TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 10, October 2018, Pages 7181–7199
https://doi.org/10.1090/tran/7241
Article electronically published on June 26, 2018

# ON THE STRICT MONOTONICITY OF THE FIRST EIGENVALUE OF THE $p$-LAPLACIAN ON ANNULI

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ABSTRACT. Let $B_1$ be a ball in $\mathbb{R}^N$ centred at the origin and let $B_0$ be a smaller ball compactly contained in $B_1$. For $p \in (1,\infty)$, using the shape derivative method, we show that the first eigenvalue of the $p$-Laplacian in annulus $B_1 \setminus \overline{B_0}$ strictly decreases as the inner ball moves towards the boundary of the outer ball. The analogous results for the limit cases as $p \to 1$ and $p \to \infty$ are also discussed. Using our main result, further we prove the nonradiality of the eigenfunctions associated with the points on the first nontrivial curve of the Fučik spectrum of the $p$-Laplacian on bounded radial domains.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $N \geq 2$. We consider the following nonlinear eigenvalue problem:

$$\left.\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\lambda \in \mathbb{R}$ and $\Delta_p$ is the $p$-Laplace operator given by $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$. A real number $\lambda$ is called an eigenvalue of (1.1) if there exists $u$ in $W_0^{1,p}(\Omega) \setminus \{0\}$ satisfying

$$\int_\Omega |\nabla u|^{p-2} \, \langle \nabla u, \nabla v \rangle \, \mathrm{d}x = \lambda \int_\Omega |u|^{p-2} \, u \, v \, \mathrm{d}x \quad \forall \, v \in W_0^{1,p}(\Omega),$$

and $u$ is said to be an eigenfunction associated with $\lambda$.

It is well known that (1.1) admits a least positive eigenvalue $\lambda_1(\Omega)$ which has the following variational characterization:

$$\lambda_1(\Omega) = \inf \left\{ \int_\Omega |\nabla u|^p \, \mathrm{d}x : u \in W_0^{1,p}(\Omega) \setminus \{0\} \text{ with } \|u\|_p = 1 \right\}.$$

In this article we consider $\Omega$ of the form $B_{R_1}(x) \setminus \overline{B_{R_0}(y)}$ with $\overline{B_{R_0}(y)} \subset B_{R_1}(x)$, where $B_r(z)$ denotes the open ball of radius $r > 0$ centred at $z \in \mathbb{R}^N$. Since the $p$-Laplacian is invariant under orthogonal transformations, it can be easily seen that

$$\lambda_1(B_{R_1}(x) \setminus \overline{B_{R_0}(y)}) = \lambda_1(B_{R_1}(0) \setminus \overline{B_{R_0}(se_1)})$$

Received by the editors November 10, 2016, and, in revised form, March 15, 2017.
2010 *Mathematics Subject Classification.* Primary 35J92, 35P30, 35B06, 49R05.
*Key words and phrases.* $p$-Laplacian, symmetries, shape derivative, Fučik spectrum, eigenvalue, eigenfunction, nonradiality.
The second author was supported by the project LO1506 of the Czech Ministry of Education, Youth and Sports.

for any $x, y \in \mathbb{R}^N$ such that $|x - y| = s$, where $e_1$ is the first coordinate vector. Let the annular region $B_{R_1}(0) \setminus \overline{B_{R_0}(se_1)}$ be denoted by $\Omega_s$ and let

$$\lambda_1(s) := \lambda_1(\Omega_s).$$

We are interested in the behaviour of $\lambda_1(s)$ with respect to $s$ (in other words, with respect to the distance between centres of the inner and outer balls). The main objective of this article is to show that $\lambda_1(s)$ is strictly decreasing on $[0, R_1 - R_0)$ for any $p > 1$.

Apparently, the first result in this direction was obtained by Hersch in [16], where he proved (in the case $N = 2$, $p = 2$ and even for more general annular domains) that $\lambda_1(s)$ attains its maximum at $s = 0$. In [23], Ramm and Shivakumar conjectured[1] that $\lambda_1(s)$ is strictly decreasing and they gave numerical results to support this claim. Later this conjecture and its higher dimensional analogue were proved independently by Harrell et al. [14] and Kesavan [19]. Their proofs mainly rely on the following expression for $\lambda_1'(s)$ obtained using the Hadamard perturbation formula (see [12, 24]):

$$\lambda_1'(s) = - \int\limits_{x \in \partial B_{R_0}(se_1)} \left| \frac{\partial u_s}{\partial n}(x) \right|^2 n_1(x) \, \mathrm{dS}(x), \tag{1.2}$$

where $u_s$ is the positive eigenfunction associated with $\lambda_1(s)$ with the normalization $\|u_s\|_2 = 1$, and $n_1$ is the first component of $n = (n_1, \dots, n_N)$, the outward unit normal to $\Omega_s$. In [14, 19], the authors used the above formula in conjunction with reflection techniques and the strong comparison principle to show that $\lambda_1'(s)$ is negative on $(0, R_1 - R_0)$. For further reading and related open problems on this topic, we refer the reader to the books [2, 15].

For general $p > 1$, it is natural to anticipate that $\lambda_1(s)$ is strictly decreasing on $[0, R_1 - R_0)$. Indeed, we have the following generalization of formula (1.2):

$$\lambda_1'(s) = -(p-1) \int\limits_{x \in \partial B_{R_0}(se_1)} \left| \frac{\partial u_s}{\partial n}(x) \right|^p n_1(x) \, \mathrm{dS}(x). \tag{1.3}$$

The above expression was derived in [8] using the Hadamard perturbation formula (shape derivative formula) for $\lambda_1'(s)$ obtained in [13]. However for $p \neq 2$, one lacks a strong comparison principle that guarantees the strict monotonicity of $\lambda_1(s)$. More precisely, the strong comparison principle that is applicable for the nonlinear nonhomogeneous problems of the following type:

$$-\Delta_p u = \lambda |u|^{p-2} u \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega. \tag{1.4}$$

Thus one cannot directly extend the ideas of [14, 19, 23] to the nonlinear case and establish the strict monotonicity of $\lambda_1(s)$ for general $p > 1$. Nevertheless, in [8], Chorwadwala and Mahadevan could show that $\lambda_1'(s) \leq 0$ for all $s \in [0, R_1 - R_0)$ using a *weak* comparison principle proved in [9] for problems of the form (1.4). However, the authors of [8] could not rule out even the possibility of $\lambda_1(s)$ being a constant, due to the absence of the *strong* comparison principle. In this article, we bypass the usage of the strong comparison principle and prove the following result.

[1] Later a proof for this conjecture using an argument attributed to M. Ashbaugh was published in arXiv:math-ph/9911040 by the same authors.

**Theorem 1.1.** *Let $p \in (1, \infty)$ and let $\lambda_1(s)$ be the first eigenvalue of $-\Delta_p$ on $\Omega_s$. Then*

$$\lambda_1'(0) = 0 \quad \text{and} \quad \lambda_1'(s) < 0 \quad \forall\, s \in (0, R_1 - R_0).$$

*In particular, $\lambda(s)$ is strictly decreasing on $[0, R_1 - R_0)$.*

For our proof, we derive another formula for $\lambda_1'(s)$ (in terms of the normal derivative of $u_s$ on the outer boundary) in the following form:

$$\lambda_1'(s) = (p-1) \int\limits_{x \in \partial B_{R_1}(0)} \left| \frac{\partial u_s}{\partial n}(x) \right|^p n_1(x) \, \mathrm{dS}(x). \tag{1.5}$$

We obtained the above expression by considering the perturbations of $\Omega_s$ generated by shifts of the outer ball. On the other hand, formula (1.3) was obtained in [8] by considering the perturbations generated by shifts of the inner ball. If we assume $\lambda_1'(s) = 0$ for some $s \in (0, R_1 - R_0)$, then formulas (1.3) and (1.5) help us to show that the first eigenfunction $u_s$ associated with $\lambda_1(s)$ is radial (up to a translation) in some annular neighbourhoods of the inner and outer boundaries of $\Omega_s$. This eventually leads to a contradiction.

Next we study the monotonicity property of the corresponding limit problems. To avoid the ambiguity, for each $p > 1$, here we denote the first eigenvalue $\lambda_1(s)$ by $\lambda_1(p, s)$. It is known that $\lim\limits_{p \to \infty} \lambda_1^{1/p}(p, s)$ and $\lim\limits_{p \to 1} \lambda_1(p, s)$ exist; see [17, 18]. We denote the limit functions as below:

$$\Lambda_\infty(s) := \lim_{p \to \infty} \lambda_1^{1/p}(p, s) \quad \text{and} \quad \Lambda_1(s) := \lim_{p \to 1} \lambda_1(p, s).$$

Now we state results analogous to Theorem 1.1.

**Theorem 1.2.** *Let $\Lambda_\infty(s)$ and $\Lambda_1(s)$ be defined as before. Then $\Lambda_\infty(s)$ and $\Lambda_1(s)$ are continuous on $[0, R_1 - R_0)$ and*

(i) *$\Lambda_\infty(s)$ is strictly decreasing on $[0, R_1 - R_0)$;*
(ii) *$\Lambda_1(s)$ is decreasing on $[0, R_1 - R_0)$. Moreover, there exists $s^* \in [0, R_1 - R_0)$ such that $\Lambda_1(0) = \Lambda_1(s^*) > \Lambda_1(s)$ for all $s \in (s^*, R_1 - R_0)$.*

We use a geometric characterization of $\Lambda_\infty(s)$ given in [17] for proving part (i), and for the existence of $s^*$ in part (ii) we use a variational characterization of $\Lambda_1(s)$ given in [18].

Finally, we study the following Fučik eigenvalue problem:

$$\left.\begin{aligned} -\Delta_p u &= \alpha (u^+)^{p-1} - \beta (u^-)^{p-1} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.6}$$

where $\alpha, \beta$ are real numbers (spectral parameters) and $u^\pm := \max\{\pm u, 0\}$. If problem (1.6) possesses a nontrivial solution for some $(\alpha, \beta)$, then we say that $(\alpha, \beta)$ belongs to the Fučik spectrum of (1.6).

In [10], the authors considered a set of critical values $c(t)$ given by

$$c(t) := \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,1]} \left( \int\limits_\Omega |\nabla u|^p \, \mathrm{d}x - t \int\limits_\Omega (u^+)^p \, \mathrm{d}x \right), \tag{1.7}$$

where

$$\Gamma := \{\gamma \in \mathcal{C}([-1,1], \mathcal{S}) : \ \gamma(-1) = -\varphi_1, \ \gamma(1) = \varphi_1\}, \tag{1.8}$$
$$\mathcal{S} := \{u \in W_0^{1,p}(\Omega) : \ \|u\|_p = 1\},$$

and $\varphi_1$ is the first eigenfunction of (1.1) with the normalization $\|\varphi_1\|_p = 1$. Note that $c(0) = \lambda_2(\Omega)$, the second eigenvalue of (1.1). Using $c(t)$, the authors gave a description of the *first nontrivial* curve $\mathscr{C}$ of the Fučik spectrum of (1.6) as the union of the points $(t + c(t), c(t))$, $t \geq 0$, and their reflections with respect to the diagonal $(t,t)$. Further, they showed that $\mathscr{C}$ is continuous and each eigenfunction associated with a point on $\mathscr{C}$ has exactly two nodal domains (see Theorem 2.1 of [11]).

In [5], Bartsch et al. conjectured that in the linear case ($p = 2$) any eigenfunction corresponding to a point on $\mathscr{C}$ is nonradial in a bounded radial domain (i.e., $\Omega$ is a ball or annulus). In the same article, they showed that the conjecture holds in a neighbourhood of $(\lambda_2(\Omega), \lambda_2(\Omega))$ (see Remark 5.2 of [5]). A complete proof of this conjecture was given by Bartsch and Degiovanni in [4] by estimating generalized Morse indices of corresponding eigenfunctions. In [6], Benedikt et al. gave a different proof for this conjecture for a ball in $\mathbb{R}^N$ with $N = 2$ and $N = 3$. In this article, we provide another proof for this conjecture for any bounded radial domain and even extend this result for general $p \in (1, \infty)$.

**Theorem 1.3.** *Let $p \in (1,\infty)$ and $\Omega$ be a bounded radial domain in $\mathbb{R}^N$, $N \geq 2$. Then any eigenfunction associated with a point on the first nontrivial curve $\mathscr{C}$ of the Fučik spectrum of the problem* (1.6) *is nonradial.*

We obtain the above result as a simple consequence of Theorem 1.1. Moreover, Theorem 1.3 gives a generalization and a simpler proof for Theorem 1.1 of [1] which states the nonradiality of second eigenfunctions of the $p$-Laplacian on a ball.

## 2. Preliminaries

In this section, we first introduce the reflections with respect to the hyperplanes and the affine hyperplanes. Then we briefly describe the shape derivative formula of [13] and derive the formulas (1.3) and (1.5) for $\lambda_1'(s)$. Finally we state some results which will be required in the latter parts of this article.

For a nonzero vector $a \in \mathbb{R}^N$, let $H_a$ be the hyperplane perpendicular to $a$, i.e.,

$$H_a = \{x \in \mathbb{R}^N : \ \langle a, x\rangle = 0\}.$$

Further, we define the half-spaces

$$\mathcal{H}_a^+ := \{x \in \mathbb{R}^N : \ \langle a, x\rangle > 0\}, \quad \mathcal{H}_a^- := \{x \in \mathbb{R}^N : \ \langle a, x\rangle < 0\}.$$

Let $\sigma_a$ be the reflection with respect to the hyperplane $H_a$, i.e.,

$$\sigma_a(x) = x - 2\frac{\langle a, x\rangle}{|a|^2} a = x\left[I - 2\frac{a^T a}{|a|^2}\right] \quad \forall x \in \mathbb{R}^N, \tag{2.1}$$

where the last expression is the matrix product of the vector $x$ and the matrix $\sigma_a = I - 2\frac{a^T a}{|a|^2}$. Let $\widetilde{\sigma}_a$ be the reflection about the affine hyperplane $se_1 + H_a$. Then $\widetilde{\sigma}_a$ is given as below:

$$\widetilde{\sigma}_a(x) = x - 2\frac{\langle a, x - se_1\rangle}{|a|^2} a = \sigma_a(x) + 2\frac{\langle a, se_1\rangle}{|a|^2} a.$$

Now we recall the set $\Omega_s = B_{R_1}(0) \setminus \overline{B_{R_0}(se_1)}$ and for each nonzero vector $a$ in $\mathbb{R}^N$, consider the following subsets of $\Omega_s$:

$$\mathcal{O}_a^+ := \Omega_s \cap \mathcal{H}_a^+; \quad \widetilde{\mathcal{O}}_a^+ := \Omega_s \cap \left(\mathcal{H}_a^+ + se_1\right);$$
$$\mathcal{O}_a^- := \Omega_s \cap \mathcal{H}_a^-; \quad \widetilde{\mathcal{O}}_a^- := \Omega_s \cap \left(\mathcal{H}_a^- + se_1\right).$$

The relation between some of the subsets of $\overline{\Omega}_s$ under the reflections are listed below:

(2.2)
$$\left.\begin{aligned}
\sigma_a(\mathcal{O}_a^+) = \mathcal{O}_a^-, \quad & \widetilde{\sigma}_a(\widetilde{\mathcal{O}}_a^+) = \widetilde{\mathcal{O}}_a^- \ \forall\, a \in \mathbb{R}^N \setminus \{0\} \text{ with } \langle a, e_1\rangle = 0;\\
\sigma_a(\mathcal{O}_a^+) \subset \mathcal{O}_a^-, \quad & \widetilde{\sigma}_a(\widetilde{\mathcal{O}}_a^+) \subset \widetilde{\mathcal{O}}_a^- \ \forall\, a \in \mathbb{R}^N \text{ with } \langle a, e_1\rangle > 0;\\
\sigma_a(\partial B_{R_0}(se_1) \cap \partial\mathcal{O}_a^+) \subset \mathcal{O}_a^-, \quad & \widetilde{\sigma}_a(\partial B_{R_1}(0) \cap \partial\widetilde{\mathcal{O}}_a^+) \subset \widetilde{\mathcal{O}}_a^- \ \forall\, a \in \mathbb{R}^N\\
& \qquad\qquad \text{with } \langle a, e_1\rangle > 0;\\
\sigma_a(\partial B_{R_1}(0) \cap \partial\mathcal{O}_a^+) = \partial B_{R_1}(0) \cap \partial\mathcal{O}_a^- & \ \forall\, a \in \mathbb{R}^N \setminus \{0\};\\
\widetilde{\sigma}_a(\partial B_{R_0}(se_1) \cap \partial\widetilde{\mathcal{O}}_a^+) = \partial B_{R_0}(se_1) \cap \partial\widetilde{\mathcal{O}}_a^- & \ \forall\, a \in \mathbb{R}^N \setminus \{0\}.
\end{aligned}\right\}$$

Now for a function $u$ defined on $\overline{\Omega}_s$ and for a vector $a \in \mathbb{R}^N \setminus \{0\}$ with $\langle a, e_1\rangle \geq 0$ we define two new functions $u_a : \overline{\mathcal{O}_a^+} \to \mathbb{R}$ and $\widetilde{u}_a : \overline{\widetilde{\mathcal{O}}_a^+} \to \mathbb{R}$ as below:

$$u_a(x) := u(\sigma_a(x)); \quad \widetilde{u}_a(x) := u(\widetilde{\sigma}_a(x)).$$

By recalling the notation $\sigma_a = I - 2\frac{a^T a}{|a|^2}$ from (2.1), for $u \in \mathcal{C}^1(\overline{\Omega_s})$ we see that

$$\nabla u_a(x) = \nabla u(\sigma_a(x))\sigma_a \ \forall x \in \overline{\mathcal{O}_a^+}; \quad \nabla \widetilde{u}_a(x) = \nabla u(\widetilde{\sigma}_a(x))\sigma_a \ \forall x \in \overline{\widetilde{\mathcal{O}}_a^+}. \tag{2.3}$$

Further, the normal vector satisfies the following relations:

$$\begin{aligned}
n(\sigma_a(x)) &= n(x)\sigma_a \quad \forall\, x \in \partial B_{R_1}(0) \cap \mathcal{O}_a^+;\\
n(\widetilde{\sigma}_a(x)) &= n(x)\sigma_a \quad \forall\, x \in \partial B_{R_0}(se_1) \cap \mathcal{O}_a^+.
\end{aligned} \tag{2.4}$$

**Shape derivative formulas.** For a smooth bounded vector field $V$ on $\mathbb{R}^N$ consider the perturbation of $\Omega_s$ given as $\widetilde{\Omega}_t = (I + tV)\Omega_s$. It is known by Theorem 3 of [13] that $\lambda_1(t, V) := \lambda_1(\widetilde{\Omega}_t)$ is differentiable at $t = 0$ and the derivative is given by

(2.5)
$$\lambda_1'(0, V) := \lim_{t\to 0} \frac{\lambda_1(t, V) - \lambda_1(0, V)}{t} = -(p-1)\int_{\partial\Omega_s} \left|\frac{\partial u_s}{\partial n}(x)\right|^p \langle V(x), n(x)\rangle \, \mathrm{dS},$$

where $n$ is the outward unit normal to $\partial\Omega_s$ and $u_s$ is the first eigenfunction corresponding to $\lambda_1(s)$ normalized as

$$u_s > 0 \text{ and } \|u_s\|_p = 1. \tag{2.6}$$

In [8], the authors considered the vector field $V$ as given below:

(2.7)
$$V(x) = \rho(x)e_1, \ \rho \in \mathcal{C}_c^\infty(B_{R_1}(0)) \text{ and } \rho(x) \equiv 1 \text{ in a neighbourhood of } B_{R_0}(se_1).$$

For this choice of $V$ and for $t$ sufficiently small, the perturbations $\widetilde{\Omega}_t$ of $\Omega_s$ are generated by the shifts of the inner ball. More precisely,

$$\widetilde{\Omega}_t = \Omega_{s+t}.$$

Therefore, one gets $\lambda_1(t,V)=\lambda_1(s+t), \lambda_1(0,V)=\lambda_1(s)$ and hence (2.5) yields

$$\lambda_1'(s) = -(p-1) \int\limits_{\partial B_{R_0}(se_1)} \left|\frac{\partial u_s}{\partial n}(x)\right|^p n_1(x)\ \mathrm{dS}, \tag{2.8}$$

where $n_1$ is the first component of $n$, the outward unit normal to $\partial\Omega_s$ on $\partial B_{R_0}(se_1)$ (i.e., the inward unit normal to $\partial B_{R_0}(se_1)$).

To derive the expression (1.5) for $\lambda'(s)$ (i.e., formula involving the normal derivative of $u_s$ on the outer boundary), we consider the perturbations of $\Omega_s$ generated by the shifts of the outer boundary. Indeed, such perturbations can be obtained by taking a vector field $V(x)=-\rho(x)e_1$ with $\rho\in\mathcal{C}^\infty(\mathbb{R}^N)$ and

(i) $\rho=0$ in a neighbourhood of the inner sphere $\partial B_{R_0}(se_1)$;
(ii) $\rho=1$ in a neighbourhood of the outer sphere $\partial B_{R_1}(0)$.

For this choice of $V$, for $t$ sufficiently close to 0, observe that

$$\widetilde{\Omega}_t = B_{R_1}(-te_1)\setminus\overline{B_{R_0}(se_1)}.$$

From the translation invariance of the $p$-Laplacian, we get

$$\lambda_1(t,V)=\lambda_1\left(B_{R_1}(0)\setminus\overline{B_{R_0}((s+t)e_1)}\right)=\lambda_1(s+t).$$

Now (2.5) yields

$$\lambda_1'(s)=\lim_{t\to 0}\frac{\lambda_1(s+t)-\lambda_1(t)}{t} = (p-1)\int\limits_{\partial B_{R_1}(0)} \left|\frac{\partial u_s}{\partial n}(x)\right|^p n_1(x)\ \mathrm{dS}, \tag{2.9}$$

where $n_1$ is the first component of $n$, the outward unit normal to $\partial\Omega_s$ on $\partial B_{R_1}(0)$ (i.e., the outward unit normal to $\partial B_{R_1}(0)$).

Next we rewrite the integral in (2.9) using certain symmetries of the domain $\Omega_s$. Set $u=u_s$ in (2.9) and express the integral as a sum of two integrals:

$$\begin{aligned}&\int\limits_{\partial B_{R_1}(0)} \left|\frac{\partial u}{\partial n}(x)\right|^p n_1(x)\ \mathrm{dS}\\ &\quad= \int\limits_{\partial B_{R_1}(0)\cap\partial\mathcal{O}_{e_1}^+} \left|\frac{\partial u}{\partial n}(x)\right|^p n_1(x)\ \mathrm{dS} + \int\limits_{\partial B_{R_1}(0)\cap\partial\mathcal{O}_{e_1}^-} \left|\frac{\partial u}{\partial n}(x)\right|^p n_1(x)\ \mathrm{dS}\,.\end{aligned} \tag{2.10}$$

From (2.3) and (2.4) we have $\frac{\partial u}{\partial n}(x')=\frac{\partial u_{e_1}}{\partial n}(x)$ and $n_1(x')=-n_1(x)$ on $\partial B_{R_1}(0)\cap\mathcal{O}_{e_1}^+$, where $x'=\sigma_{e_1}(x)$. Hence, we modify the second integral as below:

$$\begin{aligned}\int\limits_{\partial B_{R_1}(0)\cap\partial\mathcal{O}_{e_1}^-} \left|\frac{\partial u}{\partial n}(x)\right|^p n_1(x)\ \mathrm{dS} &= \int\limits_{\partial B_{R_1}(0)\cap\partial\mathcal{O}_{e_1}^+} \left|\frac{\partial u}{\partial n}(x')\right|^p n_1(x')\ \mathrm{dS}\\ &= -\int\limits_{\partial B_{R_1}(0)\cap\partial\mathcal{O}_{e_1}^+} \left|\frac{\partial u_{e_1}}{\partial n}(x)\right|^p n_1(x)\ \mathrm{dS}\,.\end{aligned} \tag{2.11}$$

Thus, by combining (2.9), (2.10), and (2.11) we get

$$\lambda_1'(s)=(p-1)\int\limits_{\partial B_{R_1}(0)\cap\partial\mathcal{O}_{e_1}^+} \left(\left|\frac{\partial u}{\partial n}\right|^p - \left|\frac{\partial u_{e_1}}{\partial n}\right|^p\right) n_1\ \mathrm{dS}\,. \tag{2.12}$$

Similarly we can rewrite formula (2.8) as below:

$$\lambda_1'(s) = -(p-1) \int\limits_{\partial B_{R_0}(se_1)\cap\partial\widetilde{\mathcal{O}}_{e_1}^+} \left( \left|\frac{\partial u}{\partial n}\right|^p - \left|\frac{\partial \widetilde{u}_{e_1}}{\partial n}\right|^p \right) n_1 \,\mathrm{dS}\,. \tag{2.13}$$

**Auxiliary results.** Next we state a few results that we require in the subsequent sections. First we recall some results about the regularity of eigenfunctions of (1.1) (cf. Theorem 1.3 of [3]).

**Proposition 2.1.** *Let $\Omega$ be a smooth domain in $\mathbb{R}^N$ and let $u$ be a first eigenfunction of* (1.1)*. Then the following assertions are satisfied*:

(i) $u \in \mathcal{C}^1(\overline{\Omega})$.

(ii) *There exists $\delta > 0$ such that $|\nabla u| > m > 0$ in $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$ for some $m$, and $u \in \mathcal{C}^2(\overline{\Omega_\delta})$.*

The following version of the strong maximum principle is due to Vazquez [25, Section 4].

**Proposition 2.2.** *Let $\Omega$ be a domain in $\mathbb{R}^N$. Let $w \in \mathcal{C}^1(\overline{\Omega})$ be a positive function satisfying*

$$-\operatorname{div}\left(a_{ij}(x)\frac{\partial w}{\partial x_j}\right) \geq 0 \text{ in } \Omega,$$

*where $a_{ij} \in W^{1,\infty}_{loc}(\Omega)$ and there exists $\alpha > 0$ such that $a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2$ $\forall \xi \in \mathbb{R}^N \setminus \{0\}$ $\forall x \in \Omega$. Then*

(i) *$w \equiv 0$ in $\Omega$ or else $w > 0$ in $\Omega$.*

(ii) *Let $x_0$ be a point on $\partial\Omega$ satisfying the interior sphere condition. If $w > 0$ in $\Omega$ and $w(x_0) = 0$, then*

$$\frac{\partial w}{\partial n}(x_0) < 0,$$

*where $n$ is the outward unit normal to $\partial\Omega$ at $x_0$.*

In the next proposition we state a weak comparison result; see Theorem 2.1 and Proposition 4.1 of [9].

**Proposition 2.3.** *Let $\Omega$ be a domain in $\mathbb{R}^N$ with Lipschitz boundary. Let $u_1, u_2 \in \mathcal{C}^1(\overline{\Omega})$ be positive weak solutions of $-\Delta_p u = \lambda u^{p-1}$ in $\Omega$. If $u_1 \geq u_2$ on $\partial\Omega$, then*

$$u_1 \geq u_2 \text{ in } \Omega \text{ and } \frac{\partial u_1}{\partial n} \leq \frac{\partial u_2}{\partial n} \text{ on } \{x \in \partial\Omega : u_1(x) = u_2(x) = 0\}.$$

## 3. Main result

In this section we give the proof of Theorem 1.1. We will be considering various annular regions apart from $\Omega_s$, for simplicity we denote them as

$$A_{r_1,r_0}(x,y) = B_{r_1}(x) \setminus \overline{B_{r_0}(y)}.$$

In particular, $A_{R_1,R_0}(0, se_1) = \Omega_s$. Throughout this section, unless otherwise specified, the eigenfunction $u_s$ is the first eigenfunction of $-\Delta_p$ on $\Omega_s$ normalized as in (2.6), namely $u_s > 0$ and $\|u_s\|_p = 1$.

The following result is proved in [8] (see Theorem 3.1) using formula (2.13). Here, for the sake of completeness, we present a proof by making use of formula (2.12).

**Lemma 3.1.** *Let $s \in [0, R_1 - R_0)$ and let $\lambda_1(s)$ be the first eigenvalue of $-\Delta_p$ on $\Omega_s$. Then $\lambda'(s) \leq 0$.*

*Proof.* By setting $u = u_s$ and noting that $\sigma_{e_1}(\mathcal{O}^+_{e_1}) \subset \mathcal{O}^-_{e_1}$ and

$$\sigma_{e_1}(\partial B_{R_0}(se_1) \cap \partial \mathcal{O}^+_{e_1}) \subset \mathcal{O}^-_{e_1},$$

we easily see that $u_{e_1}$ and $u$ weakly satisfy the following problems:

$$\left.\begin{array}{llll} -\Delta_p u_{e_1} = \lambda_1(s)\, u_{e_1}^{p-1}, & -\Delta_p u = \lambda_1(s)\, u^{p-1} & \text{in } \mathcal{O}^+_{e_1}, \\ u_{e_1} = 0, & u = 0 & \text{on } \partial B_{R_1}(0) \cap \partial \mathcal{O}^+_{e_1}, \\ u_{e_1} = u, & u = u_{e_1} & \text{on } H_{e_1} \cap \partial \mathcal{O}^+_{e_1}, \\ u_{e_1} > 0, & u = 0 & \text{on } \partial B_{R_0}(se_1) \cap \partial \mathcal{O}^+_{e_1}. \end{array}\right\} \tag{3.1}$$

Thus by applying the weak comparison principle (Proposition 2.3) we obtain $u_{e_1} \geq u$ in $\mathcal{O}^+_{e_1}$. Moreover, as $u = 0$ on $\partial B_{R_1}(0) \cap \partial \mathcal{O}^+_{e_1}$, Proposition 2.2 yields

$$\frac{\partial u_{e_1}}{\partial n} \leq \frac{\partial u}{\partial n} < 0 \text{ on } \partial B_{R_1}(0) \cap \partial \mathcal{O}^+_{e_1}. \tag{3.2}$$

Now since $n_1(x)$ is positive for $x \in \partial B_{R_1}(0) \cap \partial \mathcal{O}^+_{e_1}$, from (2.12) and (3.2) we derive that

$$\lambda_1'(s) = (p-1) \int\limits_{\partial B_{R_1}(0) \cap \partial \mathcal{O}^+_{e_1}} \left( \left|\frac{\partial u}{\partial n}\right|^p - \left|\frac{\partial u_{e_1}}{\partial n}\right|^p \right) n_1 \,\mathrm{dS} \;\leq 0.$$

This completes the proof. □

**Symmetries with respect to the hyperplanes.** First we study symmetries of the first eigenfunction of $-\Delta_p$ on $\Omega_s$. We show that for $s \in (0, R_1 - R_0)$ the associated first eigenfunction is symmetric with respect to the hyperplanes perpendicular to $H_{e_1}$.

**Lemma 3.2.** *Let $s \in (0, R_1 - R_0)$ and let $u_s$ be the first eigenfunction of $-\Delta_p$ on $\Omega_s$. If $a \in \mathbb{R}^N \setminus \{0\}$ with $\langle a, e_1 \rangle = 0$, then*

$$u_s(x) = u_s(\sigma_a(x)) \quad \forall x \in \Omega_s.$$

*In particular, for $i = 2, 3, \ldots, N$*

$$u_s(x) = u_s(\sigma_{e_i}(x)) = u_s(x_1, x_2, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_N) \quad \forall x \in \Omega_s.$$

*Proof.* Clearly for $a \neq 0$ with $\langle a, e_1 \rangle = 0$, $\mathcal{O}^+_a = \sigma_a(\mathcal{O}^-_a)$ (see (2.2)). Thus $u := u_s$ and $u_a := u_s \circ \sigma_a$ weakly satisfy the following problems, respectively:

$$\begin{array}{lll} -\Delta_p u_a = \lambda_1(s)\, u_a^{p-1}, & -\Delta_p u = \lambda_1(s)\, u^{p-1} & \text{in } \mathcal{O}^+_a, \\ u_a = u, & u = u_a & \text{on } \partial \mathcal{O}^+_a. \end{array}$$

Now by the weak comparison principle (Proposition 2.3), we obtain that $u_a \equiv u$ in $\mathcal{O}^+_a$, which implies the desired assertions. □

In the next lemma we show that $u_s$ is symmetric also with respect to $H_{e_1}$ in a neighbourhood of the outer boundary, provided $\lambda_1'(s) = 0$.

**Lemma 3.3.** *If $\lambda_1'(s) = 0$ for some $s \in (0, R_1 - R_0)$, then there exists $r_1 > 0$ such that*

$$u_s(x) = u_s(\sigma_{e_1}(x)) \quad \forall x \in A_{R_1, r_1}(0, 0).$$

*Proof.* We set $u = u_s$. Since $u \in \mathcal{C}^1(\overline{\Omega_s})$, $u > 0$, and $u$ vanishes on $\partial B_{R_1}(0)$ and $\partial B_{R_0}(se_1)$, there exists $r^* \in (R_0 + s, R_1)$ such that $\frac{\partial u}{\partial x_1}(r^* e_1) = 0$. Define

$$r_1 = \sup\left\{|x| > 0 : \langle \nabla u(x), x\rangle = 0\right\}. \tag{3.3}$$

As $\frac{\partial u}{\partial n}(x) < 0$ on $\partial B_{R_1}(0)$ (by Proposition 2.2), $\langle \nabla u(x), x\rangle < 0$ in a neighbourhood of $\partial B_{R_1}(0)$. Thus clearly $r_1 \in [r^*, R_1)$. By the construction, $A_{R_1,r_1}(0,0)$ is the maximal annular neighbourhood of $\partial B_{R_1}(0)$ on which $\langle \nabla u(x), x\rangle$ is nonvanishing. Further, by the continuity of $\nabla u$ there must exist $x_1 \in \partial B_{r_1}(0)$ such that

$$\langle \nabla u(x_1), x_1\rangle = 0. \tag{3.4}$$

Set $u_{e_1} = u \circ \sigma_{e_1}$ on $\mathcal{O}^+_{e_1}$. Now from (3.1) and Proposition 2.3, we have $u_{e_1} \geq u$ in $\mathcal{O}^+_{e_1}$. To show $u \equiv u_{e_1}$ in $A_{R_1,r_1}(0,0) \cap \mathcal{O}^+_{e_1}$ we linearize the $p$-Laplacian on the domain $A_{R_1,r}(0,0) \cap \mathcal{O}^+_{e_1}$ with $r_1 < r < R_1$ by setting $w = u_{e_1} - u$. Then $w$ weakly satisfies the following problem:

$$\begin{aligned} -\mathrm{div}(A(x)\nabla w) = \lambda\left(u_{e_1}^{p-1} - u^{p-1}\right) &\geq 0 \quad &&\text{in } A_{R_1,r}(0,0) \cap \mathcal{O}^+_{e_1},\\ w &\geq 0 &&\text{on } \partial(A_{R_1,r}(0,0) \cap \mathcal{O}^+_{e_1}), \end{aligned}$$

where the coefficient matrix $A(x) = [a_{ij}(x)]$ is given by

$$\begin{aligned} a_{ij}(x) = \int_0^1 & |(1-t)\nabla u(x) + t\nabla u_{e_1}(x)|^{p-2}\\ &\times \left[I + (p-2)\frac{[(1-t)\nabla u(x) + t\nabla u_{e_1}(x)]^T[(1-t)\nabla u(x) + t\nabla u_{e_1}(x)]}{|(1-t)\nabla u(x) + t\nabla u_{e_1}(x)|^2}\right]_{ij} \mathrm{d}t. \end{aligned}$$

Now we show that $A(x)$ is uniformly positive definite on $A_{R_1,r}(0,0) \cap \mathcal{O}^+_{e_1}$. Since $\langle \nabla u(x), x\rangle$ does not vanish on $A_{R_1,r_1}(0,0)$ and is negative near the boundary $\partial B_{R_1}(0)$, we see that $\langle \nabla u(x), x\rangle < 0$ in $A_{R_1,r}(0,0)$. By the continuity, we can find $\delta_r > 0$ such that

$$\langle \nabla u(x), x\rangle < -\delta_r \text{ in } A_{R_1,r}(0,0).$$

Notice that

$$\begin{aligned} \langle \nabla u_{e_1}(x), x\rangle &= \langle \nabla(u(\sigma_{e_1}(x))), x\rangle\\ &= \langle \nabla u(\sigma_{e_1}(x))\sigma_{e_1}, x\rangle\\ &= \langle \nabla u(\sigma_{e_1}(x)), \sigma_{e_1}(x)\rangle. \end{aligned}$$

Thus, by the above inequality we have $\langle \nabla u_{e_1}(x), x\rangle < -\delta_r$ in $A_{R_1,r}(0,0) \cap \mathcal{O}^+_{e_1}$. Therefore,

$$(1-t)\langle \nabla u(x), x\rangle + t\langle \nabla u_{e_1}(x), x\rangle < -\delta_r \;\forall t \in [0,1]\; \forall x \in A_{R_1,r}(0,0) \cap \mathcal{O}^+_{e_1}.$$

Hence, for $x \in A_{R_1,r}(0,0)$ we get

$$|(1-t)\nabla u(x) + t\nabla u_{e_1}(x)| \geq \left|\left\langle (1-t)\nabla u(x) + t\nabla u_{e_1}(x), \frac{x}{|x|}\right\rangle\right| > \frac{\delta_r}{R_1} = m_r. \tag{3.5}$$

Further, since $|\nabla u|$ is bounded in $A_{R_1,r}(0,0)$, there exists $M_r > 0$ such that

$$|(1-t)\nabla u(x) + t\nabla u_{e_1}(x)| \leq M_r. \tag{3.6}$$

Note that for each $a \in \mathbb{R}^N \setminus \{0\}$, the matrix $a^T a$ has eigenvalues $\{0, |a|^2\}$. Thus, for any $y \in \mathbb{R}^N$,

$$\min\{1, p-1\}|a|^{p-2}|y|^2 \le \left\langle |a|^{p-2}\left[I + (p-2)\frac{a^T a}{|a|^2}\right] y, y \right\rangle \le \max\{1, p-1\}|a|^{p-2}|y|^2. \tag{3.7}$$

From (3.5), (3.6), and (3.7), for $x \in A_{R_1,r}(0,0)$ and $y \in \mathbb{R}^N$ we obtain

$$\langle A(x)y, y\rangle \ge \begin{cases} m_r^{p-2}|y|^2 & \text{for } p \ge 2, \\ (p-1)M_r^{p-2}|y|^2 & \text{for } 1 < p < 2. \end{cases}$$

Thus the differential operator in (3.5) defined by means of $A(x)$ is uniformly elliptic in $A_{R_1,r}(0,0)$. Moreover, by Proposition 2.1, $a_{ij} \in \mathcal{C}^1(A_{R_1,r}(0,0))$. Hence, the strong maximum principle for (3.5) (Proposition 2.2) implies that either $w \equiv 0$ or $w > 0$ in $A_{R_1,r}(0,0) \cap \mathcal{O}_{e_1}^+$. Moreover, if $w > 0$ in $A_{R_1,r}(0,0) \cap \mathcal{O}_{e_1}^+$, then

$$\frac{\partial u_{e_1}}{\partial n} - \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} < 0 \text{ on } \partial B_{R_1}(0) \cap \partial \mathcal{O}_{e_1}.$$

Now (2.12) together with the above inequality implies that $\lambda_1'(s) < 0$, which contradicts our assumption $\lambda_1'(s) = 0$. Thus we must have $w \equiv 0$ and hence $u \equiv u_{e_1}$ in $A_{R_1,r}(0,0) \cap \mathcal{O}_{e_1}^+$. Since $r \in (r_1, R_1)$ is arbitrary, we conclude that $u(x) = u(\sigma_{e_1}(x))$ $\forall x \in A_{R_1,r_1}(0,0)$. □

Next we show that $u$ is symmetric in $A_{R_1,r_1}(0,0)$ with respect to all the hyperplanes.

**Lemma 3.4.** *Let $s$ and $r_1$ be as in Lemma* 3.3*. Then for any nonzero vector $a \in \mathbb{R}^N$*

$$u_s(x) = u_s(\sigma_a(x)) \ \forall x \in A_{R_1,r_1}(0,0).$$

*Proof.* The case $\langle a, e_1\rangle = 0$ follows from Lemma 3.2. Note that $\sigma_a(x) = \sigma_{ka}(x)$ for $k \in \mathbb{R} \setminus \{0\}$. Thus, it is enough to prove the result for $a \in A_{R_1,r_1}(0,0)$ with $\langle a, e_1\rangle > 0$. In this case we have $\sigma_a(\mathcal{O}_a^+) \subset \mathcal{O}_a^-$. Now by setting $u = u_s$ and $u_a = u_s \circ \sigma_a$ we see that $u_a$ and $u$ satisfy the following problems in $\mathcal{O}_a^+$:

$$\begin{aligned} -\Delta_p u_a &= \lambda_1(s)\, u_a^{p-1}, & -\Delta_p u &= \lambda_1(s)\, u^{p-1} && \text{in } \mathcal{O}_a^+, \\ u_a &= 0, & u &= 0 && \text{on } \partial B_{R_1}(0) \cap \partial \mathcal{O}_a^+, \\ u_a &= u, & u &= u_a && \text{on } H_a \cap \partial \mathcal{O}_a^+, \\ u_a &> 0, & u &= 0 && \text{on } \partial B_{R_0}(se_1) \cap \partial \mathcal{O}_a^+. \end{aligned}$$

Applying the weak comparison principle (Proposition 2.3), we obtain that $u_a \ge u$ in $\mathcal{O}_a^+$. As before we set $w = u_a - u$. From Lemma 3.2 and Lemma 3.3 we obtain $u(a) = u(-a)$ as below:

$$\begin{aligned} u(a_1, a_2, \dots, a_N) &= u(a_1, -a_2, \dots, a_N) \\ &= \cdots = u(a_1, -a_2, \dots, -a_N) = u(-a_1, -a_2, \dots, -a_N). \end{aligned}$$

By definition $u_a(a) = u(-a)$ and hence $w(a) = 0$. Now we proceed along the same lines as in Lemma 3.3 and see that $w$ satisfies the following problem:

$$-\mathrm{div}(A(x)w) \ge 0 \text{ in } A_{R_1,r}(0,0) \cap \mathcal{O}_a^+; \quad w \ge 0 \text{ on } \partial(A_{R_1,r}(0,0) \cap \mathcal{O}_a^+)$$

for any $r \in (r_1, R_1)$, where the coefficient matrix $A(x)$ is uniformly positive definite. By the strong maximum principle we have either $w \equiv 0$ or else $w > 0$ in $A_{R_1,r}(0,0) \cap \mathcal{O}_a^+$. Since $w(a) = 0$, we obtain $w \equiv 0$ and hence $u \equiv u_a$ in

$A_{R_1,r}(0,0)\cap\mathcal{O}_a^+$. Finally, using the reflection, we conclude that $u(x)=u(\sigma_a(x))\ \forall\, x\in A_{R_1,r_1}(0,0)$. ☐

**Theorem 3.5.** *Let $s\in(0,R_1-R_0)$ and let $u_s$ be the first eigenfunction of $-\Delta_p$ on $\Omega_s$. If $\lambda_1'(s)=0$, then $u_s$ is radial in the annulus $A_{R_1,r_1}(0,0)$, where $r_1$ is given by Lemma 3.3. Furthermore, $\nabla u_s=0$ on $\partial B_{r_1}(0)$.*

*Proof.* Let $b,c\in A_{R_1,r_1}(0,0)$ be such that $b\neq c$ and $|b|=|c|$. Then there exists a constant $k$ such that $a=k(b-c)\in A_{R_1,r_1}(0,0)$. Noting that $\sigma_a(b)=c$, from Lemma 3.4 we obtain that

$$u_s(b)=u_s(\sigma_a(b))=u_s(c).$$

Since $b$ and $c$ are arbitrary, we conclude that $u_s$ is radial in the annulus $A_{R_1,r_1}(0,0)$. Further, as $u_s$ is continuously differentiable in $A_{R_1,r_1}(0,0)$ and $\nabla u_s(x_1)\cdot x_1=0$ (see (3.4)), the radiality of $u_s$ gives $\nabla u_s=0$ on $\partial B_{r_1}(0)$. ☐

**Symmetries with respect to the affine hyperplanes passing through $se_1$.** In this subsection we prove the radiality (up to a translation of the origin) of $u_s$ in a neighbourhood of the inner boundary. Since $\widetilde{\sigma}_a(x)=\sigma_a(x)$ for $a$ such that $\langle a,e_1\rangle=0$, Lemma 3.2 holds as it is, and hence we have for $i=2,\dots,N$

$$u_s(x)=u_s(\widetilde{\sigma}_{e_i}(x))=u(x_1,x_2,\dots,x_{i-1},-x_i,x_{i+1},\dots,x_N)\ \forall x\in\Omega_s.$$

Next we prove a symmetry result along the same lines as in Lemma 3.3.

**Lemma 3.6.** *Let $s\in(0,R_1-R_0)$ and let $u_s$ be the first eigenfunction of $-\Delta_p$ on $\Omega_s$. If $\lambda_1'(s)=0$, then there exists $r_0>0$ such that*

$$u_s(x)=u(\widetilde{\sigma}_{e_1}(x))=u_s(-x_1+2s,x_2,\dots,x_N)\ \forall x\in A_{r_0,R_0}(se_1,se_1).$$

*Proof.* As it was shown in the proof of Lemma 3.3, we have $r^*\in(R_0+s,R_1)$ such that $\frac{\partial u}{\partial x_1}(r^*e_1)=0$. Define

$$r_0=\inf\left\{|x-se_1|>0:\ \langle\nabla u(x),x-se_1\rangle=0\right\}. \tag{3.8}$$

Clearly $r_0\in(R_0,R_1-s)$, since by Hopf's maximum principle $\langle\nabla u(x),x-se_1\rangle=|x-se_1|\frac{\partial u}{\partial n}(x)\neq0$ on $\partial B_{R_0}(se_1)$. By the construction, $A_{r_0,R_0}(se_1,se_1)$ is the maximal annular neighbourhood of $\partial B_{R_0}(se_1)$ on which $\langle\nabla u(x),x-se_1\rangle$ is non-vanishing. Further, by the continuity of $\nabla u$ there must exist $x_0\in\partial B_{r_0}(se_1)$ such that

$$\langle\nabla u(x_0),x_0-se_1\rangle=0.$$

As in the proof of Lemma 3.3, we linearize the $p$-Laplacian on the domain

$$A_{r,R_0}(se_1,se_1)\cap\widetilde{\mathcal{O}}_{e_1}^+$$

with $R_0<r<r_0$ by setting $w=\widetilde{u}_{e_1}-u$. Note that $\widetilde{u}_{e_1}$ and $u$ satisfy $-\Delta_p v=\lambda v^{p-1}$ in $\widetilde{\mathcal{O}}_{e_1}^+$ and $\widetilde{u}_{e_1}\geq u$ on $\partial\widetilde{\mathcal{O}}_{e_1}^+$. Thus by Proposition 2.3 we get $\widetilde{u}_{e_1}\geq u$ on $\widetilde{\mathcal{O}}_{e_1}^+$. Furthermore, $w$ weakly satisfies the following problem:

$$\begin{aligned}-\mathrm{div}(A(x)\nabla w)&=\lambda\left(\widetilde{u}_{e_1}^{p-1}-u^{p-1}\right)\geq0 &&\text{in } A_{r,R_0}(se_1,se_1)\cap\widetilde{\mathcal{O}}_{e_1}^+,\\ w&\geq0 &&\text{on } \partial(A_{r,R_0}(se_1,se_1)\cap\widetilde{\mathcal{O}}_{e_1}^+).\end{aligned}$$

By similar arguments as in Lemma 3.3, the above differential operator is uniformly elliptic on $A_{r,R_0}(se_1,se_1)\cap\widetilde{\mathcal{O}}_{e_1}^+$ and hence by the strong maximum principle we have either $w \equiv 0$ or $w > 0$ on this domain. If $w > 0$ in $A_{r,R_0}(se_1, se_1) \cap \widetilde{\mathcal{O}}^+_{e_1}$, then by the Hopf maximum principle

$$\frac{\partial \widetilde{u}_{e_1}}{\partial n} - \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} < 0 \text{ on } \partial B_{R_0}(se_1) \cap \partial \widetilde{\mathcal{O}}^+_{e_1}.$$

Now (2.13) implies that $\lambda_1'(s) < 0$, a contradiction to the assumption $\lambda_1'(s) = 0$. Thus we must have $w \equiv 0$ and hence $u \equiv \widetilde{u}_{e_1}$ in $A_{r,R_0}(se_1, se_1) \cap \widetilde{\mathcal{O}}^+_{e_1}$. Since $r \in (R_0, r_0)$ is arbitrary, we obtain the desired fact. $\square$

Next we state a lemma which is a counterpart of Lemma 3.4. The proof follows along the same lines.

**Lemma 3.7.** *Let $s \in (0, R_1 - R_0)$ and let $u_s$ be the first eigenfunction of $-\Delta_p$ on $\Omega_s$. If $\lambda_1'(s) = 0$, then for any nonzero vector $a \in \mathbb{R}^N$*

$$u_s(x) = u_s(\widetilde{\sigma}_a(x)) \ \forall x \in A_{r_0,R_1}(se_1, se_1),$$

*where $r_0$ is given by Lemma* 3.6.

The next theorem, which is a counterpart of Theorem 3.5, states that $u_s$ is radial (up to a translation of the origin) in a neighbourhood of the inner ball. The proof follows along the same lines using Lemma 3.2, Lemma 3.6, and Lemma 3.7.

**Theorem 3.8.** *Let $s \in (0, R_1 - R_0)$ and let $u_s$ be the first eigenfunction of $-\Delta_p$ on $\Omega_s$. If $\lambda_1'(s) = 0$, then $u_s$ is radial in the annulus $A_{r_0,R_0}(se_1, se_1)$. Furthermore, $\nabla u_s = 0$ on $\partial B_{r_0}(se_1)$.*

*Remark* 3.9. Let $u_0$ be a positive first eigenfunction of $-\Delta_p$ on $A_{R_1,R_0}(0,0)$. Note that $u_0$ is radial (cf. [21, Proposition 1.1]) and one can verify that $u_0$ attains its maximum on a unique sphere of radius $\bar{r} \in (R_0, R_1)$ and $u_0'(\bar{r}) = 0$. From the simplicity of the first eigenvalue, it is clear that every first eigenfunction $u$ of $-\Delta_p$ on $A_{R_1,R_0}(0,0)$ is radial and $u'(\bar{r}) = 0$.

**Lemma 3.10.** *Let $\lambda_1'(s) = 0$ for some $s \in (0, R_1 - R_0)$. Let $r_0$ and $r_1$ be given by Lemmas* 3.2 *and* 3.6*, respectively. Then $r_0 = r_1 = \bar{r}$.*

*Proof.* From the definitions of $r_0$ and $r_1$ (see (3.8) and (3.3)) it easily follows that $r_0 \le r_1$. First we show that $r_1 \le \bar{r}$. Suppose that $r_1 > \bar{r}$. For notational simplicity, we denote an annular region with centre at the origin as $A_{t_1,t_0} = A_{t_1,t_0}(0,0)$. Now consider the following function on $A_{R_1,R_0}$:

$$w_1(x) = \begin{cases} u_s(x), & x \in A_{R_1,r_1}, \\ C_1, & x \in A_{r_1,\bar{r}}, \\ u_0(x), & x \in A_{\bar{r},R_0}, \end{cases}$$

where $C_1 = u_s(x)$ for $|x| = r_1$. By multiplying with an appropriate constant we can choose $u_0$ in such a way that $u_0(x) = C_1$ for $|x| = \bar{r}$. Since $w_1$ is continuous and piecewise differentiable on $A_{R_1,R_0}$ we have $w_1 \in W_0^{1,p}(A_{R_1,R_0})$. To estimate $\|\nabla w_1\|_p^p$, we derive a few identities. Note that for any $r \in (r_1, R_1)$, $\nabla u_s$ does not vanish on $A_{R_1,r}$ and hence $u_s \in \mathcal{C}^2(A_{R_1,r})$; see Proposition 2.1. Thus $u_s \in \mathcal{C}^2(A_{R_1,r_1})$ and hence the following equation holds pointwise in $A_{R_1,r_1}$ :

$$-\Delta_p u_s = \lambda_1(s)|u_s|^{p-2}u_s.$$

Multiply the above equation by $u_s$ and integrate over $A_{R_1,r_1}$ to get

$$\int\limits_{A_{R_1,r_1}} -\Delta_p u_s\, u_s \,\mathrm{d}x = \lambda_1(s) \int\limits_{A_{R_1,r_1}} |u_s|^{p-2} u_s\, u_s \,\mathrm{d}x.$$

Now by noting that $\nabla u_s = 0$ on $\partial B_{r_1}(0)$ and $u_s = 0$ on $\partial B_{R_1}(0)$, the integration by parts gives

$$\int\limits_{A_{R_1,r_1}} |\nabla u_s|^p \,\mathrm{d}x = \lambda_1(s) \int\limits_{A_{R_1,r_1}} |u_s|^p \,\mathrm{d}x. \tag{3.9}$$

Similarly

$$\int\limits_{A_{\bar{r},R_0}} |\nabla u_0|^p \,\mathrm{d}x = \lambda_1(0) \int\limits_{A_{\bar{r},R_0}} |u_0|^p \,\mathrm{d}x. \tag{3.10}$$

Now we estimate $\|\nabla w_1\|_p^p$:

$$\int\limits_{A_{R_1,R_0}} |\nabla w_1|^p \,\mathrm{d}x = \int\limits_{A_{R_1,r_1}} |\nabla u_s|^p \,\mathrm{d}x + \int\limits_{A_{\bar{r},R_0}} |\nabla u_0|^p \,\mathrm{d}x.$$

By using (3.9) and (3.10) and inequality $\lambda_1(s) \le \lambda_1(0)$ we obtain

$$\int\limits_{A_{R_1,R_0}} |\nabla w_1|^p \,\mathrm{d}x \le \lambda_1(0) \left( \int\limits_{A_{R_1,r_1}} |u_s|^p \,\mathrm{d}x + \int\limits_{A_{\bar{r},R_0}} |u_0|^p \,\mathrm{d}x \right).$$

Next we estimate $\|w_1\|_p^p$:

$$\begin{aligned} \int\limits_{A_{R_1,R_0}} |w_1|^p \,\mathrm{d}x &= \int\limits_{A_{R_1,r_1}} |u_s|^p \,\mathrm{d}x + \int\limits_{A_{r_1,\bar{r}}} C_1^p \,\mathrm{d}x + \int\limits_{A_{\bar{r},R_0}} |u_0|^p \,\mathrm{d}x \\ &> \int\limits_{A_{R_1,r_1}} |u_s|^p \,\mathrm{d}x + \int\limits_{A_{\bar{r},R_0}} |u_0|^p \,\mathrm{d}x. \end{aligned}$$

Now combining the above estimates, we arrive at

$$\int\limits_{A_{R_1,R_0}} |\nabla w_1|^p \,\mathrm{d}x < \lambda_1(0) \int\limits_{A_{R_1,R_0}} |w_1|^p \,\mathrm{d}x,$$

a contradiction to the definition of $\lambda_1(0)$. Hence we must have $r_1 \le \bar{r}$.

Next we show that $\bar{r} \le r_0$. Suppose that $\bar{r} > r_0$. In this case, we define $w_2$ on $A_{R_1,R_0}$ as below:

$$w_2(x) = \begin{cases} u_0(x), & x \in A_{R_1,\bar{r}}, \\ C_2, & x \in A_{\bar{r},r_0}, \\ u_s(x+se_1), & x \in A_{r_0,R_0}, \end{cases}$$

where $C_2 = u_s(x)$ for $|x+se_1| = r_0$ and $u_0$ is scaled to satisfy $u_0(x) = C_2$ for $|x| = \bar{r}$. As before we see that $w_2 \in W_0^{1,p}(A_{R_1,R_0})$ and

$$\int\limits_{A_{R_1,R_0}} |\nabla w_2|^p \,\mathrm{d}x < \lambda_1(0) \int\limits_{A_{R_1,R_0}} |w_2|^p \,\mathrm{d}x,$$

which again contradicts the definition of $\lambda_1(0)$. Thus $\bar{r} \le r_0$ and we conclude that $r_0 = \bar{r} = r_1$. □

Now we give a proof of our main theorem.

*Proof of Theorem* 1.1. Suppose that there exists $s > 0$ such that $\lambda_1'(s) = 0$. Now Lemmas 3.2, 3.6, and 3.10 give $r_0$ and $r_1$ with $r_0 = r_1$. Further, from the definitions of $r_0$ and $r_1$ (see (3.8) and (3.3)) we can deduce that

$$\nabla u((r_0+s)e_1) = 0 \text{ and } \nabla u(re_1) \neq 0 \quad \forall r > r_1.$$

This is a contradiction, since $r_0 + s = r_1 + s > r_1$. Thus $\lambda_1'(s) < 0$ for all $s \in (0, R_1 - R_0)$. □

*Remark* 3.11. Note that in Theorem 1.1 we consider only the case $\overline{B_{R_0}(se_1)} \subset B_{R_1}(0)$, i.e., $s \in [0, R_1 - R_0)$. For any $s_1, s_2$ satisfying $\sqrt{R_1^2 - R_0^2} \le s_1 < s_2 \le R_1 + R_0$, it is geometrically evident that

$$B_{R_1}(0) \setminus \overline{B_{R_0}(s_1e_1)} \subsetneq B_{R_1}(0) \setminus \overline{B_{R_0}(s_2e_1)}.$$

Now the strict domain monotonicity of $\lambda_1(s)$ (cf. Lemma 5.7 of [10]) gives $\lambda_1(s_1) > \lambda_1(s_2)$. Thus $\lambda_1(s)$ is strictly decreasing on $[\sqrt{R_1^2 - R_0^2}, R_1 + R_0]$. Further, $\lambda_1(s) = \lambda_1(B_{R_1}(0))$ for $s > R_1 + R_0$.

*Remark* 3.12. It can be easily seen that the measure of the set $B_{R_1}(0) \setminus \overline{B_{R_0}(se_1)}$ strictly decreases with respect to $s \in [R_1 - R_0, \sqrt{R_1^2 - R_0^2}]$. However, nothing is known about the behaviour of $\lambda_1(B_{R_1}(0) \setminus \overline{B_{R_0}(se_1)})$ on this interval.

*Remark* 3.13. Let $\Omega_0, \Omega_1$ be any two balls in $\mathbb{R}^N$ such that $\Omega_0 \subsetneq \Omega_1, |\Omega_0| = |B_0|$ and $|\Omega_1| = |B_1|$, where $B_0$ and $B_1$ are concentric balls. Then Theorem 1.1 gives us that $\lambda_1(\Omega_1 \setminus \overline{\Omega_0}) \le \lambda_1(B_1 \setminus \overline{B_0})$. This inequality does not hold in general, if $\Omega_0$ and $\Omega_1$ are not balls. For example, consider the rectangular domains $\Omega_0$ (sides $\frac{\pi R_0}{n}$ and $R_0 n$) and $\Omega_1$ (sides $\frac{\pi R_1}{n}$ and $R_1 n$). Clearly $\lambda_1(\Omega_1 \setminus \Omega_0) \to \infty$ as $n \to \infty$ and $\lambda_1(B_1 \setminus \overline{B_0}) = \lambda_1(A_{R_1,R_0}(0,0)) < \infty$.

## 4. Limit cases $p = 1$ and $p = \infty$

In this section we prove Theorem 1.2. Recall that

$$\Lambda_\infty(s) := \lim_{p\to\infty} \lambda_1^{1/p}(p,s) \quad \text{and} \quad \Lambda_1(s) := \lim_{p\to 1} \lambda_1(p,s).$$

By Theorem 1.1, for any $p > 1$ and $0 \le s_1 < s_2 < R_1 - R_0$ it holds that $0 < \lambda_1(p, s_2) < \lambda_1(p, s_1)$ and hence we immediately deduce that

$$0 \le \Lambda_\infty(s_2) \le \Lambda_\infty(s_1) \quad \text{and} \quad 0 \le \Lambda_1(s_2) \le \Lambda_1(s_1). \tag{4.1}$$

Thus $\Lambda_1(s)$ and $\Lambda_\infty(s)$ are decreasing on $[0, R_1 - R_0)$. To show that $\Lambda_\infty(s)$ is continuous and strictly decreasing on $[0, R_1 - R_0)$, we use the following geometric characterization of $\Lambda_\infty(s)$ obtained in [17]:

$$\Lambda_\infty(s) = \frac{1}{r_{\max}},$$

where $r_{\max}$ is the radius of a maximal ball inscribed in $\Omega_s$.

*Proof of part* (i) *of Theorem* 1.2. For $s \in [0, R_1 - R_0)$, a simple calculation shows that $r_{\max} = \frac{R_1 - R_0 + s}{2}$ and hence

$$\Lambda_\infty(s) = \frac{2}{R_1 - R_0 + s}.$$

Thus one can easily see that $\Lambda_\infty(s)$ is continuous and strictly decreasing on $s \in [0, R_1 - R_0)$. ☐

*Remark* 4.1. The geometric characterization of $\Lambda_\infty(s)$ allows us to compute $\Lambda_\infty(s)$ even for $s \geq R_1 - R_0$. Indeed, the same calculation gives us

$$\Lambda_\infty(s) = \begin{cases} \frac{2}{R_1 - R_0 + s} & \text{for} \quad s \in [0, R_1 + R_0), \\ \frac{1}{R_1} & \text{for} \quad s \geq R_1 + R_0. \end{cases}$$

Clearly $\Lambda_\infty(s)$ is continuous everywhere and differentiable except at the points $s = 0$ and $s = R_1 + R_0$.

We refer the reader to [20] for related problems on the domain dependence of $\Lambda_\infty$.

Now we consider the case $p = 1$. From (4.1) we know that $\Lambda_1(s)$ is decreasing. To show the continuity of $\Lambda_1(s)$ and to prove part (ii) of Theorem 1.2, we use the following variational characterization of $\Lambda_1(s)$ given in [18]:

$$\Lambda_1(s) = h(s),$$

where $h(s)$ stands for the Cheeger constant of $\Omega_s$ which can be defined as

$$h(s) := \inf \frac{|\partial D|}{|D|}. \tag{4.2}$$

Here the infimum is taken over all Lipschitz subdomains $D$ of $\overline{\Omega}_s$ and $|\cdot|$ denotes the Hausdorff measures (coincide with the usual volume and surface area for Lipschitz domains) of dimension $N-1$ in the numerator and the dimension $N$ in the denominator. Any minimizer of (4.2) is called a Cheeger set. It is known that a Cheeger set always exists; see Theorem 8 of [18].

As in Section 2, by considering perturbations of $\Omega_s$ given by the vector field in (2.7) we apply Theorem 1.1 of [22] to conclude that $h(s)$ is continuous on $[0, R_1–R_0)$.

*Proof of part* (ii) *of Theorem* 1.2. It is known (see, for instance, [7] and also the references therein) that concentric annulus $\Omega_0$ is calibrable, (i.e., $\Omega_0$ itself is a Cheeger set of $\Omega_0$) and hence

$$h(0) = \frac{|\partial \Omega_0|}{|\Omega_0|} = N\, \frac{R_1^{N-1} + R_0^{N-1}}{R_1^N - R_0^N}.$$

On the other hand, for the eccentric annulus $\Omega_s$ with $s \in (0, R_1 - R_0)$ it is clear that

$$h(s) \leq \frac{|\partial \Omega_s|}{|\Omega_s|} = N\, \frac{R_1^{N-1} + R_0^{N-1}}{R_1^N - R_0^N} = h(0).$$

Next we show that for $s$ sufficiently close to $R_1 - R_0$ the above inequality is strict. For this we construct an appropriate subset $D$ of $\Omega_s$ satisfying $\frac{|\partial D|}{|D|} < h(0)$.

In this proof, without any ambiguity, we use $|\cdot|$ to denote the various measures such as the length, surface area, and volume of the objects lie in the appropriate spaces. Let $\varepsilon > 0$ be sufficiently small and let $B' = |OB'|\, e_1$ be the point such that $|OB'| = \sqrt{R_1^2 - \varepsilon^2}$ (see Figure 1). Then the hyperplane perpendicular to $e_1$ at $B'$ intersects with $B_{R_1}(0)$ by the $(N-1)$-dimensional ball $B_1$ of radius $|BB'| = \varepsilon$.

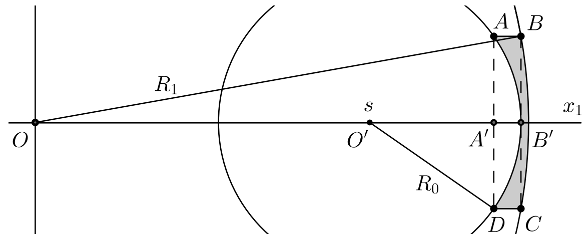


FIGURE 1. "Convex-concave lens" $ABCD_{\mathrm{lens}}$ (grey) and cylinder $ABCD_{\mathrm{cyl}}$ (dashed).

By choosing $s = s_\varepsilon = \sqrt{R_1^2 - \varepsilon^2} - R_0$, we see that the ball $B_{R_0}(se_1)$ touches $B_1$. Now consider the $N$-dimensional "convex-concave lens" $ABCD_{\mathrm{lens}}$ bounded by the spherical caps $BC_{\mathrm{cap}}$ and $AD_{\mathrm{cap}}$ of the spheres $\partial B_{R_1}(0)$ and $\partial B_{R_0}(se_1)$, respectively, and by the lateral cylindrical surface $AB_{\mathrm{lat}}$ generated by the segment $AB$ parallel to $e_1$. Let $ABCD_{\mathrm{cyl}}$ be the cylinder of radius $|BB'|$ and height $|AB|$. For simplicity, we denote the various positive constants which are independent of $\varepsilon$ by $k$. For $\varepsilon > 0$ small enough, observe that

$$\begin{aligned}
|AB| = |A'B'| &= R_0 - \sqrt{R_0^2 - \varepsilon^2} \approx k\varepsilon^2;\\
|AD_{\mathrm{cap}}| > |BC_{\mathrm{cap}}| &> |B_1| = k\varepsilon^{N-1};\\
|ABCD_{\mathrm{lens}}| < |ABCD_{\mathrm{cyl}}| &= |AB||B_1| \approx k\varepsilon^2\,\varepsilon^{N-1};\\
|AB_{\mathrm{lat}}| &= |AB||\partial B_1| \approx k\varepsilon^2\,\varepsilon^{N-2}.
\end{aligned}$$

Now by making use of the above estimates we obtain

$$\begin{aligned}
\frac{|\partial\left(\Omega_s \setminus ABCD_{\mathrm{lens}}\right)|}{|\Omega_s \setminus ABCD_{\mathrm{lens}}|} &= \frac{|\partial\Omega_s| - |AD_{\mathrm{cap}}| - |BC_{\mathrm{cap}}| + |AB_{\mathrm{lat}}|}{|\Omega_s| - |ABCD_{\mathrm{lens}}|}\\
&< \frac{|\partial\Omega_s| - 2k\varepsilon^{N-1} + k\varepsilon^N}{|\Omega_s| - k\varepsilon^{N+1}} < \frac{|\partial\Omega_s|}{|\Omega_s|}
\end{aligned}$$

for sufficiently small $\varepsilon$. Therefore, there exists $s > 0$ such that $h(s) < h(0)$. Now define

$$s^* := \inf\{s \in [0, R_1 - R_0) : \; h(0) > h(s)\}. \tag{4.3}$$

Since $h$ is continuous, the definition of $s^*$ gives $h(0) = h(s^*)$. As $h$ is decreasing, we have $h(0) \geq h(s)$ for $s \in (s^*, R_1 - R_0]$ and the equality would contradict the definition of $s^*$. Thus $h(0) > h(s)$ for $s \in (s^*, R_1 - R_0]$. □

*Remark* 4.2. Clearly $h(s) = h(0)$ for every $s \in [0, s^*]$. Thus, if $s^* > 0$, then the strict monotonicity of $\lambda_1(s)$ fails for $p = 1$. However, whether $s^* > 0$ or not is still an open question. Further, the strict monotonicity of $h(s)$ on the interval $[s^*, R_1 - R_0]$ is not answered yet. It is worth mentioning that a shape derivative formula for $h_1(\Omega)$ is obtained in [22] for $\Omega$ having just one Cheeger set. However, the uniqueness of the Cheeger set for eccentric annular regions $\Omega_s$ is not known.

## 5. Application to the Fučik spectrum

In this section we prove Theorem 1.3. To this end, we use Theorem 1.1 and the variational characterization (1.7) of $\mathscr{C}$, the first nontrivial curve of the Fučik spectrum for the eigenvalue problem (1.6); see [10]. Recall that $\mathscr{C}$ is constructed from points $(t + c(t), c(t))$, where

$$c(t) = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} \left( \int_\Omega |\nabla u|^p \, \mathrm{d}x - t \int_\Omega (u^+)^p \, \mathrm{d}x \right), \quad t \geq 0,$$

and their reflections with respect to the diagonal. See (1.8) for the definition of $\Gamma$.

*Proof of Theorem* 1.3. Let $\Omega$ be a bounded radial domain. Suppose there exist a point on $\mathscr{C}$ and a corresponding eigenfunction $u$ which is radial. Without loss of generality, we can suppose that $t \geq 0$ (otherwise we consider $-u$ instead of $u$). Thus $u$ satisfies the following problem:

$$\left.\begin{aligned} -\Delta_p u &= (t + c(t))(u^+)^{p-1} - c(t)(u^-)^{p-1} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}\right\}$$

By Theorem 2.1 of [11], we know that $u$ has exactly two nodal domains, $N^+ := \{x \in \Omega : u(x) > 0\}$ and $N^- := \{x \in \Omega : u(x) < 0\}$. Since the restriction of $u$ to each of the nodal domains is an eigenfunction of $-\Delta_p$ with a constant sign, we easily get

$$\lambda_1(N^+) = t + c(t) \text{ and } \lambda_1(N^-) = c(t). \tag{5.1}$$

Since $u$ is radial and $\Omega$ is radially symmetric, the nodal domains are also radially symmetric. Assume for definiteness that $u$ is negative near the outer boundary of $\Omega$. Thus there exists $R > 0$ such that $N^+ = \{x \in \Omega : |x| < R\}$ and $N^- = \{x \in \Omega : |x| > R\}$. If $\Omega$ is a ball, say $B_{R_1}(0)$, then $N^+ = B_R(0)$ and $N^- = A_{R_1,R}(0, 0)$. Now for $s \in (0, R_1 - R)$, by using (5.1) and Theorem 1.1 we obtain $\lambda_1(B_R(se_1)) = t + c(t)$ and $\lambda_1(A_{R_1,R}(0, se_1)) < c(t)$. Further, using the continuity of $\lambda_1(\Omega)$ (see, for instance, Theorem 1 of [13]) we can find $\widetilde{R} \in (R, R_1)$ such that

$$\lambda_1(B_{\widetilde{R}}(se_1)) < t + c(t) \text{ and } \lambda_1(A_{R_1,\widetilde{R}}(0, se_1)) < c(t).$$

If $\Omega$ is an annulus, say $A_{R_1,R_0}(0, 0)$, then we have $N^+ = A_{R,R_0}(0, 0)$ and $N^- = A_{R_1,R}(0, 0)$. Now for $0 < s < \min\{R_1 - R, R - R_0\}$ by using (5.1) and Theorem 1.1 we obtain

$$\lambda_1(A_{R,R_0}(se_1, 0)) < t + c(t) \text{ and } \lambda_1(A_{R_1,R}(0, se_1)) < c(t).$$

In either case, we have two disjoint domains $\Omega_1$ and $\Omega_2$ such that

$$\lambda_1(\Omega_1) < t + c(t) \text{ and } \lambda_1(\Omega_2) < c(t).$$

Let $u_1$ and $u_2$ be corresponding eigenfunctions. Clearly $u_1$ and $u_2$ have disjoint supports and

$$\int_\Omega |\nabla u_1|^p \, \mathrm{d}x < (t + c(t)) \int_\Omega |u_1|^p \, \mathrm{d}x \text{ and } \int_\Omega |\nabla u_2|^p \, \mathrm{d}x < c(t) \int_\Omega |u_2|^p \, \mathrm{d}x.$$

The above inequalities lead to a contradiction to the definition (1.7) of $c(t)$ by the same arguments as in the proof of Theorem 3.1 of [10]. Thus $u$ must be nonradial. This completes the proof. □

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