TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 9, September 2018, Pages 6629–6650
http://dx.doi.org/10.1090/tran/7176
Article electronically published on March 19, 2018

ON THE EXISTENCE OF F-THRESHOLDS AND RELATED LIMITS

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Dedicated to Professor Craig Huneke on the occasion of his sixty-fifth birthday

Abstract. We show the existence of F-thresholds in full generality. In addition, we study properties of standard graded algebras over a field for which F-pure threshold and F-threshold at the irrelevant maximal ideal agree. We also exhibit explicit bounds for the a-invariants and Castelnuovo-Mumford regularity of Frobenius powers of ideals in terms of F-thresholds and F-pure thresholds, obtaining the existence of related limits in certain cases.

1. Introduction

In recent years there has been intense research in algebraic geometry and commutative algebra, moving towards a better understanding of what are nowadays known as F-singularities. Particular attention has been given to F-pure [HR76, Fed83] and F-regular singularities [HH90, HH94a, HH94b, FW89]. Attached to these singularity types, there are numerical invariants that measure how good or bad a singular point is; for instance, the F-thresholds [MTW05, HMTW08], the F-pure thresholds [TW04], and the F-signature [SVdB97, HL02, Tuc12]. In this manuscript, we study these numbers, compare them, and obtain consequences regarding the singularities of the ring.

The F-thresholds were first introduced for regular rings by Mustaţă, Takagi, and Watanabe [MTW05] as a positive characteristic analogue of log-canonical thresholds. In a subsequent joint work with Huneke [HMTW08], F-thresholds were defined in general rings of prime characteristic as limits of normalized Frobenius orders, provided they exist. In the same article, the authors showed compelling relations that F-thresholds have with the Hilbert-Samuel multiplicity, tight closure, and integral closure. However, a drawback of using these methods was that the convergence of the sequence defining the F-thresholds had been shown only in some partial cases [HMTW08, HTW11, Li13]. We settle this problem by proving the existence of F-thresholds in full generality.

Theorem A (See Theorem 3.4)**.** *Let R be a Noetherian ring of prime characteristic p. Let $\mathfrak{a}, J \subseteq R$ be ideals such that $\mathfrak{a} \subseteq \sqrt{J}$. If $\nu_{\mathfrak{a}}^J(p^e) := \max\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq J^{[p^e]}\}$, then the F-threshold $c^J(\mathfrak{a}) = \lim\limits_{e\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$ exists.*

Received by the editors May 11, 2016, and in revised form, December 24, 2016.
2010 *Mathematics Subject Classification.* Primary 13A35; Secondary 13D45, 14B05.
Key words and phrases. F-thresholds, a-invariants, F-pure thresholds, Castelnuovo-Mumford regularity, strong F-regularity, F-signature.
The second author was partially supported by NSF Grant 1502282.

In the rest of this article, we study relations between F-thresholds and other invariants in prime characteristic: namely, F-pure thresholds, a-invariants, and asymptotic Castelnuovo-Mumford regularity of Frobenius powers.

The F-pure threshold of an ideal $\mathfrak{a} \subseteq R$, denoted $\operatorname{fpt}(\mathfrak{a})$, was defined by Takagi and Watanabe [TW04]. Roughly speaking, the F-pure threshold of an ideal measures its splitting order. The general expectation is that the higher the F-pure threshold, the better the singularities [HY03, BFS13]. Takagi and Watanabe [TW04] write: *"Although our first motivation was to investigate the log canonical threshold via the F-pure threshold, we find that the F-pure threshold itself is an interesting invariant in commutative algebra"*.

If $(R, \mathfrak{m}, K)$ is regular, either local or standard graded, then $\operatorname{fpt}(\mathfrak{a}) = c^{\mathfrak{m}}(\mathfrak{a})$ for any ideal $\mathfrak{a} \subseteq R$. In contrast, this is often not the case for singular rings. However, the inequality $\operatorname{fpt}(\mathfrak{a}) \leq c^{\mathfrak{m}}(\mathfrak{a})$ holds true in general. In this article, we focus on the study of the following question, asked in different settings by several researchers.

Question 1.1 ([Hir09, MOY10, HWY14])**.** What are necessary and sufficient conditions for the equality $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$ to hold?

Our first step towards an answer to Question 1.1 is a characterization of the F-pure threshold as a limit of F-thresholds (see Theorem 4.6 and Corollary 4.7). If we restrict ourselves to standard graded Gorenstein K-algebras, we are able to partially answer Question 1.1, giving a necessary condition for the equality to hold. In a sense, this result says that standard graded Gorenstein rings such that $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$ have the best possible type of F-singularities.

Theorem B (See Theorem 6.13)**.** *Let $(R, \mathfrak{m}, K)$ be a d-dimensional standard graded Gorenstein K-algebra that is F-finite and F-pure. If $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$, then R is strongly F-regular. Furthermore,*

$$s(R) \geq \frac{e(R)}{d!},$$

where $e(R)$ denotes the Hilbert-Samuel multiplicity and $s(R)$ the F-signature of R.

Using recent results of Singh, Takagi, and Varbaro [STV17], we can extend Theorem B to normal standard graded Cohen-Macaulay algebras whose anti-canonical cover is Noetherian (see Corollary 6.16). These include $\mathbb{Q}$-Gorenstein algebras.

The inequality for the F-signature in the previous theorem is particularly meaningful because lower bounds for this invariant are typically hard to produce. The first two authors proved that $\operatorname{fpt}(\mathfrak{m}) \leq -a_{\dim(R)}(R) \leq c^{\mathfrak{m}}(\mathfrak{m})$ [DSNB]. To obtain Theorem B, we need to extend these relations to the Castelnuovo-Mumford regularity and a-invariants of Frobenius powers. Recall that, for a finitely generated R-module M, the Castelnuovo-Mumford regularity of M can be defined as $\operatorname{reg}(M) = \max\{a_i(M) + i \mid i \in \mathbb{N}\}$, where $a_i(M) = \sup\{s \in \mathbb{Z} \mid [H^i_{\mathfrak{m}}(M)]_s \neq 0\}$, with the convention that $\sup(\emptyset) = -\infty$. We point out that the growth of $a_i(R/J^{[p^e]})$ and $\operatorname{reg}(R/J^{[p^e]})$ are of independent interest, since they are connected to discreteness of F-jumping coefficients [KZ14, KSSZ14, Zha15], localization of tight closure [Kat98, Hun00], and existence of the generalized Hilbert-Kunz multiplicity [DS13, Vra16]. The following theorem is the main result we prove in this direction, and it is a key ingredient in the proof of Theorem B.

Theorem C (See Theorems 5.4, 5.7, and 5.8)**.** *Let $(R, \mathfrak{m}, K)$ be a standard graded K-algebra that is F-finite and F-pure. Suppose that $J \subseteq R$ is a homogeneous ideal.*

If there exists a constant C such that $\operatorname{reg}(R/J^{[p^e]}) \leq Cp^e$ *for all* $e \gg 0$*, then* $\lim\limits_{e\to\infty} \frac{\operatorname{reg}(R/J^{[p^e]})}{p^e}$ *exists, and it is bounded below by* $\max_{i\in\mathbb{N}}\{a_i(R/J)\} + \operatorname{fpt}(\mathfrak{m})$.

Furthermore, for $t = \dim(R/J)$ *and* $D = \max\{t \in \mathbb{N} \mid [J/\mathfrak{m}J]_t \neq 0\} + 1$*, if* $H^{t-1}_{\mathfrak{m}}(R/J^{[p^e]}) \neq 0$ *for some* $e \in \mathbb{N}$*, then*

$$\lim_{e\to\infty} \frac{a_t(R/J^{[p^e]})}{p^e} \leq a_t(R/J) + c^{\mathfrak{m}}(\mathfrak{m}) \quad \textit{and} \quad \lim_{e\to\infty} \frac{a_{t-1}(R/J^{[p^e]})}{p^e} \leq D(c^J(J)+1).$$

In particular, the two limits exist.

As a consequence of Theorem C, we obtain explicit formulas for the top a-invariant of the ring modulo Frobenius powers of an ideal under the assumption that $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$ (see Proposition 6.1).

2. Notation and preliminaries

Throughout this article, R denotes a Noetherian commutative ring with identity. We write $(R,\mathfrak{m},K)$ to mean either a local ring or a standard graded K-algebra. A standard graded algebra is a positively graded algebra over a field K, generated by finitely many elements of degree one. The ideal generated by the positive degree elements, that we denote by $\mathfrak{m}$, is called the irrelevant maximal ideal. We denote by $\mu(M)$ the minimal number of generators of an R-module M, homogeneous in the graded case. We use $\lambda(M)$ to denote its length as an R-module. We make the convention that $0 \in \mathbb{N}$.

When the characteristic of R is a positive prime integer p, we can consider the Frobenius endomorphism $F : R \to R$, which raises any element of R to its p-th power. In this way, R can be viewed as an R-module by restriction of scalars via F, and we denote this module action on R by F_*R. The action is explicitly given as follows: for $r \in R$ and $F_*x \in F_*R$, we have $r \cdot F_*x = F_*(r^p x) \in F_*R$. For an integer $e \geq 1$, we can also reiterate the map F and obtain a ring endomorphism $F^e : R \to R$ which is such that $F^e(r) = r^{p^e}$ for all $r \in R$. For any R-module M, we can consider the R-module F^e_*M, whose action is induced by restriction of scalars via F^e, as illustrated above in the case $e = 1$ and $M = R$. For an ideal $J \subseteq R$, we denote by $J^{[p^e]}$ the ideal generated by $F^e(J)$, that is, the ideal generated by the p^e-th powers of elements in J. We note that $J^{[p^0]} = J$.

If R is reduced, then, for all integers $e \geq 1$, the map F^e can be identified with the R-module inclusion $R \subseteq R^{1/p^e}$, where R^{1/p^e} denotes the ring of p^e-th roots of elements in R. This viewpoint can be helpful to keep in mind, but it is not exploited further in this article.

Definition 2.1. The ring R is called F-finite if F_*R is a finitely generated R-module.

Equivalently, R is F-finite if F^e_*R is a finitely generated R-module for some (equivalently, for all) integer $e \geq 1$.

Remark 2.2. If $(R,\mathfrak{m},K)$ is local, then R is F-finite if and only if it is excellent, and $[F_*K : K] < \infty$ [Kun76, Corollary 2.6]. If $(R,\mathfrak{m},K)$ is standard graded, then R is F-finite if and only if $[F_*K : K] < \infty$ [Fed83, Lemma 1.5].

The notion of F-purity was introduced by Hochster and Roberts [HR76]. Since then, it has played a very crucial role in the theory of singularities of rings of positive characteristic.

Definition 2.3 ([HR76])**.** Let R be a Noetherian ring of prime characteristic, and let $F : R \to R$ be the Frobenius endomorphism. Then R is called *F-pure* if F is a pure morphism, that is, $F \otimes 1 : R \otimes M \to R \otimes M$ is injective for all R-modules M. The ring R is called *F-split* if F is a split monomorphism.

If R is F-finite, then R is F-split if and only if R is F-pure [HR76, Corollary 5.3 and Proposition 5.5]. More explicitly, when R is F-finite, we have that R is F-pure if and only if it is reduced, and the natural inclusion $R \subseteq F_*^e R$ of R-modules splits for some (equivalently, for all) $e \geq 1$.

We now recall the graded version of Fedder's Criterion, which characterizes F-pure rings that are quotients of regular rings. This result is needed to establish some reductions for Theorem 6.13.

Theorem 2.4 ([Fed83, Theorem 1.12])**.** *Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field of prime characteristic p. Suppose that $\deg(x_i) = 1$ and $\mathfrak{n} = (x_1, \ldots, x_n)$. Let $I \subseteq \mathfrak{n}$ be a homogeneous ideal. Under these assumptions S/I is F-pure if and only if $(I^{[p]} :_S I) \not\subseteq \mathfrak{n}^{[p]}$.*

3. The F-threshold of $\mathfrak{a}$ with respect to J

The F-thresholds are invariants of rings in positive characteristic obtained by comparing powers of an ideal $\mathfrak{a}$ with Frobenius powers of another ideal J. They were first introduced in the regular ring setting [MTW05] and, later, generalized to a wider class of rings [HMTW08]. The F-thresholds were originally defined as limits of sequences of rational numbers whenever such sequences were convergent. However, their existence remained an open problem. In this section, we show that F-thresholds exist in general.

Definition 3.1. Let R be a ring of prime characteristic p. For $\mathfrak{a}, J$ two ideals of R satisfying $\mathfrak{a} \subseteq \sqrt{J}$ and a non-negative integer e, we define

$$\nu_{\mathfrak{a}}^J(p^e) := \max\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq J^{[p^e]}\}.$$

The following lemma is well-known. We include the proof for the sake of completeness.

Lemma 3.2. *Let R be a ring of prime characteristic p, and let $\mathfrak{a}$ be an ideal. Then, for every $s, e \in \mathbb{N}$ and $r \geq (\mu(\mathfrak{a}) + s - 1)p^e$, we have that $\mathfrak{a}^r = \mathfrak{a}^{r-sp^e}\left(\mathfrak{a}^{[p^e]}\right)^s$.*

Proof. Let $u = \mu(\mathfrak{a})$, and let $f_1, \ldots, f_u$ denote a minimal set of generators for I. We proceed by induction on s. For $s = 0$, the statement is clear as $\mathfrak{a}^{r-0p^e}\left(\mathfrak{a}^{[p^e]}\right)^0 = \mathfrak{a}^r R = \mathfrak{a}^r$.

We now assume that our claim is true for s and prove it for $s+1$. Suppose that $r \geq (u+s)p^e$. Then

$$\begin{aligned}
\mathfrak{a}^r &= \mathfrak{a}^{r-sp^e}\left(I^{[p^e]}\right)^s \text{ by induction hypothesis since } r \geq (u+s-1)p^e \\
&= \mathfrak{a}^{r-(s+1)p^e}\mathfrak{a}^{[p^e]}\left(\mathfrak{a}^{[p^e]}\right)^s \text{ by the case } s = 1 \text{ because } r - sp^e \geq up^e \\
&= \mathfrak{a}^{r-(s+1)p^e}\left(\mathfrak{a}^{[p^e]}\right)^{s+1}.
\end{aligned}$$

□

We point out that the following lemma has been previously stated, without proof, for reduced rings [HTW11, Remark 1.5].

Lemma 3.3. *Let R be a ring of prime characteristic p. Let $\mathfrak{a}, J \subseteq R$ be ideals such that $\mathfrak{a} \subseteq \sqrt{J}$. Then,*

$$\frac{\nu_{\mathfrak{a}}^J(p^{e_1+e_2})}{p^{e_1+e_2}} - \frac{\nu_{\mathfrak{a}}^J(p^{e_1})}{p^{e_1}} \leq \frac{\mu(\mathfrak{a})}{p^{e_1}}$$

for every $e_1, e_2 \in \mathbb{N}$.

Proof. Taking $s = \nu_{\mathfrak{a}}^J(p^{e_1}) + 1$ in Lemma 3.2 yields

$$\begin{aligned}\mathfrak{a}^{p^{e_2}\mu(\mathfrak{a})+p^{e_2}\nu_{\mathfrak{a}}^J(p^{e_1})} &\subseteq \mathfrak{a}^{p^{e_2}\mu(\mathfrak{a})-p^{e_2}}\left(\mathfrak{a}^{[p^{e_2}]}\right)^{\nu_{\mathfrak{a}}^J(p^{e_1})+1} \\ &\subseteq \left(\mathfrak{a}^{[p^{e_2}]}\right)^{\nu_{\mathfrak{a}}^J(p^{e_1})+1} = \left(\mathfrak{a}^{\nu_{\mathfrak{a}}^J(p^{e_1})+1}\right)^{[p^{e_2}]} \\ &\subseteq \left(J^{[p^{e_1}]}\right)^{[p^{e_2}]} = J^{[p^{e_1+e_2}]}.\end{aligned}$$

Hence, $\nu_{\mathfrak{a}}^J(p^{e_1+e_2}) \leq p^{e_2}\mu(\mathfrak{a}) + p^{e_2}\nu_{\mathfrak{a}}^J(p^{e_1})$. The result follows from dividing by $p^{e_1+e_2}$. □

We now show the existence of F-thresholds in full generality.

Theorem 3.4. *Let R be a ring of prime characteristic p. If $\mathfrak{a}, J \subseteq R$ are ideals such that $\mathfrak{a} \subseteq \sqrt{J}$, then $\lim\limits_{e\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$ exists.*

Proof. From Lemma 3.3 we have

$$\frac{\nu_{\mathfrak{a}}^J(p^{e_1+e_2})}{p^{e_1+e_2}} \leq \frac{\nu_{\mathfrak{a}}^J(p^{e_1})}{p^{e_1}} + \frac{\mu(\mathfrak{a})}{p^{e_1}}.$$

Therefore,

$$\limsup_{e\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} = \limsup_{e_2\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^{e_1+e_2})}{p^{e_1+e_2}} \leq \frac{\nu_{\mathfrak{a}}^J(p^{e_1})}{p^{e_1}} + \frac{\mu(\mathfrak{a})}{p^{e_1}}.$$

Hence,

$$\limsup_{e\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} \leq \liminf_{e_1\to\infty}\left(\frac{\nu_{\mathfrak{a}}^J(p^{e_1})}{p^{e_1}} + \frac{\mu(\mathfrak{a})}{p^{e_1}}\right) = \liminf_{e\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}.$$

We conclude that $\lim\limits_{e\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$ exists. □

After Theorem 3.4, we can define F-thresholds in full generality.

Definition 3.5. Let R be a ring of prime characteristic p. Let $\mathfrak{a}, J$ be ideals of R such that $\mathfrak{a} \subseteq \sqrt{J}$. We define *the F-threshold of $\mathfrak{a}$ with respect to J* by

$$c^J(\mathfrak{a}) = \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}.$$

We recall some known properties of F-thresholds that we need in what follows.

Proposition 3.6 ([MTW05, Proposition 2.7], [HMTW08, Proposition 2.2])**.** *Let R be a ring of prime characteristic p, and let $\mathfrak{a}, I, J$ be ideals of R. Then*

(a) *If $I \supseteq J$ and $\mathfrak{a} \subseteq \sqrt{J}$, then $c^I(\mathfrak{a}) \leq c^J(\mathfrak{a})$.*
(b) *If $\mathfrak{a} \subseteq \sqrt{J}$, then $c^{J^{[p]}}(\mathfrak{a}) = p \cdot c^J(\mathfrak{a})$.*

4. A characterization of $\operatorname{fpt}(\mathfrak{a})$

In this section, we give a characterization of the F-pure threshold of a ring as the limit of certain F-thresholds. We start by recalling its definition and elementary properties. Before doing this, we need some auxiliary definitions.

Let $(R, \mathfrak{m}, K)$ be either a local ring or a standard graded K-algebra. The following ideals, introduced by Aberbach and Enescu [AE05], keep track of the R-linear homomorphisms from $F^e_* R$ to R that do not give splittings. For $e \in \mathbb{N}$, we set

$$I_e := \{f \in R \,|\, \psi(F^e_* f) \in \mathfrak{m}, \text{ for all } R\text{-linear maps } \psi : F^e_* R \to R\}.$$

In particular, $I_0 = \mathfrak{m}$.

Remark 4.1. Let $(R, \mathfrak{m}, K)$ be a local ring. If $f \notin I_e$ for some e, then there exists a map $\psi : F^e_* R \to R$ that splits the R-module inclusion $F^e_* f \cdot R \subseteq F^e_* R$. When $(R, \mathfrak{m}, K)$ is standard graded, the ideals I_e are homogeneous, and the same conclusion is true for a homogeneous element $f \notin I_e$ and homogeneous splitting maps.

Definition 4.2. Let $(R, \mathfrak{m}, K)$ be either a local ring or a standard graded K-algebra. Suppose that R is an F-pure ring. For $e \in \mathbb{N}$, we associate to the ideals I_e the following integers:

$$b_{\mathfrak{a}}(p^e) := \max\left\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq I_e\right\}.$$

Given a proper ideal $\mathfrak{a} \subseteq R$, homogeneous when R is graded, we define the *F-pure threshold of $\mathfrak{a}$ in R* as

$$\operatorname{fpt}(\mathfrak{a}) := \lim_{e \to \infty} \frac{b_{\mathfrak{a}}(p^e)}{p^e}.$$

When $\mathfrak{a} = \mathfrak{m}$, the F-pure threshold $\operatorname{fpt}(\mathfrak{m})$ is often simply denoted by $\operatorname{fpt}(R)$.

Remark 4.3. The definition presented above is not the original given by Takagi and Watanabe [TW04]. For a real number $\lambda \geq 0$, we say that $(R, \mathfrak{a}^\lambda)$ is F-pure if for every $e \gg 0$, there exists an element $f \in \mathfrak{a}^{\lfloor (p^e-1)\lambda \rfloor}$ such that the inclusion of R-modules $F^e_* f \cdot R \subseteq F^e_* R$ splits. The original definition of the F-pure threshold of $\mathfrak{a}$ is

$$\operatorname{fpt}(\mathfrak{a}) = \sup\left\{\lambda \in \mathbb{R}_{>0} \mid (R, \mathfrak{a}^\lambda) \text{ is } F\text{-pure}\right\}.$$

We refer to [DSNB, Proposition 3.10] for a proof that both definitions coincide.

Remark 4.4. If $(R, \mathfrak{m}, K)$ is a standard graded F-pure K-algebra, then $b := \frac{b_{\mathfrak{m}}(p^e)}{p^e}$ is the highest possible degree for a minimal generator of the free part of $F^e_* R$ with the natural $\frac{1}{p^e}\mathbb{N}$ grading. In other words, if $F^e_* R \cong \oplus (R(-\gamma_i)) \oplus M_e$ as $\frac{1}{p^e}\mathbb{N}$-graded modules, where M_e is a graded R-module with no free summands, then $b = \max\{\gamma_i\}$. This follows from the definition of $b_{\mathfrak{m}}(p^e)$ and by Remark 4.1.

Proposition 4.5 ([Tuc12, Lemma 4.4])**.** *Let $(R, \mathfrak{m}, K)$ be either a standard graded K-algebra or a local ring. Assume that R is F-finite. If R is an F-pure ring, then $I_e^{[p]} \subseteq I_{e+1}$ for all $e \in \mathbb{N}$.*

We now present the main result of this section. Namely, we show that the F-pure threshold of an ideal is a limit of F-thresholds. For principal ideals, this follows from the characterization of the digits in the base p-expansion of $\operatorname{fpt}(\mathfrak{a})$ [Her12, Key Lemma].

Theorem 4.6. *Let $(R, \mathfrak{m}, K)$ be either a standard graded K-algebra or a local ring. Assume further that R is F-finite and F-pure. If $\mathfrak{a}$ is an ideal of R, homogeneous in case R is graded, then*

$$\operatorname{fpt}(\mathfrak{a}) = \lim_{e\to\infty} \frac{c^{I_e}(\mathfrak{a})}{p^e}.$$

Proof. By Propositions 4.5 and 3.6(a) we have that $c^{I_{e+1}}(\mathfrak{a}) \leq c^{I_e^{[p]}}(\mathfrak{a})$ for all $e \in \mathbb{N}$. In addition, $c^{I_e^{[p]}}(\mathfrak{a}) = p \cdot c^{I_e}(\mathfrak{a})$ by Proposition 3.6(b). Hence,

$$0 \leq \frac{c^{I_{e+1}}(\mathfrak{a})}{p^{e+1}} \leq \frac{c^{I_e}(\mathfrak{a})}{p^e},$$

for all $e \in \mathbb{N}$, which shows the sequence $\{\frac{c^{I_e}(\mathfrak{a})}{p^e}\}_{e\in\mathbb{N}}$ is non-increasing and bounded below by zero. As a consequence, it does converge to a limit as e approaches infinity.

Note that, for all $e \in \mathbb{N}$, we have $b_{\mathfrak{a}}(p^e) = \max\{t \in \mathbb{N} \mid \mathfrak{a}^t \not\subseteq I_e\} = \nu_{\mathfrak{a}}^{I_e}(p^0)$. Let $s \in \mathbb{N}$ be an arbitrary integer. By taking $e_1 = 0$ and $e_2 = s$ in Lemma 3.3, we deduce that

$$\frac{\nu_{\mathfrak{a}}^{I_e}(p^s)}{p^s} - b_{\mathfrak{a}}(p^e) \leq \mu(\mathfrak{a}).$$

In addition,

$$0 \leq \frac{\nu_{\mathfrak{a}}^{I_e}(p^s)}{p^s} - b_{\mathfrak{a}}(p^e) \leq \mu(\mathfrak{a}),$$

because the sequence $\left\{\frac{\nu_{\mathfrak{a}}^{I_e}(p^s)}{p^s}\right\}_{e\in\mathbb{N}}$ is non-decreasing, as R is F-pure. Hence

$$0 \leq \frac{\nu_{\mathfrak{a}}^{I_e}(p^s)}{p^s} - b_{\mathfrak{a}}(p^e) \leq \mu(\mathfrak{a})$$

for all $e, s \in \mathbb{N}$. We take the limit as $s \to \infty$ to get

$$0 \leq c^{I_e}(\mathfrak{a}) - b_{\mathfrak{a}}(p^e) \leq \mu(\mathfrak{a}),$$

and dividing this expression by p^e gives

$$0 \leq \frac{c^{I_e}(\mathfrak{a})}{p^e} - \frac{b_{\mathfrak{a}}(p^e)}{p^e} \leq \frac{\mu(\mathfrak{a})}{p^e}.$$

Taking the limit over e gives the result. $\square$

The previous result emulates a relation showed by Tucker [Tuc12] between the Hilbert-Kunz multiplicity [Mon83] and the F-signature [SVdB97, HL02, Tuc12]:

$$s(R) = \lim_{e\to\infty} \frac{\mathrm{e}_{HK}(I_e)}{p^{ed}}.$$

As a corollary, we obtain a characterization of rings $(R, \mathfrak{m}, K)$ for which $\operatorname{fpt}(\mathfrak{a}) = c^{\mathfrak{m}}(\mathfrak{a})$, for any ideal $\mathfrak{a} \subseteq R$. This gives a first answer to Question 1.1. We study in more detail the condition $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$, for standard graded algebras, in Section 6.

Corollary 4.7. *Let $(R, \mathfrak{m}, K)$ and $\mathfrak{a} \subseteq R$ be as in Theorem 4.6. Then,* $\operatorname{fpt}(\mathfrak{a}) = c^{\mathfrak{m}}(\mathfrak{a})$ *if and only if* $c^{I_e}(\mathfrak{a}) = c^{\mathfrak{m}^{[p^e]}}(\mathfrak{a})$ *for all integers* $e \in \mathbb{N}$.

Proof. We first assume that $c^{I_e}(\mathfrak{a}) = c^{\mathfrak{m}^{[p^e]}}(\mathfrak{a})$ for all $e \in \mathbb{N}$. Then, by Proposition 3.6(b), $c^{I_e}(\mathfrak{a}) = c^{\mathfrak{m}^{[p^e]}}(\mathfrak{a}) = p^e c^{\mathfrak{m}}(\mathfrak{a})$, from which we have

$$\operatorname{fpt}(\mathfrak{a}) = \lim_{e\to\infty} \frac{c^{I_e}(\mathfrak{a})}{p^e} = \lim_{e\to\infty} \frac{p^e c^{\mathfrak{m}}(\mathfrak{a})}{p^e} = c^{\mathfrak{m}}(\mathfrak{a}).$$

Assume now that $\operatorname{fpt}(\mathfrak{a}) = c^{\mathfrak{m}}(\mathfrak{a})$. Since $\left\{\frac{c^{I_e}(\mathfrak{a})}{p^e}\right\}_{e\in\mathbb{N}}$ forms a non-increasing sequence, with first term $c^{\mathfrak{m}}(\mathfrak{a})$ and limit $\operatorname{fpt}(\mathfrak{a})$, we have that the sequence is constant. Then, $\frac{c^{I_e}(\mathfrak{a})}{p^e} = \operatorname{fpt}(\mathfrak{a}) = c^{\mathfrak{m}}(\mathfrak{a})$ for all $e \in \mathbb{N}$, and thus $c^{I_e}(\mathfrak{a}) = p^e c^{\mathfrak{m}}(\mathfrak{a}) = c^{\mathfrak{m}^{[p^e]}}(\mathfrak{a})$. □

5. Limits of a-invariants for graded rings

In this section, we investigate the growth of the a-invariants of $R/J^{[p^e]}$ for a homogeneous ideal J over a standard graded algebra $(R, \mathfrak{m}, K)$. This study is motivated by the problem of bounding the Castelnuovo-Mumford regularity of $R/J^{[p^e]}$ [Kat98, KZ14] which, in turn, is related to the localization of tight closure at one element [Kat98], the LC condition [Hun00, HH00], and the discreteness of F-jumping numbers [KZ14].

Definition 5.1 ([GW78])**.** Let $(R, \mathfrak{m}, K)$ be a standard graded K-algebra. Let M be a non-zero $\frac{1}{p^e}\mathbb{N}$-graded R-module. If $H^i_{\mathfrak{m}}(M) \neq 0$, we define the i-th a-invariant of M by

$$a_i(M) = \sup\left\{ s \in \frac{1}{p^e}\mathbb{Z} \,\middle|\, \left[H^i_{\mathfrak{m}}(M)\right]_s \neq 0 \right\}.$$

If $H^i_{\mathfrak{m}}(M) = 0$, we set $a_i(M) = -\infty$. We define the Castelnuovo-Mumford regularity of M by $\operatorname{reg}(M) = \max\{a_i(M) + i \mid i \in \mathbb{N}\}$.

We first present lower bounds for a-invariants of F-pure rings modulo Frobenius powers of an ideal.

Lemma 5.2. *Let $(R, \mathfrak{m}, K)$ be a standard graded K-algebra that is F-finite and F-pure. If $J \subseteq R$ is a homogeneous ideal and $i \in \mathbb{N}$, then*

$$\frac{a_i(R/J^{[p^s]})}{p^s} + \frac{b_{\mathfrak{m}}(p^e)}{p^{e+s}} \leq \frac{a_i(R/J^{[p^{e+s}]})}{p^{e+s}}$$

for all $e, s \in \mathbb{N}$.

Proof. For every $e \in \mathbb{N}$, let $b_e = \frac{b_{\mathfrak{m}}(p^e)}{p^e}$. By Remark 5.5, there exists a $\frac{1}{p^e}\mathbb{N}$ graded R-module, M_e, such that $F^e_* R \cong R(-b_e) \oplus M_e$ as $\frac{1}{p^e}\mathbb{N}$ graded R-modules. Let s be a non-negative integer. Applying the functor $- \bigotimes_R R/J^{[p^s]}$ to the homogeneous split inclusion $R(-b_e) \hookrightarrow F^e_* R$, we obtain that $\frac{R}{J^{[p^s]}}(-b_e)$ is a direct summand of $\frac{F^e_* R}{J^{[p^s]} F^e_* R}$. Consequently, $H^i_{\mathfrak{m}}(\frac{R}{J^{[p^s]}}(-b_e))$ splits out of $H^i_{\mathfrak{m}}(\frac{F^e_* R}{J^{[p^s]} F^e_* R})$. Looking at graded components, we conclude that

$$a_i\left(\frac{R}{J^{[p^s]}}\right) + \frac{b_{\mathfrak{m}}(p^e)}{p^e} \leq a_i\left(\frac{F^e_* R}{J^{[p^s]} F^e_* R}\right) = \frac{a_i(R/J^{[p^{e+s}]})}{p^e}$$

for all $e, s \in \mathbb{N}$, where the last step follows from the fact that

$$\frac{F^e_* R}{J^{[p^s]} F^e_* R} \cong F^e_*\left(\frac{R}{J^{[p^{e+s}]}}\right).$$

Our claim follows after dividing by p^s. $\square$

Theorem 5.3. *Let $(R,\mathfrak{m},K)$ be a standard graded K-algebra that is F-finite and F-pure. Suppose that $J \subseteq R$ is a homogeneous ideal, and let $i \in \mathbb{N}$. If there exists a constant C such that $a_i(R/J^{[p^e]}) \leq Cp^e$ for all $e \in \mathbb{N}$, then either $H^i_{\mathfrak{m}}(R/J^{[p^s]}) = 0$ for all $s \in \mathbb{N}$ or*

$$\lim_{e\to\infty} \frac{a_i(R/J^{[p^e]})}{p^e}$$

exists. Furthermore, we have inequalities

$$a_i(R/J) + \operatorname{fpt}(\mathfrak{m}) \leq \max_{s\in\mathbb{N}} \left\{ \frac{a_i(R/J^{[p^s]})}{p^s} + \frac{\operatorname{fpt}(\mathfrak{m})}{p^s} \right\} \leq \lim_{e\to\infty} \frac{a_i(R/J^{[p^e]})}{p^e}.$$

Proof. Assume that $H^i_{\mathfrak{m}}(R/J^{[p^e]}) \neq 0$ for some $e \in \mathbb{N}$. By Lemma 5.2, for any $s \in \mathbb{N}$ we have

$$(1) \qquad \frac{a_i(R/J^{[p^s]})}{p^s} + \frac{\operatorname{fpt}(\mathfrak{m})}{p^s} \leq \liminf_{e\to\infty} \frac{a_i(R/J^{[p^{e+s}]})}{p^{e+s}} = \liminf_{e\to\infty} \frac{a_i(R/J^{[p^e]})}{p^e}.$$

In particular, since there exists $e \in \mathbb{N}$ for which $a_i(R/J^{[p^e]}) > -\infty$, we have that $\liminf\limits_{e\to\infty} \frac{a_i(R/J^{[p^e]})}{p^e}$ is finite. By assumption, we have that $a_i(R/J^{[p^e]}) \leq Cp^e$ for all $e \in \mathbb{N}$. Therefore, $\limsup\limits_{e\to\infty} \frac{a_i(R/J^{[p^e]})}{p^e}$ is also finite. Taking limsup with respect to s in (1) gives

$$\limsup_{s\to\infty} \frac{a_i(R/J^{[p^s]})}{p^s} \leq \liminf_{e\to\infty} \frac{a_i(R/J^{[p^e]})}{p^e},$$

which implies that the limit exists. The last claim now follows from the fact that

$$\frac{a_i(R/J^{[p^s]})}{p^s} + \frac{\operatorname{fpt}(\mathfrak{m})}{p^s} \leq \lim_{e\to\infty} \frac{a_i(R/J^{[p^{e+s}]})}{p^{e+s}} = \lim_{e\to\infty} \frac{a_i(R/J^{[p^e]})}{p^e}$$

for all $s \in \mathbb{N}$, by Lemma 5.2. $\square$

Theorem 5.4. *Let $(R,\mathfrak{m},K)$ be a standard graded K-algebra that is F-finite and F-pure. Suppose that $J \subseteq R$ is a homogeneous ideal. If there exists a constant C such that $\operatorname{reg}(R/J^{[p^e]}) \leq Cp^e$ for all $e \in \mathbb{N}$, then*

$$\lim_{e\to\infty} \frac{\operatorname{reg}(R/J^{[p^e]})}{p^e}$$

exists, and it is bounded below by $\max\{a_i(R/J) \mid i \in \mathbb{N}\} + \operatorname{fpt}(\mathfrak{m})$.

Proof. Let $\mathcal{I} = \{i \in \mathbb{N} \mid H^i_{\mathfrak{m}}(R/J^{[p^s]}) \neq 0 \text{ for some } s \in \mathbb{N}\}$. Note that, for all $e \in \mathbb{N}$, we have $\operatorname{reg}(R/J^{[p^e]}) = \max_{i\in\mathcal{I}}\{a_i(R/J^{[p^e]}) + i\}$. In addition, since $a_i(R/J^{[p^e]}) \leq \operatorname{reg}(R/J^{[p^e]}) \leq Cp^e$ for all $e \gg 0$, by Theorem 5.3 we obtain that

$\lim\limits_{e\to\infty} \frac{a_i(R/J^{[p^e]})}{p^e}$ exists for all $i \in \mathcal{I}$. We have that

$$\begin{aligned}\lim_{e\to\infty} \frac{\operatorname{reg}(R/J^{[p^e]})}{p^e} &= \lim_{e\to\infty} \frac{\max_{i\in\mathcal{I}}\{a_i(R/J^{[p^e]})+i\}}{p^e} \\ &= \lim_{e\to\infty} \max_{i\in\mathcal{I}} \left\{ \frac{a_i(R/J^{[p^e]})+i}{p^e} \right\} \\ &= \max_{i\in\mathcal{I}} \left\{ \lim_{e\to\infty} \frac{a_i(R/J^{[p^e]})+i}{p^e} \right\} \\ &= \max_{i\in\mathcal{I}} \left\{ \lim_{e\to\infty} \frac{a_i(R/J^{[p^e]})}{p^e} \right\}.\end{aligned}$$

Therefore $\lim_{e\to\infty} \frac{\operatorname{reg}(R/J^{[p^e]})}{p^e}$ exists. The claim about the lower bound follows from the inequality in Theorem 5.3. ☐

In Theorems 5.7 and 5.8 below, we recover linear upper bounds for $a_i(R/J^{[p^e]})$ when $i \geq \dim(R/J)-1$. This type of bound has already been discovered by Zhang [Zha15, Corollary 1.3]. However, our proof is more direct and does not make use of spectral sequences. In addition, we prove the existence of $\lim\limits_{e\to\infty} \frac{a_i(R/J^{[p^e]})}{p^e}$ for $i \geq \dim(R/J)-1$, and we obtain specific lower and upper bounds for the limits.

Remark 5.5 ([DSNB, Remark 4.8]). If $(R,\mathfrak{m},K)$ is a standard graded F-pure K-algebra, then $\frac{\nu_{\mathfrak{m}}^{\mathfrak{m}}(p^e)}{p^e}$ is the highest possible degree of a minimal homogeneous generator of F^e_*R, with the natural $\frac{1}{p^e}\mathbb{N}$ grading. More specifically, we have that

$$\frac{\nu_{\mathfrak{m}}^{\mathfrak{m}}(p^e)}{p^e} = \sup\left\{ s \in \frac{1}{p^e}\mathbb{N} \;\middle|\; \left[\frac{F^e_*R}{\mathfrak{m}F^e_*R}\right]_s \neq 0 \right\}.$$

Lemma 5.6. *Let $(R,\mathfrak{m},K)$ be an F-finite standard graded K-algebra. Let $J \subseteq R$ be a homogeneous ideal, and let $t = \dim(R/J)$. Then,*

$$\frac{a_t(R/J^{[p^e]})}{p^e} \leq a_t(R/J) + \frac{\nu_{\mathfrak{m}}^{\mathfrak{m}}(p^e)}{p^e}$$

for every $e \in \mathbb{N}$.

Proof. Let $u_1,\ldots,u_\ell$ be a minimal set of homogeneous generators for F^e_*R, with degrees $\gamma_1 \leq \cdots \leq \gamma_\ell$. We note that $\gamma_\ell = \frac{\nu_{\mathfrak{m}}^{\mathfrak{m}}(p^e)}{p^e}$ by Remark 5.5. Consider the homogeneous surjection $\bigoplus R(-\gamma_i) \twoheadrightarrow F^e_*R$ obtained from this minimal generating set. Since tensor product is right exact, we have a surjection $\bigoplus R/J(-\gamma_i) \twoheadrightarrow F^e_*R/JF^e_*R$ of R/J-modules that, in turn, induces a surjective map

$$\bigoplus H^t_{\mathfrak{m}}(R/J(-\gamma_i)) \twoheadrightarrow H^t_{\mathfrak{m}}(F^e_*R/JF^e_*R).$$

We note that $H^t_{\mathfrak{m}}(R/J) \neq 0$. Then,

$$\frac{a_t(R/J^{[p^e]})}{p^e} = a_t\left(\frac{F^e_*R}{JF^e_*R}\right) \leq a_t\left(R/J\right) + \gamma_\ell,$$

where the first step follows from the fact that $\dfrac{F^e_*R}{JF^e_*R} \cong F^e_*\left(\dfrac{R}{J^{[p^e]}}\right)$. ☐

Theorem 5.7. *Let $(R, \mathfrak{m}, K)$ be an F-finite standard graded K-algebra. Let $J \subseteq R$ be a homogeneous ideal, and let $t = \dim(R/J)$. Then, there exists a constant A such that $a_t(R/J^{[p^e]}) \leq Ap^e$ for all $e \in \mathbb{N}$. In addition, if R is F-pure, the limit*

$$\lim_{e\to\infty} \frac{a_t(R/J^{[p^e]})}{p^e}$$

exists. Moreover, we have that

$$a_t(R/J) + \operatorname{fpt}(\mathfrak{m}) \leq \lim_{e\to\infty} \frac{a_t(R/J^{[p^e]})}{p^e} \leq a_t(R/J) + c^{\mathfrak{m}}(\mathfrak{m}).$$

Proof. By Lemma 5.6, for all $e \in \mathbb{N}$ we have an inequality

$$\frac{a_t(R/J^{[p^e]})}{p^e} \leq a_t(R/J) + \frac{\nu_{\mathfrak{m}}^{\mathfrak{m}}(p^e)}{p^e}.$$

Since $\{\frac{\nu_{\mathfrak{m}}^{\mathfrak{m}}(p^e)}{p^e}\}_{e\in\mathbb{N}}$ converges by Theorem 3.4, the sequence is bounded. Let A' be any upper bound. If we let $A = A' + a_t(R/J)$, then $a_t(R/J^{[p^e]}) \leq Ap^e$ for all $e \in \mathbb{N}$, as desired. If R is assumed to be F-pure, then $\lim\limits_{e\to\infty} \frac{a_t(R/J^{[p^e]})}{p^e}$ exists by Theorem 5.3. In addition, Theorem 5.3 and Lemma 5.6 yield the following inequalities:

$$a_t(R/J) + \operatorname{fpt}(\mathfrak{m}) \leq \lim_{e\to\infty} \frac{a_t(R/J^{[p^e]})}{p^e} \leq a_t(R/J) + c^{\mathfrak{m}}(\mathfrak{m}).$$

$\square$

Theorem 5.8. *Let $(R, \mathfrak{m}, K)$ be an F-finite standard graded K-algebra. Let $J \subseteq R$ be a homogeneous ideal, and let $t = \dim(R/J)$. Then, there exists a constant B such that $a_{t-1}(R/J^{[p^e]}) \leq Bp^e$ for all $e \in \mathbb{N}$. In addition, if R is F-pure and $H^{t-1}_{\mathfrak{m}}(R/J^{[p^e]}) \neq 0$ for some $e \in \mathbb{N}$, then*

$$\lim_{e\to\infty} \frac{a_{t-1}(R/J^{[p^e]})}{p^e}$$

exists. Furthermore, we have that

$$a_{t-1}(R/J) + \operatorname{fpt}(\mathfrak{m}) \leq \lim_{e\to\infty} \frac{a_{t-1}(R/J^{[p^e]})}{p^e} \leq D(c^J(J) + 1),$$

where $D = \max\{t \in \mathbb{N} \mid [J/\mathfrak{m}J]_t \neq 0\} + 1$.

Proof. Since $\{\frac{\nu_J^J(p^e)}{p^e}\}_{e\in\mathbb{N}}$ converges by Theorem 3.4, there exists an integer B' such that $\nu_J^J(p^e) \leq B'p^e$ for all $e \in \mathbb{N}$, yielding inclusions $J^{B'p^e+1} \subseteq J^{\nu_J^J(p^e)+1} \subseteq J^{[p^e]}$. For each $e \in \mathbb{N}$, the short exact sequence

$$0 \longrightarrow J^{[p^e]}/J^{B'p^e+1} \longrightarrow R/J^{B'p^e+1} \longrightarrow R/J^{[p^e]} \longrightarrow 0$$

induces the following exact sequence in local cohomology:

$$H^{t-1}_{\mathfrak{m}}(R/J^{B'p^e+1}) \longrightarrow H^{t-1}_{\mathfrak{m}}(R/J^{[p^e]}) \longrightarrow H^{t}_{\mathfrak{m}}(J^{[p^e]}/J^{B'p^e+1}).$$

For ordinary powers of an ideal, an explicit linear upper bound for the Castelnuovo-Mumford regularity in terms of the degree of minimal generators is known [TW05, Theorem 3.2] (see also [CHT99, Kod00]). As a consequence of [TW05, Theorem 3.2], there exists a constant C such that $a_{t-1}(R/J^{B'p^e+1}) \leq D'(B'p^e + 1) + C$ for all $e \gg 0$, where $D' = D-1$ is the maximal degree of a minimal homogeneous generator of J . Since $B'p^e+1 \geq C$ for e large enough, we deduce that $a_{t-1}(R/J^{B'p^e+1}) \leq D(B'p^e+1)$ for all $e \gg 0$. For a given $e \in \mathbb{N}$, if $\dim(J^{[p^e]}/J^{B'p^e+1}) < t$, then $H^t_{\mathfrak{m}}(J^{[p^e]}/J^{B'p^e+1}) = 0$, and we obtain

$$a_{t-1}(R/J^{[p^e]}) \leq a_{t-1}(R/J^{B'p^e+1}) \leq D(B'p^e+1).$$

On the other hand, if $\dim(J^{[p^e]}/J^{B'p^e+1}) = t$ and $e \gg 0$, the short exact sequence above gives

$$\begin{aligned} a_{t-1}(R/J^{[p^e]}) &\leq \max\{a_{t-1}(R/J^{B'p^e+1}), a_t(J^{[p^e]}/J^{B'p^e+1})\} \\ &\leq \max\{D(B'p^e+1), a_t(J^{[p^e]}/J^{B'p^e+1})\}. \end{aligned} \tag{2}$$

We claim that the inequality $a_t(J^{[p^e]}/J^{B'p^e+1}) \leq D(B'+1)p^e+Dp^e$ holds true for all $e \gg 0$. Let $f_1, \ldots, f_s$ be minimal homogeneous generators of J, of degrees $d_1, \ldots, d_s$. Note that $D = \max\{d_j \mid j = 1, \ldots, s\} + 1$. In addition, note that $J^{[p^e]}/J^{B'p^e+1}$ is an $R/J^{B'p^e+1}$ module generated by the residue classes of $f_1^{p^e}, \ldots, f_s^{p^e}$. We have the following surjection, which is homogeneous of degree zero:

$$\bigoplus_{j=1}^{s} \frac{R}{J^{B'p^e+1}}(-d_jp^e) \twoheadrightarrow \frac{J^{[p^e]}}{J^{B'p^e+1}}.$$

Taking local cohomology we obtain a surjection

$$\bigoplus_{j=1}^{s} H^t_{\mathfrak{m}}\left(\frac{R}{J^{B'p^e+1}}(-d_jp^e)\right) \twoheadrightarrow H^t_{\mathfrak{m}}\left(\frac{J^{[p^e]}}{J^{B'p^e+1}}\right),$$

which, in turn, gives

$$\begin{aligned} a_t(J^{[p^e]}/J^{B'p^e+1}) &\leq \max\{a_t(R/J^{B'p^e+1}) + d_jp^e \mid 1 \leq j \leq s\} \\ &\leq D(B'p^e+1) + Dp^e, \end{aligned} \tag{3}$$

as claimed. Putting (2) and (3) together, we conclude that there exists $e_0 \in \mathbb{N}$ such that $a_{t-1}(R/J^{[p^e]}) \leq D(B'p^e+1) + Dp^e \leq D(B'+2)p^e$ for all $e > e_0$. Taking

$$B = \max\left\{D(B'+2), \frac{a_{t-1}(R/J^{[p^s]})}{p^s} \mid 0 \leq s \leq e_0\right\}$$

we finally obtain that $a_{t-1}(R/J^{[p^e]}) \leq Bp^e$ for all $e \in \mathbb{N}$, as desired.

Now assume that R is F-pure and $H^{t-1}_{\mathfrak{m}}(R/J^{[p^e]}) \neq 0$ for some $e \in \mathbb{N}$. Theorem 5.3 implies that $\lim\limits_{e\to\infty} \frac{a_{t-1}(R/J^{[p^e]})}{p^e}$ exists and gives the lower bound for the limit. For the upper bound, note that we can set $B' = c^J(J)$, because $c^J(J) \geq \frac{\nu_J^J(p^e)}{p^e}$ for all $e \in \mathbb{N}$ for F-pure rings. With this choice of B', combining (2) and (3), we deduce that $a_{t-1}(R/J^{[p^e]}) \leq D(c^J(J)p^e+1) + Dp^e = D(c^J(J)+1)p^e + D$ for all $e \gg 0$. This gives

$$\lim_{e\to\infty} \frac{a_{t-1}(R/J^{[p^e]})}{p^e} \leq \lim_{e\to\infty} \frac{D(c^J(J)+1)p^e + D}{p^e} = D(c^J(J)+1).$$

□

6. The equality $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$ for standard graded rings

In this section we prove that for a Gorenstein standard graded algebra $(R, \mathfrak{m}, K)$, the equality between $\operatorname{fpt}(\mathfrak{m})$ and the so-called diagonal F-threshold $c^{\mathfrak{m}}(\mathfrak{m})$ implies that R is strongly F-regular. Throughout this section we assume that $(R, \mathfrak{m}, K)$ is a standard graded ring. We start by making some observations about a-invariants for rings satisfying $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$.

Proposition 6.1. *Let $(R, \mathfrak{m}, K)$ be a standard graded d-dimensional K-algebra that is F-finite and F-pure. Let $J \subseteq R$ be a homogeneous ideal, and let $t = \dim(R/J)$. If $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$, then for all $e \in \mathbb{N}$ we have*

$$\frac{a_t(R/J^{[p^e]})}{p^e} = a_t(R/J) - a_d(R) + \frac{a_d(R)}{p^e}.$$

Proof. In our assumptions, $\operatorname{fpt}(\mathfrak{m}) = -a_d(R) = c^{\mathfrak{m}}(\mathfrak{m})$ [DSNB, Theorem B]. From Lemmas 5.2 and 5.6, for all homogeneous ideals $\mathfrak{a} \subseteq R$ we obtain that

$$\lim_{s\to\infty} \frac{a_{\dim(R/\mathfrak{a})}(R/\mathfrak{a}^{[p^s]})}{p^s} = a_{\dim(R/\mathfrak{a})}(R/\mathfrak{a}) - a_d(R).$$

In particular, choosing $\mathfrak{a} = J^{[p^e]}$ and dividing by p^e, this implies that

$$\frac{a_t(R/J^{[p^e]}) - a_d(R)}{p^e} = \lim_{s\to\infty} \frac{a_t(R/J^{[p^{(e+s)}]})}{p^{(e+s)}} = \lim_{s\to\infty} \frac{a_t(R/J^{[p^s]})}{p^s} = a_t(R/J) - a_d(R).$$

Hence, for every $e \in \mathbb{N}$, we have $\frac{a_t(R/J^{[p^e]})}{p^e} = a_t(R/J) - a_d(R) + \frac{a_d(R)}{p^e}$. $\square$

Remark 6.2. Suppose that $(R, \mathfrak{m}, K)$ is a standard graded K-algebra. If J is an $\mathfrak{m}$-primary homogeneous ideal, then for all $e \in \mathbb{N}$,

$$\nu_{\mathfrak{m}}^J(p^e) = \max\{s \in \mathbb{N} \mid \mathfrak{m}^s \not\subseteq J^{[p^e]}\} = \max\{s \in \mathbb{N} \mid \mathfrak{m}^s(R/J^{[p^e]}) \neq 0\} = a_0(R/J^{[p^e]}).$$

The following corollary gives a formula to compute $c^J(\mathfrak{m})$ in terms of certain a-invariants.

Corollary 6.3. *Let $(R, \mathfrak{m}, K)$ be a standard graded d-dimensional K-algebra that is F-finite and F-pure. Let $J \subseteq R$ be a homogeneous $\mathfrak{m}$-primary ideal. If $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$, then*

$$\frac{\nu_{\mathfrak{m}}^J(p^e)}{p^e} = a_0(R/J) - a_d(R) + \frac{a_d(R)}{p^e}.$$

In particular, $c^J(\mathfrak{m}) = a_0(R/J) - a_d(R)$.

Proof. This follows immediately from Proposition 6.1, Remark 6.2, and the fact that $\dim(R/J) = 0$. $\square$

We recall the definition of compatible ideals, which play an important role in showing that the equality $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$ implies that R is a domain for Gorenstein rings.

Definition 6.4 ([Sch10])**.** Let $(R, \mathfrak{m}, K)$ be a reduced F-finite standard graded K-algebra. An ideal $J \subseteq R$ is said to be *compatible* if $\varphi(F_*^e J) \subseteq J$ for all integers $e \geq 1$ and all R-homomorphisms $\varphi \in \operatorname{Hom}_R(F_*^e R, R)$.

Lemma 6.5. *Let $(R,\mathfrak{m},K)$ be a d-dimensional standard graded Gorenstein K-algebra that is F-finite and F-pure. Let $J \subseteq R$ be a homogeneous compatible ideal, $\mathfrak{n} = \mathfrak{m}(R/J)$, and $t = \dim(R/J)$. If $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$, then $\operatorname{fpt}(\mathfrak{m}) = \operatorname{fpt}(\mathfrak{n}) = c^{\mathfrak{n}}(\mathfrak{n}) = c^{\mathfrak{m}}(\mathfrak{m})$. In particular, $a_d(R) = a_t(R/J)$.*

Proof. We note that $\mathfrak{m}^r \subseteq \mathfrak{m}^{[p^e]}$ implies that $\mathfrak{n}^r = \mathfrak{m}^r(R/J) \subseteq \mathfrak{m}^{[p^e]}(R/J) = \mathfrak{n}^{[p^e]}$. As a consequence, we have that $c^{\mathfrak{n}}(\mathfrak{n}) \leq c^{\mathfrak{m}}(\mathfrak{m})$. We also know that $\operatorname{fpt}(\mathfrak{m}) \leq \operatorname{fpt}(\mathfrak{n})$ [DSNB, Theorem 4.7]. Since $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$, we obtain $\operatorname{fpt}(\mathfrak{m}) = \operatorname{fpt}(\mathfrak{n}) = c^{\mathfrak{n}}(\mathfrak{n}) = c^{\mathfrak{m}}(\mathfrak{m})$. The last statement follows from the fact that $\operatorname{fpt}(\mathfrak{m}) \leq -a_d(R) \leq c^{\mathfrak{m}}(\mathfrak{m})$ and $\operatorname{fpt}(\mathfrak{n}) \leq -a_t(R/J) \leq c^{\mathfrak{n}}(\mathfrak{n})$ [DSNB, Theorem B]. ☐

The following lemma is a key ingredient in the proof of Theorem 6.13.

Lemma 6.6. *Let $(R,\mathfrak{m},K)$ be a standard graded Gorenstein K-algebra that is F-finite and F-pure. If $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$, then R is a domain.*

Proof. We proceed by way of contradiction. Let $Q_1,\ldots,Q_\ell$ be the minimal primes of R. Since R is not a domain, $\ell \geq 2$. We set $J = Q_2 \cap \cdots \cap Q_\ell$, and we note that Q_1 and J are compatible ideals [Sch10, Corollary 4.8 and Lemma 3.5]. Furthermore, we have $d := \dim(R) = \dim(R/Q_1) = \dim(R/J)$, because R is a standard graded Cohen-Macaulay K-algebra; hence it is equidimensional. In addition, since R is F-pure, it is reduced. There is a short exact sequence

$$0 \to R \to R/Q_1 \oplus R/J \to R/(Q_1+J) \to 0,$$

which induces a long exact sequence on local cohomology:

$$\cdots \to H^{d-1}_{\mathfrak{m}}(R/(Q_1+J)) \to H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R/Q_1) \oplus H^d_{\mathfrak{m}}(R/J) \to 0.$$

We point out that $H^d_{\mathfrak{m}}(R/(Q_1+J)) = 0$ because $\dim R/(Q_1+J) \leq d-1$. Let $a = a_d(R)$. Then

$$\left[H^d_{\mathfrak{m}}(R)\right]_a \to \left[H^d_{\mathfrak{m}}(R/Q_1)\right]_a \oplus \left[H^d_{\mathfrak{m}}(R/J)\right]_a$$

is surjective, and thus

$$\begin{aligned} 1 &= \dim_K \left[H^d_{\mathfrak{m}}(R)\right]_a \text{ because } R \text{ is Gorenstein} \\ &\geq \dim_K \left[H^d_{\mathfrak{m}}(R/Q_1)\right]_a + \dim_K \left[H^d_{\mathfrak{m}}(R/J)\right]_a \\ &\geq 2 \text{ because } a = a_d(R/Q_1) = a_d(R/J) \text{ by Lemma 6.5.} \end{aligned}$$

Hence, we get a contradiction, and R must be a domain. ☐

The following lemma allows us to reduce to the case of an infinite coefficient field.

Lemma 6.7. *Let $(R,\mathfrak{m},K)$ be a standard graded d-dimensional Gorenstein K-algebra that is F-finite and F-pure. Let $\overline{K}$ be the algebraic closure of K and let $\overline{\mathfrak{m}}$ be the irrelevant maximal ideal of the ring $R \otimes_K \overline{K}$. Then, $R \otimes_K \overline{K}$ is also a Gorenstein F-pure ring, $\operatorname{fpt}(\mathfrak{m}) = \operatorname{fpt}(\overline{\mathfrak{m}})$, and $c^{\mathfrak{m}}(\mathfrak{m}) = c^{\overline{\mathfrak{m}}}(\overline{\mathfrak{m}})$.*

Proof. The map $R \to R \otimes_K \overline{K} =: \overline{R}$ is faithfully flat, and the irrelevant maximal ideal $\mathfrak{m}$ of R extends to the irrelevant maximal ideal $\overline{\mathfrak{m}}$ of $\overline{R}$ under such extension. It follows that $\overline{R}$ is Cohen-Macaulay of dimension d and has the same type as R. Hence $\overline{R}$ is a Gorenstein standard graded $\overline{K}$-algebra. Moreover, we have that $\overline{R}$ is an F-pure ring as consequence of Fedder's Criterion, Theorem 2.4, since colon ideals and non-containments are preserved under faithfully flat extensions. We also note that $a_d(R) = a_d(R \otimes_K \overline{K})$, because there is a graded isomorphism $H^d_{\mathfrak{m}}(R) \otimes_K \overline{K} \cong H^d_{\overline{\mathfrak{m}}}(R \otimes_K \overline{K})$. By these observations and the fact that R and $\overline{R}$ are Gorenstein standard graded K-algebras, we see that $\operatorname{fpt}(\mathfrak{m}) = \operatorname{fpt}(\overline{\mathfrak{m}})$, since in this case the F-pure thresholds coincide with the respective a-invariants [DSNB, Theorem B]. In addition, $\nu^{\mathfrak{m}}_{\mathfrak{m}}(p^e) = \nu^{\overline{\mathfrak{m}}}_{\overline{\mathfrak{m}}}(p^e)$ for all $e \in \mathbb{N}$; therefore $c^{\mathfrak{m}}(\mathfrak{m}) = c^{\overline{\mathfrak{m}}}(\overline{\mathfrak{m}})$. □

We now recall the definitions and concepts that are relevant towards presenting Theorem 6.13. We restrict ourselves to the standard graded setting, since this is the level of generality in which we work for the rest of the article. We refer the reader to [HH90, Hun13] for more general definitions and statements.

Definition 6.8 ([HH89a, AE05])**.** Let $(R, \mathfrak{m}, K)$ be an F-finite standard graded K-algebra. We say that R is strongly F-regular if, for all homogeneous elements $c \neq 0$, there exists $e \gg 0$ such that $c \notin I_e$. Equivalently, R is strongly F-regular if $\bigcap_{e \in \mathbb{N}} I_e = (0)$.

We point out that there are several characterizations of strong F-regularity. The original definition given by Hochster and Huneke is in terms of existence of splitting maps. The definition we give is equivalent in view of Remark 4.1. For the purposes of this article, it is helpful to recall an equivalent formulation in terms of the (big) test ideal $\tau(R)$. Namely, a ring R is strongly F-regular if and only if $\tau(R) = R$ [LS01, Theorem 7.1(5)].

Another characterization of strong F-regularity can be given in terms of the F-signature, which we now introduce formally in the graded setup.

Definition 6.9 ([SVdB97, HL02, Tuc12])**.** Let $(R, \mathfrak{m}, K)$ be a d-dimensional F-finite standard graded K-algebra. *The F-signature of R* is defined by

$$s(R) = \lim_{e \to \infty} \frac{\lambda(R/I_e)}{p^{ed}}.$$

One can show that $s(R)$ equals the F-signature $s(R_{\mathfrak{m}})$ of the local ring $R_{\mathfrak{m}}$. In addition, in our assumptions, $s(R)$ also coincides with the global F-signature of R [DSPY16a]. The F-signature is an important invariant for rings of positive characteristic. For example, R is regular if and only if $s(R) = 1$ [HL02, Corollary 16], and R is strongly F-regular if and only if $s(R) > 0$ [AL03, Theorem 0.2]. See also [DSPY16b, Theorem B] for a global version of these results.

Definition 6.10 ([Mon83])**.** Let $(R, \mathfrak{m}, K)$ be a d-dimensional standard graded K-algebra, and let J be an $\mathfrak{m}$-primary homogeneous ideal. The Hilbert-Kunz multiplicity of J is defined by

$$\mathrm{e}_{HK}(J) = \lim_{e \to \infty} \frac{\lambda(R/J^{[p^e]})}{p^{ed}}.$$

This invariant measures the singularities of a ring. For instance, R is regular if and only if it is formally unmixed and $\mathrm{e}_{HK}(\mathfrak{m}) = 1$ [WY00]. Furthermore, smaller values of $e_{HK}(\mathfrak{m})$ typically imply better properties of the ring [BE04, AE08].

Remark 6.11 ([HL02, Proof of Theorem 11]). If $(R, \mathfrak{m}, K)$ is a Gorenstein graded algebra, J is a homogeneous system of parameters, and $\mathfrak{a} = (J :_R \mathfrak{m})$, we have that $s(R) = \mathrm{e}_{HK}(J) - \mathrm{e}_{HK}(\mathfrak{a})$.

Definition 6.12. Let $(R,\mathfrak{m},K)$ be a standard graded K-algebra of dimension d. We denote by $e(R)$ the Hilbert Samuel multiplicity of the irrelevant maximal ideal $\mathfrak{m}$ in R, that is,

$$e(R) = \lim_{n\to\infty} \frac{d!}{n^d}\lambda(R/\mathfrak{m}^n).$$

When $(R,\mathfrak{m},K)$ is Cohen-Macaulay, $e(R) = e(J) = \lambda(R/J)$ for any homogeneous ideal J that is a minimal reduction of $\mathfrak{m}$.

We are now ready to present the main result of this section. For this, we need the preparatory results obtained in this section and, as a crucial tool, we invoke a characterization of tight closure and integral closure for parameter ideals in terms of F-thresholds [HMTW08, Section 3].

Theorem 6.13. *Let $(R,\mathfrak{m},K)$ be a d-dimensional standard graded Gorenstein K-algebra that is F-finite and F-pure. If $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$, then R is strongly F-regular. In addition,*

$$s(R) \geq \frac{\mathrm{e}(R)}{d!}.$$

Proof. Let $\overline{K}$ be the algebraic closure of K. If $R\otimes_K \overline{K}$ is strongly F-regular, then so is R [AE03, Corollary 3.8]. Furthermore, since the closed fiber of the extension $R \to R\otimes_K \overline{K}$ is regular, we have $s(R) \geq s(R\otimes_K \overline{K})$ [Yao06, Theorem 5.4]. In light of Lemma 6.7, we can assume that $K = \overline{K}$ and, in particular, that K is infinite. As a consequence, there exists a homogeneous ideal J, generated by a homogeneous system of parameters, such that $\overline{J} = \mathfrak{m}$. If we let $\mathfrak{a} = J :_R \mathfrak{m}$, then $a_0(R/\mathfrak{a}) \leq a_0(R/J) - 1$, and we have

$$\begin{aligned} c^{\mathfrak{a}}(J) &= c^{\mathfrak{a}}(\mathfrak{m}) \\ &= a_0(R/\mathfrak{a}) - a_d(R) \text{ by Corollary 6.3} \\ &\leq a_0(R/J) - a_d(R) - 1 \\ &= c^J(\mathfrak{m}) - 1. \end{aligned}$$

Let $T = \widehat{R_{\mathfrak{m}}}$. We note that T is a domain because the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(T) = \bigoplus_i (\mathfrak{m}T)^i/(\mathfrak{m}T)^{i+1}$ is isomorphic to R, which is a domain. In this case, $c^{\mathfrak{a}}(J) = c^{\mathfrak{a}T}(JT)$ and $c^J(\mathfrak{m}) = c^{JT}(\mathfrak{m}T)$, because $\mathfrak{a}$, J, and $\mathfrak{m}$ are $\mathfrak{m}$-primary ideals. Then, $c^{\mathfrak{a}T}(JT) = c^{\mathfrak{a}}(J) \leq c^J(\mathfrak{m}) - 1 = c^{JT}(\mathfrak{m}T) - 1 \leq d-1$ [HMTW08, Theorem 3.3], and it follows that $\mathfrak{a}T \not\subseteq (JT)^*$ [HMTW08, Corollary 3.2]. Since T is Gorenstein, this means that the socle of JT does not intersect $(JT)^*$, and we conclude that $JT = (JT)^*$. In addition, for a Gorenstein ring, a parameter ideal being tightly closed is an equivalent condition to being strongly F-regular [HH89b] (see also [Hun96]). Finally, since the test ideal commutes with localization and completion for Gorenstein rings [LS01, Theorem 7.1], we have that $T = \tau(T) = \tau(R)T$. Because the test ideal is a homogeneous ideal [LS99, Lemma 4.2], we obtain that $\tau(R) = R$; hence R is a strongly F-regular ring.

We now focus on proving the inequality involving the F-signature. With the same reductions and the same notation introduced in the first part of the proof, recall that $c^{\mathfrak{a}}(J) \leq d-1$. In addition, since R is F-pure, note that $\frac{\nu_J^{\mathfrak{a}}(p^e)}{p^e} \leq c^{\mathfrak{a}}(J)$ for all $e \in \mathbb{N}$. Then, for all non-negative integers e, we have a series of containments

$$J^{p^e(d-1)+1} \subseteq J^{p^e c^{\mathfrak{a}}(J)+1} \subseteq J^{\nu_J^{\mathfrak{a}}(p^e)+1} \subseteq \mathfrak{a}^{[p^e]},$$

and therefore we get $\lambda((J^{p^e(d-1)+1}+J^{[p^e]})/J^{[p^e]}) \leq \lambda(\mathfrak{a}^{[p^e]}/J^{[p^e]})$. Consider the set

$$\mathscr{A}_e = \{(\alpha_1,\ldots,\alpha_d)\in\mathbb{N}^d \mid \alpha_i \leq p^e-1 \text{ for all } i \quad \text{and} \quad \alpha_1+\cdots+\alpha_d \geq p^e(d-1)+1\}.$$

Since R is Cohen-Macaulay and J is a parameter ideal, J is generated by a regular sequence $\underline{f} = f_1,\ldots,f_d$. The monomials in $\underline{f}$ with exponents in $\mathscr{A}_e$ induce a filtration on $(J^{p^e(d-1)+1}+J^{[p^e]})/J^{[p^e]}$:

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{|\mathscr{A}_e|} = (J^{p^e(d-1)+1}+J^{[p^e]})/J^{[p^e]},$$

with the property that $N_{i+1}/N_i \cong R/J$ for all $0 \leq i \leq |\mathscr{A}_e|-1$. As a consequence, for all $e\in\mathbb{N}$ we see that $\lambda((J^{p^e(d-1)+1}+J^{[p^e]})/J^{[p^e]}) = \lambda(R/J)|\mathscr{A}_e|$. We note that the set

$$\mathscr{B}_e = \{(\beta_1,\ldots,\beta_d)\in\mathbb{N}^d \mid \beta_1+\cdots+\beta_d \leq p^e-d-1\}$$

is in bijective correspondence with $\mathscr{A}_e$ via the function $\varphi : \mathscr{A}_e \to \mathscr{B}_e$, given by $\varphi((\alpha_1,\ldots,\alpha_d)) = (p^e-1-\alpha_1,\ldots,p^e-1-\alpha_d)$. Since $|\mathscr{A}_e| = |\mathscr{B}_e| = \binom{p^e-1}{d}$, we obtain the following relations:

$$\begin{aligned}
s(R) &= \mathrm{e}_{HK}(\mathfrak{a}) - \mathrm{e}_{HK}(J) \text{ by Remark 6.11}\\
&= \lim_{e\to\infty}\frac{\lambda(R/\mathfrak{a}^{[p^e]})}{p^{ed}} - \lim_{e\to\infty}\frac{\lambda(R/J^{[p^e]})}{p^{ed}}\\
&= \lim_{e\to\infty}\frac{\lambda(\mathfrak{a}^{[p^e]}/J^{[p^e]})}{p^{ed}} \text{ because } J\subseteq\mathfrak{a}\\
&\geq \lim_{e\to\infty}\frac{\lambda((J^{p^e(d-1)+1}+J^{[p^e]})/J^{[p^e]})}{p^{ed}}\\
&= \lim_{e\to\infty}\frac{\lambda(R/J)|\mathscr{A}_e|}{p^{ed}}\\
&= \lambda(R/J)\lim_{e\to\infty}\frac{\binom{p^e-1}{d}}{p^{ed}}\\
&= e(R)\lim_{e\to\infty}\frac{\binom{p^e-1}{d}}{p^{ed}} \text{ because } \overline{J}=\mathfrak{m}\\
&= \mathrm{e}(R)\frac{1}{d!}.
\end{aligned}$$

□

Example 6.14. Let K be a perfect field of prime characteristic p, let $n \geq 2$ be an integer, and let $S = K[x_{ij} \mid 1 \leq i,j \leq n]$. Consider the ideal $I_2(X) \subseteq X$ generated by the 2×2 minors of the matrix $X = (x_{ij})_{1\leq i,j\leq n}$. The ring $R = S/I_2(X)$ is Gorenstein of dimension $d = 2n-1$, and the a-invariant $a := a_d(R)$ is equal to $-n$. Let $\mathfrak{m}$ be the irrelevant maximal ideal of R. Since R is Gorenstein, we have that $\mathrm{fpt}(\mathfrak{m}) = -a = n$. The ring R can be viewed as a Segre product $K[X_1,\ldots,X_n] \# K[Y_1,\ldots,Y_n]$, and then, by [HWY14, Example 6.2], we have that $c^{\mathfrak{m}}(\mathfrak{m}) = n$. Since R is Gorenstein and $\mathrm{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$, Theorem 6.13 shows that R is strongly F-regular. Even though this was already known, because R is a free summand of a polynomial ring [HH89a, Theorem 3.1 (e)], Theorem 6.13 gives an alternative proof.

Remark 6.15. We think that the inequality proved in Theorem 6.13 may not provide very meaningful bounds for the Hilbert-Samuel multiplicity. In fact, there are known bounds for multiplicities for F-pure and F-rational rings [HW15], which are better in several examples. However, Theorem 6.13 is helpful to find lower bounds for the F-signature of rings as shown in Example 6.18.

Using recent results of Singh, Takagi, and Varbaro [STV17], we can extend Theorem 6.13 to a more general setting. For a standard graded F-pure normal ring $(R, \mathfrak{m}, K)$, let $X = \operatorname{Spec}(R)$, and let K_X be the canonical divisor on X. The *anti-canonical cover of* X is defined as $\mathscr{R} = \bigoplus_{n\geq 0} \mathcal{O}_X(-nK_X)$, and it is a very important object of study. It is known to be Noetherian in certain cases, which include the class of $\mathbb{Q}$-Gorenstein rings, semigroup rings, and determinantal rings. Motivated by recent results on the minimal model program [BCHM10,HM10,HX15], it is expected that $\mathscr{R}$ is Noetherian when R is strongly F-regular (see also [Kol10, Theorem 92]).

Corollary 6.16. *Let $(R, \mathfrak{m}, K)$ be a d-dimensional normal standard graded K-algebra that is Cohen-Macaulay, F-finite, and F-pure. Assume that the anti-canonical cover of R is Noetherian. If $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$, then R is Gorenstein and strongly F-regular. Furthermore, $s(R) \geq \frac{\mathrm{e}(R)}{d!}$.*

Proof. The equality $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$ forces $\operatorname{fpt}(\mathfrak{m}) = -a_d(R)$ [DSNB, Theorem B], and the latter implies that R is quasi-Gorenstein [STV17, Theorem A]. Since R is assumed to be Cohen-Macaulay, R is Gorenstein. The rest of the corollary now follows from Theorem 6.13. □

We present a class of standard graded Gorenstein rings $(R, \mathfrak{m}, K)$ that satisfy the equality $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$. In addition, we use Theorem 6.13 to find a lower bound for the F-signature. This example is possible thanks to recent computations of top socle degrees for diagonal hypersurfaces [Vra15].

Remark 6.17. Let $S = K[x_1, \ldots, x_n]$ be a polynomial over an F-finite field of positive characteristic. Let $f = x_1^b + \cdots + x_n^b$, where $b \in \mathbb{N}$, and set $R = S/fS$, with maximal ideal $\mathfrak{m}$. Suppose that $\min\{p, n\} > b$, and set $\kappa = \left\lfloor \frac{p}{b} \right\rfloor$. If $\left\lceil \frac{n\kappa - n}{2} \right\rceil \frac{1}{p} \geq 1$, then $c^{\mathfrak{m}}(\mathfrak{m}) = n - b$ [Vra15, Theorem 4.2].

Example 6.18. Let $S = K[x_1, \ldots, x_n]$ be a polynomial over an F-finite field of positive characteristic, and let $\mathfrak{m} = (x_1, \ldots, x_n)$. Let $f = x_1^b + \cdots + x_n^b$, where $b \in \mathbb{N}$, and let $R = S/fS$. Suppose that $p \equiv 1 \bmod b$, $p \geq 2(b+1)$, and $n \geq 4b$. Let $\kappa = \left\lfloor \frac{p}{b} \right\rfloor$, and note that $\kappa = \frac{p-1}{b}$. We have that

$$\left\lceil \frac{n\kappa - n}{2} \right\rceil \frac{1}{p} \geq \frac{n\kappa - n}{2} \cdot \frac{1}{p} = \frac{n}{2} \cdot \frac{\kappa - 1}{p} = \frac{n}{2} \cdot \frac{p - b - 1}{pb} = \frac{n}{2} \cdot \left(\frac{1}{b} - \frac{b+1}{pb} \right) \geq \frac{n}{2} \cdot \frac{1}{2b} \geq 1.$$

It follows that $c^{\mathfrak{m}}(\mathfrak{m}) = n - b$, by Remark 6.17. In addition, the top a-invariant of R is $a_{n-1}(R) = b - n$. In order to show that $\operatorname{fpt}(\mathfrak{m}) = n - b$, we only need to prove that R is an F-pure ring [DSNB, Theorem B], because R is Gorenstein. By Fedder's Criterion, Theorem 2.4, it suffices to show that $f^{p-1} \notin \mathfrak{m}^{[p]}$. We note that $x_1^{b\kappa} \cdots x_n^{b\kappa} = (x_1 \cdots x_n)^{p-1} \notin \mathfrak{m}^{[p]}$. Since $n\kappa \geq b\kappa = p - 1$, there exist $\gamma_1, \ldots, \gamma_n$ such that $0 \leq \gamma_i \leq \kappa$ and $\gamma_1 + \cdots + \gamma_n = p - 1$. Therefore, $x^{b\gamma_1} \cdots x_n^{b\gamma_n}$ is a monomial appearing with non-zero coefficient in f^{p-1}, and it does not belong to $\mathfrak{m}^{[p]}$. Hence, R is F-pure, and $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$. In addition, since $e(R) = b$, we conclude that $\frac{b}{(n-1)!} \leq s(R)$ by Theorem 6.13.

We conclude this article with one question motivated by Theorem B.

Question 6.19. Let $(R, \mathfrak{m}, K)$ be a standard graded K-algebra that is F-finite and F-pure. Does the equality $\operatorname{fpt}(\mathfrak{m}) = c^{\mathfrak{m}}(\mathfrak{m})$ imply that R is F-rational?

Acknowledgments

We thank Craig Huneke for helpful discussions. We also thank Daniel Hernández for inspiring conversations, in particular regarding Theorem 4.6. We thank the anonymous referee for suggesting several improvements to the contents and the exposition of this article.

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