL}_{2k}(E) \times \cdots \times \mathrm{GL}_{2k}(E))U(\hat{Q}_k)$ where $\mathrm{GL}_{2k}(E)$ occurs n times. Let $\hat{L}_k$ denote the standard parabolic subgroup of $G_{n(2k+1)}(F)$ with the Levi decomposition $\hat{L}_k = (\mathrm{GL}_{2k+1}(E) \times \cdots \times \mathrm{GL}_{2k+1}(E))U(\hat{L}_k)$ where $\mathrm{GL}_{2k+1}(E)$ occurs n times. By the same argument as in [GRS05, Lemma 3.1], we can prove the following result. Hence, we omit a proof.

Lemma A.1.

(1) *Let τ be as above. Assume that τ is a conjugate self-dual unramified representation of $\mathrm{GL}_{2n}(E)$. Then the representation θ_τ is the unramified constituent of the parabolic induction*

$$\mathrm{Ind}_{\hat{Q}_k}^{G_{2nk}(F)}(\chi_1(\det) \otimes \cdots \otimes \chi_n(\det)).$$

(2) *Let σ and τ be as above. Then the representation $\theta_{\tau,\sigma}$ is the unramified constituent of the parabolic induction*

$$\mathrm{Ind}_{\hat{L}_k}^{G_{n(2k+1)}(F)}(\chi_1(\det) \otimes \cdots \otimes \chi_n(\det)).$$

Let $U_p := U_p^{2r}(F)$ denote the unipotent radical of the standard parabolic subgroup $Q_p^{2r}(F)$ of $G_r(F)$ whose Levi part is isomorphic to $\mathrm{GL}_1(E)^p \times G_{r-p}(F)$. For $\alpha \in E^\times$, let $\psi^{p,\alpha}$ be a character of U^p defined by

$$\psi^{p,\alpha}(u) = \psi(u_{1,2} + \cdots + u_{p-1,p} + u_{p,p+1} + \alpha u_{p,2r-p}).$$

For an admissible representation (π, V_π) of $G_r(F)$, we consider linear functionals $\nu_{p,\alpha}$ on V_π satisfying $\nu_{p,a}(\pi(u)v) = \psi^{p,\alpha}(u)\nu_{p,\alpha}(v)$ for all $u \in U_p$ and $v \in V_\pi$.

Lemma A.2. *Let τ and σ be as in Lemma* A.1. *The representation $\theta_{\tau,\sigma}$ of $G_{3n}(F)$ (i.e., the case $k = 1$) has no non-zero linear functional $\nu_{n,\alpha}$ for all $\alpha \in E^\times$.*

Proof. We shall prove this lemma in a similar way as the proof of [GRS05, Lemma 3.3].

Let us set some notation. Let $\overline{\hat{L}}_3 = (\mathrm{GL}_3(E) \times \cdots \times \mathrm{GL}_3(E))Z$ denote the opposite parabolic subgroup of $\hat{L}_3$ where

$$Z := \{{}^t u : u \in U(\hat{L}_3)\} = \left\{ \begin{pmatrix} 1_3 & & & & \\ * & 1_3 & & & \\ * & * & 1_3 & & \\ \vdots & \vdots & & \ddots & \\ * & * & * & \cdots & 1_3 \end{pmatrix} \right\}.$$

Then we note that Z is an n-step nilpotent subgroup. Let $A \in \mathrm{Mat}_{3\times 3}(E)$. For $2 \le i \le n$ and $1 \le j \le i-1$, we define a matrix $\hat{A}_{i,j}$ in Z by

$$\begin{pmatrix} 1_3 & & & & & & \\ & \ddots & & & & & \\ & & 1_3 & & & & \\ & & \vdots & \ddots & & & \\ & & A & \cdots & 1_3 & & \\ & & & & \vdots & \ddots & & \\ & & & & A^* & \cdots & 1_3 & \\ & & & & & & & \ddots & \\ & & & & & & & & 1_3 \end{pmatrix},$$

which means that the (i,j)-entry (resp. $(2n+1-j, 2n+1-i)$-entry) of $\hat{A}_{i,j}$ is A (resp. A^*) when we consider $\hat{A}_{i,j}$ as a block matrix of size 3×3. Denote by $\hat{Z}_{i,j}$ the subgroup of Z consisting of $\hat{A}_{i,j}$ with $A \in \mathrm{Mat}_{3\times 3}(E)$. Similarly, we define for $n+1 \le i \le 2n$,

$$\hat{Z}_{i,j} = \left\{ \hat{A}_{i,j}; A \in \mathrm{Mat}_{3,3}(E) \text{ if } i+j \neq 2n+1 \text{ and } A \in \mathrm{Mat}^0_{3\times 3}(E) \text{ if } i+j = n+1 \right\},$$

where $\mathrm{Mat}^0_{3\times 3}(E) = \{X \in \mathrm{Mat}_{3\times 3}(E) : {}^t\bar{X} = X\}$.

For $r \in E$ and $1 \le i \le n$, we define

$$u_{i,i+1}(r) = 1_{6n} + re_{i,i+1} - \bar{r}e_{6n-i,6n-i+1} \quad \text{and} \quad u_{n,5n}(r) = 1_{6n} + re_{n,5n} - \bar{r}e_{n+1,5n+1}.$$

From part (ii) of Lemma A.1, it suffices to show our claim for the unramified principal series $\mathrm{Ind}_{\hat{L}_k(F)}^{G_{n(2k+1)}(F)}(\chi_1(\det) \otimes \cdots \otimes \chi_n(\det))$.

Let $\gamma \in \hat{L}_k(F) \backslash G_{3n}(F)/U_n$. Then by the Bruhat decomposition, we may write $\gamma = wu_w$, where w is a Weyl element and u_w is an upper triangular matrix of $G_{3n}(F)$ given by

$$\begin{pmatrix} 1_n & & \\ & u'_w & \\ & & 1_n \end{pmatrix}, \quad u'_w \in G_{2n}(F).$$

From Mackey theory, it suffices to show that we have either

$$\text{(A.1)} \qquad \gamma \hat{u}_{i,i+1}(r)\gamma^{-1} \in \hat{L}_k(F) \quad \text{or} \quad \gamma \hat{u}_{n,5n}(r)\gamma^{-1} \in \hat{L}_k(F).$$

Then in the same argument as in [GRS05, p. 201], we may suppose that $u_w = 1$ and $\gamma = w$. If (A.1) does not hold, then we should have

$$w\hat{u}_{i,i+1}(r)w^{-1} \in Z \quad \text{and} \quad w\hat{u}_{n,5n}(r)w^{-1} \in Z$$

for all $1 \le i \le n$. Then we shall deduce a contradiction.

Let us write

$$w\hat{u}_{n,5n}(r)w^{-1} = (\widehat{A_1})_{i_1,j_1}$$

with some $A_1 \in \mathrm{Mat}_{3\times 3}(E)$. After conjugating by a Weyl element, we may assume that only $(1,3)$-entry of A_1 is non-zero.

Since $\hat{u}_{n,5n}(r)$ does not commute with $\hat{u}_{n,n+1}(r)$, $w\hat{u}_{n,n+1}w^{-1}$ should be written as

$$w\hat{u}_{n,n+1}(r)w^{-1} = (\widehat{A_2})_{j_1,j_2} \quad \text{or} \quad w\hat{u}_{n,n+1}(r)w^{-1} = (\widehat{A_2})_{i_2,i_1}$$

with some $A_2 \in \mathrm{Mat}_{3\times 3}(E)$. From the definition, we have $i_1 \neq j_1$, and thus $w\hat{u}_{n,5n}(r)w^{-1}$ and $w\hat{u}_{n,n+1}(r)w^{-1}$ are in different blocks. As above, we may suppose that only the $(3,3)$-entry of A_2 is not zero.

By a similar reason as above, we may write

$$w\hat{u}_{n-1,n}(r)w^{-1} = \widehat{(A_3)}_{j_2,j_3}, \widehat{(A_3)}_{i_3,j_1}, \widehat{(A_3)}_{i_1,j_3'}, \text{ or } \widehat{(A_3)}_{i_3',i_2}$$

with some $A_3 \in \mathrm{Mat}_{3\times 3}(E)$. Since $j_1 \neq j_2$ and $i_1 \neq i_2$, $w\hat{u}_{n,n+1}(r)w^{-1}$ and $w\hat{u}_{n-1,n}(r)w^{-1}$ should be in different blocks.

Assume that $w\hat{u}_{n,5n}(r)w^{-1}$ and $w\hat{u}_{n-1,n}(r)w^{-1}$ are in the same block. Since $\hat{u}_{n,5n}(r)$ and $\hat{u}_{n-1,n}(r)$ have different entry, this is also true for $w\hat{u}_{n,5n}(r)w^{-1}$ and $w\hat{u}_{n-1,n}(r)w^{-1}$. Thus only the (k_1,k_2)-entry of A_3 is non-zero with $k_1 \neq 1$ and $k_2 \neq 3$. Then, by a direct calculation, we see that $w\hat{u}_{n-1,n}(r)w^{-1}$ and $w\hat{u}_{n,n+1}(r)w^{-1}$ commute. This is a contradiction; thus $w\hat{u}_{n,5n}(r)w^{-1}$, $w\hat{u}_{n-1,n}w^{-1}$, and $w\hat{u}_{n-1,n}(r)$ w^{-1} are in different blocks. Repeating this argument, we see that $w\hat{u}_{n,5n}(r)w^{-1}$ and $w\hat{u}_{i,i+1}(r)w^{-1}$ $(1 \leq i \leq n)$ are in different blocks.

Roots corresponding to $w\hat{u}_{n,5n}w^{-1}$, $w\hat{u}_{i,i+1}w^{-1}$ $(1 \leq i \leq n)$ form a set of simple roots of split special orthogonal group $\mathrm{SO}(2n+2) \subset \mathrm{GL}(2n+2)$. Thus Z should be a k-step nilpotent group with $k \geq n+1$ if $n \neq 1$. Hence, if $n \neq 1$, this is a contradiction, and this completes a proof of our lemma.

Suppose $n = 1$. Then Z has only one non-trivial block, and thus $w\hat{u}_{n,5n}(r)w^{-1}$ and $w\hat{u}_{1,2}(r)w^{-1}$ should be in the same block. Then these two matrices commute since Z is abelian. This contradicts our definition of $\hat{u}_{1,2}$ and $\hat{u}_{n,5n}$. □

Define a unipotent subgroup of $G_r(F)$ by

$$U_p^0 = \{u = (u_{i,j}) \in U_p : u_{p,j} = 0; p+1 \leq j \leq 2r-p\}.$$

For $a \in F^\times$, we define a character $\psi_{p,a}$ of U_p^0 by

$$\psi_{p,a}(u) = \psi(u_{1,2} + \cdots + u_{p-1,p} + a u_{p,2r-p+1}).$$

For an admissible representation (π, V_π) of $G_r(F)$, we consider linear functionals $\ell_{p,a}$ on the space V_π satisfying $\ell_{p,a}(\pi(u)v) = \psi_{p,a}(u)\ell(v)$ for all $u \in U_p^0$ and $v \in V_\pi$. In a similar argument as in the proof of the above lemma (see also the proof of [GRS05, Lemma 3.3]), we can prove the following result. We omit a proof.

Lemma A.3. *Let τ and σ be as in Lemma* A.1. *The representation θ_τ (resp. $\theta_{\tau,\sigma}$) of the group $G_{2nk}(F)$ (resp. $G_{n(2k+1)}(F)$) has no non-zero functional $\ell_{n+1,a}$ for all $a \in F^\times$.*

Finally, we shall compute a certain Jacquet module of $\theta_{\tau,\sigma}$. Let $\omega_{\psi,\eta}$ be the Weil representation of $\mathcal{F}_n := G_{2n}(F) \ltimes \mathcal{H}_{4n+1}$ defined as in (3.5) where $\mathcal{H}_{4n+1}$ denotes the Heisenberg group in $4n+1$-variable defined as in (3.4). Then we denote the twisted Jacquet module of $\theta_{\tau,\sigma}$ with respect to $\psi_{n,a}$ by

$$\mathcal{J}_{U_n^0,\psi_{n,a}}(\theta_{\tau,\sigma}).$$

Let C be the center of the Heisenberg group $\mathcal{H}_{4n+1}$. Further, we regard $C\backslash\mathcal{H}_{4n+1}$ as a subgroup of $G_{3n}(F)$ by

$$(x_1,\ldots,x_{4n};0) \mapsto \begin{pmatrix} 1_{n-1} & & & & \\ & 1 & \mathbf{x} & 0 & \\ & & 1_{4n} & \mathbf{x}' & \\ & & & 1 & \\ & & & & 1_{n-1} \end{pmatrix}$$

where $\mathbf{x} = (x_1, \dots, x_{4n})$. This twisted Jacquet module is an admissible representation of $G_{2n}(F)$. Then we consider the Jacquet module

$$\mathcal{J}_{C\backslash\mathcal{H}_{4n+1}}\left(\mathcal{J}_{U_n^0,\psi_{n,1}}(\theta_{\tau,\sigma}) \otimes \omega_{\psi^{-1},\eta^{-1}}\right). \tag{A.2}$$

As in [GRS05, Proposition 5.4], we can prove the following lemma.

Lemma A.4. *Let τ and σ be as above. Then as a representation of $G_{2n}(F)$, the Jacquet module* (A.2) *has a unique irreducible constituent, which is the unramified constituent of*

$$\mathrm{Ind}_{\hat{Q}_1}^{G_{2n}(F)}(\chi_1\eta^{-1}(\det) \otimes \cdots \otimes \chi_n\eta^{-1}(\det)). \tag{A.3}$$

Proof. It suffices to compute the Jacquet module

$$\mathcal{J}_{C\backslash\mathcal{H}_{4n+1}}\left(\mathcal{J}_{U_n^0,\psi_{n,1}}(\rho_\chi) \otimes \omega_{\psi^{-1},\eta^{-1}}\right),$$

where ρ_χ is the unramified principal series given in part (ii) of Lemma A.1. Then by the same argument as in [GRS05, pp. 218–220], we see that as a representation of $G_{2n}(F)$, this is isomorphic to

$$\mathcal{J}_{C\backslash\mathcal{H}_{4n+1}}\left(\mathcal{J}_{C,\psi}(\mathrm{ind}_{L'}^{\mathcal{F}_n}(\beta_\chi) \otimes \omega_{\psi^{-1},\eta^{-1}})\right), \tag{A.4}$$

where ind means the (unnormalized) compactly induced representation, L' is the subgroup of $\mathcal{F}_n$ consisting of elements of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \star & \star & \star & 0 \\ & g_1 & \cdots & \star & \cdots & \cdots & \star & \star \\ & & \ddots & \vdots & & & \vdots & \vdots \\ & & & & & & \vdots & \star \\ & & & g_n & & & & \\ & & & & g_n^* & & \star & 0 \\ & & & & & \ddots & \vdots & \vdots \\ & & & & & & g_1^* & 0 \\ & & & & & & & 1 \end{pmatrix},$$

and β_χ is the character of L' which maps the above matrix to

$$\prod_{i=1}^{n-1} |\det g_i|^{i-n} \cdot \prod_{i=1}^{n} \left(|a_i^{-1}| \det g_i|\right)^{\frac{3}{2}(2n-2i+1)} \chi_i(a_i^{-1} \det g_i).$$

We note that the representation (A.4) has a unique unramified constituent and has finite length. Then we may define a linear functional

$$R : \mathrm{ind}_{L'}^{\mathcal{F}_n}(\beta_\chi) \otimes \omega_{\psi^{-1},\eta^{-1}} \to \mathbb{C}$$

by

$$R(f \otimes \phi) = \int_{F^{2n+1}} f\left(\begin{pmatrix} 1 & x & & z \\ & 1_{2n} & & \\ & & 1_{2n} & x' \\ & & & 1 \end{pmatrix}\right) \omega_{\psi^{-1},\eta^{-1}}(x, 0; z)\phi(0)\, dx\, dz.$$

It is easy to check that R factors through $\mathcal{J}_{C,\psi}(\mathrm{ind}_{L'}^{\mathcal{F}_n}(\beta_\chi) \otimes \omega_{\psi^{-1},\eta^{-1}})$ and is nontrivial on the unramified constituent. Further, from the definition and the formula in (3.5), the map R satisfies

$$R\left(\rho\begin{pmatrix}1 & & \star & \\ & g & \star & \star \\ & & g^* & \\ & & & 1\end{pmatrix} f\otimes\left(\omega_{\psi^{-1},\eta^{-1}}\begin{pmatrix}g & \star \\ & g^*\end{pmatrix}\phi\right)\right)$$
$$= \delta_{\hat{Q}_1}^{1/2}(\begin{pmatrix}g & \star \\ & g^*\end{pmatrix})\prod_{i=1}^{n}\eta^{-1}\chi_i(\det g_i)R(f,\phi).$$

Therefore, R defines a $G_{2n}(F)$-invariant map from (A.4) to (A.3), and our claim readily follows. □

Acknowledgments

This paper is part of a project to give a rigorous proof of Lapid–Mao's conjecture on an explicit formula for Whittaker-Fourier coefficients of automorphic forms in the case of unitary groups. I would like to express my deep gratitude to Erez Lapid for suggesting this project to me and for helpful conversations. I thank Atsushi Ichino and Shunsuke Yamana for helpful comments. Also, I thank David Soudry for sending me his paper with Ben-Artzi [BAS] before publishing. This paper was partly written while I stayed at the Weizmann Institute of Science. I would like to thank the people at WIS for their warm hospitality. Finally, I would like to thank the anonymous referee for pointing out several inaccuracies.

References

[AKT04] U. K. Anandavardhanan, Anthony C. Kable, and R. Tandon, *Distinguished representations and poles of twisted tensor L-functions*, Proc. Amer. Math. Soc. **132** (2004), no. 10, 2875–2883, DOI 10.1090/S0002-9939-04-07424-6. MR2063106

[AR05] U. K. Anandavardhanan and C. S. Rajan, *Distinguished representations, base change, and reducibility for unitary groups*, Int. Math. Res. Not. **14** (2005), 841–854, DOI 10.1155/IMRN.2005.841. MR2146859

[Ar13] James Arthur, *The endoscopic classification of representations*, Orthogonal and symplectic groups, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013. MR3135650

[Ba95] Ehud Moshe Baruch, *Local factors attached to representations of p-adic groups and strong multiplicity one*, ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–Yale University, 1995. MR2692992

[BAS] Asher Ben-Artzi and David Soudry, *On L-functions for* $\mathrm{U}_{2k}\times\mathrm{R}_{E/F}\mathrm{GL}_m$*,* $(k<m)$, Advances in the theory of automorphic forms and their L-functions, Contemp. Math., vol. 664, Amer. Math. Soc., Providence, RI, 2016, pp. 69–104, DOI 10.1090/conm/664/13039. MR3502977

[Ch16] J. Chai, *Bessel functions and local converse conjecture of Jacquet*, preprint, arXiv:1601.05450

[CKPSS01] J. W. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro, and F. Shahidi, *On lifting from classical groups to* GL_N, Publ. Math. Inst. Hautes Études Sci. **93** (2001), 5–30, DOI 10.1007/s10240-001-8187-z. MR1863734

[CKPSS04] J. W. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro, and F. Shahidi, *Functoriality for the classical groups*, Publ. Math. Inst. Hautes Études Sci. **99** (2004), 163–233, DOI 10.1007/s10240-004-0020-z. MR2075885

[CPSS11] J. W. Cogdell, I. I. Piatetski-Shapiro, and F. Shahidi, *Functoriality for the quasisplit classical groups*, On certain L-functions, Clay Math. Proc., vol. 13, Amer. Math. Soc., Providence, RI, 2011, pp. 117–140. MR2767514

[CM93] David H. Collingwood and William M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993. MR1251060

[Fl88] Yuval Z. Flicker, *Twisted tensors and Euler products* (English, with French summary), Bull. Soc. Math. France **116** (1988), no. 3, 295–313. MR984899

[Fl91] Yuval Z. Flicker, *On distinguished representations*, J. Reine Angew. Math. **418** (1991), 139–172, DOI 10.1515/crll.1991.418.139. MR1111204

[Fl93] Yuval Z. Flicker, *On zeroes of the twisted tensor L-function*, Math. Ann. **297** (1993), no. 2, 199–219, DOI 10.1007/BF01459497. MR1241802

[FZ95] Yuval Z. Flicker and Dmitrii Zinoviev, *On poles of twisted tensor L-functions*, Proc. Japan Acad. Ser. A Math. Sci. **71** (1995), no. 6, 114–116. MR1344660

[GI16] Wee Teck Gan and Atsushi Ichino, *The Gross-Prasad conjecture and local theta correspondence*, Invent. Math. **206** (2016), no. 3, 705–799, DOI 10.1007/s00222-016-0662-8. MR3573972

[GRS03] D. Ginzburg, S. Rallis, and D. Soudry, *On Fourier coefficients of automorphic forms of symplectic groups*, Manuscripta Math. **111** (2003), no. 1, 1–16, DOI 10.1007/s00229-003-0355-7. MR1981592

[GRS05] David Ginzburg, Stephen Rallis, and David Soudry, *Construction of CAP representations for symplectic groups using the descent method*, Automorphic representations, L-functions and applications: progress and prospects, Ohio State Univ. Math. Res. Inst. Publ., vol. 11, de Gruyter, Berlin, 2005, pp. 193–224, DOI 10.1515/9783110892703.193. MR2192824

[GRS11] David Ginzburg, Stephen Rallis, and David Soudry, *The descent map from automorphic representations of* $\mathrm{GL}(n)$ *to classical groups*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011. MR2848523

[GJS12] David Ginzburg, Dihua Jiang, and David Soudry, *On correspondences between certain automorphic forms on* $\mathrm{Sp}_{4n}(\mathbb{A})$ *and* $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, Israel J. Math. **192** (2012), no. 2, 951–1008, DOI 10.1007/s11856-012-0058-4. MR3009748

[HM02] Jeffrey Hakim and Fiona Murnaghan, *Globalization of distinguished supercuspidal representations of* $\mathrm{GL}(n)$, Canad. Math. Bull. **45** (2002), no. 2, 220–230, DOI 10.4153/CMB-2002-025-x. MR1904086

[He93] Guy Henniart, *Caractérisation de la correspondance de Langlands locale par les facteurs ϵ de paires* (French, with English and French summaries), Invent. Math. **113** (1993), no. 2, 339–350, DOI 10.1007/BF01244309. MR1228128

[ILM17] Atsushi Ichino, Erez Lapid, and Zhengyu Mao, *On the formal degrees of square-integrable representations of odd special orthogonal and metaplectic groups*, Duke Math. J. **166** (2017), no. 7, 1301–1348, DOI 10.1215/00127094-0000001X. MR3649356

[JPSS83] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), no. 2, 367–464, DOI 10.2307/2374264. MR701565

[JL16] H. Jacquet and B. Liu, *On the local converse theorem for p-adic* GL_n, preprint, arXiv:1601.03656

[JL15] Dihua Jiang and Baiying Liu, *On special unipotent orbits and Fourier coefficients for automorphic forms on symplectic groups*, J. Number Theory **146** (2015), 343–389, DOI 10.1016/j.jnt.2014.03.002. MR3267118

[JNS15] Dihua Jiang, Chufeng Nien, and Shaun Stevens, *Towards the Jacquet conjecture on the local converse problem for p-adic* GL_n, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 4, 991–1007, DOI 10.4171/JEMS/524. MR3349305

[JS03] Dihua Jiang and David Soudry, *The local converse theorem for* $\mathrm{SO}(2n+1)$ *and applications*, Ann. of Math. (2) **157** (2003), no. 3, 743–806, DOI 10.4007/annals.2003.157.743. MR1983781

[Ka04] Anthony C. Kable, *Asai L-functions and Jacquet's conjecture*, Amer. J. Math. **126** (2004), no. 4, 789–820. MR2075482

[KK04] Henry H. Kim and Muthukrishnan Krishnamurthy, *Base change lift for odd unitary groups*, Functional analysis VIII, Various Publ. Ser. (Aarhus), vol. 47, Aarhus Univ., Aarhus, 2004, pp. 116–125. MR2127169

[KK05] Henry H. Kim and Muthukrishnan Krishnamurthy, *Stable base change lift from unitary groups to* GL_n, IMRP Int. Math. Res. Pap. **1** (2005), 1–52. MR2149370

[Ku94] Stephen S. Kudla, *Splitting metaplectic covers of dual reductive pairs*, Israel J. Math. **87** (1994), no. 1-3, 361–401, DOI 10.1007/BF02773003. MR1286835

[La76] Robert P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, Vol. 544, Springer-Verlag, Berlin-New York, 1976. MR0579181

[LM16] Erez Lapid and Zhengyu Mao, *On Whittaker-Fourier coefficients of automorphic forms on unitary groups: a reduction to a local identity*, Advances in the theory of automorphic forms and their L-functions, Contemp. Math., vol. 664, Amer. Math. Soc., Providence, RI, 2016, pp. 295–320, DOI 10.1090/conm/664/13109. MR3502987

[Li11] Baiying Liu, *Genericity of representations of p-adic Sp_{2n} and local Langlands parameters*, Canad. J. Math. **63** (2011), no. 5, 1107–1136, DOI 10.4153/CJM-2011-017-2. MR2865733

[MVW87] Colette Mœglin, Marie-France Vignéras, and Jean-Loup Waldspurger, *Correspondances de Howe sur un corps p-adique* (French), Lecture Notes in Mathematics, vol. 1291, Springer-Verlag, Berlin, 1987. MR1041060

[MW95] C. Mœglin and J.-L. Waldspurger, *Spectral decomposition and Eisenstein series,* Une paraphrase de l'Écriture [A paraphrase of Scripture], Cambridge Tracts in Mathematics, vol. 113, Cambridge University Press, Cambridge, 1995. MR1361168

[Mo15] Dipendra Prasad and Rainer Schulze-Pillot, *Generalised form of a conjecture of Jacquet and a local consequence*, J. Reine Angew. Math. **616** (2008), 219–236, DOI 10.1515/CRELLE.2008.023. MR3338302

[Sh90a] Freydoon Shahidi, *On multiplicativity of local factors*, Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989), Israel Math. Conf. Proc., vol. 3, Weizmann, Jerusalem, 1990, pp. 279–289. MR1159120

[Sh90b] Freydoon Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. (2) **132** (1990), no. 2, 273–330, DOI 10.2307/1971524. MR1070599

[Sh10] Freydoon Shahidi, *Eisenstein series and automorphic L-functions*, American Mathematical Society Colloquium Publications, vol. 58, American Mathematical Society, Providence, RI, 2010. MR2683009

[ST15] David Soudry and Yaacov Tanay, *On local descent for unitary groups*, J. Number Theory **146** (2015), 557–626, DOI 10.1016/j.jnt.2014.03.003. MR3267124

[Zh17a] Q. Zhang, *A local converse theorem for* $\mathrm{U}(1,1)$. MR3681684

[Zh17b] Q. Zhang, *A local converse theorem for* $\mathrm{U}(2,2)$. MR3719312

Department of Mathematics, Kobe University, 1-1, Rokkodai, Nada-ku, Kobe, Japan
Email address: kazukimorimo@gmail.com