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LARGE DEVIATION PRINCIPLE FOR SOME BETA ENSEMBLES

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Abstract. Let L be a positive line bundle over a projective complex manifold X, L^p its tensor power of order p, $H^0(X, L^p)$ the space of holomorphic sections of L^p, and N_p the complex dimension of $H^0(X, L^p)$. The determinant of a basis of $H^0(X, L^p)$, together with some given probability measure on a weighted compact set in X, induces naturally a β-ensemble, i.e., a random N_p-point process on the compact set. Physically, depending on X and the value of β, this general setting corresponds to a gas of free or interacting fermions on X and may admit an interpretation in terms of some random matrix models. The empirical measures, associated with such β-ensembles, converge almost surely to an equilibrium measure when p goes to infinity. We establish a large deviation theorem (LDT) with an effective speed of convergence for these empirical measures. Our study covers a large class of β-ensembles on a compact subset of the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ or of the Euclidean space $\mathbb{R}^n$.

1. Introduction

Let K be a metric space and let N be a positive integer. If $\boldsymbol{x} = (x_1, \ldots, x_N)$ is a point in the N-fold product K^N, then the associated *empirical measure* is the probability measure

$$\mu^{\boldsymbol{x}} := \frac{1}{N} \sum_{k=1}^{N} \delta_{x_k},$$

which is equidistributed on $x_1, \ldots, x_N$. Here, δ_x denotes the Dirac mass at x. Any probability measure ν on K^N induces a random N-point process on K, and ν is the law of this random process.

Let $\{N_p\}_{p\geq 1}$ be a sequence of positive integers such that $N_p \to \infty$ as $p \to \infty$ and let $\{\nu_p\}_{p\geq 1}$ be a sequence of probability measures on K^{N_p}. In many problems from mathematics or mathematical physics, a central question is to study the eventual convergence of the sequence $\mu^{\boldsymbol{x}^{(p)}}$ to an equilibrium measure, where $\boldsymbol{x}^{(p)}$ is the random N_p-point process on K described by the law ν_p. A significantly interesting setting considered in literature is the case of β-ensembles on a compact subset of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ or a compact subset of $\mathbb{R}^n$. We will obtain in this paper a large deviation theorem for such β-ensembles with an explicit rate of convergence. Our approach uses however techniques from complex analysis, and therefore we will first describe the general setting, which, physically, corresponds to a gas of fermions

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and may admit an interpretation in terms of some random matrix models (when $X = \mathbb{P}^1$). The reader will find in the paper of Berman [1] a detailed exposition and a list of references; see also [12] for some random matrix models. The case of β-ensembles on the unit sphere or on the real Euclidean space, mentioned above, will be obtained as a corollary; see Examples 1.5 and 1.6 below.

Let X be a compact Kähler manifold of dimension n. Let $L \to X$ be a positive line bundle endowed with a given smooth Hermitian metric h_0. We assume that the metric h_0 is positively curved; that is, the Chern form ω_0 associated with h_0 is a Kähler form on X. For simplicity, we will use the Riemannian metric on X induced by ω_0. The space of holomorphic sections of $L^p := L \otimes \cdots \otimes L$ (p times) is denoted by $H^0(X, L^p)$. Since L is ample, by Kodaira-Serre vanishing and Riemann-Roch-Hirzebruch theorems (see [21, Thms. 1.5.6 and 1.4.6]), we have

$$N_p := \dim H^0(X, L^p) = \frac{p^n}{n!}\|\omega_0^n\| + O(p^{n-1}). \tag{1.1}$$

Here, $\|\omega_0^n\|$ denotes the mass of the volume form ω_0^n. It depends only on the Chern class of L.

If L_1, L_2 are line bundles over complex manifolds X_1 and X_2 respectively, we denote by $L_1 \boxtimes L_2$ the line bundle over the product manifold $X_1 \times X_2$ defined as $L_1 \boxtimes L_2 := \pi_1^*(L_1) \otimes \pi_2^*(L_2)$, where π_1, π_2 are the natural projections from $X_1 \times X_2$ to its factors. If L_1 and L_2 are endowed with some Hermitian metrics, then $L_1 \boxtimes L_2$ carries also a metric induced by those on L_1 and L_2.

Let $S_p = (s_1, \ldots, s_{N_p})$ be a basis of $H^0(X, L^p)$. We define the section $\det S_p$ of the line bundle $(L^p)^{\boxtimes N_p} := L^p \boxtimes \cdots \boxtimes L^p$ (N_p times) over X^{N_p} by the identity

$$\det S_p(\boldsymbol{x}^{(p)}) := \sum_{\sigma \in \mathrm{Sym}_{N_p}} \mathrm{sgn}(\sigma) \bigotimes_{i=1}^{N_p} s_i(x_{\sigma(i)}) \quad \text{for} \quad \boldsymbol{x}^{(p)} = (x_1, \ldots, x_{N_p}) \in X^{N_p},$$

where Sym_{N_p} denotes the permutation group of $\{1, \ldots, N_p\}$. Note that when we change the basis S_p, this section changes only by a non-zero multiplicative constant.

Let K be a compact set in X and let ϕ be a continuous real-valued function on K. We say that the pair (K, ϕ) is a *weighted compact set.* Let μ be a probability measure on K.

Definition 1.1. Let $\beta > 0$ be a constant. *The β-ensemble* associated with the line bundle L^p, the weighted compact set (K, ϕ), and the probability measure μ is the random N_p-point process on K whose joint distribution is given by

$$\nu_p^\beta := c_{p,\beta}\|\det S_p(\boldsymbol{x}^{(p)})\|^\beta e^{-\beta p(\phi(x_1)+\cdots+\phi(x_{N_p}))} d\mu(x_1) \otimes \cdots \otimes d\mu(x_{N_p}), \tag{1.2}$$

where $c_{p,\beta}$ is the normalizing constant so that ν_p^β is a probability measure on K^{N_p}.

Observe that the constant $c_{p,\beta}$ depends also on L^p, K, ϕ, μ, but the above random point process, i.e., the measure ν_p^β, is independent of the choice of the basis S_p of $H^0(X, L^p)$. Understanding the behavior of $c_{p,\beta}$ and ν_p^β when p tends to infinity is a challenging problem which may have many applications in differential geometry and mathematical physics; see [13, 15] for references and some recent results.

In this paper, we will study the above β-ensembles when p goes to infinity. We need some assumptions on the regularity of K, ϕ, and μ. Under such conditions, we will see later that the sequence $\mu^{\boldsymbol{x}^{(p)}}$ converges almost surely to a limit $\mu_{\mathrm{eq}}(K, \phi)$, which is called the equilibrium measure of the weighted compact set (K, ϕ).

Recently, Berman [1] obtained a large deviation principle in the spirit of Donsker and Varadhan [8] using some functionals on the space of measures. In the case where $K = X$, $\phi = 0$, and μ is the Lebesgue measure on X, Carroll, Marzo, Massaneda, and Ortega-Cerdà obtained precise and optimal estimates on the expectation of the Kantorovich-Wasserstein distance between $\mu^{\boldsymbol{x}^{(p)}}$ and $\mu_{\mathrm{eq}}(K,\phi)$ when $p \to \infty$ [6]. An advantage of the latter work is that Kantorovich-Wasserstein distance gives us very explicit information about the convergence of $\mu^{\boldsymbol{x}^{(p)}}$ to $\mu_{\mathrm{eq}}(K,\phi)$. Our aim is to establish a large deviation theorem (LDT) with precise estimations in a quite general setting and in the sprit of the work by Carroll, Marzo, Massaneda, and Ortega-Cerdà. In order to state the main result, we need to introduce some more notions.

Let $\mathscr{M}(X)$ denote the space of all (Borel) probability measures on X. For $\gamma > 0$, define the distance dist_γ between two measures μ and μ' in $\mathscr{M}(X)$ by

$$\mathrm{dist}_\gamma(\mu,\mu') := \sup_{\|v\|_{\mathscr{C}^\gamma} \le 1} \big|\langle \mu - \mu', v\rangle\big|,$$

where v is a test smooth real-valued function. This distance induces the weak topology on $\mathscr{M}(X)$. By interpolation between Banach spaces (see [10, 25]), for $0 < \gamma \le \gamma'$, there exists a constant $c > 0$ such that

$$\mathrm{dist}_{\gamma'} \le \mathrm{dist}_\gamma \le c[\mathrm{dist}_{\gamma'}]^{\gamma/\gamma'}. \tag{1.3}$$

Note that dist_1 is equivalent to the classical Kantorovich-Wasserstein distance.

In Section 2 below, we will single out a very large class of compact sets K which enjoy the so-called $(\mathscr{C}^\alpha, \mathscr{C}^{\alpha'})$-regularity. We will also introduce the notion of δ-Bernstein-Markov measures which enjoy a quantified version of the Bernstein-Markov property. Here, δ is a constant such that $0 < \delta < 1$. Having in hand these natural notions, we are in the position to state the main result of the paper.

Theorem 1.2. *Let X be a complex projective manifold of dimension n. Let L be a positive line bundle over X endowed with a smooth positively curved Hermitian metric h_0. Let $\beta > 0$ and $0 < \gamma \le 2$ be constants. Let K be a $(\mathscr{C}^\alpha, \mathscr{C}^{\alpha'})$-regular compact subset of X and let ϕ be a $\mathscr{C}^\alpha$ real-valued function on K for some constants $0 < \alpha \le 2$ and $0 < \alpha' \le 1$. Let μ be a probability measure on K which is δ-Bernstein-Markov with respect to (K,ϕ) for some $0 < \delta < 1$. Then, for every $\lambda > 0$, there are $c > 0$ and Borel sets $E_p \subset K^{N_p}$ such that*

(a) $\nu_p^\beta(E_p) \le e^{-\lambda p^{n+1-\delta}}$;

(b) *if $\mu_p^{\boldsymbol{x}}$ denotes the empirical measure associated with $\boldsymbol{x} \in K^{N_p} \setminus E_p$, then*

$$\mathrm{dist}_\gamma(\mu^{\boldsymbol{x}}, \mu_{\mathrm{eq}}(K,\phi)) \le c q^\gamma.$$

Here, $q := p^{-\delta/4}$ if $\delta/4 < \alpha''$, $q := p^{-\alpha''}(\log p)^{3\alpha''}$ if $\delta/4 \ge \alpha''$, and $\alpha'' := \alpha'/(24 + 12\alpha')$.

If a sequence of points $\boldsymbol{x}^{(p)} \in K^{N_p}$ satisfies $\boldsymbol{x}^{(p)} \notin E_p$ for p large enough, then we deduce from the last theorem that $\mu^{\boldsymbol{x}^{(p)}} \to \mu_{\mathrm{eq}}(K,\phi)$ when p goes to infinity. Therefore, $\mu^{\boldsymbol{x}^{(p)}}$ converge almost surely to $\mu_{\mathrm{eq}}(K,\phi)$ when p goes to infinity. More precisely, the infinite product $\nu^\beta := \nu_1^\beta \times \nu_2^\beta \times \cdots$ is a probability measure on the space of all sequences $(\boldsymbol{x}^{(p)})_{p=1}^\infty$. With respect to this measure, the convergence $\mu^{\boldsymbol{x}^{(p)}} \to \mu_{\mathrm{eq}}(K,\phi)$ holds for almost every sequence $(\boldsymbol{x}^{(p)})_{p=1}^\infty$.

The estimate on the size of E_p is a version of LDT. Our result also implies that

$$\int_{X^{N_p}} \mathrm{dist}_\gamma\big(\mu^{\boldsymbol{x}}, \mu_{\mathrm{eq}}(K,\phi)\big) d\nu_p^\beta(\boldsymbol{x}) = O(q^\gamma). \tag{1.4}$$

This distance expectation estimate is similar to the one obtained by Carroll, Marzo, Massaneda, and Ortega-Cerdà in [6] that we mentioned above. These authors proved for $K = X$, $\phi = 0$, and μ the normalized Lebesgue measure on X that there is a constant $c > 0$ satisfying

$$c^{-1}p^{1/2} \le \int_{X^{N_p}} \mathrm{dist}_1\big(\mu^{\boldsymbol{x}}, \mu_{\mathrm{eq}}(K,\phi)\big) d\nu_p^\beta(\boldsymbol{x}) \le cp^{1/2} \tag{1.5}$$

for all p.

In order to get more concrete applications of our main result, we need the following natural class of positive Borel measures.

Definition 1.3. We say that a positive measure μ on X satisfies *the mass-density condition* with respect to a compact $W \subset X$ if there are two constants $c > 0$ and $\rho > 0$ such that

$$\mu(B(x,r)) \ge cr^\rho \qquad \text{for} \quad x \in W \quad \text{and} \quad 0 < r < 1.$$

Here, $B(x,r)$ denotes the ball in (X, ω_0) of radius r and centered at the point x.

Assume now that K is a smooth real manifold in X with piecewise smooth boundary such that the tangent space of K at each point is not contained in a complex hyperplane of the tangent space of X at that point. It was shown in [9, 24], for $0 < \alpha < 1$, that K is $(\mathscr{C}^\alpha, \mathscr{C}^{\alpha/2})$-regular and is $(\mathscr{C}^\alpha, \mathscr{C}^\alpha)$-regular when its boundary is smooth; see Theorem 2.3 below. In this case, if μ is a probability measure on K satisfying the above mass-density condition for $W = K$, we will show in Corollary 2.13 below that it satisfies the δ-Bernstein-Markov property required in Theorem 1.2. Therefore, the following result is a direct consequence of that theorem.

Corollary 1.4. *Let X, L, h_0, β, γ be as in Theorem 1.2. Let K be a smooth real manifold in X with piecewise smooth boundary such that the tangent space of K at each point is not contained in a complex hyperplane of the tangent space of X at that point. Let μ be a probability measure on K satisfying the mass-density condition with respect to K. Let ϕ be a $\mathscr{C}^\alpha$ real-valued function on K with $0 < \alpha < 1$. Then, for every $0 < \delta < 1$, the conclusion of Theorem 1.2 holds for $\alpha'' := \alpha/(48 + 24\alpha)$. Moreover, if the boundary of K is smooth, then the same statement holds for $\alpha'' := \alpha/(24 + 12\alpha)$.*

Of course, Corollary 1.4 holds when μ is given by the normalized volume form on K. It is worth noting that the assumption on the mass-density condition of the measure μ in this result can be weakened. In fact, we only need that μ satisfies the mass-density condition on a subset $W \subset K$ which satisfies a maximum principle; see Corollary 2.13 below. Finally, note that our approach also allows us to treat the case where β depends on p, but for simplicity, we will not consider this case here.

Example 1.5. Let K be the closure of an open set with piecewise smooth boundary in $\mathbb{R}^n$. Let ϕ be a $\mathscr{C}^\alpha$ real-valued function on K and let μ be a probability measure on K which satisfies the mass-density condition with respect to K. It is already interesting to consider the case where μ is the normalization of the restriction to K of the Lebesgue measure on $\mathbb{R}^n$. Denote by $\mathscr{P}_p$ the set of real polynomials of degree at most p and by N_p the dimension of $\mathscr{P}_p$. Choose a basis $(P_1,\dots,P_{N_p})$ of $\mathscr{P}_p$. Define the probability measure ν_p^β at a point $\boldsymbol{x}=(x_1,\dots,x_{N_p})$ on K^{N_p} by

$$c_{p,\beta}|\det(P_i(x_j))|^\beta e^{-\beta p(\phi(x_1)+\cdots+\phi(x_{N_p}))}e^{-\frac{1}{2}\beta p(\log(1+\|x_1\|^2)+\cdots+\log(1+\|x_{N_p}\|^2))}$$
$$\times\,\mu(x_1)\otimes\cdots\otimes\mu(x_{N_p}),$$

where $c_{p,\beta}$ is a normalizing constant so that ν_p^β is a probability measure. Here, $\det(\cdot)$ denotes the standard determinant of a square matrix. Then the conclusion of Theorem 1.2 holds for $\alpha'':=\alpha/(48+24\alpha)$. If the boundary of K is smooth, we can take $\alpha'':=\alpha/(24+12\alpha)$. The equilibrium measure $\mu_{\mathrm{eq}}(K,\phi)$ is a probability measure supported by K. Its definition is given in Section 2.

In order to obtain this result as a consequence of Theorem 1.2 and Corollary 1.4, consider $\mathbb{R}^n$ as the real part of $\mathbb{C}^n$ and $\mathbb{C}^n$ as a Zariski open set of the projective space $\mathbb{P}^n$. Denote by $[z_0:\cdots:z_n]$ the homogeneous coordinates of $\mathbb{P}^n$. We identify $\mathbb{C}^n$ with the open set $\{z_0=1\}$. Define $X:=\mathbb{P}^n$. We can identify, in the natural way, the polynomials of degree $\leq p$ on $\mathbb{R}^n$ with holomorphic sections of L^p with $L=\mathscr{O}(1)$ the tautological line bundle of $X=\mathbb{P}^n$. We consider the standard Hermitian metrics on these line bundles. So $\{P_1,\dots,P_{N_p}\}$ is identified to a basis of $H^0(X,L^p)$. If a section s in $H^0(X,L^p)$ is identified to a polynomial P, then

$$\|s(z)\|=|P(z)|e^{-\frac{1}{2}p\log(1+\|z\|^2)}\quad\text{for}\quad z\in\mathbb{C}^n.$$

So the factor involving $\log(1+\|x_i\|^2)$ in the definition of ν_p^β is due to the standard Hermitian metric of L^p. We can now apply Theorem 1.2 and Corollary 1.4 and get the LDT in this case. An interesting particular situation is the case where the weight ϕ is equal to $-\frac{1}{2}\log(1+\|\cdot\|^2)$.

Example 1.6. Let K be the closure of an open set with piecewise smooth boundary in the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$. Let ϕ be a $\mathscr{C}^\alpha$ real-valued function on K and let μ be a probability measure on K which satisfies the mass-density condition with respect to K. It is already interesting to consider the case where $\phi=0$ and μ is the normalization of the restriction to K of the Haar measure on $\mathbb{S}^n$. Consider the functions which are restrictions of (real) polynomials on $\mathbb{R}^{n+1}$ to $\mathbb{S}^n$. Denote by $\mathscr{P}_p$ the set of these functions obtained by using polynomials of degree at most p and by N_p the dimension of $\mathscr{P}_p$. Note that $\mathscr{P}_p$ is isomorphic to the quotient of the space of polynomials of degree $\leq p$ by the subspace of polynomials divisible by $x_1^2+\cdots+x_{n+1}^2-1$, where $(x_1,\dots,x_{n+1})$ is the standard coordinate system of $\mathbb{R}^{n+1}$.

Choose a basis $(P_1,\dots,P_{N_p})$ of $\mathscr{P}_p$. Define the probability measure ν_p^β on K^{N_p} by

$$\nu_p^\beta(\boldsymbol{x}):=c_{p,\beta}|\det(P_i(x_j))|^\beta e^{-\beta p(\phi(x_1)+\cdots+\phi(x_{N_p}))}\mu(x_1)\otimes\cdots\otimes\mu(x_{N_p}),$$

where $\boldsymbol{x}=(x_1,\dots,x_{N_p})$ is a point in K^{N_p} and $c_{p,\beta}$ is a normalizing constant so that ν_p^β is a probability measure. Then the conclusion of Theorem 1.2 holds for $\alpha'':=\alpha/(48+24\alpha)$. If the boundary of K is smooth, we can take $\alpha'':=\alpha/(24+12\alpha)$. The measure $\mu_{\mathrm{eq}}(K,\phi)$ is supported by K. In the case where $K=\mathbb{S}^n$ and $\phi=0$, by symmetry, this measure coincides with the Haar measure on $\mathbb{S}^n$.

In order to obtain this result as a consequence of Theorem 1.2 and Corollary 1.4, we need to complexify $\mathbb{S}^n$. Consider $\mathbb{R}^{n+1}$ as the real part of $\mathbb{C}^{n+1}$ and $\mathbb{C}^{n+1}$

as a Zariski open set of the projective space $\mathbb{P}^{n+1}$. Denote by $[z_0 : \cdots : z_{n+1}]$ the homogeneous coordinates of $\mathbb{P}^{n+1}$. We identify $\mathbb{C}^{n+1}$ with the open set $\{z_0 = 1\}$. The sphere $\mathbb{S}^n$ is then the intersection of $\mathbb{R}^{n+1}$ with the complex hypersurface $z_1^2 + \cdots + z_{n+1}^2 = z_0^2$ in $\mathbb{P}^{n+1}$. Denote by X this hypersurface. We can identify, in the natural way, the polynomials of degree $\leq p$ on $\mathbb{R}^{n+1}$ with holomorphic sections of L^p with $L = \mathscr{O}(1)$ the tautological line bundle of $\mathbb{P}^{n+1}$. As in Example 1.5, we consider the standard Hermitian metrics on these line bundles. Note that $|z_1|^2 + \cdots + |z_{n+1}|^2$ is constant on $\mathbb{S}^n$, and therefore the formula for ν_p^β is simpler than the one in Example 1.5. Observe also that a section of L^p vanishes on X if and only if it vanishes on $\mathbb{S}^n$. Therefore, $\{P_1, \ldots, P_{N_p}\}$ is identified to a basis of $H^0(X, L^p)$. We can now apply Theorem 1.2 and Corollary 1.4.

The plan of the paper is as follows. In Section 2, we discuss different notions of regularity for the weighted compact set (K, ϕ) and the measure μ. We also give criteria to check the regularity conditions used in our study. In Section 3, we prove the main theorem (Theorem 1.2), which uses an equidistribution result for almost Fekete configurations. The last result has been obtained in collaboration with Ma in the last version of [9, Remark 3.17]. For the reader's convenience, we provide here a detailed proof that we need in this paper. Note that the case of Fekete points can be seen as the limit case of β-ensembles when $\beta \to \infty$. We refer to [3, 9, 16–18, 24], the references therein, and also the end of this paper for more results on Fekete points and other configurations.

2. Pluri-regularity for weighted compact sets and measures

As we have seen in the Introduction, our study requires some regularity properties of the weighted compact set (K, ϕ) and the probability measure μ on K. In this section, we will recall some known facts and also introduce and study new notions that will be used in the proof of our main theorem. The reader will find in [7, 10, 14, 21] basic notions and results from complex geometry and pluripotential theory.

Let L be a positive (i.e., ample) holomorphic line bundle over a projective manifold X of dimension n. Fix a smooth Hermitian metric h_0 on L such that its first Chern form ω_0 is a Kähler form on X. Define $\mu^0 := \|\omega_0^n\|^{-1}\omega_0^n$, the probability measure associated with the volume form ω_0^n. Here, $\|\omega_0^n\|$ is the total mass of ω_0^n which is the integral of this volume form on X. Recall that a real-valued function on X is *quasi-p.s.h.* if it is locally the difference between a p.s.h. function and a smooth one. Let $\mathrm{PSH}(X, \omega_0)$ be the cone of ω_0*-p.s.h. functions*, i.e., the quasi-p.s.h. functions φ such that $dd^c\varphi + \omega_0 \geq 0$.

Definition 2.1. We call a *weighted compact subset of* X a data (K, ϕ), where K is a non-pluripolar compact subset of X and ϕ is a real-valued continuous function on K. The function ϕ is called *a weight* on K. The *equilibrium weight* associated with (K, ϕ) is the upper semi-continuous regularization ϕ_K^* of the function

$$\phi_K(z) := \sup\big\{\psi(z) : \psi \ \ \omega_0\text{-p.s.h. such that } \ \psi \leq \phi \text{ on } K\big\}.$$

We also call the *equilibrium measure of* (K, ϕ) the normalized Monge-Ampère measure

$$\mu_{\mathrm{eq}}(K, \phi) := \|\omega_0^n\|^{-1}(dd^c\phi_K^* + \omega_0)^n.$$

Note that the equilibrium measure $\mu_{\rm eq}(K,\phi)$ is a probability measure supported by K and $\phi_K^*=\phi_K$ almost everywhere with respect to this measure; see, e.g., [2]. The following notions are important in our study; see [9].

Definition 2.2. Denote by P_K the projection onto $\mathrm{PSH}(X,\omega_0)$ which associates ϕ with ϕ_K^*. We say that (K,ϕ) is *regular* if ϕ_K is upper semi-continuous, i.e., $P_K\phi=\phi_K$. Let $(E,\|\ \|_E)$ be a normed vector space of continuous functions on K and let $(F,\|\ \|_F)$ be a normed vector space of functions on X. We say that K is (E,F)*-regular* if (K,ϕ) is regular for $\phi\in E$ and if the projection P_K sends bounded subsets of E into bounded subsets of F.

We have the following result.

Theorem 2.3 ([9,24])**.** *Let X and L be as above. Let K be a smooth compact real manifold in X with piecewise smooth boundary. Assume that the tangent space to K at any point is not contained in a complex hyperplane of the tangent space to X at that point. Let $0<\alpha<1$ be any real number. Then K is $(\mathscr{C}^\alpha,\mathscr{C}^{\alpha/2})$-regular. Moreover, it is $(\mathscr{C}^\alpha,\mathscr{C}^\alpha)$-regular if the boundary of K is smooth.*

Consider now a real-valued function ψ on X. We can associate the line bundle L with a singular Hermitian metric $h:=e^{-2\psi}h_0$. More precisely, if v is a vector in the fiber of L over a point $x\in X$, its norms with respect to the metrics h and h_0 are related by the formula

$$|v|_h=e^{-\psi(x)}|v|_{h_0}.$$

The metrics h_0 and h induce in a canonical way metrics $h_0^{\otimes p}$ and $h^{\otimes p}$ on the power L^p of L. They are related by the formula $h^{\otimes p}=e^{-2p\psi}h_0^{\otimes p}$. Recall that for simplicity, we will use the notation $|\cdot|_{p\psi}$ instead of $|\cdot|_{h^{\otimes p}}$ for the norm of a vector in L^p with respect to the metric $h^{\otimes p}$. We also drop the subscript h_0; e.g., $|v|$ means $|v|_{h_0}$.

Consider now a weighted compact set (K,ϕ) in X. We can, in a similar way, define the metric $h=e^{-2\phi}h_0$ on L over K. Let μ be a probability measure with support in K. Consider the natural L^∞ and L^2 semi-norms on $H^0(X,L^p)$ induced by the metric h on L and the measure μ, which are defined for $s\in H^0(X,L^p)$ by

$$\|s\|_{L^\infty(K,p\phi)}:=\sup_K|s|_{p\phi}\qquad\text{and}\qquad \|s\|^2_{L^2(\mu,p\phi)}:=\int_X|s|^2_{p\phi}d\mu. \tag{2.1}$$

We will only use measures μ such that the above semi-norms are norms; i.e., there is no section $s\in H^0(X,L^p)\setminus\{0\}$ which vanishes on K or on the support of μ. The first semi-norm is a norm when K is not contained in a hypersurface of X. The second one is a norm when the support of μ is not contained in a hypersurface of X (recall that we only work with ϕ continuous). In particular, this is the case when μ is the normalized Monge-Ampère measure with continuous potentials because such a measure has no mass on hypersurfaces of X.

We need the following quantified Bernstein-Markov property; see also [2,3,19,22].

Definition 2.4. Let δ be a real number with $0<\delta<1$ and let (K,ϕ) be a weighted compact subset of X. We say that a positive measure μ on K is δ*-Bernstein-Markov* with respect to (K,ϕ) if there is a constant $A>0$ such that

$$\|s\|_{L^\infty(K,p\phi)}\le Ae^{Ap^{1-\delta}}\|s\|_{L^2(\mu,p\phi)}\quad\text{for}\quad s\in H^0(X,L^p)\quad\text{and}\quad p\ge1. \tag{2.2}$$

If μ is δ-Bernstein-Markov with respect to (K,ϕ) for all $0<\delta<1$, then we say that μ is 1-*Bernstein-Markov* with respect to (K,ϕ).

The following lemma shows that we can use the notion for other norms L^r.

Lemma 2.5. *Let δ, r be real numbers with $0 < \delta < 1$ and $r > 0$. Let (K,ϕ) be a weighted compact subset of X and μ a positive measure on K. Then μ is δ-Bernstein-Markov with respect to (K,ϕ) if and only if there is a constant $A' > 0$ such that*

$$\|s\|_{L^\infty(K,p\phi)} \leq A' e^{A' p^{1-\delta}} \|s\|_{L^r(\mu,p\phi)} \quad \textit{for} \quad s \in H^0(X,L^p) \quad \textit{and} \quad p \geq 1.$$

Proof. Assume that μ is δ-Bernstein-Markov with respect to (K,ϕ). We will only show the existence of A' as in the lemma because the converse property can be obtained in the same way. So we have property (2.2). Without loss of generality, we can assume that μ is a probability measure. If $r \geq 2$, then the L^r-norm is larger than or equal to the L^2-norm. Therefore, we can just take $A' := A$.

Assume now that $0 < r < 2$. By Hölder's inequality, we have

$$\|s\|_{L^2(\mu,p\phi)} \leq \|s\|^{r/2}_{L^r(\mu,p\phi)} \|s\|^{1-r/2}_{L^\infty(K,p\phi)}.$$

This, together with (2.2), gives us the desired property for a suitable value of A'. □

In order to get a simple criterium for a measure to have the δ-Bernstein-Markov property, we need the following notion.

Definition 2.6. A compact set W is said to satisfy *the maximum principle* relative to a weighted compact set (K,ϕ) if $W \subset K$ and

$$\sup_K(\psi - \phi) = \sup_W(\psi - \phi) \quad \text{for every} \quad \psi \in \mathrm{PSH}(X,\omega_0).$$

Clearly, $W = K$ satisfies the maximum principle relative to (K,ϕ). In general, W may be much smaller than K; see Remark 2.9 below.

Proposition 2.7. *Let (K,ϕ) be a weighted compact set and let W be a compact subset of K. Define*

$$\partial^\phi_{\omega_0} K := \{z \in K : P_K\phi(z) = \phi(z)\}.$$

Then W satisfies the maximum principle relative to (K,ϕ) if and only if $W \cap \partial^\phi_{\omega_0} K$ satisfies the same property. In particular, $\partial^\phi_{\omega_0} K$ satisfies the maximum principle relative to (K,ϕ).

Proof. Observe that the second assertion is a consequence of the first one and Definition 2.6 by taking $W = K$. We prove now the first assertion. If $W \cap \partial^\phi_{\omega_0} K$ satisfies the maximum principle relative to (K,ϕ), then clearly W satisfies the same property. Assume that W satisfies this maximum principle. It remains to prove the same property for $W \cap \partial^\phi_{\omega_0} K$.

Recall that $P_K\phi$ is upper semi-continuous and ϕ is continuous. Since $P_K\phi \leq \phi$, we deduce that

$$\partial^\phi_{\omega_0} K = \{z \in K : P_K\phi(z) \geq \phi(z)\}.$$

So it is a compact set.

Let $\psi \in \mathrm{PSH}(X,\omega_0)$ and set $m := \max_K(\psi - \phi)$. Note that ψ is also upper semi-continuous. Since W satisfies the maximum principle relative to (K,ϕ), there is a point $z_0 \in W$ such that $\psi - \phi$ attains its maximum value at z_0. We have $\psi(z_0) - m = \phi(z_0)$ and $\psi - m \leq \phi$ on K. Since $\psi - m \in \mathrm{PSH}(X,\omega_0)$, the last inequality implies that $\psi - m \leq P_K\phi$. In particular,

$$\psi(z_0) - m \leq (P_K\phi)(z_0).$$

This, combined with the equality $\psi(z_0)-m = \phi(z_0)$ and the inequality $(P_K\phi)(z_0) \leq \phi(z_0)$, implies that $(P_K\phi)(z_0) = \phi(z_0)$. Hence, $z_0 \in \partial^\phi_{\omega_0} K$, and the proposition follows. □

Remark 2.8. By [2, Prop. 2.10, Cor. 2.5], the equilibrium measure $\mu_{\rm eq}(K,\phi)$ is supported by $\partial^\phi_{\omega_0} K$, and its support also satisfies the maximum principle.

Remark 2.9. Let X be the projective space $\mathbb{P}^n$, seen as the natural compactification of $\mathbb{C}^n$. Let L be the tautological line bundle $\mathcal{O}(1)$ over $\mathbb{P}^n$. Then the holomorphic sections of $L^p = \mathcal{O}(p)$ can be identified to the complex polynomials of degree $\leq p$ on $\mathbb{C}^n$. With the standard Fubini-Study metric on $O(p)$, if a section s of L^p corresponds to a polynomial $P(z)$ of degree $\leq p$, then

$$|s(z)| = |P(z)|(1+\|z\|^2)^{-p/2}.$$

Consider a compact subset K of $\mathbb{C}^n$ and take $\phi := -\frac{1}{2}\log(1+\|z\|^2)$ on K. It is not difficult to check that the boundary of K satisfies the maximum principle relative to (K,ϕ).

Theorem 2.10. *Let X, L, h_0 be as above, let (K,ϕ) be a weighted compact subset of X, and let μ be a probability measure on K. Let $W \subset K$ be a compact set and let $0<\delta<1$ be a real number. Assume in addition the following conditions:*

(i) *the functions ϕ and $P_K\phi$ are Hölder continuous;*
(ii) *W satisfies the maximum principle relative to (K,ϕ);*
(iii) *μ satisfies the mass-density condition with respect to W; see Definition* 1.3.

Then μ is a δ-Bernstein-Markov measure with respect to (K,ϕ).

Remark 2.11. We will see in the proof of this theorem that the condition (i) can be replaced by the following much weaker condition: there are constant $c>0$ such that for $z \in X$ and $w \in K$,

$$|(P_K\phi)(z) - (P_K\phi)(w)| \leq c(1+\log^- \mathrm{dist}(z,w))^{-\frac{\delta}{1-\delta}},$$

and for $z, w \in K$,

$$|\phi(z)-\phi(w)| \leq c(1+\log^- \mathrm{dist}(z,w))^{-\frac{1}{1-\delta}},$$

where $\log^- := \max(0,-\log)$.

We are inspired by an idea of Bloom [5, Thm. 4.1]. Define $\epsilon := p^{-\delta}$ and $r := e^{-c'p^{1-\delta}}$ where $c'>0$ is a large enough constant independent of p. It follows from assumption (i) (see also Remark 2.11) that

$$\big|(P_K\phi)(z) - (P_K\phi)(z_0)\big| \leq \epsilon \quad \text{for} \quad z \in B(z_0, 2r) \quad \text{and} \quad z_0 \in K. \tag{2.3}$$

Fix $p \geq 1$ and $s \in H^0(X, L^p) \setminus \{0\}$. We need to prove inequality (2.2) for some constant $A>0$ independent of p and s. Observe that

$$dd^c \frac{1}{p}\log|s| = \frac{1}{p}[s=0] - \omega_0 \geq -\omega_0,$$

where $[s=0]$ is the current of integration on the hypersurface $\{s=0\}$. So $\frac{1}{p}\log|s|$ is ω_0-p.s.h. This, together with assumption (ii), implies the existence of a point $z_0 \in W$ such that

$$|s(z_0)|_{p\phi} = \max_{z\in K} |s(z)|_{p\phi}. \tag{2.4}$$

Lemma 2.12. *We have*

$$\big||s(z)| - |s(z_0)|\big| \leq \frac{1}{4}|s(z_0)| \quad \textit{for} \quad z \in B(z_0, r^2).$$

Proof. Consider a section $s' = cs$ where the constant c is chosen so that $\|s'\|_{L^\infty(K,p\phi)} = 1$. The last property implies the inequality $\frac{1}{p}\log|s'| \leq \phi$ on K. We have seen that $\frac{1}{p}\log|s|$ is ω_0-p.s.h. So s' satisfies a similar property. Hence, $\frac{1}{p}\log|s'| \leq P_K\phi$ on X. We then deduce the following Bernstein-Walsh type inequality:

$$|s(z)| \leq \|s\|_{L^\infty(K,p\phi)} e^{p(P_K\phi)(z)} \quad \text{for} \quad z \in X.$$

Using (2.4), we rewrite the last inequality for $z \in B(z_0, 2r)$ as

$$|s(z)| \leq |s(z_0)|_{p\phi} e^{p(P_K\phi)(z)} = |s(z_0)| e^{-p\big(\phi(z_0)-(P_K\phi)(z_0)\big)} e^{p\big((P_K\phi)(z)-(P_K\phi)(z_0)\big)}.$$

Using $\phi(z_0) \geq (P_K\phi)(z_0)$ and (2.3), we obtain

$$|s(z)| \leq |s(z_0)| e^{p\epsilon} \quad \text{for} \quad z \in B(z_0, 2r). \tag{2.5}$$

Let σ be a holomorphic frame for L on an open neighborhood U of z_0 with $|\sigma(z_0)| = 1$. Write $s = h\sigma^{\otimes p}$ with h a holomorphic function on U. Using local coordinates near z_0 and shrinking U if necessary, we may identify U with the open unit ball in $\mathbb{C}^n$. We can also assume that

$$\big||\sigma(z)| - 1\big| \leq c\|z - z_0\|$$

for some constant $c > 0$ independent of $z \in U$. For $z \in B(z_0, 2r)$, we have $\|z - z_0\| \ll p^{-1}$, and the previous inequality implies that $|\sigma^{\otimes p}(z)|$ belongs to the interval $[7/8, 9/8]$ when $z \in B(z_0, 2r)$. So the norm $|s(z)|$ is bounded below and above by $7|h(z)|/8$ and $9|h(z)|/8$ respectively.

Consider the unit vector $v := \frac{z-z_0}{\|z-z_0\|}$ in $\mathbb{C}^n$ and the following holomorphic function of one variable:

$$f(\zeta) := h(z_0 + \zeta v), \qquad \zeta \in \mathbb{D}.$$

We have for $z \in B(z_0, r)$,

$$|h(z) - h(z_0)| = |f(\|z - z_0\|) - f(0)| \leq \|z - z_0\| \sup_{|\zeta|\leq r} |f'(\zeta)|. \tag{2.6}$$

On the other hand, for $|\zeta| \leq 2r$, we have $(z_0 + \zeta v) \in B(z_0, 2r)$, and by using the definition of f, h, (2.5), and the fact that $|s(z)|$ is in-between $7|h(z)|/8$ and $9|h(z)|/8$, we obtain

$$\sup_{|\zeta|\leq 2r} |f(\zeta)| \leq c|s(z_0)| e^{p\epsilon}$$

for some constant $c > 0$. By Cauchy's formula,

$$\sup_{|\zeta|\leq r} |f'(\zeta)| \leq \frac{c}{r}|s(z_0)| e^{p\epsilon}.$$

This, together with (2.6) and the choice of ϵ, r, implies for $z \in B(z_0, r^2)$ that

$$|h(z) - h(z_0)| \leq cr|s(z_0)| e^{p\epsilon} \ll |s(z_0)|.$$

Recall that $|h(z_0)| = |s(z_0)|$ and $|s(z)|$ is bounded by $7|h(z)|/8$ and $9|h(z)|/8$. So the last inequality implies the lemma. ∎

End of the proof of Theorem 2.10. We only need to consider p large enough. We will prove that

$$|s(z)|_{p\phi} \geq \frac{1}{2}\|s\|_{L^\infty(K,p\phi)} \quad \text{for} \quad z \in K \cap B(z_0, r^2). \tag{2.7}$$

We have

$$\big||s(z)|_{p\phi} - |s(z_0)|_{p\phi}\big| \leq \big||s(z)|_{p\phi(z_0)} - |s(z_0)|_{p\phi(z_0)}\big| + \big||s(z)|_{p\phi(z)} - |s(z)|_{p\phi(z_0)}\big|.$$

Denote respectively by A_1 and A_2 the first and second terms in the last sum. By Lemma 2.12, we have

$$A_1 \leq \frac{1}{4}|s(z_0)|e^{-p\phi(z_0)}.$$

On the other hand, by Lemma 2.12 again, we have

$$A_2 = |s(z)||e^{-p\phi(z)} - e^{-p\phi(z_0)}| \leq 2|s(z_0)|e^{-p\phi(z_0)}|1 - e^{-p(\phi(z)-\phi(z_0))}|.$$

Since $z \in B(z_0, r^2)$, we deduce from assumption (i) of the theorem (see also Remark 2.11) that $|p(\phi(z) - \phi(z_0))| \leq 1/16$. Hence, $|1 - e^{-p(\phi(z)-\phi(z_0))}| \leq 1/8$. Combining the above estimates for A_1 and A_2, we obtain

$$\big||s(z)|_{p\phi} - |s(z_0)|_{p\phi}\big| \leq \frac{1}{2}|s(z_0)|_{p\phi} \quad \text{for} \quad z \in K \cap B(z_0, r^2).$$

This, combined with (2.4), implies (2.7).

Now, using (2.7) and assumption (iii), we get

$$\begin{aligned}
\int_K |s(z)|^2_{p\phi} d\mu &\geq \int_{K\cap B(z_0,r^2)} |s(z)|^2_{p\phi} d\mu \\
&\geq \Big(\min_{K\cap B(z_0,r^2)} |s(z)|^2_{p\phi}\Big)\mu(K \cap B(z_0, r^2)) \\
&\geq \frac{1}{4}cr^{2\rho}\|s\|^2_{L^\infty(K,p\phi)},
\end{aligned}$$

where $c > 0$ is the constant in Definition 1.3. Hence,

$$\|s\|_{L^\infty(K,p\phi)} \leq 2c^{-1/2}e^{\rho c' p^{1-\delta}}\|s\|_{L^2(\mu,p\phi)}.$$

So μ is δ-Bernstein-Markov with respect to (K, ϕ). □

We have the following result where condition (ii) is automatically satisfied for $W = K$. It allows us to obtain Corollary 1.4 as a direct consequence of Theorem 1.2. Note that in Corollary 1.4 we only need to assume that the measure μ satisfies the mass-density condition with respect to a compact $W \subset K$ which satisfies the maximum principle relative to (K, ϕ).

Corollary 2.13. *Let X, L, h_0 be as above, let K be a compact subset of X, let W be a compact subset of K, and let μ be a probability measure on K. Assume in addition the following conditions:*

(i) *K is $(\mathscr{C}^\alpha, \mathscr{C}^{\alpha'})$-regular for some constants $\alpha > 0$ and $\alpha' > 0$;*
(ii) *W satisfies the maximum principle relative to (K, ϕ);*
(iii) *μ satisfies the mass-density condition with respect to W.*

Then μ is a 1-Bernstein-Markov measure with respect to (K, ϕ) for every $\phi \in \mathscr{C}^\alpha(K)$.

Proof. Since $\phi \in \mathscr{C}^\alpha(K)$ and K is $(\mathscr{C}^\alpha, \mathscr{C}^{\alpha'})$-regular, (K, ϕ) satisfies the hypotheses of Theorem 2.10. According to that theorem, μ is δ-Bernstein-Markov with respect to (K, ϕ) for every $0 < \delta < 1$. The corollary follows. □

3. Almost-Fekete configurations and proof of the main result

In this section, we will give the proof of the main theorem. An important ingredient is the equidistribution of almost-Fekete points towards the equilibrium measure. This property is already mentioned in the last version of [9]; see also [20]. For the reader's convenience, we will give here some details. We also give at the end of this section another application of this result.

Theorem 3.1 ([9]). *Let X, L, h_0 be as above and let K be a compact subset of X. Let $0 < \alpha \leq 2$, $0 < \alpha' \leq 1$, and $0 < \gamma \leq 2$ be constants. Assume that K is $(\mathscr{C}^\alpha, \mathscr{C}^{\alpha'})$-regular. Let ϕ be a $\mathscr{C}^\alpha$ real-valued function on K and let $\mu_{\rm eq}(K,\phi)$ be the equilibrium measure associated with the weighted set (K,ϕ). Then, there is a constant $c > 0$ with the following property. For every $p \geq 1$ and every configuration $\boldsymbol{x} = (x_1, \ldots, x_{N_p}) \in K^{N_p}$, denote by $\mu^{\boldsymbol{x}}$ the empirical measure associated with $\boldsymbol{x}$ and let S_p be any basis of $H^0(X, L^p)$. Define*

$$\sigma_{\boldsymbol{x}} := \frac{1}{pN_p} \log \| \det S_p \|_{L^\infty(K,p\phi)} - \frac{1}{pN_p} \log \| \det S_p(\boldsymbol{x}) \|_{p\phi}.$$

Then we have for all $p > 1$,

$$\mathrm{dist}_\gamma(\mu^{\boldsymbol{x}}, \mu_{\rm eq}(K,\phi)) \leq cp^{-\alpha''\gamma} (\log p)^{3\alpha''\gamma} + c\sigma_{\boldsymbol{x}}^{\gamma/4} \quad \textit{with} \quad \alpha'' := \alpha'/(24 + 12\alpha').$$

Note that $\det S_p$ is a section of the line bundle $(L^p)^{\boxtimes N_p}$ over X^{N_p}. The given metric h_0 on L and the weight ϕ induces naturally a metric and a weight for this line bundle. So $\| \det S_p \|_{L^\infty(K,p\phi)}$ is the sup-norm of $\det S_p$ on K^{N_p}, and $\| \det S_p(\boldsymbol{x}) \|_{p\phi}$ is the norm of the value of this section at the point $\boldsymbol{x}$. Both of them are measured using the above natural metric and weight. Observe that $\sigma_{\boldsymbol{x}}$ is independent of the choice of S_p and we always have $\sigma_{\boldsymbol{x}} \geq 0$. When $\sigma_{\boldsymbol{x}} = 0$, the point $\boldsymbol{x}$ is called *a Fekete configuration of order* p of L with respect to the weighted compact set (K,ϕ). The theorem shows that if $\sigma_{\boldsymbol{x}}$ is small enough (e.g., when $\sigma_{\boldsymbol{x}} = 0$), then $\mu^{\boldsymbol{x}}$ tends to $\mu_{\rm eq}(K,\phi)$ as $p \to \infty$.

We now sketch the proof of Theorem 3.1. Recall that the *Monge-Ampère energy functional* $\mathcal{E}$, defined on bounded weights in $\mathrm{PSH}(X, \omega_0)$, is characterized by

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{E}((1-t)\phi_1 + t\phi_2) = \|\omega_0^n\|^{-1} \int_X (\phi_2 - \phi_1)(dd^c\phi_1 + \omega_0)^n.$$

So $\mathcal{E}$ is only defined up to an additive constant, but the differences such as $\mathcal{E}(\phi_1) - \mathcal{E}(\phi_2)$ are well-defined; see [2] and also (3.2) below.

Consider a non-pluripolar compact set $K \subset X$ and a continuous weight ϕ on K. Define the *energy at the equilibrium weight* of (K,ϕ) as

$$\mathcal{E}_{\rm eq}(K,\phi) := \mathcal{E}(P_K\phi).$$

This functional is also well-defined up to an additive constant. We have the following property.

Lemma 3.2 ([2], Thm. B). *The map $\phi \mapsto \mathcal{E}_{\rm eq}(K,\phi)$, defined on the affine space of continuous weights on K, is concave and Gâteaux differentiable, with directional derivatives given by integration against the equilibrium measure:*

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{E}_{\rm eq}(K, \phi + tv) = \langle v, \mu_{\rm eq}(K,\phi) \rangle \quad \textit{for every continuous function } v \textit{ on } K.$$

Let μ be a probability measure on X and let ϕ be a continuous function on the support of μ. The semi-norm $\|\cdot\|_{L^2(\mu,p\phi)}$ on $H^0(X,L^p)$ is defined as in (2.1), and recall that we only consider measures μ for which this semi-norm is a norm. Let $\mathcal{B}^2_p(\mu,\phi)$ denote the unit ball in $H^0(X,L^p)$ with respect to this norm and let $N_p := \dim H^0(X,L^p)$. Consider the $\mathcal{L}_p$-functional

$$\mathcal{L}_p(\mu,\phi) := \frac{1}{2pN_p}\log \operatorname{vol}\mathcal{B}^2_p(\mu,\phi). \tag{3.1}$$

Here, vol denotes the Lebesgue measure on the vector space $H^0(X,L^p)$ which depends on the choice of a Euclidean norm on $H^0(X,L^p)$. So the volume is only defined up to a multiplicative constant. Nevertheless, the differences such as $\mathcal{L}_p(\mu_1,\phi_1) - \mathcal{L}_p(\mu_2,\phi_2)$ are well-defined and do not depend on the choice of vol for any probability measures μ_1 and μ_2; see [2] and also (3.2) below.

Consider the norm $\|\cdot\|_{L^\infty(K,p\phi)}$ on $H^0(X,L^p)$ defined in (2.1). Let $\mathcal{B}^\infty_p(K,\phi)$ denote the unit ball in $H^0(X,L^p)$ with respect to this norm. Define

$$\mathcal{L}_p(K,\phi) := \frac{1}{2pN_p}\log \operatorname{vol}\mathcal{B}^\infty_p(K,\phi).$$

Let $\{s_1,\dots,s_{N_p}\}$ be an orthonormal basis of $H^0(X,L^p)$ with respect to the above L^2-norm; see (2.1).

Definition 3.3. We call the *Bergman function of* L^p, associated with (μ,ϕ), the function $\rho_p(\mu,\phi)$ on the support of μ given by

$$\rho_p(\mu,\phi)(x) := \sup\Big\{|s(x)|^2_{p\phi} :\quad s\in H^0(X,L^p), \|s\|_{L^2(\mu,p\phi)} = 1\Big\} = \sum_{j=1}^{N_p}|s_j(x)|^2_{p\phi},$$

and we define *the Bergman measure* associated with (μ,ϕ) by

$$\mathscr{B}_p(\mu,\phi) := N_p^{-1}\rho_p(\mu,\phi)\mu.$$

It is not difficult to obtain the identity in the definition of $\rho_p(\mu,\phi)$ and to check that $\mathscr{B}_p(\mu,\phi)$ is a probability measure. Note also that when μ is the average of N_p Dirac masses at generic points, one can easily deduce from Definition 3.3 that $\mathscr{B}_p(\mu,\phi)=\mu$, by considering sections vanishing on $\operatorname{supp}(\mu)$ except at a point. Such sections exist because $N_p = \dim H^0(X,L^p)$. In fact, this property holds for all points $x_1,\dots,x_{N_p}$ such that the section $\det S_p$ considered in the Introduction does not vanish at $(x_1,\dots,x_{N_p})$.

Lemma 3.4. *We have the following:*

(a) *The functional $\phi\mapsto\mathcal{L}_p(\mu,\phi)$ is concave on the space of all continuous weights on the support of μ.*

(b) *The directional derivative of $\mathcal{L}_p(\mu,\cdot)$ at a continuous weight ϕ on the support of μ is given by the integration against the Bergman measure $\mathscr{B}_p(\mu,\phi)$, that is,*

$$\frac{d}{dt}\mathcal{L}_p(\mu,\phi+tv)\Big|_{t=0} = \langle v,\mathscr{B}_p(\mu,\phi)\rangle, \quad \textit{with } v,\phi \textit{ continuous on the support of } \mu.$$

(c) *Let μ be a probability measure with $\operatorname{supp}(\mu)\subset K$ such that the L^2-semi-norm in (2.1) is a norm. Assume also that (K,ϕ) is a regular weighted compact set. Then*

$$\mathcal{L}_p(K,\phi) = \mathcal{L}_p(X,P_K\phi) \quad \textit{and} \quad \mathcal{L}_p(K,\phi)\le\mathcal{L}_p(\mu,\phi).$$

Proof. The concavity property of the functional $\mathcal{L}_p$ in part (a) has been established in [3, Prop. 2.4]. Part (b) has been established in [2, Lem. 5.1]. The property was stated there for smooth ϕ, but the proof also works for continuous functions; see also [4, Lem. 3.1] and [11, Lem. 2]. For part (c), see [9, Prop. 2.5, Lem. 3.4]. $\square$

From now on, in order to simplify the notation, we use the following normalization:

$$\mathcal{E}_{\mathrm{eq}}(X,0)=0 \quad \text{and} \quad \mathcal{L}_p(\mu^0,0)=0 \quad \text{for} \quad p\geq 1. \tag{3.2}$$

Here, the function identically 0 is used as a smooth strictly ω_0-p.s.h. weight. Recall also that $\mu^0=\|\omega_0^n\|^{-1}\omega_0^n$ is the probability measure associated with the volume form ω_0^n.

The following result is an immediate consequence of [9, Lem. 3.7, Prop. 3.10]. Recall that $\mathscr{C}^{k,\alpha}=\mathscr{C}^{k+\alpha}$ for $0\leq\alpha<1$ and $\mathscr{C}^{k,1}$ is the space of $\mathscr{C}^k$ functions whose partial derivatives of order k are Lipschitz.

Proposition 3.5. *Let $0<\alpha\leq 1$ and $A>0$ be constants. Let ϕ be an ω_0-p.s.h. weight of class $\mathscr{C}^{0,\alpha}$ on X such that $\|\phi\|_{\mathscr{C}^{0,\alpha}}\leq A$. Then, there is a constant $c_{A,\alpha}>0$ depending only on $X,L,\omega_0,A,$ and α such that we have for all $p>1$,*

$$\big|\mathcal{L}_p(\mu^0,\phi)-\mathcal{E}_{\mathrm{eq}}(X,\phi)\big|\leq c_{A,\alpha}(\log p)^{3\beta_\alpha}p^{-\beta_\alpha}$$

and

$$\big|\big(\mathcal{L}_p(X,\phi)-\mathcal{E}_{\mathrm{eq}}(X,\phi)\big|\leq c_{A,\alpha}(\log p)^{3\beta_\alpha}p^{-\beta_\alpha},$$

where $\beta_\alpha:=\alpha/(6+3\alpha)$.

For the following proposition, we refer to the discussion after Theorem 3.1 for the notation.

Proposition 3.6. *Let K be a compact subset of X. Let $0<\alpha\leq 2$ and $0<\alpha'\leq 1$ be constants. Assume that (K,ϕ) is a weighted compact set with $\phi\in\mathscr{C}^\alpha(K)$ such that K is $(\mathscr{C}^\alpha,\mathscr{C}^{\alpha'})$-regular. Then there is a constant $c>0$ with the following property. For $p\geq 1$ and $\boldsymbol{x}=(x_1,\ldots,x_{N_p})\in K^{N_p}$, denote by $\mu^{\boldsymbol{x}}$ the empirical measure associated with $\boldsymbol{x}$ and let S_p be a basis of $H^0(X,L^p)$. Define*

$$\sigma_{\boldsymbol{x}}:=\frac{1}{pN_p}\log\|\det S_p\|_{L^\infty(K,p\phi)}-\frac{1}{pN_p}\log\|\det S_p(\boldsymbol{x})\|_{p\phi}.$$

We have for all $p>1$,

$$|\mathcal{L}_p(\mu^{\boldsymbol{x}},\phi)-\mathcal{E}_{\mathrm{eq}}(K,\phi)|\leq c\big(p^{-1}\log p+\sigma_{\boldsymbol{x}}+\big|\mathcal{L}_p(\mu^0,P_K\phi)-\mathcal{E}_{\mathrm{eq}}(K,\phi)\big|\big).$$

Proof. Observe that $\sigma_{\boldsymbol{x}}$ does not depend on the choice of S_p. So choose S_p which is an orthonormal basis of $H^0(X,L^p)$ with respect to the L^2-norm without weight. Let μ_p be the empirical measure associated with a Fekete configuration of order p. Using identity [3, (2.4)], we get

$$\frac{1}{2pN_p}\log\frac{\operatorname{vol}\mathcal{B}_p^2(\mu^0,0)}{\operatorname{vol}\mathcal{B}_p^2(\mu_p,\phi)}=\frac{1}{pN_p}\log\|\det S_p\|_{L^\infty(K,p\phi)}-\frac{1}{2p}\log N_p$$

and

$$\frac{1}{2pN_p}\log\frac{\operatorname{vol}\mathcal{B}_p^2(\mu^0,0)}{\operatorname{vol}\mathcal{B}_p^2(\mu^{\boldsymbol{x}},\phi)}=\frac{1}{pN_p}\log\|\det S_p(\boldsymbol{x})\|_{p\phi}-\frac{1}{2p}\log N_p.$$

Subtracting the last line from the previous one and using (3.1), we obtain

$$\sigma_{\boldsymbol{x}}=\mathcal{L}_p(\mu^{\boldsymbol{x}},\phi)-\mathcal{L}_p(\mu_p,\phi).$$

On the other hand, with the normalization (3.2), [9, Prop. 3.12] tells us that there is a constant $c > 0$ satisfying

$$|\mathcal{L}_p(\mu_p, \phi) - \mathcal{E}_{\mathrm{eq}}(K,\phi)| \le c\big(p^{-1}\log p + \big|\mathcal{L}_p(\mu^0, P_K\phi) - \mathcal{E}_{\mathrm{eq}}(K,\phi)\big|\big) \quad \text{for} \quad p>1.$$

This, combined with the previous identity, implies the proposition. □

The following two lemmas were obtained in [9, Lems. 3.13 and 3.14].

Lemma 3.7. *There is a constant $c > 0$ such that for every continuous weight ϕ on K and every function v of class $\mathscr{C}^{1,1}$ on X, we have*

$$\big|\langle \mu_{\mathrm{eq}}(K, \phi + tv) - \mu_{\mathrm{eq}}(K,\phi), v\rangle\big| \le c|t|\|v\|_{L^\infty(K)}\|dd^c v\|_\infty \quad \textit{for} \quad t\in\mathbb{R}.$$

Lemma 3.8. *Let $\epsilon > 0$ and $M > 0$ be constants. Let F and G be functions defined on $[-\epsilon^{1/2}, \epsilon^{1/2}]$ such that*

(i) *$F(t) \ge G(t) - \epsilon$ and $|F(0) - G(0)| \le \epsilon$;*
(ii) *F is concave on $[-\epsilon^{1/2}, \epsilon^{1/2}]$ and differentiable at 0;*
(iii) *G is differentiable in $[-\epsilon^{1/2}, \epsilon^{1/2}]$, and its derivative G' satisfies $|G'(t) - G'(0)| \le M\epsilon^{1/2}$ for $t\in[-\epsilon^{1/2}, \epsilon^{1/2}]$. The last inequality holds when $|G'(t) - G'(0)| \le M|t|$.*

Then we have

$$|F'(0) - G'(0)| \le (2+M)\epsilon^{1/2}.$$

End of the proof of Theorem 3.1. By (1.3), we only need to consider the case $\gamma = 2$, i.e., to prove that

$$\big|\langle \mu^{\boldsymbol{x}} - \mu_{\mathrm{eq}}(K,\phi), v\rangle\big| \lesssim p^{-2\alpha''}(\log p)^{6\alpha''} + \sigma_{\boldsymbol{x}}^{1/2}$$

for every test $\mathscr{C}^2$ function v such that $\|v\|_{\mathscr{C}^2} \le 1$. Recall that $\alpha'' := \alpha'/(24+12\alpha')$.

Define

$$F(t) := \mathcal{L}_p(\mu^{\boldsymbol{x}}, \phi + tv) \qquad \text{and} \qquad G(t) := \mathcal{E}_{\mathrm{eq}}(K, \phi+tv) = \mathcal{E}_{\mathrm{eq}}(X, P_K(\phi+tv))$$

for t in a neighborhood of $0\in\mathbb{R}$. By Lemma 3.4(c),

$$\mathcal{L}_p(\mu^{\boldsymbol{x}}, \phi+tv) \ge \mathcal{L}_p(K, \phi+tv) = \mathcal{L}_p(X, P_K(\phi+tv)).$$

As $0 < \alpha \le 2$, we infer that $\phi + tv \in \mathscr{C}^\alpha(K)$. Since K is $(\mathscr{C}^\alpha, \mathscr{C}^{\alpha'})$-regular, we deduce that $P_K(\phi+tv)$ is an ω_0-p.s.h. weight on X with bounded $\mathscr{C}^{\alpha'}$-norm. This, coupled with the second inequality in Proposition 3.5, applied to $P_K(\phi+tv)$ and α' instead of α, shows that

$$F(t) - G(t) \gtrsim -p^{-4\alpha''}(\log p)^{12\alpha''}. \tag{3.3}$$

An application of the first inequality in Proposition 3.5 for α' instead of α gives

$$\big|\mathcal{L}_p(\mu^0, P_K\phi) - \mathcal{E}_{\mathrm{eq}}(K,\phi)\big| \lesssim p^{-4\alpha''}(\log p)^{12\alpha''}.$$

Consequently, applying Proposition 3.6 yields

$$|F(0) - G(0)| \lesssim p^{-4\alpha''}(\log p)^{12\alpha''} + \sigma_{\boldsymbol{x}}.$$

Recall from Lemma 3.4(a) that F is concave. Moreover, by Lemma 3.4(b), we have

$$F'(0) = \langle v, \mathscr{B}_p(\mu^{\boldsymbol{x}}, \phi)\rangle.$$

On the other hand, by Lemma 3.2, G is differentiable with

$$G'(t) = \langle v, \mu_{\mathrm{eq}}(K, \phi+tv)\rangle. \tag{3.4}$$

Finally, by Lemma 3.7, condition (iii) in Lemma 3.8 is satisfied for a suitable constant $M > 0$. Combining this and the discussion between (3.3)-(3.4), we are in the position to apply Lemma 3.8 to a constant ϵ of order $p^{-4\alpha''}(\log p)^{12\alpha''} + \sigma_{\boldsymbol{x}}$. Using the above expression for $F'(0)$ and $G'(0)$, we get

$$\big|\langle \mathscr{B}_p(\mu^{\boldsymbol{x}}, \phi), v\rangle - \langle \mu_{\mathrm{eq}}(K,\phi), v\rangle\big| \lesssim p^{-2\alpha''}(\log p)^{6\alpha''} + \sigma_{\boldsymbol{x}}^{1/2}.$$

Recall from the discussion before Lemma 3.4 that $\mathscr{B}_p(\mu^{\boldsymbol{x}}, \phi) = \mu^{\boldsymbol{x}}$. Hence, the desired estimate follows immediately. □

We continue the proof of the main theorem. We need the following result, which is a consequence of [2, Lem. 5.3].

Lemma 3.9. *Consider a probability measure μ supported on a compact set $K \subset X$ such that the L^2-semi-norm in* (2.1) *is a norm. If S_p is an orthonormal basis of $H^0(X, L^p)$ with respect to this norm, then the positive measure $\|\det S_p\|^2_{p\phi}\mu^{\otimes N_p}$ is of mass $N_p!$.*

End of the proof of Theorem 1.2. Fix an orthonormal basis S_p of $H^0(X, L^p)$ with respect to the L^2-norm induced by μ and ϕ. We first show that there is a constant $c > 0$ such that for $p \geq 1$,

$$0 \leq \log \|\det S_p\|_{L^\infty(K,p\phi)} - \log \|\det S_p\|_{L^2(\mu,p\phi)} \leq cN_p p^{1-\delta}. \tag{3.5}$$

Here, similar to the discussion after Theorem 3.1, the norm $\|\det S_p\|_{L^2(\mu,p\phi)}$ is defined using the product probability measure $\mu^{\otimes N_p}$ on $K^{N_p} \subset X^{N_p}$ together with the metric and weight for $(L^p)^{\boxtimes N_p}$, naturally induced by h_0 and ϕ.

Since μ is a probability measure, we have

$$\|\det S_p\|_{L^\infty(K,p\phi)} \geq \|\det S_p\|_{L^2(\mu,p\phi)}.$$

Now, to complete the proof of (3.5), we only need to show that

$$\log \|\det S_p\|_{L^\infty(K,p\phi)} \leq \log \|\det S_p\|_{L^2(\mu,p\phi)} + O(N_p p^{1-\delta}). \tag{3.6}$$

By (2.2), we get

$$|s(x)|^2_{p\phi} \leq A^2 e^{2Ap^{1-\delta}} \|s\|^2_{L^2(\mu,p\phi)}$$

for every section $s \in H^0(X, L^p)$, $p \geq 1$, and $x \in X$. If $x_1, \ldots, x_{N_p}$ are points in X, then for each j,

$$x \mapsto \det S_p(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_{N_p})$$

is a holomorphic section in $H^0(X, L^p)$. A successive application of the last inequality for $j = 1, 2, \ldots, N_p$ and Fubini's theorem yield

$$\|\det S_p\|^2_{L^\infty(K,p\phi)} \leq A^{2N_p} e^{2AN_p p^{1-\delta}} \|\det S_p\|^2_{L^2(\mu,p\phi)}.$$

Estimates (3.6) and (3.5) follow.

Recall that $N_p = O(p^n)$ and by Stirling's formula $N_p! \approx (N_p/e)^{N_p}\sqrt{2\pi N_p}$. Therefore, Lemma 3.9 implies that

$$\frac{1}{pN_p} \log \|\det S_p\|_{L^2(\mu,p\phi)} = \frac{1}{pN_p} \log \sqrt{N_p!} = O(p^{-1}\log p).$$

It follows from (3.5) that

$$0 \leq \frac{1}{pN_p} \log \|\det S_p\|_{L^\infty(K,p\phi)} \leq c_1 p^{-\delta} \quad \text{with some constant} \quad c_1 > 0. \tag{3.7}$$

Let $\lambda_0 > 0$ be a constant whose value will be determined later. For every $p \geq 1$, consider the set

$$E_p := \Big\{ \boldsymbol{x} \in K^{N_p} : \ \frac{1}{pN_p} \log \| \det S_p(\boldsymbol{x}) \|_{p\phi} \leq -\lambda_0 p^{-\delta} \Big\}.$$

So for $\boldsymbol{x} \in K^{N_p} \setminus E_p$, using (3.7), we obtain $\sigma_{\boldsymbol{x}} \leq (c_1 + \lambda_0) p^{-\delta}$, where as above

$$\sigma_{\boldsymbol{x}} := \frac{1}{pN_p} \log \| \det S_p \|_{L^\infty(K, p\phi)} - \frac{1}{pN_p} \log \| \det S_p(\boldsymbol{x}) \|_{p\phi}.$$

Hence, applying Theorem 3.1 yields

$$\mathrm{dist}_\gamma(\mu^{\boldsymbol{x}}, \mu_{\mathrm{eq}}(K, \phi)) \leq cp^{-\alpha''\gamma} (\log p)^{3\alpha''\gamma} + cp^{-\gamma\delta/4},$$

for some constant $c > 0$.

To complete the proof of the theorem, it remains to bound the size of E_p. Fix a constant λ as in Theorem 1.2. Consider two different cases according to the value of β.

Case 1. Assume that $\beta \geq 2$. Choose $\lambda_0 = \lambda/\beta$. We first bound the mass of $\| \det S_p \|_{p\phi}^\beta \mu^{\otimes N_p}$ from below. Recall that $\mu^{\otimes N_p}$ is a probability measure. Applying Hölder's inequality and using Lemma 3.9, we obtain

$$\int \| \det S_p \|_{p\phi}^\beta d\mu^{\otimes N_p} \geq \Big(\int \| \det S_p \|_{p\phi}^2 d\mu^{\otimes N_p} \Big)^{\beta/2} = (N_p!)^{\beta/2}.$$

Consequently, $\nu_p^\beta \leq \| \det S_p \|_{p\phi}^\beta \mu^{\otimes N_p}$. Hence, by definition of E_p, we get

$$\nu_p^\beta(E_p) \leq \int_{E_p} \| \det S_p(\boldsymbol{x}) \|_{p\phi}^\beta d\mu^{\otimes N_p}(\boldsymbol{x}) \leq \int_{E_p} e^{-\lambda p^{1-\delta} N_p} d\mu^{\otimes N_p}(\boldsymbol{x}) \leq e^{-\lambda p^{1-\delta} N_p}.$$

This completes the proof for the case $\beta \geq 2$.

Case 2. Assume that $0 < \beta \leq 2$. Combining (3.7) and Lemma 3.9, we get

$$\begin{aligned} \int_{K^{N_p}} \| \det(S) \|_{p\phi}^\beta d\mu^{\otimes N_p} &\geq e^{-(2-\beta)c_1 p^{1-\delta} N_p} \int_{K^{N_p}} \| \det(S) \|_{p\phi}^2 d\mu^{\otimes N_p} \\ &\geq e^{-(2-\beta)c_1 p^{1-\delta} N_p}. \end{aligned}$$

Consequently,

$$\nu_p^\beta \leq e^{(2-\beta)c_1 p^{1-\delta} N_p} \| \det S_p \|_{p\phi}^\beta \mu^{\otimes N_p}.$$

Hence, we infer that

$$\begin{aligned} \nu_p^\beta(E_p) &\leq e^{(2-\beta)c_1 p^{1-\delta} N_p} \int_{E_p} \| \det S_p(\boldsymbol{x}) \|_{p\phi}^\beta d\mu^{\otimes N_p}(\boldsymbol{x}) \\ &\leq e^{(2-\beta)c_1 p^{1-\delta} N_p} \int_{E_p} e^{-\beta\lambda_0 p^{1-\delta} N_p} d\mu^{\otimes N_p}(\boldsymbol{x}) \\ &\leq e^{p^{1-\delta} N_p \big((2-\beta)c_1 - \beta\lambda_0 \big)}. \end{aligned}$$

Choose $\lambda_0 \gg c_1$ and the result follows. This ends the proof of our main theorem. □

As mentioned above, Theorem 3.1 can be applied to other situations. We present now one more application. Consider the same setting as in Theorem 3.1 and a probability measure μ on K. Recall the following notion; see [3].

Definition 3.10. Let $0 < r \leq \infty$ and $0 < r' \leq \infty$. We say that $\boldsymbol{y} \in K^{N_p}$ is an *(r, r')-optimal configuration of order p* if the function in $\boldsymbol{x} \in K^{N_p}$,

$$\tau_{\boldsymbol{x}} := \sup_{s \in H^0(X, L^p) \setminus \{0\}} \frac{\|s\|_{L^r(\mu, p\phi)}}{\|s\|_{L^{r'}(\mu^{\boldsymbol{x}}, p\phi)}},$$

attains its minimum at $\boldsymbol{y}$.

We have the following elementary property; see also [3, Prop. 2.10].

Lemma 3.11. *If $\boldsymbol{y} \in K^{N_p}$ is (r, r')-optimal, then $\tau_{\boldsymbol{y}} \leq N_p^{1+1/r'}$.*

Proof. Let $\boldsymbol{x} = (x_1, \ldots, x_{N_p})$ be a Fekete configuration of order p. We only need to check that $\tau_{\boldsymbol{x}} \leq N_p^{1+1/r'}$. Choose a basis $S_p = (s_1, \ldots, s_{N_p})$ of $H^0(X, L^p)$ such that $s_i(x_j) = 0$ when $i \neq j$ and $\|s_i(x_i)\|_{p\phi} = 1$. Since $\boldsymbol{x}$ is a Fekete configuration, we have $\|\det S_p(\cdot)\|_{p\phi} \leq 1$ on K^{N_p}. This inequality on $K_i := \{x_1\} \times \cdots \times \{x_{i-1}\} \times K \times \{x_{i+1}\} \times \cdots \times \{x_{N_p}\}$ implies that $\|s_i(\cdot)\|_{p\phi} \leq 1$ on K. Finally, if s is a section in $H^0(X, L^p) \setminus \{0\}$, write $s = \lambda_1 s_1 + \cdots + \lambda_{N_p} s_{N_p}$, and we have

$$\frac{\|s\|_{L^r(\mu, p\phi)}}{\|s\|_{L^{r'}(\mu^{\boldsymbol{x}}, p\phi)}} \leq \frac{\sum |\lambda_i|}{(N_p^{-1} \sum |\lambda_i|^{r'})^{1/r'}} \leq \frac{N_p \max |\lambda_i|}{(N_p^{-1} \max |\lambda_i|^{r'})^{1/r'}} = N_p^{1+1/r'}.$$

The lemma follows. □

We deduce from Theorem 3.1 the following result, where the simple convergence of $\mu^{\boldsymbol{y}}$ when $p \to \infty$ was established in [3].

Corollary 3.12. *In the setting of Theorem* 1.2*, consider two numbers $0 < r, r' \leq \infty$. There is a constant $c > 0$ such that if $\boldsymbol{y}$ is an (r, r')-optimal configuration of order p for some $p > 1$, then*

$$\operatorname{dist}_\gamma(\mu^{\boldsymbol{y}}, \mu_{\mathrm{eq}}(K, \phi)) \leq c q^\gamma.$$

Proof. We only have to check that

$$\sigma_{\boldsymbol{x}} \leq c(p^{-1} \log \tau_{\boldsymbol{x}} + p^{-\delta}) \quad \text{for} \quad \boldsymbol{x} \in K^{N_p}$$

for some constant $c > 0$. Then, Theorem 3.1, Lemma 3.11, and the estimate $N_p = O(p^n)$ imply the result.

We can assume that $\det S_p(\boldsymbol{x}) \neq 0$ because the case $\det S_p(\boldsymbol{x}) = 0$ is trivial. So we can choose $S_p = (s_1, \ldots, s_{N_p})$ as in the proof of Lemma 3.11, but here $\boldsymbol{x}$ is no longer a Fekete configuration. By definition of $\tau_{\boldsymbol{x}}$, we have

$$\|s_i\|_{L^r(\mu, p\phi)} \leq \tau_{\boldsymbol{x}} \|s_i\|_{L^{r'}(\mu^{\boldsymbol{x}}, p\phi)} = N_p^{-1/r'} \tau_{\boldsymbol{x}} \leq \tau_{\boldsymbol{x}}.$$

Hence, it follows from Lemma 2.5 that

$$\|s_i\|_{L^\infty(K, p\phi)} \leq A' e^{A' p^{1-\delta}} \tau_{\boldsymbol{x}}.$$

Therefore, we get

$$\|\det S_p(\cdot)\|_{p\phi} \leq N_p! \big(A' e^{A' p^{1-\delta}} \tau_{\boldsymbol{x}}\big)^{N_p} \quad \text{on} \quad K^{N_p}.$$

We then deduce the desired estimate using the definition of $\sigma_{\boldsymbol{x}}$ and the fact that $\|\det S_p(\boldsymbol{x})\|_{p\phi} = 1$ by the choice of S_p. □

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