

VISCOUS DISPLACEMENT IN POROUS MEDIA: THE MUSKAT PROBLEM IN 2D

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ABSTRACT. We consider the Muskat problem describing the viscous displacement in a two-phase fluid system located in an unbounded two-dimensional porous medium or Hele-Shaw cell. After formulating the mathematical model as an evolution problem for the sharp interface between the fluids, we show that the Muskat problem with surface tension is a quasilinear parabolic problem, whereas, in the absence of surface tension effects, the Rayleigh–Taylor condition identifies a domain of parabolicity for the fully nonlinear problem. Based upon these aspects, we then establish the local well-posedness for arbitrary large initial data in H^s , $s > 2$, if surface tension is taken into account, respectively for arbitrary large initial data in H^2 that additionally satisfy the Rayleigh–Taylor condition if surface tension effects are neglected. We also show that the problem exhibits the parabolic smoothing effect and we provide criteria for the global existence of solutions.

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1. INTRODUCTION AND MAIN RESULTS

The Muskat problem is a model proposed by M. Muskat in [43] to describe the encroachment of water into an oil sand. This problem is related to the secondary phase of the oil extraction process where water injection is sometimes used to increase the pressure in the oil reservoir and to drive the oil towards the extraction well. In this paper we consider an unbounded fluid system, consisting of two immiscible and incompressible fluid phases, which moves with constant speed $|V| \geq 0$, either in a horizontal or a vertical Hele-Shaw cell (or a homogeneous porous medium). Furthermore, we assume that the flows are two-dimensional and that the velocities are asymptotically equal to $(0, V)$ far away from the origin. In a reference frame which moves with the same speed as the fluids and in the same

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direction, the Muskat problem can be formulated as an evolution problem for the pair $(f, \bar{\omega})$, where $[y = f(t, x) + Vt]$ is a parametrization for the sharp interface that separates the fluids, with f asymptotically flat for large $x \in \mathbb{R}$, and $\bar{\omega}/\sqrt{1 + f'^2}$ is the jump of the velocity at the free interface in tangential direction (see (2.6)). Mathematically, we are confronted with the following evolution problem:

$$(1.1a) \quad \begin{cases} \partial_t f(t, x) = \frac{1}{2\pi} \text{PV} \int_{\mathbb{R}} \frac{y + f'(t, x)(f(t, x) - f(t, x - y))}{y^2 + (f(t, x) - f(t, x - y))^2} \bar{\omega}(t, x - y) dy \\ \text{for } t > 0, x \in \mathbb{R}, \\ f(0) = f_0, \end{cases}$$

where f and $\bar{\omega}$ are additionally coupled through the relation

$$(1.1b) \quad \begin{aligned} & \left[\sigma \kappa(f) - \left(g(\rho_- - \rho_+) + \frac{\mu_- - \mu_+ V}{k} \right) f \right]'(t, x) \\ &= \frac{\mu_- + \mu_+}{2k} \bar{\omega}(t, x) \\ &+ \frac{\mu_- - \mu_+}{2\pi k} \text{PV} \int_{\mathbb{R}} \frac{y f'(t, x) - (f(t, x) - f(t, x - y))}{y^2 + (f(t, x) - f(t, x - y))^2} \bar{\omega}(t, x - y) dy \end{aligned}$$

for $t > 0$ and $x \in \mathbb{R}$. We denote by $(\cdot)'$ the spatial derivative ∂_x , g is the Earth's gravity, k is the permeability of the homogeneous porous medium, σ is the surface tension coefficient at the free boundary, and ρ_{\pm} is the density and μ_{\pm} the viscosity of the fluid located at $\Omega_{\pm}(t)$, where

$$\Omega_-(t) := [y < f(t, x) + Vt] \quad \text{and} \quad \Omega_+(t) := [y > f(t, x) + Vt].$$

Moreover, $\kappa(f(t))$ is the curvature of the graph $[y = f(t, x) + tV]$ and PV denotes the principal value which, depending on the regularity of the functions under the integral, is taken at zero and/or infinity. If V is positive, then the fluid $-$ expands into the region occupied by the fluid $+$ and vice versa; if V is negative, then the fluid $+$ expands into the region occupied by the fluid $-$ (see Section 2 for a rigorous derivation of (1.1)). When neglecting surface tension effects we set $\sigma = 0$ and we require that the first equation of (1.1a) and the equation (1.1b) hold also at $t = 0$.

In recent years the Muskat problem has received, due to its physical relevance, much attention especially in the field of applied mathematics. In the absence of surface tension effects the local existence of solutions was first addressed by F. Yi in [52] under the assumption that the Rayleigh–Taylor condition holds. The Rayleigh–Taylor condition [47] is a sign restriction on the jump of the pressure gradients in normal direction at the interface $[y = f_0(x)]$, and it reads

$$(1.2) \quad \partial_\nu p_- < \partial_\nu p_+ \quad \text{on } [y = f_0(x)],$$

where p_{\pm} is the pressure of the fluid \pm and ν is the outward normal at $[y = f_0(x)]$ with respect to $\Omega_-(0)$. Thereafter, questions related to the well-posedness of the

Muskat problem and other qualitative aspects of the dynamics have been studied in [5, 8, 10–22, 24–26, 32–35, 40, 48] in several physical scenarios and with various methods. These references show the Rayleigh–Taylor condition is crucial in the analysis of this problem. In the regime where the Rayleigh–Taylor condition holds with reverse inequality sign, for example if a less viscous fluid displaces a more viscous one, or when a more dense fluid sits on top of a less dense one, physical experiments evidence the occurrence of viscous fingering (cf. [36, 47]) and the Muskat problem is ill-posed (cf., e.g., [20, 25, 48]). On the other hand, it was recently shown in [26], in a bounded and periodic striplike geometry, that the Rayleigh–Taylor condition actually identifies a domain of parabolicity for the Muskat problem.

When surface tension effects are taken into consideration, it was proven in [26, 45, 46], in bounded geometries, that the Muskat problem is a quasilinear parabolic problem for arbitrary large initial data, without any kind of restrictions. Also in this setting, the solvability of the problem has been addressed in several physical scenarios with quite intricate methods [6, 24, 25, 31, 37, 50].

The first goal of this paper is to prove that the classical formulation of the Muskat problem (see Section 2) is equivalent to the evolution problem (1.1); cf. Proposition 2.2.

Our second goal is to extend the methods that have been recently applied in [40], in the particular case of fluids with equal viscosities, to the general case considered herein in order to establish the local well-posedness for the Muskat problem with and without surface tension by similar strategies and in a very general context. If the fluids have equal viscosities, the equation (1.1b) determines $\bar{\omega}$ as a function of f , and (1.1) becomes an evolution problem for f only. Surprisingly, the analysis in [40] shows that the corresponding evolution problem is of quasilinear parabolic type in both regimes, that is, for $\sigma > 0$, or when $\sigma = 0$ and the Rayleigh–Taylor condition holds. However, for $\mu_- \neq \mu_+$, the equation (1.1b) is implicit and this fact enhances the nonlinear and nonlocal character of the problem and makes the analysis more involved.

In the case when $\sigma = 0$ and the Rayleigh–Taylor condition holds, the well-posedness of the problem is still an open question. Local existence of solutions to (1.1) was first addressed in [18] for arbitrary large data in H^3 , and in three space dimensions in [19] for initial data in H^4 . Global existence is established in [48] in the periodic case and for small initial data. Quite recently, the authors of [14] have proven the existence and uniqueness of solutions which satisfy an additional energy estimate for initial data in H^2 and which are small with respect to some $H^{3/2+\varepsilon}$ -norm. In Theorem 1.2 we show that the Muskat problem without surface tension is well-posed for arbitrary large initial data in H^2 . To achieve this result we formulate (1.1) as a fully nonlinear evolution problem for f and we prove that the set of initial data for which the Rayleigh–Taylor condition holds defines, also in this geometry, a domain of parabolicity for the Muskat problem. It is worth emphasizing that the quasilinear character, present for $\mu_- = \mu_+$, is not preserved when $\mu_- \neq \mu_+$ and this makes the Muskat problem without surface tension more difficult to handle.

For $\sigma > 0$, the local well-posedness of (1.1) has been addressed in [6] for initial data in H^s , with $s \geq 6$ (see also [50] for a global existence result for small initial data in H^s , with $s \geq 6$). Exploiting the quasilinear structure of the curvature term, we show that in this regime (1.1) can be formulated as a quasilinear parabolic evolution

problem. This property enables us to establish the local well-posedness of (1.1) for arbitrary large initial data in H^s , with $s > 2$; cf. Theorem 1.1. In particular, we may choose the initial data such that the curvature is unbounded or discontinuous.

Moreover, we show that the Muskat problem features the effect of parabolic smoothing: solutions (which possess additional regularity when $\sigma = 0$) become instantly real-analytic in the time-space domain. Besides, we provide criteria for the global existence of solutions.

The first main result of this paper is the following theorem.

Theorem 1.1 (Well-posedness: with surface tension). *Let $\sigma > 0$. The evolution problem (1.1) possesses for each $f_0 \in H^s(\mathbb{R})$, $s \in (2, 3)$, a unique maximal solution $f := f(\cdot; f_0)$ satisfying*

$$f \in C([0, T_+(f_0)), H^s(\mathbb{R})) \cap C((0, T_+(f_0)), H^3(\mathbb{R})) \cap C^1((0, T_+(f_0)), L_2(\mathbb{R})),$$

with $T_+(f_0) \in (0, \infty]$, and $[(t, f_0) \mapsto f(t; f_0)]$ defines a semiflow on $H^s(\mathbb{R})$. Additionally, if

$$\sup_{[0, T_+(f_0)) \cap [0, T]} \|f(t)\|_{H^s} < \infty \quad \text{for all } T > 0,$$

then $T_+(f_0) = \infty$. Moreover, given $k \in \mathbb{N}$, it holds that

$$f \in C^\omega((0, T_+(f_0)) \times \mathbb{R}, \mathbb{R}) \cap C^\omega((0, T_+(f_0)), H^k(\mathbb{R})).^1$$

In particular, $f(t, \cdot)$ is real-analytic for each $t \in (0, T_+(f_0))$.

We emphasize that exactly the same result as in Theorem 1.1 has been achieved in [40] in the simpler case of fluids with equal viscosities.

When surface tension is neglected, that is, $\sigma = 0$, we assume that

$$(1.3) \quad \Theta := g(\rho_- - \rho_+) + \frac{\mu_- - \mu_+}{k} V \neq 0.$$

The situation when $\sigma = 0 = \Theta$ is special, because in this case the problem (1.1) possesses for each $f_0 \in H^s(\mathbb{R})$, with $s > 3/2$, a unique global solution $f(t) := f_0$ for all $t \in \mathbb{R}$; cf. Section 5. The corresponding flow is stationary with constant velocities equal to $(0, V)$ and hydrostatic pressures.

In order to discuss the well-posedness of (1.1) with $\sigma = 0 \neq \Theta$, we introduce the set of initial data for which the Rayleigh–Taylor condition holds as

$$\mathcal{O} := \{f_0 \in H^2(\mathbb{R}) : \partial_\nu p_- < \partial_\nu p_+ \text{ on } [y = f_0(x)]\}.$$

The Rayleigh–Taylor condition is reformulated later on (cf. (5.10)) where it is also proven that \mathcal{O} is an open subset of $H^2(\mathbb{R})$. Our analysis in Section 5 shows that \mathcal{O} is nonempty if and only if

$$(1.4) \quad \Theta > 0.$$

The relation (1.4) is the classical condition found within the linear theory by Saffman and Taylor [47]. In particular, if the flow takes place in a vertical Hele-Shaw cell and $V = 0$, then the less dense fluid lies above. For flows in horizontal Hele-Shaw cells the effects due to gravity are usually neglected, that is, $g = 0$, and (1.2) implies that $V \neq 0$ and that the more viscous fluid expands into the region occupied by the less viscous one.

We now come to the second main result of this paper.

¹Here and in the following C^ω stands for real-analyticity, while C^{1-} denotes local Lipschitz continuity.

Theorem 1.2 (Well-posedness: without surface tension). *Let $\sigma = 0$, $\mu_- \neq \mu_+$,² and assume that (1.4) holds. Given $f_0 \in \mathcal{O}$, the problem (1.1) possesses a solution*

$$f \in C([0, T], \mathcal{O}) \cap C^1([0, T], H^1(\mathbb{R})) \cap C^\alpha_\alpha((0, T], H^2(\mathbb{R}))$$

for some $T > 0$ and an arbitrary $\alpha \in (0, 1)$. It further holds that:

(i) *f is the unique solution to (1.1) belonging to*

$$\bigcup_{\beta \in (0, 1)} C([0, T], \mathcal{O}) \cap C^1([0, T], H^1(\mathbb{R})) \cap C^\beta_\beta((0, T], H^2(\mathbb{R})).$$

(ii) *f may be extended to a maximally defined solution*

$$f(\cdot; f_0) \in C([0, T_+(f_0)), \mathcal{O}) \cap C^1([0, T_+(f_0)), H^1(\mathbb{R})) \cap \bigcap_{\beta \in (0, 1)} C^\beta_\beta((0, T], H^2(\mathbb{R}))$$

for all $T < T_+(f_0)$, where $T_+(f_0) \in (0, \infty]$.

(iii) *The solution map $[(t, f_0) \mapsto f(t; f_0)]$ defines a semiflow on \mathcal{O} which is real-analytic in the open set $\{(t, f_0) : f_0 \in \mathcal{O}, 0 < t < T_+(f_0)\}$.*

(iv) *If $f(\cdot; f_0) : [0, T_+(f_0)) \cap [0, T] \rightarrow \mathcal{O}$ is uniformly continuous for all $T > 0$, then either*

$$T_+(f_0) < \infty \text{ and } \lim_{t \rightarrow T_+(f_0)} f(t; f_0) \in \partial\mathcal{O}, \text{ or } T_+(f_0) = \infty.$$

(v) *If $f(\cdot; f_0) \in B((0, T), H^{2+\varepsilon}(\mathbb{R}))$ for some $T \in (0, T_+(f_0))$ and $\varepsilon \in (0, 1)$, then*

$$f \in C^\omega((0, T) \times \mathbb{R}, \mathbb{R}) \cap C^\omega((0, T), H^k(\mathbb{R})) \quad \text{for each } k \in \mathbb{N}.$$

Given $T > 0$ and a Banach space \mathbb{X} , we let $B((0, T], \mathbb{X})$ [resp., $B((0, T), \mathbb{X})$] denote the Banach space of all bounded functions from $(0, T]$ [resp., $(0, T)$] into \mathbb{X} , and, given $\alpha \in (0, 1)$, we set

$$C^\alpha_\alpha((0, T], \mathbb{X}) := \left\{ u \in B((0, T], \mathbb{X}) : \sup_{s \neq t} \frac{|t^\alpha u(t) - s^\alpha u(s)|}{|t - s|^\alpha} < \infty \right\}.$$

With respect to (iv) we add the following comments. Firstly, as shown in [12, Theorem 1.1] in the case when $\mu_- = \mu_+$, there exist solutions which are not uniformly continuous in \mathcal{O} , in the sense that their slope blows up in finite time. Secondly, there exist global solutions to (1.1); cf. [16, Theorem 3.1] (see also [40, Corollary 1.4]) or [14, Theorem 2.2] (in the periodic setting). Lastly, the existence of solutions which are uniformly bounded in $H^2(\mathbb{R})$ and violate the Rayleigh–Taylor sign condition at time $T_+(f_0) < \infty$ is, to the best of our knowledge, an open issue.

The condition that $f \in B((0, T), H^{2+\varepsilon}(\mathbb{R}))$ for some $T \in (0, T_+(f_0))$ imposed at (v) is a technical assumption. Nevertheless, if $f_0 \in \mathcal{O} \cap H^3(\mathbb{R})$, our arguments can be extended to show that Theorem 1.2 still holds true if we replace \mathcal{O} by $\mathcal{O} \cap H^3(\mathbb{R})$ and $H^k(\mathbb{R})$ by $H^{k+1}(\mathbb{R})$ for $k \in \{1, 2\}$, possibly with a smaller maximal existence time $T_{+,3}(f_0)$. Hence, for solutions that start in $H^3(\mathbb{R})$, the property required at (v) is satisfied for all $T < T_{+,3}(f_0)$ and all $\varepsilon \in (0, 1)$. This additional regularity is needed for our argument because the uniqueness property in Theorem 1.2 holds only for solutions that additionally belong to the space $C^\alpha_\alpha((0, T], H^2(\mathbb{R}))$ for some

²Theorem 1.2 is still valid when $\mu_- = \mu_+$, however in this particular case its assertions can be improved; cf. [40, Theorem 1.1].

$\alpha \in (0, 1)$, and this space is not sufficiently flexible with respect to the parameter trick used in the proof of Theorem 1.2.

2. THE GOVERNING EQUATIONS AND THE EQUIVALENCE RESULT

We start by presenting the classical formulation of the Muskat problem introduced in Section 1. First of all, both fluids are taken to be incompressible, immiscible, and of Newtonian type. Since flows in porous media or Hele-Shaw cells occur at low Reynolds numbers, they are usually modeled as being two-dimensional and Darcy’s law is used instead of the conservation of momentum equation [9]. Hence, the equations of motion in the fluid layers are³

$$(2.1a) \quad \begin{cases} \operatorname{div} v_{\pm}(t) = 0 & \text{in } \Omega_{\pm}(t), \\ v_{\pm}(t) = -\frac{k}{\mu_{\pm}}(\nabla p_{\pm}(t) + (0, \rho_{\pm}g)) & \text{in } \Omega_{\pm}(t), \end{cases}$$

with $v_{\pm} := (v_{\pm}^1, v_{\pm}^2)$ denoting the velocity of the fluid \pm . These equations are supplemented by the natural boundary conditions on the free surface:

$$(2.1b) \quad \begin{cases} \langle v_+(t) | \nu(t) \rangle = \langle v_-(t) | \nu(t) \rangle & \text{on } [y = f(t, x) + Vt], \\ p_+(t) - p_-(t) = \sigma \kappa(f(t)) & \text{on } [y = f(t, x) + Vt], \end{cases}$$

where $\nu(t)$ is the unit normal at $[y = f(t, x) + Vt]$ pointing into $\Omega_+(t)$ and $\langle \cdot | \cdot \rangle$ is the Euclidean inner product on \mathbb{R}^2 . Furthermore, we impose the far-field boundary conditions

$$(2.1c) \quad \begin{cases} f(t, x) \rightarrow 0 & \text{for } |x| \rightarrow \infty, \\ v_{\pm}(t, x, y) \rightarrow (0, V) & \text{for } |(x, y)| \rightarrow \infty. \end{cases}$$

The motion of the interface $[y = f(t, x) + Vt]$ is coupled to that of the fluids through the kinematic boundary condition

$$(2.1d) \quad \partial_t f(t) = \langle v_{\pm}(t) | (-f'(t), 1) \rangle - V \quad \text{on } [y = f(t, x) + Vt],$$

and the interface at time $t = 0$ is assumed to be known,

$$(2.1e) \quad f(0) = f_0.$$

We now rewrite the classical formulation (2.1) of the Muskat problem in a coordinates system which moves with the same speed and in the same direction as the fluid system. To this end we introduce

$$\begin{cases} \tilde{v}_{\pm}(t, x, y) := v_{\pm}(t, x, y + Vt) - (0, V), \\ \tilde{p}_{\pm}(t, x, y) := p_{\pm}(t, x, y + Vt) \end{cases} \quad \text{in } \Omega_{\pm}^0(t) := \Omega_{\pm}(t) - (0, Vt).$$

³The first equation of (2.1a) expresses the fact that $\operatorname{div} v_+(t) = 0$ in $\Omega_+(t)$ and $\operatorname{div} v_-(t) = 0$ in $\Omega_-(t)$. This convention is also used in the second equation of (2.1a) and at many other places in the paper in various contexts.

It is not difficult to see that the equations (2.1) are equivalent to the following system of equations which has $(f, \tilde{v}_\pm, \tilde{p}_\pm)$ as unknowns:

$$(2.2) \quad \left\{ \begin{array}{ll} \operatorname{div} \tilde{v}_\pm(t) = 0 & \text{in } \Omega_\pm^0(t), \\ \tilde{v}_\pm(t) + (0, V) = -k\mu_\pm^{-1}(\nabla \tilde{p}_\pm(t) + (0, \rho_\pm g)) & \text{in } \Omega_\pm^0(t), \\ \langle \tilde{v}_+(t) | \nu(t) \rangle = \langle \tilde{v}_-(t) | \nu(t) \rangle & \text{on } [y = f(t, x)], \\ \tilde{p}_+(t) - \tilde{p}_-(t) = \sigma \kappa(f(t)) & \text{on } [y = f(t, x)], \\ f(t, x) \rightarrow 0 & \text{for } |x| \rightarrow \infty, \\ \tilde{v}_\pm(t, x, y) \rightarrow 0 & \text{for } |(x, y)| \rightarrow \infty, \\ \partial_t f(t) = \langle \tilde{v}_\pm(t) | (-f'(t), 1) \rangle & \text{on } [y = f(t, x)], \\ f(0) = f_0. & \end{array} \right.$$

Before stating the equivalence result (cf. Proposition 2.2), we first give a preparatory lemma, which is needed in the proof of Proposition 2.2 and also later on in the analysis (see the proof of Theorem 3.5). The proof of Lemma 2.1 is based on classical arguments used to establish the Plemelj formula and the Privalov theorem for Cauchy-type integrals defined on regular curves (see, e.g., [38]) and on Lemmas 3.1–3.2. Details of the proof can be found, in a particular case, in [40, Lemma A.2].

Lemma 2.1. *Given $f \in H^2(\mathbb{R})$ and $\bar{\omega} \in H^1(\mathbb{R})$, we set*

$$(2.3) \quad \hat{v}(x, y) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{-(y - f(s)), x - s}{(x - s)^2 + (y - f(s))^2} \bar{\omega}(s) ds \quad \text{in } \mathbb{R}^2 \setminus [y = f(x)].$$

Further, let $\Omega_-^0 := [y < f(x)]$ and $\Omega_+^0 := [y > f(x)]$, and set $\hat{v}_\pm := \hat{v}|_{\Omega_\pm^0}$. It then holds that $\hat{v}_\pm \in C(\overline{\Omega_\pm^0}) \cap C^1(\Omega_\pm^0)$ and

$$(2.4) \quad \hat{v}_\pm(x, y) \rightarrow 0 \quad \text{for } |(x, y)| \rightarrow \infty.$$

Additionally, if $\bar{\omega} \in C_0^\infty(\mathbb{R})$, then there exist a positive integer $N \in \mathbb{N}$ and a constant C such that

$$(2.5) \quad |\hat{v}_\pm(x, y)| \leq \frac{C \|\bar{\omega}\|_1}{|(x, y)|} \quad \text{for all } (x, y) \in \Omega_\pm^0 \text{ with } |(x, y)| \geq N.$$

Proof. The first two claims follow in the same way as in [40, Lemma A.2], while (2.5) is a simple exercise. □

In the particular case when $\mu_- = \mu_+$, (1.1b) gives a precise correlation between the smoothness of $\bar{\omega}$ and that of f . This correlation is for $\mu_- \neq \mu_+$ no longer obvious. We prove herein (cf. Proposition 3.6) that for $f \in H^2(\mathbb{R})$, the equation (1.1b) has a unique solution $\bar{\omega} \in H^1(\mathbb{R})$, provided that the left-hand side of (1.1b) belongs to $H^1(\mathbb{R})$. If $\sigma > 0$, the latter requirement implies that in fact $f \in H^4(\mathbb{R})$ is needed. Thanks to the parabolic smoothing in Theorem 1.2, this additional regularity is inherited by all solutions. This is one of the reasons, besides the difference in nonlinear behavior, why we separate the cases $\sigma = 0$ and $\sigma > 0$ in Proposition 2.2.

Proposition 2.2 (Equivalence of the two formulations). *Let $T \in (0, \infty]$ be given.*

(a) *Let $\sigma = 0$. The following are equivalent:*

- (i) *the problem (2.1) for $f \in C^1([0, T], L_2(\mathbb{R}))$ and*
 - $f(t) \in H^2(\mathbb{R}), \bar{\omega}(t) := \langle (v_-(t) - v_+(t))|_{[y=f(t,x)+Vt]}(1, f'(t)) \rangle \in H^1(\mathbb{R}),$
 - $v_{\pm}(t) \in C(\overline{\Omega_{\pm}(t)}) \cap C^1(\Omega_{\pm}(t)), p_{\pm}(t) \in C^1(\overline{\Omega_{\pm}(t)}) \cap C^2(\Omega_{\pm}(t))$ for all $t \in [0, T];$
- (ii) *the evolution problem (1.1) for $f \in C^1([0, T], L_2(\mathbb{R})), f(t) \in H^2(\mathbb{R}),$ and $\bar{\omega}(t) \in H^1(\mathbb{R})$ for all $t \in [0, T].$*

(b) *Let $\sigma > 0$. The following are equivalent:*

- (i) *the problem (2.1) for $f \in C^1((0, T), L_2(\mathbb{R})) \cap C([0, T], L_2(\mathbb{R}))$ and*
 - $f(t) \in H^4(\mathbb{R}), \bar{\omega}(t) := \langle (v_-(t) - v_+(t))|_{[y=f(t,x)+Vt]}(1, f'(t)) \rangle \in H^1(\mathbb{R}),$
 - $v_{\pm}(t) \in C(\overline{\Omega_{\pm}(t)}) \cap C^1(\Omega_{\pm}(t)), p_{\pm}(t) \in C^1(\overline{\Omega_{\pm}(t)}) \cap C^2(\Omega_{\pm}(t))$ for all $t \in (0, T);$
- (ii) *the Muskat problem (1.1) for $f \in C^1((0, T), L_2(\mathbb{R})) \cap C([0, T], L_2(\mathbb{R})),$ $f(t) \in H^4(\mathbb{R}),$ and $\bar{\omega}(t) \in H^1(\mathbb{R})$ for all $t \in (0, T).$*

Proof. We only prove the claim for $\sigma = 0$ (the proof of (b) is similar). We first consider the implication (i) \Rightarrow (ii). Given a set E , we denote herein by $\mathbf{1}_E$ the characteristic function of E . Assume that (f, v_{\pm}, p_{\pm}) is a solution to (2.1) on $[0, T]$ and let $t \in [0, T)$ be fixed (the time dependence is not written explicitly in this proof). It is more convenient to work here with the formulation (2.2). Stokes' theorem and the second equation of (2.1a) show that the vorticity

$$\omega := \text{rot } \tilde{v} := \partial_x \tilde{v}^2 - \partial_y \tilde{v}^1$$

defined by the global velocity field $\tilde{v} := (\tilde{v}^1, \tilde{v}^2) := \tilde{v}_- \mathbf{1}_{[y \leq f(x)]} + \tilde{v}_+ \mathbf{1}_{[y > f(x)]}$ is supported on the free boundary, that is,

$$\langle \omega, \varphi \rangle = \int_{\mathbb{R}} \bar{\omega}(x) \varphi(x, f(x)) dx \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^2),$$

where

$$(2.6) \quad \bar{\omega} := \langle (\tilde{v}_- - \tilde{v}_+)|_{[y=f(x)]}(1, f') \rangle.$$

We now claim that the velocity is given by the Biot-Savart law, that is, $\tilde{v} = \widehat{v}$ in $\mathbb{R}^2 \setminus [y = f(x)]$, where \widehat{v} is defined in (2.3) and $\bar{\omega}$ in (2.6). Indeed, according to the Plemelj formula (cf., e.g., [38]), the limits $\widehat{v}_-(x, f(x))$ and $\widehat{v}_+(x, f(x))$ of \widehat{v} at $(x, f(x))$ when we approach this point from above the interface $[y = f(x)]$ or from below, respectively, are

$$(2.7) \quad \begin{aligned} \widehat{v}_{\pm}(x, f(x)) &= \frac{1}{2\pi} \text{PV} \int_{\mathbb{R}} \frac{(-(f(x) - f(x - s)), s)}{s^2 + (f(x) - f(x - s))^2} \bar{\omega}(x - s) ds \\ &\mp \frac{1}{2} \frac{(1, f'(x)) \bar{\omega}(x)}{1 + f'^2(x)}, \quad x \in \mathbb{R}. \end{aligned}$$

Moreover, the restrictions \widehat{v}_{\pm} of \widehat{v} to Ω_{\pm}^0 belong to $C(\overline{\Omega_{\pm}^0}) \cap C^1(\Omega_{\pm}^0)$, they satisfy the first, third, and sixth equations of (2.2), and $\text{rot } \widehat{v}_{\pm} = \partial_x \widehat{v}_{\pm}^2 - \partial_y \widehat{v}_{\pm}^1 = 0$ in Ω_{\pm}^0 . We

now introduce $V_{\pm} := \tilde{v}_{\pm} - \widehat{v}_{\pm}$, we set $V := (V^1, V^2) := V_- \mathbf{1}_{[y \leq f(x)]} + V_+ \mathbf{1}_{[y > f(x)]}$, and we consider the stream functions

$$\psi_{\pm}(x, y) := \int_{f(x)}^y V_{\pm}^1(x, s) ds - \int_0^x \langle V_{\pm}(s, f(s)) | (-f'(s), 1) \rangle ds \quad \text{for } (x, y) \in \overline{\Omega}_{\pm}^0.$$

The properties of \widehat{v}_{\pm} established above together with (2.7) and Stokes' theorem show that the function $\psi := \psi_- \mathbf{1}_{[y \leq f]} + \psi_+ \mathbf{1}_{[y > f]}$ satisfies $\psi \in C(\mathbb{R}^2)$ and $\Delta\psi = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Hence, ψ is the real part of a holomorphic function $u : \mathbb{C} \rightarrow \mathbb{C}$. Since u' is also holomorphic and $u' = -(V^2, V^1)$ is bounded and vanishes for $|(x, y)| \rightarrow \infty$, it follows that $u' = 0$, hence $\tilde{v}_{\pm} = \widehat{v}_{\pm}$. Differentiating now the fourth equation of (2.2) once, the second equation of (2.2) and (2.7) lead us to

$$\begin{aligned} & \left[\sigma\kappa(f) - \left(g(\rho_- - \rho_+) + \frac{\mu_- - \mu_+}{k} V \right) f \right]'(x) \\ &= \frac{\mu_- + \mu_+}{2k} \overline{\omega}(x) \\ &+ \frac{\mu_- - \mu_+}{2\pi k} \text{PV} \int_{\mathbb{R}} \frac{f'(x)s - (f(x) - f(x-s))}{s^2 + (f(x) - f(x-s))^2} \overline{\omega}(x-s) ds \end{aligned}$$

for all $x \in \mathbb{R}$. Finally, in view of (2.7) and of the seventh equation of (2.2), we may conclude that $(f, \overline{\omega})$ is a solution to (1.1).

For the inverse implication we define $\tilde{v}_{\pm} \in C(\overline{\Omega}_{\pm}^0) \cap C^1(\Omega_{\pm}^0)$ according to (2.3) and the pressures $\tilde{p}_{\pm} \in C^1(\overline{\Omega}_{\pm}^0) \cap C^2(\Omega_{\pm}^0)$ by the formula

$$\begin{aligned} \tilde{p}_{\pm}(x, y) := c_{\pm} - \frac{\mu_{\pm}}{k} \int_0^x \tilde{v}_{\pm}^1(s, \pm d) ds - \frac{\mu_{\pm}}{k} \int_{\pm d}^y \tilde{v}_{\pm}^2(x, s) ds - \left(\rho_{\pm}g + \frac{\mu_{\pm}V}{k} \right) y \\ \text{for all } (x, y) \in \overline{\Omega}_{\pm}^0, \end{aligned}$$

where d is a positive constant satisfying $d > \|f\|_{\infty}$ and $c_{\pm} \in \mathbb{R}$. For a proper choice of c_{\pm} , the tuple $(f, \tilde{p}_{\pm}, \tilde{v}_{\pm})$ solves all the equations of (2.2) and possesses the regularity properties stated in (i). This completes the proof of (a). □

3. ON THE RESOLVENT SET OF THE ADJOINT OF THE DOUBLE LAYER POTENTIAL

In order to solve the Muskat problem (1.1), with and without surface tension, we basically follow the same strategy. The first step in our approach is to formulate the system (1.1) as an evolution problem for f . To this end, we have to address the solvability of the equation (1.1b), which is the content of this section. This issue is equivalent to inverting the linear operator $(1 + a_{\mu}\mathbb{A}(f))$, where

$$(3.1) \quad \mathbb{A}(f)[\overline{\omega}](x) := \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{yf'(x) - (f(x) - f(x-y))}{y^2 + (f(x) - f(x-y))^2} \overline{\omega}(x-y) dy,$$

and where

$$a_{\mu} := \frac{\mu_- - \mu_+}{\mu_- + \mu_+}$$

denotes the Atwood number. The operator $\mathbb{A}(f)$ can be viewed as the adjoint of the double layer potential; see, e.g., [30, 51] and Lemma 3.8. The resolvent set of $\mathbb{A}(f)$ has been studied previously in the literature (see [18, 19, 30, 41, 51] and the references therein), but mostly in bounded geometries where $\mathbb{A}(f)$ is a compact operator. With respect to our functional analytic approach to (1.1), the existing results cannot be applied, especially because the invertibility in $\mathcal{L}(H^1(\mathbb{R}))$ is established for functions

f that are too regular. For this reason we readdress this issue below, the emphasis being on finding the optimal correlation between the regularity of f and the order of the Sobolev space where the invertibility is considered; see Remark 3.7. It is important to note, in the context of the Muskat problem (1.1), that the Atwood number satisfies $|a_\mu| < 1$.

Some multilinear integral operators. We now introduce a class of multilinear singular operators which we encounter later on when solving the implicit equation (1.1b) for $\bar{\omega}$. Given $n, m \in \mathbb{N}$, with $m \geq 1$, we define the singular integral operator

$$B_{n,m}(a_1, \dots, a_m)[b_1, \dots, b_n, \bar{\omega}](x) := \text{PV} \int_{\mathbb{R}} \frac{\bar{\omega}(x-y)}{y} \frac{\prod_{i=1}^n (\delta_{[x,y]} b_i/y)}{\prod_{i=1}^m [1 + (\delta_{[x,y]} a_i/y)^2]} dy,$$

where $a_1, \dots, a_m, b_1, \dots, b_n : \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz functions and $\bar{\omega} \in L_2(\mathbb{R})$. To keep the formulas short, we set

$$\delta_{[x,y]} a := a(x) - a(x-y) \quad \text{for } x, y \in \mathbb{R}.$$

Letting H denote the Hilbert transform [49], it holds that $B_{0,1}(0) = \pi H$, and moreover

$$(3.2) \quad \pi \mathbb{A}(f) = f' B_{0,1}(f) - B_{1,1}(f)[f, \cdot].$$

We first establish the following result.

Lemma 3.1. *Let $1 \leq m \in \mathbb{N}$ and $n \in \mathbb{N}$ be given. Then:*

- (i) *Given Lipschitz functions $a_1, \dots, a_m, b_1, \dots, b_n : \mathbb{R} \rightarrow \mathbb{R}$, there exists a positive constant C , which depends only on n, m , and $\max_{i=1, \dots, m} \|a'_i\|_\infty$, such that*

$$\|B_{n,m}(a_1, \dots, a_m)[b_1, \dots, b_n, \bar{\omega}]\|_2 \leq C \|\bar{\omega}\|_2 \prod_{i=1}^n \|b'_i\|_\infty$$

for all $\bar{\omega} \in L_2(\mathbb{R})$. In particular $B_{n,m}(a_1, \dots, a_m)[b_1, \dots, b_n, \cdot] \in \mathcal{L}(L_2(\mathbb{R}))$.

- (ii) $B_{n,m} \in C^{1-}((W_\infty^1(\mathbb{R}))^m, \mathcal{L}_{n+1}((W_\infty^1(\mathbb{R}))^n \times L_2(\mathbb{R}), L_2(\mathbb{R})))$.
- (iii) *Given $r \in (3/2, 2)$ and $\tau \in (1/2, 2)$, it holds that*

$$\|B_{n,m}(a_1, \dots, a_m)[b_1, \dots, b_n, \bar{\omega}]\|_\infty \leq C \|\bar{\omega}\|_{H^\tau} \prod_{i=1}^n \|b_i\|_{H^r}$$

for all $a_1, \dots, a_m, b_1, \dots, b_n \in H^r(\mathbb{R})$ and $\bar{\omega} \in H^\tau(\mathbb{R})$, with C depending only on τ, r, n, m , and $\max_{i=1, \dots, m} \|a_i\|_{H^r}$. In particular,

$$B_{n,m} \in C^{1-}((H^r(\mathbb{R}))^m, \mathcal{L}_{n+1}((H^r(\mathbb{R}))^n \times H^\tau(\mathbb{R}), L_\infty(\mathbb{R}))).$$

Proof. Assertion (i) has been proved in [40, Remark 3.3] by exploiting a result from harmonic analysis due to T. Murai [42]. Furthermore, the local Lipschitz continuity properties stated in (ii) and (iii) follow from the estimates in (i) and (iii), respectively, via the relation

$$(3.3) \quad \begin{aligned} & B_{n,m}(\tilde{a}_1, \dots, \tilde{a}_m)[b_1, \dots, b_n, \bar{\omega}] - B_{n,m}(a_1, \dots, a_m)[b_1, \dots, b_n, \bar{\omega}] \\ &= \sum_{i=1}^m B_{n+2,m+1}(\tilde{a}_1, \dots, \tilde{a}_i, a_i, \dots, a_m)[b_1, \dots, b_n, a_i + \tilde{a}_i, a_i - \tilde{a}_i, \bar{\omega}]. \end{aligned}$$

In order to establish the estimate given in (iii), we write

$$B_{n,m}(a_1, \dots, a_m)[b_1, \dots, b_n, \bar{\omega}] = T_1 + T_2 + T_3,$$

where

$$T_1(x) := \int_{|y| \leq 1} \frac{\prod_{i=1}^n (\delta_{[x,y]} b_i / y)}{\prod_{i=1}^m [1 + (\delta_{[x,y]} a_i / y)^2]} \frac{\bar{\omega}(x - y) - \bar{\omega}(x)}{y} dy,$$

$$T_2(x) := \bar{\omega}(x) \text{PV} \int_{|y| \leq 1} \frac{1}{y} \frac{\prod_{i=1}^n (\delta_{[x,y]} b_i / y)}{\prod_{i=1}^m [1 + (\delta_{[x,y]} a_i / y)^2]} dy,$$

$$T_3(x) := \text{PV} \int_{|y| > 1} \frac{\prod_{i=1}^n (\delta_{[x,y]} b_i / y)}{\prod_{i=1}^m [1 + (\delta_{[x,y]} a_i / y)^2]} \frac{\bar{\omega}(x - y)}{y} dy.$$

A straightforward argument shows that

$$\|T_1\|_\infty \leq \frac{4}{2\tau - 1} [\bar{\omega}]_{\tau-1/2} \prod_{i=1}^n \|b'_i\|_\infty \quad \text{and} \quad \|T_3\|_\infty \leq 2\|\bar{\omega}\|_2 \prod_{i=1}^n \|b'_i\|_\infty.$$

Moreover, since $r - 1/2 \in (1, 2)$, it holds that $H^r(\mathbb{R}) \hookrightarrow \text{BC}^{r-1/2}(\mathbb{R})$, and therefore

$$(3.4) \quad \frac{|f(x + y) - 2f(x) + f(x - y)|}{|y|^{r-1/2}} \leq 4[f']_{r-3/2} \quad \text{for } f \in H^r(\mathbb{R}), x \in \mathbb{R}, y \neq 0;$$

cf. [39, Relation (0.2.2)]. Here $[\cdot]_{r-3/2}$ denotes the usual Hölder seminorm. Using the definition of the principal value together with (3.4), it follows that

$$\begin{aligned} \|T_2\|_\infty \leq \frac{8}{r - 3/2} \|\bar{\omega}\|_\infty & \left[\sum_{i=1}^n ([b'_i]_{r-3/2} \prod_{j=1, j \neq i}^n \|b'_j\|_\infty) \right. \\ & \left. + \left(\prod_{i=1}^n \|b'_i\|_\infty \right) \sum_{i=1}^m \|a'_i\|_\infty [a'_i]_{r-3/2} \right], \end{aligned}$$

and (iii) follows at once. □

We are additionally confronted in our analysis with a different type of singular integral operators. These operators, denoted by $\bar{B}_{n,m}$ with $nm \geq 1$, are extensions of the operators $B_{n,m}$ introduced above to a Sobolev space product where a lower regularity of the variable b_1 is compensated by a higher regularity of the variable $\bar{\omega}$. The extension property is a consequence of the estimate (3.5) derived below, while the estimate (3.6) plays a key role later on in the proofs of Theorems 4.4 and 5.2, when identifying the important terms that need to be estimated.

Lemma 3.2. *Let $n, m \in \mathbb{N}$ with $nm \geq 1$, $\tau \in (1/2, 1)$, and $r \in (5/2 - \tau, 2)$ be given.*

(i) *Given $a_1, \dots, a_m \in H^r(\mathbb{R})$, we let*

$$\bar{B}_{n,m}(a_1, \dots, a_m)[b_1, \dots, b_n, \bar{\omega}] := B_{n,m}(a_1, \dots, a_m)[b_1, \dots, b_n, \bar{\omega}]$$

for $\bar{\omega} \in H^1(\mathbb{R})$ and $b_1, \dots, b_n \in H^r(\mathbb{R})$. Then, there exists a constant C , depending only on n, m, r, τ , and $\max_{1 \leq i \leq m} \|a_i\|_{H^r}$, such that

$$(3.5) \quad \|\bar{B}_{n,m}(a_1, \dots, a_m)[b_1, \dots, b_n, \bar{\omega}]\|_2 \leq C \|\bar{\omega}\|_{H^\tau} \|b_1\|_{H^1} \prod_{i=2}^n \|b_i\|_{H^r}$$

and

$$(3.6) \quad \begin{aligned} & \|\overline{B}_{n,m}(a_1, \dots, a_m)[b_1, \dots, b_n, \overline{\omega}] - B_{n-1,m}(a_1, \dots, a_m)[b_2, \dots, b_n, b'_1 \overline{\omega}]\|_2 \\ & \leq C \|b_1\|_{H^r} \|\overline{\omega}\|_{H^1} \prod_{i=2}^n \|b_i\|_{H^r}. \end{aligned}$$

In particular, $\overline{B}_{n,m}(a_1, \dots, a_m)$ extends to a bounded operator

$$\overline{B}_{n,m}(a_1, \dots, a_m) \in \mathcal{L}_{n+1}(H^1(\mathbb{R}) \times (H^r(\mathbb{R}))^{n-1} \times H^r(\mathbb{R}), L_2(\mathbb{R})).$$

$$(ii) \quad \overline{B}_{n,m} \in C^{1-}((H^r(\mathbb{R}))^m, \mathcal{L}_{n+1}(H^1(\mathbb{R}) \times (H^r(\mathbb{R}))^{n-1} \times H^r(\mathbb{R}), L_2(\mathbb{R}))).$$

Proof. Similarly as in the previous lemma, the assertion (ii) is a consequence of (i), more precisely of (3.5). To establish (i) we use the formula

$$\frac{\partial}{\partial y} \left(\frac{\delta_{[x,y]} h}{y} \right) = \frac{h'(x-y)}{y} - \frac{\delta_{[x,y]} h}{y^2} \quad \text{for } x \in \mathbb{R}, y \neq 0,$$

and compute that

$$\begin{aligned} & \overline{B}_{n,m}(a_1, \dots, a_m)[b_1, \dots, b_n, \overline{\omega}](x) \\ & = \text{PV} \int_{\mathbb{R}} \frac{\delta_{[x,y]} b_1}{y^2} \frac{\prod_{i=2}^n (\delta_{[x,y]} b_i / y)}{\prod_{i=1}^m [1 + (\delta_{[x,y]} a_i / y)^2]} \overline{\omega}(x-y) dy \\ & = B_{n-1,m}(a_1, \dots, a_m)[b_2, \dots, b_n, b'_1 \overline{\omega}](x) \\ & \quad - \text{PV} \int_{\mathbb{R}} \frac{\partial}{\partial y} \left(\frac{\delta_{[x,y]} b_1}{y} \right) \frac{\prod_{i=2}^n (\delta_{[x,y]} b_i / y)}{\prod_{i=1}^m [1 + (\delta_{[x,y]} a_i / y)^2]} \overline{\omega}(x-y) dy. \end{aligned}$$

Integrating by parts, we are led to the relation

$$(3.7) \quad \begin{aligned} & \text{PV} \int_{\mathbb{R}} \frac{\partial}{\partial y} \left(\frac{\delta_{[x,y]} b_1}{y} \right) \frac{\prod_{i=2}^n (\delta_{[x,y]} b_i / y)}{\prod_{i=1}^m [1 + (\delta_{[x,y]} a_i / y)^2]} \overline{\omega}(x-y) dy \\ & = (b_1 B_{n-1,m}(a_1, \dots, a_m)[b_2, \dots, b_n, \overline{\omega}'])(x) \\ & \quad - B_{n-1,m}(a_1, \dots, a_m)[b_2, \dots, b_n, b_1 \overline{\omega}'](x) \\ & \quad + \sum_{j=2}^n \int_{\mathbb{R}} K_{1,j}(x, y) \overline{\omega}(x-y) dy - \sum_{j=1}^m \int_{\mathbb{R}} K_{2,j}(x, y) \overline{\omega}(x-y) dy, \end{aligned}$$

where, for $x \in \mathbb{R}$ and $y \neq 0$, we set

$$\begin{aligned} K_{1,j}(x, y) & := \frac{\prod_{i=2, i \neq j}^n (\delta_{[x,y]} b_i / y)}{\prod_{i=1}^m [1 + (\delta_{[x,y]} a_i / y)^2]} \frac{\delta_{[x,y]} b_j - y b'_j(x-y)}{y^2} \frac{\delta_{[x,y]} b_1}{y}, \\ K_{2,j}(x, y) & := 2 \frac{\prod_{i=2}^n (\delta_{[x,y]} b_i / y)}{[1 + (\delta_{[x,y]} a_j / y)^2] \prod_{i=1}^m [1 + (\delta_{[x,y]} a_i / y)^2]} \\ & \quad \times \frac{\delta_{[x,y]} a_j - y a'_j(x-y)}{y^2} \frac{\delta_{[x,y]} a_j}{y} \frac{\delta_{[x,y]} b_1}{y}. \end{aligned}$$

In view of Lemma 3.1 we have

$$(3.8) \quad \|B_{n-1,m}(a_1, \dots, a_m)[b_2, \dots, b_n, b'_1\bar{\omega}]\|_2 \leq C\|\bar{\omega}\|_\infty\|b_1\|_{H^1} \prod_{j=2}^n \|b'_j\|_\infty,$$

and we are left to estimate the L_2 -norm of the terms on the right-hand side of (3.7). For the integral terms we use Minkowski's integral inequality to obtain that

$$\left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_{1,j}(x, y)\bar{\omega}(x - y) dy \right|^2 dx\right)^{1/2} \leq \|\bar{\omega}\|_\infty \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |K_{1,j}(x, y)|^2 dx\right)^{1/2} dy.$$

In the following \mathcal{F} denotes the Fourier transform. Appealing to $b_1 \in BC^{\tau-1/2}(\mathbb{R})$, we have

$$\begin{aligned} \int_{\mathbb{R}} |K_{1,j}(x, y)|^2 dx &= \frac{1}{y^{7-2\tau}} [b_1]_{\tau-1/2}^2 \left(\prod_{i=2, i \neq j}^n \|b'_i\|_\infty^2\right) \int_{\mathbb{R}} |b_j - \tau_y b_j - y\tau_y b'_j|^2 dx \\ &\leq \frac{C}{y^{7-2\tau}} \|b_1\|_{H^\tau}^2 \left(\prod_{i=2, i \neq j}^n \|b'_i\|_\infty^2\right) \int_{\mathbb{R}} |\mathcal{F}b_j(\xi)|^2 |e^{iy\xi} - 1 - iy\xi|^2 d\xi, \end{aligned}$$

and together with the inequality

$$|e^{iy\xi} - 1 - iy\xi|^2 \leq C[(1 + |\xi|^2)^r y^{2r} \mathbf{1}_{(-1,1)}(y) + y^2(1 + |\xi|^2) \mathbf{1}_{[|y|\geq 1]}(y)], \quad y, \xi \in \mathbb{R},$$

we find for $2 \leq j \leq n$ that

$$\int_{\mathbb{R}} |K_{1,j}(x, y)|^2 dx \leq C\|b_1\|_{H^\tau}^2 \left(\prod_{i=2}^n \|b_i\|_{H^r}^2\right) \left[y^{2(r+\tau)-7} \mathbf{1}_{(-1,1)}(y) + \frac{1}{y^{5-2\tau}} \mathbf{1}_{[|y|\geq 1]}(y)\right].$$

Consequently, for $2 \leq j \leq n$ we have

$$(3.9) \quad \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_{1,j}(x, y)\bar{\omega}(x - y) dy \right|^2 dx\right)^{1/2} \leq C\|\bar{\omega}\|_\infty\|b_1\|_{H^\tau} \left(\prod_{i=2}^n \|b_i\|_{H^r}\right),$$

and analogously we obtain for $1 \leq j \leq m$ that

$$(3.10) \quad \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_{2,j}(x, y)\bar{\omega}(x - y) dy \right|^2 dx\right)^{1/2} \leq C\|\bar{\omega}\|_\infty\|a_j\|_{H^r}\|b_1\|_{H^\tau} \left(\prod_{i=2}^n \|b_i\|_{H^r}\right).$$

We are left with the term

$$T := b_1 B_{n-1,m}(a_1, \dots, a_m)[b_2, \dots, b_n, \bar{\omega}'] - B_{n-1,m}(a_1, \dots, a_m)[b_2, \dots, b_n, b_1\bar{\omega}'].$$

The estimate (3.6) follows by using Lemma 3.1. In order to derive (3.5) we proceed differently. The relation $\bar{\omega}'(x - y) = (\partial/\partial y)(\bar{\omega}(x) - \bar{\omega}(x - y))$ together with integration by parts leads us to

$$T(x) = \sum_{j=1}^n \int_{\mathbb{R}} K_{3,j}(x, y) dy - 2 \sum_{j=1}^m \int_{\mathbb{R}} K_{4,j}(x, y) dy,$$

where

$$K_{3,j}(x, y) := \frac{\prod_{i=1, i \neq j}^n \delta_{[x,y]} b_i / y}{\prod_{i=1}^m 1 + (\delta_{[x,y]} a_i / y)^2} \frac{\delta_{[x,y]} \bar{w}}{y} \left(\frac{\delta_{[x,y]} b_j}{y} - b'_j(x - y) \right),$$

$$K_{4,j}(x, y) := \frac{\prod_{i=1}^n \delta_{[x,y]} b_i / y}{[1 + (\delta_{[x,y]} a_j / y)^2] \prod_{i=1}^m 1 + (\delta_{[x,y]} a_i / y)^2} \times \frac{\delta_{[x,y]} \bar{w}}{y} \frac{\delta_{[x,y]} a_j}{y} \left(\frac{\delta_{[x,y]} a_j}{y} - a'_j(x - y) \right).$$

We first estimate the integral defined by the kernel $K_{4,j}$, $1 \leq j \leq m$. Minkowski's integral inequality implies that

$$\left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_{4,j}(x, y) dy \right|^2 dx \right)^{1/2} \leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |K_{4,j}(x, y)|^2 dx \right)^{1/2} dy,$$

and, since $b_1 \in BC^{r-3/2}(\mathbb{R})$, we get

$$\begin{aligned} \int_{\mathbb{R}} |K_{4,j}(x, y)|^2 dx &\leq \frac{2}{y^{7-2r}} [b_1]_{r-3/2}^2 \|a'_j\|_{\infty}^2 \left(\prod_{i=2}^n \|b'_i\|_{\infty}^2 \right) \int_{\mathbb{R}} |\bar{w} - \tau_y \bar{w}|^2 dx \\ &= \frac{C}{y^{7-2r}} \|b_1\|_{H^{r-1}}^2 \|a'_j\|_{\infty}^2 \left(\prod_{i=2}^n \|b'_i\|_{\infty}^2 \right) \int_{\mathbb{R}} |\mathcal{F}\bar{w}(\xi)|^2 |e^{iy\xi} - 1|^2 d\xi. \end{aligned}$$

Taking advantage of the inequality

$$|e^{iy\xi} - 1|^2 \leq C[(1 + |\xi|^2)^\tau y^{2\tau} \mathbf{1}_{(-1,1)}(y) + \mathbf{1}_{\{|y|\geq 1\}}(y)], \quad y, \xi \in \mathbb{R},$$

it follows that

$$\begin{aligned} \int_{\mathbb{R}} |K_{4,j}(x, y)|^2 dx &\leq C \|b_1\|_{H^{r-1}}^2 \|\bar{w}\|_{H^\tau}^2 \|a'_j\|_{\infty}^2 \left(\prod_{i=2}^n \|b'_i\|_{\infty}^2 \right) \\ &\quad \times \left[y^{2(r+\tau)-7} \mathbf{1}_{(-1,1)}(y) + \frac{1}{y^{7-2r}} \mathbf{1}_{\{|y|\geq 1\}}(y) \right], \end{aligned}$$

and therewith we obtain for $1 \leq j \leq m$ that

$$(3.11) \quad \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_{4,j}(x, y) dy \right|^2 dx \right)^{1/2} \leq C \|w\|_{H^\tau} \|b_1\|_{H^{r-1}} \|a'_j\|_{\infty} \prod_{i=2}^n \|b'_i\|_{\infty}.$$

Analogously, for $2 \leq j \leq n$, we have

$$(3.12) \quad \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_{3,j}(x, y) dy \right|^2 dx \right)^{1/2} \leq C \|w\|_{H^\tau} \|b_1\|_{H^{r-1}} \prod_{i=2}^n \|b'_i\|_{\infty},$$

while, for $j = 1$, we obtain the following estimate:

$$(3.13) \quad \begin{aligned} &\left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_{3,1}(x, y) dy \right|^2 dx \right)^{1/2} \\ &\leq C \|w\|_{H^\tau} \|b_1\|_{H^{r-1}} \prod_{i=2}^n \|b'_i\|_{\infty} \\ &\quad + \|w\|_{\infty} \|B_{n-1,m}(a_1, \dots, a_m)[b_2, \dots, b_n, b'_1]\|_2 \\ &\quad + \|B_{n-1,m}(a_1, \dots, a_m)[b_2, \dots, b_n, b'_1 w]\|_2. \end{aligned}$$

The estimate (3.5) follows from Lemma 3.1 and (3.8)–(3.13). □

Mapping properties. Using Lemmas 3.1–3.2, we now study the mapping properties of the nonlinear (with respect to f) operator \mathbb{A} defined in (3.1).

Lemma 3.3. *It holds that*

$$(3.14) \quad \mathbb{A} \in C^{1-}(H^2(\mathbb{R}), \mathcal{L}(H^1(\mathbb{R}))).$$

Proof. Given $f \in H^2(\mathbb{R})$, the relation (3.2) together with Lemma 3.1 implies that $\mathbb{A}(f) \in \mathcal{L}(L_2(\mathbb{R}))$. We next show that $\mathbb{A}(f)[\bar{\omega}] \in H^1(\mathbb{R})$, provided that $\bar{\omega} \in H^1(\mathbb{R})$. To this end we let $\{\tau_\varepsilon\}_{\varepsilon \in \mathbb{R}}$ denote the C_0 -group of right translations on $L_2(\mathbb{R})$, that is, $\tau_\varepsilon f(x) := f(x - \varepsilon)$ for $f \in L_2(\mathbb{R})$, $x, \varepsilon \in \mathbb{R}$, and we compute for $\varepsilon \in (0, 1)$ that

$$\begin{aligned} & \pi \frac{\tau_\varepsilon(\mathbb{A}(f)[\bar{\omega}]) - \mathbb{A}(f)[\bar{\omega}]}{\varepsilon} \\ &= \frac{\tau_\varepsilon f' - f'}{\varepsilon} B_{0,1}(\tau_\varepsilon f)[\tau_\varepsilon \bar{\omega}] + f' B_{0,1}(\tau_\varepsilon f) \left[\frac{\tau_\varepsilon \bar{\omega} - \bar{\omega}}{\varepsilon} \right] \\ & \quad - f' \bar{B}_{2,2}(\tau_\varepsilon f, f) \left[\frac{\tau_\varepsilon f - f}{\varepsilon}, \tau_\varepsilon f + f, \bar{\omega} \right] - \bar{B}_{1,1}(\tau_\varepsilon f) \left[\frac{\tau_\varepsilon f - f}{\varepsilon}, \tau_\varepsilon \bar{\omega} \right] \\ & \quad - B_{1,1}(\tau_\varepsilon f) \left[f, \frac{\tau_\varepsilon \bar{\omega} - \bar{\omega}}{\varepsilon} \right] + \bar{B}_{3,2}(\tau_\varepsilon f, f) \left[\frac{\tau_\varepsilon f - f}{\varepsilon}, \tau_\varepsilon f + f, f, \bar{\omega} \right]. \end{aligned}$$

The convergences

$$\begin{aligned} \tau_\varepsilon f &\xrightarrow{\varepsilon \rightarrow 0} f & \text{in } H^2(\mathbb{R}), & \quad \frac{\tau_\varepsilon f - f}{\varepsilon} &\xrightarrow{\varepsilon \rightarrow 0} -f' & \text{in } H^1(\mathbb{R}), \\ \tau_\varepsilon \bar{\omega} &\xrightarrow{\varepsilon \rightarrow 0} \bar{\omega} & \text{in } H^1(\mathbb{R}), & \quad \frac{\tau_\varepsilon \bar{\omega} - \bar{\omega}}{\varepsilon} &\xrightarrow{\varepsilon \rightarrow 0} -\bar{\omega}' & \text{in } L_2(\mathbb{R}), \end{aligned}$$

together with Lemmas 3.1–3.2 enable us to pass to the limit $\varepsilon \rightarrow 0$ in the above relation and to conclude that $\mathbb{A}(f)[\bar{\omega}] \in H^1(\mathbb{R})$ and

$$(3.15) \quad \begin{aligned} \pi(\mathbb{A}(f)[\bar{\omega}])' &= \pi \mathbb{A}(f)[\bar{\omega}'] + f'' B_{0,1}(f)[\bar{\omega}] - 2f' \bar{B}_{2,2}(f, f)[f', f, \bar{\omega}] \\ & \quad - \bar{B}_{1,1}(f)[f', \bar{\omega}] + 2\bar{B}_{3,2}(f, f)[f', f, f, \bar{\omega}]. \end{aligned}$$

The Lipschitz continuity property (3.14) is now a direct consequence of Lemmas 3.1–3.2. □

In fact, \mathbb{A} enjoys the following regularity:

$$(3.16) \quad \mathbb{A} \in C^\omega(H^r(\mathbb{R}), \mathcal{L}(L_2(\mathbb{R}))) \cap C^\omega(H^2(\mathbb{R}), \mathcal{L}(H^1(\mathbb{R}))) \quad \text{for all } r > 3/2.$$

The property (3.16) may be established by using the arguments presented in [40, Section 5] together with Lemmas 3.1–3.2. The lengthy details are left to the interested reader. Given $f \in H^2(\mathbb{R})$, we denote by $\mathbb{B}(f)$ the operator which corresponds to the right-hand side of the first equation of (1.1a), namely

$$(3.17) \quad \mathbb{B}(f) := B_{0,1}(f) + f' B_{1,1}(f)[f, \cdot].$$

For later purposes we establish the following regularity result.

Lemma 3.4. *It holds that*

$$(3.18) \quad \mathbb{B} \in C^\omega(H^r(\mathbb{R}), \mathcal{L}(L_2(\mathbb{R}))) \cap C^\omega(H^2(\mathbb{R}), \mathcal{L}(H^1(\mathbb{R}))) \quad \text{for all } r > 3/2.$$

Proof. Given $f \in H^r(\mathbb{R})$, $r > 3/2$, it follows from Lemma 3.1 that $\mathbb{B}(f) \in \mathcal{L}(L_2(\mathbb{R}))$. Furthermore, proceeding as in Lemma 3.3, we get, in view of Lemmas 3.1–3.2, that if $f \in H^2(\mathbb{R})$ and $\bar{\omega} \in H^1(\mathbb{R})$, then $\mathbb{B}(f)[\bar{\omega}] \in H^1(\mathbb{R})$ and

$$(3.19) \quad \begin{aligned} (\mathbb{B}(f)[\bar{\omega}])' &= \mathbb{B}(f)[\bar{\omega}'] - 2\bar{B}_{2,2}(f, f)[f', f, \bar{\omega}] + f''B_{1,1}(f)[f, \bar{\omega}] \\ &\quad + f'\bar{B}_{1,1}(f)[f', \bar{\omega}] - 2f'\bar{B}_{3,2}(f, f)[f', f, f, \bar{\omega}]. \end{aligned}$$

The analyticity property (3.18) follows now from Lemmas 3.1–3.2 by arguing as in [40, Section 5] (we omit again the lengthy details). \square

On the resolvent set of the adjoint of the double layer potential. We are now in the position to address the solvability of (1.1b). First, we show that, given $f \in H^r(\mathbb{R})$, with $r > 3/2$, the resolvent set of $\mathbb{A}(f)$, when we view $\mathbb{A}(f)$ as an operator in $\mathcal{L}(L_2(\mathbb{R}))$, contains the set $\{|\lambda| \geq 1\} \cap \mathbb{R}$. This property follows from the uniform bound (with respect to f and λ) established in (3.20). This bound proves crucial also when investigating the resolvent set of $\mathbb{A}(f)$, when we regard $\mathbb{A}(f)$ as an element of $\mathcal{L}(H^1(\mathbb{R}))$ and $f \in H^2(\mathbb{R})$; cf. Proposition 3.6 below.

Theorem 3.5. *Given $f \in H^r(\mathbb{R})$, $r > 3/2$, and $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$, it holds that*

$$\lambda - \mathbb{A}(f) \in \text{Isom}(L_2(\mathbb{R})).$$

Proof. Let $M > 0$ be given. We prove that there exists a constant $C = C(M) > 0$ such that for all $f \in H^r(\mathbb{R})$ with $\|f\|_{H^r} \leq M$, $\bar{\omega} \in L_2(\mathbb{R})$, and $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$ we have

$$(3.20) \quad \|\bar{\omega}\|_2 \leq C\|(\lambda - \mathbb{A}(f))[\bar{\omega}]\|_2.$$

Having established (3.20), the claim follows from the method of continuity (cf. [4, Proposition 1.1.1]) as, for each $f \in H^r(\mathbb{R})$, the spectrum of $\mathbb{A}(f)$ is compact and therefore $\lambda - \mathbb{A}(f)$ is invertible if λ is large. In view of (3.16), it suffices to prove (3.20) for $f \in C_0^\infty(\mathbb{R})$ and $\bar{\omega} \in C_0^\infty(\mathbb{R})$.

We recall from Lemma 2.1 that the restrictions $\hat{v}_\pm := \hat{v}|_{\Omega_\pm^0}$, with Ω_\pm^0 as in Lemma 2.1, of the function \hat{v} defined in (2.3) have the properties

$$(3.21) \quad \hat{v}_\pm \in C(\overline{\Omega_\pm^0}) \cap C^1(\Omega_\pm^0) \quad \text{and} \quad \text{div } \hat{v}_\pm = \text{rot } \hat{v}_\pm = 0 \quad \text{in } \Omega_\pm^0.$$

Moreover, the Plemelj formula (2.7) together with Lemmas 3.3–3.4 ensures the restrictions $F_\pm := (F_\pm^1, F_\pm^2) := \hat{v}_\pm|_{[y=f(x)]}$ satisfy

$$(3.22) \quad \langle F_\pm | (1, f') \rangle = \frac{1}{2}(\mathbb{A}(f) \mp 1)[\bar{\omega}] \in H^1(\mathbb{R}),$$

$$F^\nu := \langle F_\pm | (-f', 1) \rangle = \frac{1}{2\pi}\mathbb{B}(f)[\bar{\omega}] \in H^1(\mathbb{R}).$$

Letting τ and $\nu := (\nu_1, \nu_2)$ denote the tangent and the unit outward normal vectors at $\partial\Omega_-^0$, we write $F_\pm = F_\pm^\tau + F_\pm^\nu$, where

$$F_\pm^\tau := \langle F_\pm | \tau \rangle \tau \in H^1(\mathbb{R}, \mathbb{R}^2) \quad \text{and} \quad F_\pm^\nu := \langle F_\pm | \nu \rangle \nu = \frac{F^\nu}{\sqrt{1+f'^2}} \nu \in H^1(\mathbb{R}, \mathbb{R}^2).$$

We now introduce the bilinear form $\mathcal{B} : L_2(\mathbb{R}, \mathbb{R}^2) \times L_2(\mathbb{R}, \mathbb{R}^2) \rightarrow \mathbb{R}$ by the formula

$$\mathcal{B}(F, G) := \int_{\mathbb{R}} G^2 \langle F | (-f', 1) \rangle + F^2 \langle G | (-f', 1) \rangle - \langle F | G \rangle dx$$

for $F = (F^1, F^2)$, $G = (G^1, G^2)$, and we remark that

- (i) $\mathcal{B}(F, G) = \int_{\mathbb{R}} \langle F|G \rangle dx \quad \text{if } F = \langle F|\nu \rangle \nu, G = \langle G|\nu \rangle \nu,$
- (ii) $\mathcal{B}(F, G) = - \int_{\mathbb{R}} \langle F|\tau \rangle \langle G|\tau \rangle dx \quad \text{if } F = \langle F|\tau \rangle \tau, G = \langle G|\tau \rangle \tau,$
- (iii) $\mathcal{B}(F, G) = \int_{\mathbb{R}} F^2 \langle G|(-f', 1) \rangle dx \quad \text{if } F = \langle F|\tau \rangle \tau, G = \langle G|\nu \rangle \nu.$

We now claim that

$$(3.23) \quad \mathcal{B}(F_{\pm}, F_{\pm}) = 0.$$

In order to prove (3.23) we choose a sequence $(\varphi_n)_n \subset C_0^\infty(\mathbb{R}^2, [0, 1])$ with the property that $\varphi_n = 1$ in $[|(x, y)| \leq n]$, $\varphi_n = 0$ in $[|(x, y)| \geq n + 1]$, and $\sup_n \|\nabla \varphi_n\|_\infty < \infty$. Using Lebesgue’s dominated convergence theorem and Stokes’ theorem together with (3.21), we get

$$\begin{aligned} \mathcal{B}(F_-, F_-) &= \int_{[y=f(x)]} \left\langle \left(\frac{2F_-^1 F_-^2}{(F_-^2)^2 - (F_-^1)^2} \right) \middle| \nu \right\rangle d\sigma \\ &= \lim_{n \rightarrow \infty} \int_{[y=f(x)]} \left\langle \varphi_n \left(\frac{2\widehat{v}_-^1 \widehat{v}_-^2}{(\widehat{v}_-^2)^2 - (\widehat{v}_-^1)^2} \right) \middle| \nu \right\rangle d\sigma \\ &= \lim_{n \rightarrow \infty} \int_{\Omega_-^0} \operatorname{div} \left(\varphi_n \left(\frac{2\widehat{v}_-^1 \widehat{v}_-^2}{(\widehat{v}_-^2)^2 - (\widehat{v}_-^1)^2} \right) \right) d(x, y) \\ &= \lim_{n \rightarrow \infty} \int_{\Omega_-^0} \left\langle \nabla \varphi_n \middle| \left(\frac{2\widehat{v}_-^1 \widehat{v}_-^2}{(\widehat{v}_-^2)^2 - (\widehat{v}_-^1)^2} \right) \right\rangle d(x, y) = 0, \end{aligned}$$

the last equality being a consequence of (2.5) and of $\sup_n \|\nabla \varphi_n\|_\infty < \infty$. This proves (3.23) for F_- . The proof for F_+ is similar.

Using (3.22), (3.23), and the relations (i)–(iii), we now obtain the following Rellich formula:

$$(3.24) \quad \begin{aligned} 0 &= \mathcal{B}(F_{\pm}, F_{\pm}) = \mathcal{B}(F_{\pm}^\tau, F_{\pm}^\tau) + 2\mathcal{B}(F_{\pm}^\tau, F_{\pm}^\nu) + \mathcal{B}(F_{\pm}^\nu, F_{\pm}^\nu) \\ &= \int_{\mathbb{R}} \frac{1}{1 + f'^2} \left[|F^\nu|^2 + f' F^\nu (\mathbb{A}(f) \mp 1) [\overline{\omega}] - \frac{1}{4} |(\mathbb{A}(f) \mp 1) [\overline{\omega}]|^2 \right] dx. \end{aligned}$$

Young’s inequality and (3.24) imply that there exists a positive constant $C = C(M)$ such that

$$\|(\mathbb{A}(f) \pm 1) [\overline{\omega}]\|_2 \leq C \|F^\nu\|_2$$

for all $f \in C_0^\infty(\mathbb{R})$ with $\|f\|_{H^r} \leq M$ and $\overline{\omega} \in C_0^\infty(\mathbb{R})$. The latter inequality yields in particular

$$(3.25) \quad \|\overline{\omega}\|_2 = \frac{1}{2} \|(\mathbb{A}(f) + 1) [\overline{\omega}] - (\mathbb{A}(f) - 1) [\overline{\omega}]\|_2 \leq C \|F^\nu\|_2.$$

Let $\lambda \in \mathbb{R}$ satisfy $|\lambda| \geq 1$. Since

$$|(\mathbb{A}(f) \pm 1) [\overline{\omega}]|^2 = |(\lambda - \mathbb{A}(f)) [\overline{\omega}]|^2 - 2(\lambda \pm 1) \overline{\omega} (\lambda - \mathbb{A}(f)) [\overline{\omega}] + (\lambda \pm 1)^2 |\overline{\omega}|^2,$$

we deduce, together with (3.24), after eliminating the mixed term, that

$$\begin{aligned} & \int_{\mathbb{R}} \frac{1}{1+f'^2} [(\lambda^2 - 1)|\bar{\omega}|^2 + 4|F^\nu|^2] dx \\ &= \int_{\mathbb{R}} \frac{1}{1+f'^2} [|(\lambda - \mathbb{A}(f))[\bar{\omega}]|^2 + 4f'F^\nu(\lambda - \mathbb{A}(f))[\bar{\omega}]] dx. \end{aligned}$$

This relation together with Young’s inequality allows us to conclude that there exists a constant $C = C(M)$ with the property that

$$(\lambda^2 - 1)\|\bar{\omega}\|_2 + \|F^\nu\|_2 \leq C\|(\lambda - \mathbb{A}(f))\bar{\omega}\|_2.$$

The desired property (3.20) follows from (3.25). □

We are now in the position to study the invertibility of $\lambda - \mathbb{A}(f)$ in $\mathcal{L}(H^1(\mathbb{R}))$, when requiring additionally that $f \in H^2(\mathbb{R})$.

Proposition 3.6. *Given $f \in H^2(\mathbb{R})$ and $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$, it holds that*

$$\lambda - \mathbb{A}(f) \in \text{Isom}(H^1(\mathbb{R})).$$

Proof. As in the previous theorem, it suffices to prove that, given $M > 0$, there exists a constant $C_1 = C_1(M)$ such that for all $f \in H^2(\mathbb{R})$ with $\|f\|_{H^2} \leq M$, $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$, and $\bar{\omega} \in H^1(\mathbb{R})$ we have

$$(3.26) \quad \|\bar{\omega}\|_{H^1} \leq C_1\|(\lambda - \mathbb{A}(f))[\bar{\omega}]\|_{H^1}.$$

In view of (3.20), we are left to estimate the term $\|((\lambda - \mathbb{A}(f))[\bar{\omega}])'\|_2$. To this end, we infer from (3.15) that

$$(3.27) \quad ((\lambda - \mathbb{A}(f))[\bar{\omega}])' = (\lambda - \mathbb{A}(f))[\bar{\omega}'] - T_{0,\text{lot}}(f)[\bar{\omega}],$$

where, given $f \in H^2(\mathbb{R})$ and $\bar{\omega} \in H^1(\mathbb{R})$, we have set

$$(3.28) \quad \begin{aligned} \pi T_{0,\text{lot}}(f)[\bar{\omega}] &:= f''B_{0,1}(f)[\bar{\omega}] - 2f'\bar{B}_{2,2}(f, f)[f', f, \bar{\omega}] \\ &\quad - \bar{B}_{1,1}(f)[f', \bar{\omega}] + 2\bar{B}_{3,2}(f, f)[f', f, f, \bar{\omega}]. \end{aligned}$$

We now fix $\tau \in (1/2, 1)$ and $r \in (5/2 - \tau, 2)$. Lemmas 3.1–3.2 ensure that there exists a constant $C_0 = C_0(M)$ such that

$$(3.29) \quad \|T_{0,\text{lot}}(f)[\bar{\omega}]\|_2 \leq C_0\|\bar{\omega}\|_{H^\tau}$$

for all $f \in H^2(\mathbb{R})$ with $\|f\|_{H^2} \leq M$ and all $\bar{\omega} \in H^1(\mathbb{R})$. Using Young’s inequality, (3.20), (3.27), (3.29), and the inequality $\|\bar{\omega}\|_{H^\tau} \leq \|\bar{\omega}\|_2^{1-\tau}\|\bar{\omega}\|_{H^1}^\tau$, we conclude that

$$\|(\lambda - \mathbb{A}(f))[\bar{\omega}]\|_{H^1} \geq \frac{1}{2C}\|\bar{\omega}\|_{H^1} - C_0\|\bar{\omega}\|_{H^\tau} \geq \frac{1}{4C}\|\bar{\omega}\|_{H^1} - \tilde{C}_0\|\bar{\omega}\|_2,$$

where $C = C(M)$ denotes the constant in (3.20) and \tilde{C}_0 depends only on C and C_0 . The estimate (3.26) follows now by appealing once more to (3.20). □

Arguing as in Proposition 3.6, it can be shown that the following general result holds.

Remark 3.7. Given $f \in H^k(\mathbb{R})$, $k \geq 2$, and $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$, it holds that

$$\lambda - \mathbb{A}(f) \in \text{Isom}(H^{k-1}(\mathbb{R})).$$

We now establish an invertibility result for the double layer potential, that is, for the adjoint of $\mathbb{A}(f)$, which is needed in the proof of Theorem 1.2.

Lemma 3.8. *Given $f \in H^2(\mathbb{R})$, let $(\mathbb{A}(f))^* \in \mathcal{L}(L_2(\mathbb{R}))$ denote the adjoint operator of $\mathbb{A}(f) \in \mathcal{L}(L_2(\mathbb{R}))$. It then holds that*

$$(\mathbb{A}(f))^*[\varphi](x) = \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{(f(x) - f(x - y)) - yf'(x - y)}{y^2 + (f(x) - f(x - y))^2} \varphi(x - y) dy, \quad \varphi \in L_2(\mathbb{R}).$$

Moreover, for each $M > 0$, there exists a constant $C = C(M)$ such that

$$(3.30) \quad \|\varphi\|_{H^1} \leq C_1 \|(\lambda - (\mathbb{A}(f))^*)[\varphi]\|_{H^1}$$

for all $\|f\|_{H^2} \leq M$, $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$, and $\varphi \in H^1(\mathbb{R})$. In particular $\lambda - (\mathbb{A}(f))^* \in \text{Isom}(H^1(\mathbb{R}))$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$.

Proof. It is easy to verify that $(\mathbb{A}(f))^*$ is indeed the adjoint of $\mathbb{A}(f) \in \mathcal{L}(L_2(\mathbb{R}))$, while the property that $\lambda - \mathbb{A}(f) \in \text{Isom}(L_2(\mathbb{R}))$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$ follows from Theorem 3.5.

In addition, Lemmas 3.1–3.2 imply that $\lambda - (\mathbb{A}(f))^* \in \mathcal{L}(H^1(\mathbb{R}))$, with

$$((\lambda - (\mathbb{A}(f))^*)[\varphi])' = ((\lambda - (\mathbb{A}(f))^*)[\varphi']) - T_{0,\text{lot}}^*(f)[\varphi],$$

where

$$\begin{aligned} \pi T_{0,\text{lot}}^*(f)[\varphi] &:= -B_{0,1}(f)[f''\varphi] + \overline{B}_{1,1}(f)[f', \varphi] \\ &\quad + 2\overline{B}_{2,2}(f, f)[f', f, f'\varphi] - 2\overline{B}_{3,2}(f, f)[f', f, f, \varphi]. \end{aligned}$$

For $\|f\|_{H^2} \leq M$, Lemmas 3.1–3.2 lead us to the estimate

$$\|\pi T_{0,\text{lot}}^*(f)[\varphi]\|_2 \leq C_0 \|\varphi\|_{H^{3/4}},$$

with a constant $C_0 = C_0(M)$. Letting $C = C(M)$ denote the constant in (3.20), we additionally get

$$\|\varphi\|_2 \leq C \|(\lambda - (\mathbb{A}(f))^*)[\varphi]\|_2$$

for all $f \in H^2(\mathbb{R})$ with $\|f\|_{H^2} \leq M$, $\varphi \in L_2(\mathbb{R})$, and $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$, and therefore we may argue as in the proof of Proposition 3.6 in order to obtain the remaining claims. \square

4. THE MUSKAT PROBLEM WITH SURFACE TENSION

In this section we address the well-posedness of the Muskat problem with surface tension, and therefore we assume throughout that $\sigma > 0$. Taking advantage of Theorem 3.5 and of the structure of the curvature term, we first formulate the system (1.1) as a quasilinear evolution problem for the free boundary f only; cf. (4.5). Subsequently, we disclose the parabolic character of (4.5), and this enables us to use abstract results for quasilinear parabolic problems due to H. Amann [1–3]; see [40, Theorem 1.5] for the precise statements.

The abstract formulation. We start by solving (1.1b). For our approach it is important to point out the quasilinear structure of the equation (1.1b) which is a result of the linearity of the left-hand side of (1.1b) with respect to the highest derivative of f . Concerning this issue, it is convenient to study the solvability for \overline{w} of the equation

$$(4.1) \quad b_\mu \left[\sigma \frac{h'''}{(1 + f'^2)^{3/2}} - 3\sigma \frac{f' f'' h''}{(1 + f'^2)^{5/2}} - \Theta h' \right] = (1 + a_\mu \mathbb{A}(f))[\overline{w}],$$

where

$$b_\mu := \frac{2k}{\mu_- + \mu_+}.$$

If we replace h by f , then (4.1) is clearly equivalent to (1.1b). Since the values of the positive constants b_μ and σ are not relevant for the further analysis we set $b_\mu = \sigma = 1$.

Proposition 4.1. *Given $f \in H^2(\mathbb{R})$ and $h \in H^3(\mathbb{R})$, there exists a unique solution $\bar{\omega} := \bar{\omega}(f)[h]$ to (4.1) and*

$$(4.2) \quad \bar{\omega} \in C^\omega(H^2(\mathbb{R}), \mathcal{L}(H^3(\mathbb{R}), L_2(\mathbb{R}))).$$

Proof. Since $|a_\mu| < 1$, it follows from Theorem 3.5 that

$$\bar{\omega}(f)[h] := (1 + a_\mu \mathbb{A}(f))^{-1} \left[\frac{h'''}{(1 + f'^2)^{3/2}} - 3 \frac{f' f'' h''}{(1 + f'^2)^{5/2}} - \Theta h' \right] \in L_2(\mathbb{R})$$

is the unique solution to (4.1). Since

$$[T \mapsto T^{-1}] \in C^\omega(\text{Isom}(L_2(\mathbb{R})), \text{Isom}(L_2(\mathbb{R}))),$$

$$\left[f \mapsto \left[h \mapsto \frac{h'''}{(1 + f'^2)^{3/2}} - 3 \frac{f' f'' h''}{(1 + f'^2)^{5/2}} - \Theta h' \right] \right] \in C^\omega(H^2(\mathbb{R}), \mathcal{L}(H^3(\mathbb{R}), L_2(\mathbb{R}))),$$

the desired regularity follows from (3.16). □

For later purposes we decompose the solution operator found in Proposition 4.1 as a sum of two operators. This decomposition is very useful because $\bar{\omega}_2(f)[h]$ can be viewed as a lower order term, while the highest order term $(\bar{\omega}_1(f)[h])'$ appears as a derivative, and this enables us to use integration by parts in the arguments that follow (see the proof of Theorem 4.4).

Proposition 4.2. *Given $f \in H^2(\mathbb{R})$ and $h \in H^3(\mathbb{R})$, let*

$$\bar{\omega}_1(f)[h] := (1 + a_\mu \mathbb{A}(f))^{-1} \left[\frac{h''}{(1 + f'^2)^{3/2}} \right],$$

$$\bar{\omega}_2(f)[h] := (1 + a_\mu \mathbb{A}(f))^{-1} \left[-\Theta h' + a_\mu T_{0,\text{lot}}(f)[\bar{\omega}_1(f)[h]] \right],$$

where the mapping $T_{0,\text{lot}}$ is defined in (3.28). Then:

(i) $\bar{\omega}_1 \in C^\omega(H^2(\mathbb{R}), \mathcal{L}(H^3(\mathbb{R}), H^1(\mathbb{R}))), \bar{\omega}_2 \in C^\omega(H^2(\mathbb{R}), \mathcal{L}(H^3(\mathbb{R}), L_2(\mathbb{R}))).$

(ii)

$$\bar{\omega}(f) = \frac{d}{dx} \circ \bar{\omega}_1(f) + \bar{\omega}_2(f).$$

(iii) Given $\tau \in (1/2, 1)$, there exists a constant C such that

$$(4.3) \quad \begin{aligned} \|\bar{\omega}_1(f)[h]\|_2 &\leq C \|h\|_{H^2}, \\ \|\bar{\omega}_1(f)[h]\|_{H^\tau} + \|\bar{\omega}_2(f)[h]\|_2 &\leq C \|h\|_{H^{2+\tau}} \end{aligned}$$

for all $h \in H^3(\mathbb{R})$.

Proof. That $\bar{\omega}_1$ is well defined and $\bar{\omega}_1 \in C^\omega(H^2(\mathbb{R}), \mathcal{L}(H^3(\mathbb{R}), H^1(\mathbb{R})))$ follows from Proposition 3.6, (3.16), and the property

$$\left[f \mapsto \left[h \mapsto \frac{h''}{(1 + f'^2)^{3/2}} \right] \right] \in C^\omega(H^2(\mathbb{R}), \mathcal{L}(H^3(\mathbb{R}), H^1(\mathbb{R}))).$$

Moreover, in view of (3.27), we get that

$$(4.4) \quad (1 + a_\mu \mathbb{A}(f))[(\bar{\omega}_1(f)[h])'] = \left(\frac{h''}{(1 + f'^2)^{3/2}} \right)' - a_\mu T_{0,\text{lot}}(f)[\bar{\omega}_1(f)[h]],$$

and therewith $\bar{\omega}(f)[h] - (\bar{\omega}_1(f)[h])' = \bar{\omega}_2(f)[h]$ for all $h \in H^3(\mathbb{R})$. Invoking Proposition 4.1, we are left to establish (4.3). The estimates for $\bar{\omega}_1(f)[h]$ follow from (3.20) and (3.26) via complex interpolation; cf. (4.39). Finally, the estimate for $\|\bar{\omega}_2(f)[h]\|_2$ is a consequence of (3.20), (3.29), and of the estimate for the term $\|\bar{\omega}_1(f)[h]\|_{H^\tau}$. \square

Appealing to Proposition 4.1, we now formulate the original system (1.1), after rescaling the time appropriately, as a quasilinear evolution problem for f only, that is,

$$(4.5) \quad \partial_t f = \Phi_\sigma(f)[f], \quad t > 0, \quad f(0) = f_0,$$

where $\Phi_\sigma : H^2(\mathbb{R}) \rightarrow \mathcal{L}(H^3(\mathbb{R}), L_2(\mathbb{R}))$ is the mapping

$$\Phi_\sigma(f)[h] := \mathbb{B}(f)[\bar{\omega}(f)[h]]$$

for $f \in H^2(\mathbb{R})$ and $h \in H^3(\mathbb{R})$, and $\mathbb{B}(f)$ is the linear operator defined in (3.17). The properties (3.18) and (4.2) imply that

$$(4.6) \quad \Phi_\sigma \in C^\omega(H^2(\mathbb{R}), \mathcal{L}(H^3(\mathbb{R}), L_2(\mathbb{R}))).$$

The generator property. We now choose an arbitrary function $f \in H^2(\mathbb{R})$ which is kept fixed in the following. Our next task is to prove that $\Phi_\sigma(f)$, considered as an unbounded operator in $L_2(\mathbb{R})$ with definition domain $H^3(\mathbb{R})$, is the generator of a strongly continuous and analytic semigroup in $\mathcal{L}(L_2(\mathbb{R}))$, that is,

$$(4.7) \quad -\Phi_\sigma(f) \in \mathcal{H}(H^3(\mathbb{R}), L_2(\mathbb{R}));$$

see [4, 39] for several characterizations of such type of operators. To this end, we write the operator $\Phi_\sigma(f)$ as a sum

$$\Phi_\sigma(f) = \Phi_{\sigma,1}(f) + \Phi_{\sigma,2}(f),$$

where

$$\Phi_{\sigma,1}(f)[h] := B_{0,1}(f)[(\bar{\omega}_1(f)[h])'] + f' B_{1,1}(f)[f, (\bar{\omega}_1(f)[h])'],$$

$$\Phi_{\sigma,2}(f)[h] := \mathbb{B}(f)[\bar{\omega}_2(f)[h]]$$

for $h \in H^3(\mathbb{R})$, and where $\bar{\omega}_i(f)$, $i = 1, 2$, are the operators introduced in Proposition 4.2. For the particular choice $\tau = 3/4$ in Proposition 4.2, it follows that $\Phi_{\sigma,2}(f) \in \mathcal{L}(H^{11/4}(\mathbb{R}), L_2(\mathbb{R}))$. This property together with $[L_2(\mathbb{R}), H^3(\mathbb{R})]_{11/12} = H^{11/4}(\mathbb{R})$ (cf. (4.39) and [39, Proposition 2.4.1]) enables us to view $\Phi_{\sigma,2}(f)$ as a lower order perturbation. Hence, our task reduces to establishing the generator property for the leading order term $\Phi_{\sigma,1}(f)$. The proof of this property is technical and is based on an approach followed previously in [23, 27, 29] in the context of spaces of continuous functions, and refined recently in [26, 40].

In order to proceed we choose for each $\varepsilon \in (0, 1)$ a so-called finite ε -localization family, that is, a family

$$\{\pi_j^\varepsilon : -N + 1 \leq j \leq N\} \subset C_0^\infty(\mathbb{R}, [0, 1]),$$

with $N = N(\varepsilon) \in \mathbb{N}$ sufficiently large, such that

$$(4.8) \quad \text{supp } \pi_j^\varepsilon \text{ is an interval of length less than or equal to } \varepsilon \text{ for all } |j| \leq N - 1;$$

$$(4.9) \quad \text{supp } \pi_N^\varepsilon \subset (-\infty, -x_N] \cup [x_N, \infty) \text{ and } x_N \geq \varepsilon^{-1};$$

$$(4.10) \quad \text{supp } \pi_j^\varepsilon \cap \text{supp } \pi_l^\varepsilon = \emptyset \text{ if } \begin{cases} |j - l| \geq 2 \text{ and } |j|, |l| \in \{0, \dots, N - 1\}, \text{ or} \\ |l| \leq N - 2, j = N; \end{cases}$$

$$(4.11) \quad \sum_{j=-N+1}^N (\pi_j^\varepsilon)^2 = 1 \text{ and } \|(\pi_j^\varepsilon)^{(k)}\|_\infty \leq C\varepsilon^{-k} \text{ for all } k \in \mathbb{N}, -N + 1 \leq j \leq N.$$

Such ε -localization families can be easily constructed. Furthermore, we choose a second family

$$\{\chi_j^\varepsilon : -N + 1 \leq j \leq N\} \subset C_0^\infty(\mathbb{R}, [0, 1])$$

with the following properties:

$$(4.12) \quad \chi_j^\varepsilon = 1 \text{ on } \text{supp } \pi_j^\varepsilon;$$

$$(4.13) \quad \text{supp } \chi_j^\varepsilon \text{ is an interval with } |\text{supp } \chi_j^\varepsilon| \leq 3\varepsilon \text{ for } |j| \leq N - 1;$$

$$(4.14) \quad \text{supp } \chi_N^\varepsilon \subset [|x| \geq x_N - \varepsilon].$$

The following remark is a simple exercise.

Remark 4.3. Given $k \in \mathbb{N}$ and a finite ε -localization family $\{\pi_j^\varepsilon : -N + 1 \leq j \leq N\}$, the mapping

$$\left[h \mapsto \sum_{j=-N+1}^N \|\pi_j^\varepsilon h\|_{H^k} \right] : H^k(\mathbb{R}) \rightarrow [0, \infty)$$

defines a norm on $H^k(\mathbb{R})$ which is equivalent to the standard H^k -norm.

Let us now introduce the continuous path

$$[\tau \mapsto \Phi_{\sigma,1}(\tau f)] : [0, 1] \rightarrow \mathcal{L}(H^3(\mathbb{R}), L_2(\mathbb{R}))$$

which connects the operator $\Phi_{\sigma,1}(f)$ with $\Phi_{\sigma,1}(0)$. Recalling Proposition 4.2, we have the identity

$$\Phi_{\sigma,1}(0)[h] = \pi H[(\overline{\omega}_1(0)[h])'] = \pi H[h'''] = -\pi(\partial_x^4)^{3/4}[h],$$

where $(\partial_x^4)^{3/4}$ denotes the Fourier multiplier with symbol $[\xi \mapsto |\xi|^3]$. The following result shows that the operator $\Phi_{\sigma,1}(\tau f)$ can be approximated, in a sense to be made precise below, by Fourier multipliers $c(\partial_x^4)^{3/4}$, where c denotes negative constants.

Theorem 4.4. *Let $f \in H^2(\mathbb{R})$ and $\mu > 0$ be given. Then, there exist $\varepsilon \in (0, 1)$, a finite ε -localization family $\{\pi_j^\varepsilon : -N + 1 \leq j \leq N\}$, a constant $K = K(\varepsilon)$, and for each $j \in \{-N + 1, \dots, N\}$ and $\tau \in [0, 1]$ there exist bounded operators*

$$\mathbb{A}_{j,\tau} \in \mathcal{L}(H^3(\mathbb{R}), L_2(\mathbb{R}))$$

such that

$$(4.15) \quad \|\pi_j^\varepsilon \Phi_{\sigma,1}(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon h]\|_2 \leq \mu \|\pi_j^\varepsilon h\|_{H^3} + K \|h\|_{H^{1/4}}$$

for all $j \in \{-N + 1, \dots, N\}$, $\tau \in [0, 1]$, and $h \in H^3(\mathbb{R})$. The operators $\mathbb{A}_{j,\tau}$ are defined by

$$\mathbb{A}_{j,\tau} := -\frac{\pi}{(1 + \tau^2 f'^2(x_j^\varepsilon))^{3/2}} (\partial_x^4)^{3/4}, \quad |j| \leq N - 1,$$

where $x_j^\varepsilon \in \text{supp } \pi_j^\varepsilon$, respectively

$$\mathbb{A}_{N,\tau} := -\pi(\partial_x^4)^{3/4}.$$

Proof. We first pick a finite ε -localization family $\{\pi_j^\varepsilon : -N + 1 \leq j \leq N\}$ and an associated family $\{\chi_j^\varepsilon : -N + 1 \leq j \leq N\}$, with $\varepsilon \in (0, 1)$ to be fixed later on in the proof. It is convenient to write

$$(4.16) \quad \Phi_{\sigma,1}(\tau f)[h] = \sum_{l=0}^1 f'_{l,\tau} B_{1,1}(\tau f)[f_{l,\tau}, (\bar{\omega}_1(\tau f)[h])'], \quad h \in H^3(\mathbb{R}),$$

where, for $l \in \{0, 1\}$, $f_{l,\tau}$ denotes the Lipschitz function

$$f_{l,\tau} := (1 - l)\tau f + l \text{id}_{\mathbb{R}}.$$

Given $|j| \leq N - 1$, we further let

$$\mathbb{A}_{j,\tau}^l := -\pi \frac{(f'_{l,\tau}(x_j^\varepsilon))^2}{(1 + \tau^2 f'^2(x_j^\varepsilon))^{5/2}} (\partial_x^4)^{3/4} = \frac{(f'_{l,\tau}(x_j^\varepsilon))^2}{(1 + \tau^2 f'^2(x_j^\varepsilon))^{5/2}} B_{0,1}(0) \circ \partial_x^3,$$

respectively we set

$$\mathbb{A}_{N,\tau}^l := -\pi l^2 (\partial_x^4)^{3/4} = l^2 B_{0,1}(0) \circ \partial_x^3.$$

In the following we denote by C constants which are independent of ε (and, of course, of $h \in H^3(\mathbb{R})$, $\tau \in [0, 1]$, $l \in \{0, 1\}$, and $j \in \{-N + 1, \dots, N\}$) and the constants denoted by K may depend only upon ε . To establish (4.15) we first consider the case $|j| \leq N - 1$.

The case $|j| \leq N - 1$. Given $l \in \{0, 1\}$ and $|j| \leq N - 1$, we write

$$(4.17) \quad \pi_j^\varepsilon f'_{l,\tau} B_{1,1}(\tau f)[f_{l,\tau}, (\bar{\omega}_1(\tau f)[h])'] - \mathbb{A}_{j,\tau}^l [\pi_j^\varepsilon h] := T_1[h] + T_2[h] + T_3[h],$$

where

$$T_1[h] := \pi_j^\varepsilon f'_{l,\tau} B_{1,1}(\tau f)[f_{l,\tau}, (\bar{\omega}_1(\tau f)[h])'] - f'_{l,\tau}(x_j^\varepsilon) B_{1,1}(\tau f)[f_{l,\tau}, \pi_j^\varepsilon (\bar{\omega}_1(\tau f)[h])'],$$

$$T_2[h] := f'_{l,\tau}(x_j^\varepsilon) B_{1,1}(\tau f)[f_{l,\tau}, \pi_j^\varepsilon (\bar{\omega}_1(\tau f)[h])'] - \frac{(f'_{l,\tau}(x_j^\varepsilon))^2}{1 + \tau^2 f'^2(x_j^\varepsilon)} B_{0,1}(0) [\pi_j^\varepsilon (\bar{\omega}_1(\tau f)[h])'],$$

$$T_3[h] := \frac{(f'_{l,\tau}(x_j^\varepsilon))^2}{1 + \tau^2 f'^2(x_j^\varepsilon)} B_{0,1}(0) [\pi_j^\varepsilon (\bar{\omega}_1(\tau f)[h])'] - \mathbb{A}_{j,\tau}^l [\pi_j^\varepsilon h].$$

To begin, we compute, by using the fact that $\chi_j^\varepsilon = 1$ on $\text{supp } \pi_j^\varepsilon$, that

$$(4.18) \quad T_1[h] = \chi_j^\varepsilon (f'_{l,\tau} - f'_{l,\tau}(x_j^\varepsilon)) B_{1,1}(\tau f)[f_{l,\tau}, \pi_j^\varepsilon (\bar{\omega}_1(\tau f)[h])'] + T_{11}[h],$$

where, using integration by parts, we may reexpress $T_{11}[h]$ in the following way:

$$\begin{aligned}
 T_{11}[h] &= f'_{l,\tau} B_{1,1}(\tau f)[f_{l,\tau}, (\pi_j^\varepsilon)' \bar{\omega}_1(\tau f)[h]] + f'_{l,\tau} B_{1,1}(\tau f)[\pi_j^\varepsilon, f'_{l,\tau} \bar{\omega}_1(\tau f)[h]] \\
 &\quad - 2f'_{l,\tau} B_{2,1}(\tau f)[\pi_j^\varepsilon, f_{l,\tau}, \bar{\omega}_1(\tau f)[h]] \\
 &\quad - 2\tau^2 f'_{l,\tau} B_{3,2}(\tau f, \tau f)[\pi_j^\varepsilon, f_{l,\tau}, f, f' \bar{\omega}_1(\tau f)[h]] \\
 &\quad + 2\tau^2 f'_{l,\tau} B_{4,2}(\tau f, \tau f)[\pi_j^\varepsilon, f_{l,\tau}, f, f, \bar{\omega}_1(\tau f)[h]] \\
 &\quad + (f'_{l,\tau}(x_j^\varepsilon) - f'_{l,\tau})(1 - \chi_j^\varepsilon) B_{1,1}(\tau f)[f_{l,\tau}, (\pi_j^\varepsilon)' \bar{\omega}_1(\tau f)[h]] \\
 &\quad + (f'_{l,\tau}(x_j^\varepsilon) - f'_{l,\tau}) B_{1,1}(\tau f)[\chi_j^\varepsilon, \pi_j^\varepsilon f'_{l,\tau} \bar{\omega}_1(\tau f)[h]] \\
 &\quad - 2(f'_{l,\tau}(x_j^\varepsilon) - f'_{l,\tau}) B_{2,1}(\tau f)[\chi_j^\varepsilon, f_{l,\tau}, \pi_j^\varepsilon \bar{\omega}_1(\tau f)[h]] \\
 &\quad - 2\tau^2 (f'_{l,\tau}(x_j^\varepsilon) - f'_{l,\tau}) B_{3,2}(\tau f, \tau f)[\chi_j^\varepsilon, f_{l,\tau}, f, \pi_j^\varepsilon f' \bar{\omega}_1(\tau f)[h]] \\
 &\quad + 2\tau^2 (f'_{l,\tau}(x_j^\varepsilon) - f'_{l,\tau}) B_{4,2}(\tau f, \tau f)[\chi_j^\varepsilon, f_{l,\tau}, f, f, \pi_j^\varepsilon \bar{\omega}_1(\tau f)[h]].
 \end{aligned}$$

Combining Lemma 3.1(i) and Proposition 4.2(iii), we see that

$$\|T_{11}[h]\|_2 \leq K \|h\|_{H^2},$$

and therewith

$$(4.19) \quad \|T_1[h]\|_2 \leq C \|\chi_j^\varepsilon (f'_{l,\tau} - f'_{l,\tau}(x_j^\varepsilon))\|_\infty \|\pi_j^\varepsilon (\bar{\omega}_1(\tau f)[h])'\|_2 + K \|h\|_{H^2}.$$

We next estimate the term $\|\pi_j^\varepsilon (\bar{\omega}_1(\tau f)[h])'\|_2$. In view of (4.4) and using integration by parts, we have

$$\begin{aligned}
 (1 + a_\mu \mathbb{A}(\tau f)) [\pi_j^\varepsilon (\bar{\omega}_1(\tau f)[h])'] &= \frac{\pi_j^\varepsilon h'''}{(1 + \tau^2 f'^2)^{3/2}} - \frac{3\tau^2 \pi_j^\varepsilon f' f'' h''}{(1 + \tau^2 f'^2)^{5/2}} \\
 (4.20) \quad &\quad - a_\mu \pi_j^\varepsilon T_{0,\text{lot}}(\tau f) [\bar{\omega}_1(\tau f)[h]] \\
 &\quad - \frac{a_\mu}{\pi} T_{1j,\text{lot}}(\tau f) [\bar{\omega}_1(\tau f)[h]],
 \end{aligned}$$

where, given $f \in H^2(\mathbb{R})$, $\bar{\omega} \in H^1(\mathbb{R})$, and $j \in \{-N + 1, \dots, N\}$, we set

$$\begin{aligned}
 T_{1j,\text{lot}}(f) [\bar{\omega}] &:= \pi (\pi_j^\varepsilon \mathbb{A}(f) [\bar{\omega}'] - \mathbb{A}(f) [\pi_j^\varepsilon \bar{\omega}']) \\
 &= f' \left(B_{0,1}(f) [(\pi_j^\varepsilon)' \bar{\omega}] - B_{1,1}(f) [\pi_j^\varepsilon, \bar{\omega}] \right. \\
 (4.21) \quad &\quad \left. - 2B_{2,2}(f, f) [\pi_j^\varepsilon, f, f' \bar{\omega}] + 2B_{3,2}(f, f) [\pi_j^\varepsilon, f, f, \bar{\omega}] \right) \\
 &\quad - B_{1,1}(f) [\pi_j^\varepsilon, f' \bar{\omega}] + 2B_{2,1}(f) [\pi_j^\varepsilon, f, \bar{\omega}] - B_{1,1}(f) [f, (\pi_j^\varepsilon)' \bar{\omega}] \\
 &\quad + 2B_{3,2}(f, f) [\pi_j^\varepsilon, f, f, f' \bar{\omega}] - 2B_{4,2}(f, f) [\pi_j^\varepsilon, f, f, f, \bar{\omega}],
 \end{aligned}$$

the last identity following by using integration by parts. In view of (4.20), it follows from (3.20), (3.29), and (4.3) (with $\tau = 3/4$) that

$$(4.22) \quad \|\pi_j^\varepsilon (\bar{\omega}_1(\tau f)[h])'\|_2 \leq C \|\pi_j^\varepsilon h\|_{H^3} + K \|h\|_{H^{11/4}}.$$

The estimate (4.22) is clearly valid also for $j = N$. Choosing ε sufficiently small, it follows from (4.19) and (4.22) that

$$(4.23) \quad \|T_1[h]\|_2 \leq \frac{\mu}{3} \|\pi_j^\varepsilon h\|_{H^3} + K \|h\|_{H^{11/4}},$$

provided that ε is sufficiently small.

We now consider the term $T_2[h]$ which is decomposed as follows:

$$\begin{aligned} T_2[h] &= f'_{l,\tau}(x_j^\varepsilon)(B_{1,1}(\tau f)[f_{l,\tau}, \pi_j^\varepsilon(\overline{\omega}_1(\tau f)[h])'] \\ &\quad - B_{1,1}(\tau f'(x_j^\varepsilon)\text{id}_{\mathbb{R}})[f_{l,\tau}(x_j^\varepsilon)\text{id}_{\mathbb{R}}, \pi_j^\varepsilon(\overline{\omega}_1(\tau f)[h])']) \\ &= f'_{l,\tau}(x_j^\varepsilon)T_{21}[h] - \frac{\tau^2 f'_{l,\tau}(x_j^\varepsilon)}{1 + \tau^2 f'^2(x_j^\varepsilon)}T_{22}[h], \end{aligned}$$

with

$$\begin{aligned} T_{21}[h] &:= B_{1,1}(\tau f)[f_{l,\tau} - f'_{l,\tau}(x_j^\varepsilon)\text{id}_{\mathbb{R}}, \pi_j^\varepsilon(\overline{\omega}_1(\tau f)[h])'], \\ T_{22}[h] &:= B_{2,1}(\tau f)[f - f'(x_j^\varepsilon)\text{id}_{\mathbb{R}}, f + f'(x_j^\varepsilon)\text{id}_{\mathbb{R}}, \pi_j^\varepsilon(\overline{\omega}_1(\tau f)[h])']. \end{aligned}$$

We first estimate $T_{21}[h]$. Integrating by parts, we get

$$\begin{aligned} (4.24) \quad \|T_{21}[h]\|_2 &\leq \|\chi_j^\varepsilon B_{1,1}(\tau f)[f_{l,\tau} - f'_{l,\tau}(x_j^\varepsilon)\text{id}_{\mathbb{R}}, \pi_j^\varepsilon(\overline{\omega}_1(\tau f)[h])']\|_2 \\ &\quad + \left\| \text{PV} \int_{\mathbb{R}} \frac{[\delta_{[\cdot,y]}(f_{l,\tau} - f'_{l,\tau}(x_j^\varepsilon)\text{id}_{\mathbb{R}})/y][\delta_{[\cdot,y]}\chi_j^\varepsilon/y]}{1 + \tau^2(\delta_{[\cdot,y]}f/y)^2} \right. \\ &\quad \left. \times (\pi_j^\varepsilon(\overline{\omega}_1(\tau f)[h])')(\cdot - y) dy \right\|_2 \\ &\leq \|\chi_j^\varepsilon B_{1,1}(\tau f)[f_{l,\tau} - f'_{l,\tau}(x_j^\varepsilon)\text{id}_{\mathbb{R}}, \pi_j^\varepsilon(\overline{\omega}_1(\tau f)[h])']\|_2 + K \|h\|_{H^2}. \end{aligned}$$

Furthermore, letting $F_{l,\tau,j} \in C(\mathbb{R})$ denote the Lipschitz function with $F_{l,\tau,j} = f_{l,\tau}$ on $\text{supp } \chi_j^\varepsilon$ and $F'_{l,\tau,j} = f'_{l,\tau}(x_j^\varepsilon)$ on $\mathbb{R} \setminus \text{supp } \chi_j^\varepsilon$, we have

$$\begin{aligned} &\chi_j^\varepsilon B_{1,1}(\tau f)[f_{l,\tau} - f'_{l,\tau}(x_j^\varepsilon)\text{id}_{\mathbb{R}}, \pi_j^\varepsilon(\overline{\omega}_1(\tau f)[h])'] \\ &= \chi_j^\varepsilon B_{1,1}(\tau f)[F_{l,\tau,j} - f'_{l,\tau}(x_j^\varepsilon)\text{id}_{\mathbb{R}}, \pi_j^\varepsilon(\overline{\omega}_1(\tau f)[h])'], \end{aligned}$$

and (4.22) combined with (4.24) leads us to

$$\begin{aligned} \|T_{21}[h]\| &\leq C \|f'_{l,\tau} - f'_{l,\tau}(x_j^\varepsilon)\|_{L^\infty(\text{supp } \chi_j^\varepsilon)} \|\pi_j^\varepsilon(\overline{\omega}_1(\tau f)[h])'\|_2 + K \|h\|_{H^2} \\ &\leq \frac{\mu}{6} \|\pi_j^\varepsilon h\|_{H^3} + K \|h\|_{H^{11/4}}, \end{aligned}$$

provided that ε is sufficiently small. Since $T_{22}[h]$ can be estimated in a similar way, we conclude that

$$(4.25) \quad \|T_2[h]\|_2 \leq \frac{\mu}{3} \|\pi_j^\varepsilon h\|_{H^3} + K \|h\|_{H^{11/4}}$$

if ε is chosen sufficiently small.

We now turn to $T_3[h]$ and note that

$$T_3[h] = \frac{(f'_{l,\tau}(x_j^\varepsilon))^2}{1 + \tau^2 f'^2(x_j^\varepsilon)} B_{0,1}(0) \left[\pi_j^\varepsilon(\overline{\omega}_1(\tau f)[h])' - \frac{1}{(1 + f'^2(x_j^\varepsilon))^{3/2}} (\pi_j^\varepsilon h)''' \right],$$

and therefore

$$(4.26) \quad \|T_3[h]\|_2 \leq C \left\| \pi_j^\varepsilon(\bar{\omega}_1(\tau f)[h])' - \frac{1}{(1 + f'^2(x_j^\varepsilon))^{3/2}} \pi_j^\varepsilon h'''' \right\|_2 + K \|h\|_{H^2}.$$

Recalling (4.20), we have

$$\begin{aligned} & \pi_j^\varepsilon(\bar{\omega}_1(\tau f)[h])' - \frac{\pi_j^\varepsilon h''''}{((1 + f'^2(x_j^\varepsilon)))^{3/2}} \\ &= \left[\frac{1}{(1 + \tau^2 f'^2)^{3/2}} - \frac{1}{(1 + f'^2(x_j^\varepsilon))^{3/2}} \right] \pi_j^\varepsilon h'''' \\ & \quad - \frac{3\tau^2 \pi_j^\varepsilon f' f'' h''}{(1 + \tau^2 f'^2)^{5/2}} - a_\mu \mathbb{A}(\tau f)[\pi_j^\varepsilon(\bar{\omega}_1(\tau f)[h])'] \\ & \quad - \frac{a_\mu}{\pi} T_{1j,\text{lot}}(\tau f)[\bar{\omega}_1(\tau f)[h]] - a_\mu \pi_j^\varepsilon T_{0,\text{lot}}(\tau f)[\bar{\omega}_1(\tau f)[h]], \end{aligned}$$

and (4.26) combined with (3.29) and (4.3) (with $\tau = 3/4$) yields

$$(4.27) \quad \|T_3[h]\|_2 \leq \|\mathbb{A}(\tau f)[\pi_j^\varepsilon(\bar{\omega}_1(\tau f)[h])']\|_2 + \frac{\mu}{6} \|\pi_j^\varepsilon h\|_{H^3} + K \|h\|_{H^{11/4}},$$

provided that ε is sufficiently small. We are left with the term

$$\begin{aligned} \mathbb{A}(\tau f)[\pi_j^\varepsilon(\bar{\omega}_1(\tau f)[h])'] &= \tau f' B_{0,1}(\tau f)[\pi_j^\varepsilon(\bar{\omega}_1(\tau f)[h])'] \\ & \quad - \tau B_{1,1}(\tau f)[f, \pi_j^\varepsilon(\bar{\omega}_1(\tau f)[h])'] \\ &= \tau(T_{31}[h] + T_{32}[h]), \end{aligned}$$

where

$$\begin{aligned} T_{31}[h] &:= (f' - f'(x_j^\varepsilon)) B_{0,1}(\tau f)[\pi_j^\varepsilon(\bar{\omega}_1(\tau f)[h])'], \\ T_{32}[h] &:= B_{1,1}(\tau f)[f'(x_j^\varepsilon) \text{id}_{\mathbb{R}} - f, \pi_j^\varepsilon(\bar{\omega}_1(\tau f)[h])']. \end{aligned}$$

The term $T_{32}[h]$ may be estimated in a similar way as the term $T_{21}[h]$ above, while by integrating by parts, we obtain, similarly as in the study of $T_1[h]$, the following estimate:

$$\begin{aligned} \|T_{31}[h]\|_2 &\leq \|\chi_j^\varepsilon(f' - f'(x_j^\varepsilon)) B_{0,1}(\tau f)[\pi_j^\varepsilon(\bar{\omega}_1(\tau f)[h])']\|_2 \\ & \quad + \left\| (f' - f'(x_j^\varepsilon)) \text{PV} \int_{\mathbb{R}} \frac{\delta_{[\cdot, y]} \chi_j^\varepsilon / y}{1 + \tau^2 (\delta_{[\cdot, y]} f / y)^2} (\pi_j^\varepsilon(\bar{\omega}_1(\tau f)[h])')(\cdot - y) dy \right\|_2 \\ &\leq \|\chi_j^\varepsilon(f' - f'(x_j^\varepsilon))\|_\infty \|\pi_j^\varepsilon(\bar{\omega}_1(\tau f)[h])'\|_2 + K \|h\|_{H^2}. \end{aligned}$$

For sufficiently small ε , (4.22) leads us to

$$\|T_{31}[h]\|_2 + \|T_{32}[h]\|_2 \leq \frac{\mu}{6} \|\pi_j^\varepsilon h\|_{H^3} + K \|h\|_{H^{11/4}}$$

and, together with (4.27), we conclude that

$$(4.28) \quad \|T_3[h]\|_2 \leq \frac{\mu}{3} \|\pi_j^\varepsilon h\|_{H^3} + K \|h\|_{H^{11/4}}.$$

The desired estimate (4.15) follows for $|j| \leq N - 1$ from (4.16), (4.17), (4.23), (4.25), and (4.28).

The case $j = N$. Similarly as in the previous case, we write

$$(4.29) \quad \pi_N^\varepsilon f'_{l,\tau} B_{1,1}(f_{l,\tau}, \tau f)[(\bar{\omega}_1(\tau f)[h])'] - \mathbb{A}_{N,\tau}^l[\pi_N^\varepsilon h] := S_1[h] + S_2[h] + S_3[h],$$

with

$$S_1[h] := \pi_N^\varepsilon f'_{l,\tau} B_{1,1}(\tau f)[f_{l,\tau}, (\bar{\omega}_1(\tau f)[h])'] - l B_{1,1}(\tau f)[f_{l,\tau}, \pi_N^\varepsilon(\bar{\omega}_1(\tau f)[h])'],$$

$$S_2[h] := l B_{1,1}(\tau f)[f_{l,\tau}, \pi_N^\varepsilon(\bar{\omega}_1(\tau f)[h])'] - l^2 B_{0,1}(0)[\pi_N^\varepsilon(\bar{\omega}_1(\tau f)[h])'],$$

$$S_3[h] := l^2 B_{0,1}(0)[\pi_N^\varepsilon(\bar{\omega}_1(\tau f)[h])'] - \mathbb{A}_{N,\tau}^l[\pi_N^\varepsilon h].$$

The estimates derived when studying $T_1[h]$ together with the fact that f' vanishes at infinity imply that

$$(4.30) \quad \|S_1[h]\|_2 \leq \frac{\mu}{3} \|\pi_N^\varepsilon h\|_{H^3} + K \|h\|_{H^2}$$

for sufficiently small ε .

If $l = 0$, then $S_i = 0$, $i = 2, 3$, and we are left to consider the case $l = 1$, when $f_{l,\tau} = \text{id}_{\mathbb{R}}$. The relation (3.3) implies that

$$\begin{aligned} S_2[h] &= (B_{0,1}(\tau f) - B_{0,1}(0))[\pi_N^\varepsilon(\bar{\omega}_1(\tau f)[h])'] = -\tau^2 B_{2,1}(\tau f)[f, f, \pi_N^\varepsilon(\bar{\omega}_1(\tau f)[h])'] \\ &= \tau^2 (S_{21}[h] - S_{22}[h]), \end{aligned}$$

where

$$S_{21}[h] := \tau^2 \text{PV} \int_{\mathbb{R}} \frac{(\delta_{[\cdot,y]} f/y)^2 (\delta_{[\cdot,y]} \chi_N^\varepsilon/y)}{1 + \tau^2 (\delta_{[\cdot,y]} f/y)^2} [\pi_N^\varepsilon(\bar{\omega}_1(\tau f)[h])'](\cdot - y) dy,$$

$$S_{22}[h] := \chi_N^\varepsilon B_{2,1}(\tau f)[f, f, \pi_N^\varepsilon(\bar{\omega}_1(\tau f)[h])'].$$

Integrating by parts we get

$$\|S_{21}[h]\|_2 \leq K \|h\|_{H^2}.$$

To deal with $S_{22}[h]$, we let $F_N \in W_\infty^1(\mathbb{R})$ denote the function defined by

$$F_N(x) := \begin{cases} f(x), & |x| \geq x_N - \varepsilon, \\ \frac{x + x_N - \varepsilon}{2(x_N - \varepsilon)} f(x_N - \varepsilon) + \frac{x_N - \varepsilon - x}{2(x_N - \varepsilon)} f(-x_N + \varepsilon), & |x| \leq x_N - \varepsilon. \end{cases}$$

Since $f, f' \in C_0(\mathbb{R})$, the relation (4.9) implies that $\|F'_N\|_\infty \rightarrow 0$ for $\varepsilon \rightarrow 0$. Moreover, recalling (4.22), we find for ε sufficiently small that

$$\begin{aligned} \|S_{22}[h]\|_2 &= \|\chi_N^\varepsilon B_{2,1}(\tau f)[F_N, F_N, \pi_N^\varepsilon(\bar{\omega}_1(\tau f)[h])']\|_2 \\ &\leq C \|F'_N\|_\infty^2 \|\pi_N^\varepsilon(\bar{\omega}_1(\tau f)[h])'\|_2 \\ &\leq \frac{\mu}{3} \|\pi_N^\varepsilon h\|_{H^3} + K \|h\|_{H^{11/4}}, \end{aligned}$$

and therewith

$$(4.31) \quad \|S_2[h]\|_2 \leq \frac{\mu}{3} \|\pi_N^\varepsilon h\|_{H^3} + K \|h\|_{H^{11/4}}.$$

Finally, it holds that

$$(4.32) \quad \begin{aligned} \|S_3[h]\|_2 &= \|B_{0,1}(0)[\pi_N^\varepsilon(\bar{\omega}_1(\tau f)[h])' - (\pi_N^\varepsilon h)''']\|_2 \\ &\leq C \|\pi_N^\varepsilon(\bar{\omega}_1(\tau f)[h])' - \pi_N^\varepsilon h'''\|_2 + K \|h\|_{H^2}, \end{aligned}$$

and, since f vanishes at infinity, the arguments used to derive (4.27) show that

$$(4.33) \quad \begin{aligned} \|\pi_N^\varepsilon(\overline{\omega}_1(\tau f)[h])' - \pi_N^\varepsilon h'''\|_2 &\leq \|\mathbb{A}(\tau f)[\pi_N^\varepsilon(\overline{\omega}_1(\tau f)[h])']\|_2 \\ &\quad + \frac{\mu}{6}\|\pi_N^\varepsilon h\|_{H^3} + K\|h\|_{H^{11/4}}, \end{aligned}$$

provided that ε is sufficiently small. Furthermore, the first term on the right-hand side of (4.33) is decomposed as follows:

$$\begin{aligned} &\mathbb{A}(\tau f)[\pi_N^\varepsilon(\overline{\omega}_1(\tau f)[h])'] \\ &= \tau(f'B_{0,1}(\tau f)[\pi_N^\varepsilon(\overline{\omega}_1(\tau f)[h])'] - B_{1,1}(\tau f)[f, \pi_N^\varepsilon(\overline{\omega}_1(\tau f)[h])']) \\ &= \tau\chi_N^\varepsilon(f'B_{0,1}(\tau f)[\pi_N^\varepsilon(\overline{\omega}_1(\tau f)[h])'] - B_{1,1}(\tau f)[F_N, \pi_N^\varepsilon(\overline{\omega}_1(\tau f)[h])']) \\ &\quad - \tau f'PV \int_{\mathbb{R}} \frac{\delta_{[\cdot,y]}\chi_N^\varepsilon/y}{1 + \tau^2(\delta_{[\cdot,y]}f/y)^2} (\pi_N^\varepsilon(\overline{\omega}_1(\tau f)[h])')(\cdot - y) dy \\ &\quad + \tau PV \int_{\mathbb{R}} \frac{(\delta_{[\cdot,y]}f/y)(\delta_{[\cdot,y]}\chi_N^\varepsilon/y)}{1 + \tau^2(\delta_{[\cdot,y]}f/y)^2} (\pi_N^\varepsilon(\overline{\omega}_1(\tau f)[h])')(\cdot - y) dy, \end{aligned}$$

where F_N is the function introduced when considering $S_{22}[h]$. Integrating by parts the integral terms, we infer from Lemma 3.1(i), (4.3), (4.22), and the fact that f' vanishes at infinity that

$$\|\mathbb{A}(\tau f)[\pi_N^\varepsilon(\overline{\omega}_1(\tau f)[h])']\|_2 \leq \frac{\mu}{6}\|\pi_j^\varepsilon h\|_{H^3} + K\|h\|_{H^{11/4}}$$

for ε sufficiently small. The latter estimate together with (4.32) and (4.33) implies that

$$(4.34) \quad \|S_3[h]\|_2 \leq \frac{\mu}{3}\|\pi_j^\varepsilon h\|_{H^3} + K\|h\|_{H^{11/4}}$$

if ε is sufficiently small. Summarizing, for $j = N$, the desired estimate (4.15) follows from (4.29), (4.30), (4.31), (4.34). This completes the proof. \square

The Fourier multipliers $\mathbb{A}_{j,\tau}$ found in Theorem 4.4 are elements of the family of unbounded operators $\{\mathbb{A}_{x_0,\tau} : \tau \in [0, 1], x_0 \in \mathbb{R}\}$, where

$$\mathbb{A}_{x_0,\tau} := -\frac{\pi}{(1 + \tau^2 f'^2(x_0))^{3/2}} (\partial_x^4)^{3/4}.$$

Each operator $\mathbb{A}_{x_0,\tau}$ is the generator of a strongly continuous analytic semigroup in $\mathcal{L}(L_2(\mathbb{R}))$. Moreover, it is not difficult to prove (see for example the proof of [40, Proposition 6.3]) that there exists a constant $\kappa_0 \geq 1$ such that

$$(4.35) \quad \lambda - \mathbb{A}_{x_0,\tau} \in \text{Isom}(H^3(\mathbb{R}), L_2(\mathbb{R})),$$

$$(4.36) \quad \kappa_0\|(\lambda - \mathbb{A}_{x_0,\tau})[h]\|_2 \geq |\lambda| \cdot \|h\|_2 + \|h\|_{H^3}$$

for all $x_0 \in \mathbb{R}$, $\tau \in [0, 1]$, $\lambda \in \mathbb{C}$ with $\text{Re} \lambda \geq 1$, and $h \in H^3(\mathbb{R})$. Combining these properties with Theorem 4.4, we obtain the following generation result.

Theorem 4.5. *Given $f \in H^2(\mathbb{R})$, it holds that*

$$-\Phi_\sigma(f) \in \mathcal{H}(H^3(\mathbb{R}), L_2(\mathbb{R})).$$

Proof. The proof is similar to that of [40, Theorem 6.3], so that we only present the main steps. Using the inequality (4.36), Remark 4.3, and Theorem 4.4, we may find constants $\omega > 1$ and $\kappa \geq 1$ such that

$$(4.37) \quad \kappa \|(\lambda - \Phi_{\sigma,1}(\tau f))[h]\|_2 \geq |\lambda| \cdot \|h\|_2 + \|h\|_{H^3}$$

for all $\tau \in [0, 1]$, $\lambda \in \mathbb{C}$ with $\text{Re}\lambda \geq \omega$, and $h \in H^3(\mathbb{R})$. Additionally, taking into account that $\omega - \Phi_{\sigma,1}(0) = \omega + \pi(\partial_x^4)^{3/4}$, the relation (4.35) shows that the latter operator is an isomorphism, and (4.37) together with the method of continuity yields

$$(4.38) \quad \omega - \Phi_{\sigma,1}(f) \in \text{Isom}(H^3(\mathbb{R}), L_2(\mathbb{R})).$$

Since $\Phi_{\sigma,2}(f)$ is a lower order perturbation, the claim follows from (4.37) and (4.38). □

We now come to the proof of our first main result, which is mainly based on the abstract theory for quasilinear parabolic problems due to H. Amann (cf. [3, Section 12]) and a recent idea from the proof of [40, Theorem 1.3] where a parameter trick, also used in [7, 28, 44], is employed in a different manner.

Proof of Theorem 1.1. In virtue of (4.6) and of Theorem 4.5, we have

$$-\Phi_\sigma \in C^\omega(H^2(\mathbb{R}), \mathcal{H}(H^3(\mathbb{R}), L_2(\mathbb{R}))).$$

This relation together with the well-known interpolation property

$$(4.39) \quad [H^{s_0}(\mathbb{R}), H^{s_1}(\mathbb{R})]_\theta = H^{(1-\theta)s_0 + \theta s_1}(\mathbb{R}), \quad \theta \in (0, 1), \quad -\infty \leq s_0 \leq s_1 < \infty,$$

the choices $s \in (2, 3)$, $\bar{s} = 2$, $1 > \alpha := s/3 > \beta := 2/3 > 0$, and the relations

$$H^2(\mathbb{R}) = [L_2(\mathbb{R}), H^3(\mathbb{R})]_\beta \quad \text{and} \quad H^s(\mathbb{R}) = [L_2(\mathbb{R}), H^3(\mathbb{R})]_\alpha,$$

enables us to use abstract results from [1–3] (see [40, Theorem 1.5]) to establish, for each $f_0 \in H^s(\mathbb{R})$, $s \in (2, 3)$, the existence of a unique classical solution $f = f(\cdot; f_0)$ to (4.5) such that

$$f \in C([0, T_+(f_0)), H^s(\mathbb{R})) \cap C((0, T_+(f_0)), H^3(\mathbb{R})) \cap C^1((0, T_+(f_0)), L_2(\mathbb{R}))$$

and

$$f \in C^{(s-2)/3}([0, T], H^2(\mathbb{R})) \quad \text{for all } T < T_+(f_0).$$

Concerning the uniqueness claim, it suffices to show that any classical solution

$$f \in C([0, \tilde{T}), H^s(\mathbb{R})) \cap C((0, \tilde{T}), H^3(\mathbb{R})) \cap C^1((0, \tilde{T}), L_2(\mathbb{R})), \quad \tilde{T} \in (0, \infty],$$

to (4.5) satisfies

$$(4.40) \quad f \in C^\eta([0, T], H^2(\mathbb{R})) \quad \text{for all } T \in (0, \tilde{T}),$$

with $\eta := (s - 2)/(s + 1)$; see [40, Theorem 1.5]. Thus let $T \in (0, \tilde{T})$ be fixed. The boundedness of $f : [0, T] \rightarrow H^s(\mathbb{R})$, Theorem 3.5, Proposition 3.6, and an interpolation argument imply that

$$\sup_{t \in [0, T]} \|(1 + a_\mu \mathbb{A}(f))^{-1}\|_{\mathcal{L}(H^r(\mathbb{R}))} \leq C$$

for all $r \in (0, 1)$ (in particular for $r = s - 2$), and $\bar{\omega}_1(f)[f] = (1 + a_\mu \mathbb{A}(f))^{-1}[\kappa(f)]$ leads to

$$(4.41) \quad \sup_{t \in [0, T]} \|\bar{\omega}_1(f)[f]\|_{H^{s-2}} \leq C.$$

Arguing as in the proof of [40, Theorem 1.2], we obtain in view of (4.41) that

$$(4.42) \quad \sup_{t \in (0, T]} \|\Phi_{\sigma, 1}(f)[f]\|_{H^{-1}} \leq C.$$

In order to study the boundedness of

$$\begin{aligned} \Phi_{\sigma, 2}(f)[f] &= -\Theta \mathbb{B}(f) [(1 + a_\mu \mathbb{A}(f))^{-1} [f']] \\ &\quad + a_\mu \mathbb{B}(f) [(1 + a_\mu \mathbb{A}(f))^{-1} [T_{0, \text{lot}}(f) [\bar{\omega}_1(f)[f]]]], \end{aligned}$$

we first note that

$$\sup_{t \in [0, T]} \|(1 + a_\mu \mathbb{A}(f))^{-1} [f']\|_2 \leq C,$$

and Lemma 3.1 implies

$$(4.43) \quad \sup_{t \in [0, T]} \|\mathbb{B}(f) [(1 + a_\mu \mathbb{A}(f))^{-1} [f']]\|_2 \leq C.$$

To deal with the remaining term, we may argue as in the proof of [40, Theorem 1.2] to obtain, in virtue of (4.41), that

$$(4.44) \quad \sup_{t \in (0, T]} \|T_{0, \text{lot}}(f) [\bar{\omega}_1(f)[f]]\|_1 \leq C.$$

Given $t \in (0, T]$, it holds that $f = f(t) \in H^3(\mathbb{R})$ and by Proposition 4.2(iii) we deduce that $\bar{\omega}_3(f) := (1 + a_\mu \mathbb{A}(f))^{-1} [T_{0, \text{lot}}(f) [\bar{\omega}_1(f)[f]]] \in L_2(\mathbb{R})$ with

$$\begin{aligned} \int_{\mathbb{R}} T_{0, \text{lot}}(f) [\bar{\omega}_1(f)[f]] \varphi \, dx &= \int_{\mathbb{R}} \varphi (1 + a_\mu \mathbb{A}(f)) [\bar{\omega}_3(f)] \, dx \\ &= \int_{\mathbb{R}} \bar{\omega}_3(f) (1 + a_\mu (\mathbb{A}(f))^*) [\varphi] \, dx \end{aligned}$$

for all $\varphi \in H^1(\mathbb{R})$, where $(\mathbb{A}(f))^* \in \mathcal{L}(L_2(\mathbb{R}))$ is the adjoint of $\mathbb{A}(f)$; cf. Lemma 3.8. Moreover, since

$$\sup_{t \in [0, T]} \|(1 + a_\mu (\mathbb{A}(f))^*)^{-1}\|_{\mathcal{L}(H^1(\mathbb{R}))} \leq C,$$

the estimate (4.44) leads us to

$$\left| \int_{\mathbb{R}} \bar{\omega}_3(f) \psi \, dx \right| = \left| \int_{\mathbb{R}} T_{0, \text{lot}}(f) [\bar{\omega}_1(f)[f]] (1 + a_\mu (\mathbb{A}(f))^*)^{-1} [\psi] \, dx \right| \leq C \|\psi\|_{H^1}$$

for all $\psi \in H^1(\mathbb{R})$, hence

$$(4.45) \quad \sup_{t \in (0, T]} \|\bar{\omega}_3(f)\|_{H^{-1}} \leq C.$$

Letting $(\mathbb{B}(f))^* \in \mathcal{L}(L_2(\mathbb{R}))$ denote the adjoint of $\mathbb{B}(f)$, it is not difficult to see that

$$(\mathbb{B}(f))^* [\varphi] = -B_{0, 1}(f) [\varphi] - B_{1, 1}(f) [f, f' \varphi], \quad \varphi \in L_2(\mathbb{R}),$$

and Lemmas 3.1–3.2 yield $(\mathbb{B}(f))^* \in \mathcal{L}(H^1(\mathbb{R}))$ with

$$(4.46) \quad \sup_{t \in [0, T]} \|(\mathbb{B}(f))^*\|_{\mathcal{L}(H^1(\mathbb{R}))} \leq C.$$

Since for $t \in (0, T]$ and $\varphi \in H^1(\mathbb{R})$

$$\left| \int_{\mathbb{R}} \mathbb{B}(f) [\bar{\omega}_3(f)] \varphi \, dx \right| = \left| \int_{\mathbb{R}} \bar{\omega}_3(f) (\mathbb{B}(f))^* [\varphi] \, dx \right| \leq \|(\mathbb{B}(f))^* [\varphi]\|_{H^1} \|\bar{\omega}_3(f)\|_{H^{-1}},$$

we deduce from (4.45)–(4.46) that

$$(4.47) \quad \sup_{t \in (0, T]} \|\mathbb{B}(f)[\bar{\omega}_3(f)]\|_{H^{-1}} \leq C.$$

Gathering (4.42), (4.43), and (4.47) it follows that $f \in \text{BC}^1((0, T], H^{-1}(\mathbb{R}))$, and together with $f \in C([0, T], H^s(\mathbb{R}))$ we conclude that $f \in C^\eta([0, T], H^2(\mathbb{R}))$. This proves the uniqueness claim.

The criterion for global existence and the remaining regularity properties follow in the same way as in the particular case $\mu_- = \mu_+$ (see the proofs of [40, Theorems 1.2 and 1.3]), and the details are therefore omitted. \square

5. THE MUSKAT PROBLEM WITHOUT SURFACE TENSION

In this section we neglect the surface tension effects, that is, we set $\sigma = 0$. Since the curvature term no longer appears in (1.1b), we cannot expect to express (1.1) as a quasilinear evolution equation when $\mu_- \neq \mu_+$. As a first step we formulate the problem (1.1) as an evolution problem for the free boundary f only (cf. (5.8)), which appears to be, in the regime where $\mu_- \neq \mu_+$, of fully nonlinear type.

The abstract formulation. We start by solving the equation (1.1b). Since $\sigma = 0$, (1.1b) is equivalent to

$$(5.1) \quad -c_{\rho, \mu} f' = (1 + a_\mu \mathbb{A}(f))[\bar{\omega}],$$

where

$$c_{\rho, \mu} := \frac{2k\Theta}{\mu_- + \mu_+}$$

and where Θ is defined in (1.3). Recalling Proposition 3.6, we remark that the equation (5.1) has a unique solution $\bar{\omega} \in H^1(\mathbb{R})$, provided that the left-hand side satisfies $f' \in H^1(\mathbb{R})$ and the argument of \mathbb{A} belongs to $H^2(\mathbb{R})$. Hence, the same regularity is required from f on both sides of (5.1). This is the main reason why the Muskat problem without surface tension is, for $\mu_- \neq \mu_+$, a fully nonlinear evolution problem.

Proposition 5.1. *Given $f \in H^2(\mathbb{R})$, there exists a unique solution $\bar{\omega} := \bar{\omega}(f)$ to (5.1) and*

$$(5.2) \quad \bar{\omega} \in C^\omega(H^2(\mathbb{R}), H^1(\mathbb{R})).$$

Moreover, given $f_0 \in H^2(\mathbb{R})$ and $\tau \in (1/2, 1)$, there exists a constant C such that

$$(5.3) \quad \|\partial_f \bar{\omega}(f_0)[f]\|_2 \leq C \|f\|_{H^{1+\tau}},$$

$$(5.4) \quad \|\partial_f \bar{\omega}(f_0)[f]\|_{H^\tau} \leq C \|f\|_{H^{1+2\tau-\tau^2}},$$

$$(5.5) \quad \|\partial_f \bar{\omega}(f_0)[f]\|_{H^1} \leq C \|f\|_{H^2}$$

for all $f \in H^2(\mathbb{R})$.

Proof. Since $|a_\mu| < 1$, it follows from Proposition 3.6 that

$$\bar{\omega}(f) := -c_{\rho, \mu} (1 + a_\mu \mathbb{A}(f))^{-1} [f'] \in H^1(\mathbb{R})$$

is the unique solution to (5.1). The regularity property is a consequence of (3.16).

Now let $f_0 \in H^2(\mathbb{R})$ be fixed. Using the chain rule, we find that $\partial_f \bar{\omega}(f_0)[f]$ solves the equation

$$(5.6) \quad (1 + a_\mu \mathbb{A}(f_0))[\partial_f \bar{\omega}(f_0)[f]] = -c_{\rho,\mu} f' - a_\mu \partial \mathbb{A}(f_0)[f][\bar{\omega}(f_0)],$$

where, taking advantage of Lemma 3.1, we find that

$$(5.7) \quad \begin{aligned} \pi \partial \mathbb{A}(f_0)[f][\bar{\omega}] &= f' B_{0,1}(f_0)[\bar{\omega}] - 2f'_0 B_{2,2}(f_0, f_0)[f_0, f, \bar{\omega}] \\ &\quad - B_{1,1}(f_0)[f, \bar{\omega}] + 2B_{3,2}(f_0, f_0)[f_0, f_0, f, \bar{\omega}] \end{aligned}$$

for all $f \in H^2(\mathbb{R})$, $\bar{\omega} \in H^1(\mathbb{R})$. The estimate (5.3) is a consequence of Lemma 3.1(i) and (3.20), while (5.5) simply states that $\partial_f \bar{\omega}(f_0) \in \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R}))$. Finally, (5.4) follows from (5.3) and (5.5) via complex interpolation; cf. (4.39). \square

Appealing to Proposition 5.1, we may reformulate (1.1), after rescaling the time, as an autonomous evolution problem

$$(5.8) \quad \partial_t f = \Phi(f), \quad t \geq 0, \quad f(0) = f_0,$$

where $\Phi : H^2(\mathbb{R}) \rightarrow H^1(\mathbb{R})$ is the fully nonlinear and nonlocal operator

$$\Phi(f) := \mathbb{B}(f)[\bar{\omega}(f)], \quad f \in H^2(\mathbb{R}).$$

The regularity properties (3.18) and (5.2) ensure that

$$(5.9) \quad \Phi \in C^\omega(H^2(\mathbb{R}), H^1(\mathbb{R})).$$

In the analysis of (5.8) we have to differentiate between the cases when $\Theta = 0$ and $\Theta \neq 0$. The case $\Theta = 0$ is special, because for this choice $c_{\rho,\mu} = 0$ and therewith $\bar{\omega}(f) = 0$ for all $f \in H^r(\mathbb{R})$ with $r > 3/2$; cf. Theorem 3.5. Hence, the problem (1.1) possesses for each $f \in H^r(\mathbb{R})$ with $r > 3/2$ a unique global solution $f(t) = f_0$ for all $t \in \mathbb{R}$. In the remaining part of the paper we address the nondegenerate case when $\Theta \neq 0$. The next task it to determine the Fréchet derivative $\partial \Phi(f_0)$, $f_0 \in H^2(\mathbb{R})$, and to investigate whether this derivative is the generator of a strongly continuous and analytic semigroup in $\mathcal{L}(H^1(\mathbb{R}))$. Our analysis below shows that the operator $\partial \Phi(f_0)$ has the desired generator property, provided that f_0 is chosen such that the Rayleigh–Taylor condition holds.

The Rayleigh–Taylor condition. Given $f_0 \in H^2(\mathbb{R})$, the Rayleigh–Taylor condition may be reexpressed, in view of (1.2) and of the formulas (2.1a) and (2.7), as

$$(5.10) \quad a_{\text{RT}} := c_{\rho,\mu} + \frac{a_\mu}{\pi} B_{0,1}(f_0)[\bar{\omega}_0] + \frac{a_\mu}{\pi} f'_0 B_{1,1}(f_0)[f_0, \bar{\omega}_0] > 0,$$

where, to keep the notation short, we have set

$$(5.11) \quad \bar{\omega}_0 := \bar{\omega}(f_0) \in H^1(\mathbb{R});$$

cf. Proposition 5.1. If the fluids have equal viscosities, then $a_\mu = 0$ and the Rayleigh–Taylor condition is equivalent to the relation $\Theta = g(\rho_- - \rho_+) > 0$. Since $\Theta \neq 0$ and $\mathbb{B}(f_0)[\bar{\omega}_0] \in H^1(\mathbb{R})$ vanishes at infinity, the Rayleigh–Taylor condition implies also for $\mu_- \neq \mu_+$ that $\Theta > 0$, and due to this fact we restrict our analysis to this case. Finally, we remark that (5.10) is equivalent to

$$(5.12) \quad \inf_{\mathbb{R}} a_{\text{RT}} > 0,$$

and therefore the set \mathcal{O} of initial data that satisfy the Rayleigh–Taylor condition and that was introduced in Section 1 can be reexpressed as

$$\mathcal{O} = \left\{ f_0 \in H^2(\mathbb{R}) : \inf_{\mathbb{R}} a_{\text{RT}} > 0 \right\}.$$

Since $\bar{w}(0) = 0$, it follows that $0 \in \mathcal{O}$ and therewith all $f_0 \in H^2(\mathbb{R})$ that are sufficiently small belong to this set. We emphasize that this set may actually be very large, for example it is easy to infer from (5.10) that $\mathcal{O} = H^2(\mathbb{R})$ if $\mu_- = \mu_+$.

The Fréchet derivative. Let $f_0 \in \mathcal{O}$. Keeping (5.11) in mind, we compute that

$$(5.13) \quad \partial\Phi(f_0)[f] = \partial\mathbb{B}(f_0)[f][\bar{w}_0] + \mathbb{B}(f_0)[\partial_f \bar{w}(f_0)[f]] \quad \text{for } f \in H^2(\mathbb{R}),$$

where

$$(5.14) \quad \begin{aligned} \partial\mathbb{B}(f_0)[f][\bar{w}_0] &= -2B_{2,2}(f_0, f_0)[f_0, f, \bar{w}_0] + f' B_{1,1}(f_0)[f_0, \bar{w}_0] \\ &+ f'_0 B_{1,1}(f_0)[f, \bar{w}_0] - 2f'_0 B_{3,2}(f_0, f_0)[f, f_0, f_0, \bar{w}_0]. \end{aligned}$$

While (5.13) follows from the chain rule, the relation (5.14) can be easily derived with the help of Lemma 3.1. In order to establish the generator property for $\partial\Phi(f_0)$, we proceed in the same way as in Section 4, but now the situation is much more involved. To begin, we consider a continuous path

$$[\tau \mapsto \Psi(\tau)] : [0, 1] \rightarrow \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R})),$$

with

$$\Psi(\tau)[f] := \tau \partial\mathbb{B}(f_0)[f][\bar{w}_0] + \mathbb{B}(\tau f_0)[w(\tau)[f]] \quad \text{for } f \in H^2(\mathbb{R}),$$

and where $w : [0, 1] \rightarrow \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R}))$ is a further continuous path

$$(5.15) \quad \begin{aligned} w(\tau)[f] &:= (1 + a_\mu \mathbb{A}(\tau f_0))^{-1} \left[-c_{\rho, \mu} f' - \frac{a_\mu}{\pi} f' B_{0,1}(f_0)[\bar{w}_0] \right. \\ &\quad - \frac{a_\mu}{\pi} (1 - \tau) f' f'_0 B_{1,1}(f_0)[f_0, \bar{w}_0] \\ &\quad + 2 \frac{\tau a_\mu}{\pi} f'_0 B_{2,2}(f_0, f_0)[f_0, f, \bar{w}_0] \\ &\quad + \frac{\tau a_\mu}{\pi} B_{1,1}(f_0)[f, \bar{w}_0] \\ &\quad \left. - 2 \frac{\tau a_\mu}{\pi} B_{3,2}(f_0, f_0)[f_0, f_0, f, \bar{w}_0] \right]. \end{aligned}$$

The relations (5.6)–(5.7) and (5.13) show that $\Psi(1) = \partial\Phi(f_0)$, while, as a consequence of $\mathbb{A}(0) = 0$ and of (5.10), it holds that

$$(5.16) \quad \Psi(0)[f] = \mathbb{B}(0)[w(0)[f]] = -\mathbb{B}(0)[a_{\text{RT}} f'] = -\pi H[a_{\text{RT}} f'],$$

with H denoting the Hilbert transform again. The term on the right-hand side of (5.15) which has $(1 - \tau)$ as a multiplying factor has been introduced artificially. Due to this trick, we are able for example to write, when setting $\tau = 0$, the function a_{RT} as a multiplicative term in the argument of $\mathbb{B}(0)$ in (5.16). Moreover, this artificial term provides some useful cancellations in the proof of Theorem 5.2 which are, together with our assumption (5.12), an important ingredient when establishing the generator property for $\Psi(1)$; see Lemma 5.3 and Theorem 5.4.

We note that the operator w defined in (5.15) can be estimated in a similar manner as the Fréchet derivative $\partial_f \bar{w}(f_0)$, that is, there exists a constant C such that

$$(5.17) \quad \|w(\tau)[f]\|_2 \leq C\|f\|_{H^{7/4}},$$

$$(5.18) \quad \|w(\tau)[f]\|_{H^{3/4}} \leq C\|f\|_{H^{31/16}},$$

$$(5.19) \quad \|w(\tau)[f]\|_{H^1} \leq C\|f\|_{H^2}$$

for all $\tau \in [0, 1]$ and all $f \in H^2(\mathbb{R})$. Additionally, recalling (3.27) and (3.29), Lemmas 3.1–3.2, in particular the estimate (3.6), lead us to the relation

$$(5.20) \quad \begin{aligned} & (1 + a_\mu \mathbb{A}(\tau f_0))[(w(\tau)[f])'] \\ &= -c_{\rho,\mu} f'' - \frac{a_\mu}{\pi} f'' B_{0,1}(f_0)[\bar{w}_0] - \frac{a_\mu}{\pi} (1 - \tau) f'' f'_0 B_{1,1}(f_0)[f_0, \bar{w}_0] \\ &+ 2 \frac{\tau a_\mu}{\pi} f'_0 B_{1,2}(f_0, f_0)[f_0, f'' \bar{w}_0] + \frac{\tau a_\mu}{\pi} B_{0,1}(f_0)[f'' \bar{w}_0] \\ &- 2 \frac{\tau a_\mu}{\pi} B_{2,2}(f_0, f_0)[f_0, f_0, f'' \bar{w}_0] + T_{2,\text{lot}}(\tau)[f], \end{aligned}$$

where the lower order terms are encompassed by the term $T_{2,\text{lot}}(\tau)[f]$ and

$$(5.21) \quad \|T_{2,\text{lot}}(\tau)[f]\|_2 \leq C\|f\|_{H^{31/16}}$$

for all $\tau \in [0, 1]$ and all $f \in H^2(\mathbb{R})$.

The following theorem lies at the core of our generator result in Theorem 5.4, and its assertion is independent of whether (5.12) holds or not. Before stating the result, we point out that $B_{0,1}(0) \circ \partial_x = \pi(-\partial_x^2)^{1/2}$, where $(-\partial_x^2)^{1/2}$ is the Fourier multiplier with symbol $[\xi \mapsto |\xi|]$.

Theorem 5.2. *Let $f \in H^2(\mathbb{R})$ and $\mu > 0$ be given. Then, there exist $\varepsilon \in (0, 1)$, a finite ε -localization family $\{\pi_j^\varepsilon : -N + 1 \leq j \leq N\}$, a constant $K = K(\varepsilon)$, and for each $j \in \{-N + 1, \dots, N\}$ and $\tau \in [0, 1]$ there exist bounded operators*

$$\mathbb{A}_{j,\tau} \in \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R}))$$

such that

$$(5.22) \quad \|\pi_j^\varepsilon \Psi(\tau)[f] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\|_{H^1} \leq \mu \|\pi_j^\varepsilon f\|_{H^2} + K\|f\|_{H^{31/16}}$$

for all $j \in \{-N + 1, \dots, N\}$, $\tau \in [0, 1]$, and $f \in H^2(\mathbb{R})$. The operators $\mathbb{A}_{j,\tau}$ are defined by

$$\mathbb{A}_{j,\tau} := -\alpha_\tau(x_j^\varepsilon)(-\partial_x^2)^{1/2} + \beta_\tau(x_j^\varepsilon)\partial_x, \quad |j| \leq N - 1,$$

where $x_j^\varepsilon \in \text{supp } \pi_j^\varepsilon$,

$$\alpha_\tau := \pi\left(1 - \frac{\tau f_0'^2}{1 + f_0'^2}\right)_{a_{\text{RT}}}, \quad \beta_\tau := \tau\left(B_{1,1}(f_0)[f_0, \bar{w}_0] - a_\mu \pi \frac{\bar{w}_0}{1 + f_0'^2}\right),$$

and $\bar{w}_0 := \bar{w}(f_0)$, respectively

$$\mathbb{A}_{N,\tau} := -\pi c_{\rho,\mu}(-\partial_x^2)^{1/2}.$$

Proof. Let $\{\pi_j^\varepsilon : -N + 1 \leq j \leq N\}$ be a finite ε -localization family, with $\varepsilon \in (0, 1)$ to be fixed later on, and let $\{\chi_j^\varepsilon : -N + 1 \leq j \leq N\}$ be an associated family. As before, we denote by C constants which are independent of ε (and, of course, of

$f \in H^2(\mathbb{R})$, $\tau \in [0, 1]$, and $j \in \{-N + 1, \dots, N\}$), and the constants denoted by K may depend only upon ε .

Step 1: The lower order terms. Recalling Lemma 3.1, (3.20), (5.14), (5.17), and exploiting the embedding $H^{7/4}(\mathbb{R}) \hookrightarrow W^1_\infty(\mathbb{R})$ we find that

$$\begin{aligned}
 & \|\pi_j^\varepsilon \Psi(\tau)[f] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\|_{H^1} \\
 (5.23) \quad & \leq \|\pi_j^\varepsilon \Psi(\tau)[f] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\|_2 + \|(\pi_j^\varepsilon \Psi(\tau)[f] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon f])'\|_2 \\
 & \leq K\|f\|_{H^{7/4}} + \|\pi_j^\varepsilon (\Psi(\tau)[f])' - \mathbb{A}_{j,\tau}[(\pi_j^\varepsilon f)']\|_2.
 \end{aligned}$$

Moreover, differentiating the relations (5.14) and (5.15) once, it follows from (3.19), (3.27), (5.18), and Lemmas 3.1–3.2 that

$$\begin{aligned}
 (\Psi(\tau)[f])' &= -2\tau B_{1,2}(f_0, f_0)[f_0, f''\bar{\omega}_0] + \tau f'' B_{1,1}(f_0)[f_0, \bar{\omega}_0] + \tau f'_0 B_{0,1}(f_0)[f''\bar{\omega}_0] \\
 &\quad - 2\tau f'_0 B_{2,2}(f_0, f_0)[f_0, f_0, f''\bar{\omega}_0] + \mathbb{B}(\tau f_0)[(w(\tau)[f])'] + T_{3,\text{lot}}(\tau)[f],
 \end{aligned}$$

with

$$(5.24) \quad \|T_{3,\text{lot}}(\tau)[f]\|_2 \leq C(\|f\|_{H^{7/4}} + \|w(\tau)[f]\|_{H^{3/4}}) \leq C\|f\|_{H^{31/16}}.$$

Hence, we are left to estimate the L_2 -norm of the leading order term

$$\begin{aligned}
 & -2\tau\pi_j^\varepsilon B_{1,2}(f_0, f_0)[f_0, f''\bar{\omega}_0] + \tau\pi_j^\varepsilon f'' B_{1,1}(f_0)[f_0, \bar{\omega}_0] + \tau\pi_j^\varepsilon f'_0 B_{0,1}(f_0)[f''\bar{\omega}_0] \\
 & \quad - 2\tau\pi_j^\varepsilon f'_0 B_{2,2}(f_0, f_0)[f_0, f_0, f''\bar{\omega}_0] + \pi_j^\varepsilon \mathbb{B}(\tau f_0)[(w(\tau)[f])'] - \mathbb{A}_{j,\tau}[(\pi_j^\varepsilon f)'],
 \end{aligned}$$

and this is performed below in several steps.

Step 2. Given $|j| \leq N - 1$, we let $\mathbb{A}_j^k \in \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R}))$, $1 \leq k \leq 4$, be the operators defined by

$$\begin{aligned}
 (5.25) \quad \mathbb{A}_j^1 &:= \frac{f'_0(x_j^\varepsilon)\bar{\omega}_0(x_j^\varepsilon)}{(1 + f_0'^2(x_j^\varepsilon))^2} B_{0,1}(0) \circ \partial_x, & \mathbb{A}_j^2 &:= B_{1,1}(f_0)[f_0, \bar{\omega}_0](x_j^\varepsilon)\partial_x, \\
 \mathbb{A}_j^3 &:= \frac{f'_0(x_j^\varepsilon)\bar{\omega}_0(x_j^\varepsilon)}{1 + f_0'^2(x_j^\varepsilon)} B_{0,1}(0) \circ \partial_x, & \mathbb{A}_j^4 &:= \frac{f_0'^3(x_j^\varepsilon)\bar{\omega}_0(x_j^\varepsilon)}{(1 + f_0'^2(x_j^\varepsilon))^2} B_{0,1}(0) \circ \partial_x,
 \end{aligned}$$

with $x_j^\varepsilon \in \text{supp } \pi_j^\varepsilon$. We prove in this step that

$$\begin{aligned}
 (5.26) \quad & 2\|\pi_j^\varepsilon B_{1,2}(f_0, f_0)[f_0, f''\bar{\omega}_0] - \mathbb{A}_j^1[(\pi_j^\varepsilon f)']\|_2 \\
 & + \|\pi_j^\varepsilon f'' B_{1,1}(f_0)[f_0, \bar{\omega}_0] - \mathbb{A}_j^2[(\pi_j^\varepsilon f)']\|_2 \\
 & + \|\pi_j^\varepsilon f'_0 B_{0,1}(f_0)[f''\bar{\omega}_0] - \mathbb{A}_j^3[(\pi_j^\varepsilon f)']\|_2 \\
 & + 2\|\pi_j^\varepsilon f'_0 B_{2,2}(f_0, f_0)[f_0, f_0, f''\bar{\omega}_0] - \mathbb{A}_j^4[(\pi_j^\varepsilon f)']\|_2 \\
 & \leq \frac{\mu}{2}\|\pi_j^\varepsilon f\|_{H^2} + K\|f\|_{H^{7/4}}
 \end{aligned}$$

for all $|j| \leq N - 1$ and $f \in H^2(\mathbb{R})$, provided that $\varepsilon \in (0, 1)$ is sufficiently small.

Since $B_{1,1}(f_0)[f_0, \bar{\omega}_0] \in H^1(\mathbb{R}) \subset BC^{1/2}(\mathbb{R})$, for ε sufficiently small we have

$$\begin{aligned}
 & \|\pi_j^\varepsilon f'' B_{1,1}(f_0)[f_0, \bar{\omega}_0] - \mathbb{A}_j^2[(\pi_j^\varepsilon f)']\|_2 \\
 &= \|\pi_j^\varepsilon f'' B_{1,1}(f_0)[f_0, \bar{\omega}_0] - B_{1,1}(f_0)[f_0, \bar{\omega}_0](x_j^\varepsilon)(\pi_j^\varepsilon f)''\|_2 \\
 (5.27) \quad &\leq \|(\pi_j^\varepsilon f)''\|_2 \|\chi_j^\varepsilon(B_{1,1}(f_0)[f_0, \bar{\omega}_0] - B_{1,1}(f_0)[f_0, \bar{\omega}_0](x_j^\varepsilon))\|_\infty \\
 &\quad + K\|f\|_{H^1} \\
 &\leq \frac{\mu}{8}\|\pi_j^\varepsilon f\|_{H^2} + K\|f\|_{H^1}.
 \end{aligned}$$

The arguments used to estimate the remaining three terms in (5.26) are similar, and therefore we only present in detail those for the first term. To begin, we write

$$\pi_j^\varepsilon B_{1,2}(f_0, f_0)[f_0, f''\bar{\omega}_0] - \mathbb{A}_j^1[\pi_j^\varepsilon f] = T_1[f] + T_2[f] + T_3[f],$$

where

$$\begin{aligned}
 T_1[f] &:= \pi_j^\varepsilon B_{1,2}(f_0, f_0)[f_0, f''\bar{\omega}_0] - B_{1,2}(f_0, f_0)[f_0, \pi_j^\varepsilon f''\bar{\omega}_0], \\
 T_2[f] &:= B_{1,2}(f_0, f_0)[f_0, \pi_j^\varepsilon f''\bar{\omega}_0] - \frac{f'_0(x_j^\varepsilon)}{(1 + f_0'^2(x_j^\varepsilon))^2} B_{0,1}(0)[\pi_j^\varepsilon f''\bar{\omega}_0], \\
 T_3[f] &:= \frac{f'_0(x_j^\varepsilon)}{(1 + f_0'^2(x_j^\varepsilon))^2} (B_{0,1}(0)[\pi_j^\varepsilon f''\bar{\omega}_0] - \bar{\omega}_0(x_j^\varepsilon)B_{0,1}(0)[(\pi_j^\varepsilon f)']).
 \end{aligned}$$

The term $T_1[f]$ can be estimated in the same way as the term $T_{11}[h]$ in the proof of Theorem 4.4. Indeed, integration by parts together with Lemma 3.1 yields

$$(5.28) \quad \|T_1[f]\|_2 \leq K\|f\|_{H^{7/4}}$$

for all $-N + 1 \leq j \leq N$.

Concerning $T_2[f]$, we appeal to (3.3) and write

$$\begin{aligned}
 T_2[f] &= B_{1,2}(f_0, f_0)[f_0, \pi_j^\varepsilon f''\bar{\omega}_0] - B_{1,2}(f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}})[f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, \pi_j^\varepsilon f''\bar{\omega}_0] \\
 &= B_{1,2}(f_0, f_0)[f_0 - f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, \pi_j^\varepsilon f''\bar{\omega}_0] \\
 &\quad - \frac{B_{3,2}(f_0, f_0)[f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, f_0 - f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, f_0 + f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, \pi_j^\varepsilon f''\bar{\omega}_0]}{1 + f_0'^2(x_j^\varepsilon)} \\
 &\quad - \frac{B_{3,1}(f_0)[f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, f_0 - f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, f_0 + f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, \pi_j^\varepsilon f''\bar{\omega}_0]}{(1 + f_0'^2(x_j^\varepsilon))^2},
 \end{aligned}$$

and, since $\chi_j^\varepsilon = 1$ on $\text{supp } \pi_j^\varepsilon$, we further have $T_2[f] = T_{21}[f] - T_{22}[f]$, where

$$\begin{aligned}
 T_{21}[f] &:= \chi_j^\varepsilon B_{1,2}(f_0, f_0)[f_0 - f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, \pi_j^\varepsilon f''\bar{\omega}_0] \\
 &\quad - \frac{\chi_j^\varepsilon B_{3,2}(f_0, f_0)[f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, f_0 - f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, f_0 + f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, \pi_j^\varepsilon f''\bar{\omega}_0]}{1 + f_0'^2(x_j^\varepsilon)} \\
 &\quad - \frac{\chi_j^\varepsilon B_{3,1}(f_0)[f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, f_0 - f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, f_0 + f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}}, \pi_j^\varepsilon f''\bar{\omega}_0]}{(1 + f_0'^2(x_j^\varepsilon))^2}
 \end{aligned}$$

and

$$\begin{aligned}
 T_{22}[f] &:= \text{PV} \int_{\mathbb{R}} \frac{[\delta_{[\cdot,y]}(f_0 - f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}})/y][\delta_{[\cdot,y]}\chi_j^\varepsilon/y]}{[1 + (\delta_{[\cdot,y]}f_0/y)]^2} (\pi_j^\varepsilon f''\bar{w}_0)(\cdot - y) dy \\
 &\quad - f'_0(x_j^\varepsilon)\text{PV} \int_{\mathbb{R}} \frac{[\delta_{[\cdot,y]}(f_0 - f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}})/y][\delta_{[\cdot,y]}(f_0 + f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}})/y]}{[1 + f_0'^2(x_j^\varepsilon)][1 + (\delta_{[\cdot,y]}f_0/y)]^2} \\
 &\quad \quad \times [\delta_{[\cdot,y]}\chi_j^\varepsilon/y] (\pi_j^\varepsilon f''\bar{w}_0)(\cdot - y) dy \\
 &\quad - f'_0(x_j^\varepsilon)\text{PV} \int_{\mathbb{R}} \frac{[\delta_{[\cdot,y]}(f_0 - f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}})/y][\delta_{[\cdot,y]}(f_0 + f'_0(x_j^\varepsilon)\text{id}_{\mathbb{R}})/y]}{[1 + f_0'^2(x_j^\varepsilon)]^2 [1 + (\delta_{[\cdot,y]}f_0/y)]^2} \\
 &\quad \quad \times [\delta_{[\cdot,y]}\chi_j^\varepsilon/y] (\pi_j^\varepsilon f''\bar{w}_0)(\cdot - y) dy.
 \end{aligned}$$

Integrating by parts, we get

$$\|T_{22}[f]\|_2 \leq K\|f\|_{H^{7/4}}.$$

Furthermore, the arguments used to derive estimate (4.25) lead us to

$$\|T_{21}[f]\|_2 \leq C\|f'_0 - f'_0(x_j^\varepsilon)\|_{L^\infty(\text{supp } \chi_j^\varepsilon)}\|\pi_j^\varepsilon f''\bar{w}_0\|_2 \leq \frac{\mu}{16}\|\pi_j^\varepsilon f\|_{H^2} + K\|f\|_{H^1},$$

provided that ε is sufficiently small. Hence, for ε sufficiently small and $|j| \leq N - 1$, it holds that

$$(5.29) \quad \|T_2[f]\|_2 \leq \frac{\mu}{32}\|\pi_j^\varepsilon f\|_{H^2} + K\|f\|_{H^{7/4}}.$$

Finally, since $\bar{w}_0 \in H^1(\mathbb{R}) \subset \text{BC}^{1/2}(\mathbb{R})$ we get

$$\begin{aligned}
 \|T_3[f]\|_2 &\leq \|B_{0,1}(0)[(\pi_j^\varepsilon f)''(\bar{w}_0 - \bar{w}_0(x_j^\varepsilon))]\|_2 + K\|f\|_{H^1} \\
 &\leq C\|(\pi_j^\varepsilon f)''(\bar{w}_0 - \bar{w}_0(x_j^\varepsilon))\|_2 + K\|f\|_{H^1} \\
 &\leq C\|\chi_j^\varepsilon(\bar{w}_0 - \bar{w}_0(x_j^\varepsilon))\|_\infty\|\pi_j^\varepsilon f\|_{H^2} + K\|f\|_{H^1} \\
 &\leq \frac{\mu}{32}\|\pi_j^\varepsilon f\|_{H^2} + K\|f\|_{H^1},
 \end{aligned}$$

provided that ε is sufficiently small, and together with (5.28) and (5.29) we obtain

$$2\|\pi_j^\varepsilon B_{1,2}(f_0, f_0)[f_0, f''\bar{w}_0] - \mathbb{A}_j^1[(\pi_j^\varepsilon f)']\|_2 \leq \frac{\mu}{8}\|\pi_j^\varepsilon f\|_{H^2} + K\|f\|_{H^{7/4}}.$$

Since the other two terms in (5.26) can be estimated in the same way, we conclude, in view of (5.27), that (5.26) is satisfied.

Step 3. Given $\tau \in [0, 1]$, we let

$$c_\tau := a_{\text{RT}} - \frac{\tau a_\mu}{\pi} f'_0 B_{1,1}(f_0)[f_0, \bar{w}_0] \in H^1(\mathbb{R}).$$

Moreover, given $l \in \{0, 1\}$, $\tau \in [0, 1]$, and $|j| \leq N - 1$, let $\mathbb{A}_{j,\tau}^{5+l} \in \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R}))$ denote the operator

$$\mathbb{A}_{j,\tau}^{5+l} := \frac{f_{l,\tau}'^2(x_j^\varepsilon)}{1 + \tau^2 f_0'^2(x_j^\varepsilon)} \left[-c_\tau(x_j^\varepsilon) B_{0,1}(0) \circ \partial_x - \tau a_\mu \pi \frac{\bar{w}_0(x_j^\varepsilon)}{1 + f_0'^2(x_j^\varepsilon)} \partial_x \right],$$

where $x_j^\varepsilon \in \text{supp } \pi_j^\varepsilon$ and $f_{l,\tau} := (1-l)\tau f_0 + l \text{id}_{\mathbb{R}}$. We next prove that

$$(5.30) \quad \left\| \pi_j^\varepsilon \mathbb{B}(\tau f_0)[(w(\tau)[f])'] - \sum_{l=0}^1 \mathbb{A}_{j,\tau}^{5+l}[(\pi_j^\varepsilon f)'] \right\|_2 \leq \frac{\mu}{2} \|\pi_j^\varepsilon f\|_{H^2} + K \|f\|_{H^{31/16}}$$

for all $|j| \leq N-1$, $\tau \in [0, 1]$, and $f \in H^2(\mathbb{R})$, provided that ε is chosen sufficiently small. We begin by observing that

$$\mathbb{B}(\tau f_0) = \sum_{l=0}^1 f'_{l,\tau} B_{1,1}(\tau f_0)[f_{l,\tau}, \cdot],$$

and therefore

$$\begin{aligned} & \pi_j^\varepsilon \mathbb{B}(\tau f_0)[(w(\tau)[f])'] - \sum_{l=0}^1 \mathbb{A}_{j,\tau}^{5+l}[(\pi_j^\varepsilon f)'] \\ &= \sum_{l=0}^1 \left(\pi_j^\varepsilon f'_{l,\tau} B_{1,1}(\tau f_0)[f_{l,\tau}, (w(\tau)[f])'] - \mathbb{A}_{j,\tau}^{5+l}[(\pi_j^\varepsilon f)'] \right). \end{aligned}$$

We now decompose

$$\pi_j^\varepsilon f'_{l,\tau} B_{1,1}(\tau f_0)[f_{l,\tau}, (w(\tau)[f])'] - \mathbb{A}_{j,\tau}^{5+l}[(\pi_j^\varepsilon f)'] = T_4[f] + T_5[f] + T_6[f],$$

with

$$\begin{aligned} T_4[f] &:= \pi_j^\varepsilon f'_{l,\tau} B_{1,1}(\tau f_0)[f_{l,\tau}, (w(\tau)[f])'] - f'_{l,\tau}(x_j^\varepsilon) B_{1,1}(\tau f_0)[f_{l,\tau}, \pi_j^\varepsilon(w(\tau)[f])'], \\ T_5[f] &:= f'_{l,\tau}(x_j^\varepsilon) B_{1,1}(\tau f_0)[f_{l,\tau}, \pi_j^\varepsilon(w(\tau)[f])'] - \frac{f_{l,\tau}^{\prime 2}(x_j^\varepsilon)}{1 + \tau^2 f_0^{\prime 2}(x_j^\varepsilon)} B_{0,1}(0)[\pi_j^\varepsilon(w(\tau)[f])'], \\ T_6[f] &:= \frac{f_{l,\tau}^{\prime 2}(x_j^\varepsilon)}{1 + \tau^2 f_0^{\prime 2}(x_j^\varepsilon)} B_{0,1}(0)[\pi_j^\varepsilon(w(\tau)[f])'] - \mathbb{A}_{j,\tau}^{5+l}[(\pi_j^\varepsilon f)']. \end{aligned}$$

The arguments used to derive (4.19) combined with (5.17) imply that

$$(5.31) \quad \|T_4[f]\|_2 \leq C \|\chi_j^\varepsilon(f'_{l,\tau} - f'_{l,\tau}(x_j^\varepsilon))\|_\infty \|\pi_j^\varepsilon(w(\tau)[f])'\|_2 + K \|f\|_{H^{7/4}},$$

and we are left to estimate the term $\|\pi_j^\varepsilon(w(\tau)[f])'\|_2$. To this end we compute for $j \in \{-N+1, \dots, N\}$ that

$$(5.32) \quad \begin{aligned} (1 + a_\mu \mathbb{A}(\tau f_0))[\pi_j^\varepsilon(w(\tau)[f])'] &= \pi_j^\varepsilon(1 + a_\mu \mathbb{A}(\tau f_0))[(w(\tau)[f])'] \\ &\quad - \frac{a_\mu}{\pi} T_{1j,\text{lot}}(\tau f_0)[w(\tau)[f]], \end{aligned}$$

where $T_{1j,\text{lot}}$ are defined in (4.21). Moreover, recalling (5.20), it follows that

$$(5.33) \quad \begin{aligned} \pi_j^\varepsilon(1 + a_\mu \mathbb{A}(\tau f_0))[(w(\tau)[f])'] &= -c_\tau \pi_j^\varepsilon f'' + 2 \frac{\tau a_\mu}{\pi} f'_0 B_{1,2}(f_0, f_0)[f_0, \pi_j^\varepsilon f'' \bar{\omega}_0] \\ &\quad + \frac{\tau a_\mu}{\pi} B_{0,1}(f_0)[\pi_j^\varepsilon f'' \bar{\omega}_0] \\ &\quad - 2 \frac{\tau a_\mu}{\pi} B_{2,2}(f_0, f_0)[f_0, f_0, \pi_j^\varepsilon f'' \bar{\omega}_0] \\ &\quad + \pi_j^\varepsilon T_{2,\text{lot}}(\tau)[f] + T_{2j,\text{lot}}(\tau)[f] \end{aligned}$$

for $j \in \{-N + 1, \dots, N\}$, where

$$\begin{aligned}
 T_{2j,\text{lot}}(\tau)[f] &:= 2\frac{\tau a_\mu}{\pi} f'_0(\pi_j^\varepsilon B_{1,2}(f_0, f_0)[f_0, f''\bar{\omega}_0] - B_{1,2}(f_0, f_0)[f_0, \pi_j^\varepsilon f''\bar{\omega}_0]) \\
 &\quad + \frac{\tau a_\mu}{\pi} (\pi_j^\varepsilon B_{0,1}(f_0)[f''\bar{\omega}_0] - B_{0,1}(f_0)[\pi_j^\varepsilon f''\bar{\omega}_0]) \\
 &\quad - 2\frac{\tau a_\mu}{\pi} (\pi_j^\varepsilon B_{2,2}(f_0, f_0)[f_0, f_0, f''\bar{\omega}_0] - B_{2,2}(f_0, f_0)[f_0, f_0, \pi_j^\varepsilon f''\bar{\omega}_0]).
 \end{aligned}$$

Integrating by parts, Lemma 3.1 leads us to

$$(5.34) \quad \|T_{2j,\text{lot}}(\tau)[f]\|_2 \leq K\|f\|_{H^{7/4}}.$$

In view of (3.20), we deduce from (5.17), (5.21), and (5.32)–(5.34) that

$$(5.35) \quad \max_{-N+1 \leq j \leq N} \|\pi_j^\varepsilon (w(\tau)[f])'\|_2 \leq C\|\pi_j^\varepsilon f\|_{H^2} + K\|f\|_{H^{31/16}},$$

and (5.31) yields, for ε sufficiently small, that

$$(5.36) \quad \|T_4[f]\|_2 \leq \frac{\mu}{6}\|\pi_j^\varepsilon f\|_{H^2} + K\|f\|_{H^{31/16}}.$$

Invoking (5.35), the arguments used to deduce (4.25) show that

$$(5.37) \quad \|T_5[f]\|_2 \leq \frac{\mu}{6}\|\pi_j^\varepsilon f\|_{H^2} + K\|f\|_{H^{31/16}}$$

for all $|j| \leq N - 1$, provided that ε is sufficiently small.

Concerning the last term $T_6[f]$, we first recall that $H^2 = -\text{id}_{L_2(\mathbb{R})}$ (cf. [49]), and therewith we get

$$(B_{0,1}(0))^2 = \pi^2 H^2 = -\pi^2 \text{id}_{L_2(\mathbb{R})}.$$

Using the latter relation together with the definition of $\mathbb{A}_{j,\tau}^{5+l}$, it follows that

$$\begin{aligned}
 &\|T_6[f]\|_2 \\
 &\leq C \left\| \pi_j^\varepsilon (w(\tau)[f])' + c_\tau(x_j^\varepsilon)(\pi_j^\varepsilon f)'' - \frac{\tau a_\mu}{\pi} \frac{\bar{\omega}_0(x_j^\varepsilon)}{1 + f_0'^2(x_j^\varepsilon)} B_{0,1}(0)[(\pi_j^\varepsilon f)'] \right\|_2 \\
 (5.38) \quad &\leq C \left\| \pi_j^\varepsilon (w(\tau)[f])' + c_\tau(x_j^\varepsilon)\pi_j^\varepsilon f'' - \frac{\tau a_\mu}{\pi} \frac{\bar{\omega}_0(x_j^\varepsilon)}{1 + f_0'^2(x_j^\varepsilon)} B_{0,1}(0)[\pi_j^\varepsilon f'] \right\|_2 \\
 &\quad + K\|f\|_{H^1}.
 \end{aligned}$$

Furthermore, (3.2) and (5.33) lead us to the following identity:

$$\begin{aligned}
 & \pi_j^\varepsilon(w(\tau)[f])' + c_\tau(x_j^\varepsilon)\pi_j^\varepsilon f'' - \frac{\tau a_\mu}{\pi} \frac{\bar{\omega}_0(x_j^\varepsilon)}{1 + f_0'^2(x_j^\varepsilon)} B_{0,1}(0)[\pi_j^\varepsilon f'''] \\
 &= (c_\tau(x_j^\varepsilon) - c_\tau)\pi_j^\varepsilon f'' \\
 &+ 2 \frac{\tau a_\mu}{\pi} \left[f_0' B_{1,2}(f_0, f_0)[f_0, \pi_j^\varepsilon f'' \bar{\omega}_0] - \frac{f_0'^2(x_j^\varepsilon) \bar{\omega}_0(x_j^\varepsilon)}{(1 + f_0'^2(x_j^\varepsilon))^2} B_{0,1}[\pi_j^\varepsilon f'''] \right] \\
 &+ \frac{\tau a_\mu}{\pi} \left[B_{0,1}(f_0)[\pi_j^\varepsilon f'' \bar{\omega}_0] - \frac{\bar{\omega}_0(x_j^\varepsilon)}{1 + f_0'^2(x_j^\varepsilon)} B_{0,1}[\pi_j^\varepsilon f'''] \right] \\
 (5.39) \quad &- 2 \frac{\tau a_\mu}{\pi} \left[B_{2,2}(f_0, f_0)[f_0, f_0, \pi_j^\varepsilon f'' \bar{\omega}_0] - \frac{\tau f_0'^2(x_j^\varepsilon) \bar{\omega}_0(x_j^\varepsilon)}{(1 + f_0'^2(x_j^\varepsilon))^2} B_{0,1}[\pi_j^\varepsilon f'''] \right] \\
 &+ \pi_j^\varepsilon T_{2,\text{lot}}(\tau)[f] + T_{2j,\text{lot}}(\tau)[f] \\
 &- \frac{a_\mu}{\pi} \left[\tau f_0' \pi_j^\varepsilon B_{0,1}(\tau f_0)[(w(\tau)[f])'] - \frac{\tau f_0'(x_j^\varepsilon)}{1 + \tau^2 f_0'^2(x_j^\varepsilon)} B_{0,1}[\pi_j^\varepsilon(w(\tau)[f])'] \right] \\
 &+ \frac{a_\mu}{\pi} \left[\pi_j^\varepsilon B_{1,1}(\tau f_0)[\tau f_0, (w(\tau)[f])'] - \frac{\tau f_0'(x_j^\varepsilon)}{1 + \tau^2 f_0'^2(x_j^\varepsilon)} B_{0,1}[\pi_j^\varepsilon(w(\tau)[f])'] \right].
 \end{aligned}$$

In order to estimate the first term on the right-hand side of (5.39) we rely on the property that $c_\tau \in BC^{1/2}(\mathbb{R})$. The next three terms are of the same type as those estimated in *Step 2* above, while each of the last two expressions can be written as a sum of two terms for which we can use the arguments that led to (5.31) and (5.37). Altogether, we obtain

$$\begin{aligned}
 (5.40) \quad & \left\| \pi_j^\varepsilon(w(\tau)[f])' + c_\tau(x_j^\varepsilon)\pi_j^\varepsilon f'' - \frac{\tau a_\mu}{\pi} \frac{\bar{\omega}_0(x_j^\varepsilon)}{1 + f_0'^2(x_j^\varepsilon)} B_{0,1}(0)[\pi_j^\varepsilon f'''] \right\|_2 \\
 & \leq \frac{\mu}{6C} \|\pi_j^\varepsilon f\|_{H^2} + K \|f\|_{H^{31/16}}
 \end{aligned}$$

for all $|j| \leq N - 1$, provided that ε is sufficiently small. In (5.40) we denote by C the positive constant which appears in (5.38). Hence, recalling (5.38), we get

$$(5.41) \quad \|T_6[f]\|_2 \leq \frac{\mu}{6} \|\pi_j^\varepsilon f\|_{H^2} + K \|f\|_{H^{31/16}}.$$

The estimate (5.30) follows now from (5.31), (5.37), and (5.41).

Step 4. Gathering (5.23), (5.24), (5.26), and (5.30), we conclude that the estimate (5.22) holds for $|j| \leq N - 1$ provided that

$$(5.42) \quad \mathbb{A}_{j,\tau} = -2\tau \mathbb{A}_j^1 + \tau \mathbb{A}_j^2 + \tau \mathbb{A}_j^3 - 2\tau \mathbb{A}_j^4 + \sum_{l=0}^1 \mathbb{A}_{j,\tau}^{5+l}.$$

Exploiting the fact that $\bar{\omega}_0$ is the solution to $\bar{\omega}_0 = -c_{\rho,\mu} f_0' - a_\mu \mathbb{A}(f_0)[\bar{\omega}_0]$, it follows at once that (5.42) is satisfied. This completes the proof of (5.22) for $|j| \leq N - 1$.

Step 5. We are left to prove (5.22) for $j = N$. This estimate follows by combining arguments from the previous steps with those presented in the second part of the proof of Theorem 4.4, and therefore we omit the lengthy details. \square

We now reconsider the Fourier multipliers $\mathbb{A}_{\tau,j}$ found in Theorem 5.2 and we notice that if f_0 is chosen such that the Rayleigh–Taylor condition (5.12) holds, then there exists a constant $\eta \in (0, 1)$ with the property that

$$\eta \leq \alpha_\tau \leq 1/\eta \quad \text{and} \quad |\beta_\tau| \leq 1/\eta \quad \text{for all } \tau \in [0, 1].$$

Moreover, letting $\mathbb{A}_{\alpha,\beta}$, with $\alpha \in [\eta, 1/\eta]$ and $|\beta| \leq 1/\eta$, denote the Fourier multiplier

$$\mathbb{A}_{\alpha,\beta} := -\alpha(-\partial_x^2)^{1/2} + \beta\partial_x,$$

we may find, similarly as in [40, Proposition 4.3], a constant $\kappa_0 \geq 1$ such that

$$(5.43) \quad \kappa_0 \|(\lambda - \mathbb{A}_{\alpha,\beta})[f]\|_{H^1} \geq |\lambda| \cdot \|f\|_{H^1} + \|f\|_{H^2}$$

for all $\alpha \in [\eta, 1/\eta]$, $|\beta| \leq 1/\eta$, $\lambda \in \mathbb{C}$ with $\text{Re}\lambda \geq 1$, and $f \in H^2(\mathbb{R})$.

In order to establish the desired generation result for $\partial\Phi(f_0) = \Psi(1)$, we next show that the operator $\omega - \Psi(0)$, with $\Psi(0)$ defined in (5.16), is invertible for large $\omega > 0$. In contrast to the analysis in Section 4, where the invertibility of the translation $\omega - \Phi_{\sigma,1}(0)$, with $\omega > 0$, follows easily from the fact that this operator is a Fourier multiplier (cf. Theorem 4.5), a more involved analysis is required in order to establish the invertibility of $\omega - \Psi(0)$.

Lemma 5.3. *Given $\delta > 0$ and $h \in H^1(\mathbb{R})$, let $a := \delta + h$ and assume that*

$$\inf_{\mathbb{R}} a > 0.$$

Then, there exists $\omega_0 > 0$ such that

$$\lambda + H[a\partial_x] \in \text{Isom}(H^2(\mathbb{R}), H^1(\mathbb{R}))$$

for all $\lambda \in [\omega_0, \infty)$.

Proof. We introduce the continuous path $[\tau \mapsto B(\tau)] : [0, 1] \rightarrow \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R}))$, where

$$B(\tau) := H[a_{\tau,\delta}\partial_x] \quad \text{and} \quad a_{\tau,\delta} := (1 - \tau)\delta + \tau a,$$

and we prove that there exist constants $\omega_0 > 0$ and $C > 0$ with the property that

$$(5.44) \quad \|(\lambda + B(\tau))[f]\|_{H^1(\mathbb{R})} \geq C\|f\|_{H^2(\mathbb{R})}$$

for all $\tau \in [0, 1]$, $\lambda \in [\omega_0, \infty)$, and $f \in H^2(\mathbb{R})$. Having established (5.44), the method of continuity together with $B(1) = H[a\partial_x]$ and the observation that $\lambda + B(0)$ is an invertible Fourier multiplier (the symbol of $\lambda + B(0)$ is $m_\lambda(\xi) := \lambda + \delta|\xi|$, $\xi \in \mathbb{R}$) yields the desired claim.

We first prove that for each $\mu > 0$ there exists a finite ε -localization family $\{\pi_j^\varepsilon : -N + 1 \leq j \leq N\}$, a constant $K = K(\varepsilon)$, and for each $j \in \{-N + 1, \dots, N\}$ and $\tau \in [0, 1]$ there exist bounded operators

$$\mathbb{B}_{j,\tau} \in \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R}))$$

such that

$$(5.45) \quad \|\pi_j^\varepsilon B(\tau)[f] - \mathbb{B}_{j,\tau}[\pi_j^\varepsilon f]\|_{H^1} \leq \mu\|\pi_j^\varepsilon f\|_{H^2} + K\|f\|_{H^{7/4}}$$

for all $j \in \{-N + 1, \dots, N\}$, $\tau \in [0, 1]$, and $f \in H^2(\mathbb{R})$. The operators $\mathbb{B}_{j,\tau}$ are defined by

$$\mathbb{B}_{j,\tau} := a_{\tau,\delta}(x_j^\varepsilon)H \circ \partial_x, \quad |j| \leq N - 1,$$

with $x_j^\varepsilon \in \text{supp } \pi_j^\varepsilon$, respectively

$$\mathbb{B}_{N,\tau} := \delta H \circ \partial_x.$$

Indeed, for $|j| \leq N - 1$ we obtain by using integration by parts that

$$\begin{aligned} \|\pi_j^\varepsilon B(\tau)[f] - \mathbb{B}_{j,\tau}[\pi_j^\varepsilon f]\|_{H^1} &\leq K\|f\|_{H^{7/4}} + \|\pi_j^\varepsilon H[a_{\tau,\delta} f''] - a_{\tau,\delta}(x_j^\varepsilon)H[\pi_j^\varepsilon f'']\|_2 \\ &\leq K\|f\|_{H^{7/4}} + \|\pi_j^\varepsilon H[a_{\tau,\delta} f''] - H[\pi_j^\varepsilon a_{\tau,\delta} f'']\|_2 \\ &\quad + \|H[(a_{\tau,\delta} - a_{\tau,\delta}(x_j^\varepsilon))\pi_j^\varepsilon f'']\|_2 \\ &\leq K\|f\|_{H^{7/4}} + \|\chi_j^\varepsilon(a_{\tau,\delta} - a_{\tau,\delta}(x_j^\varepsilon))\|_\infty \|(\pi_j^\varepsilon f)''\|_2, \end{aligned}$$

and (5.45) holds for $|j| \leq N - 1$, if ε is sufficiently small. Similarly,

$$\begin{aligned} \|\pi_N^\varepsilon B(\tau)[f] - \mathbb{B}_{N,\tau}[\pi_N^\varepsilon f]\|_{H^1} &\leq K\|f\|_{H^{7/4}} + \|\pi_N^\varepsilon H[a_{\tau,\delta} f''] - \delta H[\pi_N^\varepsilon f'']\|_2 \\ &\leq K\|f\|_{H^{7/4}} + \|H[(a_{\tau,\delta} - \delta)\pi_N^\varepsilon f'']\|_2 \\ &\leq K\|f\|_{H^{7/4}} + \|\chi_N^\varepsilon h\|_\infty \|(\pi_N^\varepsilon f)''\|_2 \end{aligned}$$

and, since h vanishes at infinity, we see that (5.45) holds also for $j = N$, provided that ε is sufficiently small.

Since $\bigcup_{\tau \in [0,1]} \overline{a_{\tau,\delta}(\mathbb{R})} \subset [\eta, 1/\eta]$ for some $\eta > 0$, we may find a constant $\kappa_0 \geq 1$ such that

$$(5.46) \quad \kappa_0 \|(\lambda + \alpha H \circ \partial_x)[f]\|_{H^1} \geq \lambda \|f\|_{H^1} + \|f\|_{H^2}$$

for all $\alpha \in \bigcup_{\tau \in [0,1]} \overline{a_{\tau,\delta}(\mathbb{R})}$, $\lambda \in [1, \infty)$, and $f \in H^2(\mathbb{R})$. Choosing $\mu = 1/2\kappa_0$ in (5.45), it follows from (5.46) that for all $j \in \{-N + 1, \dots, N\}$, $\tau \in [0, 1]$, $\lambda \in [1, \infty)$, and $f \in H^2(\mathbb{R})$ we have

$$\begin{aligned} \kappa_0 \|\pi_j^\varepsilon(\lambda + B(\tau))[f]\|_{H^1} &\geq \kappa_0 \|(\lambda + \mathbb{B}_{j,\tau})[\pi_j^\varepsilon f]\|_{H^1} - \kappa_0 \|\pi_j^\varepsilon B(\tau)[f] - \mathbb{B}_{j,\tau}[\pi_j^\varepsilon f]\|_{H^1} \\ &\geq \frac{1}{2} \|\pi_j^\varepsilon f\|_{H^2} + \lambda \|\pi_j^\varepsilon f\|_{H^1} - \kappa_0 K \|f\|_{H^{7/4}} \end{aligned}$$

and, summing over $j \in \{-N + 1, \dots, N\}$, we conclude together with Young's inequality and Remark 4.3 that there exist positive constants ω_0 and C such that (5.44) holds for all $\tau \in [0, 1]$, $\lambda \in [\omega_0, \infty)$, and $f \in H^2(\mathbb{R})$. \square

We are now in the position to establish the desired generator property. To this end we recall from (5.12) that the set \mathcal{O} introduced in Section 1 is actually given by

$$\mathcal{O} = \left\{ f_0 \in H^2(\mathbb{R}) : \inf_{\mathbb{R}} \left[c_{\rho,\mu} + \frac{a_\mu}{\pi} B_{0,1}(f_0)[\overline{\omega}_0] + \frac{a_\mu}{\pi} f_0' B_{1,1}(f_0)[f_0, \overline{\omega}_0] \right] > 0 \right\},$$

and we note that the continuity of the mappings

$$[f \mapsto B_{0,1}(f)[\overline{\omega}(f)]], [f \mapsto f' B_{1,1}(f)[f, \overline{\omega}(f)]] : H^2(\mathbb{R}) \rightarrow H^1(\mathbb{R})$$

ensure that \mathcal{O} is an open subset of $H^2(\mathbb{R})$.

Theorem 5.4. *Given $f_0 \in \mathcal{O}$, it holds that*

$$-\partial\Phi(f_0) \in \mathcal{H}(H^2(\mathbb{R}), H^1(\mathbb{R})).$$

Proof. The proof follows from Theorem 5.2, the relation (5.43), Remark 4.3, and Lemma 5.3, by arguing as in the proof of Theorem 4.5. \square

Finally, we arrive at the proof of our second main result.

Proof of Theorem 1.2. Since for $\sigma = 0$ the problem (1.1) is equivalent to (5.8), the existence and uniqueness of a maximal solution to (5.8), for each $f_0 \in \mathcal{O}$, follows from [39, Theorem 8.1.1]. We note that the relation (5.9) together with Theorem 5.4 ensures that all the assumptions of [39, Theorem 8.1.1] are satisfied. Furthermore, (iii) follows from [39, Proposition 8.2.3 and Theorem 8.3.9], and (iv) from [39, Proposition 8.2.1].

In order to prove (v), we assume that $f = f(\cdot; f_0) \in B((0, T), H^{2+\varepsilon}(\mathbb{R}))$ for some $T < T_+(f_0)$ and $\varepsilon \in (0, 1)$. Given $(\lambda_1, \lambda_2) \in (0, \infty)^2$, which we view below as parameters in an associated nonlinear evolution problem, we define the function

$$f_{\lambda_1, \lambda_2}(t, x) := f(\lambda_1 t, x + \lambda_2 t), \quad x \in \mathbb{R}, 0 \leq t \leq T_{\lambda_1} := T/\lambda_1.$$

Since $f \in C([0, T], \mathcal{O}) \cap C^1([0, T], H^1(\mathbb{R})) \cap C^\varepsilon_\varepsilon((0, T], H^2(\mathbb{R}))$ (cf. Theorem 1.2(ii)), our assumption together with the translation invariance of \mathcal{O} implies that

$$(5.47) \quad f_{\lambda_1, \lambda_2} \in C([0, T_{\lambda_1}], \mathcal{O}) \cap C^1([0, T_{\lambda_1}], H^1(\mathbb{R})) \cap C^\varepsilon_\varepsilon((0, T_{\lambda_1}], H^2(\mathbb{R})).$$

Therefore, the function $u := f_{\lambda_1, \lambda_2}$ is a solution to the nonlinear evolution problem

$$(5.48) \quad \partial_t u = \Psi(u, \lambda_1, \lambda_2), \quad t \geq 0, \quad u(0) = u_0,$$

for the initial data $u_0 := f_0$, where $\Psi : \mathcal{O} \times (0, \infty)^2 \subset H^2(\mathbb{R}) \times \mathbb{R}^2 \rightarrow H^1(\mathbb{R})$ denotes the operator

$$\Psi(u, \lambda_1, \lambda_2) := \lambda_1 \Phi(u) + \lambda_2 \partial_x u.$$

Recalling (5.9), we get $\Psi \in C^\omega(\mathcal{O} \times (0, \infty)^2, H^1(\mathbb{R}))$. Moreover, given $(u_0, \lambda_1, \lambda_2) \in \mathcal{O} \times (0, \infty)^2$, the Fréchet derivative of Ψ with respect to u is given by

$$\partial_u \Psi(u_0, \lambda_1, \lambda_2) = \lambda_1 \partial \Phi(u_0) + \lambda_2 \partial_x.$$

Since $\lambda_2 \partial_x$ is a Fourier multiplier of first order with purely imaginary symbol, we may revisit the computations in the proofs of Theorem 5.2, Lemma 5.3, and Theorem 5.4, to deduce that $-\partial_u \Psi((u_0, \lambda_1, \lambda_2))$ belongs to $\mathcal{H}(H^2(\mathbb{R}), H^1(\mathbb{R}))$ for all $(u_0, \lambda_1, \lambda_2) \in \mathcal{O} \times (0, \infty)^2$. According to [39, Theorem 8.1.1 and Theorem 8.3.9], the problem (5.48) possesses for each $(u_0, \lambda_1, \lambda_2) \in \mathcal{O} \times (0, \infty)^2$ a unique maximal solution $u = u(\cdot; u_0, \lambda_1, \lambda_2)$ (that satisfies similar properties as in Theorem 1.2(i) and (ii)), the set

$$\Omega := \{(t, u_0, \lambda_1, \lambda_2) : (u_0, \lambda_1, \lambda_2) \in \mathcal{O} \times (0, \infty)^2, 0 < t < T_+((u_0, \lambda_1, \lambda_2))\}$$

is open and

$$[(t, u_0, \lambda_1, \lambda_2) \mapsto u(t; u_0, \lambda_1, \lambda_2)] : \Omega \rightarrow \mathcal{O} \quad \text{is real-analytic.}$$

Hence, for our special initial data f_0 , (5.47) yields that $T_+(f_0, \lambda_1, \lambda_2) > T_{\lambda_1}$, and moreover $u(t; u_0, \lambda_1, \lambda_2) = f_{\lambda_1, \lambda_2}(t)$ for all $t \leq T_{\lambda_1}$. Given $t_0 \in (0, T)$, we choose $\delta > 0$ such that $t_0 < T_{\lambda_1}$ for all (λ_1, λ_2) belonging to the disc $D_\delta((1, 1))$. Therewith, we conclude that in particular

$$[(\lambda_1, \lambda_2) \mapsto f_{\lambda_1, \lambda_2}(t_0)] : D_\delta((1, 1)) \rightarrow H^1(\mathbb{R})$$

is a real-analytic map. The conclusion is now immediate; see, e.g., the proof of [40, Theorem 1.3]. □

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REFERENCES

- [1] Herbert Amann, *Quasilinear parabolic systems under nonlinear boundary conditions*, Arch. Rational Mech. Anal. **92** (1986), no. 2, 153–192. MR816618
- [2] Herbert Amann, *Dynamic theory of quasilinear parabolic equations. I. Abstract evolution equations*, Nonlinear Anal. **12** (1988), no. 9, 895–919. MR960634
- [3] Herbert Amann, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, Function spaces, differential operators and nonlinear analysis (Friedrichroda, 1992), Teubner-Texte Math., vol. 133, Teubner, Stuttgart, 1993, pp. 9–126. MR1242579
- [4] Herbert Amann, *Linear and quasilinear parabolic problems. Vol. I*, Abstract linear theory, Monographs in Mathematics, vol. 89, Birkhäuser Boston, Inc., Boston, MA, 1995. MR1345385
- [5] David M. Ambrose, *Well-posedness of two-phase Hele-Shaw flow without surface tension*, European J. Appl. Math. **15** (2004), no. 5, 597–607. MR2128613
- [6] David M. Ambrose, *The zero surface tension limit of two-dimensional interfacial Darcy flow*, J. Math. Fluid Mech. **16** (2014), no. 1, 105–143. MR3171344
- [7] Sigurd B. Angenent, *Nonlinear analytic semiflows*, Proc. Roy. Soc. Edinburgh Sect. A **115** (1990), no. 1-2, 91–107. MR1059647
- [8] B. V. Bazaliy and N. Vasylyeva, *The two-phase Hele-Shaw problem with a nonregular initial interface and without surface tension* (English, with English, Russian and Ukrainian summaries), Zh. Mat. Fiz. Anal. Geom. **10** (2014), no. 1, 3–43, 152, 155. MR3236960
- [9] Jacob Bear, *Dynamics of fluids in porous media*, Dover Publications, New York, 1988.
- [10] Luigi C. Berselli, Diego Córdoba, and Rafael Granero-Belinchón, *Local solvability and turning for the inhomogeneous Muskat problem*, Interfaces Free Bound. **16** (2014), no. 2, 175–213. MR3231970
- [11] Ángel Castro, Diego Córdoba, Charles Fefferman, and Francisco Gancedo, *Breakdown of smoothness for the Muskat problem*, Arch. Ration. Mech. Anal. **208** (2013), no. 3, 805–909. MR3048596
- [12] Ángel Castro, Diego Córdoba, Charles Fefferman, Francisco Gancedo, and María López-Fernández, *Rayleigh–Taylor breakdown for the Muskat problem with applications to water waves*, Ann. of Math. (2) **175** (2012), no. 2, 909–948. MR2993754
- [13] Ángel Castro, Diego Córdoba, Charles L. Fefferman, Francisco Gancedo, and María López-Fernández, *Turning waves and breakdown for incompressible flows*, Proc. Natl. Acad. Sci. USA **108** (2011), no. 12, 4754–4759. MR2792311
- [14] C. H. Arthur Cheng, Rafael Granero-Belinchón, and Steve Shkoller, *Well-posedness of the Muskat problem with H^2 initial data*, Adv. Math. **286** (2016), 32–104. MR3415681
- [15] Peter Constantin, Diego Córdoba, Francisco Gancedo, Luis Rodríguez-Piazza, and Robert M. Strain, *On the Muskat problem: global in time results in 2D and 3D*, Amer. J. Math. **138** (2016), no. 6, 1455–1494. MR3595492
- [16] Peter Constantin, Diego Córdoba, Francisco Gancedo, and Robert M. Strain, *On the global existence for the Muskat problem*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 1, 201–227. MR2998834
- [17] Peter Constantin, Vlad Vicol, Roman Shvydkoy, and Francisco Gancedo, *Global regularity for 2D Muskat equations with finite slope*, Ann. Inst. H. Poincaré, Anal. Nonlinéaire **34** (2017), no. 4 1041–1074.
- [18] Antonio Córdoba, Diego Córdoba, and Francisco Gancedo, *Interface evolution: the Hele-Shaw and Muskat problems*, Ann. of Math. (2) **173** (2011), no. 1, 477–542. MR2753607
- [19] Antonio Córdoba, Diego Córdoba, and Francisco Gancedo, *Porous media: the Muskat problem in three dimensions*, Anal. PDE **6** (2013), no. 2, 447–497. MR3071395
- [20] Diego Córdoba and Francisco Gancedo, *Contour dynamics of incompressible 3-D fluids in a porous medium with different densities*, Comm. Math. Phys. **273** (2007), no. 2, 445–471. MR2318314
- [21] Diego Córdoba and Francisco Gancedo, *Absence of squirt singularities for the multi-phase Muskat problem*, Comm. Math. Phys. **299** (2010), no. 2, 561–575. MR2679821

- [22] Diego Córdoba Gazolaz, Rafael Granero-Belinchón, and Rafael Orive-Illera, *The confined Muskat problem: differences with the deep water regime*, Commun. Math. Sci. **12** (2014), no. 3, 423–455. MR3144991
- [23] Joachim Escher, *The Dirichlet-Neumann operator on continuous functions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), no. 2, 235–266. MR1288366
- [24] Joachim Escher, Anca-Voichita Matioc, and Bogdan-Vasile Matioc, *A generalized Rayleigh–Taylor condition for the Muskat problem*, Nonlinearity **25** (2012), no. 1, 73–92. MR2864377
- [25] Joachim Escher and Bogdan-Vasile Matioc, *On the parabolicity of the Muskat problem: well-posedness, fingering, and stability results*, Z. Anal. Anwend. **30** (2011), no. 2, 193–218. MR2793001
- [26] Joachim Escher, Bogdan-Vasile Matioc, and Christoph Walker, *The domain of parabolicity for the Muskat problem*, Indiana Univ. Math. J. (2018), no. 2, 679–737.
- [27] Joachim Escher and Gieri Simonett, *Maximal regularity for a free boundary problem*, NoDEA Nonlinear Differential Equations Appl. **2** (1995), no. 4, 463–510. MR1356871
- [28] Joachim Escher and Gieri Simonett, *Analyticity of the interface in a free boundary problem*, Math. Ann. **305** (1996), no. 3, 439–459. MR1397432
- [29] Joachim Escher and Gieri Simonett, *Classical solutions of multidimensional Hele-Shaw models*, SIAM J. Math. Anal. **28** (1997), no. 5, 1028–1047. MR1466667
- [30] E. B. Fabes, M. Jodeit Jr., and N. M. Rivière, *Potential techniques for boundary value problems on C^1 -domains*, Acta Math. **141** (1978), no. 3–4, 165–186. MR501367
- [31] Avner Friedman and Youshan Tao, *Nonlinear stability of the Muskat problem with capillary pressure at the free boundary*, Nonlinear Anal. **53** (2003), no. 1, 45–80. MR1992404
- [32] Francisco Gancedo and Robert M. Strain, *Absence of splash singularities for surface quasi-geostrophic sharp fronts and the Muskat problem*, Proc. Natl. Acad. Sci. USA **111** (2014), no. 2, 635–639. MR3181769
- [33] Javier Gómez-Serrano and Rafael Granero-Belinchón, *On turning waves for the inhomogeneous Muskat problem: a computer-assisted proof*, Nonlinearity **27** (2014), no. 6, 1471–1498. MR3215843
- [34] Rafael Granero-Belinchón, *Global existence for the confined Muskat problem*, SIAM J. Math. Anal. **46** (2014), no. 2, 1651–1680. MR3196945
- [35] Rafael Granero-Belinchón and Steve Shkoller, *Well-posedness and decay to equilibrium for the Muskat problem with discontinuous permeability*, (2016), preprint. arXiv:1611.06147.
- [36] G. M. Homsy, *Viscous fingering in porous media*, Ann. Rev. Fluid Mech. **19** (1987), 271–311.
- [37] Jiaying Hong, Youshan Tao, and Fahuai Yi, *Muskat problem with surface tension*, J. Partial Differential Equations **10** (1997), no. 3, 213–231. MR1471114
- [38] Jian Ke Lu, *Boundary value problems for analytic functions*, Series in Pure Mathematics, vol. 16, World Scientific Publishing Co., Inc., River Edge, NJ, 1993. MR1279172
- [39] Alessandra Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Progress in Nonlinear Differential Equations and their Applications, vol. 16, Birkhäuser Verlag, Basel, 1995. MR1329547
- [40] Bogdan-Vasile Matioc, *The Muskat problem in 2D: equivalence of formulations, well-posedness, and regularity results*, to appear in Anal. PDE (2018), arXiv:1610.05546.
- [41] Yves Meyer and Ronald Coifman, *Wavelets, Calderón–Zygmund and multilinear operators*; Translated from the 1990 and 1991 French originals by David Salinger, Cambridge Studies in Advanced Mathematics, vol. 48, Cambridge University Press, Cambridge, 1997. MR1456993
- [42] Takafumi Murai, *Boundedness of singular integral operators of Calderón type. VI*, Nagoya Math. J. **102** (1986), 127–133. MR846134
- [43] M. Muskat, *Two fluid systems in porous media. The encroachment of water into an oil sand*, Physics **5** (1934), 250–264.
- [44] Jan Prüss, Yuanzhen Shao, and Gieri Simonett, *On the regularity of the interface of a thermodynamically consistent two-phase Stefan problem with surface tension*, Interfaces Free Bound. **17** (2015), no. 4, 555–600. MR3450740
- [45] Jan Prüss and Gieri Simonett, *Moving interfaces and quasilinear parabolic evolution equations*, Monographs in Mathematics, vol. 105, Birkhäuser/Springer, [Cham], 2016. MR3524106
- [46] Jan Prüss and Gieri Simonett, *On the Muskat problem*, Evol. Equ. Control Theory **5** (2016), no. 4, 631–645. MR3603251

- [47] P. G. Saffman and Geoffrey Taylor, *The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid*, Proc. Roy. Soc. London. Ser. A **245** (1958), 312–329. (2 plates). MR0097227
- [48] Michael Siegel, Russel E. Caflisch, and Sam Howison, *Global existence, singular solutions, and ill-posedness for the Muskat problem*, Comm. Pure Appl. Math. **57** (2004), no. 10, 1374–1411. MR2070208
- [49] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III. MR1232192
- [50] Spencer Tofts, *On the existence of solutions to the Muskat problem with surface tension*, J. Math. Fluid Mech. **19** (2017), no. 4, 581–611. MR3714494
- [51] Gregory Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal. **59** (1984), no. 3, 572–611. MR769382
- [52] Fahuai Yi, *Local classical solution of Muskat free boundary problem*, J. Partial Differential Equations **9** (1996), no. 1, 84–96. MR1384001

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