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$\mathbb{A}^1$-EQUIVALENCE OF ZERO CYCLES ON SURFACES

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ABSTRACT. In this paper, we study $\mathbb{A}^1$-equivalence classes of zero cycles on open algebraic surfaces. We prove the logarithmic version of Mumford's theorem on zero cycles. We also prove that the log Bloch conjecture holds for surfaces with log Kodaira dimension $-\infty$.

1. INTRODUCTION

Let X be a smooth projective complex surface. Understanding the structure of the Chow group of zero cycles of degree zero $\mathrm{CH}_0(X)^0$ is important but difficult. Mumford first studied this group and proved the following theorem.

Theorem 1.1 ([Mum68]). *If $h^0(X,\Omega^2_X)>0$, the group $\mathrm{CH}_0(X)^0$ is infinite-dimensional.*

In the other direction, we have Bloch's conjecture as below.

Conjecture 1.2 ([Blo80]). *If $h^0(X,\Omega^2_X)=0$, then the Albanese morphism induces an isomorphism*

$$\mathrm{CH}_0(X)^0 \cong \mathrm{Alb}(X).$$

Bloch's conjecture has been proved for smooth projective surfaces with Kodaira dimension less than two [BKL76]. For surfaces of general type, many cases have been proved, but it is still widely open in general [Voi03, Chapter 11].

For not necessarily proper varieties, Spieß and Szamuely [SS03] observe that the right replacement for a Chow group of zero cycles is Suslin's 0-th algebraic singular homology $h_0(U)^0$ and furthermore they prove the log Roitmann's theorem for smooth quasiprojective varieties in all dimensions.

Definition 1.3. Let U be a smooth quasiprojective variety. Two zero cycles A_1, A_2 of degree n are $\mathbb{A}^1$-*equivalent* if there exists a zero cycle B of degree m such that

- A_1+B and A_2+B are effective;
- there exists a morphism $z:\mathbb{A}^1\to \mathrm{Sym}^{n+m} U$ such that $z(0)=A_1+B$, $z(1)=A_2+B$.

Definition 1.4. Suslin's zeroth homology $h_0(U)^0$ is the group of all zero cycles on U of degree 0 modulo $\mathbb{A}^1$-equivalences.

When U is a curve, $\mathbb{A}^1$-equivalence is indeed the equivalence relation of divisors defined by the modulus D as in [Ser88, V.2].

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Theorem 1.5 ([SS03, Theorem 1.1]). *Given a smooth quasiprojective variety U, the Albanese morphism*

$$alb : h_0(U)^0 \to \mathrm{Alb}(U)$$

induces an isomorphism on the torsion subgroups.

It is natural to consider Mumford's theorem for smooth quasiprojective surfaces. Consider the map

$$\sigma_d : \mathrm{Sym}^d(U) \times \mathrm{Sym}^d(U) \to h_0(U)^0,$$
$$(Z_1, Z_2) \mapsto [Z_1] - [Z_2].$$

By Lemma 3.4 below, the fiber of this map is a countable union of constructible sets. Thus we can define a dimension c_d of the general fiber of σ_d and set $\dim Im(\sigma_d) = 2d \dim U - c_d$.

Definition 1.6. We say that $h_0(U)^0$ is *infinite-dimensional* if

$$\lim_{d\to\infty} \dim Im(\sigma_d) = \infty.$$

In this paper, using logarithmic algebraic geometry, we prove the log Mumford theorem.

Theorem 1.7 (Log Mumford theorem). *Let (X, D) be a log smooth proper surface pair, and let U be its interior. If $h^0(X, \Omega^2_X(\log D)) > 0$, then the group $h_0(U)^0$ is infinite-dimensional.*

This is proved in Corollary 3.6. Our proof follows the strategy as in [Mum68] and the crucial part of the proof is the existence of induced log forms in Proposition 2.3.

Since the set of $\mathbb{A}^1$-equivalence classes of divisors on open curves is the generalized Jacobian [Ser88], we may formulate the analogue of Bloch's conjecture in the logarithmic setting.

Conjecture 1.8 (Log Bloch conjecture). *Let (X, D) be a log smooth proper surface pair, and let U be its interior. If $h^0(X, \Omega^2_X(\log D)) = 0$, then the Albanese morphism induces an isomorphism*

$$h_0(U)^0 \cong \mathrm{Alb}(U).$$

We prove a special case of log Bloch's conjecture as below.

Theorem 1.9. *The log Bloch's conjecture holds for log smooth surface pairs with log Kodaira dimension $-\infty$.*

In arbitrary dimension, if (X, D) is log rationally connected, introduced in [CZ14b, CZ14a, Zhu16], then we have the vanishing $h^0(X, \Omega_X^{\otimes m}(\log D)) = 0$ for any m. In this case, we prove that $h_0(U)^0$ vanishes as well. See Proposition 4.3. However, this is too weak to prove Theorem 1.9. There exists an $\mathbb{A}^1$-ruled surface pair with $q(X, D) = \mathrm{Alb}(U) = 0$ but not log rationally connected [Zhu16, Section 4].

Notation 1.10. In this paper, we work with (log) varieties and log pairs over complex numbers $\mathbb{C}$. We refer to [Kat89] or [Gro11, Ch. 3] for basic notions in log geometry. For any log scheme $(X, \mathcal{M}_X)$, we denote by X° the open subset with the trivial log structure and denote by $\Omega^q(X, \mathcal{M}_X)$ the sheaf of log q-forms. A *log rational curve* on a log variety $(X, \mathcal{M}_X)$ is a log morphism

$$f : (\mathbb{P}^1, \mathcal{M}_{\{\infty\}}) \to (X, \mathcal{M}_X),$$

where $\mathcal{M}_{\{\infty\}}$ is the divisorial log stricture associated to $\{\infty\}$ on $\mathbb{P}^1$.

A *log pair* (X, D) means a variety X with a reduced Weil divisor D. Let U be its interior $X - D$. We say that (X, D) is *log smooth* if X is smooth and D is a normal crossing divisor. A log pair is *proper* if the ambient variety is proper. For a log smooth pair (X, D), we use $\kappa(X, D)$ to denote the *logarithmic Kodaira dimension* and define the *log irregularity* $q(X, D) := h^0(X, \Omega^1_X(\log D))$. Since they only depend on the interior U, we may write $\kappa(U)$ and $q(U)$ as well.

2. Induced log differentials

Throughout this section, we let G be the symmetric group S_n and let $(X, \mathcal{M}_X)$ be a log smooth variety over $\mathbb{C}$. Let D be the boundary divisor $X - X^\circ$. By log smoothness, $\mathcal{M}_X$ is a divisorial log structure

$$\mathcal{M}_X = \{f \in \mathcal{O}_X | f \in \mathcal{O}^*_{X-D}\} \subset \mathcal{O}_X.$$

Let $(X^n, \mathcal{M}_{X^n})$ be the product log structure. Then $\mathcal{M}_{X^n}$ is G-invariant.

Consider the quotient map:

$$\pi : X^n \to Y := X^n/G.$$

Lemma 2.1. *Let $\mathcal{M}_Y$ be the G-invariant subsheaf $\mathcal{M}^G_{X^n}$. Then $(Y, \mathcal{M}_Y)$ is a log variety and*

$$\pi : (X^n, \mathcal{M}_{X^n}) \to (Y, \mathcal{M}_Y)$$

is a log morphism.

Proof. Since $(X^n, \mathcal{M}_{X^n})$ is a log scheme, we have

$$\mathcal{O}^*_{X^n} \subset \mathcal{M}_{X^n} \subset \mathcal{O}_X.$$

By taking the G-invariant part, we get

$$(\mathcal{O}^*_{X^n})^G \subset \mathcal{M}_Y \subset \mathcal{O}_Y.$$

Since the first term is indeed $\mathcal{O}^*_Y$, we conclude that Y is a log scheme.

The natural diagram

$$\begin{array}{ccc} \mathcal{M}_Y & \longrightarrow & \mathcal{O}_Y \\ \downarrow & & \downarrow \pi^* \\ \mathcal{M}_{X^n} & \longrightarrow & \mathcal{O}_X, \end{array}$$

where all arrows are inclusions, shows that π is a log morphism. □

Lemma 2.2. *The log variety $(Y, \mathcal{M}_Y)$ is fine and saturated.*

Proof. We know that étale locally on X, there exists a fine and saturated chart

$$P \to \mathcal{O}_X.$$

Furthermore, by choosing the defining equations of the irreducible components of D, we may assume the chart morphism factors as below:

$$P \subset \mathcal{M}_X \subset \mathcal{O}_X,$$

and $\mathcal{M}_X$ is isomorphic to $P \oplus \mathcal{O}_X^*$. This induces a G-invariant fs chart

$$P^n \to \mathcal{O}_{X^n}$$

for $(X^n, \mathcal{M}_{X^n})$ such that

$$\mathcal{M}_{X^n} \cong P^n \oplus \mathcal{O}_{X^n}^*.$$

Now taking the G-invariant part, we get

$$\mathcal{M}_Y \cong P \oplus \mathcal{O}_Y^*,$$

and actually P maps to the defining equations of the boundary divisors on Y. Therefore, $(Y, \mathcal{M}_Y)$ is a fine saturated log scheme. □

For any log smooth variety $(S, \mathcal{M}_S)$ with a morphism $f : (S, \mathcal{M}_S) \to (Y, \mathcal{M}_Y)$, let $S' = (S \times_Y X^n)$ be the fibered product with the fs log structure $\mathcal{M}_{S'}$; cf. [Ogu06, II.2.4]. Let $\widetilde{S} = (S \times_Y X^n)_{red}$ with the induced log structure from S'. We have a diagram as below:

$$\begin{array}{ccccc}
(\widetilde{S}, \mathcal{M}_{\widetilde{S}}) & \xrightarrow{\ i\ } & (S', \mathcal{M}_{S'}) & \xrightarrow{\ \widetilde{f}\ } & (X^n, \mathcal{M}_{X^n}) \\
& \searrow^{p} & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle \pi} \\
& & (S, \mathcal{M}_S) & \xrightarrow{\ f\ } & (Y, \mathcal{M}_Y).
\end{array}$$

Given a G-invariant log q-form $\omega \in \Gamma(X^n, \Omega^q(X^n, \mathcal{M}_{X^n}))$, let

$$\widetilde{\omega} = (\widetilde{f} \circ i)^*(\omega) \in \Gamma(\widetilde{S}, \Omega^q(\widetilde{S}, \mathcal{M}_{\widetilde{S}})).$$

Then $\widetilde{\omega}$ is G-invariant.

Proposition 2.3. *If S is log smooth, there exists a unique log q-form*

$$\eta_f \in \Gamma(S, \Omega^q(S, \mathcal{M}_S))$$

such that

$$p^*(\eta_f) - \widetilde{\omega} \textit{ is torsion in } \Omega^q(\widetilde{S}, \mathcal{M}_{\widetilde{S}}).$$

Remark 2.4. When S has the trivial log structure, this construction of η_f coincides with the construction in [Mum68, Section 1].

Proof. First we prove the uniqueness. Indeed, there are non-singular open dense subsets $S_0 \subset S$, $\widetilde{S_0} = p^{-1}(S_0) \subset \widetilde{S}$ with trivial log structures such that $S_0 = \widetilde{S_0}/\Gamma$ and Γ acts freely on $\widetilde{S_0}$, where Γ is a quotient group of the stabilizer group of the open subset $\widetilde{S_0}$. Thus $\widetilde{\omega}|_{\widetilde{S_0}}$ as a regular form descends to a regular form θ on S_0. By the condition in the lemma, η_f coincides with θ over S_0, thus is unique.

Let η_f be the meromorphic form extending θ on S. To prove the existence, it suffices to check that η_f as a meromorphic section of $\Omega^q(S, \mathcal{M}_S)$ is regular everywhere. Since $(S, \mathcal{M}_S)$ is log smooth, hence S is normal, it suffices to check this at points of codimension one. Hence we may assume that S is the spectrum of a local discrete valuation ring R with the fraction field K. Let T be the normalization of $\widetilde{S}$ and consider the normalization morphism

$$a : T \to \widetilde{S}.$$

The morphism p' is finite, so is the composite morphism

$$p \circ a : T \to S.$$

In particular, T is a disjoint union of local discrete valuation ring $T_i = \operatorname{Spec} R_i$ with the generic point $\operatorname{Spec} K_i$. The log structure on T is given canonically below.

Lemma 2.5. *There exists a canonical fs log structure on T by choosing*

$$\mathcal{M}_{T_i} = R_i - 0 \subset \mathcal{O}_{T_i} = R_i.$$

In particular, $(T, \mathcal{M}_T)$ is log smooth. □

Lemma 2.6. *The morphism $a : T \to \widetilde{S}$ extends to a unique log morphism:*

$$a : (T, \mathcal{M}_T) \to (\widetilde{S}, \mathcal{M}_{\widetilde{S}}).$$

Proof. We may assume that both T and $\widetilde{S}$ are irreducible. Since $(S', \mathcal{M}_{S'})$ is fine and saturated, there exists an fs chart

$$c : P \to \mathcal{O}_{S'}.$$

To show that a is a log morphism, it suffices to prove the image of the composite morphism

$$P \to \mathcal{O}_{S'} \to i_*\mathcal{O}_{\widetilde{S}} \to (i \circ a)_*\mathcal{O}_T$$

does not contain zero. Since a is the normalization map, it is enough to show the image of P in $\mathcal{O}_{\widetilde{S}}$ does not contain zero, or equivalently, none of the images of P in $\mathcal{O}_{S'}$ is nilpotent.

If there exists $p \in P$ such that $c(P)$ is nilpotent, then consider the base change of $c(p) \otimes_{\mathcal{O}_S} K$ via the following diagram is still nilpotent:

$$\begin{array}{ccc} \mathcal{O}_{S'} \otimes_R K & \longleftarrow & \mathcal{O}_{S'} \\ \uparrow & & \uparrow{\scriptstyle p'^*} \\ K & \longleftarrow & \mathcal{O}_S = R. \end{array}$$

Indeed, we have that $\mathcal{O}_{S'}$ is a flat $\mathcal{O}_S$-module and $\mathcal{O}_S$ is a principal ideal domain. Thus $\mathcal{O}_{S'}$ is torsion free. In particular, the nilpotent elements cannot be killed after tensoring with K.

This tells us the log structure on S' is non-trivial over $\operatorname{Spec} K$. On the other hand, since S° is non-empty, we have a log morphism

$$(\operatorname{Spec} K,\ \text{trivial log structure}) \to (S, \mathcal{M}_S)$$

which induces a Cartesian diagram

$$\begin{array}{ccc} S' \otimes_S \operatorname{Spec} K & \longrightarrow & (X^n, \mathcal{M}_{X^n}) \\ \downarrow & & \downarrow \\ (\operatorname{Spec} K,\ \text{trivial log structure}) & \longrightarrow & (Y, \mathcal{M}_Y). \end{array}$$

By the universal property of log fibered product, $S' \otimes_S \operatorname{Spec} K$ must have the trivial log structure. This is a contradiction. □

Now let us return to the proof of Proposition 2.3. We construct a diagram

$$\begin{array}{ccccccc} (T, \mathcal{M}_T) & \xrightarrow{a} & (\widetilde{S}, \mathcal{M}_{\widetilde{S}}) & \xrightarrow{i} & (S', \mathcal{M}_{S'}) & \xrightarrow{\widetilde{f}} & (X^n, \mathcal{M}_{X^n}) \\ & {\scriptstyle r = p\circ a}\searrow & & {\scriptstyle p}\searrow & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle \pi} \\ & & & & (S, \mathcal{M}_S) & \xrightarrow{f} & (Y, \mathcal{M}_Y) \end{array}$$

such that

- $(T, \mathcal{M}_T)$ is log smooth;
- r is finite.

Since $p^*(\eta_f) - \widetilde{\omega}$ is torsion and $(T, \mathcal{M}_T)$ is log smooth, we have

$$r^*(\eta_f) = a^*(p^*(\eta_f)) = a^*(\widetilde{\omega}).$$

Since ω as an element in $\Gamma(X^n, \Omega^1(X^n, \mathcal{M}_{X^n}))$ is regular,

$$r^*(\eta_f) = a^*(\widetilde{\omega}) = (\widetilde{f} \circ i \circ a)^* \omega$$

is a regular as an element in $\Gamma(T, \Omega^q(T, \mathcal{M}_T))$. Now the proposition is proved using the following lemma. □

Lemma 2.7. *There is a well-defined trace map*

$$tr : \Omega^1(T, \mathcal{M}_T) \to \Omega^1(S, \mathcal{M}_S)$$

such that the composite

$$\Omega^1(S, \mathcal{M}_S) \xrightarrow{r^*} \Omega^1(T, \mathcal{M}_T) \xrightarrow{tr} \Omega^1(S, \mathcal{M}_S)$$

is multiplication by the degree of r.

Proof. By construction, if $(S, \mathcal{M}_S)$ has the trivial log structure, so does $(T, \mathcal{M}_T)$. Now we can simply use the standard trace map; cf., Mumford's paper [Mum68]. From now on, we assume $(S, \mathcal{M}_S)$ has non-trivial log structure, and so does $(T, \mathcal{M}_T)$. Furthermore, since S is the spec of a local ring and log smooth, the log structure $\mathcal{M}_S$ is the canonical one as in Lemma 2.5. Let $\mathfrak{m}_S$, $\mathfrak{m}_T$ be the maximal ideals, respectively.

We claim that the morphism

$$r : (T, \mathcal{M}_T) \to (S, \mathcal{M}_S)$$

is log étale. Consider the commutative diagram given by the charts

$$\begin{array}{ccc} (T, \mathcal{M}_T) & \longrightarrow & \mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[\mathbb{N}] \\ \downarrow & & \downarrow u \\ (S, \mathcal{M}_S) & \longrightarrow & \mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[\mathbb{N}]. \end{array}$$

Here the map u is $t \mapsto t^k$, where $\mathfrak{m}_S \mathcal{O}_T = \mathfrak{m}_T^k$. This implies that the natural morphism

$$T \to S \times_{\mathbb{A}^1} \mathbb{A}^1$$

is unramified. Therefore, r is log étale.

By the universal properties of log differentials, we have a sequence

$$\Omega^1(S, \mathcal{M}_S) \otimes_{\mathcal{O}_S} \mathcal{O}_T \to \Omega^1(T, \mathcal{M}_T) \to \Omega^1_{(T, \mathcal{M}_T)|(S, \mathcal{M}_S)} \to 0.$$

Since r is log étale, the last term vanishes. Since both $(T, \mathcal{M}_T)$, $(S, \mathcal{M}_S)$ are log smooth of dimension one, by the Nakayama lemma, we have the isomorphism

$$r^* : \Omega^1(S, \mathcal{M}_S) \otimes_{\mathcal{O}_S} \mathcal{O}_T \to \Omega^1(T, \mathcal{M}_T).$$

Now the log trace map is constructed as below:

$$\Omega^1(T, \mathcal{M}_T) \xrightarrow{(r^*)^{-1}} \Omega^1(S, \mathcal{M}_S) \otimes_{\mathcal{O}_S} \mathcal{O}_T \xrightarrow{tr} \Omega^1(S, \mathcal{M}_S) ,$$

where the second map is induced by the trace map $tr : \mathcal{O}_T \to \mathcal{O}_S$. The second part of the lemma trivially follows. □

3. Log Mumford's theorem

Lemma 3.1. *Given a proper log variety $(V, \mathcal{M}_V)$ and a normal scheme T, any morphism*

$$\mathbb{A}^1 \times T \to V^\circ$$

uniquely extends to a family of log rational curves over T_0

$$(\mathbb{P}^1, \mathcal{M}_{\{\infty\}}) \times T_0 \to (V, \mathcal{M}_V),$$

where T_0 is a dense open subset of T and $\mathcal{M}_{\{\infty\}}$ is the divisorial log structure associated to $\{\infty\} \subset \mathbb{P}^1$.

Proof. Since V is proper and T is normal, we have a morphism

$$u : \mathbb{P}^1 \times T_0 \to V,$$

where T_0 is a dense open subset of T. Consider the commutative diagram

$$\begin{array}{ccc} u^{-1}\mathcal{M}_V & & \mathcal{M}_{\{\infty\}\times T_0} \\ \downarrow \alpha & & \downarrow i \\ u^{-1}\mathcal{O}_V & \xrightarrow{u^*} & \mathcal{O}_{\mathbb{P}^1 \times T_0}. \end{array}$$

To prove u extends to a log morphism, it suffices to prove that for any element $g \in \mathcal{M}_D$, $u^*(\alpha(g))$ lies in $\mathcal{M}_{\{\infty\}\times T_0} \subset u_*\mathcal{O}_{\mathbb{P}^1\times T_0}$, or equivalently, $u^*(\alpha(g))$ is invertible on $\mathbb{A}^1 \times T_0$. By assumption, the image of $\mathbb{A}^1 \times T_0$ under u factors through V°. Thus we have

$$u^*(\alpha(g))|_{\mathbb{A}^1\times T_0} = u^*(\alpha(g)|_{V^\circ})|_{\mathbb{A}^1\times T_0}.$$

Since the log structure on V° is $\mathcal{O}^*_{V^\circ}$, we have $\alpha(g)|_{V^\circ} \in \mathcal{O}^*_{V^\circ}$. In particular, $u^*(\alpha(g))$ is invertible on $\mathbb{A}^1 \times T_0$. □

Notation 3.2. Let (X, D) be a log smooth proper variety with the interior U. Let $G = S_n$. We pick a non-zero logarithmic q-form $\omega \in \Gamma(X, \Omega^q_X(\log D))$. Let $\omega^{(n)} = \sum_1^n p_i^*\omega \in \Gamma(X^n, \Omega^q(X^n, \mathcal{M}_{X^n}))$. Then $\omega^{(n)}$ is G-invariant. By Proposition 2.3, for every log smooth variety $(S, \mathcal{M}_S)$ and morphism

$$f : (S, \mathcal{M}_S) \to (Y, \mathcal{M}_Y),$$

we have an induced q-form

$$\eta_f \in \Gamma(S, \Omega^q(S, \mathcal{M}_S)).$$

Theorem 3.3. *Let T be a smooth variety. Given a morphism $f : T \to S^nU$, it extends to a morphism*

$$f : (T, \mathcal{O}^*_T) \to (Y^n, \mathcal{M}_{Y^n}).$$

If all the 0-cycles in the image $f(T)$ are $\mathbb{A}^1$-equivalent, then it follows that

$$\eta_f = 0.$$

Lemma 3.4. *$S^nU \times S^nU$ contains a countable set $Z_1, Z_2, \cdots$ of constructible sets, such that if $(A,B) \in S^nU \times S^nU$, then*

$$A \sim_{\mathbb{A}^1} B \Longleftrightarrow (A,B) \in \bigcup_{i=1}^{\infty} Z_i.$$

For each i, there is a reduced scheme W_i and a set of morphisms

$$e_i : W_i \to Z_i,$$
$$f_i : W_i \to S^mU,$$
$$g_i : W \times \mathbb{A}^1 \to S^{n+m}U$$

such that we get the equations between zero cycles:

$$g_i(w,0) = p_1(e_i(w)) + f_i(w),$$
$$g_i(w,1) = p_2(e_i(w)) + f_i(w),$$

for all $w \in W_i$ and e_i is surjective.

Proof. We observe the fact that if $A, B \in S^kU$ are joined by a chain of p $\mathbb{A}^1$-curves $E_1, \cdots, E_p$ such that

$$E_1(0) = A,$$
$$E_p(1) = B,$$
$$E_i(1) = E_{i+1}(0) = C_i, i = 1, \cdots, p.$$

Then $A + C_1 + \cdots + C_{p-1}$ and $C_1 + \cdots + C_p + B$ in $S^{pk}U$ are joined by a single $\mathbb{A}^1$-curve, whose degree is bounded by the degree of the E_i's. Therefore, for any pair (A,B), the condition $A \sim_{\mathbb{A}^1} B$ is equivalent to that there exists $C \in S^mU$ and an irreducible $\mathbb{A}^1$-curve E on $S^{n+m}U$ of bounded degree connecting $A+C$ and $B+C$.

For any l, we define $(Y^l, \mathcal{M}_{Y^l})$ the fine saturated log scheme constructed in Lemma 2.1 and Lemma 2.2 for the quotient scheme $Y^l := X^l/S_l$. Clearly, there exists a strict open immersion

$$(S^lU, \mathcal{O}^*_{S^lU}) \to (Y^l, \mathcal{M}_{Y^l}).$$

By Lemma 2.2 and Lemma 3.1, any $\mathbb{A}^1$-curve on S^lU extends uniquely to a log rational curve on $(Y^l, \mathcal{M}_{Y^l})$.

Now let $\mathcal{A}_2(Y^{n+m}, \mathcal{M}_{Y^{n+m}}; \leq p)$ be the moduli space of two-pointed stable log rational curves of degree $\leq p$ on $(Y^{n+m}, \mathcal{M}_{n+m})$; cf., [GS13, Che14, AC] and let

$$\mathcal{A}_2^\circ(Y^{n+m}, \mathcal{M}_{Y^{n+m}}; \leq p) \subset \mathcal{A}_2(Y^{n+m}, \mathcal{M}_{Y^{n+m}}; \leq p)$$

be the log trivial part which parametrize two-pointed log rational curves. We have the natural evaluation morphism

$$ev_{n+m,p} : \mathcal{A}_2^\circ(Y^{n+m}, \mathcal{M}_{Y^{n+m}}; \leq p) \to S^{m+n}U \times S^{m+n}U.$$

Define the incidence reduced subscheme

$$W_{n+m,p} \subset S^nU \times S^nU \times S^mU \times \mathcal{A}_2^\circ(Y^{n+m}, \mathcal{M}_{Y^{n+m}}; \leq p),$$
$$W_{n+m,p} = \{((A,B), C, g) | g(0) = A + C, g(1) = B + C\}.$$

Define $Z_{n+m,p}$ as the image of $W_{n+m,p}$ under the projection to $S^nU \times S^nU$, which is constructible. Define $e_{n+m,p}$, $f_{n+m,p}$ the restriction of the natural projection morphisms on $W_{n+m,p}$. The morphism $g_{n+m,p}$ is defined via the universal morphism of log rational curves on $\mathcal{A}_2^\circ(Y^{n+m}, \mathcal{M}_{Y^{n+m}}; \leq p)$. □

Remark 3.5. In the proof of Lemma 3.4, the moduli space of log rational curves are not really needed. Any other reasonable sequence of moduli spaces could also be used to define the constructible sets $\{Z_i\}$, for example, [KM99, Def. 5.1, Prop. 5.3].

Proof of Theorem 3.3. Given $f : S \to S^n U$ such that all zero cycles $f(s)$ are $\mathbb{A}^1$-equivalent, fix a base point A_0 in the image. It follows from Lemma 3.4 and Lemma 3.1 that there is a non-singular variety T, a dominant morphism $e : T \to S$, and morphisms

$$g : T \to S^m U,$$
$$h : (\mathbb{P}^1, \mathcal{M}_{\{\infty\}}) \times T \to (Y^{n+m}, \mathcal{M}_{Y^{n+m}})$$

such that:

$$h(t, 0) = g(t) + f(e(t)),$$
$$h(t, 1) = g(t) + A_0$$

for all $t \in T$.

By Proposition 2.3 and Lemma 3.1, we have induced log q-forms η_f, η_g, and η_h. By Remark 2.4, we note that η_f, η_g, $\eta_h|_{T\times\{0\}}$, and $\eta_h|_{T\times\{\infty\}}$ are indeed regular q-forms constructed by Mumford. By [Mum68, Lemma 2], we have

$$\eta_h|_{T\times\{0\}} = \eta_g + e^*(\eta_f),$$
$$\eta_h|_{T\times\{\infty\}} = \eta_g + \eta_{A_0}.$$

Now η_h is a log q-form on $(\mathbb{P}^1, \mathcal{M}_{\{\infty\}}) \times T$. Since

$$\Omega^q((\mathbb{P}^1, \mathcal{M}_{\{\infty\}}) \times T) \cong p_1^*(\Omega_T^q) + p_1^*(\Omega_T^{q-1}) \otimes p_2^*(\Omega^1(\mathbb{P}^1, \mathcal{M}_{\{\infty\}}))$$

and $\Omega^1(\mathbb{P}^1, \mathcal{M}_{\{\infty\}}) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ has no global sections, it follows that

$$\eta_h = p_1^*(\eta)$$

for some $\eta \in \Gamma(\Omega_T^q)$. Therefore,

$$\eta_h|_{T\times\{0\}} = \eta_h|_{T\times\{\infty\}}.$$

Since $\eta_{A_0} = 0$, we find $e^*(\eta_f) = 0$, hence $\eta_f = 0$. □

Now let us assume that $\dim U$=2 and $q = 2$. Let $(S^n U)_0$ be the open subset parametrizing zero cycles $\sum_{i=1}^n x_i$ such that x_i's are all distinct and $\omega(x_i) \neq 0$ for all i. The open immersion

$$f : (S^n U)_0 \to S^n U$$

induces a log morphism

$$f : (S^n U)_0 \to (Y^n, \mathcal{M}_{Y^l}).$$

The induced log 2-form is a holomorphic symplectic form. The maximal isotropic subspace of η_f is of dimension n. If $S \subset (S^n U)_0$ is a non-singular subvariety parametrizing $\mathbb{A}^1$-equivalent zero cycles, we have $\eta_f|_S = 0$, thus $\dim S \leq n$.

Corollary 3.6. *Let (X, D) be a log smooth surface with $h^0(\Omega^2_X(\log D)) > 0$ and let $(S^n U)_0$ be defined as above. Then if $S \subset (S^n U)_0$ is a subvariety consisting $\mathbb{A}^1$-equivalent zero cycles, it follows that $\dim S \leq n$.* □

4. The log Bloch conjecture

4.1. Log rationally connected varieties.

Lemma 4.1. *Let U be a smooth quasiprojective curve. For any dense open subset $V \subset U$, any point $x \in U$ is $\mathbb{A}^1$-equivalent to $A - B$, where both A and B are effective divisors supported on V.*

Proof. If $x \in V$, then the lemma is trivial. We assume that $x \notin V$. We choose the compactification (X, D) of U with $D = p_1 + \cdots + p_d$, where all p_i's are distinct. We pick an effective divisor $B \subset V$ with sufficiently high degree satisfying

- $h^0(\mathcal{O}(x + B - D)) = h^0(\mathcal{O}(x + B)) - d$;
- $\mathcal{O}(x + B - D)$ is very ample.

Let H_i be the hyperplane in $|x + B|$ parametrizing divisors containing p_i. By the above condition, the H_i's intersect transversally in $|x + B|$ and a divisor in $|x + B|$ is away from D if and only if it avoids $\bigcup_{i=1}^d H_i$. Since $\mathcal{O}(x + B - D)$ is very ample, we may choose an effective divisor $E \in H^0(\mathcal{O}(x + B - D))$ and $E \subset V \backslash B$. The base point free pencil connecting $x + B$ and $D + E$ is an $\mathbb{A}^1$-curve on the pair $(|x + B|, \bigcup_{i=1}^d H_i)$. Let A be a general element of this pencil. Then $A \sim_{\mathbb{A}^1} x + B$. Since $B \subset V$ and the pencil is base point free, A is supported on V as well. □

Lemma 4.2. *Let U be a smooth quasiprojective variety and let $V \subset U$ be a dense open subset. Then the natural map*

$$i_* : h_0(V) \to h_0(U)$$

is surjective.

Proof. By choosing a smooth curve C on U with $x \in C$ and $C \cap V \neq \emptyset$, Lemma 4.1 implies that any point $x \in U - V$ is $\mathbb{A}^1$-equivalent to $A - B$, where both A and B are effective zero cycles on V. The lemma follows. □

Proposition 4.3. *If (X, D) is log rationally connected, then $h_0(U) = \mathbb{Z}$.*

Proof. Since (X, D) is log rationally connected, let p be a general point on U and let $U' \subset U$ be a non-empty open subset of U such that any point in U' is connected by a log rational curve through p. Thus $h_0(U') = \mathbb{Z}$. By Lemma 4.2, $h_0(U)$ is isomorphic to $\mathbb{Z}$ as well. □

Remark 4.4. In general, we do not know that any pair of points in the interior of a log RC pair is connected by a log rational curve. Any pairs with such properties are called *strongly log RC pairs.*

4.2. Surface pairs with $\kappa = -\infty$.

Proof of Theorem 1.9. Let (X, D) be a proper log smooth surface pair with $\kappa(X, D) = -\infty$. By [KM98, Theorem 3.47], we run the log minimal model program on this pair

$$(X, D) = (X_0, D_0) \to (X_1, D_1) \to \cdots \to (X_k, D_k) = (X^*, D^*)$$

such that:

(1) the log Kodaira dimension remains the same, i.e., $\kappa(X_i, D_i) = -\infty$;
(2) the end product (X^*, D^*) is either
  (a) log ruled, or
  (b) a log del Pezzo surface of Picard number one, i.e., $\rho(X^*) = 1$.

If the minimal model (X^*, D^*) is a log del Pezzo surface but not log ruled, then by the works of Miyanishi-Tsunoda [MT84], Keel-McKernan [KM99], and [Zhu16, Lemma 2.1, Theorem 2.2, 2.3], (X, D) is log rationally connected. In this case, Theorem 1.9 follows from Proposition 4.3.

If the minimal model (X^*, D^*) is log ruled, then by [Zhu16, Lemma 2.1], (X, D) is log ruled. In this case, Theorem 1.9 follows from Proposition 4.5 below. ☐

Proposition 4.5. *Log Bloch's conjecture holds for log ruled surface pairs.*

First we observe the following lemma.

Lemma 4.6. *Let (X, D) be a log smooth proper surface pair with the interior U. Let X' be the surface obtained by a sequence of blow ups on X:*

$$b : X' \to X,$$

with the boundary $D' := b^{-1}(D)$. Then log Bloch's conjecture holds for (X, D) if and only if it holds for (X', D').

Proof. Let U' be the interior of (X', D'). We have a commutative diagram as below

$$\begin{array}{ccc} h_0(U')^0 & \longrightarrow & \mathrm{Alb}(U') \\ \downarrow{\scriptstyle b_*} & & \downarrow{\scriptstyle \mathrm{Alb}(b)} \\ h_0(U)^0 & \longrightarrow & \mathrm{Alb}(U). \end{array}$$

Since blowing up does not change the Albanese, it suffices to show that

$$b_* : h_0(U')^0 \to h_0(U)^0$$

is an isomorphism. This follows from the blowing up long exact sequence of Suslin's algebraic singular homology [MVW06, Proposition 14.19]. ☐

Proof of Proposition 4.5. Lemma 4.6 implies that, without loss of generality, we may always replace (X, D) by a sequence of blow ups to prove Proposition 4.5. Now assume that (X, D) is a log ruled surface. Let q be the log irregularity $h^0(\Omega^1_X(\log D))$. We first construct a log ruling on U based on the value of q.

The case when $q > 0$

Consider the Albanese morphism [Iit76]

$$a : U \to \mathrm{Alb}(U),$$

where $\mathrm{Alb}(U) = H^0(\Omega^1_X(\log D))^*/H_1(U,\mathbb{Z})$ as a semiabelian variety of dimension q. Let T_0 be the closure of the image $a(U)$ and we rename the map $a : U \to T_0$ as

$$f : U \to T_0.$$

Since $h^0(K_X + D) = 0$, T_0 is a curve on $\mathrm{Alb}(U)$. Otherwise, any nowhere vanishing log 2-form on $\mathrm{Alb}(U)$ pulls back to a non-zero log 2-form on U. By [Iit76, Corollary 1], T_0 is a smooth curve with the diagram below

$$\begin{array}{ccc} U & \xrightarrow{a} & \mathrm{Alb}(U) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{Alb}(f)}\ \cong \\ T_0 & \longrightarrow & \mathrm{Alb}(T_0), \end{array}$$

and a general fiber of f is irreducible.

Lemma 4.7. *The morphism $f : U \to T_0$ is surjective and it gives the log ruling on U, that is, a general fiber of f is a log rational curve.*

Proof. Since there are no log rational curves on the Albanese, the log ruling on U gets contracted via f. Since the general fiber of f is irreducible, f gives the log ruling. Denote the image $f(U)$ by $T_0' \subset T_0$. Since every log 1-form on T_0' pulls back to a log 1-form on U and $q(T_0) \leq q(T_0')$, we have

$$q(T_0) \leq q(T_0') \leq q(U) = q(T_0).$$

This implies $q(T_0') = q(T_0)$. Thus $T_0' = T_0$. □

The case when $q = 0$

After blowing up finitely many points on U, still denoted by U, we pick a log ruling:

$$f : U \to T_0,$$

where T_0 is a smooth curve. We may further assume f is surjective.

Lemma 4.8. *T_0 is either $\mathbb{P}^1$ or $\mathbb{A}^1$.*

Proof. Pullback of log 1-forms under f implies

$$q(T_0) \leq q(U) = 0.$$

□

In either case, we choose the smooth compactification T of T_0 and let $S = T - T_0$. After further blowing up finitely many points, still denoting the pair (X, D) and the interior U, we may assume there exists a proper flat morphism

$$f : (X, D) \to (T, S)$$

such that

(1) $f(U) = T_0$; and
(2) the morphism
$$\mathrm{Alb}(f) : \mathrm{Alb}(U) \to \mathrm{Alb}(T_0)$$
is an isomorphism.

Let T_1 be an open subset of T_0 and let $U_1 := f|_U^{-1}(T_1)$ be an open subset of U such that

$$U_1 \cong T_1 \times \mathbb{A}^1.$$

Choose a regular section $\sigma : T_1 \to U_1$. We have the diagram as below:

$$\begin{array}{ccc} T_1 & \xrightarrow{\sigma} & U \\ & \searrow^{j} & \downarrow f \\ & & T_0, \end{array}$$

where j is indeed an open immersion. It induces a diagram on algebraic singular homology

$$\begin{array}{ccc} h_0(T_1)^0 & \xrightarrow{\sigma_*} & h_0(U)^0 \\ & \searrow^{j_*} & \downarrow f_* \\ & & h_0(T_0)^0. \end{array}$$

Lemma 4.9. *We have a commutative diagram as below*

$$\begin{array}{ccc} h_0(T_1)^0 & \xrightarrow{a_1} & \mathrm{Alb}(U) \\ j_* \downarrow & & \downarrow \mathrm{Alb}(j) \\ h_0(T_0)^0 & \xrightarrow{a_0} & \mathrm{Alb}(T_0) \end{array}$$

where both a_0 and a_1 are isomorphisms. Furthermore, j_* is surjective with the kernel an algebraic torus H.

Proof. It follows from the theory of generalized Jacobians [Ser88]. ☐

Lemma 4.10. *σ_* is surjective.*

Proof. The inclusion map $i : \sigma(T_1) \to U$ factors as below:

$$\sigma(T_1) \xrightarrow{i_1} U_1 = T_1 \times \mathbb{A}^1 \xrightarrow{i_2} U\ .$$

The lemma follows from that i_{1*} induces an isomorphism on h_0 and that i_{2*} is surjective by Lemma 4.2. ☐

By Lemmas 4.9 and 4.10, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Alb}(T_1) \cong h_0(T_1)^0 & \xrightarrow{\sigma_*} \!\!\!\!\rightarrow & h_0(U)^0 & \longrightarrow & \mathrm{Alb}(U) \\ & \searrow^{j_*} & \downarrow f_* & & \cong \downarrow \mathrm{Alb}(f) \\ & & h_0(T_0)^0 & \xrightarrow[\cong]{} & \mathrm{Alb}(T_0). \end{array}$$

Proposition 4.5 follows from the following lemma. ☐

Lemma 4.11.

$$\ker(\sigma_*) = \ker(j_*).$$

Proof. We only need to prove $\ker(j_*) \subset \ker(\sigma_*)$. By Lemma 4.9, any element in $\ker(j_*)$ is of the form $A - B$ satisfying

- both A and B are effective divisors on T_1 of degree d;
- $A \sim_{\mathbb{A}^1} B$ on T_0.

Note that if $T_0 = T$ is proper, we may replace $S = \emptyset$ by $S = \{p\}$, where p is away from the support of $A+B$. Then A is $\mathbb{A}^1$-equivalent to B on the open curve $T_0 - \{p\}$ as well. So for the rest of the proof, we assume that $S \neq \emptyset$.

Recall that the morphism

$$f : (X, D) \to (T, S)$$

is log ruled. By reduced fiber theorem [Sta15, Tag 09IL], after a finite base change and also normalizing,

$$\begin{array}{ccc} (X', D') & \xrightarrow{g'} & (X, D) \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ (T', S' = g^{-1}(S)) & \xrightarrow{g} & (T, S), \end{array}$$

we may assume that all geometric fibers of f' are reduced over $T' - S'$. Since (X', D') is log ruled over (T', S'), strong approximation over complex function fields away from $S' \neq \emptyset$ [Ros02, Theorem 6.13] implies that there exists an integral section s' over $T' - S'$. The image $R_0 := g'(s'(T' - S'))$ gives an integral multisection of f which is finite of degree N over T_0. Let $u : R_0 \to T_0$ be the natural map.

Since Suslin's homology is contravariant for finite flat maps [Gei10, Section 4], the equivalence $A \sim_{\mathbb{A}^1} B$ on T_0 implies that

$$u^*(A) \sim_{\mathbb{A}^1} u^*(B)$$

on R_0. On the other hand, by our construction, for any $p \in T_1$, $f^{-1}(p)$ is a log rational curve, in particular, $u^*(p) \sim_{\mathbb{A}^1} N\sigma(p)$. Thus we have

$$u^*(A) \sim_{\mathbb{A}^1} N\sigma_*(A),$$
$$u^*(B) \sim_{\mathbb{A}^1} N\sigma_*(B).$$

It follows that

$$N(\sigma_*(A - B)) \sim_{\mathbb{A}^1} 0.$$

Since $\sigma_*(A-B)$ is torsion in $h_0(U)^0$ and it maps to zero under f_*, by [SS03, Theorem 1.1], $\sigma_*(A - B)$ is trivial on $h_0(U)^0$. □

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